Pacific Journal of Mathematics

SYMPLECTIC (−2)-SPHERES AND THE SYMPLECTOMORPHISM GROUP OF SMALL RATIONAL 4-MANIFOLDS

JUN LI AND TIAN-JUN LI

Volume 304 No. 2 February 2020

PACIFIC JOURNAL OF MATHEMATICS
Vol. 304, No. 2, 2020
dx.doi.org/10.2140/pjm.2020.304.561

SYMPLECTIC (−2)-SPHERES AND THE SYMPLECTOMORPHISM GROUP OF SMALL RATIONAL 4-MANIFOLDS

JUN LI AND TIAN-JUN LI

Let (X, ω) be a symplectic rational surface. We study the space of tamed almost complex structures $\mathcal{J}_\omega$ using a fine decomposition via smooth rational curves and a relative version of the infinite dimensional Alexander–Pontrjagin duality. This decomposition provides new understandings of both the variation and the stability of the symplectomorphism group $\mathrm{Symp}(X, \omega)$ when deforming ω. In particular, we compute the rank of $\pi_1(\mathrm{Symp}(X, \omega))$ with $\chi(X) \leq 7$ in terms of the number N_ω of (-2)-symplectic sphere classes.

1. Introduction

Let (X, ω) be a closed simply connected symplectic manifold. The symplectomorphism group with the standard C^∞-topology, denoted by $\mathrm{Symp}(X, \omega)$, is an infinite dimensional Fréchet Lie group. Let $\mathcal{J}_\omega$ be the contractible Fréchet manifold of ω-tamed almost complex structures. Finding a suitable decomposition of $\mathcal{J}_\omega$ invariant under the natural action of $\mathrm{Symp}(X, \omega)$ or its Torelli subgroup has proved useful in probing the topology of $\mathrm{Symp}(X, \omega)$ when (X, ω) is a symplectic rational surface [Gromov 1985; Abreu 1998; Abreu and McDuff 2000; Lalonde and Pinsonnault 2004; Anjos and Pinsonnault 2013]. However, the analysis is usually hard even if X is relatively simple [Anjos 2002; Anjos and Pinsonnault 2013].

Among homotopy groups of $\mathrm{Symp}(X, \omega)$, $\pi_0(\mathrm{Symp}(X, \omega))$ and $\pi_1(\mathrm{Symp}(X, \omega))$ are the most interesting ones: $\pi_0(\mathrm{Symp}(X, \omega))$ is often called the symplectic mapping class group, and $\pi_1(\mathrm{Symp}(X, \omega))$ is tied to the Hofer geometry of $\mathrm{Symp}(X, \omega)$ (see [Polterovich 2002]) and quantum cohomology (see [Seidel 1997]).

Note that $\mathrm{Symp}(X, \omega) = \mathrm{Symp}_h(X, \omega) \rtimes G(X, \omega)$, where $\mathrm{Symp}_h(X, \omega)$ is the homologically trivial part of $\mathrm{Symp}(X, \omega)$ (the Torelli part), and $G(X, \omega)$ is the image of the induced homomorphism from $\mathrm{Symp}(X, \omega)$ to $\mathrm{Aut}[H^2(X, \mathbb{Z})]$. We

MSC2010: primary 32Q65, 53C15, 53D35, 57R17; secondary 53D05, 57S05.
Keywords: symplectomorphism group, almost complex structure, rational symplectic manifold, Lagrangian root system.

also have $\pi_0(\mathrm{Symp}(X,\omega)) = \pi_0(\mathrm{Symp}_h(X,\omega)) \rtimes G(X,\omega)$, and $\pi_k(\mathrm{Symp}(X,\omega)) = \pi_k(\mathrm{Symp}_h(X,\omega))$, for all $k \geq 1$. It is shown in [Li and Wu 2012; Shevchishin 2009] that $G(X,\omega)$ is a reflection group generated by Dehn twists along Lagrangian spheres. In fact, for a symplectic rational surface (X,ω) with $\chi(X) \leq 11$, we observe in Lemma 2.23 and Section 2B5 that the collection of Lagrangian sphere classes form a root system, denoted by $\Gamma_L(X_k,\omega)$, and $G(X,\omega)$ is just the Weyl group of $\Gamma_L(X_k,\omega)$. Further, when $\chi(X) \leq 7$, $\mathrm{Symp}_h(X,\omega)$ is always path connected [Li et al. 2015]. Hence we know $\pi_0(\mathrm{Symp}(X,\omega)) = G(X,\omega)$ is just the Weyl group of the root system $\Gamma_L(X_k,\omega)$.

We aim to relate $\pi_0(\mathrm{Symp}(X,\omega))$ and $\pi_1(\mathrm{Symp}(X,\omega))$ of a rational surface X, the codimension 0 and 2 pieces of a new decomposition of $\mathcal{J}_\omega$, and self-intersection (-2)-symplectic spheres in symplectic rational surfaces. In this paper, we focus on small rational surfaces with $\chi(X) \leq 7$. In our upcoming papers [Anjos et al. 2019; Li et al. 2019; $\geq$ 2020] we will extend our program to symplectic rational 4-manifolds with larger Euler numbers.

1A. ***A fine decomposition of $\mathcal{J}_\omega$ via symplectic spheres.*** When (X,ω) is a symplectic 4-manifold, we introduce in Section 2 the following decomposition of $\mathcal{J}_\omega$ via embedded ω-symplectic spheres of self-intersection at most -2.

Let $\mathcal{S}_\omega$ denote the set of homology classes of embedded ω-symplectic spheres and K_ω the symplectic canonical class. For any $A \in \mathcal{S}_\omega$, by the adjunction formula,

$$K_\omega \cdot A = -A \cdot A - 2. \tag{1}$$

For any integer q, let

$$\mathcal{S}_\omega^{\geq q}, \quad \mathcal{S}_\omega^{>q}, \quad \mathcal{S}_\omega^{q}, \quad \mathcal{S}_\omega^{\leq q}, \quad \mathcal{S}_\omega^{<q}$$

be the subsets of $\mathcal{S}_\omega$ consisting of classes with square $\geq q$, $> q$, $= q$, $\leq q$, $< q$, respectively. For each $A \in \mathcal{S}_\omega^{<0}$ we associate the integer

$$\mathrm{cod}_A = 2(-A \cdot A - 1).$$

Then we define the prime subset $\mathcal{J}_\mathcal{C}$ labeled by a set $\mathcal{C} \subset \mathcal{S}_\omega^{\leq -2}$ as follows:

Definition 1.1. A subset $\mathcal{C} \subset \mathcal{S}_\omega^{\leq -2}$ is called admissible if

$$\mathcal{C} = \{A_1, \ldots, A_i, \ldots \mid A_i \cdot A_j \geq 0,\ \forall i \neq j\}.$$

Given an admissible subset $\mathcal{C}$, we define the real codimension of the label set $\mathcal{C}$ as

$$\mathrm{cod}(\mathcal{C}) = \sum_{A_i \in \mathcal{C}} \mathrm{cod}_{A_i} = \sum_{A_i \in \mathcal{C}} 2(-A_i \cdot A_i - 1).$$

Define the *prime subset*

$$\mathcal{J}_{\mathcal{C}} := \Big\{ J \in \mathcal{J}_\omega \;\Big|\; A \in \mathcal{S}_\omega^{\le -2} \text{ has an embedded } J\text{-holomorphic representative if and only if } A \in \mathcal{C} \Big\}.$$

The prime subset $\mathcal{J}_\varnothing$ is generally denoted by $\mathcal{J}_{\text{open}}$. And if $\mathcal{C} = \{A\}$ contains only one class A, we will use $\mathcal{J}_A$ for $\mathcal{J}_{\{A\}}$.

Notice that these prime subsets are disjoint and we have the decomposition $\mathcal{J}_\omega = \coprod_{\mathcal{C}} \mathcal{J}_{\mathcal{C}}$. We define a filtration according to the codimension of these prime subsets:

$$\cdots \subset \mathcal{X}_{2n+2} (= \mathcal{X}_{2n+1}) \subset \mathcal{X}_{2n} (= \mathcal{X}_{2n-1}) \subset \cdots \subset \mathcal{X}_2 (= \mathcal{X}_1) \subset \mathcal{X}_0 = \mathcal{J}_\omega,$$

where $\mathcal{X}_j := \coprod_{\operatorname{cod}(\mathcal{C}) \ge j} \mathcal{J}_{\mathcal{C}}$ is the union of all prime subsets having codimension no less than j.

To ensure nice properties of the decomposition into prime subsets we introduce the following condition for (X, ω).

Condition 1. *Let (X, ω) be a symplectic 4-manifold. Suppose A is a homology class in $H_2(X; \mathbb{Z})$ with $A \cdot A < 0$. Whenever A is represented by a simple J-holomorphic map $u : \mathbb{C}\mathrm{P}^1 \to (X, J)$ for some tamed J, then u is an embedding.*

We note that by [Zhang 2017], Condition 1 holds for symplectic rational surfaces with Euler number no larger than 11 (Lemma 2.8). We show that each $\mathcal{J}_{\mathcal{C}}$ is a submanifold with real codimension $\operatorname{cod}(\mathcal{C})$ under Condition 1. In Section 3, we prove the following result on $\mathcal{X}_0$, $\mathcal{X}_2$, and $\mathcal{X}_4$, which suffices for applications in this paper:

Theorem 1.2. *For a symplectic rational surface with $\chi(X) \le 8$, $\mathcal{X}_4 = \coprod_{\operatorname{cod}(\mathcal{C}) \ge 4} \mathcal{J}_{\mathcal{C}}$ and $\mathcal{X}_2 = \coprod_{\operatorname{cod}(\mathcal{C}) \ge 2} \mathcal{J}_{\mathcal{C}}$ are closed subsets in $\mathcal{X}_0 = \mathcal{J}_\omega$. Consequently, $\mathcal{X}_0 - \mathcal{X}_4$ is an open submanifold of $\mathcal{J}_\omega$ and $\mathcal{X}_2 - \mathcal{X}_4$ is a closed submanifold of $\mathcal{X}_0 - \mathcal{X}_4$.*

In [Li ≥ 2020] the first author will further prove the filtration is a stratification at every level for a symplectic rational surface with $\chi(X) \le 8$.

Inspired by [Abreu 1998], we apply a (relative) version of the Alexander–Pontrjagin duality in [Eells 1961] to get the following computation of $H_1(\mathcal{J}_{\text{open}}; \mathbb{Z})$.

Corollary 1.3. *For a symplectic rational surface with $\chi(X) \le 8$ and any abelian group G, $H^1(\mathcal{J}_{\text{open}}; G) = \bigoplus_{A_i \in \mathcal{S}_\omega^{-2}} H^0(\mathcal{J}_{A_i}; G)$.*

If we further assume that $\chi(X) \le 7$, then $\mathcal{J}_{A_i}$ is path connected for each $A_i \in \mathcal{S}_\omega^{-2}$, and $H_1(\mathcal{J}_{\text{open}}; \mathbb{Z}) = \mathbb{Z}^{N_\omega}$, where N_ω is the cardinality of $\mathcal{S}_\omega^{-2}$.

We apply Corollary 1.3 to study the topology of $\operatorname{Symp}_h(X, \omega)$, where X is a rational surface with $\chi(X) \le 7$. Notice that there is an action of $\operatorname{Symp}(X, \omega)$ on $\mathcal{J}_\omega$ which preserves the filtration. Moreover, the subgroup $\operatorname{Symp}_h(X, \omega)$ preserves each

prime subset since $\mathrm{Symp}_h(X,\omega)$ preserves the homology classes of J-holomorphic curves. This action will be used in [Li et al. 2019; $\geq$ 2020] to study the topology of $\mathrm{Symp}_h(X,\omega)$ for a rational surface X with $\chi(X)\geq 8$.

1B. ***Application to the symplectomorphism group.*** For a symplectic rational surface (X,ω) with $5\leq\chi(X)\leq 8$, the following diagram of homotopy fibrations, formulated in [Evans 2011] (in the monotone case, i.e., $[\omega]=\lambda c_1(X,\omega)$ for some $\lambda>0$) and adapted in [Li et al. 2015] for a general ω, relates $\mathcal{J}_{\mathrm{open}}$ and $\mathrm{Symp}_h(X,\omega)$:

$$\begin{array}{ccccccc} \mathrm{Symp}_c(U)\sim\mathrm{Stab}^1(D) & \longrightarrow & \mathrm{Stab}^0(D) & \longrightarrow & \mathrm{Stab}(D) & \longrightarrow & \mathrm{Symp}_h(X,\omega) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathcal{G}(D) & & \mathrm{Symp}(D) & & \mathscr{D}_0\sim\mathcal{J}_{\mathrm{open}} \end{array} \tag{2}$$

Each term above is a topological group except the term $\mathscr{D}_0\sim\mathcal{J}_{\mathrm{open}}$. Here $\sim$ means weak homotopy equivalence. We will recall each term in Section 4 and here we only explain the right tail of diagram (2):

$$\mathrm{Stab}(D)\to\mathrm{Symp}_h(X,\omega)\to\mathscr{D}_0\sim\mathcal{J}_{\mathrm{open}}. \tag{3}$$

Here $D=\bigcup D_i$ is a suitable exceptional divisor, which is a configuration of symplectic spheres as in [Evans 2011; Li et al. 2015], with $[D_i]\in\mathcal{S}_\omega^{-1}$. The space $\mathscr{D}_0$ of such configurations whose components intersect symplectic orthogonally is weakly homotopic to $\mathcal{J}_{\mathrm{open}}$, and it admits a transitive action of $\mathrm{Symp}_h(X,\omega)$. Therefore we have the above homotopy fibration (3) with $\mathrm{Stab}(D)$ as the stabilizer of the transitive action. Moreover, the (weak) homotopy type of $\mathrm{Stab}(D)$ can often be explicitly computed using the terms of the other parts of diagram (2). Hence if we can further reveal the homotopy type of $\mathcal{J}_{\mathrm{open}}$, which is very sensitive to the symplectic structure ω, we may probe at least partially the homotopy type of $\mathrm{Symp}_h(X,\omega)$ via the homotopy fibration (3).

Following this route, the full homotopy type of $\mathrm{Symp}_h(X,\omega)$ in the monotone case is determined in [Evans 2011] when $\chi(X)=6,7,8$ (the smaller χ cases follow from [Gromov 1985; Abreu 1998; Lalonde and Pinsonnault 2004]). And we show in [Li et al. 2015] that $\pi_0(\mathrm{Symp}_h)$ is trivial for $\chi(X)=7$ with a general ω (the smaller χ cases have been dealt in [Abreu 1998; Abreu and McDuff 2000; Lalonde and Pinsonnault 2004; Anjos and Pinsonnault 2013]). In addition, [Hind et al. 2016] treats similarly some noncompact cases. In this paper we continue to follow this route and systematically analyze the persistence and change of the topology of $\mathrm{Symp}(X,\omega)$ under deformations of symplectic forms (such phenomena were also discussed in [Ruan 1993; Seidel 1999; McDuff 2008]).

1B1. $\pi_1(\mathrm{Symp}_h(X,\omega))$ *and* N_ω. We are able to relate the fundamental group of $\mathrm{Symp}_h(X,\omega)$ with N_ω for a rational surface X with $\chi(X)\leq 7$. On the one hand, for

a rational surface X with $\chi(X) \le 7$, we compute $H_1(\mathcal{J}_{\text{open}}; \mathbb{Z})$ by counting (-2)-symplectic sphere classes in Corollary 1.3. On the other hand, when $5 \le \chi(X) \le 7$, we showed in [Li et al. 2015] that $\text{Stab}(D)$ is path connected for any ω. Hence we have the following portion of the long exact sequence of fibration (3):

$$\pi_1(\text{Stab}(D)) \to \pi_1(\text{Symp}_h(X, \omega)) \to \pi_1(\mathscr{D}_0) \to 1. \tag{4}$$

Note that $\pi_1(\text{Symp}_h(X, \omega))$ is abelian since $\text{Symp}_h(X, \omega)$ is a topological group. Thus we can conclude that $\pi_1(\mathscr{D}_0)$ is abelian since it is a quotient group of $\pi_1(\text{Symp}_h(X, \omega))$ by (4). Since $\mathscr{D}_0 \sim \mathcal{J}_{\text{open}}$, we have $\pi_1(\mathcal{J}_{\text{open}}) = \pi_1(\mathscr{D}_0)$ is also an abelian group, hence isomorphic to $H_1(\mathcal{J}_{\text{open}}; \mathbb{Z})$ by the Hurewicz theorem. Consequently, $\pi_1(\mathscr{D}_0) = \mathbb{Z}^{N_\omega}$.

We further examine the first map in (4). It turns out that, $\pi_1(\text{Stab}(D))$ is independent of ω by Proposition 4.5 and is isomorphic to $\pi_1(\text{Symp}_h(X, \omega_{\text{mon}}))$, where ω_{mon} is a monotone symplectic form. Moreover, Lemma 4.6 shows that $\pi_1(\text{Stab}(D))$ injects into $\pi_1(\text{Symp}_h(X, \omega))$. Since $\pi_1(\mathscr{D}_0) = \mathbb{Z}^{N_\omega}$ is free abelian, $\pi_1(\text{Symp}_h(X, \omega))$ is determined as follows, at least in the case $5 \le \chi(X) \le 7$:

Theorem 1.4. *If (X, ω) is a symplectic rational surface with $\chi(X) \le 7$,*

$$\pi_1(\text{Symp}_h(X, \omega)) = \mathbb{Z}^{N_\omega} \oplus \pi_1(\text{Symp}_h(X, \omega_{\text{mon}})). \tag{5}$$

For the case $\chi(X) \le 4$, we verify the relation (5) directly from the known computations.

1B2. *Stability and variation along the normalized reduced symplectic cone.* Notice that diffeomorphic symplectic forms have homeomorphic symplectomorphism groups. On the other hand, for a rational surface X, cohomologous symplectic forms are isotopic [Lalonde and McDuff 1996; Li and Liu 2001]. Therefore, up to homeomorphisms, $\text{Symp}(X, \omega)$ only depends on $[\omega]$. In particular, the homotopy type of $\text{Symp}(X, \omega)$ is determined by the point $[\omega]$ in the symplectic cone of X. Recall that the symplectic cone of X is the open subset of $H^2(X; \mathbb{R})$ that consists of classes of symplectic forms. The symplectic cone of $S^2 \times S^2$ is just the first quadrant in $H^2(S^2 \times S^2; \mathbb{R})$. The symplectic cone of $X_k = \mathbb{C}\text{P}^2 \# k\overline{\mathbb{C}\text{P}^2}$ is characterized (see [Biran 1997; 2001; Li and Liu 2001]) by a set of inequalities (infinitely many if $k \ge 2$). However, after identifying $H^2(X_k; \mathbb{R})$ as $\mathbb{R}^{k+1}$ via the canonical basis $\{H, E_1, \dots, E_k\}$, it is rather complicated to geometrically describe the symplectic cone in $\mathbb{R}^{k+1}$.

Here is a useful observation. It suffices to describe $\text{Symp}(X_k, \omega)$ when $[\omega]$ is in a *fundamental domain* of the symplectic cone under the homological action of $\text{Diff}^+(X_k)$ and the scaling operation.

For $X_k = \mathbb{C}\text{P}^2 \# k\overline{\mathbb{C}\text{P}^2}$, the notion of a reduced class in Definition 2.17 is useful for this purpose and we use the normalization $\omega(H) = 1$. Hence, for $X_k = \mathbb{C}\text{P}^2 \# k\overline{\mathbb{C}\text{P}^2}$,

we study the normalized reduced symplectic cone $P_k = P(X_k)$. Notice that we can identify each point in P_k as a vector $(c_1, c_2, \ldots, c_k) \in \mathbb{R}^k$, where c_i denotes the ω-area of E_i. For $S^2 \times S^2$, the normalized reduced symplectic cone can be similarly defined and $P(S^2 \times S^2)$ is the interval $[1, \infty) \subset \mathbb{R}^1$. When $\chi(X) \leq 11$, $P(X)$ is defined by finitely many (strict and nonstrict) linear inequalities in $\mathbb{R}^{\chi(X)-3}$, hence a (generally, neither closed nor open) convex polytope of dimension $\chi(X) - 3$. Notice that $P(X)$ is the disjoint union of its open faces of various dimensions, and notice that neighboring open faces have different dimensions. For instance, $P(S^2 \times S^2) = [1, \infty)$ is the disjoint union of the unique one dimensional open face, $(1, \infty)$, and the unique zero open face, $\{1\}$.

Moreover, for $3 \leq k \leq 8$, P_k has a uniform description as a cone with an open base in the $c_1 \cdots c_{k-1}$ hyperplane, with the point $M_k = \left(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}\right)$ as the vertex and k edges. Notice that M_k is the class of a monotone symplectic form on X_k. P_k has a unique zero dimensional open face M_k, k 1-dimensional open faces, each one being the interior of an edge, and generally, $\binom{k}{p}$ p-dimensional open faces. It is interesting that the k edges of P_k have a Lie theory interpretation as simple roots of a root system $R(X_k)$, well known to algebraic geometers. We further observe that the root system $R(X_k)$, $3 \leq k \leq 8$, coincides with the Lagrangian root system of $(X_k, \omega_{\text{mon}})$, where ω_{mon} is a monotone symplectic form on X_k.

We observe that, when $\chi(X) \leq 7$, both $\Gamma_L(X_k, \omega)$ and N_ω are stable on each open face of $P(X)$ and always jump between neighboring open faces (necessarily of different dimensions) with the same "amount of change." This observation leads to the following explicit stability and variation of $\pi_0(\text{Symp}(X, \omega))$ and $\pi_1(\text{Symp}(X, \omega))$.

Corollary 1.5. *For a rational surface X with $\chi(X) \leq 7$, $\pi_0(\text{Symp}(X, \omega))$ and $\pi_1(\text{Symp}(X, \omega))$ are stable on each open face of $P(X)$ and vary between neighboring open faces. Moreover, the sum $r^+[\pi_0(\text{Symp}(X, \omega))] + \text{Rank}[\pi_1(\text{Symp}(X, \omega))]$ is a constant given by $\frac{1}{2}(\chi(X) - 2)(\chi(X) - 3)$.*

Notice that in the above corollary, $\pi_1(\text{Symp}(X, \omega))$ is an abelian group and $\text{Rank}[\pi_1(\text{Symp}(X, \omega))]$ denotes the rank of its torsion-free part. When $\chi(X) \leq 7$, $\pi_0(\text{Symp}(X, \omega))$ is the Weyl group of the Lagrangian root system $\Gamma_L(X_k, \omega)$, and, by a slight abuse of notation, $r^+[\pi_0(\text{Symp}(X, \omega))]$ denotes the number of positive roots of $\Gamma_L(X_k, \omega)$.

Some notation.

- $\text{Symp}_h(X, \omega)$: the homological trivial part of symplectomorphism group of (X, ω).
- $\mathcal{J}_\omega$: the space of w-tamed almost complex structures.
- $\mathcal{C}$, $\mathcal{D}$ and $\mathfrak{D}$: subsets of $\mathcal{S}_\omega^{<0}$.

- $\mathcal{X}_{2n}$: the union of codimension no larger than $2n$ prime subsets of $\mathcal{J}_\omega$.
- $\mathcal{J}^{\mathfrak{D}}$: the set of almost complex structures J where each class in $\mathfrak{D}$ has an embedded J-holomorphic curve representative.
- N_ω: the number of ω-symplectic (-2)-sphere classes.
- $N_{\omega,L}$: the number of ω-Lagrangian sphere classes up to sign.
- $\Gamma_L(X,\omega)$: the root system of Lagrangian sphere classes associated to a symplectic 4-manifold (X,ω).
- $P(X)$: the normalized reduced symplectic cone of X, and $P_k := P(X_k)$.
- M_k: the (unique) monotone vertex of P_k for $3 \le k \le 8$.
- $r^+[\pi_0(\mathrm{Symp}(X,\omega))]$: the number of positive roots of the root system $\Gamma_L(X,\omega)$.

2. Decomposition of $\mathcal{J}_\omega$ and the reduced symplectic cone when $\chi \le 11$

In this section, we analyze the decomposition of $\mathcal{J}_\omega$ and describe the normalized reduced symplectic cone for symplectic rational surfaces with $\chi \le 11$. We often identify the degree-2 homology with degree-2 cohomology using Poincaré duality. Also recall that throughout this section, all finite-dimensional symplectic manifolds are smooth, closed and connected.

2A. *Decomposition of $\mathcal{J}_\omega$.*

2A1. *General facts on J-holomorphic curves.* We review some general facts on J-holomorphic rational curves and symplectic spheres in symplectic 4-manifolds. The presentation is similar to [Abreu 1998; Anjos and Pinsonnault 2013].

Let (X,ω) be a symplectic 4-manifold and $J \in \mathcal{J}_\omega$. A parametrized J-holomorphic curve in X is a J-holomorphic map $u : (\Sigma, j) \to (X, J)$, where (Σ, j) is a smooth, connected Riemann surface. We will always assume that u is simple, i.e., it is nonconstant and not a multiple covering. In this case, we say that $C = u(\Sigma)$ is an (unparametrized) J-holomorphic curve and denote by $[C]$ the homology class. Notice that the pairing $\omega([C])$ is positive.

Theorem 2.1 (positivity of intersection [Gromov 1985; McDuff 1994; Micallef and White 1995]). *For an almost complex 4-manifold (X, J), two distinct simple J-holomorphic curves C, C' have only finitely many intersection points. Each such point p contributes $k_p \ge 1$ to the homological intersection number $[C]\cdot[C']$, and $k_p = 1$ if and only if C and C' meet transversally at p.*

Theorem 2.2 (adjunction inequality [McDuff 1990]). *Let (X, J) be an almost complex 4-manifold with first Chern class $c_1(X, J)$ and $u : (\Sigma, j) \to X$ a simple J-holomorphic curve. Then the virtual genus of the image $C = u(\Sigma)$, defined as*

$g_v(C) = \big([C]\cdot[C] - c_1(X,J)([C])\big)/2 + 1$, *is a integer no less than* $g(\Sigma)$. *Moreover*, $g_v(C) = g(\Sigma)$ *if and only if* u *is an embedding.*

We are primarily interested in rational curves. A (parametrized) J-holomorphic rational curve in X is a J-holomorphic map $u : \mathbb{C}\mathrm{P}^1 \to (X, J)$.

Theorem 2.3 (Gromov [1985] compactness theorem for rational curves). *Let* (X, ω) *be a symplectic* 4*-manifold, and let* J_n *be a sequence in* $\mathcal{J}_\omega$ *which converges to* J_0 *in the* C^∞*-topology. Suppose* $u_n : \mathbb{C}\mathrm{P}^1 \to (X, J_n)$ *is a sequence of simple* J_n*-holomorphic rational curves such that* $[u_n(\mathbb{C}\mathrm{P}^1)] = A \in H_2(X;\mathbb{Z})$, $A \neq 0$.

Then, there is a subsequence of $\{u_n\}$, *still denoted* $\{u_n\}$, *which either converges to a simple* J_0*-holomorphic rational curve in the class* A *in the* C^∞*-topology, or weakly converges to a stable* J*-holomorphic rational curve in the class* A.

The weak convergence of $\{u_n\}$ *is described as follows.*

- *Up to a reparametrization of each* u_n, *there are finitely many simple closed loops* γ_i *in* $\mathbb{C}\mathrm{P}^1$, *and a connected finite union of Riemann spheres* $\Sigma_0 = \bigcup_\alpha \mathbb{C}\mathrm{P}^1_\alpha$ *which is obtained by collapsing each of the simple closed curves* γ_i *on* $\mathbb{C}\mathrm{P}^1$ *to a point.*
- *There is a continuous map* $u : \Sigma_0 \to X$, *called a stable rational curve in the class* A, *such that each* $u|_{\mathbb{C}\mathrm{P}^1_\alpha}$ *is a possibly multiply covered* J_0*-holomorphic rational curve, i.e.,* $u|_{\mathbb{C}\mathrm{P}^1_\alpha}$ *is the composition of a degree-*m_α *covering* $\mathbb{C}\mathrm{P}^1_\alpha \to \mathbb{C}\mathrm{P}^1$ *and a simple* J_0*-holomorphic map* $u'_\alpha : \mathbb{C}\mathrm{P}^1 \to (X, J_0)$. *Moreover,*

$$\sum_\alpha m_\alpha [u'_\alpha(\mathbb{C}\mathrm{P}^1_\alpha)] = A. \tag{6}$$

Finally, $\{u_n\}$ *converges to* u *in the complement of any fixed open neighborhood of* $\bigcup_i \gamma_i$ *in the* C^∞*-topology.*

2A2. *General facts on symplectic spheres.* We next list various facts about representing a class in $\mathcal{S}_\omega$ by simple (or stable) J-holomorphic rational curves.

For classes with self-intersection at least -1 we have the following well-known fact from the Gromov–Witten theory.

Proposition 2.4 [Li and Zhang 2011, Lemma 3.3]. *Let* (X, ω) *be a symplectic* 4*-manifold and* A *a class in* $\mathcal{S}_\omega^{-1}$ *or* $\mathcal{S}_\omega^{\geq 0}$. *Then there is a simple or stable* J*-holomorphic rational curve in the class* A *for any* $J \in \mathcal{J}_\omega$, *and by the adjunction inequality, any simple* J*-holomorphic curve in the class* A *is embedded.*

Moreover, if $\{B_i\}$ *is any collection of classes in* $\mathcal{S}_\omega^{-1}$ *or* $\mathcal{S}_\omega^{\geq 0}$, *then for a generic* $J \in \mathcal{J}_\omega$ (in a subset of second category, see also Lemma 2.12), *there is an embedded* J*-holomorphic rational curve in each class* B_i. *Consequently, by the positivity of intersection,* $B_i \cdot B_j \geq 0$ *if* $B_i \neq B_j$ *are in* $\mathcal{S}_\omega^{-1}$ *or* $\mathcal{S}_\omega^{\geq 0}$.

On the other hand, for classes with negative self-intersection, we have the uniqueness by the positivity of intersection.

Lemma 2.5. *Let (X, ω) be a symplectic 4-manifold and B a class in $\mathcal{S}_\omega^{\leq -1}$. If, for some $J \in \mathcal{J}_\omega$, there is a simple J-holomorphic rational curve in the class B, then there cannot be a distinct simple J-holomorphic curve or a stable J-holomorphic rational curve in B.*

For classes with self-intersections at most -2 we have:

Proposition 2.6 [Anjos and Pinsonnault 2013, Appendix B.1]. *Let (X, ω) be a symplectic 4-manifold. Suppose $U_\mathcal{C} \subset \mathcal{J}_\omega$ is a subset characterized by the existence of a configuration of embedded J-holomorphic rational curves $C_1 \cup C_2 \cup \cdots \cup C_N$ of negative self-intersection with $\{[C_1], [C_2], \ldots, [C_N]\} = \mathcal{C}$. Then $U_\mathcal{C}$ is a cooriented Fréchet submanifold of $\mathcal{J}_\omega$ of* (real) *codimension* $2N - 2c_1([C_1] + \cdots + [C_N])$.

Remark 2.7. Suppose (X, ω) is a symplectic rational surface with $\chi(X) \leq 8$. We will show in Proposition 3.4 that then the set $\mathcal{S}_\omega^{<0}$ is finite. Hence there are a finite number of admissible sets and each admissible set $\mathcal{C}$ is finite in this case. It should follow from Section 4 in [Zhang 2017] that the finiteness of $\mathcal{S}_\omega^{<0}$ continues to hold when $\chi(X) \leq 11$.

2A3. *Condition 1 and cusp curve decomposition.* We first recall Condition 1 for a symplectic 4-manifold (X, ω): Suppose A is a class in $H_2(X; \mathbb{Z})$ with $A \cdot A < 0$. Whenever A is represented by a simple J-holomorphic map $u : \mathbb{C}\mathrm{P}^1 \to X$ for some tamed J, then u is an embedding.

Note that, under this assumption, $\mathcal{S}_\omega^{<0}$ is the same as the set of homology classes with negative self-intersection and having a simple rational pseudoholomorphic curve representative.

Lemma 2.8 (see [Zhang 2017, Proposition 4.2]). *Condition 1 holds true for a symplectic rational surface (X, ω) with $\chi(X) \leq 11$.*

Condition 1 has crucial consequences for cusp curve decompositions and the prime sets.

Lemma 2.9. *Assume Condition 1 holds. Let $\mathcal{C}$ be an admissible set and $A \in \mathcal{S}_\omega$. For $J \in \mathcal{J}_\mathcal{C}$, suppose A is represented by a stable rational curve. Then the cusp decomposition of the class A for $J \in \mathcal{J}_\mathcal{C}$ is of the form*

$$A = \sum_\alpha r_\alpha [C_\alpha] + \sum_\beta p_\beta [C_\beta] + \sum_\gamma q_\gamma [C_\gamma], \tag{7}$$

where $r_\alpha, p_\beta, q_\gamma$ are positive integers, $C_\alpha, C_\beta, C_\gamma$ are simple J-holomorphic rational curves with $[C_\alpha]^2 \leq -2$, $[C_\beta]^2 = -1$, $[C_\gamma]^2 \geq 0$. Moreover:

- *C_α is embedded and hence $[C_\alpha] \in \mathcal{C}$.*
- *C_β is embedded and hence $[C_\beta] \in \mathcal{S}_\omega^{-1}$.*
- *$A \cdot [C_\gamma] \geq 0$.*
- *$A \neq [C_\alpha]$, $A \neq [C_\beta]$, $A \neq [C_\gamma]$ for any α, β, γ.*

Proof. The first and second bullets follow from Condition 1.

Notice that, each $[C_\gamma]$ has $[C_\gamma]^2 \geq 0$, by the positivity of intersection, $[C_\gamma]$ pairs positively with any curve class in the decomposition of A. In turn, we have $A \cdot \sum_\gamma q_\gamma [C_\gamma] \geq 0$.

The last bullet follows from the positivity of area. □

Now we examine the case $A \in \mathcal{S}_\omega^{-1}$.

Definition 2.10. Given an admissible set $\mathcal{C} \subset S_\omega^{\leq -2}$, let

$$S_\omega^{-1}(\mathcal{C}) = \{A \in S_\omega^{-1} \mid A \cdot A_i \geq 0 \text{ for any } A_i \in \mathcal{C}\}.$$

Definition 2.11. For a subset $\mathfrak{D} \subset \mathcal{S}_\omega^{-1}$, let $\mathcal{J}^{\mathfrak{D}}$ denote the set of $J \in \mathcal{J}_\omega$ such that there is an embedded J-holomorphic rational curve in each class in $\mathfrak{D}$.

Note that here $\mathfrak{D}$ is different from the admissible label set $\mathcal{C}$ in Definition 1.1 since $\mathcal{C} \subset \mathcal{S}_\omega^{\leq -2}$. We will next relate the prime set $\mathcal{J}_\varnothing = \mathcal{J}_{\text{open}}$ with $\mathcal{J}^{\mathfrak{D}}$ under Condition 1.

Lemma 2.12. *Assume Condition 1 holds. Let $\mathcal{C}$ be an admissible set and $A \in \mathcal{S}_\omega^{-1}$. If A is represented by a stable rational curve then the cusp decomposition* (7) *of the class A for $J \in \mathcal{J}_{\mathcal{C}}$ has the additional property*: $A \cdot [C_\beta] \geq 0$. *Consequently, A is represented by an embedded J-holomorphic rational curve if $A \in S_\omega^{-1}(\mathcal{C})$.*

If $J \in \mathcal{J}_{\text{open}}$, then:

- *There are no simple J-holomorphic rational curves with self-intersection less than -1.*
- *For any $A \in \mathcal{S}_\omega^{-1}$, there is an embedded J-holomorphic rational curve in A* (in fact, unique).
- *$\mathcal{J}_{\text{open}} \subset \mathcal{J}^{\mathfrak{D}}$ for any $\mathfrak{D} \subset \mathcal{S}_\omega^{-1}$.*
- *$\mathcal{J}^{\mathfrak{D}} = \mathcal{J}_{\text{open}}$ if every class in $\mathcal{S}_\omega^{\leq -2}$ pairs negatively with some element in $\mathfrak{D}$.*

Proof. Suppose $A \in \mathcal{S}_\omega^{-1}$ and A is represented by a stable rational curve. By Lemma 2.9, the cusp decomposition (7) of A satisfies $A \cdot [C_\gamma] \geq 0$. Recall that $A \in S_\omega^{-1}$, and notice that $A \neq [C_\beta]$ for any β since otherwise A is represented by C_β. By Proposition 2.4, $A \cdot \sum_\beta p_\beta [C_\beta] \geq 0$. If $A \in S_\omega^{-1}(\mathcal{C})$ then $A \cdot [C_\alpha] \geq 0$. Combining the three inequalities, we have $A \cdot A \geq 0$. But this contradicts to $A \cdot A = -1 < 0$. Therefore A has an embedded J-holomorphic curve representative for each $J \in \mathcal{J}_{\mathcal{C}}$ since, by Proposition 2.4, A is either represented by an embedded J-holomorphic rational curve or represented by a J-holomorphic stable rational curve.

Now suppose $J \in \mathcal{J}_\varnothing = \mathcal{J}_{\text{open}}$ and we verify the four statements.

The first statement follows from the definition of $\mathcal{J}_{\text{open}}$ (Definition 1.1) and Condition 1.

The second statement follows from Proposition 2.4 and the first statement. Notice that $\mathcal{S}_\omega^{-1}(\varnothing) = \mathcal{S}_\omega^{-1}$.

The third statement that $\mathcal{J}_{\text{open}} \subset \mathcal{J}^{\mathfrak{D}}$ for any $\mathfrak{D} \subset \mathcal{S}_\omega^{-1}$ follows directly from the second statement.

For the last statement, it suffices to show that $\mathcal{J}^{\mathfrak{D}} \subset \mathcal{J}_{\text{open}}$ by the third statement. If $J \in \mathcal{J}^{\mathfrak{D}}$, then there is an embedded J-holomorphic rational curve in each class in $\mathfrak{D}$. The desired conclusion follows from the positivity of intersection. □

We will further study when $\mathcal{J}_{\text{open}} = \mathcal{J}^{\mathfrak{D}}$ in Section 3D.

Remark 2.13. Two related subsets $\mathcal{J}_{\text{top}}$ and $\mathcal{J}_{\text{good}}$ were introduced in [Li and Zhang 2015]. A tamed J on a rational manifold is called *good* [Li and Zhang 2015, page 4; Friedman and Morgan 1988, when J is integrable] if

(i) there is a smooth genus one subvariety in the anticanonical class $-K_J$, and

(ii) any irreducible genus zero subvariety of negative self-intersection is a (-1)-curve.

$J \in \mathcal{J}_{\text{top}}$ [Li and Zhang 2015, Definition 2.13] if any simple J-holomorphic curve with negative self-intersection is an embedded rational curve with self-intersection -1. Clearly, $\mathcal{J}_{\text{top}} \subset \mathcal{J}_{\text{open}}$ since the definition of $\mathcal{J}_{\text{open}}$ only excludes simple J-holomorphic *rational* curves with self-intersection at most -2. However, when $\chi(X) \le 11$, $\mathcal{J}_{\text{top}} = \mathcal{J}_{\text{open}}$ by Proposition 4.2 in [Zhang 2017].

The requirement in Lemma 2.12 that $A \in S_\omega^{-1}(\mathcal{C})$ be represented by an embedded J-holomorphic rational curve could be compared with [McDuff and Opshtein 2015, Theorem 1.2.7(iii)]. Our proof is much easier since we assume Condition 1 holds.

2A4. *Condition 1 and prime submanifolds.* Recall that $\mathcal{J}_\omega$ is metrizable (see [Fujiki and Schumacher 1990]). Hence the prime subsets are also metrizable. Here is the main result of this subsection.

Proposition 2.14. *Assume Condition 1 holds. For an admissible set $\mathcal{C}$ with nonempty $\mathcal{J}_\mathcal{C}$, the prime set $\mathcal{J}_\mathcal{C}$ is a paracompact Hausdorff submanifold with* $\operatorname{cod}(\mathcal{J}_\mathcal{C}) = \operatorname{cod}_\mathcal{C} = \sum_{C_i \in \mathcal{C}} \operatorname{cod}_{C_i}$.

In particular, for each $A \in \mathcal{S}_\omega^{-k}$, $\mathcal{J}_A$ is a paracompact Hausdorff submanifold with codimension $2k-2$.

To prove this result we prepare a couple of lemmas.

Lemma 2.15. *Assume Condition 1. Suppose $\mathcal{J}_\mathcal{C}$ and $\mathcal{J}_{\mathcal{C}'}$ are two distinct prime subsets with $J_0 \in \overline{\mathcal{J}_\mathcal{C}} \cap \mathcal{J}_{\mathcal{C}'} \neq \varnothing$. Then there is a nonempty subset $\mathcal{C}_{\text{deg}} = \{A^i\} \subset \mathcal{C}$ such that $\mathcal{C} \setminus \mathcal{C}_{\text{deg}} \subset \mathcal{C}'$, and each class A^i is represented by a J_0-holomorphic stable rational curve.*

Proof. Since $J_0 \in \overline{\mathcal{J}_\mathcal{C}} \cap \mathcal{J}_{\mathcal{C}'} \neq \varnothing$, then there is a convergent sequence $\{J_n\} \subset \mathcal{J}_\mathcal{C}$ such that $\{J_n\} \to J_0 \in \mathcal{J}_{\mathcal{C}'}$. For J_0, take all the elements in $\mathcal{C}$ that are not irreducibly

J_0-holomorphic, and denote the subset by $\mathcal{C}_{\deg} = \{A^i\}$. $\mathcal{C}_{\deg}$ is nonempty since $\mathcal{C} \neq \mathcal{C}'$. Note that any class in $\mathcal{C} \setminus \mathcal{C}_{\deg}$ has an irreducible J_0-holomorphic curve representative, and hence $\mathcal{C} \setminus \mathcal{C}_{\deg} \subset \mathcal{C}'$. By Theorem 2.3, for each $A^i \in \mathcal{C}_{\deg}$, there is a J_0-holomorphic stable rational curve in the class A. □

Lemma 2.16. *Assume Condition 1. If $\mathcal{C}' \subset \mathcal{C}$ but $\mathcal{C}' \neq \mathcal{C}$, then $\overline{\mathcal{J}_{\mathcal{C}}} \cap \mathcal{J}_{\mathcal{C}'} = \varnothing$.*

Proof. We argue by contradiction. Suppose there exists some $J' \in \overline{\mathcal{J}_{\mathcal{C}}} \cap \mathcal{J}_{\mathcal{C}'}$. It follows from Lemma 2.15 and (7) that, for some $A \in \mathcal{C} \setminus \mathcal{C}'$, there is a cusp decomposition of the form

$$A = \sum_{\alpha} r_\alpha [C_\alpha] + \sum_{\beta} p_\beta [C_\beta] + \sum_{\gamma} q_\gamma [C_\gamma],$$

where r_α, p_β, q_γ are positive integers, C_α, C_β are simple J'-holomorphic rational curves with $[C_\alpha]^2 \leq -2$, $[C_\beta]^2 = -1$, $[C_\gamma]^2 \geq 0$. Moreover:

- C_α is embedded and hence $[C_\alpha] \in \mathcal{C}$.
- C_β is embedded and hence $[C_\beta] \in \mathcal{S}_\omega^{-1}$.
- $A \cdot [C_\gamma] \geq 0$.
- $\omega(A) > \omega([C_\alpha]), \omega([C_\beta]), \omega([C_\gamma])$ for any α, β, γ.

Since $J' \in \mathcal{J}_{\mathcal{C}'}$ we have $[C_\alpha] \in \mathcal{C}'$, and hence $[C_\alpha] \in \mathcal{C}$ by our assumption $\mathcal{C}' \subset \mathcal{C}$. Since $A \in \mathcal{C}$ and $[C_\alpha] \in \mathcal{C}$, we have $A \cdot [C_\alpha] \geq 0$ for each α. Notice that $A \cdot [C_\gamma] \geq 0$ for each γ.

Since $A \cdot A < 0$, there must exist β_0 such that $C_{\beta_0} \cdot A < 0$. Notice that $\omega(A) > \omega(C_{\beta_0})$. Fix $J \in \mathcal{C}$. Since $[C_{\beta_0}] \in \mathcal{S}_\omega^{-1}$, it must have J-holomorphic curve or stable curve representative. Since $A \in \mathcal{C}$, an embedded representative of $[C_{\beta_0}]$ implies $A \cdot [C_{\beta_0}] \geq 0$.

Thus we are left with the case of a J-holomorphic stable rational curve in the class $[C_{\beta_0}]$. However, since $A \in \mathcal{C}$ and $A \cdot [C_{\beta_0}] < 0$, in such a J-holomorphic stable rational curve of $[C_{\beta_0}]$ there has to be an irreducible component in the class A because otherwise we would have $A \cdot [C_{\beta_0}] \geq 0$. But this implies that $\omega([C_{\beta_0}] > \omega(A)$, contradicting to the inequality $\omega(A) > \omega(C_{\beta_0})$. □

Finally, we verify the prime subsets are submanifolds.

Proof of Proposition 2.14. First, note that $\mathcal{J}_{\mathcal{C}}$ is a subset of $U_{\mathcal{C}}$, which is a submanifold of $\mathcal{J}_\omega$ with codimension $\sum_{i \in I} \operatorname{cod}_{C_i}$ by Proposition 2.6. Then we examine $U_{\mathcal{C}} \setminus \mathcal{J}_{\mathcal{C}}$. $U_{\mathcal{C}}$ is a disjoint union of $\mathcal{J}_{\mathcal{C}'}$ where each $\mathcal{C}'$ is admissible and contains $\mathcal{C}$ as a proper subset. And the union of these $\mathcal{J}_{\mathcal{C}'}$ is relatively closed in $U_{\mathcal{C}}$ by Lemma 2.16. Hence $\mathcal{J}_{\mathcal{C}}$ is itself a submanifold of codimension $\sum_{i \in I} \operatorname{cod}_{C_i}$. The paracompactness and Hausdorff property come from the metrizable property (see [Fujiki and Schumacher 1990]). □

To further analyze the decomposition, especially $\mathcal{X}_2$ and $\mathcal{X}_4$, we will restrict to the case of a reduced symplectic form.

2B. *The cone $P(X)$, the root systems $R(X)$ and $\Gamma_L(X, \omega)$.*

2B1. *Reduced symplectic forms.* We recall the notion of reduced symplectic forms.

Definition 2.17. Let X be $\mathbb{C}\mathrm{P}^2\#k\overline{\mathbb{C}\mathrm{P}^2}$ with a standard basis $\{H, E_1, E_2, \ldots, E_k\}$ of $H_2(X;\mathbb{Z})$. A class $\nu H - \sum c_i E_i$ is called *reduced* (with respect to the basis) if

$$c_1 \geq c_2 \geq \cdots \geq c_k > 0 \quad \text{and} \quad \nu \geq c_1 + c_2 + c_3.$$

Reduced cohomology classes are defined as the Poincaré dual of reduced homology classes. A symplectic form ω on X is called reduced if $[\omega]$ is reduced. A reduced symplectic class is the class of a reduced symplectic form.

To us, the importance of the notion is the following result [Zhao et al. 2006; Li and Liu 2001; Karshon and Kessler 2017, and its MathSciNet review]:

Theorem 2.18. *For a rational surface $X = \mathbb{C}\mathrm{P}^2\#k\overline{\mathbb{C}\mathrm{P}^2}$, every class with positive square in $H^2(X;\mathbb{R})$ is equivalent to a unique reduced class under the action of* $\mathrm{Diff}^+(\mathrm{X})$. *If a symplectic form ω on X is reduced, then its canonical class is*

$$K_\omega = -3H + \sum_{i=1}^{k} E_i.$$

When $3 \leq k \leq 8$, any reduced class is represented by a symplectic form. When $k \leq 2$, any reduced class with $\nu > c_1 + c_2$ is represented by a symplectic form.

For the history of the first statement, see Weiyi Zhang's MathSciNet review of [Karshon and Kessler 2017], where a proof in this generality is also given following [Zhao et al. 2006]. The rest of the theorem follows from Theorem 3 and Proposition 3.6 in [Li and Liu 2001].

Lemma 2.19. *For any reduced symplectic form ω on $\mathbb{C}\mathrm{P}^2\#k\overline{\mathbb{C}\mathrm{P}^2}$, $k \geq 3$, E_k has the smallest ω-area in the set $\mathcal{S}_\omega^{-1}$.*

Proof. Any class in $\mathcal{S}_\omega^{-1}$ is of the form E_i, $1 \leq i \leq k$, or $A = dH - \sum_{i=1}^k a_i E_i$ is in $\mathcal{S}_\omega^{-1}$ for $d > 0$, $a_i \geq 0$ (see, e.g., [Li and Wu 2012]). By the reduced assumption, $\omega(E_1) \geq \cdots \geq \omega(E_i) \geq \cdots \geq \omega(E_k)$ and $\omega(H) \geq \omega(E_i) + \omega(E_j) + \omega(E_k)$, for all distinct i, j, k.

Suppose $A = dH - \sum_{i=1}^k a_i E_i$ is in $\mathcal{S}_\omega^{-1}$. By the adjunction formula, $K_\omega \cdot A = -1$ and hence $3d = 1 + \sum_{i=1}^n a_i$. Also by positive paring with the classes $H - E_1$ and $H - E_2$, $a_1 \leq d$ and $a_2 \leq d$. In particular, $a_1 + a_2 \leq 2d$. Therefore it is easy to see that we can write A as a sum $A = U_1 + \cdots + U_{d-1} + V$, where each U_a is of the form $H - E_p - E_q - E_r$ with no repeated E_1 or E_2, and V is of the form

$H - E_i - E_j$. Observe that $\omega(U_a) \geq 0$ and $\omega(V) \geq \omega(E_k)$ by the reduced condition. Thus the lemma would follow from such a decomposition of A. □

2B2. *$P(X_k)$ as a polyhedral cone for $3 \leq k \leq 8$.*

Definition 2.20. Let $X_k = \mathbb{C}\mathrm{P}^2 \# k\overline{\mathbb{C}\mathrm{P}^2}$. Its normalized reduced symplectic cone $P_k = P(X_k)$ is defined as the space of reduced symplectic classes having area 1 on H. We represent such a class by $(1\,|\,c_1, \ldots, c_k)$, or $(c_1, \ldots, c_k) \in \mathbb{R}^k$.

When $k \leq 8$, there is a distinguished class in P_k, $M_k = \left(1\,|\,\frac{1}{3}, \ldots, \frac{1}{3}\right) = \left(\frac{1}{3}, \ldots, \frac{1}{3}\right)$, which is called the (normalized) monotone class. When $3 \leq k \leq 8$, we will show that P_k is a polyhedral cone that is described by M_k and the following k classes of square -2:

$$l_1 = H - E_1 - E_2 - E_3, \quad l_2 = E_1 - E_2, \quad \ldots, \quad l_k = E_{k-1} - E_k. \tag{8}$$

Proposition 2.21. *For $X_k = \mathbb{C}\mathrm{P}^2 \# k\overline{\mathbb{C}\mathrm{P}^2}$, $3 \leq k \leq 8$, the normalized reduced symplectic cone P_k is a polyhedral cone in $\mathbb{R}^k$ with the vertex at M_k and the convex base in the $c_1 c_2 \cdots c_{k-1}$ (i.e., $c_k = 0$) hyperplane generated by the following k points G_i:*

$$G_1 = (0, \ldots, 0), \quad G_2 = (1, 0, \ldots, 0), \quad G_3 = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0, \ldots, 0\right),$$
$$G_4 = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, \ldots, 0\right), \quad \ldots, \quad G_k = \left(\tfrac{1}{3}, \ldots, \tfrac{1}{3}, 0\right).$$

The symplectic classes on each edge $M_k G_i$ are characterized by the property of pairing trivially with l_j for any $j \neq i$ and positively on l_i.

Consequently, the reduced symplectic classes are characterized as the symplectic classes which are positive on each E_i and nonnegative on each l_i.

Proof. When $3 \leq k \leq 8$, any reduced class of X_k is a reduced symplectic class. And a normalized reduced class satisfies

$$1 \geq c_1 + c_2 + c_3, \quad c_1 \geq c_2, \quad c_2 \geq c_3, \quad \ldots, \quad c_{k-1} \geq c_k, \quad c_k > 0. \tag{9}$$

Let Ψ be the translation moving $M_k = \left(\frac{1}{3}, \ldots, \frac{1}{3}\right)$ to 0. Under this linear translation, $(1\,|\,c_1, \ldots, c_k)$ is moved to $x = (x_1, \ldots, x_k) = \left(c_1 - \frac{1}{3}, \ldots, c_k - \frac{1}{3}\right)$, and the normalized reduced condition (9) can be written as the k homogeneous conditions:

$$0 \geq x_1 + x_2 + x_3, \quad x_1 - x_2 \geq 0, \quad x_2 - x_3 \geq 0, \quad \ldots, \quad x_{k-1} - x_k \geq 0, \quad x_k > -\tfrac{1}{3}.$$

Clearly, $\Psi(P_k)$ has only one vertex at the origin and its opposite face is open and at the hyperplane $x_k = -\frac{1}{3}$. There are k inequalities of the form $\geq$ in (9). Setting $c_k = 0$ and all of the k inequality $\geq$ to be equality except the i-th one, we obtain the k points G_i in the $c_1 \cdots c_{k-1}$ hyperplane. The rays $M_k G_i$ are clearly extremal rays.

Notice that M_k pairs trivially with each l_j, and G_i pairs trivially with each l_j for each $j \neq i$. It follows that $M_k G_i$ pairs trivially with each l_j except for $j = i$. □

We remark that the K-symplectic cone in [Li and Liu 2001] for

$$K = -3H + \sum_{i=1}^{k} E_i$$

has an explicit geometric description in [Zhang 2017, Proposition 3.2]. This $K-$symplectic cone is acted on by the Cremona group, where the Cremona group is the subgroup of $\mathrm{Aut}^+(H^2(X, \mathbb{Z}))$ preserving K. The cone of reduced symplectic classes is just the fundamental domain of the K-symplectic cone under this action, and the normalized reduced cone P_k in Proposition 2.21 is the slice in the fundamental domain with the H coefficient being 1.

Definition 2.22. A p-dimensional *open face* of P_k is defined as the *interior* of the convex hull of M_k together with $p \leq k$ points in the set $\{G_i\}$. P_k has 2^k open faces in total: a unique zero dimensional open face M_k; k one dimensional open faces, and generally, $\binom{k}{p}$ open faces of dimension p.

Our convention is to denote an open face with vertices $v_1, v_2, \ldots, v_l$ simply by $v_1 v_2 \cdots v_l$.

The top dimensional open face is the space of reduced classes given by the k strict inequalities: $\lambda := c_1 + c_2 + c_3 < 1;\ c_1 > \cdots > c_k$. An open face of codimension l is when l of those ">" is turned into "=".

Also note that, when projected onto the hyperplane $c_k = 0,\ M_k$ is sent to M_{k-1} and P_k is sent to P_{k-1}.

In Figure 1, the tetrahedron $MOAB$ describes P_3, where $M = M_3,\ O = G_1,$ $A = G_2,\ B = G_3$. There are 8 open faces:

$$M,\ MO,\ MA,\ MB,\ MOA,\ MOB,\ MAB,\ MOAB.$$

Notice that the triangle OAB is completely outside P_3. We also remark that the change from G_i to $MOAB$ notation will be used frequently.

2B3. *The root system R_k and edges of P_k.* Recall that a root system Φ in an inner product vector space $(E, \langle \cdot, \cdot \rangle)$ is a finite spanning set of nonzero vectors, called roots, that satisfies the following 2 conditions:

(a) If $\alpha \in \Phi$ then $n\alpha \in \Phi$ if and only if $n = \pm 1$.

(b) For any two roots $\alpha, \beta \in \Phi$, the number $2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$ is an integer and $\sigma_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha \in \Phi$.

Recall that, given a root system Φ in $(E, \langle \cdot, \cdot \rangle)$, a set $\triangle$ of simple roots is a subset of Φ which forms a basis of E and has the additional property that every root in

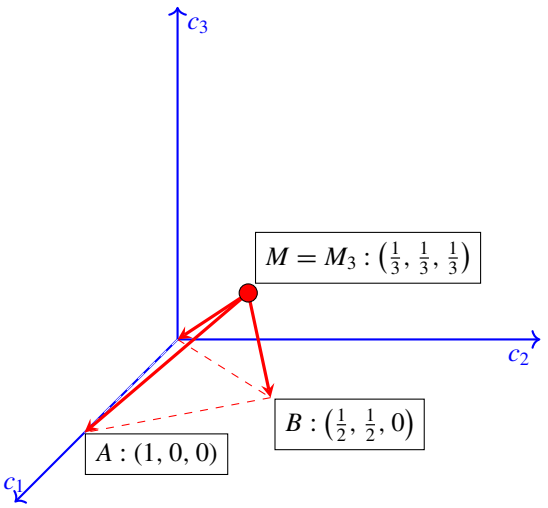


Figure 1. Normalized reduced symplectic cone of $\mathbb{C}\mathrm{P}^2\#3\overline{\mathbb{C}\mathrm{P}^2}$.

Φ is an integral linear combination of elements of $\triangle$ with the coefficients either all nonnegative or all nonpositive. Given a set $\triangle$ of simple roots, the set of roots which are nonnegative (nonpositive) integral linear combinations of elements of $\triangle$ is called the set of positive (negative) roots, and it is denoted by Φ^+ (Φ^-). Clearly, $\Phi = \Phi^+ \sqcup \Phi^-$ and $|\Phi^+| = |\Phi^-| = \frac{1}{2}|\Phi|$.

We slightly reformulate a beautiful fact in [Manin 1986] (Theorem 23.9 in pages 115–116, see also page 1 in [Leung and Zhang 2015]). For X_k with $3 \le k \le 8$, define the set

$$R(X_k) = R_k := \left\{A \in H_2(X_k, \mathbb{Z}) \mid \langle A, K_k\rangle = 0,\ \langle A, A\rangle = -2\right\}, \tag{10}$$

where $K_k = -(3H - E_1 - \cdots - E_k)$. It is straightforward to check that the root system R_k satisfies both conditions (a) and (b) and hence forms a root system. R_k as a root system is described in the this table:

k	3	4	5	6	7	8
$R(X_k)$	$\mathbb{A}_1 \times \mathbb{A}_2$	$\mathbb{A}_4$	$\mathbb{D}_5$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$\lvert R(X_k)\rvert$	8	20	40	72	126	240

Moreover,

- The classes l_i in (8) are in the root system R_k and provide a canonical choice of simple roots of R_k, which form the vertices of the Dynkin diagram.
- As remarked after Proposition 2.21, the reduced cone is the fundamental domain of some Weyl group action, and positive roots pair nonnegatively with

any vector in the Weyl chamber, which means they pair nonnegatively with an arbitrary reduced form.

- By Proposition 2.21 these simple roots l_i correspond to the edges M_kG_i of P_k in the sense that the symplectic classes on each edge M_kG_i are characterized by the property of pairing trivially with l_j for any $j \neq i$ and positively on l_i. When $k=3$, with the notation $MOAB$, l_1 corresponds to MO, l_2 corresponds to MA and l_3 corresponds to MB. We denote this correspondence $l_i \leftrightarrow MG_i$.

2B4. *Lagrangian root systems for* $3 \leq k \leq 8$. The class of a Lagrangian sphere in a symplectic 4-manifold (X, ω) has square -2 and pairs trivially with $[\omega]$ and K_ω. A class is called a Lagrangian sphere class if it is represented by an embedded Lagrangian sphere. Let $\Gamma_L(X, \omega)$ be the set of Lagrangian sphere classes. The following observation should be known to experts.

Lemma 2.23. *Suppose* (X, ω) *is a symplectic* 4*-manifold with a finite* $\Gamma_L(X, \omega)$. *Then* $\Gamma_L(X, \omega)$ *is a root system in the span of the root system* $\Gamma_L(X, \omega)$.

Proof. Condition (a) is clearly satisfied for $\Gamma_L(X, \omega)$ since any class $\alpha \in \Gamma_L(X, \omega)$ has $\alpha \cdot \alpha = -2$ and $n\alpha \cdot n\alpha = -2n^2 = -2$ only if $n = \pm 1$. As for condition (b), it is satisfied by the construction of Seidel's Lagrangian Dehn twist S_L for a Lagrangian sphere L. S_L is a symplectomorphism acting on $H_2(X; \mathbb{R})$ as a reflection through the hyperplane perpendicular to $[L]$. In particular, if α is represented by a Lagrangian sphere L and β is represented by a Lagrangian sphere L', then $S_L(L')$ is a Lagrangian sphere in the class $\sigma_\alpha(\beta)$. □

Proposition 2.24. *Suppose* ω_{mon} *is a monotone symplectic form on* X_k *with* $3 \leq k \leq 8$, *then* $\Gamma_L(X_k, \omega_{\text{mon}}) = R_k$ *as root systems.*

For a reduced symplectic form ω *on* X_k, $\Gamma_L(X_k, \omega)$ *is a subroot system of* $\Gamma_L(X_k, \omega_{\text{mon}})$, *and a canonical choice of simple roots consists of those* $l_i \in \Gamma_L(X_k, \omega_{\text{mon}})$ *which pair trivially with* $[\omega]$. $[\omega]$ *pairs nonnegatively with the positive roots of the root system* $\Gamma_L(X_k, \omega_{\text{mon}})$.

Proof. Clearly, the root system R_k contains all the Lagrangian sphere classes of $(X_k, \omega_{\text{mon}})$ since $K_{\omega_{\text{mon}}} = K_k$. On the other hand, by [Li and Wu 2012], a class A of a rational surface with $A \cdot A = -2$ is a Lagrangian sphere class if A is represented by a smoothly embedded sphere and A pairs trivially with $[\omega]$ and K_ω. Since $\chi(X_k) \leq 12$, by [Li and Li 2006], every square (-2)-class of X_k can be represented by a smoothly embedded sphere. Since $[\omega_{\text{mon}}]$ is proportional to $K_{\omega_{\text{mon}}}$, any class in the root system R_k satisfies the criterion in [Li and Wu 2012].

Now suppose that ω is a reduced symplectic form. Since $K_\omega = K_{\omega_{\text{mon}}}$ for a reduced symplectic form ω, $\Gamma_L(X_k, \omega)$ is contained in $\Gamma_L(X_k, \omega_{\text{mon}})$, and hence by Lemma 2.23, $\Gamma_L(X_k, \omega)$ is a subroot system of $\Gamma_L(X_k, \omega_{\text{mon}})$. Explicitly, suppose $[\omega]$ lies in a $p-$dimensional open face, which we denote by $F(\omega)$. If

$M_kG_{i_1}, M_kG_{i_2}, \ldots, M_kG_{i_p}$ are the edges of $F(\omega)$, then $[\omega]$ is positive on the corresponding simple roots l_{i_j} of $\Gamma_L(X_k, \omega_{\text{mon}})$ and vanish on the remaining simple roots. Therefore $[\omega]$ vanishes exactly on the positive roots of $\Gamma_L(X_k, \omega_{\text{mon}})$ generated from these simple roots l_{i_j} and is positive on the remaining positive roots of $\Gamma_L(X_k, \omega_{\text{mon}})$. In other words, $\Gamma_L(X_k, \omega)$ contain exactly the positive roots of $\Gamma_L(X_k, \omega_{\text{mon}})$ generated from these simple roots l_{i_j}. □

Let N_ω be the number of ω-symplectic (-2)-sphere classes. Note that N_ω and $\Gamma_L(X_k, \omega)$ are both invariant in any given open face. Let $N_{\omega,L}$ be the number of ω-Lagrangian sphere classes up to a change of the sign. Following from the above discussion and denoting the set of positive roots of $\Gamma_L(X_k, \omega_{\text{mon}})$ by $R^+(X_k)$, we have

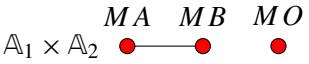
$$N_\omega + N_{\omega,L} = |R^+(X_k)| = \tfrac{1}{2}|R_k|. \tag{11}$$

We take the 3-point blow up as an example to illustrate the above correspondence between the simple roots and the edges of the cone. In this case, the simple roots are $l_1 = H - E_1 - E_2 - E_3,\ l_2 = E_1 - E_2,\ l_3 = E_2 - E_3$, and

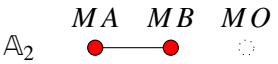
$$R^+(X_3) = \{H - E_1 - E_2 - E_3,\ E_1 - E_2, E_2 - E_3,\ E_1 - E_3\}, \quad N_\omega + N_{\omega,L} = 4.$$

We describe the Lagrangian root system on each open face of P_3, which is the tetrahedron $MOAB$ in Figure 1.

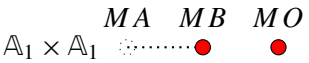
- The monotone vertex M. $N_\omega = 0$ in this case. And 3 Lagrangian simple roots $MO \leftrightarrow l_1 = H - E_1 - E_2 - E_3,\ MA \leftrightarrow l_2 = E_1 - E_2,\ MB \leftrightarrow l_3 = E_2 - E_3$ form $R(X_3)$ with the Dynkin diagram



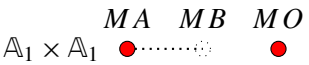
- The edge MO. $\Gamma_L = \mathbb{A}_2$, obtained from $R(X_3)$ by removing MO:



- The edge MA. $\Gamma_L = \mathbb{A}_1 \times \mathbb{A}_1$, obtained from $R(X_3)$ by removing MA:



- The edge MB. $\Gamma_L = \mathbb{A}_1 \times \mathbb{A}_1$, obtained from $R(X_3)$ by removing MB:



- Three open faces of dimension 2: $MOA,\ MOB,\ MAB$. $\Gamma_L = \mathbb{A}_1$ lattice of node MB, MA and MO respectively (by removing all other vertices).

- The top dimensional open face $MOAB$, where all 4 spherical (-2)-class are symplectic and the Lagrangian system Γ_L is $\varnothing$.

2B5. *$R(X)$, $P(X)$ and $\Gamma_L(X,\omega)$ when $\chi(X)\le 5$.* The main purpose of this subsection is to describe $R(X)$, $P(X)$ and $\Gamma_L(X,\omega)$ for rational surfaces with $\chi(X)\le 5$ and show that the Equation (11) continues to hold for these rational surfaces in Lemma 2.25. Such rational surfaces are $S^2\times S^2$ and X_k, $k=0,1,2$. We start with X_k, $k\le 2$.

• X_2, X_1, X_0. For $k=0,1,2$, the root system $R(X_k)$ is similarly defined via (10) and $K_k=-3H+\sum_{i=1}^k E_i$, and it is easy to see that

$$R(X_0)=R(X_1)=\varnothing,\qquad R(X_2)=\{\pm(E_1-E_2)\}=\mathbb{A}_1,$$

with $R^+(X_2)=\{E_1-E_2\}$.

For the normalized reduced symplectic cone, notice that P_2 can be obtained by projecting P_3 (see Figure 1) onto the plane $c_3=0$.

In fact, we can visualize P_k, $k=0,1,2$, via Figure 2, which describes the closure of P_2. We label the possible monotone points as M_1, M_2. For $k=2$, the cone P_2 is the triangle OAB with the closed edges AB and OA deleted, i.e., $\{(c_1,c_2)\mid 1>c_1+c_2>c_1\ge c_2>0\}$. For $k=1$, the cone P_1 is the projection of P_2 onto the c_1-line. Explicitly, P_1 is the horizontal open interval OA, i.e., $\{c_1\in(0,1)\}$. For $k=0$, the cone P_0 is the point O.

For X_0 or X_1, there are no Lagrangian sphere classes and the Lagrangian root system is $\varnothing$. For X_2, the Lagrangian system is empty at the 2-dimensional open face, and $\mathbb{A}_1$ at the 1-dimensional open face OB.

• $S^2\times S^2$. Let $\{F_1,F_2\}$ be the natural basis of $H_2(S^2\times S^2)$, which is also a basis of $H^2(S^2\times S^2)$ via Poincaré duality. Let $K=-2F_1-2F_2$ and use it to define the

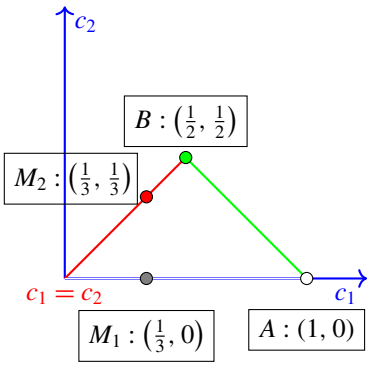


Figure 2. Normalized reduced cone when $\chi(X)\le 5$.

root system $R(S^2 \times S^2)$ via (10). Clearly, $R(S^2 \times S^2) = \{\pm(F_1 - F_2)\}$ is just $\mathbb{A}_1$, with $R^+(S^2 \times S^2) = \{F_1 - F_2\}$.

Up to the action of $\mathrm{Diff}^+(S^2 \times S^2)$ and scaling, every symplectic class is equivalent to a unique class of the form $F_1 + \mu F_2$ for some $\mu \geq 1$. Hence, the coefficient μ, or equivalently, the ratio, identifies a fundamental domain with the half-open and half-closed interval $[1, \infty)$. We simply define the normalized reduced symplectic cone $P(S^2 \times S^2)$ to be this μ interval. Via the ratio c_1/c_2, it actually appears in Figure 2 as the open interval BA together with the endpoint B. The explanation is that blowing down an $H - E_1 - E_2$ symplectic sphere in (X_2, ω) gives rise to a symplectic $S^2 \times S^2$, and this corresponds to projecting P_2 to the interval $[B, A)$. Notice that B, the image of M_2, is the class of a monotone form on $S^2 \times S^2$.

As for the Lagrangian system, there is either zero or one Lagrangian sphere class up to sign and the Lagrangian root system is $\varnothing$ or $\mathbb{A}_1$ respectively. Precisely, $\Gamma_L(S^2 \times S^2, \omega_{\mathrm{mon}}) = \mathbb{A}_1$, and $\Gamma_L(S^2 \times S^2, \omega) = \varnothing$ for any nonmonotone ω. In terms of $P(S^2 \times S^2) = [1, \infty)$, the Lagrangian system is empty on the 1-dimension open face $(1, \infty)$ and $\mathbb{A}_1$ at the 0-dimensional open face $\{1\}$. In terms of the $[B, A)$ interval representation of $P(S^2 \times S^2)$ in Figure 2, the Lagrangian system is empty on open BA and $\mathbb{A}_1$ at B.

Finally, notice that the relation (11) still holds when $\chi(X) \leq 5$. We state this fact as a lemma.

Lemma 2.25. *For any symplectic rational surface (X, ω) with $\chi(X) \leq 11$, we have*

$$N_\omega + N_{\omega,L} = |R^+(X)| = \tfrac{1}{2}|R(X)|. \tag{12}$$

This observation will be useful in proving Corollary 1.5.

Remark 2.26. When $\chi(X) \geq 12$, the normalized reduced symplectic cone is no longer a polytope. It is cut by one more quadratic equation and a K-positive linear equation. Further, the Cremona transform group is infinite, and the set of Lagrangian sphere classes forms a (generalized) root system guided by affine Kac–Moody algebra (see [Zhao et al. 2006]).

3. Level 2 stratification of $\mathcal{J}_\omega$ when $\chi \leq 8$

In this section, we prove Theorem 1.2 and Corollary 1.3 for symplectic rational 4-manifolds with Euler number no larger than 8.

For the convenience of computation, throughout this section we identify $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$, $k \geq 2$, with $S^2 \times S^2 \# (k-1)\overline{\mathbb{CP}^2}$, $k \geq 2$, and use two natural bases for H_2. Let $\{B, F, E_1', \ldots, E_{k-1}'\}$ be the natural basis of $H_2(S^2 \times S^2 \# (k-1)\overline{\mathbb{CP}^2}; \mathbb{Z})$. Then the transition from the $\{B, F, E_i'\}$ basis to the $\{H, E_i\}$ basis is explicitly given

by

$$B = H - E_2, \quad F = H - E_1, \quad E_1' = H - E_1 - E_2, \\ E_i' = E_{i+1}, \quad \text{for all } i \geq 2, \tag{13}$$

with the inverse transition given by:

$$H = B + F - E_1', \quad E_1 = B - E_1', \quad E_2 = F - E_1', \\ E_j = E_{j-1}', \quad \text{for all } j > 2. \tag{14}$$

Thus, $\nu H - c_1 E_1 - c_2 E_2 - \cdots - c_k E_k$ corresponds to

$$(\nu - c_1)B + (\nu - c_2)F - (\nu - c_1 - c_2)E_1' - c_3 E_2' - \cdots - c_k E_{k-1}'. \tag{15}$$

Note that in terms of the basis $\{B, F, E_1', \ldots, E_{k-1}'\}$, up to scaling, a reduced class is of the form $B + \mu F - \sum_{i=1}^{k-1} a_i E_i'$ with

$$\mu \geq 1 > a_1 \geq a_2 \geq \cdots \geq a_{k-1} > 0 \quad \text{and} \quad a_1 + a_2 \leq 1. \tag{16}$$

If a symplectic form ω on X is reduced, then using the basis $\{B, F, E_1', \ldots, E_{k-1}'\}$, its canonical class is

$$K_\omega = -2B - 2F + \sum_{i=1}^{k-1} E_i'.$$

3A. ***The $\mathcal{B}, \mathcal{F}, \mathcal{E}$ classification of $\mathcal{S}_\omega^{<0}$ for a reduced form.*** Here is our setting in this subsection: Suppose $X = S^2 \times S^2 \# n\overline{\mathbb{C}\mathrm{P}^2}$, $n \leq 4$, and ω is a reduced symplectic form in the class $B + \mu F - \sum_{i=1}^{n} a_i E_i'$ satisfying (16). We first make the following elementary observation, which will be crucial in this section.

Lemma 3.1. *Suppose $X = S^2 \times S^2 \# n\overline{\mathbb{C}\mathrm{P}^2}$ and ω is a reduced symplectic form in the class $B + \mu F - \sum_{i=1}^{n} a_i E_i'$. Then*

$$\sum_{k=1}^{n} (a_k)^2 \leq 1, \quad \text{if } n \leq 4. \tag{17}$$

Proof. If we consider the extreme value of the function $\sum_{k=1}^{n} (a_k)^2$ under the constrain given by the closure of the reduced condition (16), $a_i \in [0, 1]$, $a_i + a_j \leq 1$, then the extreme value can only appear at $(1, 0, \ldots, 0)$ or $\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$, meaning that $\sum_{i=1}^{n} (a_i)^2 \leq \max\left(1, \frac{n}{4}\right)$. Given $n \leq 4$, we have $\sum_{k=1}^{n} (a_k)^2 \leq 1$. □

3A1. *Constraints on simple J-holomorphic curves for a reduced form.* The following is a key lemma providing basic constraints for simple J-holomorphic curves.

Lemma 3.2. *Under the above setting, if $A = pB + qF - \sum r_i E_i' \in H_2(X; \mathbb{Z})$ is represented by a simple J-holomorphic curve C for some ω-tamed J, then $p \geq 0$.*

And if $p = 0$, then $q = 0$ or 1. If $p = 1$, then $r_i \in \{0, 1\}$. If $p > 1$, then $q \geq 1$.

Proof. We start by stating three inequalities: the area inequality (18), the adjunction inequality (19), the r_i integer inequality (20).

The area of the curve C is positive and hence

$$\omega(A) = p\mu + q - \sum_{i=1}^{n} a_i r_i > 0. \tag{18}$$

By Theorem 2.2 the virtual genus $g_v(C)$ of the simple J-holomorphic curve C (not necessarily embedded), defined by $g_v(C) = \big([C]\cdot[C] - c_1(X, J)([C])\big)/2 + 1$, is nonnegative. Since J is ω tamed and ω is reduced, $-c_1(X, J)$ is the canonical class $K_\omega = -2B - 2F + E_1' + \cdots + E_n'$. Note also that $g_v(C)$ is defined in terms of the homology class $[C] = A$ so we switch notation from $g_v(C)$ to $g_\omega(A)$. Now the adjunction inequality for the simple J-holomorphic curve C is of the following form:

$$0 \le 2g_\omega(A) = A\cdot A + K_\omega \cdot A + 2 = 2(p-1)(q-1) - \sum_{i=1}^{n} r_i(r_i - 1). \tag{19}$$

The third inequality is an estimate of the sum $-\sum_{i=1}^{n} r_i(r_i-1)$. Since each r_i is an integer, it is easy to see that

$$-\sum_{i=1}^{n} r_i(r_i - 1) \le 0, \tag{20}$$

and $-\sum_{i=1}^{n} r_i(r_i-1) = 0$ if and only if $r_i = 0$ or 1 for each i.

Now let us divide into five cases:

(i) $p = 1$, (ii) $p > 1$, (iii) $p < 0,\ q \ge 1$, (iv) $p < 0,\ q \le 0$, (v) $p = 0$.

Case (i). $p = 1$. Then $-\sum_{i=1}^{n} r_i(r_i-1) = 2g_\omega(A) \ge 0$. It follows from (20) that $r_i(r_i-1)$ has to be 0 and hence $r_i \in \{0, 1\}$.

Case (ii). $p > 1$ and $q \le 0$. Then $-\sum_{i=1}^{n} r_i(r_i-1) = 2g_\omega(A) - 2(p-1)(q-1) \ge 0 - 2(p-1)(q-1) > 0$. This is impossible. Therefore $q \ge 1$ if $p > 1$.

Case (iii). $p < 0$ and $q \ge 1$. We show that this case is impossible. Because $p \le -1$, the adjunction inequality (19) implies that

$$0 \ge -2g_\omega(A) \ge 4(q-1) + \sum_{i=1}^{n} r_i(r_i-1) \ge (q-1) + \sum_{i=1}^{n} r_i(r_i-1).$$

Applying the area Equation (18), we have

$$(q-1) + \sum_{i=1}^{n} r_i(r_i-1) > \left(\sum_{i=1}^{n} a_i r_i - \mu p - 1\right) + \sum_{i=1}^{n} r_i(r_i-1).$$

Since $-\mu p - 1 \geq 0$,

$$\left(\sum_{i=1}^{n} a_i r_i - \mu p - 1\right) + \sum_{i=1}^{n} r_i(r_i - 1) \geq \left(\sum_{i=1}^{n} a_i r_i\right) + \sum_{i=1}^{n} r_i(r_i - 1) = \sum_{i=1}^{n} r_i(r_i - 1 + a_i).$$

For any integer r_i we have $r_i(r_i - 1 + a_i) \geq 0$ due to the reduced condition $1 - a_i \in (0, 1)$. Therefore we would have $-2g_\omega(A) > 0$, which is a contradiction.

Case (iv). $p < 0$, and $q \leq 0$. We show this case is also impossible. This will follow from the following estimate, under a slightly general assumption:

$$0 \leq 2g_\omega(A) \leq 1 + |p| + |q| - p^2 - q^2, \quad \text{if } p \leq 0,\ q \leq 0. \tag{21}$$

Proof of the inequality (21). In order to estimate $-\sum_{i=1}^{n} r_i(r_i - 1)$ we rewrite the sum

$$\sum_{i=1}^{n} r_i = \sum_{k=1}^{u} r_k + \sum_{l=u+1}^{n} r_l, \tag{22}$$

where each r_k is negative and each r_l is nonnegative.

Since $p \leq 0,\ q \leq 0$, the area inequality (18) takes the following form:

$$-\sum a_i r_i > (|p| + |q|). \tag{23}$$

Note that there exists at least one negative r_i term, i.e., $u \geq 1$ in (22). An important consequence is

$$\sum_{k=1}^{u} a_k r_k \leq \sum_{i=1}^{n} a_i r_i < 0, \quad \left(\sum_{k=1}^{u} a_k r_k\right)^2 \geq \left(\sum_{i=1}^{n} a_i r_i\right)^2. \tag{24}$$

We first observe that, by the Cauchy–Schwarz inequality and equations (23), (17), we have

$$\left(\sum_{k=1}^{u} a_k r_k\right)^2 \leq \sum_{k=1}^{u} (r_k)^2 \times \sum_{k=1}^{u} (a_k)^2 \leq \sum_{k=1}^{u} (r_k)^2. \tag{25}$$

Then we do the estimate:

$$\begin{aligned}
\sum_{i=1}^{n} r_i(r_i - 1) &= \sum_{i=1}^{n} r_i^2 - \sum_{i=1}^{n} r_i = \sum_{k=1}^{u} r_k^2 - \sum_{k=1}^{u} r_k + \left(\sum_{l=u+1}^{n} r_l^2 - \sum_{l=u+1}^{n} r_l\right) \\
&\geq \sum_{k=1}^{u} r_k^2 - \sum_{k=1}^{u} r_k \quad (\text{since } x^2 - x \geq 0 \text{ for any integer}) \\
&\geq \left(\sum_{k=1}^{u} a_k r_k\right)^2 - \sum_{k=1}^{u} a_k r_k \quad \big(\text{follows from the two inequalities:} \\
&\quad -\textstyle\sum_{k=1}^{u} r_k > -\sum_{k=1}^{u} a_k r_k \quad \text{and} \quad \sum_{k=1}^{u} r_k^2 \geq \left(\sum_{k=1}^{u} a_k r_k\right)^2\big)
\end{aligned} \tag{26}$$

$$\geq \left(\sum_{i=1}^{n} a_i r_i\right)^2 - \sum_{i=1}^{n} a_i r_i \quad \text{(this crucial step follows from (24))}$$

$$> |p| + |q| + (|p| + |q|)^2.$$

Because $\sum_{i=1}^{n} r_i(r_i - 1)$ is an integer, we actually have

$$\sum_{i=1}^{n} r_i(r_i - 1) \geq 1 + |p| + |q| + (|p| + |q|)^2.$$

Now the inequality (21) follows from the inequalities (26) and (19),

$$0 \leq 2g_\omega(A) = 2pq - 2(p+q) + 2 - \sum_{i=1}^{n} r_i(r_i - 1) \leq |p| + |q| + 1 - (p^2 + q^2). \quad \square$$

With the inequality (21) established, we note that a direct consequence is that it is impossible to have $p \leq -2,\ q \leq 0$, or $p \leq 0,\ q \leq -2$: If $|p| > 1$,

$$|p| + |q| + 1 - (p^2 + q^2)$$

is clearly negative since $q^2 \geq |q|,\ p^2 > |p| + 1$; it is the same if $|q| > 1$.

So the inequality (21) leaves two cases to analyze: $p = q = -1$, or $p = -1$, $q = 0$. To deal with these two cases, as in the proof of the inequality (21), we assume that $r_k < 0$ for $1 \leq k \leq u$ and $r_l \geq 0$ for $u + 1 \leq l \leq n$. Notice that $\sum_{k=1}^{u} r_k^2 - \sum_{k=1}^{u} r_k \leq \sum_{i=1}^{n} r_i(r_i - 1)$ as shown in (26).

- $p = -1$ and $q = 0$. In this case, we have $2g_\omega(A) = 4 - \sum_{i=1}^{n} r_i(r_i - 1)$ so

$$\sum_{k=1}^{u} r_k^2 - \sum_{k=1}^{u} r_k \leq \sum_{i=1}^{n} r_i(r_i - 1) \leq 4$$

by the adjunction inequality (19). By the area inequality (18), we have $\sum_{k=1}^{u} r_k < p + q = -1$, and hence $\sum_{k=1}^{u} r_k \leq -1$. It is easy to see that $\{r_k\} = \{-1\}$ or $\{-1, -1\}$. But these possibilities are excluded by the reduced condition $a_i + a_j \leq 1 \leq \mu$ for any pair $i,\ j$ and the area inequality (18).

- $p = q = -1$. In this case, we have $2g_\omega(A) = 8 - \sum_{i=1}^{n} r_i(r_i - 1)$ so

$$\sum_{k=1}^{u} r_k^2 - \sum_{k=1}^{u} r_k \leq 8.$$

By the area inequality (18), we have $\sum_{k=1}^{u} r_k < p + q = -2$, and hence $\sum_{k=1}^{u} r_k \leq -2$. It is easy to see that $\{r_k\} = \{-1, -1, -1\},\ \{-1, -1, -1, -1\}$ or $\{-1, -2\}$. Again these possibilities are excluded by the reduced condition $a_i + a_j \leq 1 \leq \mu$ for any pair $i,\ j$ and the area inequality (18).

Case (v). $p = 0$. In this case, by the adjunction inequality (19) we have

$$-2(q-1) - \sum_{i=1}^{n} r_i(r_i - 1) \geq 0.$$

Since $-\sum_{i=1}^{n} r_i(r_i-1) \leq 0$, we have $q \leq 1$. If $p = 0,\ q \leq 0$, then we apply the inequality (21) to conclude that $1 + |q| - q^2 \geq 0$. Since $q \leq 0$, this leaves only the possibilities that $q = 0$ or $q = -1$. We exclude the case $q = -1$. If $q = -1$, then by the adjunction inequality (19) again we have $4 = -2(q-1) \geq \sum_{i=1}^{n} r_i(r_i-1)$. So $-1 \leq r_i \leq 2$ for each i and at most two r_i are negative. By the area inequality (18) we have $-\sum_{i=1}^{n} a_i r_i > -q = 1$. But this contradicts the reduced condition $a_i + a_j \leq 1 \leq \mu$ for any pair i, j. Hence we must have $q = 0$ or 1 if $p = 0$.

In conclusion, only cases (i), (ii), (v) are possible. Moreover, if $p = 0$, then $q = 0$ or 1; if $p = 1$, then $r_i \in \{0, 1\}$; if $p > 1$, then $q \geq 1$. □

For rational curves, we further have

Lemma 3.3. *Suppose $A = pB + qF - \sum r_i E_i'$ has a simple J-holomorphic rational curve representative and $A \cdot A < 0$. Then $p = 0$ or 1.*

Proof. Let us assume $p \geq 2$ and draw a contradiction. First observe that $q \geq 1$ by Lemma 3.2.

Observe also that, by the adjunction inequality (19), we have

$$\sum_{i=1}^{n} r_i(r_i - 1) = 2(p-1)(q-1). \tag{27}$$

Since $g_\omega(A) = 0$ (by Condition 1), $2g_\omega(A) - 2 = K_\omega \cdot A + A \cdot A$ and $A \cdot A < 0$, we have $-1 \leq K_\omega \cdot A = \sum_{i=1}^{n} r_i - 2p - 2q$. Namely,

$$\sum_{i=1}^{n} r_i = 2p + 2q + k, \quad k \geq -1. \tag{28}$$

Now if $p > 1, q \geq 1$, since $n \leq 4$, by the Cauchy–Schwartz inequality and (28),

$$\sum_{i=1}^{n} r_i^2 \geq \left[\sum_{i=1}^{n} r_i\right]^2 /4 \geq (2p + 2q + k)^2/4. \tag{29}$$

It follows from (28) and (29) that

$$\begin{aligned}\sum_{i=1}^{n} r_i^2 - \sum_{i=1}^{n} r_i &\geq (2p+2q+k)^2/4 - (2p+2q+k) \\ &= (p+q)^2 + (p+q)k + \frac{k^2}{4} - k - 2(p+q) \\ &= [2pq + 2 - 2(p+q)] + (p^2 + pk - 2) + (q^2 + qk - k) + \frac{k^2}{4}\end{aligned}$$

Since $p \geq 2,\ q \geq 1,\ k \geq -1$, the last 3 terms are all nonnegative, and (q^2+qk-k) is always strictly positive. To see that (q^2+qk-k) is always strictly positive, we separate into 2 cases: $k \geq 0$ and $k = -1$. If $k \geq 0$, we have

$$q^2 + qk - k = q^2 + k(q-1) \geq q^2 \geq 1.$$

If $k = -1$, we have $q^2 + qk - k = q(q+k) - k = q(q-1) + 1 \geq 1$.

Therefore we have $\sum_{i=1}^n r_i^2 - \sum_{i=1}^n r_i > 2(p-1)(q-1)$, contradicting (27). □

3A2. *The classification result.* We can explicitly write down all the classes in $\mathcal{S}_\omega^{<0}$ for a reduced symplectic form ω.

Proposition 3.4. *Any class in $\mathcal{S}_\omega^{<0}$ lies in one of the following three disjoint sets*:

$$\mathcal{B} = \left\{B - lF - \sum r_i E_i',\ l \geq -1,\ r_i \in \{0,1\}\right\};$$
$$\mathcal{F} = \left\{F - \sum r_i E_i',\ r_i \in \{0,1\}\right\};$$
$$\mathcal{E} = \left\{E_j' - \sum r_i E_i',\ j < i,\ r_i \in \{0,1\}\right\}.$$

In particular, $\mathcal{S}_\omega^{<0}$ is a finite set.

Moreover, a class in $\mathcal{B}, \mathcal{F}, \mathcal{E}$ is in $\mathcal{S}_\omega^{<0}$ if and only if it has positive ω-area. $\mathcal{S}_\omega^{<0}$

Proof. We first show that $\mathcal{S}_\omega^{<0}$ is contained in the union of $\mathcal{B}, \mathcal{F}, \mathcal{E}$. Suppose $A = pB + qF - \sum_i r_i E_i'$ is such a class. By Lemma 3.3, $p = 0$ or 1.

- $p = 1$. If $p = 1$, then $r_i = 0$ or 1 as shown in Lemma 3.2. So

$$A = B + qF - \sum r_i E_i', \quad r_i \in \{0,1\},$$

and the condition $A \cdot A < 0$ and $n \leq 4$ implies that $q \leq 1$.

- $p = 0$. In this case, we have shown that $q = 0$ or 1 in Lemma 3.2. If $p = 0$, $q = 0$, the adjunction inequality (19) is $2 - \sum_{i=1}^n r_i(r_i - 1) \geq 0$. Let x be an integer. Notice that $x(x-1) \geq 0$, and $x(x-1) = 0$ if $x = 0$ or 1. Notice also that since x is an integer, if $x(x-1) > 0$ then $x(x-1) \geq 2$, and $x(x-1) = 2$ if $x = 2$ or $x = -1$. We see there is at most one j such that $r_j \neq 0$ or $\neq 1$, and for this j, $r_j = -1$ or 2. By considering the area of such a class, we must have $r_j = -1$, and $j < i$ for any $r_i = 1$. Therefore, in this case, A can only be of the form

$$E_j' - \sum r_i E_i', i > j, \quad r_i \in \{0,1\}.$$

We are left with $q = 1$. In this case, the adjunction inequality (19) is of the form $-\sum_{i=1}^n r_i(r_i - 1) \geq 0$. So we must have $r_i = 0$ or 1. Namely,

$$A = F - \sum r_i E_i', \quad r_i \in \{0,1\}.$$

Now we show that $\mathcal{S}_\omega^{<0}$ is a finite set. By the first statement, it suffices to show that the intersection of $\mathcal{S}_\omega^{<0}$ with $\mathcal{B}$, $\mathcal{F}$, or $\mathcal{E}$ is a finite set. The sets $\mathcal{F}$ and $\mathcal{E}$ are

clearly finite since $r_i = 0$ or 1. To show that the set $\mathcal{B} \cap \mathcal{S}_\omega^{<0}$ is finite for a reduced ω it suffices to show that $[\omega]$ pairs positively with finitely many elements in $\mathcal{B}$. A reduce ω has class $[\omega] = B + \mu F - \sum a_i E_i'$ satisfying (16), and hence the pairing between $[\omega]$ and a class in $\mathcal{B}$ is $\mu - l - \sum a_i r_i \le mu - l$, which is negative if $l > \mu$.

To show that a class A in $\mathcal{B}, \mathcal{F}, \mathcal{E}$ is in $\mathcal{S}_\omega^{<0}$ if it has positive ω-area, we need to construct an ω-symplectic sphere in the class A. Notice that we can get an embedded holomorphic rational curve in the class A via small Kähler blowup. Then we apply [Dorfmeister and Li 2010, Theorem 2.11]. □

For convenience, we list the set $R_{n+1}^+ = \mathcal{B}^{-2} \sqcup \mathcal{F}^{-2} \sqcup \mathcal{E}^{-2}$ for $S^2 \times S^2 \# n\overline{\mathbb{CP}^2}$ (see Proposition 2.24):

$$
\begin{aligned}
R_1^+ &= \{B - F\},\\
R_2^+ &= \{B - F\},\\
R_3^+ &= \big\{B - E_1' - E_2',\ B - F,\ F - E_1' - E_2',\ E_1' - E_2'\big\},\\
R_4^+ &= \big\{B - E_i' - E_j',\ B - F,\ F - E_i' - E_j',\ E_j' - E_i',\ 1 \le j < i \le 3\big\},\\
R_5^+ &= \big\{B + F - E_1' - E_2' - E_3' - E_4',\ B - E_i' - E_j',\ B - F,\ F - E_i' - E_j',\\
&\qquad\qquad E_j' - E_i',\ 1 \le j < i \le 4\big\}.
\end{aligned}
$$

We remark that Proposition 3.4 overlaps with and can be derived from results in Section 4.1 in [Zhang 2017], which are in a slightly different context. In [Zhang 2017], while the almost complex structure is assumed to have the standard canonical class, the symplectic structure is not assumed to be reduced.

Remark 3.5. The following observation (coming from toric moment polytope, also carefully written in [Anjos and Pinsonnault 2013, Sections 4.2 and 4.3]) will be used in Lemma 4.7: for a symplectic rational surface X with $\chi(X) \le 7$, any $A \in \mathcal{S}_\omega^{-1} \cup \mathcal{S}_\omega^{-2}$ arises as an edge of a toric moment polytope. Consequently, there is a semifree circle action having a symplectic sphere S as a component of the fixed loci with $[S] = A$. See also Remark 4.9.

3B. ***Cusp decomposition and level 2 stratification.*** We continue to assume that $\chi(X) \le 8$ and ω is reduced.

3B1. *Cusp curve decompositions of a class in $\mathcal{S}_\omega^{<0}$.* Here is an important consequence of Lemmas 3.2 and 3.3.

Proposition 3.6. *For any stable curve in a class $A \in \mathcal{S}_\omega^{\le -2}$,*

- *there are no components with positive self-intersection,*
- *any self-intersection zero component is in the class B or kF, $k > 0$,*
- *any simple component with negative self-intersection is embedded and hence its class is in $\mathcal{S}_\omega^{<0}$.*

Proof. Suppose $A = pB + qF - \sum_i r_i E_i' \in \mathcal{S}_\omega^{\le -2}$. Then by Lemma 3.3, $p = 0$ or $p = 1$, and if $p = 1$ then $q \le 1$. We argue by contradiction to show that there are no components with positive self-intersection in any cusp curve decomposition of A. Assume the cusp decomposition (6) of A has the following form for some J,

$$A = [C'] + \sum_\alpha m_\alpha [C_\alpha],$$

where C' is a simple J-holomorphic rational curve with positive self-intersection and $[C'] = p'B + q'F - \sum_i r_i' E_i'$, $m_\alpha > 0$ and each C_α is a simple J-holomorphic rational curve. Notice that it is possible that $C' = C_\alpha$ for some α.

By Lemma 3.2, the B coefficients of A (the number p), $[C']$ (the number p') or any $[C_\alpha]$ (denoted by p_α) are all nonnegative. Moreover, since A admit some embedded J-holomorphic representative, $A \cdot A < 0$ and $g_\omega(A) = 0$, by Lemma 3.3 (or Proposition 3.4), $p = 0$ or 1. Since $[C'] \cdot [C'] > 0$, we have $p' > 0$. So we have $p = \sum_\alpha p_\alpha + p' \ge p' > 0$. Since $p = 0$ or 1, we must have $p = p' = 1,\ p_\alpha = 0$, for all α.

Now let us inspect the F coefficients q, q', q_α. First of all, we have $q \le 1$, by Proposition 3.4. For the class $[C']$, since $[C'] \cdot [C'] > 0$ we have $p'q' = q' \ge 1$. For any $[C_\alpha]$ class, since the B coefficient p_α is zero, by Lemma 3.2 the F coefficient q_α is 0 or 1. Hence $q = q' + \sum_\alpha q_\alpha \ge q' \ge 1$. Since we have also observed that $q \le 1$, we conclude that both $q = 1$ and $q' = 1,\ q_\alpha = 0$, for all α.

Since $p = q = 1,\ A \cdot A \le -2$, we must have $A = B + F - E_1' - E_2' - E_3' - E_4'$ from Proposition 3.4. In addition, since $p' = q' = 1$, by Lemma 3.2, $[C'] = B + F - \sum_i r_i' E_i',\ r_i' \in \{0, 1\}$. However, the sum of the curve class $\sum_\alpha m_\alpha [C_\alpha] = A - [C'] = -E_1' - E_2' - E_3' - E_4' + \sum_i r_i' E_i',\ r_i' \in \{0, 1\},\ 1 \le i \le 4$, has negative symplectic area. Contradiction! Hence there are no positive self-intersection components in a cusp curve decomposition (6) of a class $A \in \mathcal{S}_\omega^{\le -2}$.

Next, we analyze the possible square 0 classes in the decomposition. From the analysis above, we only need to deal with the case that either $p' = 0$ or $q' = 0$. For the case $p' = 0$, the only possible square zero classes are $kF,\ k \in \mathbb{Z}^+$. And for the case $q' = 0$, the only possible square zero class is B.

The last bullet on simple components with negative self-intersection follows from Condition 1. □

Remark 3.7. Note that the following results overlap with [Zhang 2017]: Lemma 4.1 and Proposition 4.6 in [Zhang 2017] cover Proposition 3.4 of this paper up to Cremona transformations; Lemma 4.12 in [Zhang 2017] also provided useful information about (7) for small rational surfaces, and will be useful in [Li ≥ 2020] for the general stratification.

3B2. *Level 2 stratification.* We are ready to prove Theorem 1.2, which we restate here.

Theorem 3.8. *For a symplectic rational surface* (X, ω) *with* $\chi(X) \leq 8$, $\mathcal{X}_4 = \coprod_{\mathrm{cod}(\mathcal{C})\geq 4} \mathcal{J}_{\mathcal{C}}$ *and* $\mathcal{X}_2 = \coprod_{\mathrm{cod}(\mathcal{C})\geq 2} \mathcal{J}_{\mathcal{C}}$, *are closed subsets in* $\mathcal{X}_0 = \mathcal{J}_\omega$. *Consequently*:

(i) $\mathcal{X}_0 - \mathcal{X}_4$ *is a manifold.*

(ii) $\mathcal{X}_2 - \mathcal{X}_4$ *is closed in* $\mathcal{X}_0 - \mathcal{X}_4$.

(iii) $\mathcal{X}_2 - \mathcal{X}_4$ *is a manifold.*

(iv) $\mathcal{X}_2 - \mathcal{X}_4$ *is a submanifold of* $\mathcal{X}_0 - \mathcal{X}_4$.

Proof. We can assume that ω is reduced by Theorem 2.18. Let us recall that, by Lemma 2.8, Condition 1 applies here.

We first show that $\mathcal{X}_2$ is closed in $\mathcal{X}_0$, namely, $\overline{\mathcal{X}_2} \cap (\mathcal{X}_0 - \mathcal{X}_2) = \varnothing$. We will argue by contradiction.

Suppose $\overline{\mathcal{X}_2} \cap (\mathcal{X}_0 - \mathcal{X}_2) \neq \varnothing$. Then there is a sequence $\{J_n\}$ in $\mathcal{X}_2$ converging to $J' \in (\mathcal{X}_0 - \mathcal{X}_2) = \mathcal{J}_\varnothing$. Since $\mathcal{S}_\omega^{<0}$ is finite by Proposition 3.4, which means there are only finitely many labeling sets $\mathcal{C}$ for which $\mathcal{J}_{\mathcal{C}}$ is nonempty. Thus we can assume that the sequence $\{J_n\}$ is in a single prime set $\mathcal{J}_{\mathcal{C}}$ and we can apply Lemma 2.15 here. By Condition 1, for $J' \in \mathcal{X}_0 - \mathcal{X}_2$, every simple J'-holomorphic rational curve has self-intersection at least -1 and the ones with self-intersection -1 are embedded. Suppose $A \in \mathcal{C}$. Then for each $J_n \in \mathcal{J}_{\mathcal{C}} \subset \mathcal{X}_2$ there is one embedded J_n-holomorphic rational curve in the class A. By Lemma 2.15 the class $A \in \mathcal{S}_\omega^{\leq -2}$ admits a decomposition as in Equation (7), with no class having square less than -1. Moreover, by Proposition 3.6, the decompositions can be written as

$$A = rB + \sum_i a_i F + \sum_j b_j D_j,$$

where a_i, b_j are nonnegative integers, $r \in \{0, 1\}$, and $D_j \in \mathcal{S}_\omega^{-1}$ by Condition 1. By pairing with K_ω on both sides. The left hand side is $A \cdot K_\omega \geq 0$, and the right hand side is $rB \cdot K_\omega + \sum a_i F \cdot K_\omega + \sum_j b_j D_i \cdot K_\omega \leq 0$, since $D_i \cdot K_\omega = -1$, $F \cdot K_\omega = B \cdot K_\omega = -2$. This is a contradiction.

We next show that $\mathcal{X}_4$ is closed in $\mathcal{X}_0$, namely, $\overline{\mathcal{X}_4} \cap (\mathcal{X}_0 - \mathcal{X}_4) = \varnothing$. Since $\mathcal{X}_4 \subset \mathcal{X}_2$ and $\mathcal{X}_2$ is closed in $\mathcal{X}$, it suffices to show that $\mathcal{X}_4$ is closed in $\mathcal{X}_2$. The argument is similar. By Condition 1, for each $J' \in \mathcal{X}_2 - \mathcal{X}_4$ every simple J'-holomorphic rational curve has self-intersection at least -2, there is exactly one simple J'-holomorphic rational curve with self-intersection -2, and the ones with self-intersection -1 and -2 are embedded. For each $J_n \in \mathcal{X}_4$ there is either one embedded J_n-holomorphic rational curve with self-intersection at most -3, or there are at least two embedded J_n-holomorphic rational curves with self-intersection -2.

Let us assume that $\overline{\mathcal{X}_4} \cap (\mathcal{X}_2 - \mathcal{X}_4) \neq \varnothing$, and there is a sequence $\{J_n\}$ in $\mathcal{X}_4$ converging to $J' \in \mathcal{J}_{\mathcal{C}'} \subset (\mathcal{X}_2 - \mathcal{X}_4)$ for some labeling set $\mathcal{C}'$. Notice that $\mathcal{C}'$ is of the form $\{A'\}$ for some $A' \in \mathcal{S}_\omega^{-2}$.

Again, because of the finiteness of labeling sets, we can assume $\{J_n\}$ all lie in $\mathcal{J}_{\mathcal{C}}$ for some labeling set $\mathcal{C}$. So we are again in the situation of Lemma 2.15.

(1) Assume first $\mathcal{C}$ contains a class $A \in \mathcal{S}_{\omega}^{\leq -3}$. Then by Proposition 3.6, there is a cusp curve decomposition $A = c'A' + \sum c_i A_i$ with $A_i \in \mathcal{S}_{\omega}^{-1} \sqcup \mathcal{S}_{\omega}^{0}$, $c' \geq 0$, $c_i \geq 1$. Pair the cusp curve decompositions with K_ω. We get a contradiction since $K_\omega \cdot A > 0$ while $K_\omega \cdot A' = 0$, $K_\omega \cdot A_i < 0$.

(2) Otherwise, $\mathcal{C}$ contains at least two classes in $\mathcal{S}_{\omega}^{-2}$ and one of them has to be distinct from A'. Call this class A. By Proposition 3.6, there is a cusp curve decomposition $A = c'A' + \sum c_i A_i$ with $\{A_i\} \subset \mathcal{S}_{\omega}^{-1} \sqcup \mathcal{S}_{\omega}^{0}$, $c' \geq 0$, $c_i \geq 1$. We claim that the set $\{A_i\}$ is not empty. This is true because $A \neq A'$ and $c'A'$ is not in S_{ω}^{-2} for any $c' \neq 1$. Now pair the cusp curve decompositions with K_ω. We again get a contradiction since $K_\omega \cdot A = 0$ while $K_\omega \cdot A' = 0$ and $K_\omega \cdot A_i < 0$.

So we have proved that $\mathcal{X}_2$ and $\mathcal{X}_4$ are closed in $\mathcal{J}_\omega$.

Next, let us establish the claims (i)–(iv).

(i) $\mathcal{X}_0 - \mathcal{X}_4$ is a manifold. This statement is true since $\mathcal{X}_4$ is closed in $\mathcal{X}_0$ and $\mathcal{X}_0$ is a manifold. Similarly, $\mathcal{X}_0 - \mathcal{X}_2$ is a manifold since $\mathcal{X}_2$ is also closed in $\mathcal{X}_0$. And both $\mathcal{X}_0 - \mathcal{X}_4$ and $\mathcal{X}_0 - \mathcal{X}_2$ are open submanifolds of $\mathcal{X}_0$.

(ii) $\mathcal{X}_2 - \mathcal{X}_4$ is closed in $\mathcal{X}_0 - \mathcal{X}_4$. This follows from the fact that $\mathcal{X}_2$ is closed in $\mathcal{X}_0$.

(iii) $\mathcal{X}_2 - \mathcal{X}_4$ is a manifold. This statement follows from the fact that $\mathcal{X}_2 - \mathcal{X}_4$ is a submanifold of $\mathcal{X}_0$. This latter fact follows from Proposition 2.14 and the fact that $\mathcal{X}_2 - \mathcal{X}_4$ is the disjoint union of codimension 2 prime sets $\mathcal{J}_A$ over $A \in \mathcal{S}_{\omega}^{-2}$.

(iv) $\mathcal{X}_2 - \mathcal{X}_4$ is a closed submanifold of $\mathcal{X}_0 - \mathcal{X}_4$. Since $\mathcal{X}_0 - \mathcal{X}_4$ is an open submanifold of $\mathcal{X}_0$, $\mathcal{X}_2 - \mathcal{X}_4$ is also a submanifold of $\mathcal{X}_0 - \mathcal{X}_4$.

Hence this proves that $\mathcal{X}_4 (= \mathcal{X}_3) \subset \mathcal{X}_2 (= \mathcal{X}_1) \subset \mathcal{X}_0 = \mathcal{J}_\omega$, is a level 2 stratification. □

It means that the decomposition $\mathcal{J}_\omega$ is a stratification at the first two levels with top stratum $\mathcal{J}_{\text{open}}$.

Remark 3.9. In a separate paper [Li ≥ 2020] the first author will further show that, for a symplectic rational surface (X, ω) with $\chi(X) \leq 8$, this filtration of $\mathcal{J}_\omega$ fits into the following notion of *even stratification* in the ∞-dimensional setting (For finite dimension, see, e.g., [Goresky and MacPherson 1988]):

Definition 3.10. For an ∞-dimensional real Fréchet manifold $\mathcal{X}$, a finite filtration of $\mathcal{X}$ is called an even stratification if it is a sequence of *closed* subspaces

$$\mathcal{X}_{2n+2} := \varnothing \subset \mathcal{X}_{2n} \subset \mathcal{X}_{2n-2} \cdots \subset \mathcal{X}_2 \subset \mathcal{X}_0 = \mathcal{X},$$

where $\mathcal{X}_{2i} - \mathcal{X}_{2i+2}$ is a submanifold $\mathcal{X}_0$ of real codimension $2i$ for $0 \leq i \leq n$.

There is a decomposition of $\mathcal{J}_\omega$ for $(\mathbb{C}\mathrm{P}^2\#3\overline{\mathbb{C}\mathrm{P}^2}, \omega)$ in Lemma 2.10 of [Anjos and Pinsonnault 2013] by the existence of a certain negative self-intersection curve. They have also shown that this decomposition is a stratification with finite-codimensional submanifolds as strata. Our decomposition is finer in the sense that each stratum in [Anjos and Pinsonnault 2013] is a union of prime submanifolds in our decomposition. In particular, our decomposition being a stratification as in Definition 3.10 implies their decomposition is a stratification.

3C. *Symplectic* (-2)*-spheres and* $H_1(\mathcal{J}_{\mathrm{open}})$.

3C1. *Enumerating the components of* $\mathcal{X}_2 - \mathcal{X}_4$ *by* $\mathcal{S}_\omega^{-2}$. Let N_ω denote the cardinality of $\mathcal{S}_\omega^{-2}$. We have the following crucial result about codimension 2 strata in the stratification of $\mathcal{J}_\omega$.

Proposition 3.11. *For* $X = S^2 \times S^2\#n\overline{\mathbb{C}\mathrm{P}^2}$, $n = 0, 1, 2, 3$, $\mathcal{J}_A$ *is path connected if* $A \in \mathcal{S}_\omega^{-2}$. *Hence the number of path connected components in* $\mathcal{X}_2 - \mathcal{X}_4$ *is* N_ω.

We need the following lemma, which was stated in [Li et al. 2015]:

Lemma 3.12. *For a symplectic rational manifold* (X, ω) *with* $X = S^2 \times S^2\#n\overline{\mathbb{C}\mathrm{P}^2}$, $n \geq 0$, *the group* $\mathrm{Symp}_h(X, \omega)$ *acts transitively on the space of homologous* (-2)*-symplectic spheres.*

Proof. Here we give a proof following steps sketched in [Li and Wu 2012; Borman et al. 2014]. Let S_1, S_2 be two homologous symplectic (-2)-spheres. Without loss of generality, we can assume the symplectic sphere S_i is in the homology class $[S_i] = B - F$, since we can always change basis in $H_2(X, \mathbb{R})$. For each (X, ω, S_i), by [McDuff and Opshtein 2015], there is a set $\mathcal{C}_i$ of disjoint (-1)-symplectic spheres C_i^l for $l = 1, \ldots, k$ such that

$$[C_i^l] = E_l', \quad \text{for } l = 1, \ldots n.$$

Blowing down the set $\{C_i^1, \ldots C_i^n\}$ separately, one obtains a 4-tuple $(X_i, \omega_i, \tilde{S}_i, \mathcal{B}_i)$, where (X_i, ω_i) is are symplectic $S^2 \times S^2$ with $[\omega_1] = [\omega_2]$, $\tilde{S}_i$ a symplectic sphere in X_i, and $\mathcal{B}_i = \{B_i^1, \ldots, B_i^n\}$ is a symplectic ball packing in $X_i \setminus \tilde{S}_i$ corresponding to $\mathcal{C}_i$. For any two such tuples $(X_1, \omega_1, \tilde{S}_1, \mathcal{B}_1)$ and $(X_2, \omega_2, \tilde{S}_2, \mathcal{B}_2)$, since the symplectic forms are homologous, by [Lalonde and McDuff 1996], there is a symplectomorphism Φ from $(X_1, \omega_1, \tilde{S}_1)$ to $(X_2, \omega_2, \tilde{S}_2)$, such that for a fixed l, $\mathrm{Vol}(\Phi(B_1^l)) = \mathrm{Vol}(B_2^l)$. Then according to [Abreu and McDuff 2000], we can choose this Φ such that the two symplectic spheres are isotopic, i.e., $\Phi(\tilde{S}_1) = \tilde{S}_2$. Applying Theorem 1.1 in [Borman et al. 2014], there is a compactly supported Hamiltonian isotopy ι of $(X_2, \omega_2, \tilde{S}_2)$ such that the symplectic ball packings $\Phi(\mathcal{B}_1)$ and $\mathcal{B}_2$ are connected by ι in $X_2 \setminus \tilde{S}_2$. Then $\iota \circ \Phi$ is a symplectomorphism between the tuples $(X_i, \omega_i, \tilde{S}_i, \mathcal{B}_i)$ and hence blowing up induces a symplectomorphism

$\psi : (X_1, \omega_1, \tilde{S}_1, \mathcal{B}_1) \to (X_2, \omega_2, \tilde{S}_2, \mathcal{B}_2)$. Further note that ψ preserves homology classes B, F, E_1', E_2', $\ldots$, E_n' and hence $\psi \in \mathrm{Symp}_h(X, \omega)$. □

Proof of Proposition 3.11. Applying Theorem 1.1 in [Li et al. 2015], $\mathrm{Symp}_h(X, \omega)$ is itself path connected for $X = S^2 \times S^2 \# n\overline{\mathbb{C}\mathrm{P}^2}$, $n = 0, 1, 2, 3$. Therefore, for $A \in \mathcal{S}_\omega^{-2}$, the space of symplectic (-2)-spheres in the class A is path connected by Lemma 3.12. Let $\mathcal{Z}_A$ denote the space of symplectic (-2)-spheres in the class A. Since the space of almost complex structures making a symplectic sphere pseudoholomorphic is weakly contractible [Evans 2011], and there is a unique embedded J-holomorphic sphere in the class A for each $J \in \mathcal{J}_A$, $\mathcal{J}_A$ fibers over $\mathcal{Z}_A$ with weakly contractible fibers. Hence $\mathcal{J}_A$ is weakly homotopic to $\mathcal{Z}_A$. In particular, $\mathcal{J}_A$ is path connected.

The last claim now follows from the disjoint union decomposition $\mathcal{X}_2 - \mathcal{X}_4 = \coprod_{A \in \mathcal{S}_\omega^{-2}} \mathcal{J}_A$. □

3C2. *Relative Alexander–Pontrjagin duality for regular Fréchet stratification.* Let $\mathcal{X}$ be a Hausdorff space, $\mathcal{Z} \subset \mathcal{Y}$ a closed subset of $\mathcal{X}$ such that $\mathcal{X} - \mathcal{Z}$, $\mathcal{Y} - \mathcal{Z}$ are paracompact manifolds modeled by topological linear spaces. Suppose $\mathcal{Y} - \mathcal{Z}$ is a closed cooriented submanifold of $\mathcal{X} - \mathcal{Z}$ of codimension p, then we say $(\mathcal{Y}, \mathcal{Z})$ is a closed relative submanifold of $(\mathcal{X}, \mathcal{Z})$ of codimension p. We have the following relative version of Alexander–Pontrjagin duality in [Eells 1961] when taking constant coefficient:

Theorem 3.13. *In the above situation, we have an isomorphism*

$$H^i(\mathcal{X} - \mathcal{Z}, \mathcal{X} - \mathcal{Y}; G) \cong H^{i-p}(\mathcal{Y} - \mathcal{Z}; G)$$

for any abelian coefficient group G.

Now we restate and prove Corollary 1.3:

Corollary 3.14. *For a symplectic rational surface with $\chi(X) \le 8$ and any abelian group G,*

$$H^1(\mathcal{J}_{\mathrm{open}}; G) = \bigoplus_{A_i \in \mathcal{S}_\omega^{-2}} H^0(\mathcal{J}_{A_i}; G).$$

If we further assume that $\chi(X) \le 7$, then for each $A_i \in \mathcal{S}_\omega^{-2}$, $\mathcal{J}_{A_i}$ is path connected and hence $H^1(\mathcal{J}_{\mathrm{open}}; G) = G^{N_w}$, where N_ω is the cardinality of $\mathcal{S}_\omega^{-2}$. It follows from the universal coefficient theorem that $H_1(\mathcal{J}_{\mathrm{open}}; \mathbb{Z}) = \mathbb{Z}^{N_\omega}$.

Proof. For $\mathbb{C}\mathrm{P}^2$, $\mathcal{J}_{\mathrm{open}} = \mathcal{J}_\omega$ and $N_\omega = 0$ so the statement holds trivially.

For any other X with $\chi(X) \le 8$, let $\mathcal{X} = \mathcal{X}_0$, $\mathcal{Y} = \mathcal{X}_2$, $\mathcal{Z} = \mathcal{X}_4$. By Proposition 2.14 and Theorem 3.8, the conditions in Theorem 3.13 are satisfied. Consider the cohomology long exact sequence for the pair $(\mathcal{X} - \mathcal{Z}, \mathcal{X} - \mathcal{Y})$ and and, using

Theorem 3.13, replace $H^i(\mathcal{X}-\mathcal{Z}, \mathcal{X}-\mathcal{Y}; G)$ by $H^{i-p}(\mathcal{Y}-\mathcal{Z}; G)$. Then we have the long exact sequence

$$\cdots \to H^{i-1}(\mathcal{X}-\mathcal{Y}; G) \to H^{i-p}(\mathcal{Y}-\mathcal{Z}; G) \to H^i(\mathcal{X}-\mathcal{Z}; G) \to H^i(\mathcal{X}-\mathcal{Y}; G) \to \cdots$$

Since $\mathcal{X} = \mathcal{X}_0$ is contractible, $\mathcal{Z} = \mathcal{X}_4$ is a union of submanifolds of codimension 4 or higher. Note that by Remark 3.15 (the same convention as in [Abreu et al. 2009]), each $\mathcal{X}_i$ is the inverse limit of its Banach counterparts $\{\mathcal{X}_i^p\}$, where $\mathcal{X}_0^p$ is contractible as well and $\mathcal{Z}^p = \mathcal{X}_4^p$ has codimension 4 or higher in $\mathcal{X}_0^p$. We have transversality in the Banach setting (see Chapter 2, pages 27–28 of [Lang 2002]) and hence any map into $\mathcal{X}_0^p - \mathcal{X}_4^p$ with domain being S^1 or S^2 is homotopic to a constant map since $\mathcal{X}_0^p$ is contractible and $\mathcal{Z}^p = \mathcal{X}_4^p$ has codimension 4 or higher in $\mathcal{X}_0^p$. Then we know that $\pi_1(\mathcal{X}^p - \mathcal{Z}^p) = \pi_2(\mathcal{X}^p - \mathcal{Z}^p) = 0$ in each Banach setting. Hence by the Hurewicz theorem, $H_1(\mathcal{X}^p - \mathcal{Z}^p, \mathbb{Z}) = H_2(\mathcal{X}^p - \mathcal{Z}^p, \mathbb{Z}) = 0$ for each p. Then by the universal coefficient theorem, for any abelian group G, $H^1(\mathcal{X}^p - \mathcal{Z}^p; G) = H^2(\mathcal{X}^p - \mathcal{Z}^p; G) = 0$ for each p. Then taking the inverse limit, we have $H^1(\mathcal{X}-\mathcal{Z}; G) = H^2(\mathcal{X}-\mathcal{Z}; G) = 0$ in the Fréchet setting by Remark 3.15.

Setting $i = p = 2$, we have the exact sequence

$$0 = H^1(\mathcal{X}-\mathcal{Z}; G) \to H^1(\mathcal{X}-\mathcal{Y}; G) \to H^0(\mathcal{Y}-\mathcal{Z}; G) \to H^2(\mathcal{X}-\mathcal{Z}; G) = 0,$$

namely, $H^1(\mathcal{X}-\mathcal{Y}; G) = H^0(\mathcal{Y}-\mathcal{Z}; G)$. Notice that $\mathcal{X}-\mathcal{Y} = \mathcal{J}_{\text{open}}$ and $\mathcal{Y}-\mathcal{Z}$ is the disjoint union of $\mathcal{J}_A$ for $A \in \mathcal{S}_\omega^{-2}$; we have $H^1(\mathcal{J}_{\text{open}}; G) = \bigoplus_{A_i \in \mathcal{S}_\omega^{-2}} H^0(\mathcal{J}_{A_i}; G)$.

When $\chi(X) \le 7$, the statement $H^1(\mathcal{J}_{\text{open}}; G) = G^{N_\omega}$ follows from Proposition 3.11. Then, by the universal coefficient theorem, we have

$$G^{N_\omega} \cong H^0(\mathcal{X}_2 - \mathcal{X}_4; G) \cong H^1(\mathcal{X}_0 - \mathcal{X}_2; G) \cong \text{Hom}(H_1(\mathcal{X}_0 - \mathcal{X}_2; \mathbb{Z}); G)$$

for any abelian group G. This implies that $H_1(\mathcal{X}_0 - \mathcal{X}_2; \mathbb{Z}) = \mathbb{Z}^{N_\omega}$. □

Remark 3.15. As noted in [Abreu et al. 2009, Convention part or Remark 2.2], each Fréchet manifold we work with can naturally be interpreted as the inverse limit of a sequence of Banach manifolds. In fact, Proposition 2.6 is valid and established first in the Banach settings. Moreover, it is noted in [Abreu et al. 2009] that the successive inclusions between the Banach manifolds are weak homotopy equivalences. Therefore the results about the homology and homotopy groups in the Fréchet setting can be interpreted as the corresponding results for each Banach manifold in the sequence.

Remark 3.16. An absolute version of Alexander–Pontrjagin duality in [Eells 1961] was applied by Abreu [1998] to detect the topology of $\mathcal{J}_{\text{open}}$ for $S^2 \times S^2$ with a symplectic form with ratio within $(1, 2)$. In [Li ≥ 2020], the first author will establish an Alexander–Pontrjagin duality for even stratifications as in Definition 3.10, generalizing [Eells 1961]. The following special case can also be applied to

compute the fundamental group of the symplectomorphism group of small rational 4-manifolds: Let $\mathcal{X}_0$ be a contractible paracompact Fréchet manifold which is evenly stratified by $\{\mathcal{X}_{2i}\}_{i=0}^n$ as in Definition 3.10 at the first 2 levels. Then we have the following duality on the cohomology of $\mathcal{X}_0 - \mathcal{X}_2$ and $\mathcal{X}_2 - \mathcal{X}_4$, with coefficient G being any abelian group:

$$H^1(\mathcal{X}_0 - \mathcal{X}_2; G) \cong H^0(\mathcal{X}_2 - \mathcal{X}_4; G).$$

3D. ***Characterizing $\mathcal{J}_{\text{open}}$ via subsets of $\mathcal{S}_\omega^{-1}$.*** Let X_{n+1} continue to be

$$S^2 \times S^2 \# n\overline{\mathbb{CP}^2}, \quad n \le 4.$$

We next give a characterization of $\mathcal{J}_{\text{open}}$ using a subset of classes $\mathfrak{D}'_{n+1} \subset \mathcal{S}_\omega^{-1}$. Note that the subset $\mathfrak{D}'_{n+1}$ here is different from a labeling set $\mathcal{C}$ in Definition 1.1 since $\mathcal{C} \subset \mathcal{S}_\omega^{\le -2}$. Recall that in Definition 2.11, for $\mathfrak{D} \subset \mathcal{S}_\omega^{-1}$, $\mathcal{J}^{\mathfrak{D}}$ denotes the set of J such that each class in $\mathfrak{D}$ has a J-holomorphic representative. Clearly, if $\mathfrak{D} \subset \mathfrak{C}$, then $\mathcal{J}^{\mathfrak{C}} \subset \mathcal{J}^{\mathfrak{D}}$.

Lemma 3.17. *Let X_{n+1} be $S^2 \times S^2 \# n\overline{\mathbb{CP}^2}$, $n \le 4$ with a reduced symplectic form ω. For the following configuration $\mathfrak{D}'_{n+1} \subset \mathcal{S}_\omega^{-1}(X_{n+1})$,*

$$\begin{aligned}
\mathfrak{D}'_2 &= \{B - E'_1\},\\
\mathfrak{D}'_3 &= \{B - E'_1,\ F - E'_1,\ E'_1\},\\
\mathfrak{D}'_4 &= \{B + F - E'_1 - E'_2 - E'_3,\ B - E'_1,\ F - E'_1,\ E'_2\},\\
\mathfrak{D}'_5 &= \{B + F - E'_2 - E'_3 - E'_4,\ B - E'_1,\ F - E'_1,\ E'_2,\ E'_3\},
\end{aligned}$$

we have $\mathcal{J}^{\mathfrak{D}'_{n+1}} = \mathcal{J}_{\text{open}}$. Consequently, if $\mathfrak{C} \subset \mathcal{S}_\omega^{-1}$ contains $\mathfrak{D}'_{n+1}$, then $\mathcal{J}^{\mathfrak{C}} = \mathcal{J}_{\text{open}}$.

Proof. By the 4th bullet of Lemma 2.12 we just need to check that any class in $\mathcal{S}_\omega^{\le -2}$ intersects at least one class in the configuration $\mathfrak{D}'_{n+1}$ negatively. By Proposition 3.4, $\mathcal{S}_\omega^{\le -2}$ is contained in $\mathcal{B} \sqcup \mathcal{F} \sqcup \mathcal{E}$. It is straightforward to check each class in $R_{n+1} = \mathcal{B}^{-2} \sqcup \mathcal{F}^{-2} \sqcup \mathcal{E}^{-2}$ pairs negatively with at least one class in the configuration $\mathfrak{D}'_{n+1}$. Note that R_{n+1} is the root system.

For X_2 and X_3, by Proposition 3.4, any class in $\mathcal{B}^{\le -3}$ can be written as $B - qF - r_i E'_i$, $q \ge 1$, $i \le k$, $r_i \in \{0, 1\}$, and $(B - qF - r_i E'_i) \cdot (B - E'_1) = -q - r_1 < 0$.

For X_4, the only class in $\mathcal{F}^{\le -3}$ is $F - E'_1 - E'_2 - E'_3$, and the only class in $\mathcal{E}^{\le -3}$ is $E'_1 - E'_2 - E'_3$. Each of them pairs with $B + F - E'_1 - E'_2 - E'_3$ negatively. And any class in $\mathcal{B}^{\le -3}$ can be written as either $B - E'_1 - E'_2 - E'_3$ or $B - kF - r_i E'_i$, $k \ge 1$, $r_i \in \{0, 1\}$, by Proposition 3.4. And $(B - E'_1 - E'_2 - E'_3) \cdot (B + F - E'_1 - E'_2 - E'_3) < 0$, $(B - kF - r_i E'_i) \cdot (B - E'_1) = -k - r_1 < 0$.

For X_5, $\mathcal{F}^{\le -3} = \left\{F - E'_1 - E'_2 - E'_3 - E'_4,\ F - E'_i - E'_j - E'_k,\ 1 \le i < j < k \le 4\right\}$, and $\mathcal{E}^{\le -3} = \left\{E'_1 - E'_2 - E'_3 - E'_4,\ E'_i - E'_j - E'_k,\ 1 \le i < j < k \le 4\right\}$. Each of them pairs with $B + F - E'_2 - E'_3 - E'_4$ negatively. And any class in $\mathcal{B}^{\le -3}$ can be written

as either $B-E_i'-E_j'-E_k',\ 1\le i<j<k\le 4$, or $B-kF-r_iE_i',\ k\ge 1,\ r_i\in\{0,1\}$. And $(B-E_i'-E_j'-E_k')\cdot(B+F-E_2'-E_3'-E_4')<0,\ (B-kF-r_iE_i')\cdot(B-E_1')=-k-r_1<0$.

Therefore any sphere class with square less than -1 cannot have a simultaneous J-holomorphic representative with the set $\mathfrak{D}$.

Finally, the equality $\mathcal{J}^{\mathfrak{C}}=\mathcal{J}_{\text{open}}$ follows from $\mathcal{J}_{\text{open}}=\mathcal{J}^{\mathfrak{D}'_{n+1}}\supset\mathcal{J}^{\mathfrak{C}}\supset\mathcal{J}_{\text{open}}$, where the first inclusion follows directly from Definition 2.11 and the last inclusion is the 3rd bullet of Lemma 2.12 . □

In terms of the reduced basis $\{H,E_1,\dots,E_{n+1}\},\ X_{n+1}=\mathbb{C}\mathrm{P}^2\#(n+1)\overline{\mathbb{C}\mathrm{P}^2}$, we have

$$\begin{aligned}
&\mathfrak{D}_2'=\{E_1\} && \text{for } \mathbb{C}\mathrm{P}^2\#2\overline{\mathbb{C}\mathrm{P}^2},\\
&\mathfrak{D}_3'=\{H-E_1-E_2,E_1,E_2\} && \text{for } \mathbb{C}\mathrm{P}^2\#3\overline{\mathbb{C}\mathrm{P}^2},\\
&\mathfrak{D}_4'=\{H-E_3-E_4,E_1,E_2,E_3\} && \text{for } \mathbb{C}\mathrm{P}^2\#4\overline{\mathbb{C}\mathrm{P}^2},\\
&\mathfrak{D}_5'=\{2H-E_1-E_2-E_3-E_4-E_5,E_1,E_2,E_3,E_4\} && \text{for } \mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}^2}.
\end{aligned}$$

We mention a couple of related facts that will not be used in the paper. First of all, $\mathfrak{D}'_{n+1}$ is minimal in the sense that, for any *proper* subset $\mathfrak{C}'$ of $\mathfrak{D}'_{n+1}$, $\mathcal{J}^{\mathfrak{C}'}$ is strictly bigger than $\mathcal{J}_{\text{open}}$. Notice that $\mathfrak{D}'_{n+1}$ does not contain the smallest area class E_{n+1} (see Lemma 2.19). Secondly, there are subsets $\mathcal{T}\subset\mathcal{S}_\omega^{-1}$ not containing $\mathfrak{D}'_{n+1}$ such that $\mathcal{J}^{\mathcal{T}}=\mathcal{J}_{\text{open}}$.

4. The rank of $\pi_1(\mathrm{Symp}_h(X,\omega))$ when $\chi(X)\le 7$

In this section, for a symplectic rational surface (X,ω) with $\chi(X)\le 7$, we prove Theorem 4.8, which includes both Theorem 1.4 and Corollary 1.5. Note that $\mathrm{Symp}_h(X,\omega)$ is path connected in this case.

As alluded in the introduction, we will treat the cases

$$\chi(X)\le 4 \quad\text{and}\quad 5\le\chi(X)\le 7$$

separately. In the former case, we will prove Theorem 4.8 by combining various known computations of $\pi_0(\mathrm{Symp}(X,\omega))$ and $\pi_1(\mathrm{Symp}(X,\omega))$ scattered in the literature. In the latter case we will use an appropriate configuration of exceptional spheres in [Evans 2011; Li et al. 2015] to reduce the computation of $\pi_1(\mathrm{Symp}(X,\omega))$ to $H_1(\mathcal{J}_{\text{open}};\mathbb{Z})$ and then apply Corollary 1.3.

4A. ***Configurations of exceptional spheres when $\chi(X)=5,6,7$.*** We recall the notion of a configuration of exceptional spheres (e.g., [Li et al. 2015]):

Definition 4.1. A finite collection of transversally intersected symplectic spheres $D=\bigcup D_i\subset X$ is called a (tree) configuration if

- for any pair i, j with $i \neq j$, $[D_i] \neq [D_j]$ and $[D_i] \in \mathcal{S}_\omega^{-1}$;
- the D_i are simultaneously J-holomorphic for some $J \in \mathcal{J}_\omega$;
- $D = \bigcup D_i$ is connected and its graph is a tree.

We will often use D to denote such a configuration. The homological type of D refers to the set of homology classes $\mathfrak{D} = \{[D_i]\}$. Further, a configuration is called *standard* if the components intersect ω-orthogonally at every intersection point of the configuration, *heavy* if $\mathcal{J}^{\mathfrak{D}} = \mathcal{J}_{\text{open}}$, and *full* if $H_2(X, D; \mathbb{R}) = 0$.

Denote by $\mathscr{D}$ the space of configurations associated to a fixed configuration D, and by $\mathscr{D}_0$ the subspace of standard configurations.

We then introduce various subgroups of $\text{Symp}_h(X, \omega)$ associated to a fixed configuration D that appear in the diagram (2):

- $\text{Symp}_h(X, \omega) \subset \text{Symp}(X, \omega)$ is the subgroup acting trivially on $H_2(X; \mathbb{Z})$.
- $\text{Stab}(D) \subset \text{Symp}_h(X, \omega)$ is the subgroup that fixes D setwise, but not necessarily pointwise.
- $\text{Stab}^0(D) \subset \text{Stab}(D)$ is the subgroup fixing D pointwise.
- $\text{Stab}^1(D)$ is the subgroup of $\text{Stab}^0(D)$ fixing D pointwise and acting trivially on the normal bundles of its components.
- $\text{Symp}_c(U)$ is the subgroup of $\text{Stab}^1(D)$ that have compact support in $(U, \omega|_U)$, where $U = X \setminus D$.

There are also two local groups $\text{Symp}(D)$ and $\mathcal{G}(D)$ that we will describe soon.

Recall that it suffices to assume the symplectic form is reduced since any symplectic form is diffeomorphic to a reduced form by Lemma 3.1 and diffeomorphic symplectic forms have homeomorphic symplectomorphism groups. For a reduced symplectic form on a rational surface with $\chi(X) = 5, 6, 7$, consider the configurations with the following homology types:

$$\mathbb{C}\text{P}^2\#2\overline{\mathbb{C}\text{P}^2}, \quad \mathfrak{D}_2 = \{H - E_1 - E_2, E_1, E_2\},$$
$$\mathbb{C}\text{P}^2\#3\overline{\mathbb{C}\text{P}^2}, \quad \mathfrak{D}_3 = \{H - E_1 - E_2, H - E_2 - E_3, E_1, E_2, E_3\},$$
$$\mathbb{C}\text{P}^2\#4\overline{\mathbb{C}\text{P}^2}, \quad \mathfrak{D}_4 = \{H - E_1 - E_2, H - E_3 - E_4, E_1, E_2, E_3, E_4\}.$$

Lemma 4.2. *$\mathfrak{D}_k$ is heavy and full.*

Proof. $\mathfrak{D}_k$ is full since it contains a basis of $H_2(X_k; \mathbb{Z})$. Since $\mathfrak{D}_k \supset \mathfrak{D}'_k$, by Lemma 3.17, $\mathfrak{D}_k$ is heavy. □

For such a configuration, we have the following lemma about the weak homotopy type of $\mathscr{D}$, $\mathscr{D}_0$:

Lemma 4.3. *For a configuration D with homology type $\mathfrak{D}_k$, $k = 2, 3, 4$, we have the diagram of homotopy fibrations* (2)*. In particular, both $\mathscr{D}_0$ and $\mathscr{D}$ are weakly homotopic equivalent to $\mathcal{J}_{\text{open}}$.*

Proof. This lemma has been established in [Evans 2011; Li et al. 2015] (see also [Lalonde and Pinsonnault 2004]), except for the claim that $\mathscr{D}_0$ is weakly homotopic equivalent to $\mathcal{J}_{\text{open}}$. It is shown in [Evans 2011] that $\mathscr{D}_0$ is weakly homotopic to $\mathscr{D}$, so it suffices to show that $\mathscr{D}$ is weakly homotopic to $\mathcal{J}_{\text{open}}$.

The crucial fact is that $\mathcal{J}_{\text{open}}(X_k, \omega) = \mathcal{J}^{\mathfrak{D}_k}$ for $k = 2, 3, 4$ by Lemma 4.2. So we just need to show that $\mathcal{J}^{\mathfrak{D}_k}$ is weakly homotopic to $\mathscr{D}$. Notice that $\pi : \mathcal{J}^{\mathfrak{D}_k} \to \mathscr{D}$ is a smooth surjective submersion (proof verbatim as in Proposition 4.8 in [Lalonde and Pinsonnault 2004]). Since the space of almost complex structures making a configuration pseudoholomorphic is a weakly contractible subset of $\mathcal{J}^{\mathfrak{D}_k}$ [Evans 2011], the preimage of the map π is weakly contractible everywhere. Therefore π is a fibration with weakly contractible fibers by [Meigniez 2002], and we have the desired weak homotopy equivalence between $\mathcal{J}^{\mathfrak{D}_k}$ and $\mathscr{D}$. □

We then analyze various groups associated with D that appear in (2), starting from the local groups $\mathrm{Symp}(D)$ and $\mathcal{G}(D)$.

4A1. $\mathrm{Symp}(D)$ *and* $\mathcal{G}(D)$. Given a configuration of embedded symplectic spheres $D = D_1 \cup \cdots \cup D_n \subset X$, let I be the set of intersection points of the components and k_i the cardinality of $I \cap D_i$.

The group $\mathrm{Symp}(D)$ is defined to be the direct product $\prod_{i=1}^n \mathrm{Symp}(D_i, I \cap D_i)$. Here $\mathrm{Symp}(D_i, I \cap D_i)$ denotes the group of symplectomorphisms of $D_i \cong S^2$ fixing the intersection points $I \cap D_i$. Since $\mathrm{Symp}(S^2)$ acts transitively on N-tuples of distinct points in S^2 for any N we can write $\mathrm{Symp}(D_i, I \cap D_i)$ as $\mathrm{Symp}(S^2, k_i)$ and

$$\mathrm{Symp}(D) = \prod_{i=1}^{n} \mathrm{Symp}(S^2, k_i). \tag{30}$$

As shown in [Evans 2011]:

$$\mathrm{Symp}(S^2, 1) \sim S^1; \quad \mathrm{Symp}(S^2, 2) \sim S^1; \quad \mathrm{Symp}(S^2, 3) \sim \star; \tag{31}$$

where $\sim$ means homotopy equivalence. In the rest of this paper, we will constantly use $\sim$ for weak homotopy equivalence.

Similarly, the group $\mathcal{G}(D)$ (also called the symplectic gauge group) is defined to be the product $\prod_{i=1}^n \mathcal{G}_{k_i}(D_i)$. Here $\mathcal{G}_{k_i}(D_i)$ denotes the group of gauge transformations of the symplectic normal bundle to $D_i \subset X$, which are the identity at each of the k_i intersection points. Also as shown in [Evans 2011]:

$$\mathcal{G}_0(S^2) \sim S^1; \qquad \mathcal{G}_1(S^2) \sim \star; \qquad \mathcal{G}_k(S^2) \sim \mathbb{Z}^{k-1}, \quad k > 1. \tag{32}$$

Since we assume the configuration is connected, each $k_i \geq 1$. Thus by (32), we have

$$\pi_0(\mathcal{G}(D)) \cong \bigoplus_{i=1}^{n} \pi_0(\mathcal{G}_{k_i}(S^2)) \cong \bigoplus_{i=1}^{n} \mathbb{Z}^{k_i-1}. \tag{33}$$

By (30) and (32), for D with homology type $\mathfrak{D}_k$, we have

$$\begin{aligned}
&\text{for } \mathbb{C}\mathrm{P}^2\#2\overline{\mathbb{C}\mathrm{P}^2}, \quad \mathrm{Symp}(D) \sim (S^1)^3, \quad \mathcal{G}(D) \sim \mathbb{Z},\\
&\text{for } \mathbb{C}\mathrm{P}^2\#3\overline{\mathbb{C}\mathrm{P}^2}, \quad \mathrm{Symp}(D) \sim (S^1)^5, \quad \mathcal{G}(D) \sim \mathbb{Z}^3,\\
&\text{for } \mathbb{C}\mathrm{P}^2\#4\overline{\mathbb{C}\mathrm{P}^2}, \quad \mathrm{Symp}(D) \sim (S^1)^4, \quad \mathcal{G}(D) \sim \mathbb{Z}^4.
\end{aligned}$$

4A2. $\mathrm{Symp}_c(U)$, $\mathrm{Stab}^1(D)$, $\mathrm{Stab}^0(D)$ *and* $\mathrm{Stab}(D)$. Recall Proposition 3.3 in [Li et al. 2015]:

Lemma 4.4. *For a standard configuration D with homology type $\mathfrak{D}_k$, $\mathrm{Symp}_c(U) \sim \star$.*

Consider the following fibration portion of the diagram (2):

$$\mathrm{Symp}_c(U) \sim \mathrm{Stab}^1(D) \to \mathrm{Stab}^0(D) \to \mathcal{G}(D).$$

It follows that $\mathrm{Stab}^0(D) \sim \mathcal{G}(D)$.

Proposition 4.5. *For a standard configuration D with homology type $\mathfrak{D}_k$, we have*

$$\begin{aligned}
&\textit{for } \mathbb{C}\mathrm{P}^2\#2\overline{\mathbb{C}\mathrm{P}^2}, \quad \mathrm{Stab}(D) \sim \mathbb{T}^2,\\
&\textit{for } \mathbb{C}\mathrm{P}^2\#3\overline{\mathbb{C}\mathrm{P}^2}, \quad \mathrm{Stab}(D) \sim \mathbb{T}^2,\\
&\textit{for } \mathbb{C}\mathrm{P}^2\#4\overline{\mathbb{C}\mathrm{P}^2}, \quad \mathrm{Stab}(D) \sim \star.
\end{aligned}$$

Proof. Consider the following fibration portion of the diagram (2):

$$\mathcal{G}(D) \sim \mathrm{Stab}^0(D) \to \mathrm{Stab}(D) \to \mathrm{Symp}(D). \tag{34}$$

With the computation of $\mathcal{G}(D)$ and $\mathrm{Symp}(D)$ above, we have the following homotopy fibrations for $k = 2, 3, 4$ respectively:

$$\mathbb{Z} \to \mathrm{Stab}(D) \to (S^1)^3, \quad \mathbb{Z}^3 \to \mathrm{Stab}(D) \to (S^1)^5, \quad \mathbb{Z}^4 \to \mathrm{Stab}(D) \to (S^1)^4.$$

We need to understand the connecting homomorphism

$$\pi_1(\mathrm{Symp}(D)) \to \pi_0(\mathrm{Stab}^0(D)) = \pi_0(\mathcal{G}(D)).$$

It follows from Lemma 2.9 in [Li et al. 2015] that the connecting homomorphism is surjective in each case. Therefore $\mathrm{Stab}(D)$ is path connected and $\pi_i(\mathrm{Stab}(D)) = 0$ for $i \geq 2$ in each case, and $\pi_1(\mathrm{Stab}(D)) = \mathbb{Z}^2, \mathbb{Z}^2, 0$ when $k = 2, 3, 4$. Clearly, $\mathrm{Stab}(D) \sim \mathbb{T}^2, \mathbb{T}^2, \star$ for $k = 2, 3, 4$ respectively. □

We remark that, for the monotone case, $\mathrm{Stab}(D)$ is computed in [Evans 2011].

4A3. $\pi_1(\mathscr{D}_0)=\pi_1(\mathcal{J}_{\text{open}})=\mathbb{Z}^{N_\omega}$ *and the injection* $\pi_1(\text{Stab}(D))\to\pi_1(\text{Symp}_h(X,\omega))$. Now we further move on to the last portion of the diagram (2):

$$\text{Stab}(D)\to_h (X,\omega)\to\mathscr{D}_0. \tag{35}$$

We will first compute $\pi_1(\mathscr{D}_0)$.

Lemma 4.6. *For* (X_k,ω), $k=2,3,4$, *the homomorphism*

$$\pi_1(\text{Symp}_h(X,\omega))\to\pi_1(\mathscr{D}_0)$$

is surjective and $\pi_1(\mathscr{D}_0)=\pi_1(\mathcal{J}_{\text{open}})$ *is a free abelian group with rank* N_ω.

Proof. Note that $\text{Stab}(D)$ is path connected by Proposition 4.5. So the homomorphism $\pi_1(\text{Symp}_h(X,\omega))\to\pi_1(\mathscr{D}_0)$ associated to the homotopy fibration (35) is surjective. Since $\text{Symp}_h(X,\omega)$ is a Lie group, $\pi_1(\text{Symp}_h(X,\omega))$ is abelian and hence $\pi_1(\mathscr{D}_0)$ is an abelian group as well. Then $\pi_1(\mathscr{D}_0)=H_1(\mathscr{D}_0,\mathbb{Z})$ by the Hurewicz theorem. By Lemma 4.3 and Corollary 1.3, we have $H_1(\mathscr{D}_0;\mathbb{Z})=H_1(\mathcal{J}_{\text{open}};\mathbb{Z})=\mathbb{Z}^{N_\omega}$. □

We next analyze the associated homomorphism

$$g:\pi_1(\text{Stab}(D))\to\pi_1(\text{Symp}_h(X,\omega)).$$

Lemma 4.7. *The homomorphism* $g:\pi_1(\text{Stab}(D))\to\pi_1(\text{Symp}_h(X,\omega))$ *is injective. Hence we have the exact sequence*

$$0\to\pi_1(\text{Stab}(D))\xrightarrow{g}\pi_1(\text{Symp}_h(X,\omega))\to\mathbb{Z}^{N_\omega}\to 0. \tag{36}$$

Proof. When $k=4$, the injectivity is clear since $\pi_1(\text{Stab}(D))$ is trivial in this case.

For $\mathbb{C}\text{P}^2\#k\overline{\mathbb{C}\text{P}^2}$, $k=2,3$, by Proposition 4.5, $\text{Stab}(D)\sim\mathbb{T}^2$. And the homology classes of the configuration D can be realized as the boundary of the moment polytope. We can assume that D is a toric divisor, see also Remark 3.5. Namely, there is a Hamiltonian $\mathbb{T}^2$ action on (X,ω) fixing D. In particular, we have the inclusions $\mathbb{T}^2\subset\text{Stab}(D)\subset\text{Symp}_h(X,\omega)$. Consider the induced map for π_1:

$$\mathbb{Z}^2\xrightarrow{f}\mathbb{Z}^2\xrightarrow{g}\pi_1(\text{Symp}_h(X,\omega)).$$

By [McDuff and Tolman 2010, Theorems 1.3 and 1.25], the homomorphism $\iota=g\circ f$ is injective. Therefore $\text{Im}(\iota)=g(\text{Im}(f))$ is a rank 2 free abelian group, which implies that $g:\mathbb{Z}^2\to\pi_1(\text{Symp}_h(X,\omega))$ must be injective as well.

Finally, the exactness of the sequence (36) follows from Lemma 4.6. □

4B. ***Proof of Theorem 1.4 and Corollary 1.5.*** We are ready to prove the following result, which includes both Theorem 1.4 and Corollary 1.5. Notice that $\pi_1(\text{Symp}(X,\omega))$ is always the same as $\pi_1(\text{Symp}_h(X,\omega))$, since $\text{Symp}(X,\omega)$ is the extension of $\text{Symp}_h(X,\omega)$ by the homological action, which is discrete.

Theorem 4.8. *If (X,ω) is a symplectic rational surface with $\chi(X)\le 7$, then*

$$\pi_1(\mathrm{Symp}_h(X,\omega)) = \mathbb{Z}^{N_\omega}\oplus \pi_1(\mathrm{Symp}_h(X,\omega_{\mathrm{mon}})). \tag{37}$$

$\pi_0(\mathrm{Symp}(X,\omega))$ and $\pi_1(\mathrm{Symp}(X,\omega))$ are invariant on each open face of $P(X)$ and vary between neighboring open faces. Moreover, if we define

$$Q(X) = \tfrac{1}{2}(\chi(X)-2)(\chi(X)-3),$$

then we have the formula

$$r^+[\pi_0(\mathrm{Symp}(X,\omega))] + \mathrm{Rank}[\pi_1(\mathrm{Symp}(X,\omega))] = Q(X). \tag{38}$$

Here $r^+[\pi_0(\mathrm{Symp}(X,\omega))]$ means the number of positive roots of the Weyl type group $\pi_0(\mathrm{Symp}(X,\omega))$.

Proof. There are statements in the theorem, the identity (37), the stability statement, and the identity (38).

- We first establish (37). For $X=\mathbb{C}\mathrm{P}^2\#k\overline{\mathbb{C}\mathrm{P}^2}$, $k=2,3,4$, by Lemma 4.7,

$$\pi_1(\mathrm{Symp}_h(X,\omega)) = \pi_1(\mathrm{Stab}(D))\oplus\mathbb{Z}^{N_\omega}.$$

Notice that, in the monotone case, $\mathcal{J}_{\mathrm{open}}=\mathcal{J}_{\omega_{\mathrm{mon}}}$ and so the space $\mathscr{D}_0\sim\mathcal{J}_{\mathrm{open}}\sim\mathcal{J}_{\omega_{\mathrm{mon}}}$ is contractible by Lemma 4.3. Hence $\mathrm{Symp}(X,\omega_{\mathrm{mon}})\sim\mathrm{Stab}(D)$ from the fibration (35) and we have (37) in this case.

For $\mathbb{C}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}^2}$, $S^2\times S^2$, (37) follows from the fact that $N_{\omega_{\mathrm{mon}}}=0$ and Table 1 summarizing [Gromov 1985; Abreu 1998; Abreu and McDuff 2000; Lalonde and Pinsonnault 2004], where ω_{gen} denotes a nonmonotone symplectic form. Note that the π_0, π_1 of $\mathbb{C}\mathrm{P}^2$, monotone $S^2\times S^2$ and monotone $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}^2}$ are completely done by [Gromov 1985]; the remaining cases of $S^2\times S^2$ are done by [Abreu 1998, Theorem 0.1; Abreu and McDuff 2000, Theorem 1.1]; and the remaining cases of $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}^2}$ are given by [Abreu and McDuff 2000, Theorem 1.1]. The other information in the table is easy homological information.

- Next, we verify the second statement that $\pi_0(\mathrm{Symp}(X,\omega))$ and $\pi_1(\mathrm{Symp}(X,\omega))$ are stable on each open face of $P(X)$ and vary between neighboring open faces. For X_3 and X_4, it follows from the discussions in Section 2B4 (see Proposition 2.24) which show that both N_ω and Γ_L stabilize within an open face and vary when moving to a neighboring open face.

X	$R(X)$	$\mathrm{Symp}_h(X,\omega_{\mathrm{mon}})$	$\pi_1(\mathrm{Symp}_h(X,\omega_{\mathrm{mon}}))$	$\pi_1(\mathrm{Symp}_h(X,\omega_{\mathrm{gen}}))$	$N_{\omega_{\mathrm{gen}}}$	$Q(X)$
$\mathbb{C}\mathrm{P}^2$	$\varnothing$	$\sim \mathrm{PU}_3$	$\mathbb{Z}_3$	N/A	N/A	0
$\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}^2}$,	$\varnothing$	$\sim U_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	1
$S^2\times S^2$	$\mathbb{A}_1$	$\sim \mathrm{SO}_3\times\mathrm{SO}_3$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$	1	1

Table 1. $\chi(X)\le 4$.

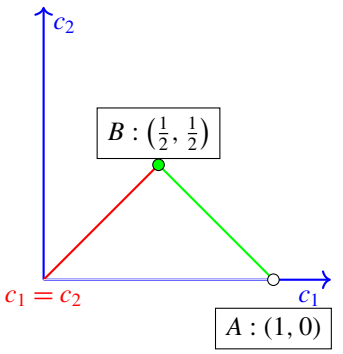


Figure 3. $P(X)$ when $\chi(X) \leq 5$.

For the remaining case with $\chi \leq 5$, it follows from the discussions and Figure 2 in Section 2B5. Notice that any normalized reduced symplectic class on such a manifold appears in Figure 2, which we redraw in Figure 3 (but with M_1 and M_2 removed).

The statement is trivial for $X_0 = \mathbb{C}\mathrm{P}^2$ since O is the only normalized reduced class.

For $X_1 = \mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}^2}$, $\pi_0(\mathrm{Symp}(X_1, \omega)) = \{1\}$ and $\pi_1(\mathrm{Symp}(X_1, \omega)) = \mathbb{Z}$ on the 1-dimensional open face OA, which is the whole cone P_1.

For $X_2 = \mathbb{C}\mathrm{P}^2\#2\overline{\mathbb{C}\mathrm{P}^2}$, the cone P_2 is the disjoint union of the two open faces OB and OAB. On the 1-dimensional open face OB, $\pi_0(\mathrm{Symp}(X_2, \omega)) = \mathbb{Z}_2$ and $\pi_1(\mathrm{Symp}(X_2, \omega)) = \mathbb{Z}^2$. On the open face OAB, $\pi_0(\mathrm{Symp}(X_2, \omega)) = \{1\}$ and $\pi_1(\mathrm{Symp}(X_2, \omega)) = \mathbb{Z}$.

For $S^2 \times S^2$, the cone is the half closed and half open interval $[B, A)$, where B is the monotone class. So the conclusion can be read off from Table 1.

• Finally, we establish (38). Note that the number of Lagrangian spherical classes $N_{\omega,L}$ is the same as $r^+[\pi_0(\mathrm{Symp}(X, \omega))]$. And by Equation (12) which is valid when $\chi(X) \leq 7$, we have $N_\omega + r^+[\pi_0(\mathrm{Symp}(X, \omega))] = |R^+(X)|$. Then (38) follows from the previously established (37) and the formula

$$\mathrm{Rank}[\pi_1(\mathrm{Symp}(X, \omega_{\mathrm{mon}}))] + |R^+(X)| = \tfrac{1}{2}(\chi(X) - 2)(\chi(X) - 3) \tag{39}$$

when $\chi(X) \leq 7$. The formula (39) can be directly verified case by case. When $\chi(X) \leq 4$, it follows from Table 1. When $\chi(X) = 5, 6, 7$, we have $|R^+(X)| = 1, 4, 10$ and $\mathrm{Rank}[\pi_1(\mathrm{Symp}(X, \omega_{\mathrm{mon}}))] = 2, 2, 0$ respectively, and hence (39) also holds in these cases. □

Remark 4.9. In [Anjos and Pinsonnault 2013], the computation of $\pi_1(\mathrm{Symp}_h(X, \omega)$ for any given form on 3-fold blow up of $\mathbb{C}\mathrm{P}^2$ is given. There the strategy is counting torus (or circle) actions. And a generating set of $\pi_1(\mathrm{Symp}_h(X, \omega)$ is

given using circle action. Note that our approach gives another (minimal) set of $\pi_1(\mathrm{Symp}_h(X,\omega)$. We give the correspondence of the two generating sets: By Remark 3.5, any (-2)-symplectic sphere in 3 fold blow up of $\mathbb{C}\mathrm{P}^2$, there is a semifree circle τ action having this (-2)-symplectic sphere as fixing locus, where τ is a generator of $\pi_1(\mathrm{Symp}_h(X,\omega)$.

4B1. *Some remarks.* When $\chi(X)\geq 8$, due to the fact that $\pi_0(\mathrm{Symp}_h(X,\omega))$ could be nontrivial, we consider the quantity

$$r^+[\pi_0(\mathrm{Symp}(X,\omega))]+\mathrm{Rank}[\pi_1(\mathrm{Symp}_h(X,\omega))]-\mathrm{Rank}[\pi_0(\mathrm{Symp}_h(X,\omega))],$$

which is conjectured to be a pure topological quantity $Q(X)$. Note that this quantity $Q(X)$ is $\frac{1}{2}(\chi(X)-2)(\chi(X)-3)$, which is the same as in Theorem 1.4 and Corollary 1.5 when $\chi(X)\leq 7$. We prove in [Li et al. 2019] that when $\chi(X)=8$ this is true, and $Q(X)=15$.

McDuff [2008, Corollary 6.9] gave a blowup approach to obtain an upper bound of the rank $\pi_1(\mathrm{Symp}(X,\omega))$ for a symplectic rational 4 manifold (X,ω). We can show that our Theorem 4.8 actually attained the upper bound via her approach when $\chi(X)\leq 7$. In the follow-up paper [Li et al. 2019], we will explore a generalization of Corollary 6.9 in [McDuff 2008] to estimate the rank of $\pi_1(\mathrm{Symp}(X_5,\omega))$ where X_5 is the 5-point blowup of the projective plane. Although the upper bound is not always optimal, it is crucial for the rank computation. As an potential application of computing the rank of $\pi_1(\mathrm{Symp})$, one can follow [Polterovich 1998] to compare the rank with the number of Hamiltonian isotopy classes of monotone Lagrangian torus to approach the Hofer diameter conjecture, which says that the diameter of $\mathrm{Ham}(X,\omega)$ with respect to the canonical bi-invariant Finsler metric (called the Hofer metric) is infinity.

There is also a general lower bound of the rank of $\pi_1(\mathrm{Symp}(X,\omega))$ given by counting circle actions in [Pedroza 2018]. In [Anjos and Pinsonnault 2013], when $\chi(X)=6$, a generating set of $\pi_1(\mathrm{Symp}(X,\omega))$ as an abelian group is given in terms of circle actions, and this set is shown to also generate the rational homotopy Lie algebra of $\mathrm{Symp}(X,\omega)$. Notice that any symplectic form on a rational 4 manifold X with $\chi(X)\leq 6$ is a toric from. Our approach gives another (minimal) generating set of $\pi_1(\mathrm{Symp}(X,\omega))$ in terms of (-2)-symplectic spheres. When $\chi(X)=6$, the two generating sets correspond via Remark 3.5: any (-2)-symplectic sphere in (X,ω) with $\chi(X)=6$ arises as a fixed point component of a Hamiltonian circle action.

Remark 4.10. Anjos and Eden [2019] (and a personal communication with Anjos, 2017) further studied some toric cases for the 4-fold blow up of $\mathbb{C}\mathrm{P}^2$, including the open face M_4OBC in Table 2. In particular, they compute the rank of $\pi_1(\mathrm{Symp}(X,\omega))\otimes\mathbb{Q}$ in this case and further prove that the group generators

of $\pi_1(\mathrm{Symp}(X,\omega))\otimes\mathbb{Q}$ generate the entire rational homotopy Lie algebra of $\mathrm{Symp}(X,\omega)$.

4B2. *Explicit computation of* $\pi_1(\mathrm{Symp}_h(X,\omega))$ *for* $X=\mathbb{C}\mathrm{P}^2\#4\overline{\mathbb{C}\mathrm{P}^2}$. Finally we apply Theorem 4.8 to explicitly compute $\pi_1(\mathrm{Symp}_h(X,\omega))$ on each open face of the normalized reduced cone P_4. Recall that by Propositions 2.21 and 2.24, the normalized reduced cone of $\mathbb{C}\mathrm{P}^2\#4\overline{\mathbb{C}\mathrm{P}^2}$ is convexly generated by 4 rays, $\{M_4O, M_4A, M_4B, M_4C\}$, with $M_4=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$, $O=(0,0,0,0)$, $A=(1,0,0,0)$, $B=\left(\frac{1}{2},\frac{1}{2},0,0\right)$, $C=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},0\right)$. The results are listed in Table 2. Note that the result on any open face containing the letter "A" is new (the other cases are also obtained by [Anjos and Eden 2019]). Recall that in the form, $\lambda=c_1+c_2+c_3$, as in Section 2B2.

Acknowledgements

We appreciate useful discussions with Silvia Anjos, Martin Pinsonnault, Weiwei Wu, Weiyi Zhang. For $\mathbb{C}\mathrm{P}^2\#4\overline{\mathbb{C}\mathrm{P}^2}$, when $[\omega]$ is in the face M_4OBC, Anjos and Eden [2019] computed the rational homotopy groups using the toric method (see Remark 4.10). Results in Section 3A overlap with results in Section 4.1 in [Zhang 2017], which are in a slightly different context. We thank the referees for their detailed and constructive comments, which greatly improved the exposition of this article. The research is supported by NSF grant DMS-1611680.

open face	Γ_L	N_ω	$\pi_1(\mathrm{Symp}_h(X,\omega))$	ω area
Point M_4	$\mathbb{A}_4$	0	trivial	monotone, $\lambda=1;\ c_1=c_2=c_3=c_4$
M_4O	$\mathbb{A}_3$	4	$\mathbb{Z}^4$	$\lambda<1;\ c_1=c_2=c_3=c_4$
M_4A	$\mathbb{A}_3$	4	$\mathbb{Z}^4$	$\lambda=1;\ c_1>c_2=c_3=c_4$
M_4B	$\mathbb{A}_1\times\mathbb{A}_2$	6	$\mathbb{Z}^6$	$\lambda=1;\ c_1=c_2>c_3=c_4$
M_4C	$\mathbb{A}_1\times\mathbb{A}_2$	6	$\mathbb{Z}^6$	$\lambda=1;\ c_1=c_2=c_3>c_4$
M_4OA	$\mathbb{A}_2$	7	$\mathbb{Z}^7$	$\lambda<1;\ c_1>c_2=c_3=c_4$
M_4OB	$\mathbb{A}_1\times\mathbb{A}_1$	8	$\mathbb{Z}^8$	$\lambda<1;\ c_1=c_2>c_3=c_4$
M_4OC	$\mathbb{A}_2$	7	$\mathbb{Z}^7$	$\lambda<1;\ c_1=c_2=c_3>c_4$
M_4AB	$\mathbb{A}_2$	7	$\mathbb{Z}^7$	$\lambda=1;\ c_1>c_2>c_3=c_4$
M_4AC	$\mathbb{A}_1\times\mathbb{A}_1$	8	$\mathbb{Z}^7$	$\lambda=1;\ c_1>c_2=c_3>c_4$
M_4BC	$\mathbb{A}_1\times\mathbb{A}_1$	8	$\mathbb{Z}^7$	$\lambda=1;\ c_1=c_2>c_3>c_4$
M_4OAB	$\mathbb{A}_1$	9	$\mathbb{Z}^8$	$\lambda<1;\ c_1>c_2>c_3=c_4$
M_4OAC	$\mathbb{A}_1$	9	$\mathbb{Z}^9$	$\lambda<1;\ c_1>c_2=c_3>c_4$
M_4OBC	$\mathbb{A}_1$	9	$\mathbb{Z}^9$	$\lambda<1;\ c_1=c_2>c_3>c_4$
M_4ABC	$\mathbb{A}_1$	9	$\mathbb{Z}^9$	$\lambda=1;\ c_1>c_2>c_3>c_4$
M_4OABC	trivial	10	$\mathbb{Z}^{10}$	$\lambda<1;\ c_1>c_2>c_3>c_4$

Table 2. Γ_L and $\pi_1(\mathrm{Symp}_h(X,\omega))$ for $\mathbb{C}\mathrm{P}^2\#4\overline{\mathbb{C}\mathrm{P}^2}$

References

[Abreu 1998] M. Abreu, "Topology of symplectomorphism groups of $S^2 \times S^2$", *Invent. Math.* **131**:1 (1998), 1–23. MR Zbl

[Abreu and McDuff 2000] M. Abreu and D. McDuff, "Topology of symplectomorphism groups of rational ruled surfaces", *J. Amer. Math. Soc.* **13**:4 (2000), 971–1009. MR Zbl

[Abreu et al. 2009] M. Abreu, G. Granja, and N. Kitchloo, "Compatible complex structures on symplectic rational ruled surfaces", *Duke Math. J.* **148**:3 (2009), 539–600. MR Zbl

[Anjos 2002] S. Anjos, "Homotopy type of symplectomorphism groups of $S^2 \times S^2$", *Geom. Topol.* **6** (2002), 195–218. MR Zbl

[Anjos and Eden 2019] S. Anjos and S. Eden, "The homotopy Lie algebra of symplectomorphism groups of 3-fold blowups of $(S^2 \times S^2, \sigma_{\mathrm{std}} \oplus \sigma_{\mathrm{std}})$", *Michigan Math. J.* **68**:1 (2019), 71–126. MR

[Anjos and Pinsonnault 2013] S. Anjos and M. Pinsonnault, "The homotopy Lie algebra of symplectomorphism groups of 3-fold blow-ups of the projective plane", *Math. Z.* **275**:1-2 (2013), 245–292. MR Zbl

[Anjos et al. 2019] S. Anjos, J. Li, T.-J. Li, and M. Pinsonnault, "Stability of the symplectomorphism group of rational surfaces", preprint, 2019, Available at https://arxiv.org/abs/1911.00961. arXiv

[Biran 1997] P. Biran, *Geometry of symplectic packing*, Ph.D. thesis, Tel-Aviv University, 1997.

[Biran 2001] P. Biran, "From symplectic packing to algebraic geometry and back", pp. 507–524 in *European Congress of Mathematics, II* (Barcelona, 2000), edited by C. Casacuberta et al., Progr. Math. **202**, Birkhäuser, Basel, 2001. MR Zbl

[Borman et al. 2014] M. S. Borman, T.-J. Li, and W. Wu, "Spherical Lagrangians via ball packings and symplectic cutting", *Selecta Math.* (*N.S.*) **20**:1 (2014), 261–283. MR Zbl

[Dorfmeister and Li 2010] J. G. Dorfmeister and T.-J. Li, "The relative symplectic cone and T^2-fibrations", *J. Symplectic Geom.* **8**:1 (2010), 1–35. MR Zbl

[Eells 1961] J. Eells, Jr., "Alexander–Pontrjagin duality in function spaces", pp. 109–129 in *Proc. Sympos. Pure Math., III*, Amer. Math. Soc., Providence, R.I., 1961. MR Zbl

[Evans 2011] J. D. Evans, "Symplectic mapping class groups of some Stein and rational surfaces", *J. Symplectic Geom.* **9**:1 (2011), 45–82. MR Zbl

[Friedman and Morgan 1988] R. Friedman and J. W. Morgan, "On the diffeomorphism types of certain algebraic surfaces, I", *J. Differential Geom.* **27**:2 (1988), 297–369. MR Zbl

[Fujiki and Schumacher 1990] A. Fujiki and G. Schumacher, "The moduli space of extremal compact Kähler manifolds and generalized Weil–Petersson metrics", *Publ. Res. Inst. Math. Sci.* **26**:1 (1990), 101–183. MR Zbl

[Goresky and MacPherson 1988] M. Goresky and R. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik (3) **14**, Springer, 1988. MR Zbl

[Gromov 1985] M. Gromov, "Pseudo holomorphic curves in symplectic manifolds", *Invent. Math.* **82**:2 (1985), 307–347. MR Zbl

[Hind et al. 2016] R. Hind, M. Pinsonnault, and W. Wu, "Symplectormophism groups of non-compact manifolds, orbifold balls, and a space of Lagrangians", *J. Symplectic Geom.* **14**:1 (2016), 203–226. MR Zbl

[Karshon and Kessler 2017] Y. Karshon and L. Kessler, "Distinguishing symplectic blowups of the complex projective plane", *J. Symplectic Geom.* **15**:4 (2017), 1089–1128. MR Zbl

[Lalonde and McDuff 1996] F. Lalonde and D. McDuff, "J-curves and the classification of rational and ruled symplectic 4-manifolds", pp. 3–42 in *Contact and symplectic geometry* (Cambridge, 1994), edited by C. B. Thomas, Publ. Newton Inst. **8**, Cambridge Univ. Press, 1996. MR Zbl

[Lalonde and Pinsonnault 2004] F. Lalonde and M. Pinsonnault, "The topology of the space of symplectic balls in rational 4-manifolds", *Duke Math. J.* **122**:2 (2004), 347–397. MR Zbl

[Lang 2002] S. Lang, *Introduction to differentiable manifolds*, 2nd ed., Springer, 2002. MR Zbl

[Leung and Zhang 2015] N. C. Leung and J. Zhang, "Cox rings of rational surfaces and flag varieties of ADE-types", *Comm. Anal. Geom.* **23**:2 (2015), 293–317. MR Zbl

[Li ≥ 2020] J. Li, "Stratification of $\mathcal{J}_w$ of (m^4, ω) when $\operatorname{kod}(m) = 0, -\infty$", in preparation.

[Li and Li 2006] B.-H. Li and T.-J. Li, "On the diffeomorphism groups of rational and ruled 4-manifolds", *J. Math. Kyoto Univ.* **46**:3 (2006), 583–593. MR Zbl

[Li and Liu 2001] T.-J. Li and A.-K. Liu, "Uniqueness of symplectic canonical class, surface cone and symplectic cone of 4-manifolds with $B^+ = 1$", *J. Differential Geom.* **58**:2 (2001), 331–370. MR Zbl

[Li and Wu 2012] T.-J. Li and W. Wu, "Lagrangian spheres, symplectic surfaces and the symplectic mapping class group", *Geom. Topol.* **16**:2 (2012), 1121–1169. MR Zbl

[Li and Zhang 2011] T.-J. Li and W. Zhang, "Additivity and relative Kodaira dimensions", pp. 103–135 in *Geometry and analysis, No. 2*, edited by L. Ji, Adv. Lect. Math. (ALM) **18**, International Press, Somerville, MA, 2011. MR Zbl

[Li and Zhang 2015] T.-J. Li and W. Zhang, "Almost Kähler forms on rational 4-manifolds", *Amer. J. Math.* **137**:5 (2015), 1209–1256. MR Zbl

[Li et al. 2015] J. Li, T.-J. Li, and W. Wu, "The symplectic mapping class group of $\mathbb{C}\mathrm{P}^2\#n\overline{\mathbb{C}\mathrm{P}^2}$ with $n \leq 4$", *Michigan Math. J.* **64**:2 (2015), 319–333. MR Zbl

[Li et al. 2019] J. Li, T.-J. Li, and W. Wu, "Symplectic -2 spheres and the symplectomorphism group of small rational 4-manifolds, II", preprint, 2019, Available at https://arxiv.org/abs/1911.11073. arXiv

[Li et al. ≥ 2020] J. Li, T.-J. Li, and W. Wu, "Braid groups and symplectomorphism mapping class group of rational surfaces", in preparation.

[Manin 1986] Y. I. Manin, *Cubic forms*, 2nd ed., North-Holland Mathematical Library **4**, North-Holland, Amsterdam, 1986. MR Zbl

[McDuff 1990] D. McDuff, "The structure of rational and ruled symplectic 4-manifolds", *J. Amer. Math. Soc.* **3**:3 (1990), 679–712. MR Zbl

[McDuff 1994] D. McDuff, "Singularities and positivity of intersections of J-holomorphic curves", pp. 191–215 in *Holomorphic curves in symplectic geometry*, edited by M. Audin and J. Lafontaine, Progr. Math. **117**, Birkhäuser, Basel, 1994. MR

[McDuff 2008] D. McDuff, "The symplectomorphism group of a blow up", *Geom. Dedicata* **132** (2008), 1–29. MR Zbl

[McDuff and Opshtein 2015] D. McDuff and E. Opshtein, "Nongeneric J-holomorphic curves and singular inflation", *Algebr. Geom. Topol.* **15**:1 (2015), 231–286. MR Zbl

[McDuff and Tolman 2010] D. McDuff and S. Tolman, "Polytopes with mass linear functions, I", *Int. Math. Res. Not.* **2010**:8 (2010), 1506–1574. MR Zbl

[Meigniez 2002] G. Meigniez, "Submersions, fibrations and bundles", *Trans. Amer. Math. Soc.* **354**:9 (2002), 3771–3787. MR Zbl

[Micallef and White 1995] M. J. Micallef and B. White, "The structure of branch points in minimal surfaces and in pseudoholomorphic curves", *Ann. of Math.* (2) **141**:1 (1995), 35–85. MR Zbl

[Pedroza 2018] A. Pedroza, "Hamiltonian loops on the symplectic one-point blow up", *J. Symplectic Geom.* **16**:3 (2018), 839–856. MR Zbl

[Polterovich 1998] L. Polterovich, "Hofer's diameter and Lagrangian intersections", *Internat. Math. Res. Notices* **4** (1998), 217–223. MR Zbl

[Polterovich 2002] L. Polterovich, "Growth of maps, distortion in groups and symplectic geometry", *Invent. Math.* **150**:3 (2002), 655–686. MR Zbl

[Ruan 1993] Y. Ruan, "Symplectic topology and extremal rays", *Geom. Funct. Anal.* **3**:4 (1993), 395–430. MR Zbl

[Seidel 1997] P. Seidel, "π_1 of symplectic automorphism groups and invertibles in quantum homology rings", *Geom. Funct. Anal.* **7**:6 (1997), 1046–1095. MR Zbl

[Seidel 1999] P. Seidel, "On the group of symplectic automorphisms of $\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n$", pp. 237–250 in *Northern California Symplectic Geometry Seminar*, edited by Y. Eliashberg et al., Amer. Math. Soc. Transl. Ser. 2 **196**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

[Shevchishin 2009] V. Shevchishin, "Secondary Stiefel–Whitney class and diffeomorphisms of rational and ruled symplectic 4-manifolds", preprint, 2009, Available at https://arxiv.org/abs/0904.0283. arXiv

[Zhang 2017] W. Zhang, "The curve cone of almost complex 4-manifolds", *Proc. Lond. Math. Soc.* (3) **115**:6 (2017), 1227–1275. MR Zbl

[Zhao et al. 2006] X. Zhao, H. Gao, and H. Qiu, "The minimal genus problem in rational surfaces $\mathbb{C}\mathrm{P}^2 \# n\overline{\mathbb{C}\mathrm{P}^2}$", *Sci. China Ser. A* **49**:9 (2006), 1275–1283. MR Zbl

Received March 22, 2019. Revised July 3, 2019.

JUN LI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MICHIGAN
ANN ARBOR, MI
UNITED STATES
lijungeo@umich.edu

TIAN-JUN LI
SCHOOL OF MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MN
UNITED STATES
tjli@math.umn.edu

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Matthias Aschenbrenner
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
matthias@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Wee Teck Gan
Mathematics Department
National University of Singapore
Singapore 119076
matgwt@nus.edu.sg

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.
STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA
UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2020 is US $520/year for the electronic version, and $705/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2020 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 304 No. 2 February 2020

Graphs admitting only constant splines 385
KATIE ANDERS, ALISSA S. CRANS, BRIANA FOSTER-GREENWOOD, BLAKE MELLOR and JULIANNA TYMOCZKO

Centers of disks in Riemannian manifolds 401
IGOR BELEGRADEK and MOHAMMAD GHOMI

The geometry of the flex locus of a hypersurface 419
LAURENT BUSÉ, CARLOS D'ANDREA, MARTÍN SOMBRA and MARTIN WEIMANN

Morse inequalities for Fourier components of Kohn–Rossi cohomology of CR covering manifolds with S^1-action 439
RUNG-TZUNG HUANG and GUOKUAN SHAO

On Seifert fibered spaces bounding definite manifolds 463
AHMAD ISSA and DUNCAN MCCOY

Regularity of quotients of Drinfeld modular schemes 481
SATOSHI KONDO and SEIDAI YASUDA

Sums of algebraic trace functions twisted by arithmetic functions 505
MAXIM KOROLEV and IGOR SHPARLINSKI

Twisted calculus on affinoid algebras 523
BERNARD LE STUM and ADOLFO QUIRÓS

Symplectic (-2)-spheres and the symplectomorphism group of small rational 4-manifolds 561
JUN LI and TIAN-JUN LI

Addendum to the article Contact stationary Legendrian surfaces in $\mathbb{S}^5$ 607
YONG LUO

The Hamiltonian dynamics of magnetic confinement in toroidal domains 613
GABRIEL MARTINS

Gluing Bartnik extensions, continuity of the Bartnik mass, and the equivalence of definitions 629
STEPHEN MCCORMICK

Decomposable Specht modules indexed by bihooks 655
LIRON SPEYER and LOUISE SUTTON

On the global well-posedness of one-dimensional fluid models with nonlocal velocity 713
ZHUAN YE

Hopf cyclic cohomology for noncompact G-manifolds with boundary 753
XIN ZHANG

0030-8730(2020)304:2;1-R