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ON THE COMPACTNESS OF COMMUTATORS OF HARDY OPERATORS

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We focus on the need for the compactness characterizations of the commutators of Hardy operators. More precisely, we prove that the commutators of Hardy operators, including the fractional Hardy operator, are compact operators on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ spaces if and only if the symbol functions of the commutators belong to $\mathrm{CVMO}(\mathbb{R}^n)$ spaces (the central $\mathrm{BMO}(\mathbb{R}^n)$ closure of $C_c^\infty(\mathbb{R}^n)$).

1. Introduction

For a locally integrable function b and an operator T, the commutator formed by T and b can be defined by

$$[b, T]f := b(Tf) - T(bf).$$

In the literature, b is also called the symbol function of $[b, T]$. The pioneer work on $[b, T]$ when T belongs to a class of nonconvolution operators and $b \in \mathrm{BMO}(\mathbb{R}^n)$ can be traced to Coifman, Rochberg and Weiss [Coifman et al. 1976], the well-known result of which is a new characterization of $\mathrm{BMO}(\mathbb{R}^n)$ via the boundedness of $[b, T]$. For any ball $B \subset \mathbb{R}^n$, $\mathrm{BMO}(\mathbb{R}^n)$ is the mean oscillation function space defined via the norm

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| \, dx \quad \text{with} \quad b_B = \frac{1}{|B|} \int_B b(x) \, dx.$$

Commutator theory, especially the boundedness of $[b, T]$ on different function spaces, is now being applied to a variety of subjects, such as the regularity of

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solutions to elliptic equations [Bramanti and Cerutti 1993; Chiarenza et al. 1993], and the characterization of function spaces [Janson 1978; Paluszyński 1995; Shi and Lu 2013; 2015]. This article results from a study of $[b, T]$ when T is the higher-dimensional Hardy operator H, which was first introduced by Christ and Grafakos [1995],

$$Hf(x) = \frac{1}{|x|^n} \int_{|y|<|x|} f(y)\,dy, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

The dual operator of H is defined as

$$H^* f(x) = \int_{|y|\geq|x|} \frac{f(y)}{|y|^n}\,dy,$$

which can be understood as

$$\int_{\mathbb{R}^n} g(x) Hf(x)\,dx = \int_{\mathbb{R}^n} f(x) H^* g(x)\,dx \quad \text{for a suitable function } g.$$

The commutators of H and H^* can be written as

$$[b, H]f(x) = \frac{1}{|x|^n} \int_{|y|<|x|} (b(x) - b(y)) f(y)\,dy, \quad x \in \mathbb{R}^n \setminus \{0\},$$

and

$$[b, H^*]f(x) = \int_{|y|\geq|x|} \frac{f(y)}{|y|^n} (b(x) - b(y))\,dy.$$

The commutator $[b, H]$ (resp. $[b, H^*]$) shares the same boundedness as that of H (resp. H^*) on $L^p(\mathbb{R}^n)$ — the usual Lebesgue space on $\mathbb{R}^n$ with the norm

$$\|f\|^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f(x)|^p\,dx \quad \text{for } 1 < p < \infty.$$

As a class of the classical average operator, the Hardy operator, the integral inequalities of which were first studied by Hardy [1920], plays an important role in probability theory [Lifshits and Linde 2002], interpolation theory [Holmstedt 1970] and the embedding theory of function spaces [Edmunds et al. 1997; Lang and Edmunds 2011]. Since 1920, a considerable amount of research has been done to estimate H and $[b, H]$; see, for example, [Andersen and Muckenhoupt 1982; Christ and Grafakos 1995; Golubov 1997; Hardy et al. 1934; Komori 2003a; Sawyer 1984; Stein and Weiss 1971]. Since the Hardy operators are centrosymmetric, the function spaces, which are characterized by the boundedness of $[b, H]$ and $[b, H^*]$, are central ones. In [Fu et al. 2007], it was shown that

$$b \in \mathrm{CBMO}^{\max\{p,p'\}}(\mathbb{R}^n) \Leftrightarrow \text{both } [b, H] \text{ and } [b, H^*] \text{ are bounded on } L^p(\mathbb{R}^n) \text{ with } \frac{1}{p} + \frac{1}{p'} = 1,$$

where the space $\mathrm{CBMO}^q(\mathbb{R}^n)$ is the central $\mathrm{BMO}(\mathbb{R}^n)$ space first introduced by Lu and Yang [1995] via the norm

$$\|b\|_{\mathrm{CBMO}^q(\mathbb{R}^n)} = \sup_{r>0}\left(\frac{1}{|B_r|}\int_{B_r}|b(x)-b_{B_r}|^q\,dx\right)^{1/q} \quad \text{with } 1 \le q < \infty.$$

Here, $B_r := B(0, r)$ is a ball with center at 0 and a radius r. $\mathrm{CBMO}^q(\mathbb{R}^n)$ can be regarded as a local version of the $\mathrm{BMO}(\mathbb{R}^n)$ space at the origin and can be understood as $\mathrm{BMO}(\mathbb{R}^n) \subset \mathrm{CBMO}^q(\mathbb{R}^n)$ for $1 < q < \infty$. It is well known that $\mathrm{BMO}(\mathbb{R}^n) = \mathrm{BMO}^q(\mathbb{R}^n)$ $(1 < q < \infty)$. However, $\mathrm{CBMO}^r(\mathbb{R}^n) \subset \mathrm{CBMO}^q(\mathbb{R}^n)$ for $1 \le q < r < \infty$ according to [Komori 2003b]. Therefore, the behavior of $\mathrm{CBMO}^q(\mathbb{R}^n)$ may be quite different from that of $\mathrm{BMO}(\mathbb{R}^n)$. For example, there is no analog of the famous John–Nirenberg inequality of $\mathrm{BMO}(\mathbb{R}^n)$ for $\mathrm{CBMO}^q(\mathbb{R}^n)$.

The earliest study on the compactness of operators can be traced to Uchiyama [1978], where the characterization of the L^p-compactness of $[b, T]$ was obtained for the case when T is the classical Calderón–Zygmund singular integral operator and $b \in \mathrm{VMO}(\mathbb{R}^n)$, the $\mathrm{BMO}(\mathbb{R}^n)$ closure of $C_c^\infty(\mathbb{R}^n)$ (the space of all functions being infinite-times continuously differential in $\mathbb{R}^n$ with compact support). Since then, the study of compactness for commutators on different function spaces and their applications (for example, the application in PDEs [Iwaniec and Sbordone 1998; Palagachev and Softova 2004]) has been a basic component of harmonic analysis. If T is the multiplication operator, then [Beatrous and Li 1993] shows the compactness of $[b, T]$ on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ when b is in an appropriately BMO space, and gives applications to Hankel-type operators on Bergman spaces. Chen and Ding [2010] proved that $[b, T]$ is a compact operator on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ if and only if $b \in \mathrm{VMO}(\mathbb{R}^n)$ with T being the parabolic singular integral. For generalized Toeplitz operators (including the singular integral operator and multiplication operator), Krantzl and Li [2001] developed the compactness theory on $L^p(\mathbb{R}^n)$. Furthermore, as applications, they formulated some characterization theorems for the compactness of $[b, T]$ on holomorphic Hardy spaces. For a Morrey space, Chen, Ding and Wang consider the compactness of $[b, T]$ when T is the Riesz potential [Chen et al. 2009] and T is the singular integral operator [Chen et al. 2012], and give some characterizations of $\mathrm{VMO}(\mathbb{R}^n)$ via the compactness of $[b, T]$. Ding and Mei [2015] showed the compactness of $[b, T]$ for bilinear Calderón–Zygmund operators, bilinear fractional integrals and bilinear pseudodifferential operators on Morrey spaces.

Generally, the existing results are all related to singular integral operators. For the average operators, one of the three most important operators in harmonic analysis due to Stein [1993], as a concept of highly independent interest, has received little attention to the best of our knowledge. The aim of this paper is to explore the compactness of $[b, H]$ and $[b, H^*]$ on the $L^p(\mathbb{R}^n)$ space. In fact, we obtain:

Theorem 1.1. *Let* $1 < p < \infty$. *Then*,

$$b \in \mathrm{CVMO}(\mathbb{R}^n) \Leftrightarrow \textit{both } [b, H] \textit{ and } [b, H^*] \textit{ are compact on } L^p(\mathbb{R}^n).$$

Here, $\mathrm{CVMO}(\mathbb{R}^n)$ *denotes the* $\mathrm{CBMO}(\mathbb{R}^n)$ *closure of* $C_c^\infty(\mathbb{R}^n)$.

Remark 1.2. Theorem 1.1 can be seen as a first work on the problem of central function space characterization via the compactness of the commutator of the classical Hardy operator. This theorem makes up for the compactness results of average integral operators and enriches the characterization theory of central function spaces via the compactness of operators.

Remark 1.3. It is easy to see that the structure of a Hardy operator is centrally symmetric, being quite different from that of other operators (including the singular integral operator, Riesz potential and bilinear operator). This might be the reason why the existing methods (for example, the John–Nirenberg inequality) used to address the singular integral operator can not be applied when considering the Hardy operator. In this paper, we prove Theorem 1.1 using some new ideas exploiting the center symmetry of the Hardy operator and function space.

Section 2 provides the proof of Theorem 1.1. In Section 3, we show the compactness characterization of the fractional Hardy operator on $L^p(\mathbb{R}^n)$.

Throughout this paper, we utilize C to express a positive constant that is independent from the main parameters, but may vary from line to line. The symbol $A \lesssim B$ means $A \le CB$. Moreover, $A \simeq B$ whenever $A \lesssim B$ and $B \lesssim A$. We denote by $\mathbb{Z}$ and $B(x, r)$ the set of all integers and the ball centered at x with radius $r > 0$, respectively, and $B_r := B(0, r)$, $B_k := B_{2^k}$ with $k \in \mathbb{Z}$ for short.

2. The compactness of Hardy operator

In this section, we verify Theorem 1.1. To do so, we need the lemmas given below. The first pertains to the properties of the $\mathrm{CBMO}(\mathbb{R}^n)$ space.

Lemma 2.1. *The space* $\mathrm{CBMO}(\mathbb{R}^n)$ *has the following properties*:

(1) $L^\infty(\mathbb{R}^n) \subset \mathrm{CBMO}(\mathbb{R}^n)$ *and* $\|b\|_{\mathrm{CBMO}(\mathbb{R}^n)} \le 2\|b\|_{L^\infty(\mathbb{R}^n)}$.

(2) *Assume that there exists* $C > 0$ *such that for all balls* $B_r \subset \mathbb{R}^n$, *there exists a constant* c *satisfying*

$$\sup_r \frac{1}{B_r} \int_{B_r} |b(x) - c|\, dx \le C.$$

Then, $b \in \mathrm{CBMO}(\mathbb{R}^n)$ *and* $\|b\|_{\mathrm{CBMO}(\mathbb{R}^n)} \le 2C$.

(3) $\|b\|_{\mathrm{CBMO}(\mathbb{R}^n)} \simeq \sup_r \inf_{c\in\mathbb{R}} 1/B_r \int_{B_r} |b(x) - c|\, dx$.

(4) *If* $b \in \mathrm{CBMO}(\mathbb{R}^n)$, *then* $b \in \mathrm{CVMO}(\mathbb{R}^n)$ *if and only if* b *satisfies the following two conditions*:

$$\lim_{r \to 0} \sup_r N(b, B_r) = 0; \tag{2-1}$$

$$\lim_{r \to \infty} \sup_r N(b, B_r) = 0, \tag{2-2}$$

where $N(b, B_r) := \inf_{c \in \mathbb{R}} 1/|B_r| \int_{B_r} |b(x) - c|\, dx$.

Proof. We prove (1)–(3) by a slight modification of [Grafakos 2009, Proposition 7.1.2]. It is easy to check (1) by the following observation:

$$\frac{1}{B_r} \int_{B_r} |b(x) - b_{B_r}|\, dx \le 2\|b\|_{L^\infty(\mathbb{R}^n)}.$$

For (2), we first note that

$$|b - b_{B_r}| \le |b - c| + |b_{B_r} - c| \le |b - c| + \frac{1}{|B_r|} \int_{B_r} |b(x) - c|\, dx,$$

which gives

$$\frac{1}{|B_r|} \int_{B_r} |b(x) - b_{B_r}|\, dx \le \frac{1}{|B_r|} \int_{B_r} |b(x) - c|\, dx + \frac{1}{|B_r|} \int_{B_r} \frac{1}{|B_r|} \int_{B_r} |b(x) - c|\, dx dy.$$

Hence $\|b\|_{\mathrm{CBMO}(\mathbb{R}^n)} \le 2C$ as desired. The proof of (3) is equivalent to the inequality

$$\tfrac{1}{2}\|b\|_{\mathrm{CBMO}(\mathbb{R}^n)} \le \sup_r \inf_{c \in \mathbb{R}} \frac{1}{|B_r|} \int_{B_r} |b(x) - c|\, dx \le \|b\|_{\mathrm{CBMO}(\mathbb{R}^n)},$$

where the lower inequality follows from (2), while the upper one is trivial.

The conditions in (4) are a characterization of the CVMO space and can be seen as a central version of [Uchiyama 1978, Lemma], which was presented in [Neri 1975] without proof. Next, we show (4), which is partly inspired by the proof of [Uchiyama 1978, Lemma]. As we will see, it requires a large modification since the $\mathrm{CBMO}(\mathbb{R}^n)$ space is central space.

The sufficiency of (4) is trivial according to the definition of the $\mathrm{CVMO}(\mathbb{R}^n)$ space. Next, we prove the necessity. To do this, it is sufficient to show that if b satisfies (2-1) and (2-2), then for any $\varepsilon > 0$, there exists $b_\varepsilon \in \mathrm{CBMO}(\mathbb{R}^n)$ such that

$$\inf_{h \in C_c^\infty(\mathbb{R}^n)} \|b_\varepsilon - h\|_{\mathrm{CBMO}(\mathbb{R}^n)} \le C\varepsilon \tag{2-3}$$

and

$$\|b_\varepsilon - b\|_{\mathrm{CBMO}(\mathbb{R}^n)} \le C\varepsilon. \tag{2-4}$$

By (2-1) and (2-2), there exist $K_1, K_2 \in \mathbb{Z}$ such that

$$\sup_k N(b, B_k) < \varepsilon \quad \text{for } k \le K_1 \tag{2-5}$$

and

$$\sup_k N(b, B_k) < \varepsilon \quad \text{for } k \geq K_2. \tag{2-6}$$

Define $b_\varepsilon'(x) = b_{B_x}$, where

$$B_x = \begin{cases} B_{K_1}, & x \in B_{K_1}; \\ B_{K_2}, & x \in B_{K_2} \setminus B_{K_1}; \\ B_k & x \in B_k \setminus B_{k-1},\ k \geq K_2 + 1. \end{cases}$$

By (2-6), there exists $K_3 > K_2$ such that

$$\sup\{|b_\varepsilon'(x) - b_\varepsilon'(y)| : x, y \in B_{K_3} \setminus B_{K_3-1}\} < \varepsilon. \tag{2-7}$$

Without loss of generality, we may assume that $K_3 = K_2 + 1$ in the following analysis.

Set

$$b_\varepsilon(x) = \begin{cases} b_\varepsilon'(x), & x \in B_{K_3}; \\ b_{B_{K_3}}, & x \in B_{K_3}^c. \end{cases}$$

Then, b_ε is our desired function. Namely, b_ε satisfies (2-3) and (2-4). We first claim that for any x and y,

$$|b_\varepsilon(x) - b_\varepsilon(y)| < C\varepsilon \tag{2-8}$$

and

$$\frac{1}{|B_r|} \int_{B_r} |b(x) - b_\varepsilon(x)|\, dx < C\varepsilon \quad \text{for } r > 0. \tag{2-9}$$

We can show (2-8) by four cases.

Case 1. $x, y \in B_{K_3}$. This case can be divided into four subcases.

Subcase 1.1 $x, y \in B_{K_3} \setminus B_{K_2}$. The formula (2-8) is an immediate consequence of (2-7).

Subcase 1.2. $x, y \in B_{K_2}$. In this subcase, (2-5) and (2-6) can be used to obtain

$$|b_\varepsilon(x) - b_\varepsilon(y)| = \begin{cases} \frac{1}{|B_{K_1}|} \int_{B_{K_1}} |b(z) - b_{B_{K_1}}|\, dz < C\varepsilon, & x, y \in B_{K_1}; \\ \frac{1}{|B_{K_2}|} \int_{B_{K_2}} |b(z) - b_{B_{K_2}}|\, dz < C\varepsilon, & x, y \in B_{K_2} \setminus B_{K_1}; \\ \frac{1}{|B_{K_1}|} \int_{B_{K_1}} |b(z) - b_{B_{K_2}}|\, dz < C\varepsilon, & x \in B_{K_1}, y \in B_{K_2} \setminus B_{K_1}; \\ \frac{1}{|B_{K_2}|} \int_{B_{K_2}} |b(z) - b_{B_{K_1}}|\, dz < C\varepsilon, & y \in B_{K_1}, x \in B_{K_2} \setminus B_{K_1}. \end{cases}$$

Subcase 1.3. $x \in B_{K_2},\ y \in B_{K_3} \setminus B_{K_2}$. We conclude from (2-6) that

$$|b_\varepsilon(x) - b_\varepsilon(y)| = \frac{1}{|B_{K_2}|} \int_{B_{K_2}} |b(z) - b_{B_{K_3}}|\, dz < C\varepsilon.$$

Subcase 1.4. $y \in B_{K_2},\ x \in B_{K_3} \setminus B_{K_2}$. This subcase can be dealt with in the same way as Subcase 1.3.

Case 2. $x, y \in B_{K_3}^c$. The formula (2-8) can be deduced from the definition of b_ε.

Case 3. $x \in B_{K_3}$, $y \in B_{K_3}^c$. Using (2-5) and (2-6) again, we can get (2-8) since

$$|b_\varepsilon(x) - b_\varepsilon(y)| = \begin{cases} \frac{1}{|B_{K_1}|}\int_{B_{K_1}} |b(z) - b_{B_{K_3}}|\,dz < C\varepsilon, & x \in B_{K_1}; \\ \frac{1}{|B_{K_2}|}\int_{B_{K_2}} |b(z) - b_{B_{K_3}}|\,dz < C\varepsilon, & x \in B_{K_2} \setminus B_{K_1}; \\ \frac{1}{|B_{K_3}|}\int_{B_{K_3}} |b(z) - b_{B_{K_3}}|\,dz < C\varepsilon, & x \in B_{K_3} \setminus B_{K_2}. \end{cases}$$

Case 4. $x \in B_{K_3}^c$, $y \in B_{K_3}$. The proof for Case 3 also works for this case.

We conclude from (2-8) that $b_\varepsilon \in \mathrm{CBMO}(\mathbb{R}^n)$ and (2-3) is obvious. Now, we are in a position to show (2-9), which can be divided into three cases.

Case 1. $B_r \subseteq B_{K_1}$. Using (2-5), we have

$$\frac{1}{|B_r|}\int_{B_r} |b(y) - b_\varepsilon(y)|\,dy = \frac{1}{|B_r|}\int_{B_r} |b(y) - b_{B_{K_1}}|\,dy < C\varepsilon.$$

Case 2. $B_r \subseteq B_{K_2}$ with $r > 2^{K_1}$. In this case, Minkowski's inequality and (2-6) are used to obtain

$$\frac{1}{|B_r|}\int_{B_r} |b(y) - b_\varepsilon(y)|\,dy \le \frac{1}{|B_{K_2}|}\int_{B_{K_2}} |b(y) - b_{B_r}|\,dy < C\varepsilon.$$

Case 3. $B_r \subseteq B_{K_2}^c$. By applying (2-6) to B_r, we obtain

$$\frac{1}{|B_r|}\int_{B_r} |b(y) - b_\varepsilon(y)|\,dy \le \frac{1}{|B_r|}\int_{B_r} |b(y) - b_{B_{K_3}}|\,dy < C\varepsilon,$$

thus finding (2-9). This shows that for any $B_r \subset \mathbb{R}^n$ with $r > 0$, $N(b - b_\varepsilon, B_r) < C\varepsilon$. This result plus Lemma 2.1(3) gives (2-4). □

The second lemma is the Fréchet–Kolmogorov theorem [Yosida 1965], which gives a characterization of compact sets.

Lemma 2.2. *Let $G \subset L^p(\mathbb{R}^n)$ and $E_\alpha = \{x \in \mathbb{R}^n : |x| > \alpha\}$. Then G is strongly precompact, if and only if,*

$$\sup_{f \in G} \|f\|_{L^p(\mathbb{R}^n)} < \infty; \tag{2-10}$$

$$\lim_{|y| \to 0} \|f(\cdot + y) - f(\cdot)\|_{L^p(\mathbb{R}^n)} = 0 \quad \textit{uniformly in } f \in G; \tag{2-11}$$

$$\lim_{\alpha \to \infty} \|f \chi_{E_\alpha}\|_{L^p(\mathbb{R}^n)} = 0 \quad \textit{uniformly in } f \in G. \tag{2-12}$$

The condition (2-12) is the L^p-boundedness of H, which was obtained by Christ and Grafakos [1995].

Lemma 2.3. *Let $1 < p < \infty$ and let f be a locally integrable function on $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that*

$$\|Hf\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}.$$

Now, Theorem 1.1 is contained in the following two theorems.

Theorem 2.4. *Let $[b, H]$ and $[b, H^*]$ be compact operators on $L^p(\mathbb{R}^n)$ with $1 < p < \infty$. Then, $b \in \mathrm{CVMO}(\mathbb{R}^n)$.*

Theorem 2.5. *If $b \in \mathrm{CVMO}(\mathbb{R}^n)$, then both $[b, H]$ and $[b, H^*]$ are compact operators on $L^p(\mathbb{R}^n)$ with $1 < p < \infty$.*

2.1. *Proof of Theorem 2.4.* Since $[b, H]$ and $[b, H^*]$ are compact operators on $L^p(\mathbb{R}^n)$, $b \in \mathrm{CBMO}^{\max\{p,p'\}}(\mathbb{R}^n)$ [Fu et al. 2007, Corollary 2.1], and hence $b \in \mathrm{CBMO}(\mathbb{R}^n)$. Without loss of generality, we may assume that $\|b\|_{\mathrm{CBMO}^{\max\{p,p'\}}(\mathbb{R}^n)} = 1$. Our task is to show (2-1) and (2-2) of Lemma 2.1. This can be done by contradiction.

We begin with the assumption that b does not satisfy (2-1). Then, it is immediate that there is a $\delta > 0$ and a sequence of balls $\{B_{r_i}\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} r_i = 0$ satisfying

$$\frac{1}{|B_{r_i}|}\int_{B_{r_i}} |b(y) - b_{B_{r_i}}|\,dy > \delta. \tag{2-13}$$

Denoting $\varepsilon_0 = 1/|B_{r_i}|\int_{B_{r_i}} \mathrm{sgn}(b(y) - b_{B_{r_i}})\,dy$, we choose the function

$$g_i(y) = \frac{1}{|B_{r_i}|^{1/p}}[\mathrm{sgn}(b(y) - b_{B_{r_i}}) - \varepsilon_0]\chi_{B_{r_i}}(y), \quad i = 1, 2, \ldots \tag{2-14}$$

to get

$$\begin{cases} g_i \in L^p(\mathbb{R}^n); \\ \mathrm{supp}\, g_i \subset B_{r_i}; \\ g_i(y)(b(y) - b_{B_{r_i}}) > 0; \\ |g_i(y)| \le 2|B_{r_i}|^{-1/p} \text{ with } y \in B_{r_i}; \\ \int_{\mathbb{R}^n} g_i(y)\,dy = 0. \end{cases} \tag{2-15}$$

Hence, $\{[b, H]g_i\}_{i=1}^{\infty}$ is a bounded set in $L^p(\mathbb{R}^n)$. Next, we pick a subsequence $\{[b, H]g_{i_m}\}_{m=1}^{\infty}$ from $\{[b, H]g_i\}_{i=1}^{\infty}$ such that $\{[b, H]g_{i_m}\}_{m=1}^{\infty}$ has no convergence subsequence in $L^p(\mathbb{R}^n)$ to show that $[b, H]$ is not a compact operator on $L^p(\mathbb{R}^n)$. This contradiction will show that b must satisfy (2-1). To do so, we first need the following estimates for $\{[b, H]g_i\}$.

Lemma 2.6. *Let $b \in \mathrm{CBMO}(\mathbb{R}^n)$, $1 < p < \infty$, and δ and g_i be defined as in* (2-13) *and* (2-14), *respectively. Then there exist constants $C_2 > C_1 > 2$, $C_3 > 0$*

and $0 < \varepsilon \ll C_2$, *which are dependent only on* n, p, δ *and* b, *such that*

$$\left(\int_{S_1} |[b, H]g_i(x)|^p\,dx\right)^{1/p} \geq C_3 \tag{2-16}$$

$$\left(\int_{S_2} |[b, H]g_i(x)|^p\,dx\right)^{1/p} \leq \frac{C_3}{4} \tag{2-17}$$

and

$$\left(\int_{S_3} |[b, H]g_i(x)|^p\,dx\right)^{1/p} \leq \frac{C_3}{4}, \tag{2-18}$$

where

$$\begin{cases} S_1 = \{x : C_1 r_i < |x| < C_2 r_i\}; \\ S_2 = \{x : |x| > C_2 r_i\}; \\ S_3 \subset S_1 \text{ with } |S_3|/|B_{r_i}| < \varepsilon^n. \end{cases}$$

Proof. For fixed r_i and $x \in (\alpha B_{r_i})^c$ with $\alpha > 2$, one gets from (2-15) and Hölder's inequality that

$$\begin{aligned} |H(b - b_{B_{r_i}})g_i)(x)| &\leq \frac{1}{|x|^n}\int_{B_{r_i}} |b(y) - b_{B_{r_i}}||g_i(y)|\,dy \\ &\leq \frac{1}{|x|^n}\left(\int_{B_{r_i}} |b(y) - b_{B_{r_i}}|^{p'}\,dy\right)^{1/p'}\left(\int_{B_{r_i}} |g_i(y)|^p\,dy\right)^{1/p} \\ &\leq C\|b\|_{\mathrm{CBMO}^{p'}(\mathbb{R}^n)}\frac{C|B_{r_i}|^{1/p'}}{|x|^n}\left(\int_{B_{r_i}} |B_{r_i}|^{-1}\,dy\right)^{1/p}. \end{aligned}$$

Namely,

$$|H(b - b_{B_{r_i}})g_i(x)| \leq \frac{C|B_{r_i}|^{1/p'}}{|x|^n}. \tag{2-19}$$

On the other hand, (2-13) and (2-15) are used to obtain

$$|H(b - b_{B_{r_i}})g_i(x)| \geq \frac{C\delta|B_{r_i}|^{1/p'}}{|x|^n}. \tag{2-20}$$

Furthermore, the fact that $\int_{\mathbb{R}^n} g_i(y)\,dy = 0$ is used to show that for $a > \max\{\alpha, 8\}$,

$$|(b(x) - b_{B_{r_i}})Hg_i(x)| = (b(x) - b_{B_{r_i}})\frac{1}{|x|^n}\int_{B_{r_i}} g_i(y)\,dy = 0. \tag{2-21}$$

Hence,

$$\left(\int_{\{|x|>ar_i\}} |(b(x) - b_{B_{r_i}})H(g_i)(x)|^p\,dx\right)^{1/p} = 0. \tag{2-22}$$

Upon setting $S_4 = \{x : ar_i < |x| < dr_i\}$ for $d > a$, (2-20) and (2-22) show that

$$\left(\int_{S_4} |[b, H]g_i(x)|^p\,dx\right)^{1/p} \ge \left(\int_{S_4} |H(b-b_{B_{r_i}})g_i(x)|^p\,dx\right)^{1/p} - \left(\int_{\{|x|>ar_i\}} |(b(x)-b_{B_{r_i}})Hg_i(x)|^p\,dx\right)^{1/p} \ge C\delta\left(a^{n(1-p)}-d^{n(1-p)}\right)^{1/p}. \tag{2-23}$$

On the other hand, combining (2-19) with (2-22), we can assert that

$$\left(\int_{\{|x|>dr_i\}} |[b, H]g_i(x)|^p\,dx\right)^{1/p} \le \left(\int_{\{|x|>dr_i\}} |H(b-b_{B_{r_i}})g_i(x)|^p\,dx\right)^{1/p} + \left(\int_{\{|x|>dr_i\}} |(b(x)-b_{B_{r_i}})Hg_i(x)|^p\,dx\right)^{1/p} \le C|B_{r_i}|^{1/p'}\left(\int_{\{|x|>dr_i\}} \frac{dx}{|x|^{np}}\right)^{1/p}.$$

Namely,

$$\left(\int_{\{|x|>dr_i\}} |[b, H]g_i(x)|^p\,dx\right)^{1/p} \le Cd^{-n/p'}. \tag{2-24}$$

It is easy to check that there exist constants $C_2 > C_1 > 2$ and $C_3 > 0$, which are dependent only on n, p, δ and b, such that the required inequalities (2-16) and (2-17) are true.

Now it remains to prove (2-18). Let $S_3 \subset S_1$ be an arbitrary measurable set. Combining (2-19) and (2-21) with the Minkowski inequality, one has

$$\left(\int_{S_3} |[b, H]g_i(x)|^p\,dx\right)^{1/p} \le \left(\int_{S_3} |H(b-b_{B_{r_i}})g_i(x)|^p dx\right)^{1/p} + \left(\int_{S_3} |(b(x)-b_{B_{r_i}})Hg_i(x)|^p dx\right)^{1/p} \le C|B_{r_i}|^{1/p'}\left(\int_{S_3} \frac{1}{|x|^{np}}\,dx\right)^{1/p} \le C\left(\frac{|S_3|}{|B_{r_i}|}\right)^{1/p} \le C\varepsilon^{n/p} \tag{2-25}$$

thanks to $|S_3| < |B_{r_i}|\varepsilon^n$, and thus (2-18) holds by taking $\varepsilon \ll C_2$. Hence, the proof of Lemma 2.6 is completed. □

Now, we can use Lemma 2.6 to choose a subsequence $\{[b, H]g_{i_m}\}_{m=1}^{\infty}$ such that $\{[b, H]g_{i_m}\}_{m=1}^{\infty}$ has no convergence subsequence in $L^p(\mathbb{R}^n)$, thus reaching (2-1). Thus, it is sufficient to verify that there exist a subsequence $\{[b, H]g_{i_m}\}_{m=1}^{\infty}$ and $\tau > 0$, independent of g_{i_m}, such that

$$\|[b, H]g_{i_m} - [b, H]g_{i_{m+k}}\|_{L^p(\mathbb{R}^n)} \geq \tau. \tag{2-26}$$

Since $\lim_{i\to\infty} r_i = 0$, we may choose a subsequence $\{B_{r_{i_m}}\}_m$ from $\{B_{r_i}\}$ such that

$$r_{i_{m+1}}/r_{i_m} < \varepsilon/C_2.$$

Here ε, C_2 are defined as in Lemma 2.6. We claim that if the function g_{i_m} is defined relative to $B_{r_{i_m}}$ as in (2-14), then $\{[b, H]g_{i_m}\}_{m=1}^{\infty}$ is our desired subsequence. Indeed, for $k, m \in \mathbb{N}$,

$$\begin{aligned}\|[b, H]g_{i_m} &- [b, H]g_{i_{m+k}}\|_{L^p(\mathbb{R}^n)} \\ &\geq \left(\int_{S_5} |[b, H]g_{i_m}(x)|^p\,dx\right)^{1/p} - \left(\int_{S_6} |[b, H]g_{i_{m+k}}(x)|^p\,dx\right)^{1/p},\end{aligned}$$

where

$$\begin{cases} S_5 = \{x : C_1 r_{i_m} < |x| < C_2 r_{i_m}\}; \\ S_6 = \{x : |x| > C_2 r_{i_{m+k}}\} \end{cases}$$

and C_1, C_2 are defined as in Lemma 2.6. Then a further use of (2-16) and (2-17) gives

$$\begin{aligned}\|[b, H]g_{i_m} &- [b, H]g_{i_{m+k}}\|_{L^p(\mathbb{R}^n)} \\ &\geq \left(\int_{S_5} |[b, H]g_{i_m}(x)|^p\,dx\right)^{1/p} - \left(\int_{S_6} |[b, H]g_{i_{m+k}}(x)|^p\,dx\right)^{1/p} \\ &\geq C_3 - \frac{C_3}{4} \\ &= \frac{3C_3}{4},\end{aligned}$$

which gives (2-26).

The proof of Theorem 2.4 is completed since the proof of (2-1) also works for (2-2).

2.2. ***Proof of Theorem 2.5.*** To prove Theorem 2.5, we first recall the following lemma which can help us to simplify the proof by considering only $b \in C_c^{\infty}(\mathbb{R}^n)$.

Lemma 2.7. *Let $[b, T]$ be the commutator of operator T and $1 < p < \infty$. If $[b, T]$ is a compact operator on $L^p(\mathbb{R}^n)$ with $b \in C_c^{\infty}(\mathbb{R}^n)$, then $[b, T]$ is also a compact operator on $L^p(\mathbb{R}^n)$ with $b \in \mathrm{CVMO}(\mathbb{R}^n)$.*

Proof. The proof of Lemma 2.7 follows by a modification of [Chen and Ding 2010, p. 2645]. Note first that for $b \in \mathrm{CBMO}(\mathbb{R}^n)$, there exists $\{b_\varepsilon\} \subset C_c^\infty(\mathbb{R}^n)$ with $\varepsilon > 0$ satisfying $\|b - b_\varepsilon\|_{\mathrm{CBMO}(\mathbb{R}^n)} < \varepsilon$. Then by the L^p-boundedness of $[b, T]$, one has

$$\|[b,T]-[b_\varepsilon,T]\|_{L^p(\mathbb{R}^n)} = \|[b-b_\varepsilon,T]\|_{L^p(\mathbb{R}^n)} \le C\|b-b_\varepsilon\|_{\mathrm{CBMO}(\mathbb{R}^n)} \le C\varepsilon. \tag{2-27}$$

Let

$$Q = \{f : f \in L^p(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{L^p(\mathbb{R}^n)} \le C\}.$$

Then the L^p-compactness of $[b_\varepsilon, T]$ implies that the set $S = \{[b_\varepsilon, T]f : f \in Q\}$ is strongly precompact. Now, it is sufficient to show that $S = \{[b, T]f : f \in Q\}$ satisfies (2-10)–(2-12). Using (2-10) and (2-27) produces

$$\sup_{f\in Q}\|[b,T]f\|_{L^p(\mathbb{R}^n)} \le \sup_{f\in Q}\|[b_\varepsilon,T]f\|_{L^p(\mathbb{R}^n)} + C\varepsilon < \infty.$$

Moreover,

$$\begin{aligned}
&\lim_{|y|\to 0}\|[b,T]f(\cdot+y)-[b,T]f(\cdot)\|_{L^p(\mathbb{R}^n)}\\
&\qquad = \lim_{|y|\to 0}\Big\|[b,T]f(\cdot+y)-[b_\varepsilon,T]f(\cdot+y)+[b_\varepsilon,T]f(\cdot+y)\\
&\qquad\qquad\qquad -[b_\varepsilon,T]f(\cdot)+[b_\varepsilon,T]f(\cdot)-[b,T]f(\cdot)\Big\|_{L^p(\mathbb{R}^n)}\\
&\qquad = \lim_{|y|\to 0}\Big\|[b_\varepsilon,T]f(\cdot+y)-[b_\varepsilon,T]f(\cdot)+[b-b_\varepsilon,T]f(\cdot+y)\\
&\qquad\qquad\qquad +[b-b_\varepsilon,T]f(\cdot)\Big\|_{L^p(\mathbb{R}^n)}\\
&\qquad \le \lim_{|y|\to 0}\|[b_\varepsilon,T]f(\cdot+y)-[b_\varepsilon,T]f(\cdot)\|_{L^p(\mathbb{R}^n)}+2\|[b-b_\varepsilon,T]f\|_{L^p(\mathbb{R}^n)}\\
&\qquad \le 2C\varepsilon \mapsto 0 \text{ uniformly for } f \in Q \text{ with } \varepsilon \to 0.
\end{aligned}$$

A similar argument gives

$$\begin{aligned}
\lim_{\alpha\to\infty}\|[b,T]\chi_{E_\alpha}\|_{L^p(\mathbb{R}^n)} &\le \lim_{\alpha\to\infty}\|[b_\varepsilon,T]f\chi_{E_\alpha}\|_{L^p(\mathbb{R}^n)} + \|[b-b_\varepsilon,T]f\|_{L^p(\mathbb{R}^n)}\\
&\le C\varepsilon \mapsto 0 \text{ uniformly for } f \in Q \text{ with } \varepsilon \to 0.
\end{aligned}$$

According to Lemma 2.2, S is a strongly precompact set in $L^p(\mathbb{R}^n)$, and hence, $[b, T]$ is a compact operator on $L^p(\mathbb{R}^n)$. □

By Lemma 2.7, we only need to consider $b \in C_c^\infty(\mathbb{R}^n)$. Namely, we will prove that the set

$$G = \{[b,H]f : f \in \mathcal{F}\} \text{ with } \mathcal{F} = \{f : f \in L^p(\mathbb{R}^n) \text{ and } \|f\|_{L^p(\mathbb{R}^n)} \le C\}$$

and

$$G^* = \{[b,H^*]f : f \in \mathcal{F}\} \text{ with } \mathcal{F} = \{f : f \in L^p(\mathbb{R}^n) \text{ and } \|f\|_{L^p(\mathbb{R}^n)} \le C\}$$

are strongly precompact on $L^p(\mathbb{R}^n)$. By Lemma 2.2, showing (2-10)–(2-12) for G and G^* is sufficient. We begin with the proof for G.

First, for $f \in \mathcal{F}$ and $b \in C_c^\infty(\mathbb{R}^n)$, the boundedness of $[b, H]$ on $L^p(\mathbb{R}^n)$ shows

$$\sup_{f\in\mathcal{F}} \|[b, H]f\|_{L^p(\mathbb{R}^n)} \le C\|b\|_{\mathrm{CBMO}(\mathbb{R}^n)} \sup_{f\in\mathcal{F}} \|f\|_{L^p(\mathbb{R}^n)} < \infty,$$

which gives (2-10).

Second, suppose that

$$\operatorname{supp} b \subset \{x : |x| \le \xi\}.$$

Then, for $0 < \varepsilon < 1$, we choose $\alpha > \xi + 1$ such that $(\xi/\alpha^n)^{n/p'} < \varepsilon$ to verify

$$\|[b, H]f\chi_{E_\alpha}\|_{L^p(\mathbb{R}^n)} < C\varepsilon \quad \text{for } E_\alpha = \{x : |x| > \alpha\}.$$

Indeed,

$$\begin{aligned}
&\biggl(\int_{|x|>\alpha} |[b, H]f(x)|^p\,dx\biggr)^{1/p}\\
&\qquad \le C\biggl(\int_{|x|>\alpha}\biggl(\frac{1}{|x|^n}\int_{|y|\le\xi}|f(y)|\,dy\biggr)^p dx\biggr)^{1/p}\\
&\qquad \le \biggl(\int_\alpha^\infty \frac{t^{n-1}dt}{t^{np}}\,dt\biggr)^{1/p}\biggl(\int_{|y|\le\xi} dy\biggr)^{1/p'}\biggl(\int_{|y|\le\xi}|f(y)|^p\,dy\biggr)^{1/p}\\
&\qquad \le C\frac{\xi^{n/p'}}{\alpha^{n/p'}}\biggl(\int_{|y|\le\xi}|f(y)|^p\,dy\biggr)^{1/p}\\
&\qquad \le C\varepsilon.
\end{aligned}$$

This means that (2-12) holds for $[b, H]$ in G uniformly.

Finally, we continue the proof by showing (2-11). To do this, it is sufficient to prove that for any $\varepsilon > 0$ and $|z|$ sufficiently small dependent only on ε, one has

$$\|[b, H]f(\cdot + z) - [b, H]f(\cdot)\|_{L^p(\mathbb{R}^n)} \le C\varepsilon \qquad \forall \quad f \in \mathcal{F}. \tag{2-28}$$

Let $0 < \varepsilon < \frac{1}{2}$ and $z \in \mathbb{R}^n$. We first rewrite $[b, H]f(x + z) - [b, H]f(x)$ as

$$\begin{aligned}
|[b, H]f(x + z) - [b, H]f(x)| &= \frac{1}{|x + z|^n}\int_{U_1}[(b(x + z) - b(y))]f(y)\,dy\\
&\quad + \frac{1}{|x + z|^n}\int_{U_2}[(b(x + z) - b(y))]f(y)\,dy\\
&\quad - \frac{1}{|x|^n}\int_{V_1}[(b(x) - b(y))]f(y)\,dy\\
&\quad - \frac{1}{|x|^n}\int_{V_2}[(b(x) - b(y))]f(y)\,dy\\
&:= I_1 + I_2 + I_3 - I_4.
\end{aligned}$$

Here,

$$\begin{cases} U_1 = \{y : |y| < |x+z| \text{ and } |x| > \varepsilon^{-1}|z|\}; \\ U_2 = \{y : |y| < |x+z| \text{ and } |x| \le \varepsilon^{-1}|z|\}; \\ V_1 = \{y : |y| < |x| \text{ and } |x| > \varepsilon^{-1}|z|\}; \\ V_2 = \{y : |y| < |x| \le \varepsilon^{-1}|z|\} \end{cases}$$

and

$$\begin{cases} I_1 = \frac{1}{|x|^n} \int_{V_1} [(b(x+z)-b(x))] f(y)\,dy; \\ I_2 = \frac{1}{|x|^n} \int_{V_1} [(b(y)-b(x+z))] f(y)\,dy - \frac{1}{|x+z|^n} \int_{U_1} [(b(y)-b(x+z))] f(y)\,dy; \\ I_3 = \frac{1}{|x|^n} \int_{V_2} [(b(y)-b(x))] f(y)\,dy; \\ I_4 = \frac{1}{|x+z|^n} \int_{U_2} [(b(y)-b(x+z))] f(y)\,dy. \end{cases}$$

Therefore, (2-28) follows from the following L^p-estimates for $I_i,\ i = 1, 2, 3, 4$. Note first that $b \in C_c^\infty(\mathbb{R}^n)$, $|b(x+z) - b(x)| \le C|z|$. So,

$$|I_1| \le C|z| H(|f|)(x) \chi_{\{|x| > \varepsilon^{-1}|z|\}}.$$

This, plus Lemma 2.3, implies that for $f \in \mathcal{F}$,

$$\|I_1\|_{L^p(\mathbb{R}^n)} \le C|z| \|H(f)\|_{L^p(\mathbb{R}^n)} \le C|z| \|f\|_{L^p(\mathbb{R}^n)} \le C|z|. \tag{2-29}$$

The fact $|b(x+z) - b(y)| \le 2\|b\|_{L^\infty(\mathbb{R}^n)} \le C$ and $|x+z| \simeq |z|$ for $|z|$ small enough allow us to obtain the following estimate for I_2:

$$\begin{aligned} |I_2| &\le \frac{C|z|}{|x|^{n+1}} \int_{V_1} |f(y)|\,dy \le \frac{C\varepsilon}{|x|^n} \int_{|y|<|x|} |f(y)|\,dy\, \chi_{\{|x| > \varepsilon^{-1}|z|\}}(x) \\ &\le C\varepsilon H(|f|)(x) \chi_{\{|x| > \varepsilon^{-1}|z|\}}(x). \end{aligned}$$

Using Lemma 2.3 again, we obtain

$$\|I_2\|_{L^p(\mathbb{R}^n)} \le C\varepsilon \|H(|f|)\|_{L^p(\mathbb{R}^n)} \le C\varepsilon \|f\|_{L^p(\mathbb{R}^n)} \le C\varepsilon. \tag{2-30}$$

For I_3, we note the fact that $|b(x) - b(y)| \le C|x-y|$ since $b \in C_c^\infty(\mathbb{R}^n)$ and that $|x-y| < 2|x|$ since $y \in V_2$; therefore,

$$\begin{aligned} |I_3| &\le \frac{C}{|x|^{n-1}} \int_{V_2} |f(y)|\,dy \le \frac{C\varepsilon^{-1}|z|}{|x|^n} \int_{V_2} |f(y)|\,dy \\ &\le C\varepsilon^{-1}|z| H(|f|)(x) \chi_{\{|x| < \varepsilon^{-1}|z|\}}(x). \end{aligned}$$

Hence,

$$\|I_3\|_{L^p(\mathbb{R}^n)} \le C\varepsilon^{-1}|z| \|f\|_{L^p(\mathbb{R}^n)} \le C\varepsilon^{-1}|z|. \tag{2-31}$$

For the term I_4, we use the fact that $|b(x+z)-b(y)| \le C|x+z-y| < C|x+z|$ since $y \in U_2$ to obtain

$$\begin{aligned}|I_4| &\le C\frac{\varepsilon^{-1}|z|+|z|}{|x+z|^n}\int_{\{|y|<|x+z|\}}|f(y)|\,dy\,\chi_{\{|x|<\varepsilon^{-1}|z|\}}(x)\\ &\le C(\varepsilon^{-1}|z|+|z|)H(|f|)(x+z)\chi_{\{|x|<\varepsilon^{-1}|z|\}}(x).\end{aligned}$$

Therefore,

$$\|I_4\|_{L^p(\mathbb{R}^n)} \le C(\varepsilon^{-1}|z|+|z|)\|f\|_{L^p(\mathbb{R}^n)} \le C(\varepsilon^{-1}|z|+|z|). \tag{2-32}$$

The desired estimate (2-28) can be obtained by (2-29)–(2-32) and by taking $|z|$ to be sufficiently small. We proceed the proof of Theorem 2.5 to show (2-10)–(2-12) for G^*. In fact, similar arguments for G can be used to deal with G^* by recalling the L^p-boundedness of $[b, H^*]$. We omit its proof here due to the similarity.

3. The compactness of the fractional Hardy operator

Let f be a locally integral function on $\mathbb{R}^n$ and $0<\alpha<n$. In [Fu et al. 2007], the n-dimensional fractional Hardy operator H_α is defined as

$$H_\alpha f(x) = \frac{1}{|x|^{n-\alpha}}\int_{|y|<|x|} f(y)\,dy, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

It is easy to check that

$$|H_\alpha f(x)| \le C M_\alpha f(x),$$

where M_α is the fractional Hardy–Littlewood maximal operator, defined by

$$M_\alpha f(x) = \sup \frac{1}{r^{n-\alpha}}\int_{\{|y-x|<r\}}|f(y)|\,dy, \quad 0<\alpha<n.$$

For the boundedness of M_α on the Lebesgue space, Lu, Ding and Yan [Lu et al. 2007, Theorem 3.2] provided the following lemma:

Lemma 3.1. *Let* $0<\alpha<n,\ 1<p\le \frac{n}{\alpha}$ *and* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. *Then there exists a constant* $C>0$ *such that*

$$\|M_\alpha f\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}.$$

For the boundedness of $[b, H_\alpha]$ on the Lebesgue space, the known result is from [Fu et al. 2007, Theorem 2.1].

Lemma 3.2. *Let* $0<\alpha<n,\ 1<p\le \frac{n}{\alpha}$ *and* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. *Then*

$$[b, H_\alpha] : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \Leftrightarrow b \in \mathrm{CBMO}^{p'}(\mathbb{R}^n).$$

This section presents the compactness characterization of $[b, H_\alpha]$ on Lebesgue space.

Theorem 3.3. *Suppose that* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then*

$$[b, H_\alpha] \text{ is compact from } L^p(\mathbb{R}^n) \text{ to } L^q(\mathbb{R}^n) \Leftrightarrow b \in \mathrm{CVMO}(\mathbb{R}^n).$$

Theorem 3.3 will be proved by the following two results.

Theorem 3.4. *Assume that* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $[b, H_\alpha]$ *is compact from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$. *Then* $b \in \mathrm{CVMO}(\mathbb{R}^n)$.

Theorem 3.5. *Let* α, p, q *be defined as in Theorem 3.4 and let* $b \in \mathrm{CVMO}(\mathbb{R}^n)$. *Then* $[b, H_\alpha]$ *is compact from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.

The proofs of Theorems 3.4 and 3.5 can be viewed as modifications of those of Theorems 2.4 and 2.5 thanks to Lemmas 3.1 and 3.2. We omit their proofs here due to their similarity.

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