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SURFACE DIFFUSION FLOW OF ARBITRARY CODIMENSION IN SPACE FORMS

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Dedicated to Professor Duanzhuang Qian on his 110th anniversary.

We first prove that a properly immersed static n-dimensional submanifold ($\Delta H = 0$) with restricted growth of the curvature at infinity in $\mathbb{F}^{n+p}(c)$ ($c \geq 0$) is totally umbilical if the Willmore functional is pinched by a positive constant depending only on n. Secondly, we obtain a global rigidity theorem for Willmore surfaces in the sphere. Thirdly, we give a lower bound on the lifespan of the surface diffusion flow in $\mathbb{F}^{2+p}(c)$. Finally, we get the longtime existence and convergence of the surface diffusion flow in $\mathbb{F}^{2+p}(c)$ ($c \geq 0$) under the small initial Willmore energy condition.

1. Introduction

An important problem in global differential geometry is the study of curvature and topology of Riemannian manifolds and submanifolds. As we know, curvature flows are powerful tools in the study of sphere theorems as in [Andrews and Baker 2010; Brendle and Schoen 2009; Hamilton 1982; Huisken 1984; Gu et al. 2017; Liu et al. 2018; Wang 2008; Gu and Xu 2012; Xu and Gu 2013], etc. For instance, Brendle and Schoen [2009] proved the remarkable differentiable $\frac{1}{4}$-pinching sphere theorem via the Ricci flow, which had been open for half a century.

Let $\mathbb{F}^{n+p}(c)$ be the $(n+p)$-dimensional complete and simply connected space form with constant curvature c. In this paper, we study the motion of an immersed submanifold with the normal velocity $-\Delta H$ in the space form $\mathbb{F}^{n+p}(c)$. More precisely, let $f : M^n \to \mathbb{F}^{n+p}(c)$ be a compact immersed submanifold. The diffusion flow for submanifolds is a fourth order flow

$$\frac{\partial f}{\partial t} = -\Delta H, \tag{1-1}$$

where $\nabla_X \phi = (D_X \phi)^\perp$ for a tangent vector field X and a normal vector field ϕ

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on M, $\Delta = \nabla_i \nabla_i$ and H is the mean curvature vector of M^n. Denote by A the second fundamental form and by $A^\circ = A - \frac{1}{n} g \otimes H$ the trace free part of the second fundamental form. When $\nabla H = 0$, we call M a submanifold with parallel mean curvature. In particular, M is minimal if $H = 0$. When $\Delta H = 0$, we call M a static submanifold for the diffusion flow which is a generalization of the submanifold with parallel mean curvature.

For $n = 2$, surface diffusion flow (1-1) in $\mathbb{R}^3$ was proposed by Mullins [1957] to describe thermal grooving in material sciences. From the view of geometric analysis it appears naturally as the gradient flow of the area functional with respect to the inner product of H^{-1}; see [Mayer 2001; Taylor and Cahn 1994]. Escher, Mayer and Simonett [Escher et al. 1998] showed that solutions that start out close to spheres with respect to the $C^{2+\beta}$-topology for hypersurfaces exist globally and converge exponentially fast to a sphere. Wheeler [2012, Chapter 3] obtained the following convergence theorem.

Theorem A. *Let $f : \Sigma \to \mathbb{R}^3$ be a compact surface. There exists an absolute constant $C_1 > 0$ such that if*

$$\int_\Sigma |A^\circ|^2 \, d\mu < C_1,$$

then the surface diffusion flow with initial data f exists smoothly for all time and converges exponentially to a round sphere as $t \to \infty$.

Willmore flow is another important fourth order flow that is closely related to the surface diffusion flow. Letting M be an n-dimensional compact submanifold in the space form, the Willmore functional is

$$\mathcal{W}(f) = \int_M |A^\circ|^n \, d\mu. \tag{1-2}$$

It is invariant under the conformal (or Moebius) transformations of the ambient space. When $n = 2$, the Willmore functional is also called Willmore energy. For a compact immersed surface $f : \Sigma \to \mathbb{R}^{2+p}$, the associated Euler–Lagrange operator is

$$W(f) = \Delta H + Q(A^\circ)H, \quad Q(A^\circ)H = A^\circ(e_i, e_j)\langle A^\circ(e_i, e_j), H\rangle. \tag{1-3}$$

When $W(f) = 0$, we call it the Willmore surface. The L_2-gradient flow of $\mathcal{W}(f)$ for surfaces, briefly called Willmore flow, is a quasilinear geometric evolution equation

$$\frac{\partial f}{\partial t} = -W(f). \tag{1-4}$$

Kuwert and Schätzle [2002] gave a lower bound on the lifespan of a smooth solution, which depends only on how much the curvature of the initial surface is concentrated, and in [Kuwert and Schätzle 2001] they proved the following theorem:

Theorem B. *Let $f : \Sigma \to \mathbb{R}^{2+p}$ be a compact surface. There exists a constant $C_2(p) > 0$ such that if*

$$\int_\Sigma |A^\circ|^2 \, d\mu < C_2(p),$$

then the Willmore flow with initial data f exists smoothly for all time and converges exponentially to a round sphere as $t \to \infty$.

Kuwert and Schätzle [2004] investigated point singularities of Willmore surfaces and obtained that the Willmore flow of spheres in $\mathbb{R}^3$ with energy less than 8π exists at all times and converges to a round sphere.

For other fourth order flows, Bernard, Wheeler and Wheeler [Bernard et al. 2019] introduced the so-called Chen's flow and investigated the lifespan theorem and finite-time singularities for such a flow. Moreover, one class of fourth order flows derived from the critical point of some constructed functionals in different background manifolds. Metzger, Wheeler and Wheeler [2013] investigated the gradient flow of $\int_\Sigma |H|^2 \, d\mu$ and proved the lifespan theorem in Riemannian 3-manifold. Link [2013] also studied this flow in bounded Riemannian manifolds. Magni [2015] studied the gradient flow of $\int_\Sigma |A|^2 \, d\mu$ and obtained a smooth convergence theorem in three-dimensional Riemannian manifolds.

The local existence of these fourth order curvature flows is standard as we can see in [Escher et al. 1998], for example. In this paper, we first obtain a gap theorem for an n-dimensional properly immersed static submanifold M^n with restricted growth of the curvature at infinity in $\mathbb{F}^{n+p}(c) (c \geq 0)$ under the pinching condition for the Willmore functional, which generalizes the gap lemma in [Wheeler 2012] to any dimension and codimension.

Theorem 1.1. *Let $f : M^n \to \mathbb{F}^{n+p}(c)$ $(c \geq 0)$ be a properly immersed submanifold with $\Delta H = 0$. There exists a positive constant $\sigma(n)$ depending only on n such that if*

$$\int_M |A^\circ|^n \, d\mu < \sigma(n),$$

$$\liminf_{\rho \to \infty} \frac{1}{\rho^4} \int_{f^{-1}(B_\rho(x))} |A|^2 \, d\mu = 0 \quad \textit{for } x \in \mathbb{F}^{n+p}(c),$$

then M is totally umbilical.

Notice that the properness of the immersion implies the completeness of M. For a complete immersion, if we choose the cutoff function on intrinsic balls, the properness of the immersion isn't essential. For higher dimensions, there are few results about the convergence of fourth order flow, and the above gap theorem provides a feasible method to handle this problem. Xu and Gu [2007] proved a gap theorem for complete surfaces with parallel mean curvature in a space form under the pinching condition $\int_M |A^\circ|^4 \, d\mu < D(|H|, c)$, where $c + |H|^2/4 > 0$. Moreover, they proposed an open problem asking if there is global rigidity for surfaces with parallel mean curvature under the pinching condition for the Willmore functional. As a consequence of Theorem 1.1, we have solved this open problem for compact surfaces in $\mathbb{F}^{2+p}(c)$ $(c \geq 0)$.

For compact Willmore surfaces in a sphere, Xu and Yang [2016] proved a global rigidity theorem under the pinching condition for $\int_M |A^\circ|^4 \, d\mu$. Motivated by the convergence result in Theorem B, we obtain the following global rigidity theorem for compact Willmore surfaces in a sphere under the pinching condition for the Willmore functional.

Theorem 1.2. *Let M be a compact Willmore surface in the unit sphere $\mathbb{S}^{2+p}$. There exists an absolute positive constant σ_1 $(= 6.23 \times 10^{-8})$ such that if*

$$\int_M |A^\circ|^2 \, d\mu < \sigma_1,$$

then M is totally umbilical.

Next we obtain a prior estimate on the lifespan of the surface diffusion flow in space forms in terms of the concentration of curvature at the initial time. Our proof is standard for the analysis of Willmore flow in Euclidean spaces and later in bounded Riemannian manifolds [Link 2013; Metzger et al. 2013].

Theorem 1.3. *Let $f : \Sigma \to \mathbb{F}^{2+p}(c)$ be a compact surface. There exist constants $\epsilon_1 > 0$, $C < \infty$ depending only on p, such that if $\rho > 0$ is chosen with*

$$\int_{f^{-1}(B_\rho(x))} |A|^2 \, d\mu \leq \epsilon \leq \epsilon_1(p) \quad \text{for any } x \in \mathbb{F}^{2+p}(c),$$

then the maximal time T for the surface diffusion flow with initial data f satisfies

$$T \geq \frac{1}{C} \rho^4,$$

and one has the estimate

$$\int_{f^{-1}(B_\rho(x))} |A|^2 \, d\mu \leq C\epsilon \quad \text{for } 0 \leq t \leq \frac{1}{C} \rho^4.$$

Then we perform a blowup at an assumed singularity and construct a static surface as a limit. Putting the above results together we get the following convergence theorem.

Theorem 1.4 (main theorem). *Let $f : \Sigma \to \mathbb{F}^{2+p}(c)$ $(c \geq 0)$ be a compact surface. There exists a positive constant $\epsilon_0(p)$ such that if*

$$\int_\Sigma |A^\circ|^2 \, d\mu < \epsilon_0(p), \tag{1-5}$$

then the surface diffusion flow with initial data f exists smoothly for all time and converges exponentially to a round sphere as $t \to \infty$.

In particular, the convergence theorem for the surface diffusion flow implies the differentiable sphere theorem and the method can be also applied to the Willmore flow in a sphere. The key ingredient of the proof is to establish regularity and stability results in a space form. As we know, the equation of the diffusion flow is a fourth order equation so that tools related to the maximum principle are not available. Here we use the Sobolev inequality due to Michael and Simon [1973] and the Hölder inequality to classify the curvature terms especially for $\int_M |A^\circ|^n \, d\mu$, then we get the gap theorem for static submanifolds under the pinching condition for the Willmore functional. However, it contradicts the blowup limit at a finite time curvature singularity and thus we obtain the convergence theorem for the surface diffusion flow in the space form $\mathbb{F}^{2+p}(c)(c \geq 0)$.

2. Preparation

Let $(\mathbb{F}^{n+p}(c), \bar{g})$ be the space form with constant curvature c. For an immersion $f : M^n \to \mathbb{F}^{n+p}(c)$, the basic geometric data associated to f is the induced metric $g = f^*\bar{g}$ and $g(X, Y) = \langle Df \cdot X, \ Df \cdot Y\rangle$ with corresponding Levi-Civita connection ∇. Let $\{e_i\}$ be a set of locally defined orthogonal bases and the summation over repeated indices is used. The second fundamental form is given by $A(X, Y) = D^2_{X,Y} f = D_X(D_Y f) - Df \cdot \nabla_X Y$ with mean curvature vector given by the trace $H = A(e_i, e_i)$ and the trace free part $A^\circ(X, Y) = A(X, Y) - \frac{1}{n} g(X, Y)H$. We have the normal connection $\nabla_X \phi = (D_X \phi)^\perp$ which acts on the normal vector field ϕ along f. In (1-1) the Laplace operator $\Delta\phi = -\nabla^*\nabla\phi$ is understood with respect to the normal connection, where ∇^* denotes the formal adjoint of ∇.

When computing tensor identities we freely use vector fields with first derivative vanishing at a given point. We define the curvature by $R^\perp(X, Y)\phi = \nabla^2_{X,Y}\phi - \nabla^2_{Y,X}\phi$ and the equations of Codazzi, Gauss and Ricci are

$$
\begin{aligned}
(\nabla_X A)(Y, Z) &= (\nabla_Y A)(X, Z),\\
\nabla H &= -\nabla^* A = -\frac{n}{n-1}\nabla^* A^\circ,\\
R(X, Y, Z, W) &= c\langle X, Z\rangle\langle Y, W\rangle - c\langle X, W\rangle\langle Y, Z\rangle\\
&\qquad + A(X, Z)A(Y, W) - A(X, W)A(Y, Z),\\
R^\perp(X, Y)\phi &= A^\circ(e_i, X)\langle A^\circ(e_i, Y), \phi\rangle - A^\circ(e_i, Y)\langle A^\circ(e_i, X), \phi\rangle.
\end{aligned}
$$

The Codazzi equation implies that ∇A and $\nabla^2 A$ can be expressed by ∇A° and $\nabla^2 A^\circ$, respectively. In particular one has inequalities

$$
|\nabla A| \leq C|\nabla A^\circ|, \quad |\nabla^2 A| \leq C|\nabla^2 A^\circ|. \tag{2-1}
$$

If ϕ, ψ are normal forms, we denote by $\phi * \psi$ any normal-valued, multilinear form depending on ϕ, ψ in a universal, bilinear way. In particular, we have the properties $|\phi*\psi| \le C|\phi|\,|\psi|$ and $\nabla(\phi*\psi) = \nabla\phi*\psi + \phi*\nabla\psi$. We use the notation P_r^m for any term of the type

$$P_r^m = \sum_{i_1+\cdots+i_r=m} \nabla^{i_1}A * \nabla^{i_2}A * \cdots * \nabla^{i_r}A. \tag{2-2}$$

Now we derive a Sobolev type inequality in $\mathbb{S}^{n+p}$.

Lemma 2.1 [Michael and Simon 1973]. *Let M^n $(n \ge 2)$ be a compact submanifold with or without boundary in the Euclidean space $\mathbb{R}^{n+p}$ with $p \ge 1$. For a nonnegative function $g \in C^1(M)$ such that $g|_{\partial M} = 0$ if $\partial M \ne 0$, we have*

$$\left[\int_M g^{\frac{n}{n-1}}\,d\mu\right]^{\frac{n-1}{n}} \le D(n)\int_M (|\nabla g| + |H|g)\,d\mu,$$

where $D(n) = 4^{n+1}\sigma_n^{-1/n}$, and σ_n is the volume of the unit ball in $\mathbb{R}^n$.

Let M^n be a compact submanifold in $\mathbb{S}^{n+p}$. We consider the composition of isometric immersions $M^n \hookrightarrow \mathbb{S}^{n+p} \subset \mathbb{R}^{n+p+1}$ and denote by $\overline{H}$ the mean curvature vector of M^n as a submanifold in $\mathbb{R}^{n+p+1}$. Then $|\overline{H}|^2 = |H|^2 + n^2$. Hence we have for any nonnegative function $f \in C^1(M)$,

$$\begin{aligned}\left[\int_M g^{\frac{n}{n-1}}\,d\mu\right]^{\frac{n-1}{n}} &\le D(n)\int_M (|\nabla g| + |\overline{H}|g)\,d\mu\\ &\le D(n)\int_M (|\nabla g| + (n+|H|)g)\,d\mu.\end{aligned}$$

3. Gap theorem

In this section, using the Sobolev inequality we get some integral estimates, and obtain the gap theorem. First, we need the following lemma.

Lemma 3.1 [Kuwert and Schätzle 2002]. *Let $f : M^n \to \mathbb{F}^{n+p}(c)$ be a complete submanifold, then for any l-linear normal form ϕ we have*

$$\begin{aligned}((\nabla\nabla^* - \nabla^*\nabla)\phi)&(X_1, X_2 \ldots X_l)\\ &= -(\nabla^* T)(X_1, X_2 \ldots X_l) + (R^\perp(e_i, X_l)\phi)(e_i, X_2 \ldots X_l)\\ &\qquad - \sum_{k=1}^{l} \phi(X_i, \ldots, R(e_i, X_l)X_k, \ldots, X_l),\end{aligned}$$

where $T(X_0, X_1 \ldots X_l) = (\nabla_{X_0}\phi)(X_1, X_2 \ldots X_l) - (\nabla_{X_1}\phi)(X_0, X_2 \ldots X_l)$.

We have the following Simons' identity in space form: $\mathbb{F}^{n+p}(c)$.

$$\Delta h_{ij} = \nabla^2 H + H\cdot h_{ip}h_{pj} - h_{ij}\cdot h_{pq}h_{pq} + h_{jq}\cdot h_{ip}h_{pq} - h_{jq}\cdot h_{pq}h_{pi} + cnA^\circ + R^\perp * A.$$

Therefore, we can write

$$\Delta A^\circ = \nabla^2 H + \frac{H^2}{n}A^\circ + A^\circ * A^\circ * A + cnA^\circ - \frac{\Delta H}{n}g. \tag{3-1}$$

Here we need to estimate the terms of $A^\circ * A^\circ * A$ and cnA°, which are different from the equation of surfaces in the Euclidean space. Taking $\phi = \nabla H$ in Lemma 3.1, and using the Gauss equation, we have

$$((\nabla\nabla^* - \nabla^*\nabla)\nabla H)(e_1) = A^\circ * A^\circ * \nabla A^\circ - \nabla H(R(e_i, e_1)e_i).$$

Since

$$\begin{aligned} -\langle R(e_i, e_1)e_i, e_k\rangle \nabla_k H &= c(n-1)\nabla_1 H + H\cdot h_{1k}\nabla_k H - h_{1i}\cdot h_{ik}\nabla_k H \\ &= c(n-1)\nabla H(e_1) + \frac{n-1}{n^2}H^2\nabla H(e_1) + A * A^\circ * \nabla A^\circ, \end{aligned}$$

we have

$$\nabla^*(\nabla^2 H) + c(n-1)\nabla H = \nabla(\nabla^*\nabla H) - \frac{n-1}{n^2}H^2\nabla H + A * A^\circ * \nabla A^\circ. \tag{3-2}$$

Taking $\phi = \nabla A^\circ$ in Lemma 3.1, we have

$$\nabla^*(\nabla^2 A^\circ) = \nabla(\nabla^*\nabla A^\circ) + A * A * \nabla A^\circ + cC\nabla A^\circ. \tag{3-3}$$

Replacing ϕ by $\nabla\phi$, we have

$$\Delta(\nabla\phi) - \nabla(\Delta\phi) = A * A * \nabla\phi + A * \nabla A * \phi + cC\nabla\phi. \tag{3-4}$$

Since the last terms of (3-2)–(3-4) are new compared with the equations in the Euclidean space, the next lemma is important in our proof.

Lemma 3.2. *If $f : M^n \to \mathbb{F}^{n+p}(c)$ is a complete submanifold with $\Delta H = F$ and $r \in C^1_c(M^n)$ satisfies $|\nabla r| \le \Lambda$, then*

$$\begin{aligned} &\int |\nabla A^\circ|^2 r^2\,d\mu + c\int |A^\circ|^2 r^2\,d\mu \\ &\qquad \le \frac{C}{\Lambda^2}\int |F|^2 r^4\,d\mu + \frac{C}{\Lambda^2}\int |A^\circ|^6 r^4\,d\mu + C\Lambda^2\int_{[r>0]} |A|^2\,d\mu, \end{aligned} \tag{3-5}$$

$$\begin{aligned} &\int |\nabla A^\circ|^2 r^4\,d\mu + c\int |A^\circ|^2 r^4\,d\mu \\ &\qquad \le C\int |\nabla H|^2 r^4\,d\mu + C\int |A^\circ|^4|A|^2 r^4\,d\mu + C\Lambda^4\int_{[r>0]} |A|^2\,d\mu. \end{aligned} \tag{3-6}$$

Proof. Multiplying (3-1) by $r^2 A^\circ$, we have

$$\begin{aligned}\int\Big(|\nabla A^\circ|^2+cn|A^\circ|^2+\frac{1}{n}H^2|A^\circ|^2\Big)r^2\,d\mu\\ =\frac{n-1}{n}\int|\nabla H|^2r^2\,d\mu+C\int|A^\circ|^3|A|r^2\,d\mu+\int A^\circ*\nabla A^\circ*r*\nabla r\,d\mu\\ \le-\frac{n-1}{n}\int\langle H,\Delta H\rangle r^2\,d\mu+\frac{C}{\Lambda^2}\int|A^\circ|^6r^4\,d\mu+\Lambda^2\int_{[r>0]}|A|^2\,d\mu+\tfrac{1}{2}\int|\nabla A^\circ|^2r^2\,d\mu\\ \le\frac{C}{\Lambda^2}\int|F|^2r^4\,d\mu+\frac{C}{\Lambda^2}\int|A^\circ|^6r^4\,d\mu+\Lambda^2\int_{[r>0]}|A|^2\,d\mu+\tfrac{1}{2}\int|\nabla A^\circ|^2r^2\,d\mu.\end{aligned}$$

Multiplying (3-1) by $r^4 A^\circ$, we get another inequality. □

Lemma 3.3. *Under the assumption of Lemma 3.2 we have for $\eta = r^4$,*

$$\begin{aligned}\int(|\nabla^2H|^2+|A|^2|\nabla A|^2+|A|^4|A^\circ|^2)\eta\,d\mu+c\int|\nabla H|^2\eta\,d\mu\\ \le C\int(|A^\circ|^2|\nabla A^\circ|^2+|A^\circ|^6)\eta\,d\mu\\ +C\int|F|^2\eta\,d\mu+\Lambda^4\int_{[r>0]}|A|^2\,d\mu.\end{aligned}$$

Proof. Multiplying (3-2) by $\nabla H\eta$, we have

$$\begin{aligned}\int(|\nabla^2H|^2+c(n-1)|\nabla H|^2)\eta\,d\mu\\ \le\int\Big(|\Delta H|^2-\frac{n-1}{n^2}H^2|\nabla H|^2\Big)\eta\,d\mu\\ +C\int|A|\,|A^\circ|\,|\nabla A^\circ|^2\eta\,d\mu+\int r^3*\nabla r*\nabla H\nabla^2H\,d\mu,\end{aligned}\tag{3-7}$$

and the proof is similar to Lemma 2.3 in [Kuwert and Schätzle 2002]. □

Proposition 3.4. *Let $f : M^n \to \mathbb{F}^{n+p}(c)$ $(c \ge 0)$ be a complete submanifold with $F = \Delta H$ and $r \in C_c^1(M)$ satisfying $s \ge 4$, and let $\Lambda = \|\nabla r\|_\infty$, then*

$$\begin{aligned}\int(|\nabla^2A|^2+|A|^2|\nabla A|^2+|A|^4|A^\circ|^2)\,r^s\,d\mu+c\int(|A^\circ|^2+|\nabla H|^2)r^s\,d\mu\\ \le C\int|F|^2r^s\,d\mu+C\Lambda^4\int_{[r>0]}|A|^2\,d\mu\\ +C\int(|A^\circ|^2|\nabla A^\circ|^2+|A^\circ|^6)r^s\,d\mu.\end{aligned}$$

Proof. Multiplying (3-3) by $\nabla A^\circ \eta,\ \ \eta = r^4$, using (3-1) we have

$$\begin{aligned}\int |\nabla^2 A^\circ|^2 \eta\, d\mu &= \int (|\Delta A^\circ|^2 + |A|^2 |\nabla A^\circ|^2 + cC|\nabla A^\circ|^2 + r^{-1} * \nabla r * \nabla^2 A^\circ * \nabla A^\circ)\eta\, d\mu \\ &\le \int (|\nabla^2 H|^2 + |A|^4 |A^\circ|^2 + |A|^2 |\nabla A^\circ|^2 + |F|^2)\eta\, d\mu \\ &\qquad + \tfrac{1}{2}\int |\nabla^2 A^\circ|^2 \eta\, d\mu + \Lambda^2 \int |\nabla A^\circ|^2 r^2\, d\mu + cC \int (|\nabla A^\circ|^2 + |A^\circ|^2)\eta\, d\mu.\end{aligned}$$

Using Lemma 3.2, we have

$$\begin{aligned}\int |\nabla^2 A^\circ|^2 \eta\, d\mu + c\int |A^\circ|^2 \eta\, d\mu &\le \int (|\nabla^2 H|^2 + |A|^2 |\nabla A^\circ|^2 + |A|^4 |A^\circ|^2 + c|\nabla H|^2)\eta\, d\mu \\ &\qquad + \int |F|^2 \eta\, d\mu + \int |A^\circ|^6 \eta\, d\mu + \Lambda^4 \int_{[r>0]} |A|^2\, d\mu.\end{aligned}$$

The assertion follows from Lemma 3.3. □

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. It follows from Proposition 3.4 that

$$\begin{aligned}\int (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^\circ|^2)\, r^s\, d\mu + c\int (|A^\circ|^2 + |\nabla H|^2) r^s\, d\mu \\ \le C\int |F|^2 r^s\, d\mu + C\Lambda^4 \int_{[r>0]} |A|^2\, d\mu + C\int (|A^\circ|^2 |\nabla A^\circ|^2 + |A^\circ|^6) r^s\, d\mu.\end{aligned}$$

Then we have

$$\int |A^\circ|^2 |\nabla A^\circ|^2 r^s\, d\mu \le C\int |A^\circ|^6 r^s\, d\mu + C\int |\nabla A^\circ|^3 r^s\, d\mu,$$

and

$$\begin{aligned}\int |\nabla A^\circ|^3 r^s\, d\mu &\le \int |A^\circ|\, |\nabla^2 A^\circ|\, |\nabla A^\circ| r^s\, d\mu + \Lambda \int |A^\circ|\, |\nabla A^\circ|^2 r^{s-1}\, d\mu \\ &\le \int (\delta |\nabla^2 A^\circ|^2 + C_\delta |A^\circ|^2 |\nabla A^\circ|^2) r^s\, d\mu \\ &\qquad + \Lambda \int |A^\circ|^{\frac{1}{2}} |\nabla A^\circ|^{\frac{1}{2}} |A^\circ|^{\frac{1}{2}} |\nabla A^\circ|^{\frac{3}{2}} r^{s-1}\, d\mu \\ &\le \int (\delta |\nabla^2 A^\circ|^2 + C_\delta |A^\circ|^6 + \delta |\nabla A^\circ|^3) r^s\, d\mu \\ &\qquad + \Lambda^{\frac{4}{3}} \int |A^\circ|^{\frac{2}{3}} |\nabla A^\circ|^2 r^{s-\frac{4}{3}}\, d\mu \\ &\le \delta \int |\nabla^2 A^\circ|^2 r^s\, d\mu + C_\delta \int |A^\circ|^6 r^s\, d\mu + \Lambda^4 \int_{[r>0]} |A|^2\, d\mu.\end{aligned} \tag{3-8}$$

Therefore,

$$\int |A^\circ|^2|\nabla A^\circ|^2 r^s\,d\mu \le \delta \int |\nabla^2 A^\circ|^2 r^s\,d\mu + C_\delta \int |A^\circ|^6 r^s\,d\mu + \Lambda^4 \int_{[r>0]} |A|^2\,d\mu.$$

When $c = 0$, the Sobolev inequality is

$$\left[\int g^{\frac{n}{n-1}}\,d\mu\right]^{\frac{n-1}{n}} \le C(n)\int (|\nabla g| + |H|g)\,d\mu.$$

If we take $g = |A^\circ|^{6(n-1)/n} r^{(n-1)/(n)s}$, then

$$\text{(3-9)}\quad \left(\int |A^\circ|^6 r^s\,d\mu\right)^{\frac{n-1}{n}}$$
$$= \left[\int (|A^\circ|^{\frac{6(n-1)}{n}} r^{\frac{n-1}{n}s})^{\frac{n}{n-1}}\,d\mu\right]^{\frac{n-1}{n}}$$
$$\le C(n)\int \left[\nabla(|A^\circ|^{\frac{6(n-1)}{n}} r^{\frac{n-1}{n}s}) + |H|\,|A^\circ|^{\frac{6(n-1)}{n}} r^{\frac{n-1}{n}s}\right] d\mu$$
$$\le \int |\nabla A^\circ|\,|A^\circ|^{\frac{5n-6}{n}} r^{\frac{n-1}{n}s}\,d\mu + \Lambda \int |A^\circ|^{\frac{6(n-1)}{n}} r^{\frac{n-1}{n}s-1}\,d\mu$$
$$+ \int |A|\,|A^\circ|^{\frac{6(n-1)}{n}} r^{\frac{n-1}{n}s}\,d\mu = a + b + c.$$

Here

$$a = \int |\nabla A^\circ|\,|A^\circ|^{\frac{n}{3(n-1)}} r^{\frac{1}{3}s} |A^\circ|^{\frac{(2n-3)(7n-6)}{3n(n-1)}} r^{\frac{2n-3}{3n}s}\,d\mu$$
$$\le \int (|\nabla A^\circ|^{\frac{3(n-1)}{n}} |A^\circ| + |A^\circ|^{\frac{7n-6}{n}}) r^{\frac{n-1}{n}s}\,d\mu$$
$$\le \left(\int_{[r>0]} |A^\circ|^n\,d\mu\right)^{\frac{1}{n}} \left[\left(\int |\nabla A^\circ|^3 r^s\,d\mu\right)^{\frac{n-1}{n}} + \left(\int |A^\circ|^6 r^s\,d\mu\right)^{\frac{n-1}{n}}\right],$$

$$b = \Lambda \int |A|^\circ |A^\circ|^{\frac{5n-6}{n}} r^{\frac{n-1}{n}s-1}\,d\mu$$
$$\le \left(\int_{[r>0]} |A^\circ|^n\,d\mu\right)^{\frac{1}{n}} \left(\Lambda^{\frac{n}{n-1}} \int |A^\circ|^{\frac{5n-6}{n-1}} r^{s-\frac{n}{n-1}}\,d\mu\right)^{\frac{n-1}{n}}$$
$$= \left(\int_{[r>0]} |A^\circ|^n\,d\mu\right)^{\frac{1}{n}} \left(\Lambda^{\frac{n}{n-1}} \int |A^\circ|^{\frac{n}{2(n-1)}} |A^\circ|^{\frac{3(3n-4)}{2(n-1)}} r^{s-\frac{n}{n-1}}\,d\mu\right)^{\frac{n-1}{n}}$$
$$\le \left(\int_{[r>0]} |A^\circ|^n\,d\mu\right)^{\frac{1}{n}} \left[\Lambda^4\left(\int_{[r>0]} |A^\circ|^2\,d\mu\right) + \left(\int |A^\circ|^6 r^s\,d\mu\right)\right]^{\frac{n-1}{n}},$$

$$\begin{aligned} c &= \int |A|\,|A^\circ|^{\frac{6(n-1)}{n}} r^{\frac{n-1}{n}s}\,d\mu \\ &\le \Big(\int_{[r>0]} |A^\circ|^n\,d\mu\Big)^{\frac{1}{n}} \Big[\int (|A|\,|A^\circ|^{\frac{5n-6}{n}})^{\frac{n}{n-1}} r^s\,d\mu\Big]^{\frac{n-1}{n}} \\ &\le \Big(\int_{[r>0]} |A^\circ|^n\,d\mu\Big)^{\frac{1}{n}} \Big(\int |A|^2|A^\circ|^4 r^s\,d\mu\Big)^{\frac{n-1}{n}}. \end{aligned}$$

When $c=1$, the Sobolev inequality is

$$\Big[\int g^{\frac{n}{n-1}}\,d\mu\Big]^{\frac{n-1}{n}} \le C(n)\int (|\nabla g|+|H|g+g)\,d\mu.$$

If we take $g=|A^\circ|^{6(n-1)/n} r^{(n-1)/(n)s}$, then the third term in the Sobolev inequality is

$$\begin{aligned} \int |A^\circ|^{\frac{6(n-1)}{n}} r^{\frac{n-1}{n}s}\,d\mu &= \int |A|^\circ |A^\circ|^{\frac{5n-6}{n}} r^{\frac{n-1}{n}s}\,d\mu \\ &\le \Big(\int_{[r>0]} |A^\circ|^n\,d\mu\Big)^{\frac{1}{n}} \Big(\int |A^\circ|^{\frac{5n-6}{n-1}} r^s\,d\mu\Big)^{\frac{n-1}{n}} \\ &\le \Big(\int_{[r>0]} |A^\circ|^n\,d\mu\Big)^{\frac{1}{n}} \Big[\Big(\int |A^\circ|^2 r^s\,d\mu\Big)+\Big(\int |A^\circ|^6 r^s\,d\mu\Big)\Big]^{\frac{n-1}{n}}. \end{aligned}$$

Combining the above estimates, we have

$$\begin{aligned} \text{(3-10)}\quad & \int (|A^\circ|^2|\nabla A^\circ|^2+|A^\circ|^6) r^s\,d\mu \\ &\le \Lambda^4(1+\|A^\circ\|_{n,[r>0]}^{\frac{n}{n-1}}) \int_{[r>0]} |A|^2\,d\mu + \delta\int |\nabla^2 A^\circ|^2 r^s\,d\mu \\ &\quad + \|A^\circ\|_{n,[r>0]}^{\frac{n}{n-1}} \int_M (|\nabla^2 A^\circ|^2+|A^\circ|^6+|A|^2|A^\circ|^4+c|A^\circ|^2) r^s\,d\mu. \end{aligned}$$

Now take the cutoff function $r(q)=\varphi(\gamma_x(f(q))/\rho)$, where γ_x is the distance function with respect to the fixed point $x\in\mathbb{F}^{2+p}$, $\varphi\in C^1(\mathbb{R})$ and $\chi_{B_{1/2}(x)}\le\varphi\le\chi_{B_1}(x)$; then $\Lambda=\frac{C}{\rho}$. From Proposition 3.4, $\|A^\circ\|_{n,[r>0]}^n<\sigma(n)$, and $s\ge 4$, we have

$$\begin{aligned} \text{(3-11)}\quad & \int (|\nabla^2 A|^2+|A|^2|\nabla A|^2+|A|^4|A^\circ|^2)\, r^s\,d\mu + c\int |A^\circ|^2 r^s\,d\mu \\ &\le \int |F|^2 r^s\,d\mu + \Big(\frac{C}{\rho}\Big)^4 \int_{[r>0]} |A|^2\,d\mu. \end{aligned}$$

Since f is a proper immersion, taking $\rho\to\infty$ and using a similar argument as in Theorem 8 of [Wheeler 2012], we finish the proof. ☐

A surface in the sphere is a Willmore surface if and only if it satisfies

$$W(f) = \Delta H + Q(A^\circ)H = 0,$$

or else we get (4-1). The estimate of $W(f)$ is easy to handle as in [Kuwert and Schätzle 2002], so we also have the gap theorem for Willmore surfaces in a sphere.

Proof of Theorem 1.2. For $n = 2$ and $c = 1$, the gradient of the second fundamental form can be controlled (see [Andrews and Baker 2010]) by

$$\tfrac{1}{4}|\nabla H|^2 \le \tfrac{1}{3}|\nabla A|^2 \le |\nabla A^\circ|^2. \tag{3-12}$$

The precise forms of (3-1), (3-2) and (3-3) are

$$\nabla^*\nabla A^\circ + 2KA^\circ = -\nabla^2 H - R^\perp(e_k,\cdot)A(e_k,\cdot) + \frac{\Delta H}{2} g(\cdot,\cdot), \tag{3-13}$$

$$\nabla^*(\nabla^2 H) + K\nabla H = \nabla(\nabla^*\nabla H) - R^\perp(e_k,\cdot)\nabla H(e_k), \tag{3-14}$$

$$\begin{aligned}\langle \nabla^*(\nabla^2 A^\circ), \nabla A^\circ\rangle \le \langle \nabla(\nabla^*\nabla A^\circ), \nabla A^\circ\rangle - 5|\nabla A^\circ|^2 \\ + 2|\nabla H|^2 + 12|A|^2|\nabla A|^2,\end{aligned} \tag{3-15}$$

where the section curvature $K = 1 + |H|^2/4 - |A^\circ|^2/2$.

Multiplying (3-13) by $r^2 A^\circ$ with $r \in C_c^1(M^2)$ satisfies $\|\nabla r\|_\infty = \Lambda$. We have

$$\begin{aligned}
&\int (|\nabla A^\circ|^2 + 2|A^\circ|^2 + \tfrac{1}{2}H^2|A^\circ|^2) r^2\, d\mu \\
&\quad \le \tfrac{1}{2}\int |\nabla H|^2 r^2\, d\mu + 2\int |A^\circ|^4 r^2\, d\mu + 6\Lambda \int |A^\circ|\,|\nabla A^\circ| r\, d\mu \\
&\quad \le -\tfrac{1}{2}\int \langle H, W\rangle r^2\, d\mu + \tfrac{1}{2}\int \langle H, Q(A^\circ)H\rangle r^2\, d\mu \\
&\qquad\qquad + \frac{1}{128\Lambda^2}\int |A^\circ|^6 r^4\, d\mu + C\Lambda^2 \int_{[r>0]} |A|^2\, d\mu + \tfrac{1}{2}\int |\nabla A^\circ|^2 r^2\, d\mu \\
&\quad \le \frac{C}{\Lambda^2}\int |W|^2 r^4\, d\mu + \tfrac{1}{2}\int H^2|A^\circ|^2 r^2\, d\mu \\
&\qquad\qquad + \frac{1}{128\Lambda^2}\int |A^\circ|^6 r^4\, d\mu + C\Lambda^2 \int_{[r>0]} |A|^2\, d\mu + \tfrac{1}{2}\int |\nabla A^\circ|^2 r^2\, d\mu,
\end{aligned}$$

and

$$\begin{aligned}&\int (|\nabla A^\circ|^2 + 4|A^\circ|^2) r^2\, d\mu \\ &\qquad \le \frac{C}{\Lambda^2}\int |W|^2 r^4\, d\mu + \frac{1}{64\Lambda^2}\int |A^\circ|^6 r^4\, d\mu + C\Lambda^2 \int_{[r>0]} |A|^2\, d\mu.\end{aligned} \tag{3-16}$$

Multiplying (3-13) by $r^4 A^\circ$, we have

$$\int (|\nabla A^\circ|^2 + |A^\circ|^2) r^4 \, d\mu \le 2 \int |\nabla H|^2 r^4 \, d\mu + 2 \int |A^\circ|^6 r^4 \, d\mu + C\Lambda^4 \int_{[r>0]} |A|^2 \, d\mu. \tag{3-17}$$

Similarly, from the proof of Proposition 3.4 we have

$$\int \left(\tfrac{3}{4}|\nabla^2 A|^2 + |A^\circ|^2 + 20|\nabla A^\circ|^2 + 40|A|^2|\nabla A|^2 + |A|^4|A^\circ|^2\right) r^4 \, d\mu \le \int (3546|A^\circ|^2|\nabla A^\circ|^2 + 5230|A^\circ|^6) r^4 \, d\mu + C \int |W|^2 r^4 \, d\mu + C\Lambda^4 \int_{[r>0]} |A|^2 \, d\mu. \tag{3-18}$$

Taking $g = |A^\circ|^3 r^2$ in the Sobolev inequality, we get

$$\begin{aligned} \frac{1}{D^2(2)} \int |A^\circ|^6 r^4 \, d\mu &\le \left[\int (3|A^\circ|^2|\nabla A^\circ| r^2 + 2\Lambda |A^\circ|^3 r + |H|\,|A^\circ|^3 r^2 + 2|A^\circ|^3 r^2) \, d\mu \right]^2 \\ &\le 5 \int_{[r>0]} |A^\circ|^2 \, d\mu \int (9|A^\circ|^2|\nabla A^\circ|^2 + 2|A^\circ|^6 + 2|A|^2|A^\circ|^4 + |A^\circ|^2) r^4 \, d\mu \\ &\quad + C\Lambda^4 \left(\int_{[r>0]} |A^\circ|^2 \, d\mu \right)^2. \end{aligned} \tag{3-19}$$

Thus

$$\left(\frac{1}{5D^2(2)} - 2\sigma_1 \right) \int |A^\circ|^6 r^4 \, d\mu \le \int_{[r>0]} |A^\circ|^2 \, d\mu \int (9|A^\circ|^2|\nabla A^\circ|^2 + 2|A|^2|A^\circ|^4 + |A^\circ|^2) r^4 \, d\mu + C\Lambda^4 \left(\int_{[r>0]} |A^\circ|^2 \, d\mu \right)^2. \tag{3-20}$$

Taking $g = |A^\circ||\nabla A^\circ| r^2$ in the Sobolev inequality, we also obtain

$$\frac{1}{5D^2(2)} \int |A^\circ|^2|\nabla A^\circ|^2 r^4 \, d\mu \le \int_{[r>0]} |A^\circ|^2 \, d\mu \int \left(\tfrac{5}{2}|\nabla^2 A^\circ|^2 + 2|A|^2|\nabla A^\circ|^2 + 4|\nabla A^\circ|^2\right) r^4 \, d\mu + C\Lambda^4 \left(\int_{[r>0]} |A^\circ|^2 \, d\mu \right)^2. \tag{3-21}$$

It follows from (3-18), (3-20) and (3-21) and the assumption $\|A^\circ\|_2^2 < 6.23\times 10^{-8}$ that

$$\int (|\nabla^2 A|^2+|A|^2|\nabla A|^2+|A|^4|A^\circ|^2)\, r^s\, d\mu+\int |A^\circ|^2 r^s\, d\mu \le C\int |W|^2 r^s\, d\mu + C\Lambda^4\int_{[r>0]}|A|^2\, d\mu.$$

We take the cutoff function $r(q)=\varphi(\gamma_x(f(q))/(\rho))$, where γ_x is a distance function with respect to the fixed point $x\in\mathbb{S}^{2+p}$, $\varphi\in C^1(\mathbb{R})$ and $\chi_{B_{1/2}(x)}\le\varphi\le\chi_{B_1}(x)$; then $\Lambda=C/\rho$. Taking $\rho\to\infty$, we finish the proof. □

4. Lifespan theorem

In this section, we give a lower bound on the lifespan of a smooth solution of the surface diffusion flow, which depends only on how much the curvature of the initial surface is concentrated. First, we recall these equations from [Kuwert and Schätzle 2002], and we can see the proof in [Link 2013].

Lemma 4.1. *Let $f: M^n\times[0,T)\to\mathbb{F}^{n+p}(c)$ be a smooth variation with normal velocity $\partial_t f=-V$, then*

$$\begin{aligned}
\partial_t g(X\ Y) &= 2\langle A(X\ Y), V\rangle,\\
\partial_t(d\mu) &= \langle H, V\rangle\, d\mu,\\
\partial_t^\perp(\nabla_X\phi) &= \nabla_X\partial_t^\perp\phi - A(X\ e_i)\langle\nabla_{e_i}V,\ \phi\rangle-\nabla_{e_i}V\langle A(X\ e_i),\phi\rangle,\\
\partial_t^\perp(\nabla_X Y) &= \langle(\nabla_{e_i}A)(X,\ Y),\ V\rangle e_i-\langle A(X\ Y),\nabla_{e_i}V\rangle e_i\\
&\qquad +\langle A(X\ e_i),\nabla_Y V\rangle e_i+\langle A(Y\ e_i),\nabla_X V\rangle e_i,\\
\partial_t^\perp A(X\ Y) &= -\nabla^2_{X,Y}V + A(e_i,X)\langle A(e_i,Y),V\rangle - cg(X\ Y)V,\\
\partial_t^\perp H &= -\Delta V - A(e_i,e_j)\langle A(e_i,e_j),V\rangle - ncV.
\end{aligned}$$

Using Gauss equation $2R_{1212}=2c+H^2-|A|^2$ and Lemma 4.1, the first variation formula for the Willmore functional in space forms in the normal direction -ϕ is

$$\begin{aligned}
\text{(4-1)}\quad &\frac{d}{d\epsilon}\mathcal{W}(f-\epsilon\phi)|_{\epsilon=0}\\
&=\frac{d}{d\epsilon}\int_\Sigma |A^\circ|^2\, d\mu|_{\epsilon=0}=\frac{d}{d\epsilon}\int_\Sigma\Big(\frac{|H|^2}{2}+2c\Big)\, d\mu|_{\epsilon=0}\\
&=\int_\Sigma\langle H,-\Delta\phi-A(e_i,e_j)\langle A(e_i,e_j),\phi\rangle-2c\phi+2c\phi\rangle\, d\mu+\tfrac12\int_\Sigma\langle H^3,\phi\rangle\, d\mu\\
&=-\int_\Sigma\langle\phi,\Delta H+Q(A^\circ)H\rangle\, d\mu,
\end{aligned}$$

which proves $\mathrm{grad}_{L^2}\mathcal{W}(f)=\Delta H+Q(A^\circ)H$.

The following lemmas will be needed for computing the evolution of derivatives of the curvature.

Lemma 4.2. *Let ϕ be an $(l-1)$-form with normal values along a variation $f : M^n \times I \to \mathbb{F}^{n+p}(c)$ with normal velocity $\partial_t f = -V$. If $\partial_t^\perp \phi + \Delta^2 \phi = Y$, then $\psi = \nabla\phi$ satisfies an equation*

$$\partial_t^\perp \psi + \Delta^2 \psi = \nabla Y + \sum_{i+j+k=3} \nabla^i A * \nabla^j A * \nabla^k \phi + A * \nabla V * \phi + \nabla A * V * \phi + C(\Delta(\nabla\phi) + \nabla(\Delta\phi)). \tag{4-2}$$

Proof. Let $X_1, \dots, X_l$ be independent of t and such that $\nabla X_k = 0$ at a given point and a given time. Using Lemma 4.1 we have

$$\begin{aligned}(\partial_t^\perp \psi)(X_1, \dots, X_l) &= \partial_t^\perp((\nabla_{X_1}\phi)(X_2, \dots, X_l)) \\ &= \partial_t^\perp(\nabla_{X_1}\phi(X_2, \dots, X_l)) - \partial_t^\perp \sum_{k=2}^{l} \phi(X_2, \dots, \nabla_{X_1} X_k, \dots, X_l) \\ &= (\nabla(\partial_t^\perp \phi))(X_1, \dots, X_l) + A * \nabla V * \phi + \nabla A * V * \phi.\end{aligned}$$

So,

$$\partial_t^\perp \psi + \Delta^2 \psi - \nabla Y = \Delta^2(\nabla\phi) - \nabla(\Delta^2\phi) + A * \nabla V * \phi + \nabla A * V * \phi. \tag{4-3}$$

Using (3-4), we have

$$\begin{aligned}\Delta^2(\nabla\phi) - \nabla(\Delta^2\phi) &= \Delta(\Delta(\nabla\phi) - \nabla(\Delta\phi)) + \Delta(\nabla(\Delta\phi)) - \nabla(\Delta(\Delta\phi)) \\ &= \Delta(A * A * \nabla\phi + A * \nabla A * \phi + C\nabla\phi) + A * A * \nabla(\Delta\phi) \\ &\qquad + A * \nabla A * (\Delta\phi) + C\nabla(\Delta\phi) \\ &= \sum_{i+j+k=3} \nabla^i A * \nabla^j A * \nabla^k \phi + C(\Delta(\nabla\phi) + \nabla(\Delta\phi)).\end{aligned}$$ □

Lemma 4.3. *Let $f : \Sigma \times [0, T) \to \mathbb{F}^{2+p}(c)$ be a compact surface diffusion flow, then*

$$\partial_t^\perp(\nabla^m A) + \Delta^2(\nabla^m A) = P_3^{m+2}(A) + P_1^{m+2}(A),$$

for any $m \in \mathbb{N}_0$.

Proof. We proceed by induction on m, starting with $m = 0$,

$$\begin{aligned}&\Delta(\nabla^2 H) - \nabla^2(\Delta H) \\ &\quad = (\nabla\nabla^* - \nabla^*\nabla)\nabla(\nabla H) + \nabla(\nabla\nabla^* - \nabla^*\nabla)\nabla H \\ &\quad = A * A * \nabla^2 H + A * \nabla A * \nabla H + \nabla(A * A * \nabla H + A * \nabla A * \nabla H) + C\nabla^2 H,\end{aligned}$$

and

$$\partial_t^{\perp} A = -\nabla^2(\Delta H) + A * A * \Delta H - c\Delta H g = -\Delta(\nabla^2 H) + P_3^2(A) + P_1^2(A) \\ = -\Delta^2(A) + P_3^2(A) + P_1^2(A). \tag{4-4}$$

Now let $m \geq 1$ and conclude from (4-2), that we have

$$\partial_t^{\perp}(\nabla^m A) + \Delta^2(\nabla^m A) \\ = \nabla(P_3^{m+1}(A) + P_1^{m+1}(A)) + \sum_{i+j+k=3} \nabla^i A * \nabla^j A * \nabla^k(\nabla^{m-1} A) \\ + A * \nabla(\Delta H) * \nabla^{m-1} A + \nabla A * \Delta H * \nabla^{m-1} A + \nabla^3(\nabla^{m-1} A),$$

which yields the result. □

In the following we assume $r = \tilde{r} \circ f$, where $\tilde{r} \in C_c^1(B_\rho)$ and $\|\tilde{r}\|_{C^2} \leq C < \infty$. This implies

$$|\nabla r| \leq C, \quad |\nabla^2 r| \leq C(1 + |A|), \tag{4-5}$$

and for a ball $B_\rho = B_\rho(x_0) \subset \mathbb{F}^{2+p}(c)$ we use the notion

$$\Sigma_\rho(x_0) = f^{-1}(B_\rho(x_0)). \tag{4-6}$$

We denote by $\epsilon_1(p)$ the upper bound of the concentration of curvature for the initial surface, which is a positive constant depending only on the codimension p.

The next lemma can be proved much as in [Kuwert and Schätzle 2002].

Lemma 4.4. *Let $f : \Sigma \times [0, T) \to \mathbb{F}^{2+p}(c)$ be a compact surface diffusion flow, then for $\phi = \nabla^m A$ with $m \in \mathbb{N}$ and $s \geq 2m + 4$, we have*

$$\frac{d}{dt}\int |\phi|^2 r^s \, d\mu + \tfrac{3}{4}\int |\nabla^2 \phi|^2 r^s \, d\mu \\ \leq \int (P_3^{m+2}(A) + P_5^m(A) + P_1^{m+2}(A)) * \phi \, r^s \, d\mu + C\int |A|^2 r^{s-4-2m} \, d\mu.$$

Remark 4.5. Suppose $\chi_{B_{\rho(x)}} \leq \tilde{r} \leq \chi_{B_{2\rho(x)}}$ and $\|D^j \tilde{r}\|_{L^\infty} \leq C'\rho^{-j}$ for $j = 1, 2$. Then we have $C = C'/(\rho^{4+2m})$ in Lemma 4.4.

Proposition 4.6. *Let $f : \Sigma \times [0, T) \to \mathbb{F}^{2+p}(c)$ be a compact surface diffusion flow, then for $\phi = \nabla^m A$ with $m \in \mathbb{N}$ and $s \geq 2m + 4$, we have*

$$\frac{d}{dt}\int |\phi|^2 r^s \, d\mu + \tfrac{1}{2}\int |\nabla^2 \phi|^2 r^s \, d\mu \\ \leq C(\|A\|_{\infty\,[r>0]}^4 + 1)\|A\|_{2\,[r>0]}^2 + C\|A\|_{\infty\,[r>0]}^4 \int |\phi|^2 r^s \, d\mu. \tag{4-7}$$

Proof. According to Lemma 4.4, we only need to show

$$\int (P_3^{m+2}(A)+P_5^m(A)+P_1^{m+2}(A))*\phi\, r^s\, d\mu+C\int_{[r>0]}|A|^2\,d\mu$$
$$\le \tfrac{1}{4}\int|\nabla^2\phi|^2r^s\,d\mu+C(\|A\|^4_{\infty\ [r>0]}+1)\|A\|^2_{2\ [r>0]}+C\|A\|^4_{\infty\ [r>0]}\int|\phi|^2\, r^s\,d\mu,$$

and the proof is similar to Proposition 4.5 in [Kuwert and Schätzle 2002] with

$$\int P_1^{m+2}(A)*\phi r^s\,d\mu\le\tau\int|\nabla^2\phi|^2r^s\,d\mu+C(\tau)\|A\|^2_{2\ [r>0]}. \tag{4-8}$$

It follows from Corollaries 5.3 and 5.5 in [Kuwert and Schätzle 2002]. □

Since the Sobolev inequality is similar when in $\mathbb{F}^{2+p}(c)$ for $c\le 0$, in the next lemma we just check the situation $c>0$ (see [Kuwert and Schätzle 2002]).

Lemma 4.7. *For any normal l-form ϕ on Σ and r as in (4-5),*

$$\|\phi\|^4_{\infty\ [r=1]}\le C\|\phi\|^2_{2\ [r>0]}(\|\nabla^2\phi\|^2_{2\ [r>0]}+\|\phi^2|A|^4\|_{1\ [r>0]}+\|\phi\|^2_{2\ [r>0]}). \tag{4-9}$$

Moreover, if $\phi=A$ and $\|A\|^2_{2\ [r>0]}\le\epsilon_1(p)$ for some ϵ_1 small enough depending on the constants in (4-5), then

$$\|A\|^4_{\infty\ [r=1]}\le C\|A\|^2_{2\ [r>0]}(\|\nabla^2 A\|^2_{2\ [r>0]}+\|A\|^2_{2\ [r>0]}). \tag{4-10}$$

Proposition 4.8. *Let $f:\Sigma\times[0,T]\to\mathbb{F}^{2+p}(c)$ be a compact surface diffusion flow, then there exists an $\epsilon_1(p)>0$ such that if*

$$\epsilon=\sup_{[0,T]}\int_{[r>0]}|A|^2\,d\mu\le\epsilon_1,$$

then for any $t\in[0,T]$ we have

$$\int_{[r=1]}|A|^2\,d\mu+\frac{1}{4}\int_0^t\int_{[r=1]}(|\nabla^2A|^2+|A|^2|\nabla A|^2+|A|^6)\,d\mu\,dt \le\int_{[r_0>0]}|A_0|^2\,d\mu_0+C\epsilon t. \tag{4-11}$$

Proof. We know from (3-10) that

$$\int(|A|^2|\nabla A|^2+|A|^6)r^4\,d\mu$$
$$\le C\int_{[r>0]}|A|^2\,d\mu+\tfrac{1}{4}\int_M|\nabla^2A|^2r^4\,d\mu$$
$$+\|A\|^2_{2,[r>0]}\int_M(|\nabla^2A|^2+|A|^2|\nabla A|^2+|A|^6+|A|^2)r^4\,d\mu.$$

Choosing $m = 0$ in Lemma 4.4, we get

$$\frac{d}{dt}\int |A|^2 r^4\,d\mu + \tfrac{3}{4}\int |\nabla^2 A|^2 r^4\,d\mu \le \int (P_3^2(A) + P_5^0(A) + P_1^2(A)) * A\, r^4\,d\mu + C\int_{[r>0]} |A|^2\,d\mu.$$

Combining these two inequalities, we have

$$\begin{aligned}\frac{d}{dt}\int |A|^2 r^4\,d\mu &+ \tfrac{3}{4}\int (|\nabla^2 A|^2 + |A|^2|\nabla A|^2 + |A|^6) r^4\,d\mu \\ &\le C\int (|A|^3|\nabla^2 A| + |A|^2|\nabla A|^2 + |A|^6 + |\nabla^2 A|\,|A|) r^4 + C\int_{[r>0]} |A|^2\,d\mu \\ &\le \|A\|^2_{2,[r>0]}\int (|\nabla^2 A|^2 + |A|^2|\nabla A|^2 + |A|^6) r^4\,d\mu + \tfrac{1}{2}\int |\nabla^2 A|^2\,d\mu \\ &\qquad + C\int_{[r>0]} |A|^2\,d\mu + C\Big(\int_{[r>0]} |A|^2\,d\mu\Big)^2.\end{aligned}$$

Since $\int_{[r>0]} |A|^2\,d\mu \le \epsilon$, it yields

$$\frac{d}{dt}\int |A|^2 r^4\,d\mu + \tfrac{1}{4}\int (|\nabla^2 A|^2 + |A|^2|\nabla A|^2 + |A|^6) r^4\,d\mu \le C\epsilon. \tag{4-12}$$

The proposition follows by integrating over $[0, t]$. □

This proposition is the local estimate for the flow, from which we can see the variation of curvature is locally small. Furthermore, we have the following estimate as Proposition 3.4 in [Kuwert and Schätzle 2001] for the surface diffusion flow in $\mathbb{F}^{2+p}(c)$ $(c \ge 0)$ with small initial Willmore energy.

Proposition 4.9. *Let $f : \Sigma \times [0, T] \to \mathbb{F}^{2+p}(c)$ $(c \ge 0)$ be a compact surface diffusion flow with $\int_\Sigma |A|^2\,d\mu \le \aleph$, then there exist constants $\epsilon_0(p) \ge 0$ and $c_1 = C(p)/\aleph > 0$, such that if $\rho > 0$ is chosen with*

$$\int_{\Sigma_\rho} |A^\circ|^2\,d\mu \le \epsilon < \epsilon_0(p) \text{ at time } t = 0 \text{ for all } \Sigma_\rho \subset \mathbb{F}^{2+p},$$

then for any time $0 \le t < t_1 = \min\{c_1\rho^4, T\}$, we have

$$\int_{\Sigma_\rho} |A^\circ|^2\,d\mu + \int_0^t \int_{\Sigma_\rho} (|\nabla^2 A|^2 + |A|^2|\nabla A|^2 + |A|^6 + c|\nabla H|^2)\,d\mu dt \le C(\epsilon + \aleph\rho^{-4} t).$$

Applying Proposition 4.6, Lemma 4.7 and Proposition 4.8, we get the higher order derivative estimate and the proof is similar to Proposition 4.6 in [Kuwert and Schätzle 2002].

Proposition 4.10. *Let $f : \Sigma \times [0, T] \to \mathbb{F}^{2+p}(c)$ be a compact surface diffusion flow. If*

$$\sup_{[0,T]} \int_{[r>0]} |A|^2 \, d\mu \le \epsilon_1(p),$$

where $\epsilon_1(p)$ is small enough, then

$$\|\nabla^m A\|_{\infty\ [r=1]} \le C(m, T, \alpha_0(m+2)),$$

where $\alpha_0(m) = \sum_{j=0}^{m} \|\nabla^j A_0\|_{2\ [r_0>0]}$.

Furthermore, we get the following proposition:

Proposition 4.11 (interior estimates). *Let $f : \Sigma \times (0, T] \to \mathbb{F}^{2+p}(c)$ be a compact surface diffusion flow satisfying the condition*

$$\sup_{(0,T]} \int_{\Sigma_\rho(x)} |A|^2 \, d\mu \le \epsilon < \epsilon_1(p) \quad \textit{for } x \in \mathbb{F}^{2+p}(c),$$

where $T \le C\rho^4$, then for any $k \in \mathbb{N}_0$, at time $t \in (0, T]$, we have the estimates

$$\|\nabla^k A\|_{L^\infty(\Sigma_{\rho/2}(x))} \le c(k)\sqrt{\epsilon}\, t^{-\frac{k+1}{4}},$$
$$\|\nabla^k A\|_{L^2(\Sigma_{\rho/2}(x))} \le c(k)\sqrt{\epsilon}\, t^{-\frac{k}{4}}.$$

These are the higher order derivative estimate, which are localized in time. The proof follows similar to Theorem 3.5 in [Kuwert and Schätzle 2001].

Now we prove the lifespan theorem for the surface diffusion flow.

Proof of Theorem 1.3. We may assume that $\rho = 1$ by rescaling. Put

$$\epsilon(t) = \sup_{x \in \mathbb{F}^{2+p}} \int_{\Sigma_1(x)} |A|^2 \, d\mu.$$

By a trivial covering argument, we get

$$\epsilon(t) \le \Gamma \cdot \sup_{x \in \mathbb{F}^{2+p}} \int_{\Sigma_{\frac{1}{2}}(x)} |A|^2 \, d\mu,$$

for some $\Gamma = \Gamma(p)$. The function $\epsilon : [0, T) \to \mathbb{R}$ is continuous by the compactness of $f(\Sigma \times [0, t])$ for $t < T$. Now let $\lambda > 0$ be a parameter, and define

$$t_0 : \sup\{0 \le t \le \min(T, \lambda) : \epsilon(\tau) \le 3\Gamma\epsilon \text{ for } 0 \le \tau < t\}.$$

The continuity of $\epsilon(t)$ implies $t_0 \ge 0$ and

$$\epsilon(t_0) = 3\Gamma\epsilon \text{ if } t_0 < \min(T, \lambda). \tag{4-13}$$

Fix a cutoff function $\tilde{r} \in C^2(\mathbb{F}^{2+p})$ with $\|\tilde{r}\|_{C^2(\mathbb{F}^{2+p})} \le C$ and $\chi_{B_{1/2}(x)} \le \tilde{r} \le \chi_{B_1(x)}$, then $r = \tilde{r} \circ f$ satisfies condition (4-5). Thus, it follows from Proposition 4.8 that

$$\int_{\Sigma_{\frac{1}{2}}(x)} |A|^2(t)\, d\mu \le \int_{\Sigma_1(x)} |A|^2(0)\, d\mu_0 + C\Gamma\epsilon t \le 2\epsilon \quad \text{for } 0 \le t \le t_0.$$

Taking $\lambda = (C\Gamma)^{-1}$, we conclude that

$$\epsilon(t) \le 2\Gamma\epsilon, \quad 0 \le t \le t_0,$$

and (4-13) implies $t_0 = \min(T, (C\Gamma)^{-1})$. Now if $t_0 = (C\Gamma)^{-1}$, we prove the proposition with a contraction for

$$t_0 = T. \tag{4-14}$$

We can apply Proposition 4.10 to obtain

$$\|\nabla^m A\|_\infty \le C(p, m, f_0).$$

With the same argument as in [Magni 2015], we can get

$$\|\partial^m f\|_\infty, \|\partial^m \partial_t f\|_\infty \le C(p, m, f_0).$$

Then $f(t)$ converges in $C^m(\Sigma)$ as $t \to T$ to a smooth function $f(T)$. By short time existence, we can extend the flow f to an interval $[0, T+\delta)$, which is contrary to the maximality of T. Hence it contradicts with (4-14). This proves Theorem 1.3. □

5. Blowup analysis

In this section, we rescale the surface diffusion flow in $\mathbb{F}^{2+p}(c)$ $(c \ge 0)$ at an assumed singularity, thereby constructing a static surface as a limit. First we need the following local area bound due to L. Simon [1993] in the Euclidean space, which has been generalized by F. Link [2013] in Riemannian manifolds.

Lemma 5.1. *Let $f : \Sigma \to \mathbb{F}^{2+p}(c)$ be a compact surface. Then for $0 < \rho < \infty$ and $\Sigma_\rho = \Sigma_\rho(x_0)$, one has*

$$\frac{\mu(\Sigma_\rho)}{\rho^2} \le C\left(\int_\Sigma |A^\circ|^2\, d\mu + 4\pi\chi(\Sigma)\right). \tag{5-1}$$

Moreover, the global area estimate is given as follows:

Proposition 5.2 (area estimate). *Let $f : \Sigma \times [0, T] \to \mathbb{F}^{2+p}(c)$ be a compact surface diffusion flow with $\int_\Sigma |A^\circ|^2\, d\mu \le \epsilon \le \epsilon_0(p)$, where $\epsilon_0(p)$ is as in Proposition 4.9, then*

$$(1 - C\epsilon)\mu_0(\Sigma) \le \mu(\Sigma) \le \mu_0(\Sigma). \tag{5-2}$$

Proof. Using Lemma 4.1, we have

$$\frac{d}{dt}\int d\mu = -\int |\nabla H|^2\, d\mu < 0,$$

so, $\mu(\Sigma) \le \mu_0(\Sigma)$, and the area is decreasing. On the other hand, by the Sobolev inequalities we have

$$\int |\nabla H|^2\, d\mu \le C(n)\mu(\Sigma)\int |\nabla^2 H|^2 + |H|^2|\nabla H|^2 + c|\nabla H|^2\, d\mu.$$

We obtain

$$\frac{d}{dt}\mu(\Sigma) \ge -C\mu(\Sigma)\int |\nabla^2 H|^2 + |H|^2|\nabla H|^2 + c|\nabla H|^2\, d\mu.$$

Using Proposition 4.9 with $\rho = \infty$ implies

$$\mu(\Sigma) \ge \mu_0(\Sigma)e^{-C\int_0^t\int_\Sigma |\nabla^2 H|^2+|H|^2|\nabla H|^2+c|\nabla H|^2\, d\mu dt} \ge \mu_0(\Sigma)(1-C\epsilon). \qquad \square$$

Now we state the required compactness theorem, which was originally proved by Langer [1985] for surfaces. Recently it has been generalized by Breuning [2015] to any dimension and codimension in the Euclidean space and even to Riemannian setting, where the ambient space can be isometrically embedded into some Euclidean space with bounded second fundamental form. Here we use a simplified version from [Cooper 2011; Magni 2015].

Theorem 5.3 [Cooper 2011, Theorem 1.2]. *Given a compact surface Σ, a sequence of complete Riemannian manifolds $\{(N^{2+p}, \bar{g}_j)\}_{j\in\mathbb{N}}$ with uniformly bounded geometry and two sequences of points $\{q_j\}_{j\in\mathbb{N}} \subset \Sigma$ and $\{x_j\}_{j\in\mathbb{N}} \subset N^{2+p}$, let $f_j : (\Sigma, g_j) \to (N^{2+p}, \bar{g}_j)$ be a sequence of isometric proper immersions such that $f_j(q_j) = x_j$. Suppose that*

$$\mu_j(\Sigma_R(x_j)) \le C(R) \quad \text{for any } R > 0, \tag{5-3}$$

$$\|\nabla^k A_j\|_{L^\infty(\Sigma_R)} \le C_k(R) \quad \text{for any } R > 0 \text{ and any } k \in \mathbb{N}_0. \tag{5-4}$$

Then there exist a surface $\hat{\Sigma}$, a complete Riemannian manifold $(M^{2+p}, \bar{g})$ and two points $q \in \hat{\Sigma}$ and $x \in M^{2+p}$ such that:

- *There exists an increasing exhaustion $\{U_j\}_{j\in\mathbb{N}}$ of $\hat{\Sigma}$ made of open relatively compact sets, and there are diffeomorphisms $\phi_j : U_j \to \Sigma$ with $\phi_j(q) = q_j$, such that for any $R > 0$ we have $\Sigma_R^{g_j}(q_j) \subset \phi_j(U_j)$, for all $j \ge j_0(R)$.*
- *There is an increasing exhaustion $\{V_j\}_{j\in\mathbb{N}}$ of M^{2+p} made of open relatively compact sets, and diffeomorphisms $\psi_j : V_j \to N^{2+p}$ with $\psi_j(x) = x_j$, such that for any $R > 0$ we have $B_R^{\bar{g}_j}(x_j) \subset \psi_j(V_j)$, for all $j \ge j_0(R)$.*
- *$\phi_j(U_j) \subset \psi_j(V_j)$ and $\psi_j^* \bar{g}_j \to \bar{g}$ smoothly.*

- *There exists a proper immersion* $\hat{f} : \hat{\Sigma} \to M^n$ *such that* $\psi_j^{-1} \circ f_j \circ \phi_j \to \hat{f}$ *smoothly with respect to a global isometric embedding of* $(M^{2+p}, \bar{g})$ *into a suitable Euclidean space* $\mathbb{R}^K$*. The immersion* $\hat{f}$ *also satisfies* (5-3) *and* (5-4) *with respect to* x.

Here, the sequence of proper immersions f_j converging as in Theorem 5.3 to a proper immersion $\hat{f} : \hat{\Sigma} \to M^{2+p}$, will be denoted by $f_j \to \hat{f}$. Let $f : \Sigma \times [0, T) \to \mathbb{F}^{2+p}(c)$ be a surface diffusion flow defined on a compact surface Σ, where $0 < T \le \infty$. Define

$$\aleph(r, t) = \sup_{x \in \mathbb{F}^{2+p}} \int_{\Sigma_r(x)} |A(t)|^2 \, d\mu_t, \qquad \Sigma_r(x) = f^{-1}(B_r(x)).$$

Choose an arbitrary sequence $r_j \searrow 0$ and assume concentration in the sense that for all j,

$$t_j = \inf\{t \ge 0 : \aleph(r_j, t) > \epsilon_2\} < T, \tag{5-5}$$

where $\epsilon_2 = \epsilon_1 / C$, and ϵ_1 and C are the constants from the lifespan theorem. Clearly,

$$\int_{\Sigma_{r_j}(x)} |A(t_j)|^2 \, d\mu_{t_j} \le \epsilon_2 \quad \text{for any } x \in \mathbb{F}^{2+p}(c).$$

On the other hand, choosing an appropriate sequence of balls at times $\tau \searrow t_j$, we find a point $x_j \in \mathbb{F}^{2+p}$ satisfying

$$\int_{f^{-1}(\overline{B_{r_j}(x_j)})} |A(t_j)|^2 \, d\mu_{t_j} \ge \epsilon_2.$$

Now we rescale by considering

$$f_j : (\Sigma, g_j) \times [\, -r_j^{-4} t_j, \ r_j^{-4}(T - t_j)) \to (\mathbb{F}^{2+p}, \bar{g}_j),$$
$$f_j(p, t) = f(p, \ t_j + r_j^4 t),$$

where $\bar{g}_j = r_j^{-2} \bar{g}$ and $g_j = f_j(\,\cdot\,, t)^* \bar{g}_j$. From Theorem 5.2 in [Magni 2015], we know that the rescaled flows converge locally smoothly on $\hat{\Sigma} \times \mathbb{R}$ to a static surface diffusion flow represented by a static properly immersed surface $\hat{f} : \hat{\Sigma} \to \mathbb{R}^{2+p}$, open sets with the property $\aleph_j(1, t) \le \epsilon_2$, for all $t \le 0$ and

$$\int_{\hat{f}^{-1}(\overline{B_1(0)})} |\hat{A}|^2 d\hat{\mu} \ge \epsilon_2. \tag{5-6}$$

More precisely, the lifespan theorem yields $r_j^{-4}(T - t_j) \ge c_0$ and in fact

$$\aleph_j(1, t) \le \epsilon_1 \quad \text{for } 0 < t \le c_0.$$

We may now apply Proposition 4.11 on parabolic cylinders $B_1(x) \times (t-1, t]$ to obtain

$$\|\nabla^k A\|_{L^\infty} \le c(k) \quad \text{for } r_j^{-4} + 1 \le t \le c_0.$$

Then we can apply the compactness theorem to the sequence

$$f_j = f_j(\,\cdot\,, 0) : \Sigma \to \mathbb{F}^{2+p},$$

thus obtaining a limit immersion $\hat{f}_0 : \hat{\Sigma} \to \mathbb{R}^{2+p}$. Then by reparametrization $f_j(\phi_j, \cdot)$ is a surface diffusion flow and has initial data converging locally in C^k to the immersion $\hat{f}_0$. By standard estimates for geometric evolution equations in [Kuwert and Schätzle 2002], we deduce the locally smooth convergence $f_j \to \hat{f}$, where

$$\hat{f} : \hat{\Sigma} \times [0, c_0] \to \mathbb{R}^{2+p}$$

is a surface diffusion flow with initial data $\hat{f}_0$.

In order to get the static blowup limit, we need the monotonicity of the Willmore functional.

Lemma 5.4. *Let $f : \Sigma \times (0, T] \to \mathbb{F}^{2+p}(c)$ be a compact surface diffusion flow with small initial Willmore energy satisfying Proposition 5.2; we have*

$$\begin{aligned} \frac{d}{dt}\int |A^\circ|^2\,d\mu &\le \frac{1}{2}\frac{d}{dt}\int |A|^2\,d\mu \\ &\le \frac{1}{2}\frac{d}{dt}\int |H|^2\,d\mu \le -\frac{1}{2}\int |\Delta H|^2\,d\mu. \end{aligned}$$

Proof. The first two inequalities are obvious. From (3-7) and (3-8) with $r = 1$, we know

$$\begin{aligned} c\int |\nabla H|^2\,d\mu &\le C\int |A\|A^\circ\|\nabla A^\circ|^2\,d\mu \le C\int |A|^3|A^\circ|^3 + |\nabla A^\circ|^3\,d\mu \\ &\le C\int |A|^4|A^\circ|^2\,d\mu + \delta\int |\nabla^2 A^\circ|^2\,d\mu + C_\delta\int |A^\circ|^6\,d\mu. \end{aligned}$$

From (3-11) we know

$$\int (|\nabla^2 A|^2 + |A|^2|\nabla A|^2 + |A|^4|A^\circ|^2)\,d\mu + c\int |A^\circ|^2\,d\mu \le \int |\Delta H|^2\,d\mu,$$

and (3-10) implies

$$\begin{aligned} \int |A|^2|A^\circ|^4\,d\mu + c\int |\nabla H|^2\,d\mu &\le \int (\delta|A|^4|A^\circ|^2 + C_\delta|A^\circ|^6 + c|\nabla H|^2)\,d\mu \\ &\le (\delta + c\epsilon + \epsilon C_\delta)\int |\Delta H|^2\,d\mu. \end{aligned}$$

So,

$$\frac{1}{2}\frac{d}{dt}\int |H|^2\,d\mu = -\int (\Delta H + H|A^\circ|^2 + 2cH)(\Delta H)\,d\mu$$
$$\leq -\frac{3}{4}\int |\Delta H|^2\,d\mu + 2\int |A|^2|A^\circ|^4\,d\mu + 2c\int |\nabla H|^2\,d\mu$$
$$\leq -\frac{1}{2}\int |\Delta H|^2\,d\mu.$$

□

Proposition 5.5. *Let $f : \Sigma \times (0, T] \to \mathbb{F}^{2+p}(c)$ be a compact surface diffusion flow with small initial Willmore energy, then the blow up $\hat{f}$ which is constructed above is static.*

Proof. Let $U \Subset \hat{\Sigma}$ be an open set and ϕ_j be the diffeomorphisms in Theorem 5.3, then from Lemma 5.4 and the scale invariance of $\|A\|_2^2$,

$$\int_0^{c_0}\int_U |\Delta H(f_j(\phi_j, t))|^2\,d\mu_{f_j}(\phi_j, \cdot)\,dt = \int_0^{c_0}\int_{\phi_j(U)} |\Delta H_j|^2\,d\mu_j\,dt$$
$$\leq \int_\Sigma |A_j(0)|^2\,d\mu_j - \int_\Sigma |A_j(c_0)|^2\,d\mu_j = \int_\Sigma |A(t_j)|^2\,d\mu - \int_\Sigma |A(t_j + r_j^4 c_0)|^2\,d\mu,$$

and it converges to zero as $j \to \infty$, Therefore $\Delta H(\hat{f}) \equiv 0$ and the blow up $\hat{f}$ is static, which means that $\hat{f}(\,\cdot\,, t) = \hat{f}_0$. Furthermore (5-6) implies

$$\int_{\hat{f}^{-1}(\overline{B_1(0)})} |\hat{A}|^2 d\hat{u} \geq \epsilon_2 > 0.$$

Thus $\hat{f}$ is not totally geodesic. □

Lemma 5.6. *Let $\hat{f} : \hat{\Sigma} \to \mathbb{R}^{2+p}$ be the blowup constructed above. If $\hat{\Sigma}$ contains a compact component C, then in fact $\hat{\Sigma} = C$ and Σ is diffeomorphic to C.*

Proposition 5.7 (nontriviality of the blowup). *Let $\hat{f} : \hat{\Sigma} \to \mathbb{R}^{2+p}$ be the blowup of a compact surface diffusion flow as constructed above. Then none of the components of $\hat{f}$ is compact. In particular, the blowup has a component which is a noncompact nonumbilical surface with $\Delta H \equiv 0$.*

Proof. Assume that there is a compact umbilical component of $\hat{f}$, then Lemma 5.6 implies $\hat{f}$ has no further components. It follows that, up to diffeomorphism $\phi_i : \hat{\Sigma} \to \Sigma$, we have

$$\int_\Sigma |A^\circ(t_j)|^2\,d\mu = \int_\Sigma |A_j^\circ(0)|^2\,d\mu_j \to 0,$$
$$\mu(t_j)(\Sigma) = r_j \mu_j(0)(\Sigma) \to 0.$$

This contradicts the area estimate. □

Now we prove the longtime existence. Combining the previous theorems, we can finally rule out concentration of curvature in finite time.

Proposition 5.8. *Let $f : \Sigma \times [0, T) \to \mathbb{F}^{2+p}(c)$ be a compact surface diffusion flow, then there exists a constant $\epsilon_0(p)$ such that if*

$$\int_\Sigma |A^\circ|^2 \, d\mu|_{t=0} < \epsilon_0 \le \sigma(2), \tag{5-7}$$

then $T = \infty$ and there exists a radius $r_0 > 0$ such that

$$\int_{\Sigma_{r_0}(x)} |A|^2 \, d\mu < \epsilon_2, \tag{5-8}$$

for all $x \in \mathbb{F}^{2+p}$, $t \in [0, +\infty)$, where $\epsilon_2 > 0$ as in (5-5).

Proof. First, if curvature concentrates at time T, then we perform a blowup as above at T and get a static surface $\hat{f}$ with small tracefree curvature. Now the gap theorem implies $\hat{f}$ must be a plane or sphere, which contradicts the nontriviality of the blowup. There does not exist a finite time when curvature concentrates, so $T = \infty$.

Then we claim there exists a radius $r_0 > 0$, such that

$$\int_{\Sigma_{r_0}(x)} |A|^2 \, d\mu < \epsilon_2 \quad \text{for all } x \in \mathbb{F}^{2+p}, \quad t \in [0, \infty),$$

where $\epsilon_2 > 0$ is as in(5-5).

If not, in the same way we can perform a blowup as above, and get a static surface $\hat{f} : \hat{\Sigma} \to \mathbb{R}^{2+p}$ with

$$\int_{\hat{f}^{-1}(\overline{B_1(0)})} |\hat{A}|^2 d\hat{\mu} \ge \epsilon_2,$$

whereas

$$\int_{\hat{\Sigma}} |\hat{A}^\circ|^2 d\hat{\mu} \le \epsilon_0,$$

due to the scale invariance of $\|A^\circ\|_2^2$. Now the gap theorem implies $\hat{f}$ must be totally umbilical, which contradicts the nontriviality of the blowup. □

Using a similar argument as in Proposition 5.5, we get the smooth convergence from Propositions 4.11 and 5.8.

Proof of Theorem 1.4. For any sequence $t_j \to \infty$, there exist $\phi_j \in \mathrm{Diff}(\Sigma)$ such that, after passing to a subsequence, $f(\phi_j, t_j)$ converges smoothly to a static surface. The gap theorem implies it is a union of planes and spheres. After excluding several components as in [Kuwert and Schätzle 2001] and investigating the asymptotic behavior as in [Wheeler 2012], we get the global existence and exponential convergence for the surface diffusion flow with small initial Willmore energy. □

In particular, the convergence theorem implies a differentiable sphere theorem.

Corollary 5.9. *Let $f : \Sigma \to \mathbb{F}^{2+p}(c)$ $(c \geq 0)$ be a compact surface. If there exists a positive constant $\epsilon_0(p)$ such that*

$$\int_\Sigma |A^\circ|^2 \, d\mu \ < \epsilon_0(p),$$

then Σ is diffeomorphic to the unit sphere.

The proof of Theorem 1.4 also applies to the Willmore flow in a sphere.

6. Open problems

In this section, we propose several problems for the convergence of the curvature flow. Applying the Morse theory of submanifolds, Shiohama and Xu [2000] obtained a topological sphere theorem for n-dimensional compact submanifolds in $\mathbb{F}^{n+p}(c)$ $(c \geq 0)$ under the pinching condition for the Willmore functional. Thus, a natural problem is whether or not M^n is diffeomorphic to $\mathbb{S}^n$ under the pinching condition for the Willmore functional. More precisely, we propose the following problem.

Conjecture 6.1. *Let $f : M^n \to \mathbb{F}^{n+p}(c)$ be a compact submanifold. There exists a positive constant $C_3(n)$ such that if*

$$\int_M |A^\circ|^n \, d\mu < C_3(n), \tag{6-1}$$

then the diffusion flow for submanifolds with initial data f exists smoothly for all time and converges to a round sphere as $t \to \infty$. In particular, M is diffeomorphic to $\mathbb{S}^n$.

Liu, Xu, Ye and Zhao [2018] investigated the convergence of the mean curvature flow of compact n-dimensional submanifolds in $\mathbb{R}^{n+p}$. They proved if the initial submanifold satisfies some suitable integral curvature conditions, then along the mean curvature flow it will shrink to a round point in finite time.

T. J. Willmore [1968] proved the Willmore inequality about the total mean curvature $\int |H|^2 \, d\mu$ for compact surfaces in $\mathbb{R}^3$, and then B. Y. Chen [1971] obtained a general version of the Willmore inequality for compact submanifolds in $\mathbb{R}^{n+p}$, as follows:

Theorem 6.2. *If M is an n-dimensional compact submanifold in $\mathbb{R}^{n+p}$, then*

$$n^n \operatorname{Vol}(\mathbb{S}^n) \leq \int_M |H|^n \, d\mu,$$

where the equality holds if and only if $M^n = \mathbb{S}^n(r)$.

Therefore, the total mean curvature for compact submanifolds in $\mathbb{R}^{n+p}$ has a natural lower bound. Applying the Morse theory of submanifolds, Xu [2007] obtained a topological sphere theorem under the total mean curvature pinching condition. For other topological sphere theorems we can see [Shiohama and Xu 1994; 2000]. Similarly, we have the following problem for the diffusion flow under the total mean curvature pinching condition.

Conjecture 6.3. *Let $f : M^n \to \mathbb{R}^{n+p}$ be a compact submanifold. There exists a positive constant $C_4(n)$ such that if*

$$\int_M |H|^n \, d\mu < n^n \operatorname{Vol}(\mathbb{S}^n) + C_4(n),$$

then the diffusion flow for submanifolds with initial data f exists smoothly for all time and converges to a round sphere as $t \to \infty$. In particular, M is diffeomorphic to $\mathbb{S}^n$.

For surfaces in $\mathbb{R}^3$, the small Willmore energy assumption means a pinching condition for $\int |H|^2 \, d\mu$ and the convergence of the surface diffusion flow has answered these two problems. The Willmore conjecture verified by Marques and Neves [2014] says the integral of the square of the mean curvature of a torus in $\mathbb{R}^3$ is at least $8\pi^2$. In fact, they proved the integral inequality of the square of the mean curvature for any compact surfaces with genus greater than or equal to one. Thus, a strong version of the problem related to the Marques and Neves's theorem is: what is the best pinching constant in Conjecture 6.3 for compact surfaces in $\mathbb{R}^3$?

Conjecture 6.4. *Let $f : \Sigma \to \mathbb{R}^3$ be a compact surface satisfying*

$$\int_\Sigma |H|^2 \, d\mu < 8\pi^2,$$

then the surface diffusion flow with initial data f exists smoothly for all time and converges to a round sphere as $t \to \infty$. In particular, Σ is diffeomorphic to $\mathbb{S}^2$.

Conjecture 6.5. *Let $f : \Sigma \to \mathbb{R}^3$ be a compact surface satisfying*

$$\int_\Sigma |H|^2 \, d\mu < 8\pi^2,$$

then the Willmore flow with initial data f exists smoothly for all time and converges to a round sphere as $t \to \infty$. In particular, Σ is diffeomorphic to $\mathbb{S}^2$.

For higher dimensions, there are seldom results about the diffusion flow. However it provides a feasible method to handle the above problems so we can study the gradient flow of some constructed functionals. Li [2002] investigated the rigidity of the Willmore submanifold, which is the critical point of the Willmore functional for n-dimensional compact submanifolds in a sphere. As the Willmore flow for surfaces, we can also study the higher-dimensional Willmore flow, which is the gradient flow of the Willmore functional for n-dimensional compact submanifolds in space forms. More precisely, we will study the following higher-dimensional Willmore flow

$$\frac{\partial f}{\partial t} = -\operatorname{grad} \mathcal{W}(f), \tag{6-2}$$

where grad $\mathcal{W}(f)$ is the gradient vector field of the Willmore functional (1-2). We propose the following problems.

Conjecture 6.6. *Let $f : M^n \to \mathbb{F}^{n+p}(c)$ be a compact submanifold. There exists a positive constant $C_5(n)$ such that if*

$$\int_M |A^\circ|^n \, d\mu < C_5(n), \tag{6-3}$$

then the Willmore flow for submanifolds with initial data f exists smoothly for all time and converges to a round sphere as $t \to \infty$. In particular, M is diffeomorphic to $\mathbb{S}^n$.

Conjecture 6.7. *Let $f : M^n \to \mathbb{R}^{n+p}$ be a compact submanifold. There exists a positive constant $C_6(n)$ such that if*

$$\int_M |H|^n \, d\mu < n^n \operatorname{Vol}(\mathbb{S}^n) + C_6(n),$$

then the Willmore flow for submanifolds with initial data f exists smoothly for all time and converges to a round sphere as $t \to \infty$. In particular, M is diffeomorphic to $\mathbb{S}^n$.

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