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We prove boundedness of commutators associated to some anisotropic singular integral on weighted Lebesgue spaces and on generalized weighted Morrey spaces. As an application, we obtain the boundedness of those operators on weighted Triebel–Lizorkin space.

1. Introduction and main results

In harmonic analysis, there are a number of important inequalities of the form

$$\int_{\mathbb{R}^n} |Tf|^p w(x)\,dx \le C \int_{\mathbb{R}^n} |Sf(x)|^p w(x)\,dx, \tag{1-1}$$

where T is a singular integral operator and S is an operator which is, in principle, easier to handle (e.g., a maximal operator), and w is in some class of weights. This type of estimate plays an important role in many fields of analysis, including partial differential equation theory. For instance, the estimate (1-1) can be used to study the regularity of the elliptic equation with discontinuous coefficients; see for example [Chiarenza et al. 1991].

Definition 1.1. A Calderon–Zygmund kernel (C–Z kernel) is a function

$$K : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$$

such that:

(i) $K \in C^\infty(\mathbb{R}^n \backslash \{0\})$.

(ii) K is homogeneous of degree $-n$.

(iii) $\int_{S^{n-1}} K(\sigma)\,d\sigma = 0$ and $\int_{S^{n-1}} |K(\sigma)|^2\,d\sigma < \infty$, where $S^{n-1}$is the unit sphere in $\mathbb{R}^n$.

Theorem 1.2. *Let K be a C-Z kernel and $\varepsilon > 0$. Then the operator*

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x-y) f(y)\,dy$$

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is well defined in L^p and for any $f \in L^p$, there exists $Tf \in L^p$ such that

$$\lim_{\varepsilon\to 0} \|T_\varepsilon f - Tf\|_{L^p} = 0.$$

Furthermore T is a bounded operator in L^p, $1 < p < \infty$, i.e., for some positive constant $c = c(n, p)$,

$$\|Tf\|_{L^p} \le c\|K\|_{L^2(S^{n-1})}\|f\|_{L^p}, \quad \text{for all } f \in L^p.$$

The operator T will be called a Calderon–Zygmund (C-Z) singular operator and we will use the notation

$$Tf(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} K(x-y) f(y)\, dy.$$

In their work F. Chiarenza, M. Frasca and P. Longo [Chiarenza et al. 1991] have considered more general operators that are not necessarily of convolution type,

$$Tf(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} K(x, x-y) f(y)\, dy.$$

They established, under some suitable conditions, the continuity of such operators in nonweighted L^p spaces. More precisely, they obtained the following result.

Theorem 1.3. *Let $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be such that*

(i) *$K(x, \cdot)$ is a C–Z kernel for a.e $x \in \mathbb{R}^n$,*

(ii) $\sup_{y\in S^{n-1}} |\partial_y^\beta K(x, y)| = M < +\infty$ *for all multi-indices $\beta < r$ with*

$$r > \frac{3n}{4} - \frac{1}{2}.$$

For $f \in L^p$, $1 < p < \infty$, and $\varphi \in BMO$, set

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x, x-y) f(y)\, dy$$

and

$$C_\varepsilon[\varphi, f](x) = \varphi(x) T_\varepsilon f(x) - T_\varepsilon(\varphi f)(x) = \int_{|x-y|>\varepsilon} K(x, x-y)[\varphi(x) - \varphi(y)] f(y)\, dy.$$

Then for any $f \in L^p$ there exist Tf and $C[\varphi, f] \in L^p$ such that

$$Tf(x) = \lim_{\varepsilon\to 0} T_\varepsilon f(x), \qquad C[\varphi, f](x) = \lim_{\varepsilon\to 0} C_\varepsilon[\varphi, f](x)$$

in L^p. Moreover, there exists a constant $c = c(n, p, M)$ such that

$$\|Tf\|_{L^p} \le c\|f\|_{L^p}$$

and

$$\|C[\varphi, f]\|_{L^p} \le c\|\varphi\|_{BMO}\|f\|_{L^p}.$$

Introducing a new metric ρ, Fabes and Rivière [1966] studied the continuity of T in $L^p(\mathbb{R}^n)$, where $\mathbb{R}^n$ is endowed with the topology induced by the metric ρ. Using this metric, L. Softova [2006] showed that T and its corresponding commutator operators are continuous in generalized Morrey spaces.

In this work we study the continuity of the N-th commutator operators associated to Calderon–Zygmund singular operators on weighted Lebesgue and Morrey spaces by using the metric ρ. As an application we conclude by showing that the operator T is continuous in inhomogeneous weighted Triebel–Lizorkin spaces, which contain many classical spaces, such as BMO spaces, Hardy spaces, and fractional Sobolev spaces.

In what follows, ρ will be a metric as defined in Section 2.

Definition 1.4. We say that $K : \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ is a variable kernel with mixed homogeneity if

(i) $K(x, \cdot) \in C^\infty(\mathbb{R}^n \backslash \{0\})$,

(ii) for any $\lambda > 0$, $\beta_i \ge 1$, $\beta = \sum_{i=1}^n \beta_i$

$$K(x, \lambda^{\beta_1} y_1, \dots, \lambda^{\beta_n} y_n) = \lambda^{-\beta} K(x, y),$$

(iii) $\int_{S^{n-1}} K(x, \sigma)\, d\sigma = 0$ and $\int_{S^{n-1}} |K(x, \sigma)|^2\, d\sigma < \infty$,

(iv) $\sup_{y \in S^{n-1}} |\partial_y^\gamma K(x, y)| = M < +\infty$ for all multi-indices $\gamma < r$ with

$$r > \frac{3n}{4} - \frac{1}{2}.$$

The N-th commutator operators, $N = 1, 2, \dots$ are defined inductively by

$$C_\varepsilon^N[\varphi, f] = T_\varepsilon^N f(x) = \varphi(x) T_\varepsilon^{N-1} f(x) - T_\varepsilon^{N-1}(\varphi f)(x)$$

with

$$T_\varepsilon^0 f(x) = T_\varepsilon f(x) = \int_{\rho(x-y)>\varepsilon} K(x, x-y) f(y)\, dy.$$

Note that for every $N \in \mathbb{N}_0$,

$$T_\varepsilon^N f(x) = \int_{\rho(x-y)>\varepsilon} K(x, x-y)[\varphi(x) - \varphi(y)]^N f(y)\, dy.$$

Throughout this work, φ is assumed to be in BMO, C denotes a constant that does not depend on f but it is not the same at each occurrence, and the kernel of the operator T is as in Definition 1.4. Our main results are the following.

Theorem 1.5. *If* $0 < p < \infty$ *and* $w \in A_\infty$, *then*

$$\int_{\mathbb{R}^n} |T^N f|^p w(x)\,dx \le C\|\varphi\|_{BMO}^{Np} \int_{\mathbb{R}^n} |M^{N+1} f(x)|^p w(x)\,dx. \tag{1-2}$$

In particular, *if* $1 < p < \infty$ *and* $w \in A_p$, *then*

$$\int_{\mathbb{R}^n} |T^N f|^p w(x)\,dx \le C\|\varphi\|_{BMO}^{Np} \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx, \tag{1-3}$$

where $M^k f(x)$ *is the Hardy–Littlewood maximal operator iterated k-times.*

Note that $M^{N+1} f(x) \approx M_{L(\log L)^N} f(x)$ for each $x \in \mathbb{R}^n$ and $N = 0, 1, 2, \dots$.

Remark 1.6. An interesting consequence of the estimate (1-2) is the following sharp weighted estimate.

Let w be a weight. Then

$$\int_{\mathbb{R}^n} |T^N f|^p w(x)\,dx \le C\|\varphi\|_{BMO}^{Np} \int_{\mathbb{R}^n} |f(x)|^p M^{[(N+1)p]+1} w(x)\,dx \tag{1-4}$$

for all $1 < p < \infty$, where $[p]$ stands for the integer part of p. See [Lorente et al. 2008; Pérez 1995b].

Also, the weighted vector-valued inequality in [Cruz-Uribe et al. 2004] and the weighted vector-valued inequality for the maximal operator in [Andersen and John 1981] lead to the following important result.

Corollary 1.7. *If* $0 < p, q < \infty$ *and* $w \in A_\infty$, *then*

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |T^N f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)} \le C\|\varphi\|_{BMO}^{Np} \Big\| \Big(\sum_{j\in\mathbb{Z}} |M^{N+1} f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)}.$$

In particular, if $1 < p, q < \infty$ and $w \in A_p$, then

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |T^N f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)} \le C\|\varphi\|_{BMO}^{Np} \Big\| \Big(\sum_{j\in\mathbb{Z}} |f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)}.$$

The estimate (1-2) can also by applied to obtain an other important estimate in rearrangement invariant Banach spaces. We recall that a Banach function space X is said to be rearrangement invariant if whenever $f, g \in X$ are equimeasurable, then $\|f\|_X = \|g\|_X$ and that two functions are equimeasurable if $w_f(t) = w_g(t)$, $t > 0$, where $w_f(t) = w(\{x \in \mathbb{R}^n : |f(x)| > t\})$ is the distribution of f. Given a Banach function space X and $0 < r < \infty$, f is said to be in X^r if and only if

$$\|f\|_{X^r(w)} = \||f|^r\|_{X(w)}^{\frac{1}{r}} < \infty.$$

The definition of the Boyd indices involves the norm of the dilation operator in X and gives information about the localization of X in terms of interpolation properties; see [Bennett and Sharpley 1988, Chapter 1] for more details. For instance, L^p, $L^{p,q}$ and $L^p(\log L)^r$ have Boyd indices $q_X = p$.

Using estimate (1-2) and [Curbera et al. 2006, Theorem 2.1] we obtain the following weighted vector-valued inequality.

Theorem 1.8. *Let X be a rearrangement invariant Banach function space with upper power indices $0 < q_X < \infty$ and such that X^r is a Banach space for some $r > 1$. Then for all $1 < q < \infty$ and for all $w \in A_\infty$,*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |T^N f_j|^q \right)^{\frac{1}{q}} \right\|_{X(w)} \le C \|\varphi\|_{BMO}^{Np} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{\frac{1}{q}} \right\|_{X(w)}.$$

2. Preliminaries

We begin this section by recalling some definitions and some classical results in harmonic analysis. Good references are [García-Cuerva and Rubio de Francia 1985], and [Stein 1993] in the context of the Euclidean spaces. In the context of the noneuclidean spaces we refer, for instance, to Y. Sawano [2006].

Let $\alpha_1, \dots, \alpha_n$ be real numbers, where $\alpha_i \ge 1$, and set $\alpha = \sum_1^n \alpha_i$. Following Fabes and Rivière [1966], the function $F(x, \rho) = \sum_1^n \rho^{-2\alpha_i} x_i^2$, considered for any fixed $x \in \mathbb{R}^n \setminus \{0\}$, is a decreasing function of ρ and therefore there is a unique solution $\rho(x)$ of the equation $F(x, \rho) = 1$. Set $\rho(0) = 0$ and define $d(x, y) = \rho(x - y)$. It can be proved (see [Fabes and Rivière 1966]) that d is actually a distance. The balls with respect to ρ, centered at some point x and of radius r, are defined by

$$Q_r(x) = \{x \in \mathbb{R}^n : d(x - y) < r\}.$$

Introducing the polar type change of variables

$$\begin{cases} x_1 = \rho^{\alpha_1} \cos\varphi_1 \cdots \cos\varphi_{n-2} \, \cos\varphi_{n-1}, \\ x_2 = \rho^{\alpha_2} \cos\varphi_1 \cdots \sin\varphi_{n-1}, \\ \quad\vdots \\ x_n = \rho^{\alpha_n} \sin\varphi_1, \end{cases}$$

we find $dx = \rho^{\alpha-1} d\rho d\sigma$ with $d\sigma$ the surface measure on the Euclidean unit sphere S^{n-1}. Therefore we can compute

$$\mu(Q_r(x)) = c_n r^\alpha \quad \text{for all } r > 0,$$

where $d\mu = dx = \rho^{\alpha-1} d\rho d\sigma$. In particular $(\mathbb{R}^n, d, dx)$ is a homogeneous space. In what follows the balls considered are balls with respect to ρ. To simplify notation we put $(\mathbb{R}^n, d, dx) = \mathbb{R}^n$.

We say that w is a weight if w is an a.e. positive locally integrable function in $\mathbb{R}^n$. Given E a measurable set, we set $|E| = \mu(E)$ and $w(E) = \int_E w(x)\,dx$. Let $0 < p < \infty$. Then f is said to be in $L^p(w)$ if and only if

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{\frac{1}{p}} < \infty.$$

The weak-L^p spaces, denoted by $L^{p,\infty}(w)$ are defined as follows: $f \in L^{p,\infty}(w)$ if and only if

$$\|f\|_{L^{p,\infty}(w)} = \sup_{\lambda>0}(\lambda[w\{x \in \mathbb{R}^n : |f(x)| > \lambda\}]^{\frac{1}{p}}) < \infty.$$

When $w = 1$ we drop the subscription w. The Hardy–Littlewood maximal function Mf and the sharp maximal function $M^\# f$ are defined, respectively, for a locally integrable function f by

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\,dy, \tag{2-1}$$

$$M^\# f(x) = \sup_{x\in Q}\inf_{c\in\mathbb{R}} \frac{1}{|Q|}\int_Q |f(y) - c|\,dy \approx \sup \frac{1}{|Q|}\int_Q |f(y) - f_Q|\,dy, \tag{2-2}$$

where the supremum is taken over all balls containing x, and f_Q denotes the average of f over the ball Q. For $\delta > 0$, we set

$$M_\delta f(x) = (M|f|^\delta(x))^{\frac{1}{\delta}}, \tag{2-3}$$

$$M_\delta^\# f(x) = (M^\#|f|^\delta(x))^{\frac{1}{\delta}}. \tag{2-4}$$

Lemma 2.1 (Kolmogorov's inequality). *Let* $0 < r < q < \infty$ *and define*

$$N_{q,r}(f) = \sup_E \frac{1}{w(E)^{\frac{1}{s}}}\left(\int_E |f(x)|^r w(x)\,dx\right)^{\frac{1}{r}}, \qquad \frac{1}{s} = \frac{1}{r} - \frac{1}{q},$$

where the sup *is taken for all measurable sets with* $0 < w(E) < \infty$. *Then*

$$\|f\|_{L^{q,\infty}(w)} \le N_{q,r}(f) \le \frac{q}{q-r}\|f\|_{L^{q,\infty}(w)}. \tag{2-5}$$

A nonnegative locally integrable function w is said to be in the Muckenhoupt classes A_p if there exists a constant $C_p > 0$ such that for all balls Q,

$$\frac{1}{|Q|}\int_Q w\,dy\left(\frac{1}{|Q|}\int_Q w^{1-p'}\,dy\right)^{p-1} \le C_p,$$

when $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, and for $p = 1$,

$$\frac{1}{|Q|}\int_Q w\,dy \le C_1 w(x),$$

for a.e. $x \in Q$, or equivalently $Mw(x) \le C_1 w(x)$ for a.e. $x \in \mathbb{R}^n$, and we set $A_\infty = \bigcup_{p\geq 1} A_p$. It is well known that Muckenhoupt classes characterize the boundedness of the Hardy–Littlewood maximal M on the weighted Lebesgue's space; see [García-Cuerva and Rubio de Francia 1985, Chapter 3]. Namely,

$$M : L^p(w) \to L^p(w) \tag{2-6}$$

if and only if $w \in A_p$ when $1 < p < \infty$, and

$$M : L^1(w) \to L^{1,\infty}(w), \tag{2-7}$$

if and only if $w \in A_1$.

Another important result that we will use in this work is the Fefferman–Stein's inequality; see, for instance, [García-Cuerva and Rubio de Francia 1985, p. 410].

Theorem 2.2 (Fefferman–Stein's inequality). *Let $w \in A_\infty$ and let f be such that $Mf \in L^r(w)$ for some $0 < r < \infty$. Then for every p such that $r < p < \infty$,*

$$\int_{\mathbb{R}^n} |Mf(x)|^p w(x)\, dx \leq c \int_{\mathbb{R}^n} |M^\# f(x)|^p w(x)\, dx. \tag{2-8}$$

Lemma 2.1 and Theorem 2.2 play a principal role in the studies of many singular integral operators since most of these operators are "controlled" by the operator M.

Another class of functions that plays an important role in harmonic analysis and in partial differential equation theory is the class of functions of bounded mean oscillation noted by *BMO*, i.e., $\varphi \in BMO$ if

$$\sup_Q \frac{1}{|Q|} \int_Q |\varphi(y) - \varphi_Q|\, dy. < A < \infty \tag{2-9}$$

The smallest constant for which (2-9) is satisfied is taken to be the norm of φ in the space *BMO*, and is denoted by $\|\varphi\|_{BMO}$.

One of most important properties of *BMO* is the John–Nirenberg inequality,

$$\left(\frac{1}{|Q|} \int_Q |\varphi(y) - \varphi_Q|^p\, dy\right)^{1/p} \leq c_p \|\varphi\|_{BMO}. \tag{2-10}$$

One has also

$$|\varphi_{2^k Q} - \varphi_Q| \leq ck\|\varphi\|_{BMO}. \tag{2-11}$$

See for instance [Stein 1993, Chapter IV].

Remark 2.3. The above results are well known in the literature in weighted Euclidean spaces, and by standard argument they are still valid in weighted space of homogeneous type; see for instance [Hu et al. 2007; Strömberg and Torchinsky 1989].

3. Proof of Theorem 1.3

As in [Chiarenza et al. 1991], the proof of Theorem 1.5 is based essentially in the expansion of the kernel K into spherical harmonics. Therefore we need to recall first some definitions and results related to this last notion. A good reference is [Neri 1971].

Denote by $\Pi_m$the set of all real polynomials in $x \in \mathbb{R}^n$, $n \geq 2$, which are homogeneous of degree m. It is well known that Π_m is a finite-dimensional vector space of dimension $g_m = C^{n-1}_{m+n-1}$.

Solid harmonics of degree m are polynomials $P \in \Pi_m$ which satisfy $\Delta P = 0$.

The set of all solid harmonics of degree m, denoted by S_m, is a subspace of Π_m of dimension

$$d_m = g_m - g_{m-2} = C^{n-1}_{m+n-1} - C^{n-1}_{m+n-3}.$$

The restrictions of solid harmonics to the unit sphere are called spherical harmonics of degree m and we denote by Q_m the set of all spherical harmonics of degree m.

The vector space Q_m can be seen as a linear subspace of the Hilbert space $L_2(S^{n-1})$, with inner product

$$\langle f, g\rangle = \int_{S^{n-1}} fg\, d\sigma.$$

With respect to this inner product, we can construct in each Q_m an orthonormal basis $Y_{km},\ k = 1, \dots, d(m)$. Moreover, we have:

Theorem 3.1. *The collection* $\{Y_{km}(z)\},\ k \in \{1, \dots, d_m\},\ m \in \mathbb{N}$, *is a complete orthonormal system of spherical harmonics on* $L^2(S^{n-1})$.

On the other hand if we denote by L the operator defined by $Lf = |x|^2 \Delta f$ then:

Lemma 3.2 [Neri 1971, Chapter III]. (a) $d_m \leq c(n) m^{n-2}$,

(b) $\left|\left(\frac{\partial}{\partial x}\right)^\alpha \left[|z|^m Y_{km}(z/|z|)\right]\right| \leq C(\alpha, n) m^{(n-2)/2+|\alpha|} |z|^{m-|\alpha|}$,

(c) $L^r Y_{km} = (-m)^r (m+n-2)^r Y_{km}$ *for all* $r \in \mathbb{N}$,

(d) *if* $f, g \in C^{2r}(R^n \setminus \{0\})$ *are homogeneous of degree zero, then*

$$\int_{S^{n-1}} f L^r g\, d\sigma = \int_{S^{n-1}} f L^r g\, d\sigma.$$

By the completeness of $\{Y_{km}\}$ in $L_2(S^{n-1})$, we can write (see [Neri 1971, Chapters III and IV])

$$K(x, z) = \rho(x - z)^{-\alpha} \sum_{m=1}^{+\infty} \sum_{k=1}^{d_m} a_{km}(x) Y_{km}(z'), \tag{3-1}$$

where

$$a_{km}(x) = \int_{S^{n-1}} K(x, z) Y_{km}(z)\, d\sigma_z$$

and $z' = z/\rho(z)$. The last equality, Lemma 3.2 and the definition of the kernel K imply

$$\|a_{km}\|_{L^\infty} \le C(n) M_r m^{-2r}. \tag{3-2}$$

Lemma 3.3 (pointwise Hörmander's condition [Softova 2006]). *Let Q and $2Q$ be balls centered at x_0 and put $K_{km}(x) = Y_{km}(x')/\rho^\alpha(x)$. Then*

$$|K_{km}(x-y) - K_{km}(x_0-y)| \le C(n) m^{\frac{n}{2}} \frac{\rho(x_0-x)}{\rho(x_0-y)^{\alpha+1}} \tag{3-3}$$

for each $x \in Q$ and $y \notin 2Q$.

Define for $\varepsilon > 0$, $N = 0, 1, 2, \dots$ and $\varphi \in BMO$, the operators $T^N_{km\varepsilon}$ by

$$T^N_{km\varepsilon} f(x) = \int_{\rho(x-y)>\varepsilon} K_{km}(x-y)[\varphi(x)-\varphi(y)]^N f(y)\,dy$$

and set

$$T_{km\varepsilon} f(x) = T^0_{km\varepsilon} f(x) = \int_{\rho(x-y)>\varepsilon} K_{km}(x-y) f(y)\,dy.$$

Lemma 3.4 [Tao 1999]. *T_{km} is a weak $(1, 1)$ with*

$$\left|\{x \in \mathbb{R}^n : |T_{km} f(x)| > \lambda\}\right| \le \frac{C(n) m^{\frac{n}{2}}}{\lambda} \int_{\mathbb{R}^n} |f(x)|\,dx. \tag{3-4}$$

Lemma 3.5. *For any $0 < \delta < \eta < 1$, there exists a constant*

$$C = C(n, \delta, \eta)$$

such that

$$M^\#_\delta(T_{km} f)(x) \le C m^{\frac{n}{2}} M(f)(x) \tag{3-5}$$

and

$$M^\#_\delta(T^N_{km} f)(x) \le C m^{n/2} \Big[|\varphi|^N_{BMO} M^{N+1}(f)(x) + \sum_{j=0}^{N-1} \|\varphi\|^{N-j}_{BMO} M_\eta(T^j_{km} f)(x) \Big]. \tag{3-6}$$

Proof of Lemma 3.5. To prove the inequality (3-5) we proceed as in [Pérez 1995a].

Case 1. $N = 0$. Fix a ball Q containing x and put

$$f = f\chi_{2Q} + f\chi_{\mathbb{R}^n \setminus 2Q} = f_1 + f_2,$$

then

$$T_{km} f = T_{km} f_1 + T_{km} f_2$$

and it suffices to prove (3-5) for f_1 and f_2. For $0 < \delta < 1$, for all α, $\beta \in \mathbb{R}$,

$$\left| |\alpha|^\delta - |\beta|^\delta \right| \le |\alpha - \beta|^\delta.$$

This implies

$$\left(\frac{1}{|Q|}\int_Q \big||T_{km}f(y)|^\delta - (T_{km}f_2)_Q^\delta\big|\,dy\right)^{\frac{1}{\delta}} \le C_\delta\left(\frac{1}{|Q|}\int_Q |T_{km}f(y)-(T_{km}f_2)_Q|^\delta\,dy\right)^{\frac{1}{\delta}}$$
$$\le C_\delta\left(\frac{1}{|Q|}\int_Q |T_{km}f_1|^\delta\,dy\right)^{\frac{1}{\delta}} + C_\delta\left(\frac{1}{|Q|}\int_Q |T_{km}f_2(y)-(T_{km}f_2)_Q|^\delta\,dy\right)^{\frac{1}{\delta}}$$
$$\le C_\delta(I+II).$$

To estimate I we use the weak type (1, 1) and the Kolmogorov inequality (2-5):

$$\left(\frac{1}{|Q|}\int_Q |T_{km}f_1|^\delta\,dy\right)^{\frac{1}{\delta}} \le \frac{1}{1-\delta}\frac{1}{|Q|}\|T_{km}f_1\|_{L^{1,\infty}}.$$

Hence,

$$I \le \frac{C(n,\delta)m^{\frac{n}{2}}}{|Q|}\int_{\mathbb{R}^n}|f_1|\,dy \le \frac{C(n,\delta)m^{\frac{n}{2}}}{|Q|}\int_{2Q}|f|\,dy \le C(n,\delta)m^{\frac{n}{2}}M(f)(x). \tag{3-7}$$

Also for $y\in Q$,

$$|T_{km}f_2(y)-(T_{km}f_2)_Q| \le \frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n\setminus 2Q}|K_{km}(y-\xi)-K_{km}(z-\xi)|\,|f(\xi)|\,d\xi dz$$
$$\le J_1+J_2$$

with

$$J_1 = \frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n\setminus 2Q}|K_{km}(y-\xi)-K_{km}(x-\xi)|\,|f(\xi)|\,d\xi dz$$

and

$$J_2 = \frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n\setminus 2Q}|K_{km}(x-\xi)-K_{km}(z-\xi)|\,|f(\xi)|\,d\xi dz.$$

The estimate (3-3) leads to

$$J_1 \le cm^{\frac{n}{2}}\frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n\setminus 2Q}\frac{\rho(x-z)}{\rho(x-\xi)^{\alpha+1}}|f(\xi)|\,d\xi dz$$
$$\le cm^{\frac{n}{2}}\left|\sum_{k=1}^{+\infty}2^{-k}\frac{1}{|2^kQ|}\right|\int_{2^{k+1}Q}|f(\xi)|\,d\xi$$
$$\le cm^{\frac{n}{2}}M(f)(x).$$

The same argument can be applied to obtain the same inequality for J_2. It follows that

$$II \le c m^{\frac{n}{2}} M(f)(x). \tag{3-8}$$

Case 2. $k = 1, 2 \dots$. We expand $T^N_{km} f$ as follows: for an arbitrary constant λ we can write

$$T^N_{km} f(x) = \sum_{j=0}^{N-1} C_{j,N} (\varphi(x) - \lambda)^{N-j} T^j_{km} f(x) + T_{km}((\lambda - \varphi)^N f)(x).$$

As before, we fix a ball Q containing x and put $f = f\chi_{2Q} + f\chi_{\mathbb{R}^n \setminus 2Q} = f_1 + f_2$. Then, with $\lambda = \varphi_{2Q}$ and $c = (T_{km}(\varphi - \varphi_{2Q})^N f_2)_Q$ we have

$$\begin{aligned}
\left(\frac{1}{|Q|}\int_Q \big||T^N_{km} f(y)|^\delta - |c|^\delta\big|\, dy\right)^{\frac{1}{\delta}} &\le \left(\frac{1}{|Q|}\int_Q |T^N_{km} f(y) - c|^\delta\, dy\right)^{\frac{1}{\delta}} \\
&\le C\Bigg[\sum_{j=0}^{N-1}\left(\frac{1}{|Q|}\int_Q |\varphi(y)-\lambda|^{(N-j)\delta} |T^j_{km} f(y)|^\delta\, dy\right)^{\frac{1}{\delta}} \\
&\qquad + \left(\frac{1}{|Q|}\int_Q |T_{km}((\varphi-\lambda)^N f_1)(y)|^\delta\, dy\right)^{\frac{1}{\delta}} \\
&\qquad + \left(\frac{1}{|Q|}\int_Q |T_{km}((\varphi-\lambda)^N f_2)(y) - c|^\delta\, dy\right)^{\frac{1}{\delta}}\Bigg] \\
&\le C(I + II + III).
\end{aligned}$$

To estimate I we use Hölder's inequality with $q = \frac{\eta}{\delta} > 1$ and John–Nirenberg's inequality (2-10),

$$\begin{aligned}
I &\le \sum_{j=0}^{N-1}\left(\frac{1}{|Q|}\int_Q |(\varphi(y) - \lambda)^{(N-j)\delta q'}|\, dy\right)^{\frac{1}{\delta q'}} \left(\frac{1}{|Q|}\int_Q |T^j_{km} f(y)|^{\delta q}\right)^{\frac{1}{\delta q}} dy \\
&\le C \sum_{j=0}^{N-1} \|\varphi\|^{N-j}_{BMO} M_\eta(T^j_{km} f)(x).
\end{aligned}$$

To estimate II we again use Hölder's inequality and the weak type (1, 1) of T_{km}:

$$\begin{aligned}
II &\le C(n) m^{\frac{n}{2}} \frac{1}{|Q|}\int_{2Q} |(\varphi(y) - \lambda)^N f|\, dy \\
&\le C(n) m^{\frac{n}{2}} \|\varphi - \lambda\|^N_{\operatorname{Exp} L(2Q)} \|f\|_{L \log L^m(2Q)} \\
&\le C(n) m^{\frac{n}{2}} \|\varphi\|^N_{BMO} M^{N+1} f(x).
\end{aligned}$$

To estimate the last term *III* we use the Jensen's inequality to obtain

$$\begin{aligned}
III &\le \frac{1}{|Q|}\int_Q |T_{km}(\varphi(y)-\varphi_{2Q})^N f_2)(y)-(T(\varphi-\varphi_{2Q})^N)_Q|\,dy\\
&\le \frac{1}{|Q|^2}\int_Q\int_Q\int_{\mathbb{R}^n\setminus 2Q}|K_{km}(y-\xi)-K_{km}(y-\xi)|\,|\varphi(\xi)-\varphi_{2Q}|^N|f(\xi)|\,d\xi\,dz\,dy\\
&\le \frac{cm^{n/2}}{|Q|^2}\int_Q\int_Q\int_{\mathbb{R}^n\setminus 2Q}\frac{\rho(y-z)}{\rho(x-z)^{\alpha+1}}|\varphi(\xi)-\varphi_{2Q}|^N|f(\xi)|\,d\xi\,dz\,dy\\
&\le cm^{\frac{n}{2}}\sum_{j=1}^{\infty}\frac{2^{-j}}{|2^{j+1}Q|}\int_{2^{j+1}Q}|\varphi(\xi)-\varphi_{2Q}|^N|f(\xi)|\,d\xi\\
&\le cm^{\frac{n}{2}}\sum_{j=1}^{\infty}\frac{2^{-j}}{|2^{j+1}Q|}\int_{2^{j+1}Q}|\varphi(\xi)-\varphi_{2^{j+1}Q}|^N|f(\xi)|\,d\xi\\
&\qquad+cm^{\frac{n}{2}}\sum_{j=1}^{\infty}\frac{2^{-j}}{|2^{j+1}Q|}\int_{2^{j+1}Q}|\varphi_{2^{j+1}Q}-\varphi_{2Q}|^N|f(\xi)|\,d\xi\\
&\le IV+V.
\end{aligned}$$

Using again the Hölder and John–Nirenberg inequalities, we get

$$\begin{aligned}
IV &\le cm^{n/2}\sum_{j=1}^{\infty}2^{-j}\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|\varphi(\xi)-\varphi_{2^{j+1}Q}|^N|f(\xi)|\,d\xi\\
&\le cm^{n/2}\|\varphi\|_{BMO}^N M^{N+1}(f)(x)\sum_{j=1}^{\infty}2^{-j}\le cm^{n/2}\|\varphi\|_{BMO}^N M^{N+1}(f)(x).
\end{aligned}$$

On the other hand, the inequality

$$|\varphi_{2^{j+1}Q}-\varphi_{2Q}|\le cj\|\varphi\|_{BMO}$$

implies

$$\begin{aligned}
V &\le Cm^{\frac{n}{2}}\|\varphi\|_{BMO}^N\sum_{j=0}^{\infty}\frac{j^N2^{-j}}{|2^{j+1}Q|}\int_{2^{j+1}Q}|f(\xi)|\,d\xi\\
&\le Cm^{\frac{n}{2}}\|\varphi\|_{BMO}^N M(f)(x).
\end{aligned}$$

□

Corollary 3.6. *For any* $0<p<\infty,\ w\in A_\infty$,

$$\int_{\mathbb{R}^n}|T_{km}^N f|^p w(x)\,dx\le cm^{\frac{pn}{2}}\|\varphi\|_{BMO}^{Np}\int_{\mathbb{R}^n}|M^{N+1}f(x)|^p w(x)\,dx. \tag{3-9}$$

Proof. Let f be a bounded function with compact support, i.e., $f\in L_c^\infty$. Then by [García-Cuerva and Rubio de Francia 1985, Theorem 3.1, p. 411], $M(T_{km}f)\in L^r(w)$ whenever $w\in A_r$ and $1<r<\infty$. Let $w\in A_\infty$; then $w\in A_r$ for some $1 < r < \infty$. Let $0 < r_0 < r$. Then for all $0 < \delta < r_0/r$ we have $w \in A_{r_0/\delta}$ and then $M(T_{km} f) \in L^{r_0/\delta}(w)$. It follows from the Fefferman–Stein's inequality (2-8) and Lemma 3.5 that for all $w \in A_\infty$ and all $0 < p < \infty$,

$$\begin{aligned}\int_{\mathbb{R}^n} |T_{km} f|^p w(x)\,dx &\le \int_{\mathbb{R}^n} (M(|T_{km} f|)^\delta)^{p/\delta} w(x)\,dx \\ &\le \int_{\mathbb{R}^n} (M^\#(|T_{km} f|^\delta))^{\frac{p}{\delta}} w(x)\,dx \\ &\le cm^{\frac{pn}{2}} \int_{\mathbb{R}^n} |Mf(x)|^p w(x)\,dx.\end{aligned}$$

Iterating over the last inequality, we obtain

$$\int_{\mathbb{R}^n} |T^N_{km} f|^p w(x)\,dx \le cm^{\frac{pn}{2}} \|\varphi\|^{Np}_{BMO} \int_{\mathbb{R}^n} |M^{N+1} f(x)|^p w(x)\,dx.$$

Since L^∞_c is dense in $L^p(w)$, the last inequality is true in the whole space $L^p(w)$. □

Denote $T^\star_{km} f(x) = \sup_{\varepsilon>0} |T_{km\varepsilon} f(x)|$.

Corollary 3.7 (Coltar's inequality). *For any* $0 < \delta \le 1$,

$$T^\star_{km} f(x) \le C(M_\delta(T_{km} f)(x) + m^{\frac{n}{2}} Mf(x))$$

where $C = C(n, \delta)$.

Proof. We adapt here the proof given in [Duoandikoetxea 2001, pp. 103–105]. First assume $0 < \delta < 1$ and fix $\varepsilon > 0$. Let $Q = B(x, \varepsilon/2)$ and define $f = f\chi_{2Q} + f\chi_{\mathbb{R}^n\setminus 2Q} = f_1 + f_2$. Then

$$T_{km\varepsilon} f(x) = T_{km} f_2(x).$$

If $y \in Q$, then it follows from the proof of the estimation (3-8) that

$$\begin{aligned}|T_{km} f_2(y) - T_{km} f_2(x)| &\le |T_{km} f_2(y) - (T_{km} f_2)_Q| + |T_{km} f_2(x) - (T_{km} f_2)_Q| \\ &\le Cm^{\frac{n}{2}} M(f)(x).\end{aligned}$$

It follows that

$$|T_{km\varepsilon} f(x)| \le C(m^{\frac{n}{2}} Mf(x) + |T_{km} f(y)| + |T_{km} f_1(y)|)$$

and then

$$|T_{km\varepsilon} f(x)|^\delta \le C(m^{\frac{\delta n}{2}} Mf(x)^\delta + |T_{km} f(y)|^\delta + |T_{km} f_1(y)|^\delta).$$

Integrate in y over Q, divide by $|Q|$ and raise to the power $1/\delta$, to get

$$|T_{km\varepsilon} f(x)|^\delta \le C\left(m^{\frac{n}{2}} Mf(x) + M(|T_{km} f(x)|^\delta)^{1/\delta} + \left(\frac{1}{|Q|}\int_Q |T_{km} f_1|^\delta\,dy\right)^{\frac{1}{\delta}}\right).$$

Use the estimate (3-7) to obtain

$$|T_{km\varepsilon} f(x)|^\delta \le C(m^{\frac{n}{2}} Mf(x) + M(|T_{km} f(x)|^\delta)^{\frac{1}{\delta}}).$$

The proof of Coltar's inequality when $\delta = 1$ is as in [Duoandikoetxea 2001]. □

By standard argument we obtain Proposition 3.8 as a consequence.

Proposition 3.8. *$T^\star_{km}$ is strong (p, p), $1 < p < \infty$, and is weak $(1, 1)$ with*

$$\int_{\mathbb{R}^n} |T^\star_{km} f(x)|^p \, dx \le c(n, p) m^{\frac{n}{2}} \int_{\mathbb{R}^n} |f(x)|^p \, dx \tag{3-10}$$

and

$$\left|\{x \in \mathbb{R}^n : |T^\star_{km} f(x)| > \lambda\}\right| \le \frac{C(n) m^{\frac{n}{2}}}{\lambda} \int_{\mathbb{R}^n} |f(x)| \, dx. \tag{3-11}$$

Now we will prove the following result.

Theorem 3.9. *For any $0 < p < \infty$, $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |S^\star_{km,N} f|^p w(x) \, dx \le cm^{\frac{pn}{2}} \|\varphi\|^{Np}_{BMO} \int_{\mathbb{R}^n} |M^{N+1} f(x)|^p w(x) \, dx \tag{3-12}$$

with $S_{km,j} = T^j_{km}$.

To prove Theorem 3.9, we need a pointwise estimate of $S^\star_{km,j} f$. To obtain this, we use some technical results used in [Rubio de Francia et al. 1986].

Lemma 3.10. *Let $\psi \in C^1(\mathbb{R})$ such that*

$$\operatorname{supp}(\psi) \subset \left\{t \in \mathbb{R} : \tfrac{1}{2} \le t \le 3\right\}, \quad 0 \le \psi(t) \le 1,$$

and put $\varphi(x) = 1 - \psi(x)$, then both the kernels

$$\psi_{km\varepsilon}(x-y) = K_{km}(x-y)\psi\left(\frac{\rho(x-y)}{\varepsilon}\right),$$

$$\varphi_{km\varepsilon}(x-y) = K_{km}(x-y)\varphi\left(\frac{\rho(x-y)}{\varepsilon}\right)$$

satisfy the Hörmander's condition uniformly in ε and are bounded uniformly in ε by $(Cm^n/2)/(\rho^\alpha(x-y))$.

Proof. We have from the definition of ρ,

$$\begin{aligned}\partial_j \rho(x) &= \frac{x_j \rho^{-2\alpha_j}(x)}{\sum_1^n \alpha_i \rho^{-2\alpha_i}(x) x_i^2} \rho(x) \le C|x_j| \rho^{-2\alpha_j+1}(x) \\ &\le C\rho^{-\alpha_j+1}(x),\end{aligned}$$

since $|x_j| \le \rho(x)^{\alpha_j}$ and $\sum_1^n \alpha_i \rho^{-2\alpha_i}(x) x_i^2 \ge \min \alpha_i$. Let φ_ε be either $\psi\big(\frac{\cdot}{\varepsilon}\big)$ or $\varphi\big(\frac{\cdot}{\varepsilon}\big)$, $y \in Q = Q(x_0, r)$ and $z \notin 2Q$. Using the last estimate, we obtain

$$\begin{aligned}|\varphi_\varepsilon(z-y) - \varphi_\varepsilon(z-x_0)| &\le C\frac{|y-x_0|}{\varepsilon}\varphi'\left(\frac{\rho(c)}{\varepsilon}\right)|\nabla\rho(c)| \\ &\le C|y-x_0|\frac{|\nabla\rho(c)|}{\rho(c)} \\ &\le C\rho(y-x_0)^\alpha \rho^{-\alpha}(c),\end{aligned}$$

with $c = (1-t)(z-y) + t(z-x_0)$ for some $t \in (0, 1)$. On the other hand, one can check that $2\rho(c) \ge \rho(z-x_0)$. Thus

$$\begin{aligned}|\varphi_\varepsilon(z-y) - \varphi_\varepsilon(z-x_0)| &\le C\rho(y-x_0)^\alpha \rho^{-\alpha}(z-x_0) \\ &\le C\rho(y-x_0)^{\alpha-1}\rho^{-\alpha+1}(z-x_0)\frac{\rho(y-x_0)}{\rho(z-x_0)} \\ &\le C\frac{\rho(y-x_0)}{\rho(z-x_0)}.\end{aligned}$$

By this and the properties of the kernel K_{km}, we have

$$\begin{aligned}&|\varphi_{km\varepsilon}(z-y) - \varphi_{km\varepsilon}(z-x_0)| \\ &\quad\le |\varphi_\varepsilon(z-y)|\,|(K_{km}(z-y) - K_{km}(z-x_0)| + |K_{km}(z-x_0)|\,|\varphi_\varepsilon(z-y) - \varphi_\varepsilon(z-x_0)| \\ &\quad\le Cm^{n/2}\frac{\rho(y-x_0)}{\rho(z-x_0)^{\alpha+1}}.\end{aligned}$$

□

Define

$$\begin{aligned}\Psi^\star_{km,j} f(x) &= \sup_{\varepsilon>0} \Psi_{km\varepsilon,j} f(x) \\ &= \sup_{\varepsilon>0} \int_{\mathbb{R}^n} |\psi_{km\varepsilon}(x-y)[\varphi(x)-\varphi(y)]^j f(y)|\,dy, \end{aligned} \tag{3-13}$$

$$\begin{aligned}\Phi^\star_{km,j} f(x) &= \sup_{\varepsilon>0} |\Phi_{km\varepsilon,j} f(x)| \\ &= \sup_{\varepsilon>0} \left| \int_{\mathbb{R}^n} \varphi_{km\varepsilon}(x-y)[\varphi(x)-\varphi(y)]^j f(y)\,dy \right|. \end{aligned} \tag{3-14}$$

Then we can see that

$$S^\star_{km,j} f(x) \le \Psi^\star_{km,j} f(x) + \Phi^\star_{km,j} f(x). \tag{3-15}$$

Lemma 3.11. *For any* $0 < \delta < \eta < 1$, *there exists a constant* $C = C(n, \delta, \eta)$ *such that*

$$M^\#_\delta(\Psi^\star_{km} f)(x) \le Cm^{\frac{n}{2}} M(f)(x)$$

and

$$M^{\#}_{\delta}(\Psi^{\star}_{km,N}f)(x) \le Cm^{n/2}\Big[\|\varphi\|^{N}_{BMO}M^{N+1}(f)(x) + \sum_{j=0}^{N-1}\|\varphi\|^{N-j}_{BMO}M_{\eta}(\Psi^{\star}_{km,j}f)(x)\Big]$$

with $\Psi^{\star}_{km} = \Psi^{\star}_{km,0}$.

Lemma 3.12. *For any* $0 < \delta < \eta < 1$, *there exists a constant* $C = C(n, \delta, \eta)$ *such that*

$$M^{\#}_{\delta}(\Phi^{\star}_{km}f)(x) \le Cm^{\frac{n}{2}}M(f)(x)$$

and

$$M^{\#}_{\delta}(\Phi^{\star}_{km,N}f)(x) \le Cm^{n/2}\Big[\|\varphi\|^{N}_{BMO}M^{N+1}(f)(x) + \sum_{j=0}^{N-1}\|\varphi\|^{N-j}_{BMO}M_{\eta}(\Phi^{\star}_{km,j}f)(x)\Big]$$

with $\Phi^{\star}_{km} = \Phi^{\star}_{km,0}$.

Proof. We only prove Lemma 3.11, the proof of Lemma 3.12 is similar. We note first that

$$\begin{aligned}\Psi^{\star}_{km}f(x) &\le \sup_{\varepsilon>0}\int_{\varepsilon/2<\rho(y-x)<3\varepsilon}|K_{km}(x-y)f(y)|\,dy\\ &\le \sup_{\varepsilon>0}Cm^{n/2}\varepsilon^{-\alpha}\int_{\rho(y-x)<3\varepsilon}|f(y)|\,dy \le Cm^{n/2}Mf(x).\end{aligned}$$

It follows that $\Psi^{\star}_{km}$ is of weak type (1, 1) and satisfies

$$\big|\{x\in\mathbb{R}^n : |\Psi^{\star}_{km}f(x)| > \lambda\}\big| \le \frac{C(n)m^{\frac{n}{2}}}{\lambda}\int_{\mathbb{R}^n}|f(x)|\,dx. \tag{3-16}$$

Put $Q = B(x, r)$ and

$$f = f\chi_{2Q} + f\chi_{\mathbb{R}^n\setminus 2Q} = f_1 + f_2.$$

The Kolmogorov inequality and (3-16) tell us that

$$\left(\frac{1}{|Q|}\int_{Q}|\Psi^{\star}_{km}f_1(y)|^{\delta}\,dy\right)^{1/\delta} \le Cm^{n/2}Mf(x). \tag{3-17}$$

On the other hand, since $T^{\star}_{km}$ is finite almost everywhere, we can choose $x_0 \in Q$ such that $\Psi^{\star}_{km}f_2(x_0) < \infty$. From the definition of the supremum, for any $y \in Q$,

$$\begin{aligned}|\Psi^{\star}_{km}f_2(y) - \Psi^{\star}_{km}f_2(x_0)| &\le \sup_{\varepsilon>0}|\Psi_{km\varepsilon}f_2(y) - \Psi_{km\varepsilon}f_2(x_0)|\\ &\le \sup_{\varepsilon>0}\int_{\mathbb{R}^n\setminus 2Q}|(\psi_{km\varepsilon}(y-z) - \psi_{km\varepsilon}(x_0-z))|\,dz.\end{aligned}$$

For any $\varepsilon > 0$ we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus 2Q} &|(\psi_{km\varepsilon}(y-z)-\psi_{km\varepsilon}(x_0-z))||f(z)|\,dz\\ &\le \sum_{j=1}^{\infty}\int_{2^{j+1}Q\setminus 2^jQ}|(\psi_{km\varepsilon}(y-z)-\psi_{km\varepsilon}(x_0-z))|\,|f(z)|\,dz\\ &\le Cm^{n/2}\sum_{j=1}^{\infty}\int_{2^{j+1}Q)\setminus 2^jQ}\frac{\rho(x_0-y)}{\rho(x_0-z)^{\alpha+1}}|f(z)|\,dz\\ &\le Cm^{n/2}\sum_{j=1}^{\infty}\frac{r}{(2r)^{j(\alpha+1)}}\int_{2^{j+1}Q)\setminus 2^jQ}|f(z)|\,dz\\ &\le Cm^{n/2}Mf(x).\end{aligned}$$

To prove the second estimate in Lemma 3.11, we proceed, with a slight modification, as in the proof of the estimate (3-6). Set $\lambda=\varphi_{2Q}$ and $c=c_\varepsilon,\ \varepsilon>0$, with

$$c_\varepsilon=\Psi_{km\varepsilon}(\varphi-\lambda)^N)f_2)(x_0)$$

and $x_0\in Q$ such that

$$\Psi^\star_{km}(\varphi-\lambda)^N)f_2)(x_0)<\infty.$$

Then with the same notation as before

$$\begin{aligned}\biggl(\frac{1}{|Q|}\int_Q\bigl||\Psi_{km\varepsilon,N}f(y)|^\delta-|c_\varepsilon|^\delta\bigr|\,dy\biggr)^{\frac{1}{\delta}}&\\ \le C\biggl[\sum_{j=0}^{N-1}&\biggl(\frac{1}{|Q|}\int_Q|\varphi(y)-\lambda|^{(N-j)\delta}|\Psi_{km\varepsilon,j}f(y)|^\delta\,dy\biggr)^{\frac{1}{\delta}}\\ &+\biggl(\frac{1}{|Q|}\int_Q|\Psi_{km\varepsilon}((\varphi-\lambda)^Nf)(y)|^\delta\,dy\biggr)^{\frac{1}{\delta}}\\ &+\biggl(\frac{1}{|Q|}\int_Q|\Psi_{km\varepsilon}((\varphi-\lambda)^Nf)(y)-c_\varepsilon|^\delta\,dy\biggr)^{\frac{1}{\delta}}\biggr].\end{aligned}$$

Since $\Psi_{km\varepsilon}$ is nonnegative and

$$\bigl||a+b|-|c|\bigr|\le|a|+|b-c|,$$

it follows that

$$\begin{aligned}|\Psi_{km\varepsilon}((\varphi-\lambda)^Nf)(y)-c_\varepsilon|&\\ =\bigl||\Psi_{km\varepsilon}&((\varphi-\lambda)^Nf_1)(y)+\Psi_{km\varepsilon}((\varphi-\lambda)^Nf_2)(y)|-|c_\varepsilon|\bigr|\\ \le|\Psi_{km\varepsilon}&((\varphi-\lambda)^Nf_1)(y)|+|\Psi_{km\varepsilon}((\varphi-\lambda)^Nf_2)(y)-c_\varepsilon|.\end{aligned}$$

This implies

$$\begin{aligned}\Big(\frac{1}{|Q|}\int_Q \big||\Psi_{km\varepsilon,N}f(y)|^\delta-|c_\varepsilon|^\delta\big|\,dy\Big)^{\frac{1}{\delta}}&\\ \le C\Big[\sum_{j=0}^{N-1}\Big(\frac{1}{|Q|}\int_Q|\varphi(y)-\lambda|^{(N-j)\delta}|\Psi_{km\varepsilon,j}f(y)|^\delta\,dy\Big)^{\frac{1}{\delta}}&\\ +\Big(\frac{1}{|Q|}\int_Q|\Psi_{km\varepsilon}((\varphi-\lambda)^N f_1)(y)|^\delta\,dy\Big)^{\frac{1}{\delta}}&\\ +\Big(\frac{1}{|Q|}\int_Q|\Psi_{km\varepsilon}((\varphi-\lambda)^N f_2)(y)-c_\varepsilon|^\delta\,dy\Big)^{\frac{1}{\delta}}\Big]&\\ \le C(I+II+III).&\end{aligned}$$

By definition of $\Psi^\star_{km,j}$, we have

$$\begin{aligned}I&\le\sum_{j=0}^{N-1}\Big(\frac{1}{|Q|}\int_Q|\varphi(y)-\lambda|^{(N-j)\delta}|\Psi^\star_{km,j}f(y)|^\delta\,dy\Big)^{\frac{1}{\delta}}\\ &\le C\sum_{j=0}^{N-1}\|\varphi\|_{BMO}^{N-j}M_\eta(\Psi^\star_{km,j}f)(x).\end{aligned}$$

Arguing as in the proof of Lemma 3.5 we obtain

$$\begin{aligned}II&\le Cm^{\frac{n}{2}}\|\varphi\|_{BMO}^N M^{N+1}f(x)\\ III&\le Cm^{\frac{n}{2}}(\|\varphi\|_{BMO}^N M^{N+1}f(x)+\|\varphi\|_{BMO}Mf(x)).\end{aligned}$$

All those estimations are uniformly in ε. □

Proof of Theorem 3.9. The proof is an immediate consequence of Lemmas 3.11 and 3.12. □

Proof of Theorem 1.5. Note that Theorem 3.9 implies the convergence of $T^j_{km\varepsilon}$ in $L^p(w)$ with $w\in A_\infty$. Using the series expansion of $K(x,y)$, Lemma 3.2, estimation (3-2) and Theorem 3.9 we get

$$\begin{aligned}\|(T^N)^\star f\|_{L^p(w}&\le C\|\varphi\|_{BMO}^N\|M^{N+1}f\|_{L^p(w)}\sum_{m=1}^{+\infty}m^{n/2-2r+n-2}\\ &\le C\|\varphi\|_{BMO}^N\|M^{N+1}f\|_{L^p(w)}\end{aligned}$$

since $r>\frac{3n}{4}-\frac{1}{2}>\frac{3n}{4}-1$. We conclude that the series is absolutely convergent in $L^p(w)$. Set

$$\lim_{\varepsilon\to0}T^N_{km\varepsilon}f=T^N_{km}f,\quad T^Nf=\sum_{m=1}^{+\infty}\sum_{k=1}^{d_m}a_{km}T^N_{km}f;$$

then we have

$$\|T^N f\|_{L^p(w)} \le C\|\varphi\|_{BMO}^N \|M^{N+1} f(x)\|_{L^p(w)}.$$

The second inequality of Theorem 1.5 is obvious. □

4. Weighted estimates in Morrey space

Let $h(x,r)$ be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$ satisfying

$$h(x,2r) \le Ch(x,r) \quad \text{for all } r \in \mathbb{R}^+, \tag{4-1}$$

where $1 \le C = C_h < 2^\alpha$ is a constant independent of r and x.

Definition 4.1. We say, for $0 < p < \infty$, that a locally integrable function f is in the generalized weighted Morrey space $L_h^p(w)$ whenever

$$\|f\|_{L_h^p(w)} = \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{h(x,r)} \int_{B(x,r)} |f(y)|^p w(y)\, dy \right)^{\frac{1}{p}} < \infty.$$

Theorem 4.2. *For any* $1 < p < \infty,\ w \in A_\infty$,

$$\|T^N f\|_{L_h^p(w)} \le C\|\varphi\|_{BMO}^N \|f\|_{L_h^p(w)}. \tag{4-2}$$

Theorem 4.2 is a direct consequence of following lemma.

Lemma 4.3. *Let* $w \in A_\infty$ *and* $0 < p < \infty$; *then*

$$\|Mf\|_{L_h^p(w)} \le C\|M^\# f\|_{L_h(w)}^p \tag{4-3}$$

and for a sufficiently small $\delta > 0$,

$$\|M_\delta f\|_{L_h^p(w)} \le C\|f\|_{L_h(w)}^p. \tag{4-4}$$

If $0 < \delta \le p \le \infty$ *and* $w \in A_{p/\delta}$, *then*

$$\|M_\delta f\|_{L_h^p(w)} \le C\|f\|_{L_h(w)}^p. \tag{4-5}$$

Proof. Let $w \in A_\infty$. Then there is some $q > 1$ such that $w \in A_q$. The Jones factorization theorem tells that $w \in A_q$ if and only if there exist $w_1, w_2 \in A_1$ such that $w = w_1^{1-q} w_2$; see [García-Cuerva and Rubio de Francia 1985]. On the other hand, $w_2 \in A_1$ if and only $w_2 = k(x)u_2(x)$ with $u_2 \in A_1$ and k is a positive function such that $k,\ \frac{1}{k} \in L^\infty$. Thus, we can write, to simplify notation, $w = w_1^{1-q} w_2$ with $w_2,\ \frac{1}{w_2,} \in L^\infty$. Now, fix a ball Q centered at x, with radius r. Then we may assume $M(\chi_Q w_2)$ is still in A_1 with A_1 constant independent of Q.

Using the Fefferman–Stein's inequality, we obtain

$$\begin{aligned}
\int_Q |Mf(z)|^p w(z)\,dz &= \int_Q |Mf(z)|^p w_1^{1-q}(z) w_2(z)\,dz \\
&\le \int_{\mathbb{R}^n} |Mf(z)|^p w_1^{1-q}(z) M(\chi_Q w_2)(z)\,dz \\
C &\le \int_{\mathbb{R}^n} |M^\# f(z)|^p w_1^{1-q}(z) M(\chi_Q w_2)(z)\,dz \\
C &\le \int_{2Q} |M^\# f(z)|^p w_1^{1-q}(z) M(\chi_{2Q} w_2)(z)\,dz \\
&\qquad + C\int_{\mathbb{R}^n\setminus 2Q} |M^\# f(z)|^p w_1^{1-q}(z) M(\chi_Q w_2)(z)\,dz \\
&= J_1 + J_2.
\end{aligned}$$

Since $w_2 \in A_1$, we have

$$\begin{aligned}
J_1 &\le C\int_{2Q} |M^\# f(z)|^p w_1^{1-q}(z) M w_2(z)\,dz \\
&\le C\int_{2Q} |M^\# f(z)|^p w_1^{1-q}(z) w_2(z)\,dz \\
&\le C\frac{h(x,2r)}{h(x,2r)}\int_{2Q} |M^\# f(z)|^p w(z)\,dz \le C h(x,r)\|M^\# f\|^p_{L^p_h(w)}.
\end{aligned}$$

To estimate J_2, we observe that if B is any ball in $\mathbb{R}^n$, then for $z \in (2Q)^c,\ z \in B$, and $Q \cap B \neq \varnothing$, we have $\rho(z-x)^\alpha \le C|B|$. Thus

$$\frac{1}{|B|}\int_{Q\cap B} w_2(y)\,dy \le C\rho(z-x)^{-\alpha}\int_Q w_2(y)\,dy. \tag{4-6}$$

It follows from (4-6) that

$$\begin{aligned}
J_2 &\le C\int_{\mathbb{R}^n\setminus 2Q} |M^\# f(z)|^p w_1^{1-q}(z) \sup_{z\in B}\left(\frac{1}{|B|}\int_Q w_2(y)\,dy\right) dz \\
&\le C\int_{\mathbb{R}^n\setminus 2Q} |M^\# f(z)|^p w_1^{1-q}(z) \rho(z-x)^{-\alpha}|Q| w_2(z) \\
&\le C\sum_{j=0}^{\infty}\int_{2^{j+2}Q\setminus 2^{j+1}Q} |M^\# f(z)|^p w_1^{1-q}(z)\frac{|Q|}{|2^{j+2}Q|} w_2(z)\,dz \\
&\le C\sum_{j=0}^{\infty} 2^{-j\alpha}\int_{2^{j+2}Q} |M^\# f(z)|^p w_1^{1-q}(z) w_2(z)\,dz \\
&\le C\sum_{j=0}^{\infty} 2^{-j\alpha}\int_{2^{j+2}Q} |M^\# f(z)|^p w_1^{1-q}(z) w_2(z)\,dz
\end{aligned}$$

$$\le C\sum_{j=0}^{\infty} 2^{-j\alpha} h(x, 2^{j+2}r)\frac{1}{h(x, 2^{j+2}r)}\int_{2^{j+2}Q} |M^{\#} f(z)|^p w(z)\,dz$$
$$\le C\|f\|^p_{L^p_h(w)}\sum_{j=0}^{\infty}(2^{-\alpha}C_h)^j h(x,r) \le Ch(x,r)\|M^{\#} f\|^p_{L^p_h(w)}.$$

Therefore,

$$\frac{1}{h(x,r)}\int_Q |Mf(z)|^p w(z)\,dz \le C\|M^{\#} f\|^p_{L^p_h(w)}.$$

This last estimate implies (4-3). To prove (4-4) we choose $\delta > 0$ small enough so that $w \in A_q$ with $q = p/\delta$. It follows from the Fefferman–Stein inequality that

$$\begin{aligned}\int_Q |M(f^\delta)(z)|^{\frac{p}{\delta}} w(z)\,dz &= \int_Q |M(f^\delta)(z)|^{\frac{p}{\delta}} w_1^{1-q}(z) w_2(z)\,dz\\ &\le \int_{\mathbb{R}^n} |M(f^\delta)(z)|^{\frac{p}{\delta}} w_1^{1-q}(z) M(\chi_Q w_2)(z)\,dz\\ &\le C\int_{\mathbb{R}^n} |f(z)|^p w_1^{1-q}(z) M(\chi_Q w_2)(z)\,dz\\ &\le Ch(x,r)\|f\|^p_{L^p_h(w)}.\end{aligned}$$

The proof of (4-5) is similar. □

5. Application

Definition 5.1. Let ν be in the Schwartz space with supp $\hat{\nu}$ contained in an annulus about the origin and

$$\sum_{j\in\mathbb{Z}} \hat{\nu}(2^{-j}\xi) = 1 \quad \text{for all } \xi \ne 0.$$

Let $w \in A_\infty$, $0 < p, q \le \infty$, and $\gamma \in \mathbb{R}$.

- The homogeneous Triebel–Lizorkin space $\dot{F}^{\gamma,q}_p$ is the set of all distributions f (modulo polynomials) such that

$$\|f\|_{\dot{F}^{\gamma,q}_{p,w}} = \left\| \left(\sum_{j\in\mathbb{Z}} 2^{j\gamma q} |\nu_{2^{-j}} \star f|^q\right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty,$$

- the inhomogeneous Triebel–Lizorkin space $F^{\gamma,q}_p$ is the set of all distributions f (modulo polynomials) such that

$$\|f\|_{F^{\gamma,q}_{p,w}} = \|\mu \star f\|_{p,w} + \left\| \left(\sum_{j\ge 1} 2^{j\gamma q} |\nu_{2^{-j}} \star f|^q\right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty,$$

where μ is a Schwartz function satisfying

$$\hat{\mu}(\xi)+\sum_{j\geq 1}\hat{\nu}(2^j\xi)=1 \quad \text{for all } \xi\in\mathbb{R}^n.$$

Here, $\hat{\nu}$ denotes the Fourier transform of ν and $\nu_t(x)=t^{-n}\nu\left(\frac{x}{t}\right)$. Note that the Triebel–Lizorkin spaces are independent of the choices of ν; see for instance [Qui 1982].

Proposition 5.2. *Let $1<p,q<\infty$ and let T be as in Theorem 1.3. If T is a convolution operator then*

$$\|Tf\|_{\dot{F}^{\gamma,q}_{p,w}}\leq C\|f\|_{\dot{F}^{\gamma,q}_{p,w}}$$

and

$$\|Tf\|_{F^{\gamma,q}_{p,w}}\leq C\|f\|_{F^{\gamma,q}_{p,w}}.$$

Proof. Since T commutes with convolution,

$$\begin{aligned}\left\|\left(\sum_j 2^{j\gamma q}|\nu_{2^{-j}}\star Tf|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)}&=\left\|\left(\sum_j 2^{j\gamma q}|T(\nu_{2^{-j}}\star f)|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)}\\&\leq C\left\|\left(\sum_j 2^{j\gamma q}|\nu_{2^{-j}}\star f|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)}.\end{aligned}$$

In the last step we have used Corollary 1.7. It follows that

$$\|Tf\|_{\dot{F}^{\gamma,q}_{p,w}}\leq C\|f\|_{\dot{F}^{\gamma,q}_{p,w}}.$$

A similar argument leads to

$$\|Tf\|_{F^{\gamma,q}_{p,w}}\leq C\|f\|_{F^{\gamma,q}_{p,w}}.$$ □

Proposition 5.3. *Let $1<p,q<\infty$ and let T be a C–Z operator with kernel K satisfying*

$$\sup_{x\in\mathbb{R}^n}\sup_{y\in S^{n-1}}|\partial_x^\sigma\partial_y^\beta K(x,y)|=M<+\infty$$

for all multi-indices β,σ such that $|\beta|+|\sigma|<r$, with $r>\frac{3n}{4}-\frac{1}{2}$. Then

$$\|Tf\|_{F^{\gamma,q}_{p,w}}\leq C\|f\|_{F^{\gamma,q}_{p,w}}.$$

The proof of Proposition 5.3 is based on the following lemmas.

Lemma 5.4 (boundedness of the multiplication operator [Rychkov 2001]). *Let $a\in C^N(\mathbb{R}^n)$ and $w\in A_\infty$. Assume that*

$$\|\partial^\sigma a\|_\infty\leq C_N \quad \textit{for all } |\sigma|\leq N,$$

then we have

$$\|af\|_{F^{\beta,r}_{p,w}} \le C_N \|f\|_{F^{\beta,r}_{p,w}} \quad \text{for all } f \in F^{\beta,r}_{p,w}.$$

Lemma 5.5. *Let $r \in \mathbb{N}_0$ and assume*

$$\sup_{x\in\mathbb{R}^n} \sup_{y\in S^{n-1}} |\partial_x^\sigma \partial_y^\beta K(x,y)| = M < +\infty \tag{5-1}$$

for all multi-indices β, σ such that $|\beta| + |\sigma| < r$. Then

$$\|\partial^\sigma a_{km}\|_{L^\infty} \le C(n) M_r m^{-2r}.$$

Proof. We have

$$\begin{aligned} a_{km}(x) &= (-m)^{-r}(m+n-2)^{-r} \int_{S^{n-1}} K(x,z) L^r Y_{km}\, d\sigma_z \\ &= (-m)^{-r}(m+n-2)^{-r} \int_{S^{n-1}} L^r K(x,z) Y_{km}\, d\sigma_z. \end{aligned}$$

Then Hölder's inequality and (5-1) lead to

$$\|\partial^\sigma a_{km}\|_{L^\infty} \le C(n) M_r m^{-2r} \left(\int_{S^{n-1}} |Y_{km}|^2 \, d\sigma_z \right)^{\frac{1}{2}} \le C(n) M_r m^{-2r}. \qquad \square$$

Proof of Proposition 5.3. We conclude using Proposition 5.2, Lemmas 5.4 and 5.5 and arguing as in the proof of Theorem 1.5. □

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