Pacific Journal of Mathematics

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Volume 306 No. 1 May 2020

PACIFIC JOURNAL OF MATHEMATICS
Vol. 306, No. 1, 2020
dx.doi.org/10.2140/pjm.2020.306.185

A POSITIVE MASS THEOREM FOR MANIFOLDS WITH BOUNDARY

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We derive a positive mass theorem for asymptotically flat manifolds with boundary whose mean curvature satisfies a sharp estimate involving the conformal Green's function. The theorem also holds if the conformal Green's function is replaced by the standard Green's function for the Laplacian operator. As an application, we obtain an inequality relating the mass and harmonic functions that generalizes H. Bray's mass-capacity inequality in his proof of the Riemannian Penrose conjecture.

1. Introduction

One of the fundamental results in mathematical relativity is the Riemannian positive mass theorem:

Theorem 1.1 [Schoen and Yau 2017, Theorem 5.3]. *Suppose* (M^n, g) *is an* n*-dimensional,* $n \geq 3$*, complete, asymptotically flat manifold with nonnegative scalar curvature. Then the ADM mass of* (M, g) *is nonnegative and is zero if and only if* (M^n, g) *is isometric to the Euclidean space* $(\mathbb{R}^n, \delta)$.

This result was first proven in low dimensions by R. Schoen and S.-T. Yau [1979a; 1979b] using minimal surface methods and later by E. Witten [1981] for spin manifolds exploiting the Bochner–Weitzenböck formula for the Dirac operator (see also [Parker and Taubes 1982; Bartnik 1986]). Recently, Schoen and Yau [2017] extended their arguments to arbitrary dimensions (see [Lohkamp 2015; 2016] for a different approach). The minimal surface method applies to the case in which the manifold has nonempty boundary with nonpositive mean curvature, i.e., the mean curvature vector vanishes or points toward the infinity (see [Schoen and Yau 1979a]). If the boundary has positive mean curvature, the positivity of the mass is a more subtle question as the boundary no longer acts as a barrier for minimal surfaces. For instance, consider a manifold M obtained by cutting out a rotationally symmetric ball in a negative-mass Schwarzschild manifold. In this case, M has

MSC2010: primary 53Z05; secondary 83C99.
Keywords: positive mass theorem with boundary.

negative mass and a zero area singularity (see [Bray and Jauregui 2013]) is shielded by the boundary due to the relatively large positive mean curvature of ∂M.

The spinor method also adapts to the setting of manifolds with boundary. If the boundary mean curvature is nonpositive, positivity of the mass was shown in [Gibbons et al. 1983] (see also [Herzlich 1998]). Refining the spinor analysis, M. Herzlich [1997] relaxed the requirement of nonpositive mean curvature and showed that a condition $H \leq 4\sqrt{\pi/|\partial M|}$ in dimension 3 implies the positivity of the mass. For higher-dimensional spin manifolds, Herzlich [2002] proved a similar result.

In this work, we consider asymptotically flat manifolds of dimensions $n \geq 3$, with boundary allowed to have positive mean curvature. We show that if the mean curvature satisfies an upper estimate involving the conformal Green's function, then the manifold has nonnegative mass. More precisely, we have:

Theorem 1.2. *Let (M^n, g) be an n-dimensional, asymptotically flat manifold with nonnegative scalar curvature R, with boundary Σ. Let H be the mean curvature of Σ with respect to the ∞-pointing unit normal ν. Let u be the conformal Green's function given by*

$$\begin{cases} \Delta u - \frac{n-2}{4(n-1)} R u = 0 & \text{in } M, \\ u \to 0 & \text{at } \infty, \\ u = 1 & \text{at } \Sigma. \end{cases}$$

Then the estimate

$$H(x) \leq -\frac{n-1}{n-2} \nabla_\nu u(x) \quad \text{for all } x \in \Sigma \tag{1-1}$$

implies positivity of the ADM mass of (M^n, g); moreover, (M, g) has zero mass if and only if (M^n, g) is isometric to $(\mathbb{R}^n, \delta)$ minus a round ball.

By the assumption $R \geq 0$ and the maximum principle, we have an immediate corollary:

Corollary 1.3. *The same result as above holds true if one replaces u with the standard Green's function*

$$\begin{cases} \Delta v = 0 & \text{in } M, \\ v \to 0 & \text{at } \infty, \\ v = 1 & \text{at } \Sigma, \end{cases}$$

and requires

$$H(x) \leq -\frac{n-1}{n-2} \nabla_\nu v(x) \quad \text{for all } x \in \Sigma. \tag{1-2}$$

H. Bray [2001] proved the following theorem which plays a key role in his proof of the Riemannian Penrose inequality.

Theorem 1.4. *Let (M^3, g) be a complete, asymptotically flat 3-manifold with nonnegative scalar curvature, with boundary Σ which has zero mean curvature. Let $\varphi(x)$ be a function on (M^3, g) which satisfies*

$$\begin{cases} \Delta\varphi = 0 & \text{in } M, \\ \varphi \to 1 & \text{at } \infty, \\ \varphi = 0 & \text{at } \Sigma. \end{cases}$$

Then

$$\mathfrak{m} \geq \mathcal{C}, \tag{1-3}$$

where $\mathfrak{m}$ is the ADM mass of (M^3, g) and $\mathcal{C} > 0$ is the constant in the asymptotic expansion

$$\varphi(x) = 1 - \frac{\mathcal{C}}{|x|} + o(|x|^{-1}) \quad \text{as } x \to \infty.$$

Moreover, equality in (1-3) *holds if and only if (M^3, g) is isometric to a spatial Schwarzschild manifold outside its horizon.*

Using Corollary 1.3, we obtain the following generalization of Bray's theorem.

Theorem 1.5. *Let (M^n, g) be an n-dimensional, asymptotically flat manifold, with nonnegative scalar curvature, with boundary Σ. Let H be the mean curvature of Σ with respect to the ∞-pointing unit normal ν. Let $\phi(x)$ be a function on (M, g) which satisfies*

$$\begin{cases} \Delta\phi = 0 & \text{in } M, \\ \phi \to 1 & \text{at } \infty, \\ \phi = c & \text{at } \Sigma, \end{cases}$$

where $c > -1$ is a constant and $c \neq 1$. If

$$\frac{2c}{1-c^2}\frac{\partial\phi}{\partial\nu} \geq \frac{n-2}{n-1}H, \tag{1-4}$$

then

$$\mathfrak{m} \geq C, \tag{1-5}$$

where $\mathfrak{m}$ is the ADM mass of (M^n, g) and C is the constant in

$$\phi = 1 - \frac{C}{|x|^{n-2}} + o(|x|^{2-n}) \quad \text{as } x \to \infty.$$

Moreover, equality in (1-5) *holds if and only if (M^n, g) is isometric to the exterior region outside a rotationally symmetric sphere in an n-dimensional spatial Schwarzschild manifold, which is*

$$\left(\mathbb{R}^n/\{|x| < r_0\}, \left(1 + \frac{\mathfrak{m}}{2|x|^{n-2}}\right)^{4/(n-2)}\delta_{ij}\right)$$

for some constants $r_0 > 0$.

Remark 1.6. The mass $\mathfrak{m}$ of the Schwarzschild manifold in the rigidity statement of Theorem 1.5 can be arbitrary, and in particular $\mathfrak{m}$ can be negative. Moreover, if $\mathfrak{m} > 0$, r_0 can be arbitrary, meaning that Σ can be either outside the Schwarzschild horizon or inside the Schwarzschild horizon.

Remark 1.7. If $H \leq 0$, one can take $c = 0$. In this case, Theorem 1.5 reduces to Theorem 1.4.

Remark 1.8. An equivalent formulation of Theorem 1.5 shows Corollary 1.3 corresponds to a version of Theorem 1.5 if $c = 1$. See Theorem 5.1 in Section 5 for details.

Remark 1.9. In application, given a harmonic function ϕ that goes to 1 at infinity on an asymptotically flat manifold with nonnegative scalar curvature, one can consider the level sets $\Sigma_c = \phi^{-1}(c)$. As long as condition (1-4) holds at some Σ_c, one will have an estimate of the mass in terms of ϕ.

We now outline the proof of Theorem 1.2. The idea is simply to show the manifold (M, g) in Theorem 1.2 admits a legitimate compact fill-in $(\Omega, \tilde{g})$ so that, if $(\Omega, \tilde{g})$ is glued to (M, g), the resulting manifold satisfies the assumptions of the positive mass theorem. To produce such a fill-in, one conformally deforms (M, g) using the given function u (or v). The idea of conformally deforming an asymptotically flat manifold with boundary to produce a compact piece appeared in the pioneer work of G. Bunting and A. Masood-ul-Alam [1987] and was used by Bray [2001] in his proof of the Riemannian Penrose inequality. A recent application of this idea to obtain capacity estimates was given by C. Mantoulidis, L.-F. Tam and Miao [2019].

Once Theorem 1.2 is proved, Theorem 1.5 follows by considering a conformally deformed metric $\tilde{g} = \left(\frac{1}{2}(1+\phi)\right)^{4/(n-2)} g$ on M and applying Corollary 1.3 to $(M^n, \tilde{g})$.

The rest of this paper is organized as follows. We provide some background material in Section 2. In Section 3, we prove Theorem 1.2 for harmonically flat metrics. The general case of Theorem 1.2 is proved in Section 4. In Section 5, we use Corollary 1.3 to prove Theorem 1.5. In Section 6, we give some examples and discussions.

2. Prerequisites

We start by recalling the definition of a manifold being asymptotically flat (see [Schoen and Yau 2017; Bray and Lee 2009] for instance). Here we only consider manifolds with one end, and the general case can be treated in the same fashion.

Definition 2.1. Let $n \geq 3$. A Riemannian manifold (M^n, g) is asymptotically flat if there is a compact set $K \subset M$ such that M/K is diffeomorphic to $\mathbb{R}^n$ minus a ball and in this coordinate chart, g satisfies

$$g = \delta + \mathcal{O}(|x|^{-\tau}), \quad \partial g = \mathcal{O}(|x|^{-\tau-1}), \quad \partial^2 g = \mathcal{O}(|x|^{-\tau-2}),$$

and $R = \mathcal{O}(|x|^{-q})$, where $\tau > \frac{1}{2}(n-2)$ and $q > n$. Here R is the scalar curvature of g.

On an asymptotically flat (M^n, g), the ADM mass [Arnowitt et al. 1959] m is given by

$$\mathfrak{m} = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} (g_{ij,j} - g_{jj,i})\nu^i,$$

where ω_{n-1} is the volume of the unit sphere in $\mathbb{R}^n$, and the unit normal ν and volume integral are with respect to the Euclidean metric. The fact that m is a geometric invariant of (M, g) was shown by Bartnik [1986] and by Chruściel [1986] independently.

We next address some regularity questions regarding the positive mass theorem. Originally the result was proved for smooth manifolds, but since then there have been also several singular cases, proven in [Miao 2002; Shi and Tam 2002; Lee 2013; 2018; McFeron and Székelyhidi 2012; Lee and LeFloch 2015; Li and Mantoulidis 2019]. Combined with Theorem 1.1, results and proofs in [Miao 2002; McFeron and Székelyhidi 2012; Shi and Tam 2018] in particular give the following theorems:

Theorem 2.2 [Miao 2002, Theorem 1; McFeron and Székelyhidi 2012, Theorem 2]. *Let (M^n, g) be an asymptotically flat manifold. Suppose $\Sigma^{n-1} \subset M^n$ is a closed hypersurface which divides M into an exterior part M_1 and an interior part M_2. Suppose that the metric g is smooth up to Σ on both sides of Σ and has nonnegative scalar curvature away from Σ. Let H_1 and H_2 be the mean curvature of Σ with respect to the unit normal pointing to infinity in (M_1, g) and (M_2, g), respectively. If*

$$H_1 \le H_2,$$

then the ADM mass of (M^n, g) is nonnegative and is zero if and only if (M_1, g) is isometric to $(\mathbb{R}^n / \Omega, \delta)$ for a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary and (M_2, g) is isometric to (Ω, δ).

Theorem 2.3 [Shi and Tam 2018, Theorem 7.2]. *Let (M^n, g) be an asymptotically flat manifold such that $g \in W^{1,p}_{\text{loc}}$ for some $p > n$ and is smooth with nonnegative scalar curvature away from a point q. Then $m \ge 0$ and $m = 0$ only if M^n is diffeomorphic to $\mathbb{R}^n$ and g is flat away from q.*

3. The harmonically flat case

As in [Bray 2001], we first prove Theorem 1.2 for the case $g = \xi^{4/(n-2)}\delta$ near ∞, where ξ is a Euclidean harmonic function. This property is often known as (M^n, g) being harmonically flat (see [Bray 2001]).

Proof. Let $(\tilde{M}^n, \tilde{g})$ be the one point compactification of $(M^n, u^{4/(n-2)}g)$ where u is the conformal Green's function from Theorem 1.2. Note that in particular

$\tilde{g}|_\Sigma = g|_\Sigma$ and by the maximum principle $u > 0$. Also, we are able to extend $\tilde{g}$ smoothly to the point at ∞ as in Bray's proof of Theorem 1.4, also see the detailed exposition in Section 2 of [Mantoulidis et al. 2019]. This argument is based on the removable singularity theorem for harmonic functions and thus crucially requires the manifold to be harmonically flat. Note that in this case u is harmonic near ∞. In the general setting this conformal blow-down may produce a singular metric which must be smoothed which is performed in Section 4 following [Shi and Tam 2018] and [Mantoulidis et al. 2019].

The conformally transformed manifold $\tilde{M}$ has zero scalar curvature $\tilde{R}$ due to the well-known formula

$$\tilde{R} = u^{-4/(n-2)} R - \frac{4(n-1)}{n-2} u^{-4/(n-2)-1} \Delta u. \tag{3-6}$$

Moreover, the mean curvature transforms under conformal transformation and change of unit normal via

$$\tilde{H} = -u^{-2/(n-2)} H - \frac{2(n-1)}{n-2} u^{-n/(n-2)} \nabla_\nu u. \tag{3-7}$$

Note that we have to change the sign of the unit normal of $\tilde{\Sigma}$ so that it corresponds with the normal of Σ after gluing $\tilde{M}$ into M as carried out below. In combination with $u|_\Sigma = 1$, (3-7) leads to

$$\tilde{H} = -H - \frac{2(n-1)}{n-2} \nabla_\nu u. \tag{3-8}$$

Next we glue M and $\tilde{M}$ together along Σ to obtain a manifold $\hat{M}$ with corner Σ. More precisely, we define $(\hat{M}, \hat{g})$ by $\hat{M} = (M \sqcup \tilde{M})/_{\Sigma \sim \tilde{\Sigma}}$ and $\hat{g}|_M = g$, $\hat{g}|_{\tilde{M}} = \tilde{g}$. Then $\hat{M}$ is asymptotically flat, has nonnegative scalar curvature and no boundary. Furthermore, due to (1-1) and (3-8), we have

$$\tilde{H} = -H - \frac{2(n-1)}{n-2} \nabla_\nu u \geq H.$$

Thus all the assumptions of the positive mass theorem with corners, Theorem 2.2, are satisfied and we obtain $\hat{\mathfrak{m}} \geq 0$. Since we did not change the asymptotic behavior of the metric during the construction of $(\hat{M}, \hat{g})$ we have $\mathfrak{m} = \hat{\mathfrak{m}} \geq 0$ as desired.

Next, we apply the rigidity part of the positive mass theorem with corners [Miao 2002; McFeron and Székelyhidi 2012] to deduce that in the case of $\mathfrak{m} = 0$, $(\hat{M}, \hat{g})$ is isometric to $(\mathbb{R}^n, \delta)$ where δ is the standard metric. Thus (M^n, g) is isometric to $(R^n/\Omega, \delta)$, and $(\tilde{M}^n, \tilde{g})$ is isometric to (Ω, δ) where $\Omega \subset \mathbb{R}^n$ is some smooth, open set. Our goal is to show that $\Omega = B_\rho(x_0)$, $\rho > 0$, where $x_0 \in \tilde{M}$ is the compactified point from ∞. We begin by observing that the function

$$\tilde{u} := \frac{1}{u}$$

is harmonic on $\tilde{M}$ with $\tilde{u}|_\Sigma = 1$ and

$$\nabla_{\tilde{\nu}}\tilde{u} = -\nabla_\nu \tilde{u} = \nabla_\nu u.$$

Again, note how the unit normal of Σ changes sign after the conformal transformation. Therefore, we can define $\hat{u} \in C^1(\hat{M})$ by

$$\begin{cases} \hat{u}(x) = u(x) & \text{for } x \in M, \\ \hat{u}(x) = \tilde{u}(x) & \text{for } x \in \tilde{M}. \end{cases}$$

Since $\hat{u}$ is harmonic outside Σ and C^1 across Σ, it follows from standard PDE arguments that $\hat{u}$ is also harmonic across Σ. Thus, the uniqueness of solutions to the Laplace equation implies

$$\hat{u}(x) = \frac{\rho^{n-2}}{|x - x_0|^{n-2}}$$

for some $\rho > 0$ which results in $\Sigma = S_\rho(x_0)$ as desired. Alternatively, we could also directly apply Theorem 5.1 of [Mantoulidis et al. 2019] to $(\tilde{M}, \tilde{g}, \tilde{u})$. □

4. The general case

In this section, we prove Theorem 1.2 in the general case. Let u be the conformal Green's function from Theorem 1.2, i.e., $\mathcal{L}u = 0$ where

$$\mathcal{L} = \Delta - \frac{n-2}{4(n-1)}R$$

is the conformal Laplacian with respect to g. As in Appendix A of [Mantoulidis et al. 2019] we have the following asymptotic behavior of u at ∞:

Lemma 4.1. *There exists a constant $\mathcal{D}$ such that*

$$u(x) = \frac{\mathcal{D}}{|x|^{n-2}} + \mathcal{O}(|x|^{-\gamma}),$$

where $\gamma := \min(q - 2, n + \tau - 2, n - 1) > n - 2$.

Proof. Let $\mathbb{R}^n / B_{R_1}(0)$ be the asymptotically flat end. We start by constructing a barrier to the conformal Green's function. For this purpose let $\psi = ar^{-n+2} - r^{-n+2-\epsilon}$ with $a, \epsilon > 0$. Then as in the proof of Lemma A2 in [Mantoulidis et al. 2019],

$$\mathcal{L}\psi = (-n + 2 - \epsilon)(-n + 1 - \epsilon)r^{-n-\epsilon} + \mathcal{O}(|x|^{-\kappa}),$$

where $\kappa := \min(n + \tau, q) > n$. Choosing $\epsilon < \min(\tau, n - q)$ there is an $R_2 \geq R_1$ independent of a such that $\Delta\psi \leq 0$ on $\mathbb{R}^3 / B_{R_2}(0)$. Moreover, we set $a \gg 1$ such that $\psi > u$ on the complement of $\mathbb{R}^n / B_{R_2}(0)$ in M. Hence, by the maximum principle, $u \leq \psi \leq ar^{-n+2}$ and similarly $u \geq -ar^{-n+2}$.

By the Schauder estimates and Theorem 6.2 in [Gilbarg and Trudinger 1998],

$$|u| \le C|x|^{-\kappa}, \quad |\nabla u| \le C|x|^{-\kappa-1}, \quad |\nabla^2 u| \le C|x|^{-\kappa-2},$$

where C denotes some constant which may change in the following computations from line to line. We extend u smoothly onto $\mathbb{R}^n$ such that $u \le C$ in $B_{R_0}(0)$ with $R_0 \le R_1$. Let $f = \hat{\Delta} u = \mathcal{O}(|x|^{-\kappa})$, where $\hat{\Delta}$ is the Euclidean Laplacian, and define

$$w(x) := -\frac{1}{n(n-2)\alpha_n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} f(y)\, dy,$$

where α_n is the volume of the unit ball in $\mathbb{R}^n$. This is well defined due to the decay properties of $f = \hat{\Delta} u$. Also, observe that $\hat{\Delta} w = f$. Next, as in the proof of Lemma A2 in Appendix A of [Mantoulidis et al. 2019], denoting $|x| = r$, we compute

$$\int_{B_r(x)/B_r(0)} \frac{1}{|x-y|^{n-2}} f(y)\, dy \le C r^{-\kappa} \int_{B_r(x)/B_r(0)} \frac{1}{|x-y|^{n-2}}\, dy \le C r^{-\kappa+2} \tag{4-9}$$

and

$$\begin{aligned} \int_{\mathbb{R}^n/(B_r(0)\cup B_r(x))} \frac{1}{|x-y|^{n-2}} f(y)\, dyi &\le C r^{-n+2} \int_{\mathbb{R}^3/(B_r(0)\cup B_r(x))} |y|^{-n-\tau} \\ &\le C r^{-\kappa+2}. \end{aligned} \tag{4-10}$$

Further, we additionally split the integral over $B_r(0)$ into $B_r(0)/B_{R_0}(0) \cup B_{R_0}(0)$:

$$\begin{aligned} &\int_{B_r(0)/B_{R_0}(0)} \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x|^{n-2}} \right) f(y)\, dy \\ &\quad = \int_{B_{R_0}(0)/B_\rho(0)} \frac{|x|^{2n-4} - |x-y|^{2n-4}}{|x|^{n-2}|x-y|^{n-2}(|x|^{n-2} + |x-y|^{n-2})} f(y)\, dy \\ &\quad \le \sum_{j=1}^{2n-4} \int_{B_r(0)/B_{R_0}(0)} C \frac{|y|^{-\kappa+j}}{|x|^{n-2+j}} \le C r^{-n+1}. \end{aligned} \tag{4-11}$$

Similarly, we have

$$\begin{aligned} &\int_{B_{R_0}(0)} \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x|^{n-2}} \right) f(y)\, dy \\ &\quad = \int_{B_{R_0}(0)} \frac{|x|^{2n-4} - |x-y|^{2n-4}}{|x|^{n-2}|x-y|^{n-2}(|x|^{n-2} + |x-y|^{n-2})} f(y)\, dy \\ &\quad \le \sum_{j=1}^{2n-4} \int_{B_{R_0}(0)} C \frac{|y|^k}{|x|^{n-2+k}} \le C r^{-n+1}. \end{aligned} \tag{4-12}$$

Lastly, we note

$$\frac{1}{|x|^{n-2}} \int_{\mathbb{R}^n / B_r(0)} f(y)dy \leq Cr^{-\kappa+2}. \tag{4-13}$$

Combining equations (4-9)–(4-13), we obtain

$$-n(n-2)\alpha_n w = \frac{\int_{\mathbb{R}^n} \hat{\Delta} u}{r^{n-2}} + \mathcal{O}(|x|^{-\min(q-2, n+\tau-2, n-1)}).$$

In particular, by the maximum principle, we have $w = u$. As in [Mantoulidis et al. 2019], Schauder theory [Gilbarg and Trudinger 1998, Theorem 6.3] gives the higher order estimates. □

Without loss of generality we may assume $R_1 = 1$, i.e., the asymptotically flat end is diffeomorphic to $\mathbb{R}^n / B_1(0)$. Next, we introduce the Kelvin transform $\mathcal{K}$ and prove the following elementary lemma:

Lemma 4.2. *Let (M^n, g) be asymptotically flat, i.e., $g_{ij} = \delta_{ij} + \sigma_{ij}$ where $\sigma_{ij} = \mathcal{O}_2(|x|^{-\tau}) = \mathcal{O}_2(|y|^{\tau})$ and $\mathcal{K}(x) := x/|x|^2$. Then for $y = \mathcal{K}(x)$ and $h_{ij} = g(\partial_i y, \partial_j y)$, we have*

$$h_{ij} = |y|^{-4}\delta_{ij} + \mathcal{O}(|y|^{\tau-4}).$$

Moreover, we have

$$\partial_k h_{ij} = -4|y|^{-5}\partial_k |y| \delta_{ij} + \mathcal{O}(|y|^{\tau-5}).$$

Proof. We compute

$$\begin{aligned} h_{ij} &= g\left(\frac{\partial_i x}{|x|^2} - 2\frac{\langle \partial_i x, x\rangle x}{|x|^4}, \frac{\partial_j x}{|x|^2} - 2\frac{\langle \partial_j x, x\rangle x}{|x|^4}\right) \\ &= |y|^{-4}[\delta_{ij} + \sigma_{ij} + |y|^{-4}(4y_i y_j y_k y_l \sigma^{kl} - 2|y|^2(y_i y^k \sigma_{jk} + y_j y^k \sigma_{ik}))] \\ &= |y|^{-4}\delta_{ij} + \mathcal{O}(|y|^{\tau-4}). \end{aligned}$$

The second statement follows analogously. □

Next, we study the metric $\tilde{g} = u^{4/(n-2)} g$. A priori $\tilde{g}$ is defined on $\mathbb{R}^n / B_1(0)$ but by inverting the coordinates via the Kelvin transform $\mathcal{K}$ we may view $\tilde{g}$ as metric on $B_1(0)$. Thereby the point ∞ of the one point compactification corresponds to the origin under the Kelvin transform and we wish to extend $\tilde{g}$ there. This is done in the following lemma which is based on Lemma 6.1 in [Mantoulidis et al. 2019]. Moreover, we also obtain some regularity for $\tilde{g}$ which is necessary in order to perform the smoothing procedure below.

Lemma 4.3. *The metric $\tilde{g}$ extends continuously across the origin to a $W^{1,p}$ metric for some $p > n$.*

Proof. We compute the coordinates $y = \mathcal{K}(x)$ for h_{ij},

$$u^{4/(n-2)} h_{ij} = \mathcal{D}\delta_{ij} + \mathcal{O}(|y|^{\gamma+2-n}).$$

Hence h is continuous in the origin. Next, we compute in a similar fashion

$$\begin{aligned}\partial_k(u^{4/(n-2)}h_{ij}) &= (4\mathcal{D}|y|^{n-7}+\mathcal{O}(|y|^{\gamma-5}))(\mathcal{D}|y|^{n-2}+\mathcal{O}(|y|^{\gamma}))^{(6-n)/(n-2)}\partial_k|y| \\ &\quad +(-4\mathcal{D}|y|^{n-7}+\mathcal{O}(|y|^{\gamma-5})(\mathcal{D}|y|^{n-2}+\mathcal{O}(|y|^{\gamma}))^{(6-n)/(n-2)}\partial_k|y| \\ &= \mathcal{O}(|y|^{\gamma-n+1}).\end{aligned}$$

Thus $u^4 h_{ij} \in W^{1,p}(B_1(0))$ for $p = n/(\min(q-n-1, \tau-1)) > n$. □

Now we are in a position to verify the following proposition, which is known to people who are familiar with the work in [Miao 2002; Shi and Tam 2002; 2018; Lee 2013; McFeron and Székelyhidi 2012; Li and Mantoulidis 2019; Mantoulidis et al. 2019].

Proposition 4.4. *Suppose (M^n, g) has corner singularity along a hypersurface Σ with $H_1 \le H_2$ as in Theorem 2.2. Also, suppose there is a point singularity $q \in M_2$ where g is in $W^{1,p}$ for some $p > n$ near q. If g has nonnegative scalar curvature away from Σ and $\{q\}$, then $\mathfrak{m} \ge 0$ with equality if and only if (M_1, g) is isometric to $(\mathbb{R}^n/\Omega, \delta)$ for a bounded domain Ω with smooth boundary.*

Proof. Due to the assumption $H_1 \le H_2$ at Σ, we may exactly follow Proposition 3.1 in [Miao 2002] to approximate g by a family of smooth metrics g_δ such that $g_\delta(x) = g(x)$ for $\operatorname{dist}(\Sigma, x) \ge \delta$ and the scalar curvature R_δ satisfies

$$R_\delta(x) \ge -C \tag{4-14}$$

for $\operatorname{dist}(\Sigma, x) < \delta$. Equation (4-14) in particular shows that the integral of the negative part of the scalar curvature can be made arbitrarily small during the approximation process.

Near the point singularity q, we approximate g as in [Shi and Tam 2018, Lemma 4.1] and [Mantoulidis et al. 2019, Lemma 3.6] to obtain smooth metrics $\{g_\epsilon\}$ so that $g_\epsilon = g$ outside $B_\epsilon(\{q\})$, and $\|g_\epsilon\|_{W^{1,p}(B_\epsilon(\{q\}))} \le C$ where C is independent of ϵ. By Lemma 3.7 in [Mantoulidis et al. 2019], the uniform $W^{1,p}$ bound on g_ϵ implies that the integral of the negative part of R_{g_ϵ} over $B_\epsilon(\{q\})$ becomes arbitrarily small. (We note that Lemma 3.7 in [Mantoulidis et al. 2019] is stated in a slightly more general version and for our purpose the result already follows from the estimate on the term $\boldsymbol{I}_B$ in Lemma 3.7 in [Mantoulidis et al. 2019].)

Since both approximations g_ϵ and g_δ are local, we can perform them simultaneously to approximate g by a smooth metric with $\int_M R^-$ becoming arbitrarily small. Here R^- denotes the negative part of the scalar curvature.

Now we proceed as usual and conformally transform M such that $R \geq 0$ everywhere; see [Schoen and Yau 1979b] and Section 4 in [Miao 2002]. Thereby the mass converges as in [Miao 2002] and we may deduce that the mass is nonnegative by the standard positive mass theorem.

If the mass is zero, we need to apply the argument from [McFeron and Székelyhidi 2012]. This is due to [McFeron and Székelyhidi 2012] relying on Ricci flow which has the advantage that the mass stays constant during the smoothing process so we can apply the rigidity statement of the positive mass theorem to the flow solution with initial data g (see also Section 7 of [Shi and Tam 2018]). This shows the rigidity of the proposition. □

Proof of Theorem 1.2. Due to Lemma 4.3, the conformally filled-in manifold $(\hat{M}, \hat{g})$ constructed in Section 3 satisfies the conditions of Proposition 4.4. Hence Theorem 1.2 follows from Proposition 4.4 in the same way that the harmonically flat case is proven in Section 3. □

5. Application

In this section, we prove Theorem 1.5, which generalizes Bray's result in Theorem 1.4.

Proof of Theorem 1.5. By the maximum principle, $\phi > -1$ on M. Define

$$w = \frac{2}{1+\phi} \quad \text{and} \quad \tilde{g} = w^{-4/(n-2)} g.$$

$(M^n, \tilde{g})$ is asymptotically flat with nonnegative scalar curvature. The fact $\Delta w^{-1} = 0$ implies

$$\tilde{\Delta} w = 0.$$

Moreover, $w \to 1$ at ∞ and $w = 2/(1+c)$ at Σ. Next, define

$$v = \frac{1+c}{1-c}(w-1).$$

Then

$$\tilde{\Delta} v = 0, \quad v \to 0 \text{ at } \infty, \quad \text{and} \quad v = 1 \text{ at } \Sigma.$$

Let $\tilde{H}$ be the mean curvature of Σ in $(M, \tilde{g})$ with respect to the ∞-pointing unit normal $\tilde{\nu}$. By (3-7), it follows that

$$\tilde{H} = w^{2/(n-2)}\left[H + \frac{2(n-1)}{(n-2)} w \nabla_\nu w^{-1}\right],$$
$$\tilde{\nu} = w^{2/(n-2)} \nu,$$

and

$$\nabla_{\tilde{\nu}} v = w^{2/(n-2)} \nabla_\nu v.$$

Hence, at Σ, by (1-4),

$$\begin{aligned}\tilde{H}+\frac{n-1}{n-2}\nabla_\nu v &= w^{2/(n-2)}\left[H+\frac{(n-1)}{(n-2)}(2w\nabla_\nu w^{-1}+\nabla_\nu v)\right]\\ &= w^{2/(n-2)}\left[H-\frac{(n-1)}{(n-2)}\frac{2c}{(1-c^2)}\nabla_\nu v\right]\\ &\le 0.\end{aligned}$$

Thus, by Corollary 1.3,

$$\tilde{\mathfrak{m}}\ge 0, \tag{5-15}$$

where $\tilde{\mathfrak{m}}$ is the ADM mass of $(M^n,\tilde{g})$. Since by the definition of mass $\tilde{\mathfrak{m}}$ and $\mathfrak{m}$ are related by $\tilde{\mathfrak{m}}=\mathfrak{m}-C$, we conclude from (5-15) that

$$\mathfrak{m}\ge C.$$

If $\mathfrak{m}=C$, then $\tilde{\mathfrak{m}}=0$. By Corollary 1.3, $(M^n,\tilde{g})$ is isometric to $(\mathbb{R}^n/\{|x|<r_0\},\delta)$ for some constant $r_0>0$. In this case, w is a Euclidean harmonic function that goes to 1 at ∞ and equals a positive constant $2/(1+c)$ at $\Sigma=\{|x|=r_0\}$. Hence,

$$w=1+\frac{\mathfrak{m}}{2|x|^{n-2}},$$

where $\mathfrak{m}$ is a constant satisfying $\mathfrak{m}=(1-c)/(1+c)2r_0^{n-2}$. □

It is worth pointing out the following equivalent form of Theorem 1.5.

Theorem 5.1. *Let (M^n,g) be an n-dimensional, asymptotically flat manifold, with nonnegative scalar curvature, with boundary Σ. Let $\varphi(x)$ be a function on (M,g) which satisfies*

$$\begin{cases}\Delta\varphi=0 & \text{in } M,\\ \varphi\to 1 & \text{at } \infty,\\ \varphi=0 & \text{at } \Sigma.\end{cases}$$

Let H be the mean curvature of Σ in (M^n,g) with respect to the ∞-pointing normal ν. If there is a constant $c>-1$ such that

$$\frac{2c}{1+c}\nabla_\nu\varphi\ge\frac{n-2}{n-1}H, \tag{5-16}$$

then

$$\mathfrak{m}\ge(1-c)\mathcal{C}, \tag{5-17}$$

where $\mathfrak{m}$ is the ADM mass of (M^n,g) and $\mathcal{C}$ is the constant in

$$\varphi=1-\frac{\mathcal{C}}{|x|^{n-2}}+o(|x|^{2-n})\quad \text{as } x\to\infty.$$

Moreover, equality in (5-17) *holds if and only if* (M^n, g) *is isometric to an* n*-dimensional spatial Schwarzschild manifold outside a rotationally symmetric sphere, that is,*

$$\left(\mathbb{R}^n/\{|x| < r_0\}, \left(1 + \frac{\mathfrak{m}}{2|x|^{n-2}}\right)^{4/(n-2)} \delta_{ij}\right) \quad \textit{for some constants } r_0 > 0.$$

Proof. If $c \neq 1$, the theorem follows directly from Theorem 1.5 by letting $\varphi = 1/(1-c)(\phi - c)$. If $c = 1$, the theorem reduces to Corollary 1.3. □

Remark 5.2. For a constant $c > -1$ satisfying (5-16) to exist, one needs to have

$$2\nabla_\nu \varphi > \frac{n-2}{n-1} H.$$

If $n = 3$, this coincides with the condition in [Mantoulidis et al. 2019, Theorem 1.5].

6. Examples and discussions

Example 6.1 (boundary with $H \leq 0$). Every manifold (M^n, g) whose boundary ∂M has nonpositive mean curvature satisfies condition (1-1) and thus has positive mass m. More precisely, $m \geq \mathcal{C} > 0$ by Theorem 1.5.

Example 6.2 (regions in Schwarzschild manifold). Let

$$(M^n, g) = \left(\mathbb{R}^n/B_{r_0}(0), \left(1 + \frac{m}{2r^{n-2}}\right)^{4/(n-2)} \delta\right)$$

be a spacelike slice in Schwarzschild spacetime of mass m. Note, m is allowed to be negative. We begin by computing the mean curvature of the boundary $S_{r_0}(0)$ using (3-7):

$$H = (2^{n/(n-2)} r_0^{n-1} - 2^{2/(n-2)} m r_0) \frac{n-1}{(2r_0^{n-2} + m)^{n/(n-2)}}.$$

Next, we observe that the Green's function is given by

$$u(r) = \frac{2r_0^{n-2} + m}{2r^{n-2} + m}. \tag{6-18}$$

Combining this with the observation $|\partial_r|_g = (1 + m/(2r^{n-2}))^{2/(n-2)}$, at S_{r_0} we have

$$-\frac{n-1}{n-2} \nabla_\nu u = 2^{n/(n-2)} r_0^{n-1} \frac{n-1}{(2r_0^{n-2} + m)^{n/(n-2)}}.$$

Thus, as predicted by the rigidity statement of Theorem 1.5, we have

$$H = -\frac{2c(n-1)}{(1+c)(n-2)} \nabla_\nu u$$

for

$$c = \frac{2r_0^{n-2}}{2r_0^{n-2} + m}.$$

Moreover, $2c/(c+1)$ can approach 1 while maintaining negative mass which shows that Theorem 1.2 is sharp in this sense.

Example 6.3 (conformal minimal boundary). Suppose that $g = \phi^{4/(n-2)}\tilde{g}$ such that Σ is a minimal surface with respect to $\tilde{g}$ where ϕ solves

$$\begin{cases} \Delta\phi = 0 & \text{in } M, \\ \phi \to 2 & \text{at } \infty, \\ \phi = 1 & \text{at } \Sigma. \end{cases}$$

Observe that by (3-6), $\tilde{R}$ also has nonnegative scalar curvature. Note that $v\phi$ is a harmonic function on $(M^n, \tilde{g})$ satisfying $(\phi v)|_\Sigma = 1$ and $(\phi v)(x) \to 0$ for $|x| \to \infty$. Precisely the same properties hold true for the function $(2-\phi)$ and thus, by the uniqueness property for harmonic functions, we deduce $\phi v = 2 - \phi$. Therefore, at Σ we have

$$\nabla_\nu v = -\frac{2}{v^2}\nabla_\nu \phi = -2\nabla_\nu \phi.$$

Next, using (3-7) we compute

$$H = \tilde{H}\phi^{-2/(n-2)} + \frac{2(n-1)}{n-2}\phi^{-n/(n-2)}\nabla_\nu\phi = \frac{2(n-1)}{n-2}\nabla_\nu\phi = -\frac{n-1}{n-2}\nabla_\nu v.$$

Hence condition (1-2) is satisfied with equality and we have $m \geq 0$. Conversely, suppose we have $H \equiv -(n-1)/(n-2)\nabla_\nu v$. Then, we define $\phi := 2/(1+v)$ and $\tilde{g} = \phi^{-4/(n-2)}g$. Applying (3-7) again, we obtain

$$\tilde{H} = H\phi^{2/(n-2)} + \frac{2(n-1)}{n-2}\phi^{n/(n+2)}\nabla_\nu(\phi^{-1}) = H + \frac{n-1}{n-2}\nabla_\nu v = 0.$$

Thus Σ is a minimal surface with respect to $\tilde{g}$. Indeed, this calculation shows that condition (1-2) holds if and only if $\tilde{H} \leq 0$ under the above conformal transformation. In particular, this suggests Bray's result and proof of Theorem 1.4 in [Bray 2001] can also be applied to derive Corollary 1.3 and Theorem 1.5 in the harmonically flat case.

Next, we want to relate Theorem 1.2, Theorem 1.5 and Example 6.3 to the Riemannian Penrose conjecture. This conjecture was shown by G. Huisken and T. Ilmanen [2001] (one black hole) and by Bray [2001] (multiple black holes). The latter proof has been extended up to dimension 7 by Bray and D. Lee [2009].

One of the key steps in Bray's proof is to prove the mass-capacity estimate, Theorem 1.4. There an asymptotically flat manifold with nonnegative scalar curvature is first reflected across its horizon, then conformally transformed via the Green's function so the positive mass theorem may be applied. In view of the positive mass theorem for manifolds with boundary, the reflection argument can

now be dropped. More precisely, the asymptotically flat manifold can be directly conformally transformed via the Green's function as presented in Example 6.3.

As it turns out, Theorem 1.4 is the reason the proof of the Penrose inequality breaks down in higher dimensions as discovered by Bray. This is due to minimal hypersurfaces being no longer regular which then in particular complicates the reflection argument. Therefore, showing a positive mass theorem for manifolds with singular boundary may be considered a strategy for proving the higher-dimensional Penrose inequality.

We also recall that Herzlich [1997; 2002] showed a positive mass theorem for asymptotically flat manifolds with boundary ∂M whose mean curvature satisfies an upper bound which depends on the area of ∂M in dimension 3 and depends on the Yamabe invariant $\mathcal{Y}$ of ∂M in higher dimensions. More precisely, the latter case assumes

$$H \leq |\Sigma|^{-1/(n-1)} \sqrt{\frac{n-1}{n-2} \mathcal{Y}(\Sigma)}.$$

The proofs in [Herzlich 1997; 2002] make use of estimates of the first eigenvalue of the Dirac operator by C. Bär [1992] and by O. Hijazi [1986; 1991]. Results in [Herzlich 1997; 2002] have been extended to the asymptotically hyperbolic setting by Hijazi, S. Montiel and S. Raulot [Hijazi et al. 2015]. It is plausible that Theorem 1.2 or Corollary 1.3 may have an analogue in the asymptotically hyperbolic setting.

Acknowledgements

This work was initiated when Hirsch was visiting the University of Miami (UM) in October 2018. He is grateful for the hospitality of the department of mathematics at UM. He also wants to thank H. Bray for many encouraging discussions and interest in this work. Miao's research is partially supported by Simons Foundation Collaboration Grant for Mathematicians #585168.

References

[Arnowitt et al. 1959] R. Arnowitt, S. Deser, and C. W. Misner, "Dynamical structure and definition of energy in general relativity", *Phys. Rev.* (2) **116** (1959), 1322–1330. MR Zbl

[Bär 1992] C. Bär, "Lower eigenvalue estimates for Dirac operators", *Math. Ann.* **293**:1 (1992), 39–46. MR Zbl

[Bartnik 1986] R. Bartnik, "The mass of an asymptotically flat manifold", *Comm. Pure Appl. Math.* **39**:5 (1986), 661–693. MR Zbl

[Bray 2001] H. L. Bray, "Proof of the Riemannian Penrose inequality using the positive mass theorem", *J. Differential Geom.* **59**:2 (2001), 177–267. MR Zbl

[Bray and Jauregui 2013] H. L. Bray and J. L. Jauregui, "A geometric theory of zero area singularities in general relativity", *Asian J. Math.* **17**:3 (2013), 525–559. MR Zbl

[Bray and Lee 2009] H. L. Bray and D. A. Lee, "On the Riemannian Penrose inequality in dimensions less than eight", *Duke Math. J.* **148**:1 (2009), 81–106. MR Zbl

[Bunting and Masood-ul Alam 1987] G. L. Bunting and A. K. M. Masood-ul Alam, "Nonexistence of multiple black holes in asymptotically Euclidean static vacuum space-time", *Gen. Relativity Gravitation* **19**:2 (1987), 147–154. MR Zbl

[Chruściel 1986] P. Chruściel, "Boundary conditions at spatial infinity from a Hamiltonian point of view", pp. 49–59 in *Topological properties and global structure of space-time* (Erice, 1985), edited by P. G. Bergmann and V. De Sabbata, NATO Adv. Sci. Inst. Ser. B Phys. **138**, Plenum, New York, 1986. MR Zbl

[Gibbons et al. 1983] G. W. Gibbons, S. W. Hawking, G. T. Horowitz, and M. J. Perry, "Positive mass theorems for black holes", *Comm. Math. Phys.* **88**:3 (1983), 295–308. MR

[Gilbarg and Trudinger 1998] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, revised 2nd ed., Springer, 1998. MR Zbl

[Herzlich 1997] M. Herzlich, "A Penrose-like inequality for the mass of Riemannian asymptotically flat manifolds", *Comm. Math. Phys.* **188**:1 (1997), 121–133. MR Zbl

[Herzlich 1998] M. Herzlich, "The positive mass theorem for black holes revisited", *J. Geom. Phys.* **26**:1-2 (1998), 97–111. MR Zbl

[Herzlich 2002] M. Herzlich, "Minimal surfaces, the Dirac operator and the Penrose inequality", pp. 9–16 in *Séminaire de Théorie Spectrale et Géométrie*, Sémin. Théor. Spectr. Géom. **20**, Univ. Grenoble I, Saint-Martin-d'Hères, 2002. MR Zbl

[Hijazi 1986] O. Hijazi, "A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors", *Comm. Math. Phys.* **104**:1 (1986), 151–162. MR Zbl

[Hijazi 1991] O. Hijazi, "Première valeur propre de l'opérateur de Dirac et nombre de Yamabe", *C. R. Acad. Sci. Paris Sér. I Math.* **313**:12 (1991), 865–868. MR Zbl

[Hijazi et al. 2015] O. Hijazi, S. Montiel, and S. Raulot, "A positive mass theorem for asymptotically hyperbolic manifolds with inner boundary", *Internat. J. Math.* **26**:12 (2015), art. id. 1550101, 17 pp. MR Zbl

[Huisken and Ilmanen 2001] G. Huisken and T. Ilmanen, "The inverse mean curvature flow and the Riemannian Penrose inequality", *J. Differential Geom.* **59**:3 (2001), 353–437. MR Zbl

[Lee 2013] D. A. Lee, "A positive mass theorem for Lipschitz metrics with small singular sets", *Proc. Amer. Math. Soc.* **141**:11 (2013), 3997–4004. MR Zbl

[Lee and LeFloch 2015] D. A. Lee and P. G. LeFloch, "The positive mass theorem for manifolds with distributional curvature", *Comm. Math. Phys.* **339**:1 (2015), 99–120. MR Zbl

[Li and Mantoulidis 2019] C. Li and C. Mantoulidis, "Positive scalar curvature with skeleton singularities", *Math. Ann.* **374**:1-2 (2019), 99–131. MR Zbl

[Lohkamp 2015] J. Lohkamp, "Skin structures in scalar curvature geometry", preprint, 2015. arXiv

[Lohkamp 2016] J. Lohkamp, "The higher dimensional positive mass theorem I", preprint, 2016. arXiv

[Mantoulidis et al. 2019] C. Mantoulidis, P. Miao, and L.-F. Tam, "Capacity, quasi-local mass, and singular fill-ins", *J. Reine Angew. Math.* (online publication December 2019).

[McFeron and Székelyhidi 2012] D. McFeron and G. Székelyhidi, "On the positive mass theorem for manifolds with corners", *Comm. Math. Phys.* **313**:2 (2012), 425–443. MR Zbl

[Miao 2002] P. Miao, "Positive mass theorem on manifolds admitting corners along a hypersurface", *Adv. Theor. Math. Phys.* **6**:6 (2002), 1163–1182. MR

[Parker and Taubes 1982] T. Parker and C. H. Taubes, "On Witten's proof of the positive energy theorem", *Comm. Math. Phys.* **84**:2 (1982), 223–238. MR Zbl

[Schoen and Yau 1979a] R. Schoen and S. T. Yau, "On the proof of the positive mass conjecture in general relativity", *Comm. Math. Phys.* **65**:1 (1979), 45–76. MR Zbl

[Schoen and Yau 1979b] R. M. Schoen and S. T. Yau, "Complete manifolds with nonnegative scalar curvature and the positive action conjecture in general relativity", *Proc. Nat. Acad. Sci. U.S.A.* **76**:3 (1979), 1024–1025. MR Zbl

[Schoen and Yau 2017] R. Schoen and S.-T. Yau, "Positive scalar curvature and minimal hypersurface singularities", preprint, 2017. arXiv

[Shi and Tam 2002] Y. Shi and L.-F. Tam, "Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature", *J. Differential Geom.* **62**:1 (2002), 79–125. MR Zbl

[Shi and Tam 2018] Y. Shi and L.-F. Tam, "Scalar curvature and singular metrics", *Pacific J. Math.* **293**:2 (2018), 427–470. MR Zbl

[Witten 1981] E. Witten, "A new proof of the positive energy theorem", *Comm. Math. Phys.* **80**:3 (1981), 381–402. MR Zbl

Received March 19, 2019. Revised July 6, 2019.

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

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