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MABUCHI METRICS AND PROPERNESS OF THE MODIFIED DING FUNCTIONAL

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In this paper, we study Mabuchi metrics on Fano manifolds. We prove that Mabuchi metrics exist if and only if the modified Ding functional is proper modulo the automorphism group. As an application, we establish a criterion for the existence of Mabuchi metrics on Fano group compactifications.

1. Introduction

The existence of canonical metrics has been a fundamental and longstanding problem in Kähler geometry. On Fano manifolds, Kähler–Einstein metrics have been studied extensively. The most remarkable progress is the resolution of Yau–Tian–Donaldson conjecture which relates the existence of Kähler–Einstein metrics to the K-stability of the Fano manifold [Tian 2015; Chen et al. 2015a; 2015b; 2015c]. It has been known early in the 1980s that the existence of Kähler–Einstein metrics fails when the Fano manifold has nonvanishing Futaki invariant. In this case, other canonical metrics, such as extremal metrics and Kähler–Ricci solitons have attracted much attention.

Mabuchi [2001a; 2001b; 2002; 2003] studied a generalized Kähler–Einstein metric, which is neither an extremal metric nor a Kähler–Ricci soliton. Following [Yao 2017], we call this metric the *Mabuchi metric* for simplicity. Let M be a compact Fano manifold of complex dimension n. Let

$$\omega = \sqrt{-1} g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j \in 2\pi c_1(M)$$

be a Kähler metric and h_ω be its Ricci potential. ω is a Mabuchi metric if

$$X_\omega := -\sqrt{-1} g^{i\bar{j}} \frac{\partial e^{h_\omega}}{\partial \bar{z}^j} \frac{\partial}{\partial z^i} \tag{1-1}$$

is holomorphic [Mabuchi 2001a]. The uniqueness of Mabuchi metrics has been proved in [Mabuchi 2003]. Recently, Donaldson [2017] introduced a new GIT

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(geometric invariant theory) picture, in which the corresponding moment map is given by the Ricci potential. Then Yao [2017] observed that in this picture X_ω is holomorphic if and only if ω is a critical point of the norm square of the moment map, given by the following energy

$$\mathscr{E}^D(\omega) = \int_M (e^{h_\omega} - 1)^2 \omega^n. \tag{1-2}$$

This brings new interest in the study of Mabuchi metrics. On toric Fano manifolds, the notion of relative Ding stability has been introduced by Yao [2017]. He has also established the existence of Mabuchi metrics when the toric Fano manifold is relatively Ding stable. The purpose of this paper is to discuss the existence of Mabuchi metrics on general Fano manifolds through properness of energy functionals.

According to [Mabuchi 2001a], if ω is a Mabuchi metric, then (1-1) coincides with the extremal vector field [Futaki and Mabuchi 1995]. To state the main results, we first recall notions on the extremal vector field. Denote by $\mathrm{Aut}_0(M)$ the identity component of its holomorphic transformation group. Its Lie algebra $\eta(M)$ consists of all holomorphic vector fields on M. $\mathrm{Aut}_0(M)$ admits a semidirect decomposition

$$\mathrm{Aut}_0(M) = \mathrm{Aut}_r(M) \propto R_u,$$

where $\mathrm{Aut}_r(M) \subset \mathrm{Aut}_0(M)$ is a reductive group and R_u is the unipotent radical of $\mathrm{Aut}_0(M)$. Denote by $\eta_r(M)$ the reductive part of $\eta(M)$. For any $v \in \eta(M)$, let K_v be the one parameter group generated by the image part $\mathrm{Im}(v)$. For a Kähler metric $\omega_0 \in 2\pi c_1(M)$, by Hodge theorem, there is a unique normalized potential given by

$$i_v\omega_0 = \sqrt{-1}\bar\partial\theta_v(\omega_0), \quad \int_M \theta_v(\omega_0)\omega_0^n = 0. \tag{1-3}$$

Then $\theta_v(\omega)$ is real valued if and only if ω is K_v-invariant. For any

$$\phi \in \mathscr{H}_v(\omega_0) := \{\phi \in C^\infty(M) \mid \omega_\phi := \omega_0 + \sqrt{-1}\partial\bar\partial\phi > 0,\ \phi \text{ is } K_v\text{-invariant}\},$$

the normalized potential $\theta_v(\omega_\phi) = \theta_v(\omega_0) + v(\phi)$. Denoted by $\mathrm{Fut}(v)$ the Futaki invariant of $v \in \eta(M)$. The extremal vector field X is the holomorphic vector field uniquely determined by [Futaki and Mabuchi 1995]

$$\mathrm{Fut}_X(v) := \mathrm{Fut}(v) + \int_M \theta_v(\omega_0)\theta_X(\omega_0)\omega_0^n = 0, \quad \text{for all } v \in \eta(M). \tag{1-4}$$

Moreover, $X \in \eta_c(M)$, the center of $\eta_r(M)$ and K_X lies in a compact Lie group.

From now on, we assume that ω_0 is K_X-invariant unless otherwise claimed. As pointed by Mabuchi [2003], both $\min_M \theta_X(\omega_\phi)$ and $\max_M \theta_X(\omega_\phi)$ are independent of the choice of $\omega_\phi \in 2\pi c_1(M)$. For convenience, we write

$$c_X := \min_M\{1 - \theta_X(\omega_\phi)\}, \quad C_X := \max_M\{1 - \theta_X(\omega_\phi)\}.$$

By [Mabuchi 2001a], Mabuchi metrics exist only if $c_X > 0$, and $\omega_\phi \in 2\pi c_1(M)$ is a Mabuchi metric if

$$\mathrm{Ric}(\omega_\phi) - \omega_\phi = \sqrt{-1}\partial\bar{\partial}\log(1-\theta_X(\omega_\phi)). \tag{1-5}$$

Tian [1997] introduced the notion of properness of energy functionals as an analytic characterization of existence of Kähler-Einstein metrics. When the automorphism group of M is not discrete, a notion of properness modulo a subgroup of $\mathrm{Aut}_0(M)$ was reformulated [Cao et al. 2005; Darvas and Rubinstein 2017; Tian 1996; Zhou and Zhu 2008]. In particular, Darvas and Rubinstein [2017] established a properness principle and solved Tian's properness conjecture. It is natural to ask the analogous problem for Mabuchi metrics. By [Mabuchi 2001b], the Mabuchi metric is a critical point of the following *modified Ding functional*

$$\mathfrak{D}_X(\phi) = -\frac{1}{V}\int_0^1\int_M \dot{\phi}_s(1-\theta_X(\omega_{\phi_s}))\,\omega_{\phi_s}^n \wedge ds - \log\left(\frac{1}{V}\int_M e^{h_0-\phi}\omega_0^n\right), \tag{1-6}$$

where $V = \int_M \omega_0^n$, $\{\phi_s\}_{s\in[0,1]}$ is any smooth path in $\mathcal{H}_X(\omega_0)$ joining 0 and ϕ, and h_0 is the Ricci potential of ω_0, normalized by

$$\int_M e^{h_0}\omega_0^n = \int_M \omega_0^n.$$

In view of [Cao et al. 2005; Darvas and Rubinstein 2017; Tian 1997; 1996; Zhou and Zhu 2008], we have the following definition of properness.

Definition 1.1. Suppose H^c is a reductive subgroup (which is the complexification of a compact Lie group H) of $\mathrm{Aut}_0(M)$ which contains K_X. The modified Ding functional $\mathfrak{D}_X(\cdot)$ is said to be *proper modulo H^c* if there exists an increasing function $f(t) \geq -c$ for $t \in \mathbb{R}$ and some constant $c \geq 0$ such that $\lim_{t\to+\infty} f(t) = +\infty$ and

$$\mathfrak{D}_X(\phi) \geq \inf_{\sigma\in H^c} f(I_X(\phi_\sigma) - J_X(\phi_\sigma)),$$

where ϕ_σ is defined by $\sigma^*(\omega_\phi) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi_\sigma$. I_X and J_X are modified Aubin functionals (see Section 2B).

Our first main result is the following properness theorem.

Theorem 1.2. *Suppose $c_X > 0$ and $\mathrm{Aut}_0^X(M)$ is the centralizer of K_X^c in $\mathrm{Aut}_0(M)$. Then M admits Mabuchi metrics if and only exists $C, C' > 0$, such that*

$$\mathfrak{D}_X(\phi) \geq C \inf_{\sigma\in\mathrm{Aut}_0^X(M)} J_X(\phi_\sigma) - C', \quad \textit{for all } \phi \in \mathcal{H}_X(\omega_0).$$

Remark 1.3. One can show that the existence of Mabuchi metric implies the properness of $\mathfrak{D}_X(\cdot)$ modulo the automorphism group of M following the arguments for Kähler–Ricci solitons [Cao et al. 2005]. However, the theorem gives an optimal properness can be obtained by using the properness principle of Darvas and Rubinstein [2017].

Remark 1.4. Suppose ω_0 is a Mabuchi metric on M. We can define

$$\Lambda_{1,X} = \Big\{ u \in C^\infty(M) \;\Big|\; \triangle_{\omega_0} u - \frac{X}{1-\theta_X(\omega_0)} u = -u \Big\}. \tag{1-7}$$

Then by the similar argument as in [Wang et al. 2016, Lemma 3.2], one can show that the properness modulo $\mathrm{Aut}_0^X(M)$ is equivalent to the properness for Kähler potentials that are perpendicular to $\Lambda_{1,X}$ with respect to the weighted inner product

$$(\varphi, \psi) = \int_M \varphi\psi(1-\theta_X(\omega_0))\omega_0^n.$$

The properness condition can be verified for some special Fano manifolds. A characterization for the properness of the modified Ding functional on toric Fano manifolds has been given by [Nakamura 2017]. We consider more general Fano group compactifications by using the ideas of [Li et al. 2018], in which the modified K-energy is discussed. Let G be a connected, complex reductive group of dimension n, we call M a *(biequivariant) compactification of* G if it admits a holomorphic $G \times G$ action on M with an open and dense orbit isomorphic to G as a $G\times G$-homogeneous space [Alexeev and Katzarkov 2005; Delcroix 2017a]. (M, L) is called a *polarized compactification* of G if L is a $G\times G$-linearized ample line bundle on M. In particular, when $L = -K_M$, we call M a *Fano group compactification*. We establish the criterion for the existence of Mabuchi metrics on Fano group compactifications.

Theorem 1.5. *Let $(M, -K_M)$ be a Fano compactification of G and P be the associated polytope. Then M admits Mabuchi metrics if and only if $c_X > 0$ and*

$$\boldsymbol{b}_X - 4\rho \in \Xi, \tag{1-8}$$

where

$$\boldsymbol{b}_X = \frac{1}{V}\int_{2P_+} y[1-\theta_X(y)]\pi(y)\,dy,$$

$$\pi(y) = \prod_{\alpha\in\Phi_+} \langle \alpha, y\rangle^2, \quad V = \int_{2P_+} \pi(y)\,dy,$$

Ξ is the relative interior of the cone generated by positive roots Φ_+, $\rho = \frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha$ and $\theta_X(y)$ is the normalized potential of X viewed as a function on $2P_+$, which will be described in Lemma 4.2 below. For notation on group compactifications, see Section 2C.

The paper is organized as follows: In Section 2, we first review some preliminaries on energy functionals and the definition of properness modulo an automorphism group. Then we recall basic properties of polarized compactifications. Theorem 1.2 will be proved in Section 3. In Section 4, we obtain Theorem 1.5. The sufficient part will be proved by the verification of properness of the modified Ding functional.

2. Preliminaries

In this section, we first review the notions of energy functionals associated to Mabuchi metrics. Then we recall the basic knowledge for group compactifications for later use.

2A. ***Reduction to the complex Monge–Ampère equations.*** It is clear that (1-5) is equivalent to the following equation

$$\omega_\phi^n(1-\theta_X(\omega_\phi)) = \omega_0^n e^{h_0-\phi}. \tag{2-1}$$

We consider the following continuity path

$$\omega_{\phi_t}^n(1-\theta_X(\omega_{\phi_t})) = \omega_0^n e^{h_0-t\phi_t}, \quad t\in[0,1]. \tag{2-2}$$

Let $\mathfrak{I} := \{t\in[0,1] \mid \text{(2-2) has a solution for } t\}$. Then $\mathfrak{I}$ is open by the implicit function theorem. For the starting point $t=0$, we have

Theorem 2.1. *When $c_X>0$,* (2-2) *has a solution at $t=0$.*

Since we did not find a reference for this result, we give a proof of it for completeness in the Appendix. Hence, $0\in\mathfrak{I}$ and there exists an $\epsilon_0>0$ such that (2-2) has a solution for $t\in[0,\epsilon_0]$. For the closedness of $\mathfrak{I}$, it suffices to establish the C^0-estimate of (2-2). The following lemmas will be used later.

Lemma 2.2. *Let ϕ_t be a solution of* (2-2) *at t. Then the first eigenvalue of $-L_t$ for*

$$L_t := \triangle_{\omega_{\phi_t}} - \frac{X}{1-\theta_X(\omega_{\phi_t})} + t \tag{2-3}$$

is nonnegative for $t\in[0,1]$ and equals 0 *only if $t=1$. Consequently, we have the following weighted Poincaré inequality*:

$$\int_M |\bar\partial\psi|^2_{\omega_{\phi_t}}(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n \geq t\left[\int_M \psi^2(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n - \frac{1}{V}\left(\int_M \psi(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n\right)^2\right]. \tag{2-4}$$

for any K_X-invariant $\psi\in C^{1,\alpha}$.

Remark 2.3. We remark that L_t is self-dual on the space of real-valued K_X-invariant functions, equipped with the weighted inner product (see [Mabuchi 2003, Lemma 2.1])

$$\langle f,g\rangle_t := \int_M f L_t(g)(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n.$$

Proof of Lemma 2.2. Without loss of generality, we may choose a local coframe $\{\Theta^i\}_{i=1}^n$ such that $\omega_{\phi_t} = \sqrt{-1}\sum_{i=1}^n \Theta^i \wedge \bar{\Theta}^i$. Suppose $L_t\psi = -\lambda\psi$. Then

$$\begin{aligned}(2\text{-}5)\quad \lambda \int_M &\psi_{,i}\psi_{,\bar{i}}(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n \\ &= -\int_M \Big[\Big(\triangle_{\omega_{\phi_t}} - \frac{X}{1-\theta_X(\omega_{\phi_t})} + t\Big)\psi\Big]_{,i}\psi^{,i}(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n \\ &= -\int_M \psi_{,j\bar{j}i}\psi_{,\bar{i}}(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n + \int_M X^j_{,i}\psi_{,\bar{i}}\psi_{,j}\omega_{\phi_t}^n + \int_M X^i\psi_{,ij}\psi_{,\bar{j}}\omega_{\phi_t}^n \\ &\quad + \int_M \frac{X(\psi)\theta_X(\omega_{\phi_t})_{,i}}{[1-\theta_X(\omega_{\phi_t})]^2}\psi_{,\bar{i}}(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n - t\int_M \psi_{,i}\psi_{,\bar{i}}(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n,\end{aligned}$$

here and below, we denote $\phi_{,i}$ for covariant derivatives with respect to ω_{ϕ_t}, similar conventions are used for covariant derivatives of other tensors.

By the Ricci identity and integration by parts, we have

$$\begin{aligned}-\int_M &\psi_{,j\bar{j}i}\psi_{,\bar{i}}(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n \\ &= -\int_M \psi_{,ij\bar{j}}\psi_{,\bar{i}}(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n + \int_M \psi_{,j}\psi_{,\bar{i}}\,\mathrm{Ric}_{i\bar{j}}(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n \\ &= \int_M \psi_{,ij}\psi_{,\bar{i}\bar{j}}(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n - \int_M X^i\psi_{,ij}\psi_{,\bar{j}}\omega_{\phi_t}^n \\ &\qquad + \int_M \psi_{,j}\psi_{,\bar{i}}\,\mathrm{Ric}_{i\bar{j}}(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n.\end{aligned}$$

Substituting this into (2-5) and using (2-2), it follows that

$$\begin{aligned}\lambda\int_M &\psi_{,i}\psi_{,\bar{i}}(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n \\ &= \int_M \psi_{,ij}\psi_{,\bar{i}\bar{j}}(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n + (1-t)\int_M \psi_{,\bar{i}}\psi_{,j}g_{i\bar{j}}(0)(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n,\end{aligned}$$

where $\omega_0 = \sqrt{-1}g_{i\bar{j}}(0)\Theta^i \wedge \bar{\Theta}^j$. □

2B. *Energy functionals.* Recall that the Aubin's functionals are given by

$$I(\phi) = \int_M \phi(\omega_0^n - \omega_\phi^n), \quad J(\phi) = \int_0^1\int_M \dot{\phi}_s(\omega_0^n - \omega_{\phi_s}^n)\wedge ds,$$

where $\{\phi_s\}_{s\in[0,1]}$ is any smooth path in $\mathcal{H}_X(\omega_0)$ joining 0 and ϕ. It is known [Tian 2012] that

$$(2\text{-}6)\qquad 0 \le \frac{1}{n}J(\phi) \le I(\phi) - J(\phi) \le nJ(\phi).$$

To deal with Mabuchi metrics, the following modified functionals were introduced in [Mabuchi 2003]:

$$I_X(\phi) = \int_M \phi[(1-\theta_X(\omega_0))\omega_0^n - (1-\theta_X(\omega_\phi))\omega_\phi^n],$$

$$J_X(\phi) = \int_0^1 \int_M \dot{\phi}_s\big[(1-\theta_X(\omega_0))\omega_0^n - (1-\theta_X(\omega_{\phi_s}))\omega_{\phi_s}^n\big] \wedge ds.$$

By [Mabuchi 2003, Remark A.1.9], when $c_X > 0$,

$$0 \le I_X(\phi) \le (n+2)(I_X(\phi) - J_X(\phi)) \le (n+1)I_X(\phi). \tag{2-7}$$

Lemma 2.4. *There are positive constants* $c_1, c_2 > 0$ *such that*

$$c_1 I(\phi) \le I_X(\phi) - J_X(\phi) \le c_2 I(\phi). \tag{2-8}$$

Proof. Take a path $\phi_s = s\phi$. Then

$$\begin{aligned}\frac{d}{ds}[I_X(\phi_s) - J_X(\phi_s)] &= -s\int_M \phi\cdot\left(\triangle_{\omega_{\phi_s}} - \frac{X}{1-\theta_X(\omega_{\phi_s})}\right)\phi\cdot(1-\theta_X(\omega_{\phi_s}))\,\omega_{\phi_s}^n \\ &= s\int_M |\partial\phi|^2_{\omega_{\phi_s}}(1-\theta_X(\omega_{\phi_s}))\,\omega_{\phi_s}^n,\end{aligned}$$

Note that

$$\frac{d}{ds}[I(\phi_s) - J(\phi_s)] = s\int_M |\partial\phi|^2_{\omega_{\phi_s}}\,\omega_{\phi_s}^n.$$

When $c_X > 0$, it follows that

$$0 \le c_X\frac{d}{ds}[I(\phi_s) - J(\phi_s)] \le \frac{d}{ds}[I_X(\phi_s) - J_X(\phi_s)] \le C_X\frac{d}{ds}[I(\phi_s) - J(\phi_s)].$$

Thus the lemma follows from (2-6). □

For convenience, we write the modified Ding functional (1-6) as $\mathcal{D}_X(\phi) = \mathcal{N}(\phi) + \mathcal{D}_X^0(\phi)$, where

$$\mathcal{N}(\phi) = -\log\left(\frac{1}{V}\int_M e^{h_0-\phi}\omega_0^n\right), \tag{2-9}$$

$$\mathcal{D}_X^0(\phi) = -\frac{1}{V}\int_0^1\int_M \dot{\phi}_s(1-\theta_X(\omega_{\phi_s}))\,\omega_{\phi_s}^n \wedge ds. \tag{2-10}$$

It is known that $\mathcal{N}(\cdot)$ is convex with respect to geodesics [Berndtsson 2015]. In the later proof of Theorem 1.2, we need the convexity of $\mathcal{D}_X(\cdot)$.

Lemma 2.5. *The functional* $\mathcal{D}_X^0(\cdot)$ *satisfies*:

(1) *When* $c_X > 0$, $\mathcal{D}_X^0(\cdot)$ *is monotonic, that is for any* $\phi_0 \le \phi_1$, $\mathcal{D}_X^0(\phi_0) \ge \mathcal{D}_X^0(\phi_1)$.

(2) $\mathcal{D}_X^0(\cdot)$ *is affine along any* $C^{1,1}$*-geodesic connecting two smooth potentials in* $\mathcal{H}_X(\omega_0)$.

Proof. To see (1), by definition we have

$$\mathcal{D}^0_X(\phi_1)=\mathcal{D}^0_X(\phi_0)-\frac{1}{V}\int_0^1\int_M\dot\phi_s(1-\theta_X(\omega_{\phi_s}))\,\omega^n_{\phi_s}\wedge ds,$$

where ϕ_s is any smooth path in $\mathcal{H}_X(\omega_0)$ joining ϕ_0 and ϕ_1. Take in particular $\phi_s=s(\phi_1-\phi_0)+\phi_0$ and note that $c_X>0$; we have

$$\mathcal{D}^0_X(\phi_1)=\mathcal{D}^0_X(\phi_0)-\frac{1}{V}\int_0^1\int_M(\phi_1-\phi_0)(1-\theta_X(\omega_{\phi_s}))\,\omega^n_{\phi_s}\wedge ds\le\mathcal{D}^0_X(\phi_0).$$

Next we prove (2). Let $\{\phi_t\}$ be the $C^{1,1}$-geodesic connecting $\phi_0,\phi_1\in\mathcal{H}_X(\omega_0)$. By [Chen 2000], $\{\phi_t\}$ can be approximated by a family of smooth ϵ-geodesic $\{\phi^\epsilon_t\mid t\in\Omega\}$ in $\mathcal{H}_X(\omega_0)$ connecting ϕ_0 and ϕ_1, satisfying

$$\left(\frac{\partial^2}{\partial\tau\partial\bar\tau}\phi^\epsilon_t-\left|\bar\partial\left(\frac{\partial\phi^\epsilon_t}{\partial\tau}\right)\right|^2_{\omega_{\phi_t}}\right)(\omega_0+\sqrt{-1}\partial\bar\partial\phi^\epsilon_t)^n=\epsilon\cdot\omega_0^n,\tag{2-11}$$

on $M\times\Omega$, where $\Omega:=[0,1]\times S^1\subset\mathbb{C}$ and $t=\mathrm{Re}(\tau)$. For each ϵ, we have

$$\frac{\partial}{\partial\tau}\mathcal{D}^0_X(\phi^\epsilon_t)=-\frac{1}{V}\int_M\frac{\partial}{\partial\tau}\phi^\epsilon_t(1-\theta_X(\omega_{\phi^\epsilon_t}))\,\omega^n_{\phi^\epsilon_t}.$$

It follows that

$$\begin{aligned}\frac{\partial^2}{\partial\tau\partial\bar\tau}\mathcal{D}^0_X(\phi^\epsilon_t)=&-\frac{1}{V}\int_M\frac{\partial^2\phi^\epsilon_t}{\partial\tau\partial\bar\tau}(1-\theta_X(\omega_{\phi^\epsilon_t}))\,\omega^n_{\phi^\epsilon_t}\\&+\frac{1}{V}\int_M\frac{\partial\phi^\epsilon_t}{\partial\tau}\frac{\partial\theta_X(\omega_{\phi^\epsilon_t})}{\partial\bar\tau}\,\omega^n_{\phi^\epsilon_t}\\&-\frac{\sqrt{-1}}{V}\int_M\frac{\partial\phi^\epsilon_t}{\partial\tau}(1-\theta_X(\omega_{\phi^\epsilon_t}))\,n\omega^{n-1}_{\phi^\epsilon_t}\wedge\partial\bar\partial\frac{\partial\phi^\epsilon_t}{\partial\bar\tau}.\end{aligned}\tag{2-12}$$

Recall that $\theta_X(\omega_{\phi^\epsilon_t})=\theta_X(\omega_0)+X(\phi^\epsilon_t)$. One gets

$$\frac{1}{V}\int_M\frac{\partial\phi^\epsilon_t}{\partial\tau}\frac{\partial\theta_X(\omega_{\phi^\epsilon_t})}{\partial\bar\tau}\,\omega^n_{\phi^\epsilon_t}=\frac{1}{V}\int_M\frac{\partial\phi^\epsilon_t}{\partial\tau}X^i\left(\frac{\partial\phi^\epsilon_t}{\partial\bar\tau}\right)_{,i}\omega^n_{\phi^\epsilon_t}.$$

On the other hand, by integration by parts, we have

$$\begin{aligned}&\frac{\sqrt{-1}}{V}\int_M\frac{\partial\phi^\epsilon_t}{\partial\tau}(1-\theta_X(\omega_{\phi^\epsilon_t}))\,n\omega^{n-1}_{\phi^\epsilon_t}\wedge\partial\bar\partial\frac{\partial\phi^\epsilon_t}{\partial\bar\tau}\\&=\frac{\sqrt{-1}}{V}\left[\int_M\bar\partial\frac{\partial\phi^\epsilon_t}{\partial\tau}(1-\theta_X(\omega_{\phi^\epsilon_t}))\,n\omega^{n-1}_{\phi^\epsilon_t}\wedge\partial\frac{\partial\phi^\epsilon_t}{\partial\bar\tau}-\int_M\frac{\partial\phi^\epsilon_t}{\partial\tau}\bar\partial\theta_X(\omega_{\phi^\epsilon_t})\,n\omega^{n-1}_{\phi^\epsilon_t}\wedge\partial\frac{\partial\phi^\epsilon_t}{\partial\bar\tau}\right]\\&=-\frac{1}{V}\int_M\left|\bar\partial\left(\frac{\partial\phi^\epsilon_t}{\partial\tau}\right)\right|^2_{\omega_{\phi_t}}(1-\theta_X(\omega_{\phi^\epsilon_t}))\,\omega^n_{\phi^\epsilon_t}+\frac{1}{V}\int_M\frac{\partial\phi^\epsilon_t}{\partial\tau}X^i\left(\frac{\partial\phi^\epsilon_t}{\partial\bar\tau}\right)_{,i}\omega^n_{\phi^\epsilon_t}.\end{aligned}$$

Plugging these into (2-12), by (2-11), we have

$$\frac{\partial^2}{\partial\tau\partial\bar\tau}\mathcal{D}^0_X(\phi^\epsilon_t)=-\epsilon<0.$$

Thus $\mathfrak{D}^0_X(\cdot)$ is concave along ϕ^ϵ_t. Let $\epsilon \to 0$. Then $\sqrt{-1}\partial\bar{\partial}_\tau\mathfrak{D}^0_X(\phi^\epsilon_t)$ converges weakly to $\sqrt{-1}\partial\bar{\partial}_\tau\mathfrak{D}^0_X(\phi_t)$ as Monge–Ampère measures. It follows that

$$\sqrt{-1}\partial\bar{\partial}_\tau\mathfrak{D}^0_X(\phi_t) = 0;$$

thus $\mathfrak{D}^0_X(\phi_t)$ is affine as desired. □

Remark 2.6. Indeed, one can improve Lemma 2.5(2) to any bounded geodesic in finite energy spaces $\mathcal{E}^1_{K_X}(M)$. See [Berman and Witt Nyström 2014, Proposition 2.17], where we take $g(t) = 1 - t$ in their settings.

2C. ***Group compactifications.*** As an application of Theorem 1.2, we will study the existence of Mabuchi metrics on group compactifications by testing properness of the modified Ding functional. The existence of Kähler–Einstein metrics on these manifolds has been solved by [Delcroix 2017a] by using the continuity method, while the properness of K-energy was studied in [Li et al. 2018]. We will prove Theorem 1.5 by ideas therein later. In this subsection, we recall some facts of group compactifications from [Delcroix 2017a; Li et al. 2018].

2C1. *Notation on Lie groups.* Choose a maximal compact subgroup K of G such that G is its complexification. Let T be a chosen maximal torus of K and T^c its complexification. Then T^c is the maximal algebraic torus of G. Denote their Lie algebras by the corresponding Fraktur lower case letters. Assume that Φ is the root system of (G, T^C) and W is the Weyl group. Choose a set of positive roots Φ_+. Set $\rho = \frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha$ and let Ξ be the relative interior of the cone generated by Φ_+. Let J be the complex structure of G. Then

$$\mathfrak{g} = \mathfrak{k} \oplus J\mathfrak{k}.$$

Set $\mathfrak{a} = J\mathfrak{t}$, it can be decomposed as a toric part and a semisimple part:

$$\mathfrak{a} = \mathfrak{a}_t \oplus \mathfrak{a}_{ss},$$

where $\mathfrak{a}_t := \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{a}$ and $\mathfrak{a}_{ss} := \mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}]$. We extend the Killing form on $\mathfrak{a}_{ss}$ to a scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}$ such that $\mathfrak{a}_t$ is orthogonal to $\mathfrak{a}_{ss}$. The positive roots Φ_+ define a positive Weyl chamber $\mathfrak{a}_+ \subset \mathfrak{a}$, and a positive Weyl chamber $\mathfrak{a}^*_+$ on $\mathfrak{a}^*$, where

$$\mathfrak{a}^*_+ := \{y \mid \alpha(y) := \langle\alpha, y\rangle > 0,\ \forall\alpha \in \Phi_+\},$$

it coincides with the dual of $\mathfrak{a}_+$ under $\langle\cdot,\cdot\rangle$. For later use, we fix a Lebesgue measure dy on $\mathfrak{a}^*$ which is normalized by the lattice of the characters of T^c.

2C2. *$K\times K$-invariant Kähler metrics.* Let Z be the closure of T^c in M. It is known that $(Z, L|_Z)$ is a polarized toric manifold with a W-action, and $L|_Z$ is a W-linearized ample toric line bundle on Z [Alexeev and Brion 2004a; 2004b; Alexeev and Katzarkov 2005; Delcroix 2017a]. Let $\omega_0 \in 2\pi c_1(L)$ be a $K\times K$-invariant

Kähler form induced from (M, L) and P be the polytope associated to $(Z, L|_Z)$, which is defined by the moment map associated to ω_0. Then P is a W-invariant Delzant polytope in $\mathfrak{a}^*$. By the $K\times K$-invariance, for any

$$\phi \in \mathcal{H}_{K\times K}(\omega_0) := \left\{\phi \in C^\infty(M) \mid \omega_\phi > 0,\ \phi \text{ is } K\times K\text{-invariant}\right\},$$

the restriction of ω_ϕ on Z is a toric Kähler metric. It induces a smooth strictly convex function ψ_ϕ on $\mathfrak{a}$, which is W-invariant [Azad and Loeb 1992; Delcroix 2017a].

By the KAK-decomposition [Knapp 1996, Theorem 7.39], for any $g \in G$, there are $k_1, k_2 \in K$ and $x \in \mathfrak{a}$ such that $g = k_1 \exp(x) k_2$. Here x is uniquely determined up to a W-action. This means that x is unique in $\bar{\mathfrak{a}}_+$. Thus there is a bijection between smooth $K\times K$-invariant functions Ψ on G and smooth W-invariant functions on $\mathfrak{a}$ which is given by

$$\Psi(\exp(\cdot)) = \psi(\cdot) : \mathfrak{a} \to \mathbb{R}.$$

Clearly when a W-invariant ψ is given, Ψ is well-defined. In the following, we will not distinguish ψ and Ψ. The following KAK-integral formula can be found in [Knapp 1986, Proposition 5.28] (see also [Hu and Yan 2005]).

Proposition 2.7. *Let dV_G be a Haar measure on G and dx the Lebesgue measure on $\mathfrak{a}$. Then there exists a constant $C_H > 0$ such that for any $K\times K$-invariant, dV_G-integrable function ψ on G,*

$$\int_G \Psi(g)\, dV_G = C_H \int_{\mathfrak{a}_+} \psi(x) \boldsymbol{J}(x)\, dx,$$

where $\boldsymbol{J}(x) = \prod_{\alpha\in\Phi_+} \sinh^2 \alpha(x)$.

Without loss of generality, we can normalize $C_H = 1$ for simplicity.

Next we recall local holomorphic coordinates on G used in [Delcroix 2017a]. By the standard Cartan decomposition, we can decompose $\mathfrak{g}$ as

$$\mathfrak{g} = (\mathfrak{t} \oplus \mathfrak{a}) \oplus \left(\textstyle\bigoplus_{\alpha\in\Phi} V_\alpha\right),$$

where $V_\alpha = \{X \in \mathfrak{g} \mid \mathrm{ad}_H(X) = \alpha(H)X,\ \forall H \in \mathfrak{t}\oplus\mathfrak{a}\}$, the root space of complex dimension 1 with respect to α. By [Helgason 1978], one can choose $X_\alpha \in V_\alpha$ such that $X_{-\alpha} = -\iota(X_\alpha)$ and $[X_\alpha, X_{-\alpha}] = \alpha^\vee$, where ι is the Cartan involution and $\alpha^\vee$ is the dual of α by the Killing form. Let $E_\alpha := X_\alpha - X_{-\alpha}$ and $E_{-\alpha} := J(X_\alpha + X_{-\alpha})$. Denote by $\mathfrak{k}_\alpha$, $\mathfrak{k}_{-\alpha}$ the real line spanned by E_α, $E_{-\alpha}$, respectively. Then we have the Cartan decomposition of $\mathfrak{k}$:

$$\mathfrak{k} = \mathfrak{t} \oplus \left(\textstyle\bigoplus_{\alpha\in\Phi_+} (\mathfrak{k}_\alpha \oplus \mathfrak{k}_{-\alpha})\right).$$

Denote by r the dimension of T, and choose a real basis $\{E_1^0, \dots, E_r^0\}$ of $\mathfrak{t}$. Then $\{E_1^0, \dots, E_r^0\}$ together with $\{E_\alpha, E_{-\alpha}\}_{\alpha\in\Phi_+}$ forms a real basis of $\mathfrak{k}$, which is indexed by $\{E_1, \dots, E_n\}$, which can also be regarded as a complex basis of $\mathfrak{g}$. For any $g \in G$, we define local coordinates $\{z^i_{(g)}\}_{i=1,\dots,n}$ on a neighborhood of g by

$$(z^i_{(g)}) \to \exp(z^i_{(g)} E_i) g.$$

It is easy to see that $\theta^i|_g = dz^i_{(g)}|_g$, where θ^i is the dual of E_i, which is a right-invariant holomorphic 1-form. Thus $\bigwedge_{i=1}^n (dz^i_{(g)} \wedge d\bar{z}^i_{(g)})|_g$ is also a right-invariant (n, n)-form, which defines a Haar measure dV_G.

The derivations of the $K \times K$-invariant function ψ in the above local coordinates was computed by Delcroix [2017a, Theorem 1.2] as follows.

Lemma 2.8. *Let ψ be a $K \times K$ invariant function on G. Then for any $x \in \mathfrak{a}_+$,*

$$E^0_i(\psi)|_{\exp(x)} = d\psi(\mathrm{Im}(E^0_i))|_x, \quad 1 \le i \le r, \qquad E_{\pm\alpha}(\psi)|_{\exp(x)} = 0.$$

Lemma 2.9. *Let ψ be a $K \times K$-invariant function on G. Then for any $x \in \mathfrak{a}_+$, the complex Hessian matrix of ψ in the above coordinates is diagonal by blocks, and equal to*

$$\mathrm{Hess}_{\mathbb{C}}(\psi)(\exp(x)) = \begin{pmatrix} \frac{1}{4}\mathrm{Hess}_{\mathbb{R}}(\psi)(x) & 0 & & 0 \\ 0 & M_{\alpha_{(1)}}(x) & & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & M_{\alpha_{((n-r)/2)}}(x) \end{pmatrix}, \tag{2-13}$$

where $\Phi_+ = \{\alpha_{(1)}, \dots, \alpha_{((n-r)/2)}\}$ is the set of positive roots and

$$M_{\alpha_{(i)}}(x) = \frac{1}{2}\langle \alpha_{(i)}, \nabla\psi(x)\rangle \begin{pmatrix} \coth \alpha_{(i)}(x) & \sqrt{-1} \\ -\sqrt{-1} & \coth \alpha_{(i)}(x) \end{pmatrix}.$$

By (2-13) in Lemma 2.9, we see that a ψ induced by some ω_ϕ is convex on $\mathfrak{a}$. The complex Monge–Ampère measure is given by $\omega^n = (\sqrt{-1}\partial\bar{\partial}\psi_\phi)^n = \mathrm{MA}_{\mathbb{C}}(\psi_\phi)\, dV_G$, where

$$\mathrm{MA}_{\mathbb{C}}(\psi_\phi)(\exp(x)) = \frac{1}{2^{r+n}} \mathrm{MA}_{\mathbb{R}}(\psi_\phi)(x) \frac{1}{\boldsymbol{J}(x)} \prod_{\alpha \in \Phi_+} \langle \alpha, \nabla\psi_\phi(x)\rangle^2. \tag{2-14}$$

2C3. *Legendre functions.* By the convexity of ψ_ϕ on $\mathfrak{a}$, the gradient $\nabla\psi_\phi$ defines a diffeomorphism from $\mathfrak{a}$ to the interior of the dilated polytope $2P$.[1] Let $P_+ := P \cap \bar{\mathfrak{a}}^*_+$. Then by the W-invariance of ψ_ϕ and P, the restriction of $\nabla\psi_\phi$ on $\mathfrak{a}_+$ is a diffeomorphism from $\mathfrak{a}_+$ to the interior of $2P_+$. Let u_G be the standard Guillemin function on $2P$ [Guillemin 1994]. Set

$$\mathscr{C}_W = \big\{ u \mid u \text{ is strictly convex},\ u - u_G \in C^\infty(\overline{2P}) \text{ and } u \text{ is } W\text{-invariant} \big\}.$$

[1] We remark that the moment map is given by $\frac{1}{2}\nabla\psi_\phi$, whose image is P.

It is known that for any $K\times K$-invariant $\omega=\sqrt{-1}\partial\bar\partial\psi\in 2\pi c_1(L)$, its Legendre function u is given by

$$u(y(x))=x^i y_i(x)-\psi(x),\quad y_i(x)=\psi_{,i}(x)=\frac{\partial\psi}{\partial x_i} \tag{2-15}$$

and is a function in $\mathscr{C}_W$ (see [Abreu 1998]). By a similar argument to that in [Guan 1999] for toric manifolds, we have:

Lemma 2.10. *For any $\phi_0,\phi_1\in\mathscr{H}_{K\times K}(\omega_0)$, there exists a geodesic $\{\phi_t\}_{t\in[0,1]}$ in $\mathscr{H}_{K\times K}(\omega_0)$ joining them, and the Legendre function of ψ_ϕ is given by*

$$u_{\phi_t}=(1-t)u_{\phi_0}+tu_{\phi_1}.$$

3. Proof of the properness theorem

We always assume $c_X>0$ in this section.

3A. We first prove the properness modulo an arbitrary reductive subgroup $H^c$of $\mathrm{Aut}_0(M)$ which contains K_X implies the existence of Mabuchi metrics. It will be proved by steps as for Kähler–Ricci solitons [Cao et al. 2005; Tian and Zhu 2000].

First, we have:

Lemma 3.1. *Let ϕ_t be a solution of* (2-2) *at t. If $I_X(\phi_t)$ is uniformly bounded, then there is a uniform constant C such that*

$$|\phi_t|\le C,\quad \textit{for all } t\in[0,1].$$

Proof. This estimate was essentially obtained in [Mabuchi 2003]. Here we will give a different proof following the arguments of [Tian and Zhu 2000]. In view of Kołodziej's L^∞-estimate [1998] for the complex Monge–Ampère equation, it suffices to obtain the L^p-estimate of $e^{-t\phi_t}$ for some $p>1$.

By the assumption, $0\le I_X(\phi_t)\le C_1$ for some uniform C_1. By (2-2), we have

$$\int_M e^{h_0-t\phi_t}\omega_0^n=\int_M(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n=\int_M e^{h_0}\omega_0^n,$$

thus

$$\inf_M\phi_t\le 0\le\sup_M\phi_t.$$

While by (2-2),

$$-t\int_M\phi_t\omega_{\phi_t}^n=-t\int_M\phi_t\frac{e^{h_0-t\phi_t}}{1-\theta_X(\omega_{\phi_t})}\,\omega_0^n\ge -C_2t\int_{\{\phi_t\ge 0\}}\phi_t e^{-t\phi_t}\omega_0^n\ge -C_3.$$

Thus

$$t\int_M\phi_t\,\omega_0^n\le C_4. \tag{3-1}$$

Let $\Gamma(\cdot,\cdot)$ be the Green function of ω_0. Then by $\triangle_{\omega_0}\phi_t > -n,\ \Gamma + C_\Gamma \geq 0$ for some $C_\Gamma > 0$. By (3-1) and Green's formula, we have

$$t\sup_M \phi_t \leq \frac{t}{V}\int_M \phi_t\,\omega_0^n - \frac{t}{V}\min_M\biggl(\int_M (\Gamma(x,\cdot)+C_\Gamma)\triangle_{\omega_0}\phi_t\,\omega_0^n\biggr) \leq C_5. \tag{3-2}$$

By the boundedness of $I_X(\phi_t)$, we have

$$-\frac{1}{V}\int_M \phi_t\,\omega_{\phi_t}^n \leq C_1 - \frac{1}{V}\int_M \phi_t\,\omega_0^n \leq C_6. \tag{3-3}$$

Moreover,

$$\begin{aligned} -t\int_{\{\phi_t\leq 0\}} \phi_t\,\omega_{\phi_t}^n &= -t\int_M \phi_t\,\omega_{\phi_t}^n + t\int_{\{\phi_t\geq 0\}} \phi_t\,\omega_{\phi_t}^n \\ &\leq tVC_6 + t\int_{\{\phi_t\geq 0\}} \phi_t \frac{e^{h_0-t\phi_t}}{1-\theta_X(\omega_{\phi_t})}\,\omega_0^n \leq C_7. \end{aligned} \tag{3-4}$$

By (3-2), there is a uniform $C > 0$ such that $\hat{\phi}_t := \phi_t - C/t \leq -1$. By (3-4), it follows that

$$-t\int_M \phi_t(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n \leq -t\int_{\{\phi_t\leq 0\}} \phi_t(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n \leq C_X C_7,$$

and consequently,

$$-t\int_M \hat{\phi}_t(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n \leq C_8. \tag{3-5}$$

On the other hand,

$$\begin{aligned} \int_M \Bigl|\bar{\partial}(-\hat{\phi}_t)^{\frac{p+1}{2}}\Bigr|^2_{\omega_{\phi_t}} \omega_{\phi_t}^n &= \frac{n(p+1)^2}{4p}\int_M (-\hat{\phi}_t)^p(\omega_{\phi_t}^n - \omega_{\phi_t}^{n-1}\wedge\omega_0) \\ &\leq \frac{n(p+1)^2}{4p}\int_M (-\hat{\phi}_t)^p\,\omega_{\phi_t}^n. \end{aligned}$$

Recall that $0 < c_X < 1-\theta_X(\omega_{\phi_t}) < C_X$. Combining the above inequality with (2-4),

$$\begin{aligned} &\int_M (-\hat{\phi}_t)^{p+1}(1-\theta_X(\omega_{\phi_t}))\omega_{\phi_t}^n \\ &\leq \frac{Cp}{t}\int_M (-\hat{\phi}_t)^p(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n + \frac{1}{V}\biggl(\int_M (-\hat{\phi}_t)^{(p+1)/2}(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n\biggr)^2 \\ &\leq \frac{Cp}{t}\int_M (-\hat{\phi}_t)^p(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n \\ &\qquad + \frac{1}{V}\biggl(\int_M (-\hat{\phi}_t)^p(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n\biggr)\biggl(\int_M (-\hat{\phi}_t)(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n\biggr) \\ &\leq \frac{C'p}{t}\int_M (-\hat{\phi}_t)^p(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n, \end{aligned}$$

where we used (3-5) in the last line. By iteration and using (3-5), we have

$$\int_M (-\hat{\phi}_t)^{p+1}(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n \le \frac{C'^p(p+1)!}{t^p}\int_M (-\hat{\phi}_t)(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n \le \frac{c^{p+1}(p+1)!}{t^{p+1}}.$$

Thus for $0<\epsilon<1/c$,

$$\int_M e^{-t\epsilon\hat{\phi}_t}(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n = \sum_{p=0}^{+\infty}\frac{(t\epsilon)^p}{p!}\int_M (-\hat{\phi}_t)^p(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n \le \frac{1}{1-c\epsilon}.$$

It follows that

$$\int_M e^{-t(1+\epsilon)\phi_t}\omega_0^n = \int_M e^{-t(1+\epsilon)\phi_t}e^{-h_0-t\phi_t}(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n \le C_9\int_M e^{-t\epsilon\hat{\phi}_t}(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n \le C.$$

Then the lemma then follows from Kołodziej's result. □

Lemma 3.2 [Li 2007]. *Fix $\epsilon_0\in(0,1)$. Then the modified Ding functional $\mathfrak{D}_X(\phi_t)$ is uniformly bounded from above for $t>\epsilon_0$.*

The proof follows from the above and the next lemmas.

Lemma 3.3. *For any solution ϕ_t of* (2-2) *with $t<1$,*

$$\min_{\sigma\in H^c}\{I_X((\phi_t)_\sigma)-J_X((\phi_t)_\sigma)\} = I_X(\phi_t)-J_X(\phi_t).$$

Proof. We will use the argument of Tian [2012] to prove this lemma. For any $Y\in\mathfrak{h}^c$, let $\sigma(s)$ be the one parameter group generated by $\mathrm{Re}(Y)$ with $\sigma(0)=\mathrm{id}$. For a solution ϕ_t of (2-2), set $\phi_{t\,s}=(\phi_t)_{\sigma(s)}$. Note that (2-2) is equivalent to

$$h_t+(1-t)\phi_t=\log(1-\theta_X(\omega_{\phi_t}))+c_t,$$

where h_t is the normalized Ricci potential of ω_{ϕ_t} and c_t is a constant depending on t. Thus

$$\begin{aligned}(3\text{-}6)\quad &\frac{\partial}{\partial s}\Big|_{s=0}(I_X-J_X)(\phi_{t\,s})\\ &=\int_M \frac{\partial}{\partial s}\Big|_{s=0}\phi_{t\,s,\bar{k}}\phi_{t,i}g^{i\bar{k}}(1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n\\ &=-\frac{1}{1-t}\int_M Y^i\left[h_{t,i}+\frac{\theta_X(\omega_{\phi_t})_{,i}}{1-\theta_X(\omega_{\phi_t})}\right](1-\theta_X(\omega_{\phi_t}))\,\omega_{\phi_t}^n\\ &=-\frac{1}{1-t}\int_M Y(h_t)\omega_{\phi_t}^n+\frac{1}{1-t}\left(\int_M Y(h_t)\theta_X(\omega_{\phi_t})\,\omega_{\phi_t}^n-\int_M Y^i\theta_X(\omega_{\phi_t})_{,i}\,\omega_{\phi_t}^n\right).\end{aligned}$$

Recall that $\theta_Y(\omega_{\phi_t})$ satisfies

$$\triangle_{\omega_{\phi_t}}\theta_Y(\omega_{\phi_t})+Y(h_t)+\theta_Y(\omega_{\phi_t})=\text{const.},$$

thus

$$\int_M Y(h_t)\theta_X(\omega_{\phi_t})\,\omega_{\phi_t}^n=-\int_M\theta_X(\omega_{\phi_t})\theta_Y(\omega_{\phi_t})\,\omega_{\phi_t}^n-\int_M\theta_X(\omega_{\phi_t})\triangle_{\omega_{\phi_t}}\theta_Y(\omega_{\phi_t})\,\omega_{\phi_t}^n.$$

Substituting this into (3-6) and by integration by parts, yields

$$\frac{\partial}{\partial s}\Big|_{s=0}(I_X-J_X)(\phi_{t\,s})=-\frac{1}{1-t}\int_M Y(h_t)\,\omega_{\phi_t}^n-\frac{1}{1-t}\int_M\theta_X(\omega_{\phi_t})\theta_Y(\omega_{\phi_t})\,\omega_{\phi_t}^n \\ =0. \tag{3-7}$$

The last equality follows from (1-4). This shows that $s=0$ is a critical point of $(I_X-J_X)(\phi_{t\,s})$.

To prove the lemma, it suffices to show that $(I_X-J_X)(\phi_{t\,s})$ is convex with respect to s. It is direct to check that

$$\frac{\partial^2}{\partial s^2}\phi_{t\,s}=\left|\bar\partial\left(\frac{\partial}{\partial s}\phi_{t\,s}\right)\right|^2_{\omega_{\phi_{t\,s}}}. \tag{3-8}$$

Thus $\phi_{t\,s}$ gives a geodesic in the space of Kähler potentials. In the following, we denote $\phi_s=\phi_{t\,s}$ for fixed t for simplicity and $\omega_{\phi_{t\,s}}=\sqrt{-1}g_{i\bar j}(s)dz^i\wedge d\bar z^j$. Then

$$\frac{\partial}{\partial s}\triangle_{\omega_{\phi_s}}\phi_s=-g^{i\bar k}g^{\bar jl}\cdot\phi_{s,\bar kl}\phi_{s,\bar ji}+\triangle_{\omega_{\phi_s}}\dot\phi_s. \tag{3-9}$$

Note that

$$\frac{d}{ds}(I_X-J_X)(\phi_s)=-\int_M\dot\phi_s\triangle_{\omega_{\phi_s}}\phi_s(1-\theta_X(\omega_{\phi_s}))\,\omega_{\phi_s}^n+\int_M\dot\phi_s X(\phi_s)\,\omega_{\phi_s}^n. \tag{3-10}$$

We want to differentiate the above equality. For the first term, we have by (3-9)

$$\begin{aligned}&\frac{d}{ds}\int_M\dot\phi_s\triangle_{\omega_{\phi_s}}\phi_s(1-\theta_X(\omega_{\phi_s}))\,\omega_{\phi_s}^n\\&=\int_M\ddot\phi_s\triangle_{\omega_{\phi_s}}\phi_s(1-\theta_X(\omega_{\phi_s}))\,\omega_{\phi_s}^n-\int_M X^i\dot\phi_{s,i}\triangle_{\omega_{\phi_s}}\phi_s\dot\phi_s\,\omega_{\phi_s}^n\\&\quad+\int_M\dot\phi_s\triangle_{\omega_{\phi_s}}\dot\phi_s\triangle_{\omega_{\phi_s}}\phi_s(1-\theta_X(\omega_{\phi_s}))\,\omega_{\phi_s}^n+\int_M\dot\phi_s\triangle_{\omega_{\phi_s}}\dot\phi_s(1-\theta_X(\omega_{\phi_s}))\,\omega_{\phi_s}^n\\&\quad-\int_M\dot\phi_s(\dot\phi_{s,l\bar k}\phi_{s,i\bar j})g^{i\bar k}g^{l\bar j}(1-\theta_X(\omega_{\phi_s}))\,\omega_{\phi_s}^n.\end{aligned}$$

Substituting (3-8) into the first term and by integration by parts, it follows that

$$\begin{aligned}
(3\text{-}11)\quad & \frac{d}{ds}\int_M \dot\phi_s \triangle_{\omega_{\phi_s}} \phi_s (1-\theta_X(\omega_{\phi_s}))\,\omega^n_{\phi_s} \\
&= -\int_M \dot\phi_s \dot\phi_{s,i} (\triangle_{\omega_{\phi_s}}\phi_s)_{,\bar k} g^{i\bar k}(1-\theta_X(\omega_{\phi_s}))\,\omega^n_{\phi_s} \\
&\qquad - \int_M \dot\phi_s \dot\phi_{s,l\bar k}\phi_{s,i\bar j} g^{i\bar k} g^{l\bar j}(1-\theta_X(\omega_{\phi_s}))\,\omega^n_{\phi_s} \\
&\qquad\qquad + \int_M \dot\phi_s \triangle_{\omega_{\phi_s}} \dot\phi_s (1-\theta_X(\omega_{\phi_s}))\,\omega^n_{\phi_s} \\
&= \int_M \dot\phi_{s,l}\dot\phi_{s,\bar k}\phi_{s,i\bar j} g^{l\bar j} g^{i\bar k}(1-\theta_X(\omega_{\phi_s}))\,\omega^n_{\phi_s} - \int_M \dot\phi_s X^i \dot\phi_{s,l}\phi_{s,i\bar j} g^{l\bar j}\,\omega^n_{\phi_s} \\
&\qquad\qquad + \int_M \dot\phi_s \triangle_{\omega_{\phi_s}} \dot\phi_s (1-\theta_X(\omega_{\phi_s}))\,\omega^n_{\phi_s} \\
&= \int_M \dot\phi_{s,l}\dot\phi_{s,\bar k}(\phi_{s,i\bar j} - g_{i\bar j}) g^{i\bar k} g^{l\bar j}(1-\theta_X(\omega_{\phi_s}))\,\omega^n_{\phi_s} \\
&\qquad\qquad - \int_M \dot\phi_s X^i \dot\phi_{s,l}\phi_{s,i\bar j} g^{l\bar j}\,\omega^n_{\phi_s} + \int_M X^i \dot\phi_{s,i}\dot\phi_s\,\omega^n_{\phi_s},
\end{aligned}$$

where $g_{i\bar j} = g_{i\bar j}(s)$. The second term in (3-10) gives

$$\frac{d}{ds}\int_M \dot\phi_s X^i \phi_{s,i}\,\omega^n_{\phi_s} = \int_M \ddot\phi_s X^i \phi_{s,i}\,\omega^n_{\phi_s} + \int_M \dot\phi_s X^i \dot\phi_{s,i}\,\omega^n_{\phi_s} + \int_M \dot\phi_s X^i \phi_{s,i} \triangle_{\omega_{\phi_s}} \dot\phi_s\,\omega^n_{\phi_s}.$$

Substituting (3-8) into the above equality and by integration by parts again, we have

$$(3\text{-}12)\quad \frac{d}{ds}\int_M \dot\phi_s X^i \phi_{s,i}\,\omega^n_{\phi_s} = -\int_M \dot\phi_s \dot\phi_{s,l} (X^i\phi_{s,i})_{,\bar j} g^{l\bar j}\,\omega^n_{\phi_s} + \int_M \dot\phi_s X^i \dot\phi_{s,i}\,\omega^n_{\phi_s}.$$

Combining (3-10)–(3-12), we get

$$\frac{d^2}{ds^2}(I_X - J_X)(\phi_s) = \int_M \dot\phi_{s,\bar k}\dot\phi_{s,l} g_{i\bar j}(0) g^{i\bar k} g^{l\bar j}(1-\theta_X(\omega_{\phi_s}))\,\omega^n_{\phi_s} \ge 0. \qquad \square$$

3B. The converse part of Theorem 1.2 can be proved by using the properness principle of [Darvas and Rubinstein 2017]. Since the pluripotential theory for Mabuchi will be used, we first recall some results in [Berman and Witt Nyström 2014].

Let T be the (closed) torus generated by the imaginary part of X. Let $\mathrm{PSH}_T(\omega_0)$ be the set of T-invariant ω_0-plurisubharmonic functions. According to [Berman and Witt Nyström 2014], for any continuous nonnegative function g and $\phi \in \mathrm{PSH}_T(\omega_0)$, one can define a g-Monge–Ampère measure $\mathrm{MA}_g(\phi)$ and the measure has weak continuity. When ϕ is smooth, $\mathrm{MA}_g(\phi) = \mathrm{MA}(\phi) g(m_\phi)$, where m_ϕ is the moment map of the torus action with respect to ω_ϕ. In particular, when $g(t) = e^t$, it corresponds to the case of Kähler–Ricci solitons; when $g(t) = 1 - t$, it corresponds to the case of Mabuchi metrics. Based on the pluripotential theory of g-Monge–Ampère measure, the existence and uniqueness theory for Kähler–Ricci solitons is obtained by a variational approach. The approach also applies to Mabuchi metrics. Write $\omega_0 = \sqrt{-1}\partial\bar{\partial}\psi_0$ locally. Following [Berman and Witt Nyström 2014], we call $\phi \in \mathrm{PSH}_T(\omega_0)$ a *weak Mabuchi metric* if

$$\mathrm{MA}_g(\phi) = e^{-(\psi_0+\phi)}. \tag{3-13}$$

Let $\mathcal{E}^1_T(M)$ be the set of T-invariant metrics with finite energy [Berman et al. 2013]. Then by the same argument as in [Berman and Witt Nyström 2014] with the modified Ding energy $\mathcal{D}_X(\cdot)$ in this paper, we have

Proposition 3.4. *$\phi \in \mathcal{E}^1_T(M)$ is a weak Mabuchi metric if and only if it minimizes the modified Ding functional $\mathcal{D}_X(\cdot)$.*

Proof of the converse part of Theorem 1.2. We take $\mathcal{R} = \mathcal{H}_X(\omega_0)$, $d = d_1$ as in [Darvas and Rubinstein 2017, Definition 4.2], $F = \mathcal{D}_X(\cdot)$ and $G = \mathrm{Aut}^X_0(M)$ in the setting of [Darvas and Rubinstein 2017, §3].

Suppose ω_0 is the Mabuchi metric and $\mathrm{Iso}_0(M, \omega_0)$ is the identity component of the corresponding isometry group. By a Calabi–Matsushima type theorem of Mabuchi [2002], we have

$$\mathfrak{aut}^X(M) = \mathfrak{iso}(M, \omega_0) \oplus J\mathfrak{iso}(M, \omega_0), \tag{3-14}$$

where $\mathfrak{aut}^X(M)$ and $\mathfrak{iso}(M, \omega_0)$ are Lie algebras of $\mathrm{Aut}^X_0(M)$ and $\mathrm{Iso}_0(M, \omega_0)$, respectively. We will check that $\mathcal{D}_X(\cdot)$, $\mathrm{Aut}^X(M)$ satisfy (P1)–(P7) in the Hypothesis 3.2 of [Darvas and Rubinstein 2017], which are enough for the "existence ⇒ properness" direction:

(P1) This is confirmed by [Berndtsson 2015, Theorem 1.1] and Lemma 2.5(2).

(P2) This can be shown by using Lemma 2.5(1) and Lemmas 5.15, 5.20, 5.29 of [Darvas and Rubinstein 2017], where we replace $AM_X(\cdot)$ and $F^X(\cdot)$ in [Darvas and Rubinstein 2017] by $-\mathcal{D}^0_X(\cdot)$ and $\mathcal{D}_X(\cdot)$, respectively. The monotonicity of $-\mathcal{D}^0_X(\cdot)$ is confirmed by Lemma 2.5(1). The proof then follows exactly those in [Darvas and Rubinstein 2017].

(P3) This can be proved similarly to [Berman and Witt Nyström 2014, Theorem 1.3] by using Proposition 3.4, we will finish it in the proof of (P5).

(P4) This is [Darvas and Rubinstein 2017, Lemma 5.9].

(P5) We mainly use the arguments of [Berman and Witt Nyström 2014] by taking $g(t) = 1 - t$ in the g-Monge–Ampère equation there. By Corollary 2.9 and Theorem 2.18 of [Berman and Witt Nyström 2014], we see that any weak solution ω_ϕ is locally bounded and minimizes $\mathcal{D}_X(\cdot)$. Hence any two weak solution ω_{ϕ_0}

and ω_{ϕ_1} can be connected by a bounded geodesic $\phi_t \subset \mathrm{PSH}_T(\omega_0)$. By Remark 2.6 and [Berndtsson 2015, Theorem 1.2], $\mathcal{D}_X(\phi_t)$ is affine and there is a family of $\{\sigma_t\} \subset \mathrm{Aut}_0(M)$ such that $\omega_{\phi_t} = \sigma_t^*(\omega_{\phi_0})$. In particular, we can take $\omega_{\phi_0} = \omega_0$, the *smooth* Mabuchi metric on M. Thus for any t, ω_{ϕ_t} is a *smooth* Mabuchi metric. This confirms (P3).

It remains to show that $\{\sigma_t\} \subset \mathrm{Aut}_0^X(M)$. Write $\omega_0 = \sqrt{-1}\partial\bar{\partial}\psi_0$ locally and denoted by $\mathrm{MA}(\phi)$ the Monge–Ampére operator of $\psi_0 + \phi$. Since ω_{ϕ_t} is smooth, (3-13) reduces to

$$\mathrm{MA}(\phi_t)(1 - \theta_X(\omega_{\phi_t})) = e^{-\psi_0 - \phi_t},$$

which is equivalent to

$$\sigma_t^*(\mathrm{MA}(\phi_0)(1 - \theta_{\sigma_{t*}^{-1}X}(\omega_{\phi_0})) = \sigma_t^*(e^{-\psi_0 - \phi_0})$$

since $\omega_{\phi_t} = \sigma_t^*(\omega_{\phi_0})$. Thus

$$\sigma_{t*}^{-1} X = X$$

for all t which proves $\{\sigma_t\} \subset \mathrm{Aut}_0^X(M)$. Thus (P5) is proved.

(P6) This can be shown exactly as in [Darvas and Rubinstein 2017, Theorem 8.1], by using (3-14) instead of [Darvas and Rubinstein 2017, Proposition 6.10];

(P7) This follows from the cocycle condition of $\mathcal{D}_X(\cdot)$.

The theorem then follows from the second part of [Darvas and Rubinstein 2017, Theorem 3.4]. □

4. Existence criterion on Fano group compactifications

In this section, we will prove Theorem 1.5. Let M be a group compactification and ω_0 be a $K \times K$-invariant Kähler metric in $2\pi c_1(M)$. Assume $\omega_0 = \sqrt{-1}\partial\bar{\partial}\psi_0$ on G. For $\phi \in \mathcal{H}_{K\times K}(\omega_0)$, we will write ψ_ϕ in short for $\psi_0 + \phi$ and u_ϕ the Legendre function of ψ_ϕ.

4A. ***Reduction of the modified Ding functional.*** We will give a formula of $\mathcal{D}_X(\phi)$ in terms of ϕ and u_ϕ. First, we compute the Futaki invariant of a vector field in $\mathfrak{z}(\mathfrak{g})$.

Lemma 4.1. *Let Y be a vector field of the form*

$$Y = \sqrt{-1} Y^i E_i^0, \quad 1 \le i \le r \tag{4-1}$$

for some $Y^i \in \mathbb{C}$ such that $\alpha_i Y^i = 0$ for any $\alpha \in \Phi$. Then

$$\mathrm{Fut}(Y) = -V \cdot Y^i \boldsymbol{b}_i, \tag{4-2}$$

where $\boldsymbol{b} = (1/V)\int_{2P_+} y\pi(y)\,dy$ is the barycenter of $2P_+$ with respect to the measure $\pi(y)\,dy$.

Proof. Since $Y \in \mathfrak{z}(\mathfrak{g})$, it is $K\times K$-invariant, so is its potential. Recall that

$$\operatorname{Fut}(Y) = -\int_M \hat{\theta}_Y(\omega_0)\,\omega_0^n, \tag{4-3}$$

where $\hat{\theta}_Y(\omega_0)$ is the potential of Y normalized by

$$\int_M \hat{\theta}_Y(\omega_0)e^{h_0}\omega_0^n = 0. \tag{4-4}$$

By $\omega_0 = \sqrt{-1}\partial\bar{\partial}\psi_0$ and Lemma 2.8, it is not hard to see that

$$\hat{\theta}_Y(\omega_0) = Y^i \frac{\partial}{\partial x^i}\psi_0 + C, \quad \text{for all } x \in \mathfrak{a}_+, \tag{4-5}$$

where C is a constant determined by (4-4). On the other hand, we have

$$\begin{aligned}\int_{\mathfrak{a}_+} Y^i \frac{\partial}{\partial x^i}\psi_0 e^{h_0}\det(\psi_{0,ij}) \prod_{\alpha\in\Phi_+}\langle\alpha,\nabla\psi_0\rangle^2\,dx &= \int_{\mathfrak{a}_+} Y^i \frac{\partial}{\partial x^i}\psi_0 e^{-\psi_0}\boldsymbol{J}(x)\,dx \\ &= -\int_{\mathfrak{a}_+} Y^i \frac{\partial}{\partial x^i}(e^{-\psi_0}\boldsymbol{J}(x))\,dx.\end{aligned} \tag{4-6}$$

Here we used the fact that

$$Y^i \frac{\partial}{\partial x^i}\boldsymbol{J}(x) = 2\boldsymbol{J}(x)\sum_{\alpha\in\Phi_+} Y^i\alpha_i\cdot\coth\langle\alpha,x\rangle \equiv 0.$$

Note that when M is Fano, $4\rho \in \operatorname{Int}(2P_+)$ (see [Delcroix 2017b, Remark 4.10] or [Li et al. 2018, §3.2]), by [Delcroix 2017a, Proposition 2.10]. Hence, we have

$$e^{-\psi_0}\boldsymbol{J}(x) = e^{4\rho(x)-\psi_0}\prod_{\alpha\in\Phi_+}\Big(\frac{1-e^{-2\alpha(x)}}{2}\Big)^2 \to 0, \quad x\to\infty \text{ in } \mathfrak{a}_+.$$

Also recall the fact that $\boldsymbol{J}(x) = 0$ on $\partial(\mathfrak{a}_+)$. By integration by parts in (4-6), we see that

$$\int_{\mathfrak{a}_+} Y^i \frac{\partial}{\partial x^i}\psi_0 e^{h_0}\det(\psi_{0,ij}) \prod_{\alpha\in\Phi_+}\langle\alpha,\nabla\psi_0\rangle^2\,dx = 0.$$

Thus by Proposition 2.7, we get $C = 0$ in (4-5). Equation (4-2) then follows from (4-3). □

Then we use (1-4) to determine the potential of the extremal vector field X.

Lemma 4.2. *Under the coordinates chosen in Section 2C, the extremal field X, when restricted on Z, can be expressed as*

$$X = \sqrt{-1}X^i E_0^i, \quad 1 \le i \le r \tag{4-7}$$

for some $X^i \in \mathbb{R}$ *such that* $\alpha(X) = 0$, *for all* $\alpha \in \Phi$. *Furthermore, the* X^i *are determined by the condition*

$$\int_{2P_+} v^i y_i (1 - \theta_X(y))\pi(y)\, dy = 0, \quad \textit{for all } v \in \mathfrak{z}(\mathfrak{g}), \tag{4-8}$$

where $\theta_X(y) = X^i y_i - X^i \boldsymbol{b}_i$.

Proof. Since the Futaki invariant is a character on $\eta_r(M)$, it suffices to consider (1-4) for all $v \in \mathfrak{z}(\eta_r(M)) \subset \mathfrak{z}(\mathfrak{g})$. We may assume X is of the form (4-7). Since K_X lies in a compact group, we have $X^i \in \mathbb{R}$.

For $\phi \in \mathcal{H}_{K\times K}(\omega_0)$ and $v \in \eta_c(M)$, $\theta_v(\omega_\phi)$ is $K\times K$-invariant, so it can be written as

$$\theta_v(\omega_\phi) = v^i \frac{\partial \psi_\phi}{\partial x^i} + c_v,$$

where v^i and c_v are constants with $v^i \alpha_i = 0$ for any $\alpha \in \Phi_+$. By the second equality of (1-3), the potential is determined by

$$\theta_v(y) = v^i y_i - v^i \boldsymbol{b}_i. \tag{4-9}$$

Let $r_z = \dim(\mathfrak{z}(\mathfrak{g}))$ and suppose $E_1^0, \dots, E_{r_z}^0$ is a basis of $\mathfrak{z}(\mathfrak{g})$. We claim that the extremal vector field X is given by $X = \sum_1^{r_z} \sqrt{-1} X^i E_i^0 \in \mathfrak{z}(\mathfrak{g})$ such that

$$\boldsymbol{b}_i = \frac{1}{V}\left(\int_{2P_+} y_i y_j \pi(y)\, dy - V \boldsymbol{b}_i \boldsymbol{b}_j\right) X^j, \quad 1 \le i, j \le r_z. \tag{4-10}$$

In view of (4-9) and Lemma 4.1, it is direct to check that X given by (4-10) satisfies (1-4). Hence X must be extremal by the uniqueness. To see that (4-10) has a unique solution, it suffices to check that the matrix (a_{ij}) given by

$$a_{ij} = \frac{1}{V}\int_{2P_+} y_i y_j \pi(y)\, dy - \boldsymbol{b}_i \boldsymbol{b}_j$$

is invertible. In fact, for any vector $v = (v^i)$, consider the convex function $f_v(y) = (v^i y_i)^2$. By Jensen's inequality,

$$v^i v^j a_{ij} = \frac{1}{V}\int_{2P_+} [v(y)]^2 \pi(y)\, dy - [v(\boldsymbol{b})]^2 \ge 0,$$

with equality if and only if $f_v(y)$ is affine on $2P_+$. However, this forces $v = 0$, thus $(a_{ij}) > 0$. □

Proposition 4.3. *For* $\phi \in \mathcal{H}_{K\times K}(\omega_0)$, *the modified Ding functional is given by*

$$\mathcal{D}_X(\phi) = \mathcal{L}_X(u_\phi) + \mathcal{F}(u_\phi) + \text{const.},$$

where u_ϕ is the Legendre function of ψ_ϕ and

$$\mathcal{L}_X(u_\phi) = \frac{1}{V}\int_{2P_+} u_\phi(y)\pi(y)[1-\theta_X(y)]\,dy - u_\phi(4\rho), \tag{4-11}$$

$$\mathcal{F}(u_\phi) = -\log\biggl(\int_{\mathfrak{a}_+} e^{-\psi_\phi}\boldsymbol{J}(x)\,dx\biggr) + u_\phi(4\rho). \tag{4-12}$$

Proof. By (1-6), Proposition 2.7 and (2-14), it follows that[2]

$$\begin{aligned}\mathcal{D}_X^0(\phi) &= -\frac{1}{V}\int_0^1\int_{\mathfrak{a}_+} \dot\phi_s[1-\theta_X(\omega_{\phi_s})]\det(\psi_{\phi_s,ij})\prod_{\alpha\in\Phi_+}\langle\alpha,\nabla\psi_{\phi_s}\rangle^2\,dx\wedge ds + \text{const.},\\ \mathcal{N}(\phi) &= -\log\biggl(\frac{1}{V}\int_{\mathfrak{a}_+} e^{-\psi_\phi}\boldsymbol{J}(x)\,dx\biggr).\end{aligned} \tag{4-13}$$

By differentiation with Legendre transformations, we have $\dot u_s(y_s(x)) = -\dot\psi_s(x)$. Then by (4-13), $\mathcal{D}^0(\phi)$ equals

$$\int_0^1\int_{2P_+} \dot u_s[1-\theta_X(y)]\pi(y)\,dy\wedge ds = \int_{2P_+} u_\phi[1-\theta_X(y)]\pi(y)\,dy + \text{const.} \qquad \square$$

Remark 4.4. The functionals

$$\mathcal{L}_X(u) = \frac{1}{V}\int_{2P_+} u(y)\pi(y)[1-\theta_X(y)]\,dy - u(4\rho),$$

$$\mathcal{F}(u) = -\log\biggl(\int_{\mathfrak{a}_+} e^{-\psi}\boldsymbol{J}(x)\,dx\biggr) + u(4\rho),$$

where ψ is the Legendre function of u, are well-defined on

$$\mathcal{C}'_W = \bigl\{u \in C^0(\overline{2P})\cap C^\infty(2P) \mid u \text{ is strictly convex, } W\text{-invariant}\bigr\}.$$

First, we deal with the case when $u \in C^\infty(\overline{2P})\cap\mathcal{S}_W$. In this case, we can choose the function U_δ constructed in [Donaldson 2002, Proposition 3.3.11]. Then $u_\delta = u + U_\delta \in \mathcal{C}_W$ and u_δ converges uniformly to u on $\overline{2P}$. On the other hand, let ψ, ψ_δ be the Legendre functions of u, u_δ. It follows ψ_δ converges uniformly to ψ. Hence, by definition,

$$\mathcal{L}_X(u_\delta)\to\mathcal{L}_X(u) \quad\text{and}\quad \mathcal{F}(u_\delta)\to\mathcal{F}(u) \qquad \text{as } \delta\to 0^+. \tag{4-14}$$

Thus $\mathcal{L}_X(u)$ and $\mathcal{F}(u)$ are well-defined.

For an arbitrary $u \in \mathcal{C}'_W$, consider $u^c(y) = u(cy)|_{2P}$ for $c \in (0,1)$. Then $u^c \in C^\infty(\overline{2P})\cap\mathcal{C}'_W$. Let ψ^c be the Legendre functions of u^c. As before, as $c\to 1^-$, we

[2] Since we have assumed $C_H = 1$, it follows that $V := \int_M \omega_0^n = \int_{2P_+}\pi(y)\,dy$ by Proposition 2.7. Similarly $\int_{2P_+}[1-\theta_X(y)]\pi(y)\,dy = V$.

have that u^c and ψ^c converge uniformly to u and ψ, respectively. Again, we have

$$\mathcal{L}_X(u^c) \to \mathcal{L}_X(u) \quad \text{and} \quad \mathcal{F}(u^c) \to \mathcal{F}(u) \qquad \text{as } c \to 1^-. \tag{4-15}$$

Hence, $\mathcal{L}_X(u)$ and $\mathcal{F}(u)$ are well-defined. Moreover, by (4-14) and (4-15), it follows that

$$\inf_{u \in \mathcal{C}'_W} \mathcal{D}_X(u) = \inf_{u \in \mathcal{C}_W} \mathcal{D}_X(u) \tag{4-16}$$

as proved in [Donaldson 2002, Proposition 3.3.11] for K-energy.

4B. *The linear part.* In this part, we deal with the linear part $\mathcal{L}_X(\cdot)$. First, we introduce the spaces of normalized functions. Let O be the origin of $\mathfrak{a}^*$. Note that $\mathfrak{a}_t^*$ is the fixed point set of the W-action. Thus $\nabla u(O) \in \mathfrak{a}_t$ for any $u \in \mathcal{C}_W$. We normalize $u \in \mathcal{C}_W$ by

$$\hat{u}(y) = u(y) - \langle \nabla u(O), y \rangle - u(O).$$

Clearly $\hat{u} \in \mathcal{C}_W$ and

$$\min_{2P} \hat{u} = \hat{u}(O) = 0. \tag{4-17}$$

The subset of normalized functions in $\mathcal{C}_W$ will be denoted by $\hat{\mathcal{C}}_W$.

Proposition 4.5. *Under the assumption* $c_X > 0$ *and* (1-8), *there exists a constant* $\lambda > 0$ *such that*

$$\mathcal{L}_X(u) \geq \lambda \int_{2P_+} u\pi(y)[1 - \theta_X(y)]\,dy, \quad \textit{for all } u \in \hat{\mathcal{C}}_W. \tag{4-18}$$

Proof. Suppose the proposition is not true. Then there is a sequence $\{u_k\} \subset \hat{\mathcal{C}}_W$ such that

$$\mathcal{L}_X(u_k) \to 0 \quad \text{and} \quad \int_{2P_+} u_k\pi(y)[1 - \theta_X(y)]\,dy = 1. \tag{4-19}$$

By $c_X > 0$ and the argument of [Li et al. 2018, Lemma 6.1], the second equality implies there is a subsequence (still denoted by $\{u_k\}$) which converges locally uniformly to some $u_\infty \in \hat{\mathcal{C}}_W$.

For any $u \in \mathcal{C}_W$, by convexity, we have

$$u - \langle \nabla u(\boldsymbol{b}_X), y - \boldsymbol{b}_X \rangle - u(\boldsymbol{b}_X) \geq 0. \tag{4-20}$$

Thus

$$
\begin{aligned}
(4\text{-}21)\quad \mathcal{L}_X(u) &= \frac{1}{V}\int_{2P_+}[u-\langle\nabla u(\boldsymbol{b}_X), y-\boldsymbol{b}_X\rangle - u(\boldsymbol{b}_X)]\pi(y)[1-\theta_X(y)]\,dy \\
&\qquad + \frac{1}{V}\int_{2P_+}[\langle\nabla u(\boldsymbol{b}_X), y-\boldsymbol{b}_X\rangle + u(\boldsymbol{b}_X)]\pi(y)[1-\theta_X(y)]\,dy - u(4\rho) \\
&= \frac{1}{V}\int_{2P_+}[u-\langle\nabla u(\boldsymbol{b}_X), y-\boldsymbol{b}_X\rangle - u(\boldsymbol{b}_X)]\pi(y)[1-\theta_X(y)]\,dy + u(\boldsymbol{b}_X) \\
&\qquad - u(4\rho) \\
&\geq \frac{1}{V}\int_{2P_+}[u-\langle\nabla u(\boldsymbol{b}_X), y-\boldsymbol{b}_X\rangle - u(\boldsymbol{b}_X)]\pi(y)[1-\theta_X(y)]\,dy \\
&\qquad + \langle\nabla u(4\rho), \boldsymbol{b}_X-4\rho\rangle \\
&\geq 0,
\end{aligned}
$$

where the last inequality follows from (1-8), (4-20) and the fact that $\nabla u(4\rho)\in\mathfrak{a}_+$. Applying the above inequality to u_k, by (4-19), we have

$$(4\text{-}22)\quad 0\leq \int_{2P_+}[u_k-\langle\nabla u_k(\boldsymbol{b}_X), y-\boldsymbol{b}_X\rangle - u_k(\boldsymbol{b}_X)]\pi(y)[1-\theta_X(y)]\,dy \to 0,$$

$$(4\text{-}23)\quad 0\leq \langle\nabla u_k(4\rho), \boldsymbol{b}_X-4\rho\rangle \to 0.$$

By (4-22), we see that u_∞ must be affine linear. Since $u_k(O)=0$, we have $u_\infty(y)=\xi^i y_i$ for some $(\xi^i)\in\bar{\mathfrak{a}}_+$. Since u_∞ is normalized and O lies in the interior of $2P_+\cap\mathfrak{a}_+^*$, it holds that $\xi\in\mathfrak{a}_{ss}$. Otherwise u_∞ is not nonnegative. Substituting u_∞ into (4-23), we see that $\langle\xi, \boldsymbol{b}_X-4\rho\rangle=0$. But $\xi\in\bar{\mathfrak{a}}_+$ and $\boldsymbol{b}_X-4\rho\in\Xi$. Hence $\xi^i=0$ and consequently $u_\infty(y)\equiv 0$.

Since $u_k(4\rho)\to u_\infty(4\rho)=0$, by (4-11) and the second line of (4-19), we have $\mathcal{L}_X(u_k)\to 1$ by the second line of (4-19), which is a contradiction. □

Yao [2017] uses (4-18) to define the "uniform relative Ding stability" in the toric case. In [Yao 2017], it is shown the condition $c_X>0$ is a necessary condition of (4-18). Since those arguments can be generalized to group compactifications with no difficulties, we omit the details.

Proposition 4.6. *Inequality* (4-18) *can not hold if* $c_X\leq 0$.

4C. *Sufficiency.* We first show the sufficient part of Theorem 1.5 using Theorem 1.2. It suffices to prove the following theorem.

Theorem 4.7. *If* $c_X>0$ *and* (1-8) *hold, then the modified Ding functional is proper modulo* $Z(G)$. *Consequently, M admits Mabuchi metrics, by Theorem 1.2.*

First we have the following lemma on the nonlinear part.

Lemma 4.8. *For any $\phi \in \mathcal{H}_{K\times K}(\omega_0)$, let*

$$\tilde{\psi}_\phi := \psi_\phi - 4\rho_i x^i, \quad x \in \mathfrak{a}_+.$$

Then

$$\mathcal{F}(u_\phi) = -\log\bigg(\int_{\mathfrak{a}_+} e^{-(\tilde{\psi}_\phi - \inf_{\mathfrak{a}_+}\tilde{\psi}_\phi)} \prod_{\alpha\in\Phi_+}\bigg(\frac{1-e^{-2\alpha_i x^i}}{2}\bigg)^2 dx\bigg). \tag{4-24}$$

Consequently, for any $c > 0$,

$$\mathcal{F}(u_\phi) \geq \mathcal{F}\bigg(\frac{u_\phi}{1+c}\bigg) - n\cdot\log(1+c). \tag{4-25}$$

Proof. Since ψ_ϕ is convex, so is $\tilde{\psi}_\phi$. Thus if $x^* \in \mathfrak{a}_+$ satisfies $\nabla\psi_\phi(x^*) = 4\rho$, then

$$\tilde{\psi}_\phi(x) \geq \tilde{\psi}_\phi(x^*) = \inf_{\mathfrak{a}_+}\tilde{\psi}_\phi.$$

By the definition of Legendre transformation, we have

$$\psi_\phi(x) + u_\phi(4\rho) = \psi_\phi(x) + 4x^{*i}\rho_i - \psi_\phi(x^*) = \psi_\phi(x) - \inf_{\mathfrak{a}_+}\tilde{\psi}_\phi.$$

Substituting this into (4-12), it follows that

$$\begin{aligned}\mathcal{F}(u_\phi) &= -\log\bigg(\int_{\mathfrak{a}_+} e^{-(\psi_\phi + u_\phi(4\rho))}\boldsymbol{J}(x)\,dx\bigg)\\ &= -\log\bigg(\int_{\mathfrak{a}_+} e^{-(\tilde{\psi}_\phi - \inf_{\mathfrak{a}_+}\tilde{\psi}_\phi)}e^{-4\rho_i x^i}\boldsymbol{J}(x)\,dx\bigg)\\ &= -\log\bigg(\int_{\mathfrak{a}_+} e^{-(\tilde{\psi}_\phi - \inf_{\mathfrak{a}_+}\tilde{\psi}_\phi)}\prod_{\alpha\in\Phi_+}\Big(\frac{1-e^{-2\alpha_i x^i}}{2}\Big)^2 dx\bigg).\end{aligned}$$

This proves (4-24).

Then we prove (4-25). For $u_c(y) = \frac{1}{1+c}u(y)$, its Legendre function $\psi_c(x) = \frac{1}{1+c}\psi((1+c)x)$ satisfies $\tilde{\psi}_c(x) = \frac{1}{1+c}\tilde{\psi}((1+c)x)$. In particular,

$$-\inf_{\mathfrak{a}_+}\tilde{\psi}_c(x) = -\frac{1}{1+c}\inf_{\mathfrak{a}_+}\tilde{\psi}.$$

By the above relations and (4-24), one gets

$$\begin{aligned}&\mathcal{F}(u_c)\\ &\quad= -\log\bigg(\int_{\mathfrak{a}_+} e^{-\frac{1}{1+c}(\tilde{\psi}_\phi((1+c)x) - \inf_{\mathfrak{a}_+}\tilde{\psi}_\phi)}\prod_{\alpha\in\Phi_+}\Big(\frac{1-e^{-2\alpha_i x^i}}{2}\Big)^2 dx\bigg)\\ &\quad= -\log\bigg(\int_{\mathfrak{a}_+} e^{-\frac{1}{1+c}(\tilde{\psi}_\phi(x) - \inf_{\mathfrak{a}_+}\tilde{\psi}_\phi)}\prod_{\alpha\in\Phi_+}\Big(\frac{1-e^{-\frac{2}{1+c}\alpha_i x^i}}{2}\Big)^2 dx\bigg) + r\cdot\log(1+c).\end{aligned}$$

Note that $\#\Phi = (n-r)/2$. Combining the above inequality and relations

$$\log(1+c) \geq \log(1-e^{-t}) - \log(1-e^{-t/(1+c)}) \geq 0, \quad \text{for all } t \geq 0, c \geq 0$$

and

$$\mathcal{F}(u) \geq -\log\biggl(\int_{\mathfrak{a}_+} e^{-\frac{1}{1+c}(\tilde{\psi}_\phi(x) - \inf_{\mathfrak{a}_+} \tilde{\psi}_\phi)} \prod_{\alpha \in \Phi_+} \Bigl(\frac{1-e^{-2\alpha_i x^i}}{2}\Bigr)^2 \, dx\biggr),$$

we have (4-25). □

Proposition 4.9. *Suppose $c_X > 0$ and* (1-8) *holds. Then there are positive constants c and C such that*

$$\mathcal{D}_X(u) \geq c \int_{2P_+} u[1-\theta_X(y)]\pi(y)\,dy - C, \quad \textit{for all } u \in \hat{\mathcal{C}}_W. \tag{4-26}$$

Proof. Let $\omega_0 = \sqrt{-1}\partial\bar{\partial}\psi_0$ be a fixed smooth background metric with h_0 its normalized Ricci potential. Let u_0 be the Legendre function of ψ_0. Define a function A by

$$A(y) = \frac{V}{\int_{\mathfrak{a}_+} e^{-\psi_0} \boldsymbol{J}(x)\,dx} e^{h_0(\nabla u_0(y))}, \quad y(x) = \nabla\psi_0(x).$$

It is clear that

$$\int_{\mathfrak{a}_+} e^{-\psi_0} \boldsymbol{J}(x)\,dx = \int_M e^{h_0} \omega_0^n = V.$$

Hence, A is a bounded smooth function.

Let

$$\mathcal{D}_A(u_\phi) := \mathcal{D}_A^0(u_\phi) + \mathcal{N}(\phi), \quad \text{for all } \phi \in \mathcal{H}_{K\times K}(\omega_0),$$

where

$$\mathcal{D}_A^0(u) := \frac{1}{V}\int_{2P_+} uA(y)\pi(y)\,dy.$$

It is obvious that u_0 is a critical point of $\mathcal{D}_A(\cdot)$. On the other hand, along any geodesic, $\mathcal{D}_A^0(\cdot)$ is affine by Lemma 2.10 and $\mathcal{N}(\cdot)$ is convex by [Berndtsson 2015, Theorem 1.1]. Hence,

$$\mathcal{D}_A(u) \geq \mathcal{D}_A(u_0), \quad \text{for all } u \in \mathcal{C}_W.$$

Furthermore, by a similar argument as in Remark 4.4, we can extend $\mathcal{D}_A(\cdot)$ to a functional defined on $\mathcal{C}'_W$. Moreover, analogous to (4-16), we have

$$\min_{u \in \mathcal{C}'_W} \mathcal{D}_A(u) = \mathcal{D}_A(u_0). \tag{4-27}$$

Rewrite $\mathcal{D}_A(\cdot) = \mathcal{L}_A(\cdot) + \mathcal{F}(\cdot)$, where

$$\mathcal{L}_A(u) := \frac{1}{V}\int_{2P_+} uA(y)\pi(y)\,dy - u(4\rho).$$

By Proposition 4.5 and the boundedness of A, it is clear that for any $\delta > 0$

$$\begin{aligned}\mathcal{L}_X(u) - \mathcal{L}_A(u) &= \int_{2P_+} u(1-\theta_X(y) - A(y))\pi(y)\,dy \\ &\le C_A \int_{2P_+} u[1-\theta_X(y)]\pi(y)\,dy \\ &\le \frac{C_A(1+\delta)}{\lambda}\mathcal{L}_X(u) - C_A\delta \int_{2P_+} u[1-\theta_X(y)]\pi(y)\,dy,\end{aligned}$$

for all $u \in \hat{\mathscr{C}}_W$ and some constant $C_A > 0$. Thus

$$\mathcal{L}_X(u) \ge \frac{\lambda}{\lambda + C_A(1+\delta)}\left[\mathcal{L}_A(u) + C_A\delta \int_{2P_+} u[1-\theta_X(y)]\pi(y)\,dy\right],$$

for all $u \in \hat{\mathscr{C}}_W$. Hence, taking $C = C_A(1+\delta)/\lambda$, we have for any $u \in \hat{\mathscr{C}}_W$,

$$\begin{aligned}&\mathcal{D}_X(u) \\ &\ge \mathcal{L}_A\left(\frac{u}{1+C}\right) + \mathcal{F}(u) + \frac{C_A\delta}{1+C}\int_{2P_+} u[1-\theta_X(y)]\pi(y)\,dy \\ &\ge \mathcal{L}_A\left(\frac{u}{1+C}\right) + \mathcal{F}\left(\frac{u}{1+C}\right) + \frac{C_A\delta}{1+C}\int_{2P_+} u[1-\theta_X(y)]\pi(y)\,dy \\ &\quad - n\log(1+C) \\ &= \mathcal{D}_A\left(\frac{u}{1+C}\right) + \frac{C_A\delta}{1+C}\int_{2P_+} u[1-\theta_X(y)]\pi(y)\,dy - n\log(1+C),\end{aligned} \tag{4-28}$$

where we used (4-25). By using (4-27)

$$\mathcal{D}_A\left(\frac{u}{1+C}\right) \ge \mathcal{D}_A(u_0).$$

Thus, combining with (4-28), we have

$$\mathcal{D}_X(u) \ge \frac{C_A\delta}{1+C}\int_{2P_+} u[1-\theta_X(y)]\pi(y)\,dy + \big(\mathcal{D}_A(u_0) - n\log(1+C)\big). \qquad \square$$

To use Theorem 1.2, we introduce the following normalization: for any $\phi \in \mathcal{H}_{K\times K}(\omega_0)$, let u_ϕ be the Legendre function of ψ_ϕ. Take a $v \in \eta_c(M)$ such that $\mathrm{Re}(v) = -\nabla u_\phi(O)$. Let $\sigma_v(t)$ be the one parameter group generated by $\mathrm{Re}(v)$. Then $\sigma_v(t) \in Z(G)$. It follows that

$$(\sigma_v(1))^*\omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\hat{\phi}$$

induces a $K\times K$-invariant Kähler potential $\hat{\phi}$. Since we may also normalize $\psi_{\hat{\phi}}$ so that $\psi_{\hat{\phi}}(O) = 0$, the Legendre function $u_{\hat{\phi}}$ of $\psi_{\hat{\phi}}$ is given by

$$u_{\hat{\phi}}(y) = u_\phi(y) - \langle \nabla u_\phi(O), y\rangle - u_\phi(O), \tag{4-29}$$

which satisfies $u_{\hat{\phi}} \in \hat{\mathscr{C}}_W$. Then we have:

Lemma 4.10. *Under the above normalization, we have* $\mathfrak{D}_X(u_{\hat{\phi}}) = \mathfrak{D}_X(u_\phi)$.

Proof. Let $a^i = -\mathrm{Re}(u_{\phi,i}(O))$; then $(a^i) \in \mathfrak{a}_t$ and consequently $\alpha(a) = 0$ for all $\alpha \in \Phi$. On the other hand, we have

$$\psi_{\hat{\phi}}(x) = \psi_\phi(x-a) + u_\phi(O).$$

Taking the change of variables $x \to (x-a)$ in (4-24), by the above relations, we see that $\mathscr{F}(u_\phi) = \mathscr{F}(u_{\hat{\phi}})$. By $(a^i) \in \mathfrak{a}_t$ and (4-8), $\mathscr{L}_X(a^i y_i - u_\phi(O)) = 0$. Hence, by (4-29) $\mathscr{L}_X(u_\phi) = \mathscr{L}_X(u_{\hat{\phi}})$. The lemma is proved. □

The following lemma is analogous to [Li et al. 2018, Lemma 4.14; Wang et al. 2016, Lemma 3.4], we omit the proof.

Lemma 4.11. *There exists a uniform* $C_J > 0$ *such that*

$$\left| J_X(\hat{\phi}) - \int_{2P_+} u_{\hat{\phi}}[1-\theta_X(y)]\pi(y)\,dy \right| \le C_J, \quad \textit{for all } \phi \in \mathcal{H}_{K\times K}(\omega_0),$$

where $u_{\hat{\phi}} \in \hat{\mathscr{C}}_W$ *and* $\psi_{\hat{\phi}}$ *is the Legendre function of* $u_{\hat{\phi}}$.

Proof of Theorem 4.7. For any $\phi \in \mathcal{H}_{K\times K}(\omega_0)$, there exists $\sigma \in Z(G)$ such that

$$\sigma^*\omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\hat{\phi}$$

as above. Applying Proposition 4.9, we have

$$\mathfrak{D}_X(u_{\hat{\phi}}) \ge \delta \int_{2P_+} u_{\hat{\phi}}\pi\,dy - C_\delta.$$

Thus by Proposition 4.3 and Lemmas 4.10, 4.11,

$$\mathfrak{D}_X(\phi) = \mathfrak{D}_X(\hat{\phi}) = \mathfrak{D}_X(u_{\hat{\phi}}) \ge \delta\cdot J_X(\hat{\phi}) - C_J - C_\delta.$$

The theorem then follows from (2-7). □

4D. *Necessity.* To complete the proof of Theorem 1.5, we will show that (1-8) is also a necessary condition of the existence of Mabuchi metrics. It is equivalent to show that

(4-30) $$\langle \xi, \boldsymbol{b}_X - 4\rho\rangle > 0, \quad \text{for all } \xi \in \mathfrak{a}_+.$$

We will adopt the method used in [Delcroix 2017a].

By the $K\times K$-invariance, (2-1) can be reduced to the following Monge–Ampére equation on $\mathfrak{a}_+$,

(4-31) $$\det(\psi_{0,ij} + \phi_{ij}) \prod_{\alpha\in\Phi_+} \langle \alpha, \nabla(\psi_0+\phi)\rangle^2 = C\cdot \frac{e^{-(\psi_0+\phi-\log \boldsymbol{J})}}{1-\theta_X(\omega_0) - X(\phi)}.$$

Suppose ϕ is a solution, for any $\xi \in \mathfrak{a}_+$, we have

$$\begin{aligned}
0 &= -\int_{\mathfrak{a}_+} \xi^i \frac{\partial}{\partial x^i} e^{-(\psi_0+\phi-\log \boldsymbol{J})} \\
&= \int_{\mathfrak{a}_+} \xi^i e^{-(\psi_0+\phi-\log \boldsymbol{J})} \frac{\partial(\psi_0+\phi-\log \boldsymbol{J})}{\partial x^i} \\
&= \int_{\mathfrak{a}_+} \xi^i \det(\psi_{0,ij}+\phi_{,ij}) \Big(\prod_{\alpha\in\Phi_+} \langle \alpha, \nabla(\psi_0+\phi)\rangle^2\Big)[1-\theta_X(\omega_\phi)] \frac{\partial(\psi_0+\phi-\log \boldsymbol{J})}{\partial x^i} \\
&< V\int_{2P_+} \xi^i (y_i-4\rho_i)\pi(y)[1-\theta_X(y)]\,dy \\
&= V\cdot\langle \xi, \boldsymbol{b}_X-4\rho\rangle,
\end{aligned}$$

where in the fourth line we used the fact that for any $\xi, x \in \mathfrak{a}_+$

$$-\xi^i \frac{\partial}{\partial x^i} \log \boldsymbol{J} = -2\sum_{\alpha\in\Phi_+} \alpha(\xi)\cdot \coth\alpha(x) < -2\sum_{\alpha\in\Phi_+} \alpha(\xi) = -4\rho(\xi).$$

Then we have (4-30).

Appendix: Proof of Theorem 2.1

In this appendix, we solve (2-2) at $t = 0$. Following [Zhu 2000] for the Kähler–Ricci soliton case, we introduce the following path,

$$(1-\theta_X(\omega_{\phi_t}))^t \omega_{\phi_t}^n = e^{h_0}\omega_0^n, \quad t \in [0, 1]. \tag{A-1}$$

Set $\mathfrak{I} := \{t \in [0, 1] \mid \text{(A-1) has a solution for } t\}$. The Calabi–Yau theorem implies that $0 \in \mathfrak{I}$. We shall prove $\mathfrak{I}$ is both open and closed in [0, 1].

Openness. Define a functional

$$J_t(\phi) = \int_0^1 \int_M \dot\phi_s (1-\theta_X(\omega_{\phi_s}))^t\, \omega_{\phi_s}^n \wedge ds,$$

where ϕ_s is any smooth path joining ϕ and 0 in $\mathscr{H}_X(\omega_0)$. It is standard to show that $J_t(\cdot)$ is well-defined. Thus by taking $\phi_s = s\phi$, we have

$$J_t(\phi) = \int_0^1 \int_M \phi (1-\theta_X(\omega_{s\phi_s}))^t\, \omega_{s\phi}^n \wedge ds.$$

Replacing h_0 by $h_0 + J_t(\phi_t)$ in (A-1), we have the operator L_t defined by

$$L_t(\psi) := \triangle_{\omega_{\phi_t}} \psi - \frac{tX(\psi)}{1-\theta_X(\omega_{\phi_t})} - \int_M \psi(1-\theta_X(\omega_{\phi_t}))^t\, \omega_{\phi_t}^n, \quad \text{for all } \psi \in \mathscr{H}_X(\omega_0).$$

Then for any K_X-invariant smooth real functions f and g, it is easy to see

$$\int_M L_t(f)g(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n = \int_M fL_t(g)(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n. \tag{A-2}$$

Lemma A.1. *Suppose ϕ_t is a smooth solution of* (A-1) *for some $t \in [0, 1)$. Then, the first eigenvalue of L_t is positive.*

Proof. Suppose λ is the first eigenvalue and ψ is an eigenfunction. Then by $L_t\psi = -\lambda\psi$,

$$\lambda\int_M \psi(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n = \int_M \psi(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n\cdot\int_M \psi(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n.$$

By the assumption $c_X > 0$, if $\psi \equiv c$ for some constant $c \neq 0$, so $\lambda > 0$. Thus we may assume that $\psi \not\equiv$ const. below.

As before, we may choose a local coframe $\{\Theta^i\}$ such that

$$\omega_{\phi_t} = \sqrt{-1}\sum_{i=1}^n \Theta^i\wedge\bar\Theta^i.$$

By (3-2) and integration by parts, it follows that

$$\begin{aligned}
\lambda\int_M &\psi_{,i}\psi_{,\bar i}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n \\
&= -\int L_t(\psi)_{,i}\psi_{,\bar i}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n \\
&= -\int_M \psi_{,j\bar j i}\psi_{,\bar i}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n + t\int_M \frac{X^j_{,i}\psi_{,j}\psi_{,\bar i}}{1-\theta_X(\omega_{\phi_t})}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n \\
&\qquad + t\int_M \frac{\bar X^{\bar j}X^i\psi_{,i}\psi_{,\bar j}}{(1-\theta_X(\omega_{\phi_t}))^2}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n \\
&\qquad\qquad + \int_M \frac{X^j\psi_{,\bar i}\psi_{,ij}}{1-\theta_X(\omega_{\phi_t})}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n.
\end{aligned} \tag{A-3}$$

By the Ricci identity and integration by parts, the first term on the right-hand side

$$\begin{aligned}
-\int_M &\psi_{,j\bar j i}\psi_{,\bar i}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n \\
&= -\int_M (\psi_{,ij\bar j} - R_{j\ i\bar j}^{p}\psi_{,p})\psi_{,\bar i}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n \\
&= \int_M \mathrm{Ric}_{i\bar p}\,\psi_{,p}\psi_{,\bar i}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n + \int_M \psi_{,ij}\psi_{,\bar i\bar j}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n \\
&\qquad - t\int_M \psi_{,ij}\psi_{,\bar i}X^j(1-\theta_X(\omega_{\phi_t}))^{t-1}\omega_{\phi_t}^n.
\end{aligned}$$

Plugging the above equality into (A-3), one gets

$$\text{(A-4)}\quad \begin{aligned}\lambda\int_M \psi_{,i}\psi_{,\bar{i}}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n &= \int_M\left(\mathrm{Ric}_{i\bar{j}}+\frac{tX_{\bar{j},i}}{1-\theta_X(\omega_{\phi_t})}\right)\psi_{,j}\psi_{,\bar{i}}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n\\ &\quad+\int_M \psi_{,ij}\psi_{,\bar{i}\bar{j}}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n+t\int_M\frac{\bar{X}^{\bar{j}}X^i\psi_{,i}\psi_{,\bar{j}}}{(1-\theta_X(\omega_{\phi_t}))^2}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n.\end{aligned}$$

On the other hand, by (A-1),

$$\mathrm{Ric}_{i\bar{j}}(\omega_{\phi_t})=g_{i\bar{j}}(t)-t\left[\frac{X_{i,\bar{j}}}{1-\theta_X(\omega_{\phi_t})}+\frac{\bar{X}_{,i}X_{,\bar{j}}}{(1-\theta_X(\omega_{\phi_t}))^2}\right].$$

Plugging this into (A-4), one gets

$$\begin{aligned}\lambda\int_M \psi_{,i}\psi_{,\bar{i}}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n=\int_M g_{i\bar{j}}(0)\psi_{,\bar{i}}\psi_{,j}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n&\\ +\int_M \psi_{,ij}\psi_{,\bar{i}\bar{j}}(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n&.\end{aligned}$$

Since $\psi\not\equiv$ const., it must hold that $\lambda>0$. □

The openness then follows from the above lemma and implicit function theorem.

Closedness. For the closeness, it suffices to establish the a priori estimates for (A-1).

First, we prove the C^0-estimate.

Proposition A.2. *Let ϕ_t be a solution of* (A-1) *at t. Then there exists a uniform constant C such that $|\phi_t|\le C$.*

Proof. Consider the equation

$$\text{(A-5)}\qquad \det(g_{i\bar{j}}(t))(1-\theta_X(\omega_{\phi_t}))^t=\det(g_{i\bar{j}}(0))e^{h_0+J_t(\phi_t)}.$$

By integration,

$$\int_M(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n=e^{J_t(\phi_t)}V,$$

we have

$$J_t(\phi_t)=\log\int_M(1-\theta_X(\omega_{\phi_t}))^t\omega_{\phi_t}^n-\log V.$$

This implies

$$\text{(A-6)}\qquad t\log c_X\le J_t(\phi_t)\le t\log C_X.$$

Equation (A-5) can be rewritten as

$$\det(g_{i\bar{j}}(t))=\det(g_{i\bar{j}}(0))e^{\hat{f}_t},$$

where $\hat{f}_t := h_0 + J_t(\phi_t) - t\log(1-\theta_X(\omega_{\phi_t}))$. Let $\hat{\phi}_t = \phi_t - c_t$. Then $\sup_M \hat{\phi}_t = -1$. Since

$$|\hat{f}_t| \le \|h_0\|_{C^0} + 2\max\{|\log c_X|, |\log C_X|\},$$

by the argument for the C^0-estimate in [Tian 1996], we see that $|\hat{\phi}_t| \le C'$ for some uniform $C' > 0$. On the other hand,

$$\text{(A-7)}\quad c_t \int_0^1 \int_M (1-\theta_X(\omega_{s\phi_t}))^t \omega^n_{s\phi_t} \wedge ds \\ = J_t(\phi_t) - \int_0^1 \int_M \hat{\phi}_t (1-\theta_X(\omega_{s\phi_t}))^t \omega^n_{s\phi_t} \wedge ds.$$

Combining (A-6), (A-7) and the fact that $0 < c_x \le 1-\theta_X(\omega_{s\phi_t}) \le C_X$, one gets a uniform constant $\hat{C}$ such that $|c_t| \le \hat{C}$. This implies

$$|\phi_t| \le |\hat{\phi}_t| + \hat{C} \le C' + \hat{C}. \qquad \square$$

Next we consider the C^2-estimate.

Proposition A.3. *Let $\phi = \phi_t$ be a solution of* (A-1) *at t. Then there exist two uniform positive constants C and c such that*

$$n + \triangle_{\omega_0}\phi \le Ce^{c(\phi_t - \inf_M \phi_t)}.$$

Proof. Following the computations of [Zhu 2000, Section 6], at the point where $(n+\triangle_{\omega_0}\phi)e^{-c\phi}$ takes its maximum, we have

$$\triangle_{\omega_0}\log(1-\theta_X(\omega_\phi)) = -\frac{\triangle_{\omega_0}\theta_X(\omega_\phi)}{1-\theta_X(\omega_\phi)} - \frac{|\partial\theta_X(\omega_\phi)|^2}{(1-\theta_X(\omega_\phi))^2} \le C_1(n+\triangle_{\omega_0}\phi) + C_2$$

for some constants C_1, C_2 independent of ϕ. Then as in [Zhu 2000, (6.2)], we see that at this point,

$$\text{(A-8)}\quad \triangle_{\omega_\phi}((n+\triangle_{\omega_0}\phi)e^{-c\phi}) \\ \ge e^{-c\phi}\big(\triangle_{\omega_0}(h_0 - t\log(1-\theta_X(\omega_\phi))) - n^2 \inf_{l\ne k} R_{i\bar{i}l\bar{l}}\big) \\ + (c + \inf_{l\ne i} R_{i\bar{i}l\bar{l}})(n+\triangle_{\omega_0}\phi)e^{-c\phi}\left(\sum_i \frac{1}{1+\phi_{,i\bar{i}}}\right) - cn(n+\triangle_{\omega_0}\phi)e^{-c\phi} \\ \ge -e^{-c\phi}(C_3 + cC_4(n+\triangle_{\omega_0}\phi)) + C_5 e^{-c\phi}(n+\triangle_{\omega_0}\phi)^{n/(n-1)}$$

for sufficiently large constant c and some uniform constants C_3–C_5. The proposition then follows from (A-8) in a standard way. $\square$

The higher order estimates then follow from nonlinear elliptic equation theory and we omit the details.

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