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ALGEBRAIC AND GEOMETRIC PROPERTIES OF FLAG BOTT–SAMELSON VARIETIES AND APPLICATIONS TO REPRESENTATIONS

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We define and study flag Bott–Samelson varieties which generalize both Bott–Samelson varieties and flag varieties. Using a birational morphism from an appropriate Bott–Samelson variety to a flag Bott–Samelson variety, we compute the Newton–Okounkov bodies of flag Bott–Samelson varieties as generalized string polytopes, which are applied to give polyhedral expressions for irreducible decompositions of tensor products of G-modules. Furthermore, we show that flag Bott–Samelson varieties degenerate into flag Bott manifolds with higher rank torus actions, and we describe the Duistermaat–Heckman measures of the moment map images of flag Bott–Samelson varieties with torus actions and invariant closed 2-forms.

1. Introduction

Bott–Samelson varieties provide fruitful connections between representation theory and algebraic geometry. They are nonsingular towers of $\mathbb{C}P^1$-fibrations, and studied in [Bott and Samelson 1958; Demazure 1974; Hansen 1973] to find resolutions of singularities of Schubert varieties. Moreover, the set of global sections of a holomorphic line bundle over a Bott–Samelson variety is the dual of a *generalized Demazure*

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module. This leads to worthwhile connections between representation theory and algebraic geometry exemplified by the character formulas of Demazure modules [Andersen 1985; Kumar 1987], the standard monomial theory [Lakshmibai et al. 1979; Lakshmibai and Seshadri 1991; Lakshmibai and Raghavan 2008; Seshadri 2007], and the theory of Newton–Okounkov bodies [Fujita and Naito 2017; Kaveh 2015].

On the other hand, for a given Bott–Samelson variety, it is presented by Grossberg and Karshon [1994] that there is a complex one-parameter family of smooth varieties, which are all diffeomorphic, such that a generic fiber is the Bott–Samelson variety and the special fiber is a nonsingular toric variety, called a *Bott manifold*. It should be noted that the Bott manifold is toric, while the Bott–Samelson variety is not toric in general. Using this connection, [Grossberg and Karshon 1994] also introduces a combinatorial object, called a *Grossberg–Karshon twisted cube*, which is used to compute the multiplicities of generalized Demazure modules (see [Grossberg and Karshon 1994, Theorem 3]).

One of the primary goals of this paper is to generalize the notion of Bott–Samelson varieties and to extend its rich connections with representation theory. Moreover, the generalization also supports the Grossberg–Karshon type degeneration into flag Bott manifolds. Indeed, we consider a *flag Bott–Samelson variety* (see Definition 2.1) which is a nonsingular projective tower of products of full flag manifolds. Moreover, under a certain condition, the flag Bott–Samelson variety is a desingularization of a Schubert variety. Because of the definition, both the full flag varieties and Bott–Samelson varieties are flag Bott–Samelson varieties. Hence we may regard flag Bott–Samelson varieties as the generalization of both flag varieties and Bott–Samelson varieties.

This notion of flag Bott–Samelson varieties is not new. Actually, in [Jantzen 2003], flag Bott–Samelson varieties are treated in a more general setting without naming them. Indeed, flag Bott–Samelson varieties are iterated flag manifold fibrations but Jantzen [2003] considers iterated Schubert varieties fibrations. Perrin [2007] uses these varieties to obtain small resolutions of Schubert varieties. In fact, they are called *generalized Bott–Samelson varieties* (see [Brion and Kannan 2019a; 2019b]). Moreover, flag Bott–Samelson varieties are generalized Bott–Samelson varieties (see Remark 2.2).

Let G be a simply connected semisimple algebraic group of rank n over $\mathbb{C}$. Bott–Samelson varieties $Z_{\boldsymbol{i}}$ are parametrized by words $\boldsymbol{i} = (i_1, \dots, i_r)$, where $i_1, \dots, i_r$ are elements in the set $[n] := \{1, \dots, n\}$. On the other hand, flag Bott–Samelson varieties $Z_{\mathcal{I}}$ are parametrized by sequences $\mathcal{I} = (I_1, \dots, I_r)$ of subsets of $[n]$. Even though the class of flag Bott–Samelson varieties is much larger than that of Bott–Samelson varieties, for each flag Bott–Samelson variety, there exists a Bott–Samelson variety such that there is a birational morphism from the Bott–Samelson variety to the flag Bott–Samelson variety (see Proposition 2.7).

Using the above mentioned birational morphism, we provide Theorem 2.20, which describes the set of holomorphic sections of a holomorphic line bundle over the flag Bott–Samelson variety in terms of generalized Demazure modules. The theory of Newton–Okounkov bodies of projective varieties has been used to present a connection between representation theory and algebraic geometry (see Section 2C for the definition of Newton–Okounkov bodies). The description of holomorphic sections of flag Bott–Samelson varieties is used to compute their Newton–Okounkov bodies. Indeed, using the result of Newton–Okounkov bodies of Bott–Samelson varieties by Fujita [2018], we obtain Theorem 2.22, which shows that the Newton–Okounkov bodies of flag Bott–Samelson varieties with an appropriate valuation agree with generalized string polytopes up to sign.

One of the fundamental questions in group representation theory is to find the multiplicities of irreducible representations in the tensor product of two representations. Berenstein and Zelevinsky [2001] describe the multiplicities in terms of the numbers of lattice points in some explicit rational convex polytope. In Theorem 3.19 we give a different description of the multiplicities using the integral lattice points of the Newton–Okounkov bodies, hence generalized string polytopes, of flag Bott–Samelson varieties. We notice that our results give concrete constructions of convex bodies, appearing in [Kaveh and Khovanskii 2012a], which encode multiplicities of irreducible representations.

As is mentioned before, we degenerate the complex structures of flag Bott–Samelson varieties. The notion of Bott manifolds is generalized to that of *flag Bott manifolds* in terms of iterated flag manifold fibrations described in [Kaji et al. 2020; Kuroki et al. 2020]. More precisely, a flag Bott manifold is the total space of an iterated flag manifold fibrations which are taken by the full flag fibration of a sum of line bundles (see Definition 4.1). In general, a flag Bott manifold is not toric but admits an action of a certain torus. For a given flag Bott–Samelson variety, we provide a complex one-parameter family of smooth varieties, which are all diffeomorphic, such that a generic fiber is the flag Bott–Samelson variety and the special fiber is a flag Bott manifold (see Corollary 4.7). Moreover, when the Levi subgroup L_{I_k} of the parabolic subgroup P_{I_k} is of type A, we explicitly describe such flag Bott manifolds in Theorem 4.10 in terms of the Chern classes of the line bundles used in the construction of the flag Bott manifold.

For a given flag Bott–Samelson variety, there exists a Bott–Samelson variety which is birationally equivalent to the flag Bott–Samelson variety. Moreover, using the result of Grossberg and Karshon [1994], and our one-parameter family, we obtain two manifolds: a flag Bott manifold and a Bott manifold, and a map between them. We study a relation between these manifolds. Actually, considering torus actions on these manifolds, we describe the Duistermaat–Heckman measure of the flag Bott manifold with a certain closed 2-form using a Grossberg–Karshon twisted cube in Theorem 5.5.

This paper is organized as follows. In Section 2, we provide the definition of flag Bott–Samelson varieties and their properties. In particular, we investigate a relation between flag Bott–Samelson varieties and Bott–Samelson varieties. Moreover, we describe the set of holomorphic sections of a line bundle over a flag Bott–Samelson variety using generalized Demazure modules in Theorem 2.20. Using this association, we describe the Newton–Okounkov bodies of flag Bott–Samelson varieties in Theorem 2.22. In Section 3, we give an application of Newton–Okounkov bodies of flag Bott–Samelson varieties to representation theory. Indeed, we provide a way to compute the multiplicities of representations in the tensor product of representations counting certain lattice points in the Newton–Okounkov bodies of flag Bott–Samelson varieties in Theorem 3.19. In Section 4, we present a Grossberg–Karshon type degeneration of complex structures on flag Bott–Samelson varieties, and explicitly describe the corresponding flag Bott manifold when all Levi subgroups of parabolic subgroups P_{I_k} are of type A in Theorem 4.10. In Section 5, we study torus actions on flag Bott manifolds which are obtained by the degeneration of flag Bott–Samelson manifolds. Moreover, we describe the Duistermaat–Heckman measure of flag Bott manifolds using Grossberg–Karshon twisted cubes in Theorem 5.5.

2. Newton–Okounkov bodies of flag Bott–Samelson varieties

2A. ***Definition of flag Bott–Samelson varieties.*** In this subsection we introduce flag Bott–Samelson varieties which generalize both Bott–Samelson varieties and flag varieties, and study their properties. We notice that the notion of flag Bott–Samelson varieties is already considered in Jantzen's book [2003, II.13] without naming it.

Let G be a simply connected semisimple algebraic group of rank n over $\mathbb{C}$. Choose a Cartan subgroup H, and let $\mathfrak{g} = \mathfrak{h} \oplus \sum_\alpha \mathfrak{g}_\alpha$ be the decomposition of the Lie algebra $\mathfrak{g}$ of G into root spaces, where $\mathfrak{h}$ is the Lie algebra of H. Let $\Delta \subset \mathfrak{h}^*$ denote the roots of G. Choose a set of positive roots $\Delta^+ \subset \Delta$, and let $\Sigma = \{\alpha_1, \dots, \alpha_n\} \subset \Delta^+$ denote the simple roots. Let $\Delta^- := -\Delta^+$ be the set of negative roots. Let B be the Borel subgroup whose Lie algebra is $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. Let $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ denote the coroots, and $\{\varpi_1, \dots, \varpi_n\}$ the fundamental weights which are characterized by the relation $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$. Here, δ_{ij} denotes the Kronecker symbol. Let $s_i \in W$ denote the simple reflection in the Weyl group W of G corresponding to the simple root α_i.

For a subset I of $[n] := \{1, \dots, n\}$, define the subtorus $H_I \subset H$ as

$$H_I := \{h \in H \mid \alpha_i(h) = 1 \text{ for all } i \in I\}^0. \tag{2-1}$$

Here, for a group G, G^0 is the connected component which contains the identity element of G. Then the centralizer $C_G(H_I) = \{g \in G \mid gh = hg \text{ for all } h \in H_I\}$ of

H_I is a connected reductive subgroup of G whose Weyl group is isomorphic to $W_I := \langle s_i \mid i \in I \rangle$. We set $L_I := C_G(H_I)$ for simplicity. Then the Borel subgroup B_I of L_I is $B \cap L_I$ (see [Springer 1998, §8.4.1]). Let Δ_I be the subset $\Delta \cap \mathrm{span}_{\mathbb{Z}}\{\alpha_i \mid i \in I\}$ of Δ. The set of roots $\Delta^+ \setminus \Delta_I$ defines the unipotent subgroup U_I of G satisfying the condition

$$\mathrm{Lie}(U_I) = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_\alpha.$$

The *parabolic subgroup* P_I of G corresponding to I is defined to be $P_I := L_I U_I$. The subgroup L_I is called a *Levi subgroup* of P_I.

Note that the Lie algebra of the parabolic subgroup P_I is

$$\mathrm{Lie}(P_I) = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Delta^- \cap \Delta_I} \mathfrak{g}_\alpha.$$

Moreover the parabolic subgroup P_I can be described that

$$P_I = \bigcup_{w \in W_I} BwB = \overline{Bw_I B} \subset G,$$

where w_I be the longest element in W_I (see [Springer 1998, Theorem 8.4.3]).

We now define the flag Bott–Samelson variety using a sequence of parabolic subgroups. Let $\mathcal{I} = (I_1, \dots, I_r)$ be a sequence of subsets of $[n]$, and let $\boldsymbol{P}_{\mathcal{I}} = P_{I_1} \times \cdots \times P_{I_r}$. Define a right action Θ of $B^r = \underbrace{B \times \cdots \times B}_{r}$ on $\boldsymbol{P}_{\mathcal{I}}$ as

$$\Theta\big((p_1, \dots, p_r), (b_1, \dots, b_r)\big) = \big(p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{r-1}^{-1} p_r b_r\big) \tag{2-2}$$

for $(p_1, \dots, p_r) \in \boldsymbol{P}_{\mathcal{I}}$ and $(b_1, \dots, b_r) \in B^r$.

Definition 2.1. Let $\mathcal{I} = (I_1, \dots, I_r)$ be a sequence of subsets of $[n]$. The *flag Bott–Samelson variety* $Z_{\mathcal{I}}$ is defined to be the orbit space

$$Z_{\mathcal{I}} := \boldsymbol{P}_{\mathcal{I}} / \Theta.$$

For instance, suppose that $\mathcal{I} = ([n])$. Then we have $\boldsymbol{P}_{\mathcal{I}} = G$ and the action Θ is the right multiplication of B. Therefore the flag Bott–Samelson variety $Z_{\mathcal{I}}$ is the flag variety G/B. Moreover, for the case when $|I_k| = 1$ for all k, the flag Bott–Samelson variety is a *Bott–Samelson variety*, see [Bott and Samelson 1958] for the definition of a Bott–Samelson variety. In this case we use a sequence $(i_1, \dots, i_r)$ of elements of $[n]$ rather than $(\{i_1\}, \dots, \{i_r\})$, and we write $Z_{(i_1,\dots,i_r)}$ for the corresponding Bott–Samelson variety.

For the subsequence $\mathcal{I}' = (I_1, \dots, I_{r-1})$ of $\mathcal{I}$, there is a fibration structure on the flag Bott–Samelson variety $Z_{\mathcal{I}}$:

$$P_{I_r}/B \hookrightarrow Z_{\mathcal{I}} \xrightarrow{\pi} Z_{\mathcal{I}'}, \tag{2-3}$$

where the projection map $\pi : Z_{\mathcal{I}} \to Z_{\mathcal{I}'}$ is defined as

$$\pi([p_1, \ldots, p_{r-1}, p_r]) = [p_1, \ldots, p_{r-1}].$$

On the other hand, we have that

$$Z_{\mathcal{I}} = P_{I_1} \times^B Z_{(I_2,\ldots,I_r)}.$$

Remark 2.2. For a finite sequence $\hat{w} = (w_1, \ldots, w_r)$ of elements of W, Perrin [2007] considers a tower $\widehat{X}(\hat{w})$ of Schubert varieties $X(w_1), \ldots, X(w_r)$ fibrations and call it a *generalized Bott–Samelson variety*. Let $\mathcal{I} = (I_1, \ldots, I_r)$ be a sequence of subsets of $[n]$. When we take w_k to be the longest element in W_{I_k} for $1 \le k \le r$, the generalized Bott–Samelson variety $\widehat{X}(w_1, \ldots, w_r)$ is the flag Bott–Samelson variety $Z_{\mathcal{I}}$. Indeed, using the notation in [Perrin 2007, §5.2], we obtain $P^{w_k} = P_{[n]\setminus I_k}$, $P_{w_k} = P_{I_k}$, $G_{w_k} = L_{I_k}$. Therefore, $P^{w_k} \cap G_{w_k}$ is a Borel subgroup B_{I_k} of L_{I_k} and $P_{w_k} \cap L_{I_k} = L_{I_k}$. Thus, we obtain

$$\begin{aligned}
\widehat{X}(w_1, \ldots, w_r) &= \overline{(P_{w_1} \cap G_{w_1}) w_1 (P^{w_1} \cap G_{w_1})} \times^{(P^{w_1} \cap G_{w_1})} \widehat{X}(w_2, \ldots, w_r) \\
&= \overline{B_{I_1} w_1 L_{I_1}} \times^{B_{I_1}} \widehat{X}(w_2, \ldots, w_r) \\
&= L_{I_1} \times^{B_{I_1}} \widehat{X}(w_2, \ldots, w_r) \\
&\simeq P_{I_1} \times^B \widehat{X}(w_2, \ldots, w_r).
\end{aligned}$$

Continuing this procedure, we get $\widehat{X}(w_1, \ldots, w_r) \simeq Z_{\mathcal{I}}$. This shows that flag Bott–Samelson varieties are generalized Bott–Samelson varieties. Because we are considering sequences $\hat{w} = (w_1, \ldots, w_r)$ consisting of longest elements, not all generalized Bott–Samelson varieties are flag Bott–Samelson varieties.

Let $w_k \in W_{I_k}$ be the longest element in W_{I_k} for $1 \le k \le r$. Consider the following subset of $\boldsymbol{P}_{\mathcal{I}}$:

$$\boldsymbol{P}'_{\mathcal{I}} := B w_1 B \times \cdots \times B w_r B \subset \boldsymbol{P}_{\mathcal{I}}.$$

One can check that $\boldsymbol{P}'_{\mathcal{I}}$ is closed under the action Θ of B^r in (2-2), so we consider the orbit space

$$Z'_{\mathcal{I}} := \boldsymbol{P}'_{\mathcal{I}} / \Theta.$$

It is known that flag Bott–Samelson varieties $Z_{\mathcal{I}}$ have following properties (see [Jantzen 2003, II.13] for details).

Proposition 2.3. *Let $\mathcal{I} = (I_1, \ldots, I_r)$ be a sequence of subsets of $[n]$. Then the flag Bott–Samelson variety $Z_{\mathcal{I}}$ has following properties*:

(1) *$Z_{\mathcal{I}}$ is a smooth projective variety.*

(2) *$Z'_{\mathcal{I}}$ is a dense open subset in $Z_{\mathcal{I}}$.*

(3) *$Z'_{\mathcal{I}} \simeq \mathbb{C}^{\sum_{k=1}^{r} \ell(w_k)}$, where $\ell(w_k)$ is the length of the element w_k.*

Consider the multiplication map

$$\eta : Z_{\mathcal{I}} \to G/B, \quad [p_1, \ldots, p_r] \mapsto p_1 \cdots p_r \tag{2-4}$$

which is a well-defined morphism. The following proposition says that certain flag Bott–Samelson varieties are birationally equivalent to Schubert varieties via the map η.

Proposition 2.4 [Jantzen 2003, II.13.5]. *Let $\mathcal{I} = (I_1, \ldots, I_r)$ be a sequence of subsets of $[n]$, and let $w_k \in W_{I_k}$ be the longest element in $W_{I_k} = \langle s_i \mid i \in I_k \rangle$. Set $w = w_1 \cdots w_r$. If $\ell(w) = \ell(w_1) + \cdots + \ell(w_r)$, then the morphism η induces an isomorphism between $Z'_{\mathcal{I}}$ and $BwB/B \subset G/B$. Indeed, the morphism η maps $Z_{\mathcal{I}}$ birationally onto its image $X(w) := \overline{BwB/B} \subset G/B$.*

Example 2.5. Let $G = \mathrm{SL}(4)$.

(1) Suppose that $\mathcal{I}_1 = (\{1\}, \{2\}, \{1\}, \{3\})$. Then we have $w_1 = s_1$, $w_2 = s_2$, $w_3 = s_1$, $w_4 = s_3$, and $w = s_1 s_2 s_1 s_3$, which is a reduced decomposition. Hence the morphism η gives a birational morphism between $Z_{\mathcal{I}_1}$ and $X(s_1 s_2 s_1 s_3)$.

(2) Let $\mathcal{I}_2 = (\{1, 2\}, \{3\})$. Then we have that $w_1 = s_1 s_2 s_1$, $w_2 = s_3$, and $w = w_1 w_2 = s_1 s_2 s_1 s_3$. Again, this is a reduced decomposition, so the morphism η gives a birational morphism between $Z_{\mathcal{I}_2}$ and $X(s_1 s_2 s_1 s_3)$.

Remark 2.6. Example 2.5 gives two different choices of flag Bott–Samelson varieties each of which has a birational morphism onto the same Schubert variety $X(s_1 s_2 s_1 s_3)$. For a given Schubert variety $X(w)$, there are different choices of flag Bott–Samelson varieties which define birational morphisms onto $X(w)$, and there are several studies about such different choices. See, for example, [Elnitsky 1997; Escobar et al. 2018; Tenner 2006].

We now define a multiplication map between two flag Bott–Samelson varieties. Let

$$\mathcal{J} = \left(J_{1,1}, \ldots, J_{1,N_1}, \ldots, J_{r,1}, \ldots, J_{r,N_r}\right) \tag{2-5}$$

be a sequence of subsets of $[n]$ such that $J_{k,l} \subset I_k$ for $1 \le l \le N_k$ and $1 \le k \le r$. Since each $J_{k,l}$ is contained in I_k, we have $P_{J_{k,l}} \subset P_{I_k}$ by the definition of parabolic subgroups. Hence we have a multiplication map

$$\eta_{\mathcal{J},\mathcal{I}} : Z_{\mathcal{J}} \to Z_{\mathcal{I}} \tag{2-6}$$

defined as

$$[(p_{k,l})_{1 \le k \le r, 1 \le l \le N_k}] \mapsto \left[\prod_{l=1}^{N_1} p_{1,l}, \ldots, \prod_{l=1}^{N_r} p_{r,l}\right].$$

The following proposition describes a birational morphism between two flag Bott–Samelson varieties.

Proposition 2.7. *Let* $\mathcal{I} = (I_1, \dots, I_r)$ *be a sequence of subsets of* $[n]$, *and let* $\mathcal{J} = \bigl(J_{1,1}, \dots, J_{1,N_1}, \dots, J_{r,1}, \dots, J_{r,N_r}\bigr)$ *be a sequence of subsets of* $[n]$ *such that* $J_{k,1}, \dots, J_{k,N_k} \subset I_k$ *for* $1 \le k \le r$. *Let* $w_{k,l}$, *respectively* v_k, *be the longest element in* $W_{J_{k,l}}$, *respectively in* W_{I_k}. *Suppose that*

$$w_{k,1} \cdots w_{k,N_k} = v_k \quad \textit{and} \quad \ell(w_{k,1}) + \cdots + \ell(w_{k,N_k}) = \ell(v_k) \quad \textit{for } 1 \le k \le r.$$

Then the multiplication map $\eta_{\mathcal{J},\mathcal{I}} : Z_{\mathcal{J}} \to Z_{\mathcal{I}}$ *in* (2-6) *induces an isomorphism between dense open subsets* $Z'_{\mathcal{J}} \xrightarrow{\sim} Z'_{\mathcal{I}}$.

There always exists a sequence $(i_{k,1}, \dots, i_{k,N_k}) \in [n]^{N_k}$ which is a reduced word for the longest element in W_{I_k} for $1 \le k \le r$. Concatenating such sequences we get a sequence $\boldsymbol{i} = (i_{k,l})_{1 \le k \le r, 1 \le l \le N_k} \in [n]^{N_1 + \cdots + N_r}$. Hence for a given flag Bott–Samelson variety $Z_{\mathcal{I}}$ one can always find a Bott–Samelson variety $Z_{\boldsymbol{i}}$ which is birationally isomorphic to $Z_{\mathcal{I}}$.

Proof of Proposition 2.7. First we recall from [Bourbaki 2002, VI. §1, Corollary 2 of Proposition 17; Jantzen 2003, II.13.1] that for a reduced decomposition $w = s_{i_1} \cdots s_{i_N} \in W$, the subgroup $U(w) \subset G$ is defined to be

$$U(w) := U_{\alpha_{i_1}} \cdot U_{s_{i_1}(\alpha_{i_2})} \cdot U_{s_{i_1} s_{i_2}(\alpha_{i_3})} \cdots U_{s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N})}.$$

Moreover, we have an isomorphism

$$\psi(w) : U_{\alpha_{i_1}} \times U_{\alpha_{i_2}} \times \cdots \times U_{\alpha_{i_N}} \xrightarrow{\sim} U(w) \tag{2-7}$$

which is defined to be $(u_1, \dots, u_N) \mapsto u_1 s_{i_1} u_2 s_{i_2} \cdots u_N s_{i_N} w^{-1}$. Also we have another isomorphism $\psi_{\mathcal{I}}$ between varieties:

$$\psi_{\mathcal{I}} : U(v_1) \times \cdots \times U(v_r) \xrightarrow{\sim} Z'_{\mathcal{I}} \tag{2-8}$$

which sends $(g_1, \dots, g_r)$ to $[g_1 v_1, \dots, g_r v_r]$ (see [Jantzen 2003, II.13.5]).

Because of the assumption, the concatenation $w_{k,1} \cdots w_{k,N_k}$ is a reduced decomposition of the element v_k. Hence we have an isomorphism induced by (2-7):

$$\psi_k : U(w_{k,1}) \times \cdots \times U(w_{k,N_k}) \xrightarrow{\sim} U(v_k)$$

which maps $(u_1, \dots, u_{N_k})$ to $u_1 w_{k,1} u_2 w_{k,2} \cdots u_{N_k} w_{k,N_k} v_k^{-1}$ for $1 \le k \le r$. Combining isomorphisms ψ_k and (2-8) we have the following commutative diagram:

$$\begin{tikzcd}
Z'_{\mathcal{J}} \arrow[d, "\eta_{\mathcal{J},\mathcal{I}}"'] & U(w_{1,1}) \times \cdots \times U(w_{1,N_1}) \times \cdots \times U(w_{r,1}) \times \cdots \times U(w_{r,N_r}) \arrow[l, "\psi_{\mathcal{J}}"{above}, "\sim"{below}] \arrow[d, "\psi_1 \times \cdots \times \psi_r", "\wr"'] \\
Z'_{\mathcal{I}} & U(v_1) \times \cdots \times U(v_r) \arrow[l, "\psi_{\mathcal{I}}"{above}, "\sim"{below}]
\end{tikzcd}$$

Hence the result follows. □

Example 2.8. Let $G = \mathrm{SL}(4)$, and let $\mathcal{I} = (\{1,2\},\{3\})$. Then $w_1 = s_1 s_2 s_1$, respectively $w_2 = s_3$, is a reduced decomposition of the longest element of $W_{\{1,2\}}$, respectively $W_{\{3\}}$. Then we have the birational morphism $\eta_{(1,2,1,3),\mathcal{I}} : Z_{(1,2,1,3)} \to Z_{\mathcal{I}}$. Together with the birational morphism η described in Example 2.5(2), we can see that three varieties $Z_{(1,2,1,3)}$, $Z_{\mathcal{I}}$, and $X(s_1 s_2 s_1 s_3)$ are birationally equivalent:

$$Z_{(1,2,1,3)} \to Z_{\mathcal{I}} \to X(s_1 s_2 s_1 s_3).$$

On the other hand, we have another reduced decomposition $w_1' = s_2 s_1 s_2$ of the longest element of $W_{\{1,2\}}$. This also gives the birational morphism

$$\eta_{(2,1,2,3),\mathcal{I}} : Z_{(2,1,2,3)} \to Z_{\mathcal{I}}.$$

Hence we have the following diagram whose maps are all birational morphisms:

$$\begin{tikzcd}
Z_{(1,2,1,3)} \arrow[dr] & & \\
& Z_{\mathcal{I}} \arrow[r] & X(s_1 s_2 s_1 s_3) \\
Z_{(2,1,2,3)} \arrow[ur] & &
\end{tikzcd}$$

2B. ***Line bundles over flag Bott–Samelson varieties.*** Let $\mathcal{I}$ be a sequence of subsets of $[n]$. In this subsection we study line bundles over a flag Bott–Samelson variety $Z_{\mathcal{I}}$ and their pullbacks in Proposition 2.10. For an integral weight $\lambda \in \mathbb{Z}\varpi_1 + \cdots + \mathbb{Z}\varpi_n$, we have the homomorphism $e^{\lambda} : H \to \mathbb{C}^*$. We extend it to the homomorphism $e^{\lambda} : B \to \mathbb{C}^*$ by composing with the homomorphism

$$\Upsilon : B \to H \tag{2-9}$$

induced by the canonical projection of Lie algebras $\mathfrak{b} \to \mathfrak{h}$ as in [Jantzen 2003, II.1.8]. Suppose that $\lambda_1, \dots, \lambda_r$ are integral weights. Define a representation $\mathbb{C}_{\lambda_1,\dots,\lambda_r}$ of $B^r = B \times \cdots \times B$ (r factors) on $\mathbb{C}$ as

$$(b_1, \dots, b_r) \cdot v = e^{\lambda_1}(b_1) \cdots e^{\lambda_r}(b_r) v.$$

From this we can build a line bundle over $Z_{\mathcal{I}}$ by setting

$$\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r} = \boldsymbol{P}_{\mathcal{I}} \times_{B^r} \mathbb{C}_{-\lambda_1,\dots,-\lambda_r}, \tag{2-10}$$

where an action of B^r is defined as

$$\begin{aligned}(p_1, \dots, p_r, w) \cdot (b_1, \dots, b_r) \\ = \bigl(\Theta((p_1, \dots, p_r), (b_1, \dots, b_r)), e^{\lambda_1}(b_1) \cdots e^{\lambda_r}(b_r) w\bigr).\end{aligned} \tag{2-11}$$

For simplicity, we use the following notation:

$$\mathcal{L}_{\mathcal{I},\lambda} := \mathcal{L}_{\mathcal{I},0,\dots,0,\lambda}. \tag{2-12}$$

Remark 2.9. Recall from [Fulton 1998, Example 19.1.11(d)] that for a flag bundle X over Y, the cycle map $\mathrm{cl}_X : A_k(X) \to H_{2k}(X)$ is an isomorphism if and only if cl_Y is an isomorphism. Moreover, the cycle map is isomorphic for an arbitrary flag manifold. Since a flag Bott–Samelson variety is an iterated bundle of flags P_{I_k}/B over a point, the cycle map $\mathrm{cl}_{Z_{\mathcal{I}}} : A_k(Z_{\mathcal{I}}) \to H_{2k}(Z_{\mathcal{I}})$ is an isomorphism. On the other hand, since flag Bott–Samelson varieties are smooth (see Proposition 2.3(1)), we obtain the following isomorphisms

$$\mathrm{Pic}(Z_{\mathcal{I}}) \xrightarrow{\cong} A_{(\dim_{\mathbb{C}} Z_{\mathcal{I}})-1}(Z_{\mathcal{I}}) \xrightarrow[\mathrm{cl}_{Z_{\mathcal{I}}}]{\cong} H_{2(\dim_{\mathbb{C}} Z_{\mathcal{I}})-2}(Z_{\mathcal{I}}) \xrightarrow{\cong} H^2(Z_{\mathcal{I}}). \tag{2-13}$$

Here, the first isomorphism comes from [Fulton 1998, Example 2.1.1] and the last isomorphism is obtained by the Poincaré duality. Indeed, $c_1 : \mathrm{Pic}(Z_{\mathcal{I}}) \to H^2(Z_{\mathcal{I}})$ is the isomorphism (2-13). When the Levi subgroup L_{I_k} of P_{I_k} has Lie type A for all k, we present the association (2-13) using a certain generator of $H^2(Z_{\mathcal{I}})$ in (4-3), and we present the first Chern class of the line bundle $\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ in Proposition 4.9.

Specifically when a flag Bott–Samelson variety is a usual Bott–Samelson variety, we will choose the weights to be of special form. We recall a description of the Picard group of $Z_{\boldsymbol{i}}$ from [Lauritzen and Thomsen 2004]. For given an integer vector $\boldsymbol{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$, we define a sequence of weights $\lambda_1, \dots, \lambda_r$ associated to the word $\boldsymbol{i} = (i_1, \dots, i_r)$ and the vector $\boldsymbol{a}$ by setting

$$\lambda_1 := a_1 \varpi_{i_1}, \dots, \lambda_r := a_r \varpi_{i_r}.$$

For such λ_j we use the notation

$$\mathcal{L}_{\boldsymbol{i},\boldsymbol{a}} := \mathcal{L}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r}. \tag{2-14}$$

Since a Bott–Samelson variety is an iterated sequence of projective bundles, the Picard group of Bott–Samelson variety $Z_{\boldsymbol{i}}$ is a free abelian group of rank r by [Hartshorne 1977, Exercise II.7.9]. Indeed, the association between $\boldsymbol{a} \in \mathbb{Z}^r$ and $\mathcal{L}_{\boldsymbol{i},\boldsymbol{a}}$ gives an isomorphism between $\mathbb{Z}^r$ and $\mathrm{Pic}(Z_{\boldsymbol{i}})$ (see [Lauritzen and Thomsen 2004, §3.1]).

Let $\boldsymbol{i} = (i_{k,l})_{1 \le k \le r, 1 \le l \le N_k} \in [n]^{N_1 + \cdots + N_r}$ be a sequence such that $(i_{k,1}, \dots, i_{k,N_k})$ is a reduced word for the longest element in W_{I_k} for $1 \le k \le r$. Recall from Proposition 2.7 that we have the birational morphism $\eta_{\boldsymbol{i},\mathcal{I}} : Z_{\boldsymbol{i}} \to Z_{\mathcal{I}}$. The following proposition describes the pullback bundle $\eta_{\boldsymbol{i},\mathcal{I}}^* \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ under the morphism $\eta_{\boldsymbol{i},\mathcal{I}}$ in terms of an integer vector.

Proposition 2.10. *Let $\mathcal{I}$, $\boldsymbol{i}$, and $\lambda_1, \dots, \lambda_r$ be as above. The pullback bundle $\eta_{\boldsymbol{i},\mathcal{I}}^* \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ over the Bott–Samelson variety $Z_{\boldsymbol{i}}$ is isomorphic to the line bundle $\mathcal{L}_{\boldsymbol{i},\boldsymbol{a}}$ for the integer vector*

$$\boldsymbol{a} = \bigl(\boldsymbol{a}_1(1), \dots, \boldsymbol{a}_1(N_1), \dots, \boldsymbol{a}_r(1), \dots, \boldsymbol{a}_r(N_r)\bigr) \in \mathbb{Z}^{N_1} \oplus \cdots \oplus \mathbb{Z}^{N_r}$$

given by

$$\boldsymbol{a}_k(l) = \begin{cases} \langle \lambda_k, \alpha_s^\vee \rangle + \displaystyle\sum_{\substack{k<j\le r \\ s\notin\{i_{t,u} \mid k<t\le j,\, 1\le u\le N_t\}}} \langle \lambda_j, \alpha_s^\vee \rangle & \text{if } l = \max\{q \mid i_{k,q} = s\}, \\ 0 & \text{otherwise}. \end{cases}$$

Example 2.11. Let $G = \mathrm{SL}(4)$, $\mathcal{I} = (\{1, 2\}, \{3\})$ and $\boldsymbol{i} = (1, 2, 1, 3)$. Consider the line bundle $\mathcal{L}_{\mathcal{I},\lambda_1,\lambda_2}$. Then the pullback line bundle $\eta^*_{\boldsymbol{i},\mathcal{I}}\mathcal{L}_{\mathcal{I},\lambda_1,\lambda_2}$ corresponds to the integer vector

$$\begin{aligned} \boldsymbol{a} &= (\boldsymbol{a}_1(1), \boldsymbol{a}_1(2), \boldsymbol{a}_1(3), \boldsymbol{a}_2(1)) \\ &= \big(0, \langle \lambda_1, \alpha_2^\vee \rangle + \langle \lambda_2, \alpha_2^\vee \rangle, \langle \lambda_1, \alpha_1^\vee \rangle + \langle \lambda_2, \alpha_1^\vee \rangle, \langle \lambda_2, \alpha_3^\vee \rangle\big). \end{aligned}$$

Remark 2.12. It is known from [Lauritzen and Thomsen 2004, Theorem 3.1, Corollary 3.3] that the line bundle $\mathcal{L}_{\boldsymbol{i},\boldsymbol{a}}$ is very ample, respectively generated by global sections, if and only if $\boldsymbol{a} \in \mathbb{Z}^{|\boldsymbol{i}|}_{>0}$, respectively $\boldsymbol{a} \in \mathbb{Z}^{|\boldsymbol{i}|}_{\ge 0}$. Suppose that $\boldsymbol{i}$ is a sequence satisfying the condition in Proposition 2.10. As we saw in Example 2.11, we cannot ensure that the pullback line bundle $\eta^*_{\boldsymbol{i},\mathcal{I}}\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ is very ample even if the weights $\lambda_1, \dots, \lambda_r$ are regular dominant weights.

Proof of Proposition 2.10. By the definition of pullback line bundles, we have

$$\eta^*_{\boldsymbol{i},\mathcal{I}}\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r} = \big\{(p, q) \in Z_{\boldsymbol{i}} \times \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r} \mid \eta_{\boldsymbol{i},\mathcal{I}}(p) = \pi_{\mathcal{I},\lambda_1,\dots,\lambda_r}(q)\big\},$$

where $\pi_{\mathcal{I},\lambda_1,\dots,\lambda_r} : \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r} \to Z_{\mathcal{I}}$. In other words,

$$\eta^*_{\boldsymbol{i},\mathcal{I}}\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r} = \left\{ ([(p_{k,l})_{1\le k\le r, 1\le l\le N_k}], [p_1, \dots, p_r, w]) \;\middle|\; \left[\prod_{l=1}^{N_1} p_{1,l}, \dots, \prod_{l=1}^{N_r} p_{r,l}\right] = [p_1, \dots, p_r] \text{ in } Z_{\mathcal{I}} \right\}. \tag{2-15}$$

Define the line bundle $\mathcal{L}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r}$ on $Z_{\boldsymbol{i}}$ by

$$\begin{aligned} \mathcal{L}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r} &:= \mathcal{L}_{\boldsymbol{i},\underbrace{0,\dots,0,\lambda_1}_{N_1},\underbrace{0,\dots,0,\lambda_2}_{N_2},\dots,\underbrace{0,\dots,0,\lambda_r}_{N_r}} \\ &= \big(\boldsymbol{P}_{\boldsymbol{i}} \times \mathbb{C}_{0,\dots,0,\lambda_1,0,\dots,0,\lambda_2,\dots,0,\dots,0,\lambda_r}\big)/B^{N_1+\dots+N_r}. \end{aligned}$$

Claim 1. $$\eta^*_{\boldsymbol{i},\mathcal{I}}\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r} \cong \mathcal{L}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r}.$$

Consider a well-defined morphism $f : \eta^*_{\boldsymbol{i},\mathcal{I}}\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r} \to \mathcal{L}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r}$ given by

$$f\big([(p_{k,l})_{k,l}], [p_1, \dots, p_r, w]\big) := [(p_{k,l})_{k,l}, Cw]. \tag{2-16}$$

Here, the value C is defined as follows. Because of the description in (2-15), for each element $([(p_{k,l})_{k,l}], [p_1, \dots, p_r, w])$ in the pullback bundle $\eta^*_{\boldsymbol{i},\mathcal{I}}\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$, there exist $b_1, \dots, b_r \in B$ such that

$$p_1 b_1 = \prod_{l=1}^{N_1} p_{1,l}, \quad b_1^{-1} p_2 b_2 = \prod_{l=1}^{N_2} p_{2,l}, \quad \dots, \quad b_{r-1}^{-1} p_r b_r = \prod_{l=1}^{N_r} p_{r,l}. \tag{2-17}$$

Using these elements $b_1, \dots, b_r$, the value C is defined by

$$C := e^{\lambda_1}(b_1) e^{\lambda_2}(b_2) \cdots e^{\lambda_r}(b_r). \tag{2-18}$$

On the other hand, we have a well-defined morphism

$$g : \mathcal{L}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r} \to \eta_{\boldsymbol{i},\mathcal{I}}^{*} \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$$

defined by

$$g([(p_{k,l})_{k,l}, w]) := \left([(p_{k,l})_{k,l}], \left[\prod_{l=1}^{N_1} p_{1,l}, \prod_{l=1}^{N_2} p_{2,l}, \dots, \prod_{l=1}^{N_r} p_{r,l}, w \right] \right). \tag{2-19}$$

We claim that both compositions $f \circ g$ and $g \circ f$ are identities. First we consider the composition $f \circ g$:

$$\begin{aligned} &f \circ g([(p_{k,l})_{k,l}, w]) \\ &\qquad = f\left([(p_{k,l})_{k,l}], \left[\prod_{l=1}^{N_1} p_{1,l}, \prod_{l=1}^{N_2} p_{2,l}, \dots, \prod_{l=1}^{N_r} p_{r,l}, w \right] \right) \quad \text{(by (2-19))} \\ &\qquad = ([(p_{k,l})_{k,l}, w]). \end{aligned}$$

Here, the last equality holds because all the elements $b_1, \dots, b_r$ satisfying the equations (2-17) are the identity element, and so $C = 1$. For the composition $g \circ f$, we obtain

$$\begin{aligned} &g \circ f\big([(p_{k,l})_{k,l}], [p_1, \dots, p_r, w]\big) \\ &\qquad = g([(p_{k,l})_{k,l}, Cw]) \\ &\qquad = \left([(p_{k,l})_{k,l}], \left[\prod_{l=1}^{N_1} p_{1,l}, \prod_{l=1}^{N_2} p_{2,l}, \dots, \prod_{l=1}^{N_r} p_{r,l}, Cw \right] \right) \\ &\qquad = \big([(p_{k,l})_{k,l}], [p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{r-1}^{-1} p_r b_r, Cw]\big) \quad \text{(by (2-17))} \\ &\qquad = \big([(p_{k,l})_{k,l}], [p_1, p_2, \dots, p_r, e^{\lambda_1}(b_1^{-1}) \cdots e^{\lambda_r}(b_r^{-1}) Cw]\big) \\ &\qquad = \big([(p_{k,l})_{k,l}], [p_1, \dots, p_r, w]\big) \quad \text{(by (2-18))}. \end{aligned}$$

Accordingly, f is an isomorphism, and Claim 1 holds.

Claim 2. $\mathcal{L}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r} \cong \mathcal{L}_{\boldsymbol{i},\boldsymbol{a}}$, *where* $\boldsymbol{a}$ *is the integer vector given in the statement of the proposition.*

To present a concrete isomorphism, we set

$$
\begin{aligned}
&k(j,s) := \max\bigl(\bigl\{k \mid 1 \le k \le j,\ i_{k,l} = s \text{ for some } 1 \le l \le N_k\bigr\} \cup \{0\}\bigr),\\
&m(j,s) := \max\{q \mid i_{k(j,s),q} = s\}
\end{aligned}
\tag{2-20}
$$

for $1 \le j \le r$ and $s \in [n]$. We define certain products $\zeta(j,s)$ of $p_{k,l}$ using $k(j,s)$ as follows:

$$
\zeta(j,s) := \biggl(\prod_{m=m(j,s)+1}^{N_{k(j,s)}} p_{k(j,s),m}\biggr)\biggl(\prod_{k=k(j,s)+1}^{j} \prod_{l=1}^{N_k} p_{k,l}\biggr).
$$

Here, $p_{0,l}$ is the identity element.

We denote the integral weight λ_k by $d_{k,1}\varpi_1 + \cdots + d_{k,n}\varpi_n$ using integers $d_{k,j}$ for $1 \le k \le r$. We consider the following morphism:

$$
f_2 : \mathcal{L}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r} \to \mathcal{L}_{\boldsymbol{i},\boldsymbol{a}}, \quad [(p_{k,l})_{k,l}, w] \mapsto [(p_{k,l})_{k,l}, C'w],
\tag{2-21}
$$

where the value C' is defined to be

$$
C' := \prod_{s=1}^{n}\prod_{j=1}^{r} e^{d_{j,s}\varpi_s}(\zeta(j,s))^{-1}.
\tag{2-22}
$$

We note that if $I \subset [n]$ and $s \notin I$, then the map $e^{\varpi_s} : B \to \mathbb{C}^*$ is naturally extended to $e^{\varpi_s} : P_I \to \mathbb{C}^*$ by setting $e^{\varpi_s}(\exp(\mathfrak{g}_\alpha)) = \{1\}$ for all $\alpha \in \Delta^- \cap \Delta_I$. Hence $e^{d_{j,s}\varpi_s}(\zeta(j,s))$ is defined.

If the map f_2 is well-defined, then we obtain Claim 2 because the inverse of f_2 is attained by multiplying $(C')^{-1}$. Therefore, to prove Claim 2, it is enough to show that f_2 is well-defined. Suppose that

$$
\bigl[(b_{k,l-1}^{-1} p_{k,l} b_{k,l})_{k,l}, e^{\lambda_1}(b_{1,N_1}) \cdots e^{\lambda_r}(b_{r,N_r})w\bigr]
$$

is another representative of the element $[(p_{k,l})_{k,l}, w]$ in $\mathcal{L}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r}$. Here, we use the convention that $b_{k,0} = b_{k-1,N_{k-1}}$ and $b_{0,l}$ is the identity element. To show the well-definedness of f_2, we have to prove that the following equality holds:

$$
\begin{aligned}
&\biggl(\prod_{k=1}^{r}\prod_{l=1}^{N_k} e^{\boldsymbol{a}_k(l)\varpi_{i_{k,l}}}(b_{k,l})\biggr)\biggl(\prod_{s=1}^{n}\prod_{j=1}^{r} e^{d_{j,s}\varpi_s}(\zeta(j,s))^{-1}\biggr)\\
&\qquad = \biggl(\prod_{j=1}^{r}\prod_{s=1}^{n} e^{d_{j,s}\varpi_s}(\zeta(j,s)')^{-1}\biggr)\biggl(\prod_{k=1}^{r} e^{\lambda_k}(b_{k,N_k})\biggr).
\end{aligned}
\tag{2-23}
$$

Here, $\zeta(j,s)'$ is defined by

$$\begin{aligned}\zeta(j,s)' &= \Bigg(\prod_{m=m(j,s)+1}^{N_{k(j,s)}} b_{k(j,s),m-1}^{-1} p_{k(j,s),m} b_{k(j,s),m}\Bigg)\Bigg(\prod_{k=k(j,s)+1}^{j}\prod_{l=1}^{N_k} b_{k,l-1}^{-1} p_{k,l} b_{k,l}\Bigg)\\ &= b_{k(j,s),m(j,s)}^{-1}\zeta(j,s) b_{j,N_j}.\end{aligned}$$

Furthermore, since the weight λ_j is the sum of $d_{j,s}\varpi_s$, we have that

$$\prod_{j=1}^{r}\prod_{s=1}^{n} e^{d_{j,s}\varpi_s}(b_{j,N_j}) = \prod_{j=1}^{r} e^{\lambda_j}(b_{j,N_j}).$$

Therefore, the right hand side of (2-23) becomes

$$\Bigg(\prod_{j=1}^{r}\prod_{s=1}^{n} e^{d_{j,s}\varpi_s}(b_{k(j,s),m(j,s)})\Bigg)\Bigg(\prod_{s=1}^{n}\prod_{j=1}^{r} e^{d_{j,s}\varpi_s}(\zeta(j,s))^{-1}\Bigg).$$

This implies that to show the equality (2-23), it is enough to show that

$$\prod_{k=1}^{r}\prod_{l=1}^{N_k} e^{\boldsymbol{a}_k(l)\varpi_{i_{k,l}}}(b_{k,l}) = \prod_{s=1}^{n}\prod_{j=1}^{r} e^{d_{j,s}\varpi_s}(b_{k(j,s),m(j,s)}). \tag{2-24}$$

The left hand side of (2-24) is written as

$$\prod_{k=1}^{r}\prod_{l=1}^{N_k} e^{\boldsymbol{a}_k(l)\varpi_{i_{k,l}}}(b_{k,l}) = \prod_{s=1}^{n}\prod_{\substack{1\le k\le r,\ 1\le l\le N_k\\ i_{k,l}=s}} e^{\boldsymbol{a}_k(l)\varpi_s}(b_{k,l}).$$

Using this observation, we verify the equality (2-24) by showing

$$\prod_{\substack{1\le k\le r,\ 1\le l\le N_k\\ i_{k,l}=s}} e^{\boldsymbol{a}_k(l)\varpi_s}(b_{k,l}) = \prod_{j=1}^{r} e^{d_{j,s}\varpi_s}(b_{k(j,s),m(j,s)}) \tag{2-25}$$

for all $s\in[n]$. Let s be an arbitrary index in $[n]$. If s does not appear in $(i_{k,l})_{k,l}$, then $k(j,s)=0$ for all j, and so the equality (2-25) holds. Otherwise, let $1\le j_1<\cdots<j_x\le r$ be the indices such that $s\in\{i_{j,1},\dots,i_{j,N_j}\}$ if and only if $j\in\{j_1,\dots,j_x\}$. By the definition of the number $k(j,s)$, we have that

$$\begin{aligned}&0=k(1,s)=\cdots=k(j_1-1,s),\\ &j_u=k(j_u,s)=\cdots=k(j_{u+1}-1,s) \quad \text{for } 1\le u\le x.\end{aligned}$$

Here, we set $j_{x+1}=r+1$. Therefore, we have that

$$\prod_{j=1}^{r} e^{d_{j,s}\varpi_s}(b_{k(j,s),m(j,s)}) = \prod_{u=1}^{x} e^{(d_{j_u,s}+\cdots+d_{j_{u+1}-1,s})\varpi_s}(b_{j_u,m(j_u,s)}).$$

On the other hand, by the definition of the integer vector $\boldsymbol{a}$, if $i_{k,l} = s$, then $k \in \{j_1, \dots, j_x\}$ and we have that

$$\boldsymbol{a}_k(l) = \begin{cases} d_{j_u,s} + d_{j_u+1,s} + \cdots + d_{j_{u+1}-1,s} & \text{if } k = j_u \text{ and } l = m(j_u, s), \\ 0 & \text{otherwise.} \end{cases}$$

Accordingly, we obtain the equality (2-25) for all $s \in [n]$, and we get the equality (2-24). Therefore, the morphism f_2 is a well-defined isomorphism. This proves Claim 2. By combining Claims 1 and 2, the result follows. □

For the reader's convenience, we provide an example for explaining notation $C, k(j, s), C'$ in the proof of Proposition 2.10.

Example 2.13. Let $G = \mathrm{SL}(4)$. Suppose that $\mathcal{I}$ and $\boldsymbol{i}$ are given as in Example 2.11. Then for an element $([p_{1,1}, p_{1,2}, p_{1,3}, p_{2,1}], [p_1, p_2, w])$ in $\eta^*_{\boldsymbol{i},\mathcal{I}} \mathcal{L}_{\mathcal{I},\lambda_1,\lambda_2}$ the value C in (2-18) is given by

$$C = e^{\lambda_1}\big(p_1^{-1} p_{1,1} p_{1,2} p_{1,3}\big) e^{\lambda_2}\big(p_2^{-1} p_1^{-1} p_{1,1} p_{1,2} p_{1,3} p_{2,1}\big).$$

Moreover the indices $k(j, s)$ in (2-20) are computed by

$$k(1,1) = 1, \quad k(1,2) = 1, \quad k(1,3) = 0, \quad k(2,1) = 1, \quad k(2,2) = 1, \quad k(2,3) = 2.$$

Hence the value C' in (2-22) is

$$C' = e^{d_{1,2}\varpi_2}(p_{1,3})^{-1} e^{d_{1,3}\varpi_3}(p_{1,1}p_{1,2}p_{1,3})^{-1} e^{d_{2,1}\varpi_1}(p_{2,1})^{-1} e^{d_{2,2}\varpi_2}(p_{1,3}p_{2,1})^{-1},$$

where $\lambda_k = d_{k,1}\varpi_1 + d_{k,2}\varpi_2 + d_{k,3}\varpi_3$ for $k = 1, 2$.

2C. *Newton–Okounkov bodies of flag Bott–Samelson varieties.*

Here we study the Newton–Okounkov bodies of flag Bott–Samelson varieties in Theorem 2.22. First we recall the definition and background of Newton–Okounkov bodies. We refer the reader to [Fujita and Naito 2017; Harada and Kaveh 2015; Kaveh 2015; Kaveh and Khovanskii 2012b] for more details. Let R be a $\mathbb{C}$-algebra without nonzero zero-divisors, and fix a total order $<$ on $\mathbb{Z}^r$, $r \geq 1$, respecting the addition.

Definition 2.14. A map $v : R \setminus \{0\} \to \mathbb{Z}^r$ is called a *valuation* on R if the following conditions hold. For every $f, g \in R \setminus \{0\}$ and $c \in \mathbb{C} \setminus \{0\}$,

(1) $v(f \cdot g) = v(f) + v(g)$,

(2) $v(cf) = v(f)$, and

(3) $v(f + g) \geq \min\{v(f), v(g)\}$ unless $f + g = 0$.

Moreover we say the valuation v has *one-dimensional leaves* if it satisfies that if $v(f) = v(g)$ then there exists a nonzero constant $\lambda \in \mathbb{C}$ such that $v(g - \lambda f) > v(g)$ or $g - \lambda f = 0$.

Let X be a projective variety of dimension r over $\mathbb{C}$ equipped with a line bundle $\mathcal{L}$ which is generated by global sections. Fix a valuation v which has one-dimensional leaves on the function field $\mathbb{C}(X)$ of X. Using the valuation v one can associate a semigroup $S \subset \mathbb{N} \times \mathbb{Z}^r$ as follows. Fix a nonzero element $\tau \in H^0(X, \mathcal{L})$. We use τ to identify $H^0(X, \mathcal{L})$ with a finite-dimensional subspace of $\mathbb{C}(X)$ by mapping

$$H^0(X, \mathcal{L}) \to \mathbb{C}(X), \quad \sigma \mapsto \sigma/\tau.$$

Similarly we have the map

$$H^0(X, \mathcal{L}^{\otimes k}) \to \mathbb{C}(X), \quad \sigma \mapsto \sigma/\tau^k.$$

Using these identifications we define the semigroup:

$$S = S(v, \tau) = \bigcup_{k>0} \bigl\{(k, v(\sigma/\tau^k)) \mid \sigma \in H^0(X, \mathcal{L}^{\otimes k}) \setminus \{0\}\bigr\} \subset \mathbb{N} \times \mathbb{Z}^r,$$

and denote by $C = C(v, \tau) \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^r$ the smallest real closed cone containing $S(v, \tau)$. Now we have the definition of Newton–Okounkov body:

Definition 2.15. The *Newton–Okounkov body associated to* $(X, \mathcal{L}, v, \tau)$ is defined to be

$$\{\boldsymbol{x} \in \mathbb{R}^r \mid (1, \boldsymbol{x}) \in C(v, \tau)\}.$$

We denote the Newton–Okounkov body by $\Delta(X, \mathcal{L}, v, \tau)$.

If we take another section $\tau' \in H^0(X, \mathcal{L}) \setminus \{0\}$ then

$$\Delta(X, \mathcal{L}, v, \tau') = \Delta(X, \mathcal{L}, v, \tau) + v(\tau/\tau').$$

Hence the Newton–Okounkov body $\Delta(X, \mathcal{L}, v, \tau)$ does not fundamentally depend on the choice of the nonzero section $\tau \in H^0(X, \mathcal{L}) \setminus \{0\}$. Accordingly, we sometimes denote the Newton–Okounkov body by $\Delta(X, \mathcal{L}, v)$.

Remark 2.16. If we choose a very ample line bundle $\mathcal{L}$ in the construction, then it is known in [Harada and Kaveh 2015, Theorem 3.9] that the Newton–Okounkov body has maximal dimension, i.e., it has real dimension r. Since we do not necessarily assume that the line bundle $\mathcal{L}$ is very ample in this paper, the real dimension of the Newton–Okounkov body may be less than r.

There are many possible valuations with one-dimensional leaves. We recall one of them introduced in [Kaveh 2015]. One can construct a valuation on the function field $\mathbb{C}(X)$ using a regular system of parameters $u_1, \dots, u_r$ in a neighborhood of a smooth point p on X. Fix a total ordering on $\mathbb{Z}^r$ respecting the addition. Let f be a polynomial in $u_1, \dots, u_r$. Suppose that $c_k u_1^{k_1} \cdots u_r^{k_r}$ is the term in f with the largest exponent $k = (k_1, \dots, k_r)$. Then

$$v(f) := (-k_1, \dots, -k_r)$$

defines a valuation on $\mathbb{C}(X)$, called the *highest term valuation* with respect to the parameters $u_1, \dots, u_r$.

Example 2.17. Let $X = Z_{\boldsymbol{i}}$ be the Bott–Samelson variety determined by a word $\boldsymbol{i} = (i_1, \dots, i_r)$. Let f_i be a nonzero element in $\mathfrak{g}_{-\alpha_i}$. Then the following map $\Phi_{\boldsymbol{i}} : \mathbb{C}^r \to Z_{\boldsymbol{i}}$ defines a coordinate system as in [Fujita 2018, §2.3; Kaveh 2015, §2.2]:

$$\Phi_{\boldsymbol{i}} : (t_1, \dots, t_r) \mapsto (\exp(t_1 f_{i_1}), \dots, \exp(t_r f_{i_r})) \mod B^r.$$

We denote the highest term valuation with respect to the lexicographic order on $\mathbb{Z}^r$ by $v_{\boldsymbol{i}}^{\text{high}}$.

There are some results on computing the Newton–Okounkov bodies using the valuation $v_{\boldsymbol{i}}^{\text{high}}$. We recall a result of Kaveh [2015]:

Example 2.18. Let $X = G/B$ be the full flag variety, and let $\mathcal{L}$ be the line bundle over X given by a dominant weight λ. Suppose that $\boldsymbol{i} = (i_1, \dots, i_m)$ is a reduced word for the longest element in the Weyl group W of G. Then the Bott–Samelson variety $Z_{\boldsymbol{i}}$ and the full flag variety G/B are birational by Proposition 2.4. Hence their function fields are isomorphic, i.e., $\mathbb{C}(Z_{\boldsymbol{i}}) \cong \mathbb{C}(G/B)$. Using the valuation $v_{\boldsymbol{i}}^{\text{high}}$ in Example 2.17, Kaveh [2015, Corollary 4.2] proves that the Newton–Okounkov body $\Delta(G/B, \mathcal{L}, v_{\boldsymbol{i}}^{\text{high}})$ can be identified with the string polytope.

The following lemma directly comes from the definition of Newton–Okounkov bodies.

Lemma 2.19. *Let $f : X \to Y$ be a birational morphism between varieties of dimension r, and let $\mathcal{L}$ be a line bundle on Y generated by global sections. Suppose that the canonical morphism $H^0(Y, \mathcal{L}^{\otimes k}) \to H^0(X, f^*\mathcal{L}^{\otimes k})$ is an isomorphism for every $k > 0$. Then their Newton–Okounkov bodies coincide, i.e.,*

$$\Delta(X, f^*\mathcal{L}, v, f^*\tau) = \Delta(Y, \mathcal{L}, v, \tau)$$

for any valuation $v : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^r$ and $\tau \in H^0(Y, \mathcal{L}) \setminus \{0\}$. Here v is regarded also as a valuation on $\mathbb{C}(Y)$ under the isomorphism $\mathbb{C}(Y) \cong \mathbb{C}(X)$.

Now we define left actions of P_{I_1} on $Z_{\mathcal{I}}$ and $\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ by

$$\begin{aligned} p \cdot [p_1, \dots, p_r] &:= [pp_1, p_2, \dots, p_r], \\ p \cdot [p_1, \dots, p_r, v] &:= [pp_1, p_2, \dots, p_r, v] \end{aligned} \tag{2-26}$$

for $p, p_1 \in P_{I_1}$, $p_2 \in P_{I_2}, \dots, p_r \in P_{I_r}$, and $v \in \mathbb{C}$. Since the projection $\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r} \twoheadrightarrow Z_{\mathcal{I}}$ is compatible with these actions, it follows that the space $H^0(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r})$ of global sections has the natural P_{I_1}-module structure.

Theorem 2.20. *Let $\mathcal{I} = (I_1, \dots, I_r)$ be a sequence of subsets of $[n]$, and let $\boldsymbol{i} = (i_{k,l})_{1 \le k \le r,\, 1 \le l \le N_k} \in [n]^{N_1+\cdots+N_r}$ be a sequence such that $(i_{k,1}, \dots, i_{k,N_k})$ is a reduced word for the longest element in W_{I_k} for $1 \le k \le r$. Let $\eta_{\boldsymbol{i},\mathcal{I}} : Z_{\boldsymbol{i}} \to Z_{\mathcal{I}}$ be the birational morphism in Proposition 2.7. Then for integral weights $\lambda_k :=$*

$d_{k,1}\varpi_1 + \cdots + d_{k,n}\varpi_n$ for $1 \le k \le r$, and the corresponding integer vector $\boldsymbol{a}$ given in Proposition 2.10,

(1) *the canonical morphism $H^0(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}) \to H^0(Z_{\boldsymbol{i}}, \mathcal{L}_{\boldsymbol{i},\boldsymbol{a}})$ is an isomorphism.*

(2) *The isomorphism in* (1) *induces the B-module isomorphism*

$$H^0(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}) \cong H^0(Z_{\boldsymbol{i}}, \mathcal{L}_{\boldsymbol{i},\boldsymbol{a}}) \otimes \mathbb{C}_{-\mu},$$

where μ is the weight defined by

$$\mu = \sum_{j=1}^{r} \sum_{s \in [n] \setminus \{i_{k,l} \mid 1 \le k \le j, 1 \le l \le N_k\}} d_{j,s}\varpi_s.$$

To prove the theorem, we recall the following lemma.

Lemma 2.21 [Jantzen 2003, II.14.5.(a)]. *Let $\varphi : Y \to X$ be a dominant and projective morphism of noetherian and integral schemes such that φ induces an isomorphism $\mathbb{C}(X) \xrightarrow{\sim} \mathbb{C}(Y)$ of function fields. If X is normal, then $\varphi_*\mathcal{O}_Y = \mathcal{O}_X$.*

Proof of Theorem 2.20. (1) Because of Propositions 2.3 and 2.7, the morphism $\eta = \eta_{\boldsymbol{i},\mathcal{I}} : Z_{\boldsymbol{i}} \to Z_{\mathcal{I}}$ satisfies all the conditions in Lemma 2.21. Hence we have that

$$\eta_*\mathcal{O}_{Z_{\boldsymbol{i}}} = \mathcal{O}_{Z_{\mathcal{I}}}. \tag{2-27}$$

Then we have the following:

$$\begin{aligned}\eta_*(\eta^*\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}) &= \eta_*(\mathcal{O}_{Z_{\boldsymbol{i}}} \otimes_{\mathcal{O}_{Z_{\boldsymbol{i}}}} \eta^*\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}) \\ &\cong \eta_*\mathcal{O}_{Z_{\boldsymbol{i}}} \otimes_{\mathcal{O}_{Z_{\mathcal{I}}}} \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r} \quad \text{(by [Hartshorne 1977, Exercise II.5.1(d)])} \\ &= \mathcal{O}_{Z_{\mathcal{I}}} \otimes_{\mathcal{O}_{Z_{\mathcal{I}}}} \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r} \quad \text{(by (2-27))} \\ &= \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}.\end{aligned}$$

Taking global sections we have an isomorphism between $H^0(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r})$ and $H^0(Z_{\boldsymbol{i}}, \eta^*_{\boldsymbol{i},\mathcal{I}}\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r})$ as $\mathbb{C}$-vector spaces. And the later one is isomorphic to $H^0(Z_{\boldsymbol{i}}, \mathcal{L}_{\boldsymbol{i},\boldsymbol{a}})$ as $\mathbb{C}$-vector spaces by Proposition 2.10.

(2) Note that there is a bijective correspondence between the set $H^0(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r})$ of holomorphic sections and the set of morphisms $f : \boldsymbol{P}_{\mathcal{I}} \to \mathbb{C}$ satisfying

$$f((p_1, \dots, p_r) \cdot (b_1, \dots, b_r)) = e^{\lambda_1}(b_1) \cdots e^{\lambda_r}(b_r) f(p_1, \dots, p_r) \tag{2-28}$$

for $(p_1, \dots, p_r) \in \boldsymbol{P}_{\mathcal{I}}$ and $(b_1, \dots, b_r) \in B^r$. Indeed, a morphism f defines a section $[p_1, \dots, p_r] \mapsto [p_1, \dots, p_r, f(p_1, \dots, p_r)]$. Using C and C' defined in the proof of Proposition 2.10 (see (2-18) and (2-22)), for a morphism f satisfying

(2-28), we associate a morphism $\tilde{f} : \boldsymbol{P_i} \to \mathbb{C}$

$$\tilde{f}((p_{k,l})_{k,l}) = C'Cf\left(\prod_{l=1}^{N_1} p_{1,l}, \ldots, \prod_{l=1}^{N_r} p_{r,l}\right)$$

which also gives a section in $H^0(Z_{\boldsymbol{i}}, \mathcal{L}_{\boldsymbol{i},\boldsymbol{a}})$. Actually, this association is the isomorphism in (1).

On the other hand, the left action of P_{I_1} on $Z_{\mathcal{I}}$ and that of $P_{i_{1,1}}$ on $Z_{\boldsymbol{i}}$ given in (2-26) define actions of B on the sets of holomorphic sections. For $b \in B$, $f : \boldsymbol{P}_{\mathcal{I}} \to \mathbb{C}$, and $\tilde{f} : \boldsymbol{P_i} \to \mathbb{C}$, we set

$$\begin{aligned}(b \cdot f)(p_1, \ldots, p_r) &:= f(b^{-1}p_1, p_2, \ldots, p_r),\\ (b \cdot \tilde{f})((p_{k,l})_{k,l}) &:= \tilde{f}(b^{-1}p_{1,1}, p_{1,2}, \ldots, p_{1,N_1}, \ldots, p_{r,N_r}).\end{aligned}$$

Recall from (2-22) that C' is the product of $e^{d_{j,s}\varpi_s}(\zeta(j,s))^{-1}$. For each $s \in [n]$ and $j \in [r]$, by (2-20), the following three conditions are equivalent:

- $p_{1,1}$ is involved in $\zeta(j,s)$;
- $k(j,s) = 0$;
- $s \in [n] \setminus \{i_{1,1}, \ldots, i_{1,N_1}, \ldots, i_{j,N_j}\}$.

Using this observation, we obtain that

$$\begin{aligned}&(b \cdot \tilde{f})((p_{k,l})_{k,l})\\ &\quad= \tilde{f}(b^{-1}p_{1,1}, p_{1,2}, \ldots, p_{1,N_1}, \ldots, p_{r,N_r})\\ &\quad= \left(\prod_{j=1}^{r} \prod_{s\in[n]\setminus\{i_{k,l} \mid 1\le k\le j, 1\le l\le N_k\}} e^{d_{j,s}\varpi_s}(b)\right) C'Cf\left(b^{-1}\prod_{l=1}^{N_1} p_{1,l}, \ldots, \prod_{l=1}^{N_r} p_{r,l}\right)\\ &\quad= e^{\mu}(b)\big(\widetilde{b \cdot f}((p_{k,l})_{k,l})\big),\end{aligned}$$

where C and C' are values determined by $(p_{k,l})_{k,l}$, and μ is the weight given in the statement. This proves the desired equality $\widetilde{b \cdot f} = e^{-\mu}(b)(b \cdot \tilde{f})$. □

As a direct consequence of Theorem 2.20(1) and Lemma 2.19 we have the following theorem.

Theorem 2.22. *Let $\mathcal{I} = (I_1, \ldots, I_r)$ be a sequence of subsets of $[n]$, and let $\boldsymbol{i} = (i_{k,l})_{1\le k\le r, 1\le l\le N_k} \in [n]^{N_1+\cdots+N_r}$ be a sequence such that $(i_{k,1}, \ldots, i_{k,N_k})$ is a reduced word for the longest element in W_{I_k} for $1 \le k \le r$. Let $\eta_{\boldsymbol{i},\mathcal{I}} : Z_{\boldsymbol{i}} \to Z_{\mathcal{I}}$ be the birational morphism defined in Proposition 2.7. Then for integral dominant weights λ_k, $1 \le k \le r$, a valuation v on $\mathbb{C}(Z_{\mathcal{I}})$, and a nonzero section $\tau \in H^0(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\ldots,\lambda_r})$, we have the equality*

$$\Delta(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}, v, \tau) = \Delta(Z_{\boldsymbol{i}}, \eta_{\boldsymbol{i},\mathcal{I}}^*\mathcal{L}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}, v, \eta_{\boldsymbol{i},\mathcal{I}}^*\tau).$$

Remark 2.23. Even if the line bundle $\mathcal{L} = \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ is very ample, the pullback bundle $\eta_{\boldsymbol{i},\mathcal{I}}^* \mathcal{L}$ is not necessarily very ample when $Z_{\mathcal{I}}$ is not a Bott–Samelson variety (see Remark 2.12). Therefore the real dimension of Newton–Okounkov body $\Delta(Z_{\boldsymbol{i}}, \eta_{\boldsymbol{i},\mathcal{I}}^* \mathcal{L}, v)$ can possibly be smaller than the complex dimension of $Z_{\boldsymbol{i}}$ as is mentioned in Remark 2.16. However, by Theorem 2.22, when $\mathcal{L}$ is very ample, we can see that

$$\dim_{\mathbb{R}} \Delta(Z_{\boldsymbol{i}}, \eta_{\boldsymbol{i},\mathcal{I}}^* \mathcal{L}, v) = \dim_{\mathbb{R}} \Delta(Z_{\mathcal{I}}, \mathcal{L}, v) = \dim_{\mathbb{C}} Z_{\mathcal{I}} = \dim_{\mathbb{C}} Z_{\boldsymbol{i}}$$

for any valuation v which has one-dimensional leaves.

By Theorem 2.22 and [Fujita 2018, Corollary 5.4], we have the following corollary.

Corollary 2.24. *Suppose that the line bundle $\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ constructed by weights $\lambda_1, \dots, \lambda_r$ is very ample. Then, the Newton–Okounkov body $\Delta(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}, v_{\boldsymbol{i}}^{\mathrm{high}})$ is a rational convex polytope of real dimension equal to the complex dimension of $Z_{\mathcal{I}}$.*

3. Applications to representation theory

In this section, we give applications of Newton–Okounkov bodies of flag Bott–Samelson varieties to representation theory, using the theory of generalized string polytopes introduced in [Fujita 2018]. We restrict ourselves to a specific class of flag Bott–Samelson varieties $Z_{\mathcal{I}}$, that is, to the case of a sequence $\mathcal{I} = (I_1, \dots, I_r)$ of subsets of $[n]$ such that $I_1 = [n]$. In this case, we have $P_{I_1} = P_{[n]} = G$. Hence the space $H^0(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r})$ of global sections has a natural G-module structure. Let

$$\chi(H) := \mathbb{Z}\varpi_1 + \cdots + \mathbb{Z}\varpi_n$$

be the character lattice, and let

$$\chi_+(H) := \mathbb{Z}_{\geq 0}\varpi_1 + \cdots + \mathbb{Z}_{\geq 0}\varpi_n$$

be the set of integral dominant weights. Fix nonzero elements $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ for $i \in [n]$. For $\lambda \in \chi_+(H)$, let $V(\lambda)$ denote the irreducible highest weight G-module over $\mathbb{C}$ with the highest weight λ, and let $v_\lambda \in V(\lambda)$ be a highest weight vector. Recall that every finite-dimensional irreducible G-module is isomorphic to $V(\lambda)$ for some $\lambda \in \chi_+(H)$, see [Humphreys 1975, §31.3], and that every finite-dimensional G-module is completely reducible, that is, isomorphic to a direct sum of irreducible G-modules (see [Humphreys 1975, §14.3]). For $\lambda_1, \dots, \lambda_r \in \chi_+(H)$, we denote by $\tau_{\mathcal{I},\lambda_1,\dots,\lambda_r} \in H^0(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r})$ the section corresponding to $\tau_{\boldsymbol{i},\boldsymbol{a}} \in H^0(Z_{\boldsymbol{i}}, \mathcal{L}_{\boldsymbol{i},\boldsymbol{a}})$ under the isomorphism in Theorem 2.20 (1), where $\tau_{\boldsymbol{i},\boldsymbol{a}}$ is the section defined in

[Fujita 2018, §2.3]. Let $\pi_{\geq 2} : \mathbb{R}^{N_1+\cdots+N_r} \twoheadrightarrow \mathbb{R}^{N_2+\cdots+N_r}$ be the canonical projection given by $\pi_{\geq 2}((x_{k,l})_{1\leq k\leq r, 1\leq l\leq N_k}) := (x_{k,l})_{2\leq k\leq r, 1\leq l\leq N_k}$, and set

$$\hat{\Delta}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r} := \pi_{\geq 2}\bigl(-\Delta(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}, v_{\boldsymbol{i}}^{\mathrm{high}}, \tau_{\mathcal{I},\lambda_1,\dots,\lambda_r})\bigr).$$

Since $\Delta(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}, v_{\boldsymbol{i}}^{\mathrm{high}}, \tau_{\mathcal{I},\lambda_1,\dots,\lambda_r})$ is a rational convex polytope, the image $\hat{\Delta}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r}$ is also a rational convex polytope. The following is the main result in this section.

Theorem 3.1. *Let $\mathcal{I} = (I_1, \dots, I_r)$ be a sequence of subsets of $[n]$ such that $I_1 = [n]$, and fix $\boldsymbol{i} = (i_{1,1}, \dots, i_{1,N_1}, \dots, i_{r,1}, \dots, i_{r,N_r}) \in [n]^{N_1+\cdots+N_r}$ such that $(i_{k,1}, \dots, i_{k,N_k})$ is a reduced word for the longest element in W_{I_k} for $1 \leq k \leq r$. For $\lambda_1, \dots, \lambda_r \in \chi_+(H)$, write*

$$H^0(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r})^* \simeq \bigoplus_{\nu\in\chi_+(H)} V(\nu)^{\oplus c^{\nu}_{\mathcal{I},\lambda_1,\dots,\lambda_r}}$$

as a G-module. Then, the multiplicity $c^{\nu}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ equals the cardinality of

$$\Bigl\{\boldsymbol{x} = (x_{k,l})_{2\leq k\leq r, 1\leq l\leq N_k} \in \hat{\Delta}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r} \cap \mathbb{Z}^{N_2+\cdots+N_r} \Bigm| \lambda_1 + \cdots + \lambda_r - \sum_{2\leq k\leq r, 1\leq l\leq N_k} x_{k,l}\alpha_{i_{k,l}} = \nu\Bigr\}.$$

Remark 3.2. Since $\Delta(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}, v_{\boldsymbol{i}}^{\mathrm{high}}, \tau_{\mathcal{I},\lambda_1,\dots,\lambda_r}) = \Delta(Z_{\boldsymbol{i}}, \mathcal{L}_{\boldsymbol{i},\boldsymbol{a}}, v_{\boldsymbol{i}}^{\mathrm{high}}, \tau_{\boldsymbol{i},\boldsymbol{a}})$ by Theorem 2.22, it is natural to ask why we consider not only $Z_{\boldsymbol{i}}$ but also $Z_{\mathcal{I}}$. The reason is that the space $H^0(Z_{\boldsymbol{i}}, \mathcal{L}_{\boldsymbol{i},\boldsymbol{a}})$ of global sections does not have a natural G-module structure because $Z_{\boldsymbol{i}}$ is not a G-variety. The theory of flag Bott–Samelson varieties gives a natural framework to relate the usual Bott–Samelson variety $Z_{\boldsymbol{i}}$ with G-modules.

In order to prove Theorem 3.1, we use the theory of crystal bases, see [Kashiwara 1995] for a survey on this topic. Lusztig [1990; 1991; 1993] and Kashiwara [1991] constructed a specific $\mathbb{C}$-basis of $V(\lambda)$ via the quantized enveloping algebra associated with $\mathfrak{g}$. This is called (the specialization at $q = 1$ of) the *lower global basis* (= the *canonical basis*), and denoted by $\{G^{\mathrm{low}}_{\lambda}(b) \mid b \in \mathcal{B}(\lambda)\} \subset V(\lambda)$. See, for example, [Kashiwara 1995, §12] for the definition of $G^{\mathrm{low}}_{\lambda}(b)$. In this manuscript, we put "low" to emphasize that we are considering the *lower* global basis while Kashiwara [1995] denoted it by $G_{\lambda}(b)$. The index set $\mathcal{B}(\lambda)$ is endowed with specific maps

$$\mathrm{wt} : \mathcal{B}(\lambda) \to \chi(H), \quad \varepsilon_i, \varphi_i : \mathcal{B}(\lambda) \to \mathbb{Z}_{\geq 0},$$
$$\tilde{e}_i, \tilde{f}_i : \mathcal{B}(\lambda) \to \mathcal{B}(\lambda) \cup \{0\} \quad \text{for } i \in [n],$$

which have the following properties:

$$\begin{aligned}
\mathrm{wt}(b_\lambda) &= \lambda, \\
\mathrm{wt}(\tilde{e}_i b) &= \mathrm{wt}(b) + \alpha_i && \text{if } \tilde{e}_i b \neq 0, \\
\mathrm{wt}(\tilde{f}_i b) &= \mathrm{wt}(b) - \alpha_i && \text{if } \tilde{f}_i b \neq 0, \\
\varepsilon_i(b) &= \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^k b \neq 0\}, \\
\varphi_i(b) &= \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^k b \neq 0\}, \\
e_i \cdot G_\lambda^{\mathrm{low}}(b) &\in \mathbb{C}^* G_\lambda^{\mathrm{low}}(\tilde{e}_i b) + \sum_{\substack{b' \in \mathcal{B}(\lambda);\ \mathrm{wt}(b') = \mathrm{wt}(b) + \alpha_i \\ \varphi_i(b') > \varphi_i(b) + 1}} \mathbb{C} G_\lambda^{\mathrm{low}}(b'), \\
f_i \cdot G_\lambda^{\mathrm{low}}(b) &\in \mathbb{C}^* G_\lambda^{\mathrm{low}}(\tilde{f}_i b) + \sum_{\substack{b' \in \mathcal{B}(\lambda);\ \mathrm{wt}(b') = \mathrm{wt}(b) - \alpha_i \\ \varepsilon_i(b') > \varepsilon_i(b) + 1}} \mathbb{C} G_\lambda^{\mathrm{low}}(b')
\end{aligned}$$

for $i \in [n]$ and $b \in \mathcal{B}(\lambda)$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $b_\lambda \in \mathcal{B}(\lambda)$ is defined as $G_\lambda^{\mathrm{low}}(b_\lambda) \in \mathbb{C}^* v_\lambda$, called the *highest element*. We call $\mathcal{B}(\lambda)$ the *crystal basis* for $V(\lambda)$, which satisfies the axiom of *crystals*, see [Kashiwara 1993, Definition 1.2.1] for the definition of crystals. The operations $\tilde{e}_i$ and $\tilde{f}_i$ are called the *Kashiwara operators*.

Definition 3.3 (see [Kashiwara 1995, §4.2]). The *crystal graph* of a crystal $\mathcal{B}$ is the $[n]$-colored, directed graph with vertex set $\mathcal{B}$ whose directed edges are given by: $b \xrightarrow{i} b'$ if and only if $b' = \tilde{f}_i b$.

In this paper, we identify a crystal $\mathcal{B}$ with its crystal graph. By [Kashiwara 1991, Theorem 3], for a G-module $V = V(\nu_1) \oplus \cdots \oplus V(\nu_M)$, the crystal graph of the corresponding crystal basis $\mathcal{B}(V)$ is the disjoint union of the crystal graphs $\mathcal{B}(\nu_1), \ldots, \mathcal{B}(\nu_M)$.

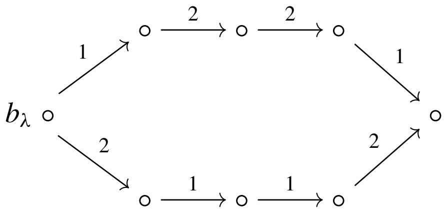
Proposition 3.4 (see [Kashiwara 1993, Proposition 3.2.3]). *Let $\boldsymbol{i} = (i_1, \ldots, i_r) \in [n]^r$ be a reduced word for $w \in W$, and $\lambda \in \chi_+(H)$. Then, the subset*

$$\mathcal{B}_w(\lambda) := \left\{ \tilde{f}_{i_1}^{x_1} \cdots \tilde{f}_{i_r}^{x_r} b_\lambda \mid x_1, \ldots, x_r \in \mathbb{Z}_{\geq 0} \right\} \setminus \{0\} \subset \mathcal{B}(\lambda)$$

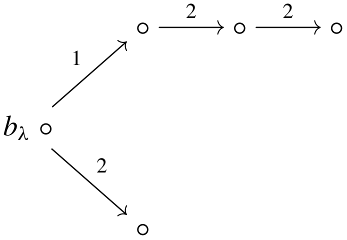
is independent of the choice of a reduced word $\boldsymbol{i}$.

The subset $\mathcal{B}_w(\lambda)$ is called a *Demazure crystal.*

Example 3.5. Let $G = \mathrm{SL}(3)$, and $\lambda = \alpha_1 + \alpha_2 = \varpi_1 + \varpi_2$. Then, the crystal graph of $\mathcal{B}(\lambda)$ is given as follows:



In addition, for $w = s_2 s_1 \in W$, the following directed graph gives the Demazure crystal $\mathcal{B}_w(\lambda)$:



The following is an immediate consequence of [Kashiwara 1993, Proposition 3.2.3].

Lemma 3.6. *Let $\boldsymbol{i} = (i_1, \dots, i_N) \in [n]^N$ be a reduced word for the longest element $w_0 \in W$. Then, the following equalities hold for all $\lambda \in \chi_+(H)$:*

$$\mathcal{B}(\lambda) = \mathcal{B}_{w_0}(\lambda) = \left\{ \tilde{f}_{i_1}^{x_1} \cdots \tilde{f}_{i_N}^{x_N} b_\lambda \mid x_1, \dots, x_N \in \mathbb{Z}_{\geq 0} \right\} \setminus \{0\}.$$

In particular, the following equality holds for all $w \in W$:

$$\left\{ \tilde{f}_{i_1}^{x_1} \cdots \tilde{f}_{i_N}^{x_N} b \mid x_1, \dots, x_N \in \mathbb{Z}_{\geq 0},\ b \in \mathcal{B}_w(\lambda) \right\} \setminus \{0\} = \mathcal{B}(\lambda).$$

For two crystals $\mathcal{B}_1, \mathcal{B}_2$, we can define another crystal $\mathcal{B}_1 \otimes \mathcal{B}_2$, called the *tensor product* of $\mathcal{B}_1$ and $\mathcal{B}_2$, see [Kashiwara 1993, §1.3] for the definition. For $\lambda_1, \dots, \lambda_r \in \chi_+(H)$, the tensor product $\mathcal{B}(\lambda_1) \otimes \cdots \otimes \mathcal{B}(\lambda_r)$ is identical to the crystal basis for the tensor product module $V(\lambda_1) \otimes \cdots \otimes V(\lambda_r)$ by [Kashiwara 1991, Theorem 1]. Let us recall the definitions of generalized Demazure crystals and generalized string polytopes.

Definition 3.7 (see [Lakshmibai et al. 2002, §1.2]). Let $\boldsymbol{i} = (i_1, \dots, i_r) \in [n]^r$ be an arbitrary word, and $\boldsymbol{a} = (a_1, \dots, a_r) \in \mathbb{Z}_{\geq 0}^r$. We define

$$\mathcal{B}_{\boldsymbol{i},\boldsymbol{a}} \subset \mathcal{B}(a_1 \varpi_{i_1}) \otimes \cdots \otimes \mathcal{B}(a_r \varpi_{i_r})$$

to be the subset

$$\Big\{ \tilde{f}_{i_1}^{x_1} \big(b_{a_1 \varpi_{i_1}} \otimes \tilde{f}_{i_2}^{x_2} \big(b_{a_2 \varpi_{i_2}} \otimes \cdots \otimes \tilde{f}_{i_{r-1}}^{x_{r-1}} (b_{a_{r-1} \varpi_{i_{r-1}}} \otimes \tilde{f}_{i_r}^{x_r} (b_{a_r \varpi_{i_r}})) \cdots \big) \big) \mid x_1, \dots, x_r \in \mathbb{Z}_{\geq 0} \Big\} \setminus \{0\};$$

this is called a *generalized Demazure crystal.*

Definition 3.8 [Fujita 2018, Definition 4.4]. Let $\boldsymbol{i} = (i_1, \dots, i_r) \in [n]^r$ be an arbitrary word, and $\boldsymbol{a} = (a_1, \dots, a_r) \in \mathbb{Z}_{\geq 0}^r$. For $b \in \mathcal{B}_{\boldsymbol{i},\boldsymbol{a}}$, we set $b(1) := b$,

$$\begin{aligned} x_1 &:= \max\{x \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_{i_1}^{x} b(1) \neq 0\}, \quad & \tilde{e}_{i_1}^{x_1} b(1) &= b_{a_1 \varpi_{i_1}} \otimes b(2), \\ x_2 &:= \max\{x \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_{i_2}^{x} b(2) \neq 0\}, \quad & \tilde{e}_{i_2}^{x_2} b(2) &= b_{a_2 \varpi_{i_2}} \otimes b(3), \\ &\ \vdots \end{aligned}$$

$$x_r := \max\{x \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_{i_r}^{x} b(r) \neq 0\},$$

and define the *generalized string parametrization* $\Omega_{\boldsymbol{i}}(b)$ of b with respect to $\boldsymbol{i}$ by $\Omega_{\boldsymbol{i}}(b) := (x_1, \ldots, x_r)$.

Definition 3.9 [Fujita 2018, Definition 4.7]. For an arbitrary word $\boldsymbol{i} \in [n]^r$ and $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$, define a subset $\mathcal{S}_{\boldsymbol{i},\boldsymbol{a}} \subset \mathbb{Z}_{>0} \times \mathbb{Z}^r$ by

$$\mathcal{S}_{\boldsymbol{i},\boldsymbol{a}} := \bigcup_{k>0} \{(k, \Omega_{\boldsymbol{i}}(b)) \mid b \in \mathcal{B}_{\boldsymbol{i},k\boldsymbol{a}}\},$$

and denote by $\mathcal{C}_{\boldsymbol{i},\boldsymbol{a}} \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^r$ the smallest real closed cone containing $\mathcal{S}_{\boldsymbol{i},\boldsymbol{a}}$. Let us define a subset $\Delta_{\boldsymbol{i},\boldsymbol{a}} \subset \mathbb{R}^r$ by

$$\Delta_{\boldsymbol{i},\boldsymbol{a}} := \{\boldsymbol{x} \in \mathbb{R}^r \mid (1, \boldsymbol{x}) \in \mathcal{C}_{\boldsymbol{i},\boldsymbol{a}}\};$$

this is called the *generalized string polytope* associated to $\boldsymbol{i}$ and $\boldsymbol{a}$.

The following is a fundamental property of generalized string polytopes.

Proposition 3.10 (see [Fujita 2018, Corollaries 4.16, 5.4(3)]). *The generalized string polytope $\Delta_{\boldsymbol{i},\boldsymbol{a}}$ is a rational convex polytope, and the equality $\Omega_{\boldsymbol{i}}(\mathcal{B}_{\boldsymbol{i},\boldsymbol{a}}) = \Delta_{\boldsymbol{i},\boldsymbol{a}} \cap \mathbb{Z}^r$ holds.*

Fujita proved the following relation between the generalized string polytope and a Newton–Okounkov body of the Bott–Samelson variety $Z_{\boldsymbol{i}}$.

Theorem 3.11 (see [Fujita 2018, Corollary 5.3]). *Let $Z_{\boldsymbol{i}}$ be the Bott–Samelson variety determined by a word $\boldsymbol{i} \in [n]^r$, and let $\mathcal{L}_{\boldsymbol{i},\boldsymbol{a}}$ be the line bundle on $Z_{\boldsymbol{i}}$ determined by an integer vector $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ as in (2-14). Then we have that*

$$\Delta(Z_{\boldsymbol{i}}, \mathcal{L}_{\boldsymbol{i},\boldsymbol{a}}, v_{\boldsymbol{i}}^{\text{high}}, \tau_{\boldsymbol{i},\boldsymbol{a}}) = -\Delta_{\boldsymbol{i},\boldsymbol{a}}.$$

Remark 3.12. The combinatorial structure of generalized string polytopes is quite complicated that even their real dimensions are not easy to be determined. By Remark 2.23, Theorem 3.11 determines the dimensions of generalized string polytopes of the type $\Delta(Z_{\boldsymbol{i}}, \eta_{\boldsymbol{i},\mathcal{I}}^{*}\mathcal{L}, v_{\boldsymbol{i}}^{\text{high}}, \tau_{\boldsymbol{i},\boldsymbol{a}})$, where $\mathcal{I}$ is a sequence of subsets of $[n]$ and $\mathcal{L}$ is a very ample line bundle over $Z_{\mathcal{I}}$.

Let $\mathcal{I} = (I_1, \ldots, I_r)$ be a sequence of subsets of $[n]$, and fix a sequence $\boldsymbol{i} = (i_{k,l})_{1\leq k\leq r, 1\leq l\leq N_k} \in [n]^{N_1+\cdots+N_r}$ such that $(i_{k,1}, \ldots, i_{k,N_k})$ is a reduced word for the longest element in W_{I_k} for $1 \leq k \leq r$. Given $\lambda_1, \ldots, \lambda_r \in \chi_+(H)$, we denote the dual P_{I_1}-module $H^0(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\ldots,\lambda_r})^*$ by $V_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$, and define $\mathcal{B}_{\boldsymbol{i},\lambda_1,\ldots,\lambda_r} \subset \mathcal{B}(\lambda_1) \otimes \cdots \otimes \mathcal{B}(\lambda_r)$ to be the set of elements of the form

$$\tilde{f}_{i_{1,1}}^{x_{1,1}} \cdots \tilde{f}_{i_{1,N_1}}^{x_{1,N_1}} \bigl(b_{\lambda_1} \otimes \cdots \otimes \tilde{f}_{i_{r-1,1}}^{x_{r-1,1}} \cdots \tilde{f}_{i_{r-1,N_{r-1}}}^{x_{r-1,N_{r-1}}} \bigl(b_{\lambda_{r-1}} \otimes \tilde{f}_{i_{r,1}}^{x_{r,1}} \cdots \tilde{f}_{i_{r,N_r}}^{x_{r,N_r}}(b_{\lambda_r})\bigr) \cdots \bigr) \tag{3-1}$$

for some $x_{1,1}, \ldots, x_{1,N_1}, \ldots, x_{r,1}, \ldots, x_{r,N_r} \in \mathbb{Z}_{\geq 0}$.

Proposition 3.13. *For $\lambda_1, \dots, \lambda_r \in \chi_+(H)$, let $\boldsymbol{a} \in \mathbb{Z}^{N_1+\cdots+N_r}$ be the integer vector such that $\mathcal{L}_{\boldsymbol{i},\boldsymbol{a}} \simeq \eta_{\boldsymbol{i},\mathcal{I}}^* \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ as given in Proposition 2.10, and let $\mu \in \chi_+(H)$ be the weight defined in Theorem 2.20*(2).

1. *The B-module $V_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ is naturally isomorphic to $\mathbb{C}_\mu \otimes V_{\boldsymbol{i},\boldsymbol{a}}$, where $V_{\boldsymbol{i},\boldsymbol{a}}$ is the generalized Demazure module defined in* [Lakshmibai et al. 2002, §1.1].
2. *There is a natural bijective map*
$$\mathcal{B}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r} \xrightarrow{\sim} b_\mu \otimes \mathcal{B}_{\boldsymbol{i},\boldsymbol{a}}$$
compatible with the crystal structures.
3. *The crystal graph of $\mathcal{B}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r}$ is identical to that of $\mathcal{B}_{\boldsymbol{i},\boldsymbol{a}}$.*

Proof. (1) The assertion is an immediate consequence of Theorem 2.20 and [Lakshmibai et al. 2002, Theorem 6].

(2) For $\lambda, \mu \in \chi_+(H)$, the crystal basis $\mathcal{B}(\lambda+\mu)$ can be regarded as a connected component of $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ by identifying $b_{\lambda+\mu}$ with $b_\lambda \otimes b_\mu$ (see [Kashiwara 1995, §4.5]). If we identify b_λ with $b_{\lambda - \langle \lambda, \alpha_i^\vee \rangle \varpi_i} \otimes b_{\langle \lambda, \alpha_i^\vee \rangle \varpi_i}$ for $i \in [n]$ and $\lambda \in \chi_+(H)$, then the definition of tensor product crystals implies that

$$\tilde{f}_i^a b_\lambda = b_{\lambda - \langle \lambda, \alpha_i^\vee \rangle \varpi_i} \otimes \tilde{f}_i^a b_{\langle \lambda, \alpha_i^\vee \rangle \varpi_i} \quad \text{for all } a \in \mathbb{Z}_{\geq 0}$$

(see [Fujita 2018, Appendix A]). Hence it follows that

$$\begin{aligned} &\tilde{f}_{i_{1,1}}^{x_{1,1}} \cdots \tilde{f}_{i_{1,N_1}}^{x_{1,N_1}} (b_{\lambda_1} \otimes b) \\ &\qquad = b_{\lambda_1 - \sum_{1 \leq l \leq N_1} \mu_l} \otimes \tilde{f}_{i_{1,1}}^{x_{1,1}} \big(b_{\mu_1} \otimes \tilde{f}_{i_{1,2}}^{x_{1,2}} \big(b_{\mu_2} \otimes \cdots \otimes \tilde{f}_{i_{1,N_1}}^{x_{1,N_1}} (b_{\mu_{N_1}} \otimes b) \cdots \big)\big) \end{aligned}$$

for $b \in \mathcal{B}_{\boldsymbol{i}_{\geq 2},\lambda_2,\dots,\lambda_r}$ and $x_{1,1}, \dots, x_{1,N_1} \in \mathbb{Z}_{\geq 0}$, where

$$\mu_l := \begin{cases} \langle \lambda_1, \alpha_{i_{1,l}}^\vee \rangle \varpi_{i_{1,l}} & \text{if } l = \max\{1 \leq q \leq N_1 \mid i_{1,q} = i_{1,l}\}, \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq l \leq N_1$, and $\boldsymbol{i}_{\geq 2} := (i_{k,l})_{2 \leq k \leq r, 1 \leq l \leq N_k}$. By repeating this deformation, all the elements of the form (3-1) can be naturally written as elements in $b_\mu \otimes \mathcal{B}_{\boldsymbol{i},\boldsymbol{a}}$. This proves part (2).

(3) Let us prove that $\tilde{e}_i(b_\mu \otimes b) = b_\mu \otimes \tilde{e}_i b$ for all $i \in [n]$ and $b \in \mathcal{B}_{\boldsymbol{i},\boldsymbol{a}}$. By the definition of $\mathcal{B}_{\boldsymbol{i},\boldsymbol{a}}$, we have

$$\mathrm{wt}(b) - \mathrm{wt}(b') \in \sum_{j \in \{i_{k,l} \mid 1 \leq k \leq r,\ 1 \leq l \leq N_k\}} \mathbb{Z}\alpha_j$$

for all $b, b' \in \mathcal{B}_{\boldsymbol{i},\boldsymbol{a}}$. Hence $\mathcal{B}_{\boldsymbol{i},\boldsymbol{a}}$ does not have edges labeled by

$$j \notin \{i_{k,l} \mid 1 \leq k \leq r,\ 1 \leq l \leq N_k\}.$$

From this, we may assume that $i \in \{i_{k,l} \mid 1 \leq k \leq r,\ 1 \leq l \leq N_k\}$. Then, we have $\langle \mu, \alpha_i^\vee \rangle = 0$ by the definition of μ, which implies by the definition of tensor product crystals that $\tilde{e}_i(b_\mu \otimes b) = b_\mu \otimes \tilde{e}_i b$. Thus, we have proved that the crystal graph of $b_\mu \otimes \mathcal{B}_{\boldsymbol{i},\boldsymbol{a}}$ is identical to that of $\mathcal{B}_{\boldsymbol{i},\boldsymbol{a}}$. Then, part (3) follows immediately from part (2). □

Proposition 3.13 implies that all the results in [Lakshmibai et al. 2002] for $V_{\boldsymbol{i},\boldsymbol{a}}$ and $\mathcal{B}_{\boldsymbol{i},\boldsymbol{a}}$ are applicable also for $V_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ and $\mathcal{B}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r}$.

Proposition 3.14. *The set $\mathcal{B}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r}$ depends only on $\mathcal{I}, \lambda_1, \dots, \lambda_r$, that is, does not depend on the choice of $\boldsymbol{i}$.*

Proof. We proceed by induction on r. If $r = 1$, then the assertion is an immediate consequence of Proposition 3.4. Assume that $r \geq 2$, and that $\mathcal{B}_{\boldsymbol{i}_{\geq 2},\lambda_2,\dots,\lambda_r}$ is independent of the choice of $\boldsymbol{i}_{\geq 2}$. By [Lakshmibai et al. 2002, Theorem 2] and Proposition 3.13, it follows that $b_{\lambda_1} \otimes \mathcal{B}_{\boldsymbol{i}_{\geq 2},\lambda_2,\dots,\lambda_r}$ is a disjoint union of Demazure crystals. Hence it suffices to prove that for each connected component $\mathcal{B}_v(\lambda)$ of $b_{\lambda_1} \otimes \mathcal{B}_{\boldsymbol{i}_{\geq 2},\lambda_2,\dots,\lambda_r}$ the set

$$\mathcal{B}_{v,i_{1,1},\dots,i_{1,N_1}}(\lambda) := \left\{ \tilde{f}_{i_{1,1}}^{x_1} \cdots \tilde{f}_{i_{1,N_1}}^{x_{N_1}} b \mid x_1, \dots, x_{N_1} \in \mathbb{Z}_{\geq 0},\ b \in \mathcal{B}_v(\lambda) \right\} \setminus \{0\}$$

does not depend on the choice of $(i_{1,1}, \dots, i_{1,N_1})$. We define $v_1, \dots, v_{N_1} \in W$ inductively by

$$v_1 := \begin{cases} s_{i_{1,N_1}} v & \text{if } \ell(s_{i_{1,N_1}} v) > \ell(v), \\ v & \text{if } \ell(s_{i_{1,N_1}} v) < \ell(v), \end{cases}$$
$$v_l := \begin{cases} s_{i_{1,N_1-l+1}} v_{l-1} & \text{if } \ell(s_{i_{1,N_1-l+1}} v_{l-1}) > \ell(v_{l-1}), \\ v_{l-1} & \text{if } \ell(s_{i_{1,N_1-l+1}} v_{l-1}) < \ell(v_{l-1}). \end{cases}$$

Then, we deduce by [Kashiwara 1993, Proposition 3.2.3 (iii)] that $\mathcal{B}_{v,i_{1,1},\dots,i_{1,N_1}}(\lambda) = \mathcal{B}_{v_{N_1}}(\lambda)$. In addition, it follows by [Kashiwara 1993, Lemma 3.2.1 and Proposition 3.2.3 (i)] that

$$\sum_{x_1,\dots,x_{N_1} \in \mathbb{Z}_{\geq 0}} f_{i_{1,1}}^{x_1} \cdots f_{i_{1,N_1}}^{x_{N_1}} \Biggl(\sum_{b \in \mathcal{B}_v(\lambda)} \mathbb{C} G_\lambda^{\mathrm{low}}(b) \Biggr) = \sum_{b \in \mathcal{B}_{v_{N_1}}(\lambda)} \mathbb{C} G_\lambda^{\mathrm{low}}(b).$$

From these, we have

$$\sum_{b \in \mathcal{B}_{v,i_{1,1},\dots,i_{1,N_1}}(\lambda)} \mathbb{C} G_\lambda^{\mathrm{low}}(b) = \sum_{x_1,\dots,x_{N_1} \in \mathbb{Z}_{\geq 0}} f_{i_{1,1}}^{x_1} \cdots f_{i_{1,N_1}}^{x_{N_1}} \Biggl(\sum_{b \in \mathcal{B}_v(\lambda)} \mathbb{C} G_\lambda^{\mathrm{low}}(b) \Biggr);$$

the right hand side does not depend on the choice of $(i_{1,1}, \dots, i_{1,N_1})$ by [Kashiwara 1993, Proposition 3.2.5(v)], which implies that the set $\mathcal{B}_{v,i_{1,1},\dots,i_{1,N_1}}(\lambda)$ is also independent. This proves the proposition. □

We denote $\mathcal{B}_{\boldsymbol{i},\lambda_1,\ldots,\lambda_r}$ by $\mathcal{B}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$, which is also called a *generalized Demazure crystal*. By definition, we have

$$\mathcal{B}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}=\bigl\{\tilde{f}_{i_{1,1}}^{x_1}\cdots\tilde{f}_{i_{1,N_1}}^{x_{N_1}}(b_{\lambda_1}\otimes b)\mid x_1,\ldots,x_{N_1}\in\mathbb{Z}_{\geq 0},\ b\in\mathcal{B}_{(I_2,\ldots,I_r),\lambda_2,\ldots,\lambda_r}\bigr\}\setminus\{0\}.$$

Assume that $I_1=[n]$, and hence that $(i_{1,1},\ldots,i_{1,N_1})$ is a reduced word for $w_0\in W$. By [Lakshmibai et al. 2002, Theorem 2] and Proposition 3.13, the set $b_{\lambda_1}\otimes\mathcal{B}_{(I_2,\ldots,I_r),\lambda_2,\ldots,\lambda_r}$ is a disjoint union of Demazure crystals. Hence the second assertion of Lemma 3.6 implies that each connected component of $\mathcal{B}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$ is of the form $\mathcal{B}(\nu)$ for some $\nu\in\chi_+(H)$. Note that the character of $V_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$ equals the formal character of $\mathcal{B}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$ by [Lakshmibai et al. 2002, Theorem 5 and Corollary 10] and Proposition 3.13. Since finite-dimensional G-modules are characterized by their characters, we obtain the following.

Proposition 3.15. *Let $\mathcal{I}=(I_1,\ldots,I_r)$ be a sequence of subsets of $[n]$ such that $I_1=[n]$. Then, the generalized Demazure crystal $\mathcal{B}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$ is isomorphic to the crystal basis for the G-module $V_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$. In particular, if $\mathcal{B}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$ is the disjoint union of $\mathcal{B}(\nu_1),\ldots,\mathcal{B}(\nu_M)$, then $V_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$ is isomorphic to $V(\nu_1)\oplus\cdots\oplus V(\nu_M)$.*

Since, by Proposition 3.13(3), the crystal graph of $\mathcal{B}_{\boldsymbol{i},\boldsymbol{a}}$ is identical to that of $\mathcal{B}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$, the generalized string parametrization $\Omega_{\boldsymbol{i}}$ of $\mathcal{B}_{\boldsymbol{i},\boldsymbol{a}}$ can be regarded as a parametrization of $\mathcal{B}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$. We denote $\Delta_{\boldsymbol{i},\boldsymbol{a}}$ by $\Delta_{\boldsymbol{i},\lambda_1,\ldots,\lambda_r}$. Then, we have $\hat{\Delta}_{\boldsymbol{i},\lambda_1,\ldots,\lambda_r}=\pi_{\geq 2}(\Delta_{\boldsymbol{i},\lambda_1,\ldots,\lambda_r})$ by Theorems 2.22, 3.11.

Proof of Theorem 3.1. By Proposition 3.15, the multiplicity $c^{\nu}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$ equals the number of connected components of $\mathcal{B}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$ isomorphic to $\mathcal{B}(\nu)$. For $b\in\mathcal{B}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$, we write $\Omega_{\boldsymbol{i}}(b)=(x_{1,1},\ldots,x_{1,N_1},\ldots,x_{r,1},\ldots,x_{r,N_r})$. By the definition of $\Omega_{\boldsymbol{i}}$, we have

$$x_{1,l}=\max\bigl\{x\in\mathbb{Z}_{\geq 0}\mid \tilde{e}_{i_{1,l}}^{x}\tilde{e}_{i_{1,l-1}}^{x_{1,l-1}}\cdots\tilde{e}_{i_{1,1}}^{x_{1,1}}b\neq 0\bigr\}\tag{3-2}$$

for $1\leq l\leq N_1$. Let $\mathcal{C}_b$ denote the connected component of $\mathcal{B}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$ containing b. Since $I_1=[n]$, it follows that $(i_{1,1},\ldots,i_{1,N_1})$ is a reduced word for $w_0\in W$. So we deduce by [Kashiwara 1993, Proposition 3.2.3] that $\tilde{e}_{i_{1,N_1}}^{x_{1,N_1}}\cdots\tilde{e}_{i_{1,1}}^{x_{1,1}}b$ is the highest element in $\mathcal{C}_b$. Hence

$$\bigl(0,\ldots,0,x_{2,1},\ldots,x_{2,N_2},\ldots,x_{r,1},\ldots,x_{r,N_r}\bigr)\tag{3-3}$$

is the generalized string parametrization of the highest element. In particular, the surjective map $\mathcal{B}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}\twoheadrightarrow\hat{\Delta}_{\boldsymbol{i},\lambda_1,\ldots,\lambda_r}\cap\mathbb{Z}^{N_2+\cdots+N_r}$ given by $b\mapsto\pi_{\geq 2}(\Omega_{\boldsymbol{i}}(b))$ induces a bijective map

$$\Psi:\{\text{connected components of }\mathcal{B}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}\}\xrightarrow{\sim}\hat{\Delta}_{\boldsymbol{i},\lambda_1,\ldots,\lambda_r}\cap\mathbb{Z}^{N_2+\cdots+N_r}.$$

In addition, for a connected component $\mathcal{C}$ of $\mathcal{B}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}$, the weight of the highest element in $\mathcal{C}$ is determined by $\Psi(\mathcal{C})$ due to the definition of generalized string parametrizations (see Definition 3.8). Indeed, if

$$\Psi(\mathcal{C}) = \bigl(x_{2,1}, \dots, x_{2,N_2}, \dots, x_{r,1}, \dots, x_{r,N_r}\bigr),$$

then the weight of the highest element in $\mathcal{C}$ is given by

$$\lambda_1 + \cdots + \lambda_r - \sum_{2 \le k \le r,\ 1 \le l \le N_k} x_{k,l}\alpha_{i_{k,l}}$$

since the generalized string parametrization of this element is given by (3-3). By these reasons, we deduce the assertion of the theorem. □

The following is an immediate consequence of the proof of Theorem 3.1.

Corollary 3.16. *Let $\mathcal{I} = (I_1, \dots, I_r)$ be a sequence of subsets of $[n]$ such that $I_1 = [n]$. Then, the number of connected components of $\mathcal{B}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ equals the cardinality of*

$$\hat{\Delta}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r} \cap \mathbb{Z}^{N_2+\cdots+N_r}.$$

Let $\pi_1 : \mathbb{R}^{N_1+\cdots+N_r} \twoheadrightarrow \mathbb{R}^{N_1}$ denote the canonical projection given by

$$\bigl(x_{1,1}, \dots, x_{1,N_1}, \dots, x_{r,1}, \dots, x_{r,N_r}\bigr) \mapsto (x_{1,1}, \dots, x_{1,N_1}).$$

Proposition 3.17. *For $\boldsymbol{x} \in \hat{\Delta}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r} \cap \mathbb{Z}^{N_2+\cdots+N_r}$, the set $\pi_1(\pi_{\ge 2}^{-1}(\boldsymbol{x}) \cap \Delta_{\boldsymbol{i},\lambda_1,\dots,\lambda_r})$ is identical to the string polytope for the connected component $\Psi^{-1}(\boldsymbol{x})$ of $\mathcal{B}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ with respect to the reduced word $(i_{1,1}, \dots, i_{1,N_1})$ for $w_0 \in W$; see* [Kaveh 2015, Definition 3.5; Littelmann 1998, §1] *for the definition of string polytopes.*

Proof. Recall that, by Proposition 3.10, $\Omega_{\boldsymbol{i}} : \mathcal{B}_{\mathcal{I},\lambda_1,\dots,\lambda_r} \to \Delta_{\boldsymbol{i},\lambda_1,\dots,\lambda_r} \cap \mathbb{Z}^{N_1+\cdots+N_r}$ is bijective. Hence by the definition of Ψ, we obtain the following bijective map:

$$\begin{aligned} \Psi^{-1}(\boldsymbol{x}) &\to \pi_{\ge 2}^{-1}(\boldsymbol{x}) \cap \Delta_{\boldsymbol{i},\lambda_1,\dots,\lambda_r} \cap \mathbb{Z}^{N_1+\cdots+N_r}, \\ b &\mapsto \Omega_{\boldsymbol{i}}(b). \end{aligned}$$

In addition, we see by (3-2) that $\pi_1(\Omega_{\boldsymbol{i}}(b))$ is the string parametrization of $b \in \Psi^{-1}(\boldsymbol{x})$ with respect to the reduced word $(i_{1,1}, \dots, i_{1,N_1})$; see [Littelmann 1998, §1; Kaveh 2015, Definition 3.2] for the definition of string parametrizations. From these, we obtain the assertion of the proposition. □

Remark 3.18. Kaveh and Khovanskii [2012a] gave a general framework to describe multiplicities of irreducible representations by using the Newton–Okounkov bodies. Our results give concrete constructions of convex bodies appearing in [Kaveh and Khovanskii 2012a]. Indeed, by the proof of Theorem 3.1 and [Fujita 2018, Theorem 5.2], it is not hard to prove that the rational convex polytope $\hat{\Delta}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r}$ is identical to the multiplicity convex body $\hat{\Delta}_G(A)$ in [Kaveh and Khovanskii 2012a,

§4.1] for the valuation $v_{\boldsymbol{i}}^{\text{high}}$, where

$$A := \bigoplus_{k\geq 0} H^0(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\ldots,\lambda_r}^{\otimes k}).$$

From this and Proposition 3.17, we deduce that the generalized string polytope $\Delta_{\boldsymbol{i},\lambda_1,\ldots,\lambda_r}$ equals the string convex body $\tilde{\Delta}(A)$ in [Kaveh and Khovanskii 2012a, §5.2].

In representation theory, it is a fundamental problem to determine the G-module structure of the tensor product module $V(\lambda)\otimes V(\mu)$, which is equivalent to determining the multiplicity $c_{\lambda,\mu}^{\nu}$ of $V(\nu)$ in $V(\lambda)\otimes V(\mu)$. Berenstein and Zelevinsky [2001, Theorems 2.3, 2.4] describes the multiplicity $c_{\lambda,\mu}^{\nu}$ as the number of lattice points in some explicit rational convex polytope. In the following, we see that Theorem 3.1 gives a different approach to such polyhedral expressions for $c_{\lambda,\mu}^{\nu}$. Let us consider the case $\mathcal{I} = ([n], [n])$. In this case, the flag Bott–Samelson variety $Z_{\mathcal{I}}$ is identical to $G\times_B G/B$, and the following map is an isomorphism of varieties:

$$Z_{\mathcal{I}} \xrightarrow{\sim} G/B\times G/B, \quad [g_1, g_2]\mapsto (g_1B/B, g_1g_2B/B);$$

the inverse map is given by $(g_1B/B, g_2B/B)\mapsto [g_1, g_1^{-1}g_2]$. It is easily seen that under the isomorphism $Z_{\mathcal{I}}\simeq G/B\times G/B$, the G-action on $Z_{\mathcal{I}}$ coincides with the diagonal action on $G/B\times G/B$, and the line bundle $\mathcal{L}_{\mathcal{I},\lambda,\mu}$ corresponds to the direct product of $\mathcal{L}_\lambda$ and $\mathcal{L}_\mu$, where $\mathcal{L}_\nu$ denotes the line bundle $\mathcal{L}_{([n]),\nu}$ over G/B for $\nu\in\chi_+(H)$. Hence we obtain the following isomorphisms of G-modules:

$$\begin{aligned} H^0(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda,\mu})^* &\simeq H^0(G/B\times G/B, \mathcal{L}_\lambda\times\mathcal{L}_\mu)^* \\ &\simeq H^0(G/B, \mathcal{L}_\lambda)^*\otimes H^0(G/B, \mathcal{L}_\mu)^* \\ &\simeq V(\lambda)\otimes V(\mu), \end{aligned}$$

by the Borel–Weil theorem (see [Jantzen 2003, Corollary II.5.6]). If we write

$$V(\lambda)\otimes V(\mu)\simeq \bigoplus_{\nu\in\chi_+(H)} V(\nu)^{\oplus c_{\lambda,\mu}^{\nu}}$$

as a G-module, then we obtain the following by Theorem 3.1:

Theorem 3.19. *Let $\mathcal{I} = ([n], [n])$, and let $(i_1,\ldots,i_N), (j_1,\ldots,j_N)\in [n]^N$ be reduced words for $w_0\in W$. Then, the tensor product multiplicity $c_{\lambda,\mu}^{\nu}$ equals the cardinality of*

$$\left\{(y_1,\ldots,y_N)\in\hat{\Delta}_{\boldsymbol{i},\lambda,\mu}\cap\mathbb{Z}^N \;\middle|\; \lambda+\mu-\sum_{1\leq l\leq N} y_l\alpha_{j_l} = \nu\right\},$$

where $\boldsymbol{i} := (i_1,\ldots,i_N, j_1,\ldots,j_N)$.

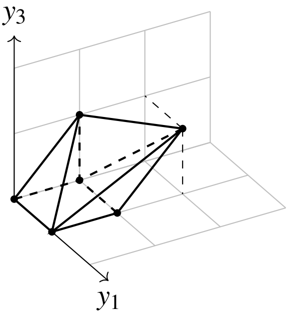


Figure 1. The polytope $\hat{\Delta}_{\boldsymbol{i},\lambda,\mu}$ in Example 3.20.

Example 3.20. Let $G = \mathrm{SL}(3)$, $\mathcal{I} = ([2], [2])$, and $\boldsymbol{i} = (1, 2, 1, 1, 2, 1)$. By [Fujita 2018, Corollary 4.15], the generalized string polytope $\Delta_{\boldsymbol{i},\lambda,\mu}$ is identical to the set of $(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6_{\geq 0}$ satisfying the following inequalities:

$$\begin{aligned} 0 \leq y_3 &\leq \min\{\lambda_2, \mu_1\}, \\ y_3 \leq y_2 &\leq y_3 + \mu_2, \\ y_2 - \lambda_2 \leq y_1 &\leq \min\{\lambda_1, y_2 - 2y_3 + \mu_1\}, \\ \max\{y_3 - \lambda_2, -y_1 + y_2 - \lambda_2\} \leq x_3 &\leq -2y_1 + y_2 - 2y_3 + \lambda_1 + \mu_1, \\ x_3 \leq x_2 &\leq x_3 + y_1 - 2y_2 + y_3 + \lambda_2 + \mu_2, \\ 0 \leq x_1 &\leq x_2 - 2x_3 - 2y_1 + y_2 - 2y_3 + \lambda_1 + \mu_1, \end{aligned}$$

where $\lambda_i := \langle \lambda, \alpha_i^\vee \rangle$ and $\mu_i := \langle \mu, \alpha_i^\vee \rangle$ for $i = 1, 2$. Hence the polytope $\hat{\Delta}_{\boldsymbol{i},\lambda,\mu}$ is identical to the set of $(y_1, y_2, y_3) \in \mathbb{R}^3_{\geq 0}$ satisfying the following inequalities:

$$\begin{aligned} 0 \leq y_3 &\leq \min\{\lambda_2, \mu_1\}, \\ y_3 \leq y_2 &\leq y_3 + \mu_2, \\ y_2 - \lambda_2 \leq y_1 &\leq \min\{\lambda_1, y_2 - 2y_3 + \mu_1\}. \end{aligned}$$

We deduce by Theorem 3.19 that the tensor product multiplicity $c^\nu_{\lambda,\mu}$ equals the cardinality of $(y_1, y_2, y_3) \in \hat{\Delta}_{\boldsymbol{i},\lambda,\mu} \cap \mathbb{Z}^3$ such that $\lambda + \mu - (y_1 + y_3)\alpha_1 - y_2\alpha_2 = \nu$.

If $\lambda = \mu = \varpi_1 + \varpi_2$, then the polytope $\hat{\Delta}_{\boldsymbol{i},\lambda,\mu}$ is identical to the set of $(y_1, y_2, y_3) \in \mathbb{R}^3_{\geq 0}$ satisfying the following inequalities:

$$0 \leq y_3 \leq 1, \quad y_3 \leq y_2 \leq y_3 + 1, \quad y_2 - 1 \leq y_1 \leq \min\{1, y_2 - 2y_3 + 1\};$$

see Figure 1. Hence we deduce that

$$V(\varpi_1 + \varpi_2)^{\otimes 2} \simeq V(2\varpi_1 + 2\varpi_2) \oplus V(3\varpi_1) \oplus V(3\varpi_2) \oplus V(\varpi_1 + \varpi_2)^{\oplus 2} \oplus V(0).$$

Theorem 3.1 can be applied to a more general class of representations than [Berenstein and Zelevinsky 2001]. We next consider the case $\mathcal{I} = ([n], [n], \dots, [n])$ (an r-tuple). In this case, we have

$$Z_{\mathcal{I}} = \underbrace{G \times_B G \times_B \cdots \times_B G}_{r} / B,$$

and this is isomorphic to $(G/B)^r := G/B \times G/B \times \cdots \times G/B$ (r factors) as follows:

$$Z_{\mathcal{I}} \xrightarrow{\sim} (G/B)^r, \quad [g_1, g_2, \dots, g_r] \mapsto \bigl(g_1 B/B, g_1 g_2 B/B, \dots, g_1 g_2 \cdots g_r B/B\bigr);$$

the inverse map is given by

$$\bigl(g_1 B/B, g_2 B/B, \dots, g_r B/B\bigr) \mapsto \bigl[g_1, g_1^{-1} g_2, g_2^{-1} g_3, \dots, g_{r-1}^{-1} g_r\bigr].$$

As in the case $r = 2$, under the isomorphism $Z_{\mathcal{I}} \simeq (G/B)^r$, the G-action on $Z_{\mathcal{I}}$ coincides with the diagonal action on $(G/B)^r$, and the line bundle $\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ corresponds to the direct product of $\mathcal{L}_{\lambda_1}, \dots, \mathcal{L}_{\lambda_r}$. From this, we have the following isomorphisms of G-modules:

$$\begin{aligned} H^0(Z_{\mathcal{I}}, \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r})^* &\simeq H^0((G/B)^r, \mathcal{L}_{\lambda_1} \times \cdots \times \mathcal{L}_{\lambda_r})^* \\ &\simeq H^0(G/B, \mathcal{L}_{\lambda_1})^* \otimes \cdots \otimes H^0(G/B, \mathcal{L}_{\lambda_r})^* \\ &\simeq V(\lambda_1) \otimes \cdots \otimes V(\lambda_r). \end{aligned}$$

If we write

$$V(\lambda_1) \otimes \cdots \otimes V(\lambda_r) \simeq \bigoplus_{\nu \in \chi_+(H)} V(\nu)^{\oplus c^{\nu}_{\lambda_1,\dots,\lambda_r}}$$

as a G-module, then Theorem 3.1 implies the following.

Corollary 3.21. *Let $\mathcal{I} = ([n], [n], \dots, [n])$, an r-tuple, and take reduced words $(i_{k,1}, \dots, i_{k,N}) \in [n]^N$, $1 \le k \le r$, for $w_0 \in W$. Then, the multiplicity $c^{\nu}_{\lambda_1,\dots,\lambda_r}$ equals the cardinality of*

$$\Bigl\{ \boldsymbol{x} = (x_{k,l})_{2 \le k \le r, 1 \le l \le N} \in \hat{\Delta}_{\boldsymbol{i},\lambda_1,\dots,\lambda_r} \cap \mathbb{Z}^{(r-1)N} \Bigm| \lambda_1 + \cdots + \lambda_r - \sum_{2 \le k \le r, 1 \le l \le N} x_{k,l} \alpha_{i_{k,l}} = \nu \Bigr\},$$

where $\boldsymbol{i} := (i_{1,1}, \dots, i_{1,N}, \dots, i_{r,1}, \dots, i_{r,N})$.

The following gives an application to $Z_{\mathcal{I}}$ for general $\mathcal{I}$ which does not necessarily start with $[n]$:

Corollary 3.22. *Let $\mathcal{I} = (I_1, \dots, I_r)$ be a sequence of subsets of $[n]$, and set $I_0 := [n]$. Fix $\boldsymbol{i}_0 = (i_{k,l})_{0 \le k \le r, 1 \le l \le N_k} \in [n]^{N_0 + \cdots + N_r}$ such that $(i_{k,1}, \dots, i_{k,N_k})$ is a* *reduced word for the longest element in* W_{I_k} *for* $0 \le k \le r$*. Then, the number of connected components of* $\mathcal{B}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ *equals the cardinality of*

$$\hat{\Delta}_{\boldsymbol{i}_0,0,\lambda_1,\dots,\lambda_r} \cap \mathbb{Z}^{N_1+\dots+N_r}.$$

Proof. We set $\mathcal{I}_0 := (I_0, I_1, \dots, I_r)$. By the definition of tensor product crystals, the bijective map $\mathcal{B}_{\mathcal{I},\lambda_1,\dots,\lambda_r} \xrightarrow{\sim} b_0 \otimes \mathcal{B}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$, $b \mapsto b_0 \otimes b$, is compatible with their crystal structures, where we mean by $b_0 \in \mathcal{B}(0)$ the element b_λ for $\lambda = 0$. Hence we may identify $b_0 \otimes \mathcal{B}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ with $\mathcal{B}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$. This implies by the definition that the crystal basis $\mathcal{B}_{\mathcal{I}_0,0,\lambda_1,\dots,\lambda_r}$ is obtained from $\mathcal{B}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ by actions of $\tilde{f}_i$, $i \in [n]$. By [Lakshmibai et al. 2002, proof of Theorem 2] and Proposition 3.13, all connected components of $\mathcal{B}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ are Demazure crystals in connected components of $\mathcal{B}(\lambda_1) \otimes \dots \otimes \mathcal{B}(\lambda_r)$. Hence they are not joined by $\tilde{f}_i$, $i \in [n]$, since they have different highest elements. From these, the crystal basis $\mathcal{B}_{\mathcal{I}_0,0,\lambda_1,\dots,\lambda_r}$ has the same number of connected components as $\mathcal{B}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$, which implies the assertion of the corollary by Corollary 3.16. ☐

4. Flag Bott–Samelson varieties and flag Bott towers

In this section, we study complex structures on the flag Bott–Samelson variety $Z_{\mathcal{I}}$, and its relation with a flag Bott tower in Theorem 4.10. We first recall flag Bott manifolds introduced in [Kuroki et al. 2020]. Let M be a complex manifold and E a holomorphic vector bundle over M. The *associated flag bundle* $\mathcal{F}\ell(E) \to M$ is a fiber bundle obtained from E by replacing each fiber E_p over a point $p \in M$ by the full flag manifold $\mathcal{F}\ell(E_p)$.

Definition 4.1 [Kuroki et al. 2020, Definition 2.1]. A *flag Bott tower* $\{F_k\}_{0 \le k \le r}$ of height r (or an r*-stage flag Bott tower*) is a sequence,

$$F_r \xrightarrow{p_r} F_{r-1} \xrightarrow{p_{r-1}} \cdots \xrightarrow{p_2} F_1 \xrightarrow{p_1} F_0 = \{\text{a point}\}$$

of manifolds $F_k = \mathcal{F}\ell\big(\bigoplus_{l=1}^{m_k+1} \xi_k^{(l)}\big)$, where $\xi_k^{(l)}$ is a holomorphic line bundle over F_{k-1} for each $1 \le l \le m_k + 1$ and $1 \le k \le r$. We call F_k the k*-stage flag Bott manifold* of the flag Bott tower.

For example, the flag manifold $\mathcal{F}\ell(\mathbb{C}^{m+1}) = \mathcal{F}\ell(m+1)$ is a 1-stage flag Bott manifold, and the product of flag manifolds $\mathcal{F}\ell(m_1 + 1) \times \dots \times \mathcal{F}\ell(m_r + 1)$ is an r-stage flag Bott manifold. Also an r-stage Bott manifold is an r-stage flag Bott manifold (see [Grossberg and Karshon 1994] for the definition of Bott manifolds). We call two flag Bott towers $\{F_k\}_{0 \le k \le r}$ and $\{F'_k\}_{0 \le k \le r}$ *isomorphic* if there is a collection of diffeomorphisms $\varphi : F_k \to F'_k$ which commutes with the maps $p_k : F_k \to F_{k-1}$ and $p'_k : F'_k \to F'_{k-1}$.

Remark 4.2. In [Kaji et al. 2020], an iterated flag bundle whose fibers are not only full flag manifolds of type A but also other flag manifolds of general Lie type is considered. We recall their construction briefly. For $1 \le k \le r$, let K_k be a compact connected Lie group, $T_k \subset K_k$ a maximal torus, and $Z_k \subset K_k$ the centralizer of a circle subgroup of T_k. Recall from [Kaji et al. 2020, Definition 3.1] that an *r-stage flag Bott tower $\{F_k\}_{0 \le k \le r}$ of general Lie type associated to $\{(K_k, Z_k)\}_{0 \le k \le r}$* is defined recursively:

1. F_0 is a point.
2. F_k is the flag bundle over F_{k-1} with fiber K_k/Z_k associated to a map

$$f_k : F_{k-1} \to BK_k,$$

where f_k factors through BT_k.

Here, the map f_k induces the flag bundle $F_k \to F_{k-1}$ from the universal flag bundle $K_k/Z_k \hookrightarrow BZ_k \to BK_k$.

$$\begin{tikzcd}
K_k/Z_k \arrow[rr, equal] \arrow[d, hook] & & K_k/Z_k \arrow[d, hook] \\
F_k \arrow[rr] \arrow[d] & & BZ_k \arrow[d] \\
F_{k-1} \arrow[rr, "f_k"] \arrow[dr] & & BK_k \\
& BT_k \arrow[ur] &
\end{tikzcd} \tag{4-1}$$

Because the map f_k factors through BT_k, the bundle F_k is the associated K_k/Z_k-flag bundle of the sum of complex line bundles over F_{k-1}. A flag Bott tower defined in Definition 4.1 is a flag Bott tower of general Lie type associated to $\{(U(m_k+1), T^{m_k+1})\}_{0 \le k \le r}$.

Lemma 4.3. *Let M be a complex manifold and E a holomorphic vector bundle over M. Let $\mathcal{L}$ be a holomorphic line bundle over M. Then we have that $\mathcal{F}\ell(E) \cong \mathcal{F}\ell(E \otimes \mathcal{L})$ as differentiable manifolds.*

Proof. It is well-known that for a holomorphic vector bundle $E \to M$ over a smooth manifold M and a holomorphic line bundle $\mathcal{L} \to M$, there is a diffeomorphism $\mathbb{P}(E \otimes \mathcal{L}) \cong \mathbb{P}(E)$ (see, for example, [Choi et al. 2010, Lemma 2.1]). Since the induced flag bundle is a sequence of projective bundles as shown in [Bott and Tu 1982, Proposition 21.15], we have a diffeomorphism $\mathcal{F}\ell(E) \cong \mathcal{F}\ell(E \otimes \mathcal{L})$. $\square$

The flag manifold $\mathcal{F}\ell(m+1)$ and an orbit space $\mathrm{GL}(m+1)/B_{\mathrm{GL}(m+1)}$ can be identified. Similarly, an r-stage flag Bott manifold F_r can also be considered as an orbit space. We briefly review the orbit space construction of [Kuroki et al. 2020, §2.2]. Recall from [Kuroki et al. 2020, Lemma 2.12] that for a given Bott tower $\{F_k\}_{0\le k\le r}$ such that $\mathcal{F}\ell(m_k+1)\hookrightarrow F_k\to F_{k-1}$, there is a surjective group homomorphism:

$$\psi:\mathbb{Z}^{m_1+1}\times\cdots\times\mathbb{Z}^{m_k+1}\twoheadrightarrow \operatorname{Pic}(F_k)\quad\text{for } 1\le k\le r. \tag{4-2}$$

We briefly explain the geometric meaning of the homomorphism (4-2). For the flag bundle $\mathcal{F}\ell(E)\xrightarrow{p} M$ obtained by a vector bundle E of rank n over a complex manifold M, consider the universal flag of bundles $0\subset E_1\subset E_2\subset\cdots\subset E_n=p^*E$ on $\mathcal{F}\ell(E)$. Then every element of $\operatorname{Pic}(\mathcal{F}\ell(E))$ can be written as a polynomial in $x_i=c_1(E_i/E_{i-1})$ for $1\le i\le n$ with coefficients in $\operatorname{Pic}(M)$ (see, for example, [Fulton 1998, Example 3.3.5]). Because F_k is an iterated flag bundle, applying this procedure recurrently, we obtain the homomorphism (4-2). Moreover, for $\xi\in\operatorname{Pic}(F_k)$, if we have $\xi=\psi(\boldsymbol{a}_1,\dots,\boldsymbol{a}_k)$, where $\boldsymbol{a}_j$ is an integer vector $(\boldsymbol{a}_j(1),\dots,\boldsymbol{a}_j(m_j+1))\in\mathbb{Z}^{m_j+1}$ for $1\le j\le k$, then

$$c_1(\xi)=\sum_{j=1}^{k}\sum_{l=1}^{m_j+1}\boldsymbol{a}_j(l)x_{j,l}. \tag{4-3}$$

Here, $x_{j,l}$ is the first Chern class of the quotient bundle $E_{j,l}/E_{j,l-1}$ obtained by the universal flag of bundles $0\subset E_{j,1}\subset E_{j,2}\subset\cdots\subset E_{j,m_j+1}$ on F_j.[1]

Suppose that $c_1(\xi_l^{(k)})$ is determined by a set of integer vectors

$$\{\boldsymbol{a}_{k,j}^{(l)}\in\mathbb{Z}^{m_j+1}\}_{1\le l\le m_k+1,\,1\le j<k\le r}.$$

Then

$$\psi(\boldsymbol{a}_{k,1}^{(l)},\boldsymbol{a}_{k,2}^{(l)},\dots,\boldsymbol{a}_{k,k-1}^{(l)})=\xi_k^{(l)}\to F_{k-1}$$

for each $1\le l\le m_k+1$ and $2\le k\le r$. Using this set of integer vectors, we define a right action Φ_k of $B_{\mathrm{GL}(m_1+1)}\times\cdots\times B_{\mathrm{GL}(m_k+1)}$ on $\mathrm{GL}(m_1+1)\times\cdots\times\mathrm{GL}(m_k+1)$ as

$$\begin{aligned}\Phi_k(&(g_1,\dots,g_k),(b_1,\dots,b_k))\\ &:=\bigl(g_1b_1,\Lambda_{2,1}(b_1)^{-1}g_2b_2,\Lambda_{3,1}(b_1)^{-1}\Lambda_{3,2}(b_2)^{-1}g_3b_3,\dots,\\ &\qquad\Lambda_{k,1}(b_1)^{-1}\Lambda_{k,2}(b_2)^{-1}\cdots\Lambda_{k,k-1}(b_{k-1})^{-1}g_kb_k\bigr)\end{aligned}$$

---

[1] The classes $x_{j,l}$ generate the cohomology $H^2(F_j;\mathbb{Z})$ with the relations $x_{j,1}+\cdots+x_{j,m_j+1}=c_1(\xi_1^{(j)})+\cdots+c_1(\xi_{m_j+1}^{(j)})$ for $1\le j\le k$ (see [Fulton 1998, Example 3.3.5] or [Kaji et al. 2020, Corollary 2.4]).

for $1 \le k \le r$. Here $\Lambda_{k,j}$ is a homomorphism $B_{\mathrm{GL}(m_j+1)} \to H_{\mathrm{GL}(m_k+1)}$ which sends $b \in B_{\mathrm{GL}(m_j+1)}$ to

$$\operatorname{diag}\bigl(\Upsilon(b)^{\boldsymbol{a}_{k,j}^{(1)}}, \Upsilon(b)^{\boldsymbol{a}_{k,j}^{(2)}}, \dots, \Upsilon(b)^{\boldsymbol{a}_{k,j}^{(m_k+1)}}\bigr) \in H_{\mathrm{GL}(m_k+1)},$$

where $\Upsilon : B_{\mathrm{GL}(m_j+1)} \to H_{\mathrm{GL}(m_j+1)}$ is the canonical projection in (2-9), and

$$h^{\boldsymbol{a}} := h_1^{\boldsymbol{a}(1)} h_2^{\boldsymbol{a}(2)} \cdots h_{m+1}^{\boldsymbol{a}(m+1)}$$

for $h = \operatorname{diag}(h_1, \dots, h_{m+1}) \in H_{\mathrm{GL}(m+1)}$ and $\boldsymbol{a} = (\boldsymbol{a}(1), \dots, \boldsymbol{a}(m+1)) \in \mathbb{Z}^{m+1}$. Now we can describe the flag Bott manifold F_r as an orbit space as follows:

Proposition 4.4 [Kuroki et al. 2020, Propositions 2.8 and 2.11]. *Let $\{F_k\}_{0 \le k \le r}$ be a flag Bott tower. Suppose that $c_1(\xi_l^{(k)})$ is determined by a set of integer vectors $\{\boldsymbol{a}_{k,j}^{(l)} \in \mathbb{Z}^{m_j+1}\}_{1 \le l \le m_k+1, 1 \le j < k \le r}$ and let Φ_k be the action determined by these integer vectors. Then the flag Bott tower $\{F_k\}_{0 \le k \le r}$ is isomorphic to*

$$\bigl\{\bigl(\mathrm{GL}(m_1+1) \times \cdots \times \mathrm{GL}(m_k+1)\bigr)/\Phi_k\bigr\}_{0 \le k \le r}$$

as flag Bott towers.

A Bott–Samelson variety has a family of complex structures which gives a toric degeneration (see [Grossberg and Karshon 1994, §3.4; Pasquier 2010]). Now we study a family of complex structures on a given flag Bott–Samelson variety. Since the simple roots are linearly independent elements in $\mathfrak{h}^*$, there exist $q \in \mathbb{Z}_{>0}$ and an injective homomorphism $\lambda : \mathbb{C}^* \to H$ such that

$$e^{\alpha}(\lambda(t)) = t^q \tag{4-4}$$

for all simple roots α and $t \in \mathbb{C}^*$. Here $e^{\alpha} : H \to \mathbb{C}^*$ is a character induced from $\alpha : \mathfrak{h} \to \mathbb{C}$. For example, when $G = \mathrm{SL}(2k+1)$ and $q = 1$, consider the homomorphism $\lambda : \mathbb{C}^* \to H$ defined by

$$\lambda : t \mapsto \operatorname{diag}\bigl(t^k, t^{k-1}, \dots, t, 1, t^{-1}, \dots, t^{-k+1}, t^{-k}\bigr). \tag{4-5}$$

Then this homomorphism satisfies the condition on (4-4). We define $\Upsilon_t : B \to B$ by

$$\Upsilon_t : b \mapsto \lambda(t) b (\lambda(t))^{-1}$$

for $t \in \mathbb{C}^*$. It is proved in [Grossberg and Karshon 1994, Proposition 3.5] that $\Upsilon = \lim_{t \to 0} \Upsilon_t$, where $\Upsilon : B \to H$ is the homomorphism in (2-9). We put $\Upsilon_0 := \Upsilon$.

Example 4.5. Suppose that $G = \mathrm{SL}(3)$ and $q = 1$. Considering the homomorphism $\lambda : \mathbb{C}^* \to H$ defined in (4-5), the homomorphism $\Upsilon_t : B \to B$ is given by

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} \mapsto \begin{bmatrix} b_{11} & tb_{12} & t^2 b_{13} \\ 0 & b_{22} & tb_{23} \\ 0 & 0 & b_{33} \end{bmatrix}.$$

Hence we have that $\lim_{t\to 0} \Upsilon_t = \Upsilon$.

We use the homomorphism $\Upsilon_t : B \to B$ to construct a family of complex structures on the flag Bott–Samelson manifold $Z_{\mathcal{I}} = \boldsymbol{P}_{\mathcal{I}}/B^r$. For $t \in \mathbb{C}$, we define a right action Θ_t of B^r on $\boldsymbol{P}_{\mathcal{I}}$ as

$$\begin{aligned}(4\text{-}6)\quad \Theta_t((p_1, \dots, p_r), (b_1, \dots, b_r)) \\ = \bigl(p_1 b_1, \Upsilon_t(b_1)^{-1} p_2 b_2, \dots, \Upsilon_t(b_{r-1})^{-1} p_r b_r\bigr)\end{aligned}$$

for $(p_1, \dots, p_r) \in \boldsymbol{P}_{\mathcal{I}}$ and $(b_1, \dots, b_r) \in B^r$. Then Θ_1 coincides with the right action in (2-2) because $\lambda(1) = e \in H$ and hence $\Upsilon_1 = \mathrm{Id}_B$. Again we consider the family of orbit spaces

$$Z^t_{\mathcal{I}} := \boldsymbol{P}_{\mathcal{I}}/\Theta_t$$

for $t \in \mathbb{C}$. The holomorphic line bundle $\mathcal{L}^t_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ over $Z^t_{\mathcal{I}}$ can be defined in a way similar to $\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ in (2-10) for integral weights $\lambda_1, \dots, \lambda_r$. Set $\mathcal{L}^t_{\mathcal{I},\lambda} := \mathcal{L}^t_{\mathcal{I},0,\dots,0,\lambda}$ for simplicity.

Proposition 4.6. *For a given sequence $\mathcal{I} = (I_1, \dots, I_r)$, the manifolds $Z^t_{\mathcal{I}}$ are all diffeomorphic for $t \in \mathbb{C}$.*

Proof. We use the similar argument to the proof of Proposition 3.7 in [Grossberg and Karshon 1994]. Let K_{I_j} be the maximal compact subgroup of P_{I_j}. Let T be the maximal compact torus in G, i.e., $T = (S^1)^n$. Recall that $K_{I_j} \cap B = T$. Define a right action of $T^{(r)} := T \times T \times \cdots \times T$ (r factors) on $K_{\mathcal{I}} := K_{I_1} \times \cdots \times K_{I_r}$ as

$$(4\text{-}7)\qquad (g_1, \dots, g_r) \cdot (a_1, \dots, a_r) = (g_1 a_1, a_1^{-1} g_2 a_2, \dots, a_{r-1}^{-1} g_r a_r).$$

Let $X_{\mathcal{I}}$ be the orbit space

$$(4\text{-}8)\qquad X_{\mathcal{I}} := (K_{I_1} \times \cdots \times K_{I_r})/(T \times \cdots \times T).$$

The inclusion map

$$K_{\mathcal{I}} = K_{I_1} \times \cdots \times K_{I_r} \hookrightarrow \boldsymbol{P}_{\mathcal{I}} = P_{I_1} \times \cdots \times P_{I_r}$$

is $T^{(r)}$-equivariant with respect to the $T^{(r)}$-action of (4-7) on $K_{\mathcal{I}}$ and the restricted $T^{(r)}$-action of (4-6) on $\boldsymbol{P}_{\mathcal{I}}$ via the inclusion $T^{(r)} \hookrightarrow B^r$ because $\Upsilon_t(a) = a$ for all $a \in T$. Therefore we get a map

$$(4\text{-}9)\qquad f^t_{\mathcal{I}} : X_{\mathcal{I}} \to Z^t_{\mathcal{I}}.$$

Since, for all k, the inclusion $K_{I_k} \hookrightarrow P_{I_k}$ induces a diffeomorphism $K_{I_k}/T \cong P_{I_k}/B$, the map $f^t_{\mathcal{I}}$ is a diffeomorphism. □

The manifold $Z^t_{\mathcal{I}}$ has a fibration structure, similar to a flag Bott–Samelson manifold in (2-3):

$$(4\text{-}10)\qquad P_{I_r}/B \hookrightarrow Z^t_{\mathcal{I}} \xrightarrow{\pi} Z^t_{\mathcal{I}'},$$

where $\mathcal{I}' = (I_1, \dots, I_{r-1})$ is the subsequence of $\mathcal{I}$ and π is the first $r-1$ coordinates projection for all $t \in \mathbb{C}$.

Let $\mathcal{I} = (I_1, \dots, I_{r-1}, I_r)$ and $\mathcal{I}' = (I_1, \dots, I_{r-1})$. We note that the orbit space $X_{\mathcal{I}}$ has a bundle structure.

$$\begin{tikzcd}
X_{\mathcal{I}} = \boldsymbol{P}_{\mathcal{I}'} \times_T (K_{I_r}/T) \arrow[d] & K_{I_r}/T \arrow[l, hook'] \\
X_{\mathcal{I}'} &
\end{tikzcd}$$

Because the structure group T of this bundle is an abelian group, the map f_k inducing the flag bundle $X_{\mathcal{I}} \to X_{\mathcal{I}'}$ from the universal flag bundle factors through BT.

$$\begin{tikzcd}
K_{I_r}/T \arrow[rr, equal] \arrow[d, hook] & & K_{I_r}/T \arrow[d, hook] \\
X_{\mathcal{I}} \arrow[rr] \arrow[d] & & BT \arrow[d] \\
X_{\mathcal{I}'} \arrow[rr, "f_k"] \arrow[dr] & & BK_{I_r} \\
& BT \arrow[ur] &
\end{tikzcd}$$

Continuing this procedure, we obtain the following corollary.

Corollary 4.7. *The manifold $X_{\mathcal{I}}$ is an r-stage flag Bott tower of general Lie type associated to $\{(K_{I_j}, T)\}_{0 \le j \le r}$, and so are $Z_{\mathcal{I}}^t$ for all $t \in \mathbb{C}$* (see Remark 4.2 for the definition of flag Bott towers of general Lie type).

For the remaining part of this section, we consider the case when the Levi subgroup L_{I_k} of the parabolic subgroup P_{I_k} has Lie type A, that is, the flag Bott tower $X_{\mathcal{I}}$ is a flag Bott manifold whose fibers are all full flag manifolds of Lie type A. Moreover, we describe the line bundles appearing in the construction explicitly (see Theorem 4.10). We can always take an enumeration $I_k = \{u_{k,1}, \dots, u_{k,m_k}\}$ so that

$$\langle \alpha_{u_{k,s}}, \alpha_{u_{k,t}}^\vee \rangle = \begin{cases} 2 & \text{if } s = t, \\ -1 & \text{if } s - t = \pm 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4-11}$$

Proposition 4.8. *Let $Z_{\mathcal{I}}$ be a flag Bott–Samelson manifold. Let $\mathcal{I}' = (I_1, \dots, I_{r-1})$ be the subsequence of $\mathcal{I}$. Assume that the Levi subgroup L_{I_k} of the parabolic subgroup P_{I_k} has Lie type A_{m_k} for all $1 \le k \le r$. Then the manifold $Z_{\mathcal{I}}^0$ is diffeomorphic to the induced flag bundle over $Z_{\mathcal{I}'}^0$:*

$$Z_{\mathcal{I}}^0 \cong \mathcal{F}\ell\big(\mathcal{L}_{\mathcal{I}', \chi_1}^0 \oplus \cdots \oplus \mathcal{L}_{\mathcal{I}', \chi_{m_r}}^0 \oplus \underline{\mathbb{C}}\big),$$

where $\chi_j = \alpha_{u_{r,j}} + \cdots + \alpha_{u_{r,m_r}} \in \mathfrak{h}^*$ *for* $1 \le j \le m_r$, $\mathcal{L}^0_{\mathcal{I}',\chi} = \mathcal{L}^0_{\mathcal{I}',0,\dots,0,\chi}$, *and* $\underline{\mathbb{C}}$ *is the trivial line bundle.*

Before proving the proposition, we observe the following. Suppose that the Levi subgroup L_I of the parabolic subgroup P_I for a subset $I \subset [n]$ has Lie type A_m. Then we can label the elements of I as $u_1, \dots, u_m$ which satisfy the relation (4-11). Also we have the group homomorphism $F : \mathrm{SL}(m+1) \to L_I \hookrightarrow P_I$. Then the map F induces the homomorphism $F_* : \mathfrak{h}_{\mathrm{SL}(m+1)} \to \mathfrak{h}$. We label the coroots of $\mathrm{SL}(m+1)$ as $\beta_1^\vee, \beta_2^\vee, \dots, \beta_m^\vee$ so that F_* sends $\beta_l^\vee$ to $\alpha_{u_l}^\vee$ for $1 \le l \le m$. Then we have that

$$\langle F^*\lambda, \beta_l^\vee \rangle = \langle \lambda, F_*\beta_l^\vee \rangle = \langle \lambda, \alpha_{u_l}^\vee \rangle$$

for a weight $\lambda \in \mathfrak{h}^*$ and $1 \le l \le m$. Here, we note that $F^*\lambda = \lambda \circ F$ for $\lambda \in \mathfrak{h}^*$. Let $\varpi_1, \varpi_2, \dots, \varpi_m \in \mathfrak{h}^*_{\mathrm{SL}(m+1)}$ be the fundamental weights. Then the pullback $F^*\lambda$ is given by

$$F^*\lambda = \sum_{l=1}^{m} \langle \lambda, \alpha_{u_l}^\vee \rangle \varpi_l \in \mathfrak{h}^*_{\mathrm{SL}(m+1)}. \tag{4-12}$$

Proof of Proposition 4.8. We write $I = I_r$, $m = m_r$, and $u_j = u_{r,j}$ for $1 \le j \le m$. Note that we have $P_I = L_I U_I$ (see Section 2A). Since we have an isomorphism of varieties

$$P_I/B = (L_I U_I)/B = L_I/(B \cap L_I) = L_I/B_I,$$

we get a diffeomorphism

$$F_1 : \mathrm{SL}(m+1)/B_{\mathrm{SL}(m+1)} \to P_I/B.$$

Moreover, the map which sends an element g in $\mathrm{SL}(m+1)$ to a full flag $(V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_m)$, where $V_l = \langle c_1, \dots, c_l \rangle$ and c_l is the l-th column vector of g, descends to a diffeomorphism

$$F_2 : \mathrm{SL}(m+1)/B_{\mathrm{SL}(m+1)} \to \mathcal{F}\ell(m+1).$$

The map F_2 is equivariant with respect to the following actions of the torus $H_{\mathrm{SL}(m+1)}$: each element

$$h = \mathrm{diag}(h_1, h_2, \dots, h_{m+1}) \in H_{\mathrm{SL}(m+1)}$$

acts on $\mathrm{SL}(m+1)/B_{\mathrm{SL}(m+1)}$ by the left multiplication, and on $\mathcal{F}\ell(m+1)$ as the induced action from the representation space $\mathbb{C}^{m+1}$ with weights

$$\big(\varpi_1, -\varpi_1 + \varpi_2, \dots, -\varpi_{m-1} + \varpi_m, -\varpi_m\big), \tag{4-13}$$

namely $h \cdot v = (h_1 v_1, h_2 v_2, \dots, h_{m+1} v_{m+1})$ for $v = (v_1, \dots, v_{m+1}) \in \mathbb{C}^{m+1}$. On the other hand, the map F_1 is equivariant with respect to the left multiplication actions of $H_{\mathrm{SL}(m+1)}$ and of H via the homomorphism $H_{\mathrm{SL}(m+1)} \to H$ given by the map F.

By the relation (4-12) between weights in $\mathfrak{h}^*$ and $\mathfrak{h}^*_{\mathrm{SL}(m+1)}$, we have the following:

$$\begin{aligned} F^*(\chi_j) &= F^*(\alpha_{u_j} + \cdots + \alpha_{u_m}) \\ &= \sum_{l=1}^{m} \langle \alpha_{u_j} + \cdots + \alpha_{u_m}, \alpha_{u_l}^\vee \rangle \varpi_l \\ &= -\varpi_{j-1} + \varpi_j + \varpi_m, \end{aligned}$$

where $\varpi_0 = 0$ for $1 \le j \le m$. Here the third equality follows by considering the Cartan matrix of $\mathrm{SL}(m+1)$. The $H_{\mathrm{SL}(m+1)}$-representation on $\mathbb{C}^{m+1}$ with weights (4-13) becomes an H-representation on $\mathbb{C}^{m+1}$ with weights

$$\bigl(\chi_1 - \chi', \chi_2 - \chi', \ldots, \chi_m - \chi', -\chi'\bigr),$$

where χ' is a weight which maps to ϖ_m under the map F^* such that $F_2 \circ F_1^{-1}$ is equivariant with respect to the actions of elements in $H \setminus F(H_{\mathrm{SL}(m+1)})$. This proves that $F_2 \circ F_1^{-1}$ is a left H-equivariant diffeomorphism

$$F_2 \circ F_1^{-1} : P_I/B \to \mathcal{F}\ell\bigl(\mathbb{C}_{\chi_1 - \chi'} \oplus \cdots \oplus \mathbb{C}_{\chi_m - \chi'} \oplus \mathbb{C}_{-\chi'}\bigr).$$

We notice that the construction of twisted product is functorial, i.e., for a topological group G and a right G-space X, if $f : Y \to Y'$ is an equivariant map of left G-spaces then we have the induced map $X \times_G Y \to X \times_G Y'$, see, for example, [Bredon 1972, §II.2]. Since the unipotent part of B acts trivially on P_I/B and $\mathcal{F}\ell(\mathbb{C}_{\chi_1 - \chi'} \oplus \cdots \oplus \mathbb{C}_{\chi_m - \chi'} \oplus \mathbb{C}_{-\chi'})$, the left H-equivariant diffeomorphism $F_2 \circ F_1^{-1}$ induces a diffeomorphism

$$\boldsymbol{P}_{\mathcal{I}}/\Theta_0 \cong \mathcal{F}\ell\bigl(\mathcal{L}^0_{\mathcal{I}', \chi_1 - \chi'} \oplus \cdots \oplus \mathcal{L}^0_{\mathcal{I}', \chi_m - \chi'} \oplus \mathcal{L}^0_{\mathcal{I}', -\chi'}\bigr).$$

Moreover we have that

$$\begin{aligned} &\mathcal{F}\ell\bigl(\mathcal{L}^0_{\mathcal{I}', \chi_1 - \chi'} \oplus \cdots \oplus \mathcal{L}^0_{\mathcal{I}', \chi_m - \chi'} \oplus \mathcal{L}^0_{\mathcal{I}', -\chi'}\bigr) \\ &\qquad\qquad = \mathcal{F}\ell\bigl((\mathcal{L}^0_{\mathcal{I}', \chi_1} \oplus \cdots \oplus \mathcal{L}^0_{\mathcal{I}', \chi_m} \oplus \underline{\mathbb{C}}) \otimes \mathcal{L}^0_{\mathcal{I}', -\chi'}\bigr). \end{aligned}$$

Then by Lemma 4.3, we are done. □

By Proposition 4.8, we can conclude that $Z^0_{\mathcal{I}}$ is an r-stage flag Bott manifold. For given integral weights $\lambda_1, \ldots, \lambda_r$, consider the line bundle $\mathcal{L}^0_{\mathcal{I}, \lambda_1, \ldots, \lambda_r}$ over a flag Bott manifold $Z^0_{\mathcal{I}}$. By (4-2) there is a set of integer vectors $\{\boldsymbol{a}_k \in \mathbb{Z}^{m_k+1}\}_{1 \le k \le r}$ determined by $c_1(\mathcal{L}^0_{\mathcal{I}, \lambda_1, \ldots, \lambda_r})$. Indeed, we have

$$\psi(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r) \cong \mathcal{L}^0_{\mathcal{I}, \lambda_1, \ldots, \lambda_r}.$$

The following proposition computes these integer vectors in terms of integral weights $\lambda_1, \dots, \lambda_r$ and a sequence $\mathcal{I}$ of subsets of $[n]$.

Proposition 4.9. *Let $\mathcal{I} = (I_1, \dots, I_r)$ be a sequence of subsets of $[n]$. Assume that the Levi subgroup L_{I_k} of the parabolic subgroup P_{I_k} has Lie type A_{m_k} for all $1 \le k \le r$. For given integral weights $\lambda_1, \dots, \lambda_r \in \mathbb{Z}\varpi_1 + \cdots + \mathbb{Z}\varpi_n$, the first Chern class of the line bundle $\mathcal{L} = \mathcal{L}^0_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ is given by integer vectors $\boldsymbol{a}_k = (\boldsymbol{a}_k(1), \dots, \boldsymbol{a}_k(m_k+1)) \in \mathbb{Z}^{m_k+1}$ for $1 \le k \le r$, where*

$$\boldsymbol{a}_k(l) = \langle \lambda_k + \cdots + \lambda_r, \alpha^\vee_{u_{k,l}} + \cdots + \alpha^\vee_{u_{k,m_k}} \rangle \quad \textit{for } 1 \le l \le m_k,$$

$$\boldsymbol{a}_k(m_k+1) = 0.$$

Here, we take an enumeration $I_k = \{u_{k,1}, \dots, u_{k,m_k}\}$ which satisfies (4-11). Indeed, $\mathcal{L}$ is isomorphic to the line bundle $\psi(\boldsymbol{a}_1, \dots, \boldsymbol{a}_r)$.

Proof. Since the Levi subgroup L_{I_k} of P_{I_k} is Lie type A_{m_k}, we have a Lie group homomorphism $F_k : \mathrm{SL}(m_k+1) \to P_{I_k}$. For each $1 \le k \le r$, consider the homomorphism $\psi_k : \mathrm{SL}(m_k+1) \to P_{I_1} \times \cdots \times P_{I_r}$ defined as

$$p \mapsto (e, \dots, e, \underbrace{F_k(p)}_{k\text{-th}}, e, \dots, e)$$

and consider

$$\varphi_k : B_{\mathrm{SL}(m_k+1)} \to \underbrace{B \times \cdots \times B}_{r} = B^r \tag{4-14}$$

which sends b to

$$\bigl(e, \dots, e, \underbrace{F_k(b)}_{k\text{-th}}, \underbrace{F_k(h)}_{(k+1)\text{-th}}, \dots, \underbrace{F_k(h)}_{r\text{-th}}\bigr),$$

where $h = \Upsilon(b)$. Then the map ψ_k is φ_k-equivariant, namely, for $b \in B_{\mathrm{SL}(m_k+1)}$ and $g \in \mathrm{SL}(m_k+1)$ we have that

$$\begin{aligned}
&\Theta_0(\psi_k(g), \varphi_k(b)) \\
&\qquad = \Theta_0\bigl((e, \dots, e, F_k(g), e, \dots, e), (e, \dots, e, F_k(b), F_k(h), \dots, F_k(h))\bigr) \\
&\qquad = \bigl(e, \dots, e, F_k(g)F_k(b), \Upsilon(F_k(b))^{-1}F_k(h), e, \dots, e\bigr) \\
&\qquad = (e, \dots, e, F_k(gb), e, e, \dots, e) \\
&\qquad = \psi_k(gb).
\end{aligned}$$

Here the third equality comes from the fact that F_k is a homomorphism and $\Upsilon(F_k(b)) = F_k(\Upsilon(b))$.

Under the map (4-14) the weight $(\lambda_1, \dots, \lambda_r)$ of H^r pulls back to the weight

$$\sum_{l=1}^{m_k} \langle \lambda_k + \cdots + \lambda_r, \alpha^\vee_{u_{k,l}} \rangle \varpi_l \in \mathfrak{h}^*_{\mathrm{SL}(m_k+1)} \tag{4-15}$$

by (4-12). The integer vector $\boldsymbol{a}_k \in \mathbb{Z}^{m_k+1}$ is completely determined by the weight in (4-15) because of the construction of a flag Bott manifold (see [Kuroki et al. 2020, §2.2]). Indeed, the integer vector $\boldsymbol{a}_k \in \mathbb{Z}^{m_k+1}$ should satisfy the equality

$$\sum_{l=1}^{m_k} \langle \lambda_k + \cdots + \lambda_r, \alpha^\vee_{u_{k,l}} \rangle \varpi_l = \sum_{l=1}^{m_k+1} \boldsymbol{a}_k(l) \varepsilon_l, \tag{4-16}$$

where $\varepsilon_i \in \mathfrak{h}^*_{\mathrm{SL}(m_k+1)}$ sends $\mathrm{diag}(h_1, \ldots, h_{m_k+1})$ in $\mathfrak{h}_{\mathrm{SL}(m_k+1)}$ to h_i. Using the identification $\varpi_l = \varepsilon_1 + \cdots + \varepsilon_l$, we have that

$$\begin{aligned} \sum_{l=1}^{m_k} &\langle \lambda_k + \cdots + \lambda_r, \alpha^\vee_{u_{k,l}} \rangle \varpi_l \\ &= \langle \lambda_k + \cdots + \lambda_r, \alpha^\vee_{u_{k,1}} \rangle \varepsilon_1 + \langle \lambda_k + \cdots + \lambda_r, \alpha^\vee_{u_{k,2}} \rangle (\varepsilon_1 + \varepsilon_2) \\ &\qquad + \cdots + \langle \lambda_k + \cdots + \lambda_r, \alpha^\vee_{u_{k,m_k}} \rangle (\varepsilon_1 + \cdots + \varepsilon_{m_k}) \\ &= \big\langle \lambda_k + \cdots + \lambda_r, \alpha^\vee_{u_{k,1}} + \alpha^\vee_{u_{k,2}} + \cdots + \alpha^\vee_{u_{k,m_k}} \big\rangle \varepsilon_1 \\ &\qquad + \big\langle \lambda_k + \cdots + \lambda_r, \alpha^\vee_{u_{k,2}} + \cdots + \alpha^\vee_{u_{k,m_k}} \big\rangle \varepsilon_2 \\ &\qquad + \cdots + \langle \lambda_k + \cdots + \lambda_r, \alpha^\vee_{u_{k,m_k}} \rangle \varepsilon_{m_k}. \end{aligned} \tag{4-17}$$

Comparing (4-16) and (4-17), we obtain the assertion of the proposition. □

By combining Propositions 4.8 and 4.9, we can prove the following theorem:

Theorem 4.10. *Suppose that $\mathcal{I} = (I_1, \ldots, I_r)$ is a sequence of subsets of $[n]$ such that the Levi subgroup L_{I_k} of the parabolic subgroup P_{I_k} has Lie type A_{m_k} for all $1 \le k \le r$. Take an enumeration $I_k = \{u_{k,1}, \ldots, u_{k,m_k}\}$ which satisfies (4-11). Then the manifold $Z^0_{\mathcal{I}}$ is an r-stage flag Bott manifold which is determined by*

$$\big\{\boldsymbol{a}^{(l)}_{k,j} = \big(\boldsymbol{a}^{(l)}_{k,j}(1), \boldsymbol{a}^{(l)}_{k,j}(2), \ldots, \boldsymbol{a}^{(l)}_{k,j}(m_j+1)\big)\big\}_{1 \le l \le m_k+1,\, 1 \le j < k \le r}$$

in the sense of Proposition 4.4, where $\boldsymbol{a}^{(l)}_{k,j}(p)$ is

$$\big\langle \alpha_{u_{k,l}} + \cdots + \alpha_{u_{k,m_k}}, \alpha^\vee_{u_{j,p}} + \cdots + \alpha^\vee_{u_{j,m_j}} \big\rangle$$

if $1 \le l \le m_k$ and $1 \le p \le m_j$, and 0 otherwise.

Proof. Consider the subsequence $\mathcal{I}_k := (I_1, \ldots, I_k)$ of the sequence $\mathcal{I}$ for all $1 \le k \le r$. Recall from Proposition 4.8 that the flag Bott manifold $Z^0_{\mathcal{I}_k}$ is the induced flag bundle over $Z^0_{\mathcal{I}_{k-1}}$:

$$Z^0_{\mathcal{I}_k} = \mathcal{F}\ell\big(\mathcal{L}^0_{\mathcal{I}_{k-1},\chi_1} \oplus \cdots \oplus \mathcal{L}^0_{\mathcal{I}_{k-1},\chi_{m_k}} \oplus \underline{\mathbb{C}}\big),$$

where $\chi_l = \alpha_{u_{k,l}} + \cdots + \alpha_{u_{k,m_k}}$ for $1 \le l \le m_k$. By Proposition 4.9, the integer vectors $\{\boldsymbol{a}_{k,j}^{(l)} \in \mathbb{Z}^{m_j+1}\}_{1\le j\le k-1}$ which define the line bundle $\mathcal{L}^0_{\mathcal{I}_{k-1},\chi_l}$ are given by

$$\begin{aligned}\boldsymbol{a}_{k,j}^{(l)}(p) &= \langle \chi_l, \alpha^\vee_{u_{j,p}} + \cdots + \alpha^\vee_{u_{j,m_j}} \rangle && \text{(by Proposition 4.9)}\\ &= \big\langle \alpha_{u_{k,l}} + \cdots + \alpha_{u_{k,m_k}}, \alpha^\vee_{u_{j,p}} + \cdots + \alpha^\vee_{u_{j,m_j}} \big\rangle && \text{(by the definition of } \chi_l)\end{aligned}$$

for $1 \le l \le m_k$ and $1 \le p \le m_j$. Moreover we have $\boldsymbol{a}_{k,j}^{(l)}(p) = 0$ if $l = m_k + 1$ or $p = m_j + 1$ by Proposition 4.9. Hence the result follows. □

Example 4.11. Let $G = \mathrm{SL}(4)$. Consider the sequence $\mathcal{I} = (\{1, 2\}, \{1, 2\})$. Hence $u_{1,1} = 1,\ u_{1,2} = 2,\ u_{2,1} = 1,\ u_{2,2} = 2$. The manifold $Z^0_{\mathcal{I}}$ is a 2-stage flag Bott manifold with $F_2 = \mathcal{F}\ell(\xi_2^{(1)} \oplus \xi_2^{(2)} \oplus \underline{\mathbb{C}})$, where line bundles $\xi_2^{(1)}$ and $\xi_2^{(2)}$ are determined by the following integer vectors:

$$\begin{aligned}\boldsymbol{a}_{2,1}^{(1)} &= \big(\langle \alpha_1 + \alpha_2, \alpha_1^\vee + \alpha_2^\vee \rangle, \langle \alpha_1 + \alpha_2, \alpha_2^\vee \rangle, 0\big) = (2, 1, 0),\\ \boldsymbol{a}_{2,1}^{(2)} &= \big(\langle \alpha_2, \alpha_1^\vee + \alpha_2^\vee \rangle, \langle \alpha_2, \alpha_2^\vee \rangle, 0\big) = (1, 2, 0).\end{aligned}$$

Remark 4.12. Suppose that the flag Bott–Samelson variety $Z_{\mathcal{I}}$ is a Bott–Samelson variety, i.e., $m_1 = \cdots = m_r = 1$. Then integer vectors $\{\boldsymbol{a}_{k,j}^{(l)} \in \mathbb{Z}^2\}_{l\in[2],1\le j<k\le r}$ determining the flag Bott tower $Z^0_{\mathcal{I}}$ is

$$\boldsymbol{a}_{k,j}^{(l)} = \begin{cases} (\langle \alpha_{u_{k,1}}, \alpha^\vee_{u_{j,1}} \rangle, 0) & \text{if } l = 1,\\ (0, 0) & \text{if } l = 2\end{cases}$$

by Theorem 4.10. This computation of $\boldsymbol{a}_{k,j}^{(1)}(1)$ for $1 \le j < k \le r$ coincides with the known result in [Grossberg and Karshon 1994, §3.7].

5. Torus actions and Duistermaat–Heckman measure

Let $\mathcal{I} = (I_1, \ldots, I_r)$ be a sequence of subsets of $[n]$ such that $|I_k| = m_k$. In this section we study torus actions on the manifold $Z^0_{\mathcal{I}}$. We define a torus invariant closed 2-form induced from a given complex line bundle, and we consider the Duistermaat–Heckman measure of the flag Bott–Samelson manifold using a Bott–Samelson variety $Z_{\boldsymbol{i}}$ admitting the birational morphism $\eta_{\boldsymbol{i},\mathcal{I}} : Z_{\boldsymbol{i}} \to Z_{\mathcal{I}}$ (see Theorem 5.5).

We first study torus actions on $Z^0_{\mathcal{I}}$. Let T be the maximal compact torus of G contained in H. Define an action of $T^{(r)}$ on $Z^0_{\mathcal{I}}$ as

$$\begin{aligned}(s_1, \ldots, s_r) \cdot [p_1, \ldots, p_r] &= [s_1 p_1, s_1^{-1} s_2 p_2, \ldots, s_{r-1}^{-1} s_r p_r]\\ &= [s_1 p_1 s_1^{-1}, \ldots, s_r p_r s_r^{-1}].\end{aligned} \tag{5-1}$$

This action is smooth but not effective. We now find the subtorus which acts trivially on $Z^0_{\mathcal{I}}$. Define a subtorus $T_I \subset T$ for a subset $I \subset [n]$ as

$$T_I := \big\{ s \in T \mid \alpha_i(s) = 1 \text{ for all } i \in I \big\}^0$$

which is similar to (2-1). Here, we consider a simple root $\alpha \in \chi(H)$ as a homomorphism $T \to S^1$. For a given sequence $\mathcal{I} = (I_1, \dots, I_r)$ of subsets of $[n]$, we define the subtorus $T_{\mathcal{I}}$ of $T^{(r)}$ as

$$T_{\mathcal{I}} := T_{I_1} \times \cdots \times T_{I_r}.$$

Similarly, we set $T_{\boldsymbol{i}} := T_{\{i_1\}} \times \cdots \times T_{\{i_r\}}$ for a sequence $\boldsymbol{i} = (i_1, \dots, i_r) \in [n]^r$. Then the following proposition comes from (5-1).

Proposition 5.1. *The torus $T_{\mathcal{I}}$ acts trivially on $Z^0_{\mathcal{I}}$.*

By Proposition 5.1, we have the torus action on $Z^0_{\mathcal{I}}$:

$$T^{(r)}/T_{\mathcal{I}} \curvearrowright Z^0_{\mathcal{I}}. \tag{5-2}$$

Note that $T^{(r)}/T_{\mathcal{I}} \cong (S^1)^{m_1+\cdots+m_r}$.

Suppose that

$$\boldsymbol{i} = (i_{k,l})_{1\le k\le r,\, 1\le l\le N_k} \in [n]^{N_1+\cdots+N_r}$$

is a sequence such that $(i_{k,1}, \dots, i_{k,N_k})$ is a reduced word for the longest element in W_{I_k} for $1 \le k \le r$. From now on, we ignore the complex structure on the flag Bott–Samelson manifold $Z_{\mathcal{I}}$ and regard it as a smooth manifold. Therefore we can identify $Z_{\mathcal{I}}$ with $Z^0_{\mathcal{I}}$ and $Z_{\boldsymbol{i}}$ with $Z^0_{\boldsymbol{i}}$ by Proposition 4.6. Using the observation (5-2), we have the torus action on the Bott–Samelson manifold $Z_{\boldsymbol{i}}$:

$$(S^1)^N \cong T^{(N)}/T_{\boldsymbol{i}} \curvearrowright Z_{\boldsymbol{i}},$$

where $N := N_1 + N_2 + \cdots + N_r$. We denote $\widetilde{\boldsymbol{T}} := (T^{(N)})/T_{\boldsymbol{i}}$ and $\boldsymbol{T} := (T^{(r)})/T_{\mathcal{I}}$ for simplicity.

Lemma 5.2. *There is a homomorphism $A : \boldsymbol{T} \to \widetilde{\boldsymbol{T}}$ such that the map $\eta_{\boldsymbol{i},\mathcal{I}} : Z_{\boldsymbol{i}} \to Z_{\mathcal{I}}$ is equivariant with respect to the action of $\boldsymbol{T}$, i.e.,*

$$\eta_{\boldsymbol{i},\mathcal{I}}(A(t)\cdot x) = t \cdot \eta_{\boldsymbol{i},\mathcal{I}}(x)$$

for any $t \in \boldsymbol{T}$ and $x \in Z_{\boldsymbol{i}}$.

Proof. Define an inclusion map $\iota : T^{(r)} \hookrightarrow T^{(N)}$ as

$$(a_1, \dots, a_r) \overset{\iota}{\mapsto} (\underbrace{a_1, \dots, a_1}_{N_1}, \dots, \underbrace{a_k, \dots, a_k}_{N_k}, \dots, \underbrace{a_r, \dots, a_r}_{N_r}).$$

Then we have the action $T^{(r)} \curvearrowright Z_{\boldsymbol{i}}$ via the inclusion ι and the map $\eta_{\boldsymbol{i},\mathcal{I}} : Z_{\boldsymbol{i}} \to Z_{\mathcal{I}}$ is equivariant with respect to the action of $T^{(r)}$ by the definition of torus action in (5-1).

We claim that $\iota(T_{\mathcal{I}}) \subset T_{\boldsymbol{i}}$. For an element $(a_1, \dots, a_r) \in T^{(r)}$, we have that

$$(a_1, \dots, a_r) \in T_{\mathcal{I}} \Leftrightarrow a_k \in T_{I_k} \quad \text{for all } 1 \le k \le r.$$

Hence we have $a_k \in T_{i_{k,1}}, \dots, a_k \in T_{i_{k,N_k}}$ since $\{i_{k,1}, \dots, i_{k,N_k}\} = I_k$ for all $1 \le k \le r$. This gives that $\iota(T_{\mathcal{I}}) \subset T_{\boldsymbol{i}}$ as claimed. We thus have the homomorphism

$$A : T^{(r)}/T_{\mathcal{I}} \to T^{(N)}/T_{\boldsymbol{i}} \tag{5-3}$$

induced from the inclusion ι. Moreover the projection map $Z_{\boldsymbol{i}} \to Z_{\mathcal{I}}$ is equivariant with respect to the action of $\boldsymbol{T}$ because of the $T^{(r)}$-equivariance of the projection. □

We set $A_k : T/T_{I_k} \to T^{(N_k)}/T_{(i_{k,1},\dots,i_{k,N_k})}$ for $1 \le k \le r$. By the definition of $T_{(i_{k,1},\dots,i_{k,N_k})}$, the torus $T^{(N_k)}/T_{(i_{k,1},\dots,i_{k,N_k})}$ has dimension N_k. Suppose that $\{f_{k,1}, \dots, f_{k,N_k}\}$ is the standard basis of $\mathrm{Lie}((S^1)^{N_k})^* \cong \mathbb{R}^{N_k}$. Then it is known from [Grossberg and Karshon 1994, §3.7] that the pullback of $f_{k,l}$ is $\alpha_{i_{k,l}}$ for $1 \le l \le N_k$. Since the homomorphism A can be identified with $A_1 \times \cdots \times A_r$, the Lie algebra homomorphism $(dA)^* : \mathbb{R}^N \to \mathbb{R}^{m_1+\cdots+m_r}$ maps $f_{k,l}$ to $\alpha_{i_{k,l}}$ for $1 \le k \le r$ and $1 \le l \le N_k$.

Example 5.3. Recall from Example 2.8 that we have a morphism $\eta_{(1,2,1,3),\mathcal{I}}$ from $Z_{(1,2,1,3)}$ to $Z_{\mathcal{I}}$, where $\mathcal{I} = (\{1,2\},\{3\})$. Suppose that $A : T^{(2)}/T_{\mathcal{I}} \to T^{(4)}/T_{(1,2,1,3)}$ is the homomorphism in Lemma 5.2. Then the Lie algebra homomorphism $(dA)^* : \mathbb{R}^4 \to \mathbb{R}^3$ is defined using the integer matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We now consider Duistermaat–Heckman measures corresponding to flag Bott–Samelson manifolds. We recall definitions from [Audin 2004]. Suppose that M is an oriented, compact manifold of real dimension $2d$ with an action of a compact torus T. Let ω be a presymplectic form, i.e., a T-invariant closed not necessarily nondegenerate 2-form. Then we call the manifold (M, ω, T) *presymplectic T-manifold.* A *moment map* on (M, ω, T) is defined to be a map $\Phi : M \to \mathrm{Lie}(T)^*$ such that

$$\langle d\Phi, \xi\rangle = -\iota(\xi_M)\omega \quad \text{for all } \xi \in \mathrm{Lie}(T),$$

where ξ_M is the vector field on M which generates the action of the one-parameter subgroup $\{\exp(t\xi) \mid t \in \mathbb{R}\}$ of T. Note that the Liouville measure on M is defined to be $\int_A \omega^d/d!$ for an open subset $A \subset M$, and its push-forward $\Phi_*\omega^d/d!$ is called the *Duistermaat–Heckman measure* in $\mathrm{Lie}(T)^*$.

Consider the line bundle $\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r}$ over $Z_{\mathcal{I}}$ determined by integral weights $\lambda_1, \dots, \lambda_r$. Then we have an integer vector $\boldsymbol{a} = (\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(r)}) \in \mathbb{Z}^{N_1} \oplus \cdots \oplus \mathbb{Z}^{N_r}$ such that $\eta^*\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r} = \mathcal{L}_{\boldsymbol{i},\boldsymbol{a}}$ by Proposition 2.10. Let $\omega'_{\boldsymbol{i}}$, respectively $\omega'_{\mathcal{I}}$, be a closed 2-form corresponding to the first Chern class of the line bundle $\mathcal{L}_{\boldsymbol{i},\boldsymbol{a}} \to Z_{\boldsymbol{i}}$, respectively $\mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r} \to Z_{\mathcal{I}}$. By taking averages of $\omega'_{\boldsymbol{i}}$ and $\omega'_{\mathcal{I}}$ by corresponding torus actions we have the following two 2-forms:

$$\omega_{\boldsymbol{i}} := \int_{a\in\widetilde{\boldsymbol{T}}} (a^*\omega'_{\boldsymbol{i}})\,da \quad \text{and} \quad \omega_{\mathcal{I}} := \int_{t\in\boldsymbol{T}} (t^*\omega'_{\mathcal{I}})\,dt. \tag{5-4}$$

Then the form $\omega_{\boldsymbol{i}}$, respectively $\omega_{\mathcal{I}}$, is a $\widetilde{\boldsymbol{T}}$-invariant, respectively $\boldsymbol{T}$-invariant, closed 2-form on $(Z_{\boldsymbol{i}}, \widetilde{\boldsymbol{T}})$, respectively $(Z_{\mathcal{I}}, \boldsymbol{T})$. Since compact tori $\widetilde{\boldsymbol{T}}$ and $\boldsymbol{T}$ are connected, we have that

$$[\omega_{\boldsymbol{i}}] = [\omega'_{\boldsymbol{i}}] \quad \text{in } H^2(Z_{\boldsymbol{i}};\mathbb{R}), \qquad [\omega_{\mathcal{I}}] = [\omega'_{\mathcal{I}}] \quad \text{in } H^2(Z_{\mathcal{I}};\mathbb{R}) \tag{5-5}$$

(see [Guillemin et al. 2002, Corollary B.13]).

Grossberg and Karshon [1994] proved that the Duistermaat–Heckman measure of the presymplectic manifold $(Z_{\boldsymbol{i}}, \omega_{\boldsymbol{i}}, \widetilde{\boldsymbol{T}})$ can be computed by considering a combinatorial object, called a *Grossberg–Karshon twisted cube*. We use it to compute the Duistermaat–Heckman measure of the presymplectic manifold $(Z_{\mathcal{I}}, \omega_{\mathcal{I}}, \boldsymbol{T})$.

We recall from [Grossberg and Karshon 1994, §2.5] the definition of Grossberg–Karshon twisted cubes. Let $\boldsymbol{i} = (i_1, \dots, i_N)$ be a sequence of elements in $[n]$ and $\boldsymbol{a} = (a_1, \dots, a_N) \in \mathbb{Z}^N$. A Grossberg–Karshon twisted cube is a pair $(C(\boldsymbol{i}, \boldsymbol{a}), \rho)$, where $C(\boldsymbol{i}, \boldsymbol{a})$ is a subset of $\mathbb{R}^N$ and $\rho : \mathbb{R}^N \to \mathbb{R}$ is a density function with support equal to $C(\boldsymbol{i}, \boldsymbol{a})$. We define the following functions on $\mathbb{R}^N$:

$$\begin{aligned} A_N(x) &= A_N(x_1, \dots, x_N) = -\langle a_N \varpi_{i_N}, \alpha^\vee_{i_N}\rangle, \\ A_\ell(x) &= A_\ell(x_1, \dots, x_N) \\ &= -\langle a_\ell \varpi_{i_\ell} + \cdots + a_N \varpi_{i_N}, \alpha^\vee_{i_\ell}\rangle - \sum_{j>\ell} \langle \alpha_{i_j}, \alpha^\vee_{i_\ell}\rangle x_j \quad \text{for } 1 \le \ell \le N-1. \end{aligned}$$

We also define a function sign : $\mathbb{R} \to \{\pm 1\}$ as $\operatorname{sign}(x) = -1$ for $x \le 0$ and $\operatorname{sign}(x) = 1$ for $x > 0$.

Definition 5.4. Let $C(\boldsymbol{i}, \boldsymbol{a})$ be the following subset of $\mathbb{R}^N$:

$$C(\boldsymbol{i}, \boldsymbol{a}) := \bigl\{x = (x_1, \dots, x_N) \in \mathbb{R}^N \mid A_j(x) \le x_j \le 0 \text{ or } 0 < x_j < A_j(x) \\ \text{for } 1 \le j \le N\bigr\}.$$

We define a density function $\rho : \mathbb{R}^N \to \mathbb{R}$ whose support is $C(\boldsymbol{i}, \boldsymbol{a})$ and $\rho(x) = (-1)^N \operatorname{sign}(x_1) \cdots \operatorname{sign}(x_N)$ on the set $C(\boldsymbol{i}, \boldsymbol{a})$. We call the pair $(C(\boldsymbol{i}, \boldsymbol{a}), \rho)$ the *Grossberg–Karshon twisted cube associated to* $\boldsymbol{i}$ *and* $\boldsymbol{a}$. Also we define a measure

$$m_{C(\boldsymbol{i},\boldsymbol{a})} = \rho(\alpha)|d\alpha|,$$

where $|d\alpha|$ is the Lebesgue measure in $\mathbb{R}^N$.

Now we have the following theorem.

Theorem 5.5. *Let* $(Z_{\mathcal{I}}, \omega_{\mathcal{I}}, \boldsymbol{T})$ *be as above, and let* $\Phi : Z_{\mathcal{I}} \to \mathbb{R}^{m_1+\cdots+m_r}$ *be a moment map of* $(Z_{\mathcal{I}}, \omega_{\mathcal{I}}, \boldsymbol{T})$. *Then there is a Grossberg–Karshon twisted cube* $(C(\boldsymbol{i}, \boldsymbol{a}), \rho)$ *and an affine projection* $L : \mathbb{R}^N \to \mathbb{R}^{m_1+\cdots+m_r}$ *such that the Duistermaat–Heckman measure in* $\mathrm{Lie}(\boldsymbol{T})^* \cong \mathbb{R}^{m_1+\cdots+m_r}$ *is* $L_* m_{C(\boldsymbol{i},\boldsymbol{a})}$.

To give a proof, we need the following theorem.

Theorem 5.6 [Grossberg and Karshon 1994, Theorem 2]. *Let* $\widetilde{\Phi} : Z_{\boldsymbol{i}} \to \mathbb{R}^N$ *be a moment map of* $(Z_{\boldsymbol{i}}, \omega_{\boldsymbol{i}}, \widetilde{\boldsymbol{T}})$. *Then the Duistermaat–Heckman measure in* $\mathrm{Lie}(\widetilde{\boldsymbol{T}})^* \cong \mathbb{R}^N$ *coincides with the measure* $m_{C(\boldsymbol{i},\boldsymbol{a})}$ *for the Grossberg–Karshon twisted cube* $C(\boldsymbol{i}, \boldsymbol{a})$.

Proof of Theorem 5.5. Suppose that $\boldsymbol{i} \in [n]^N$ defines a Bott–Samelson manifold $Z_{\boldsymbol{i}}$ which has a birational morphism $\eta : Z_{\boldsymbol{i}} \to Z_{\mathcal{I}}$. For given weights $\lambda_1, \dots, \lambda_r$, let $\boldsymbol{a} \in \mathbb{Z}^N$ be an integer vector such that $\eta^* \mathcal{L}_{\mathcal{I},\lambda_1,\dots,\lambda_r} = \mathcal{L}_{\boldsymbol{i},\boldsymbol{a}}$. Consider the pullback of $\omega_{\mathcal{I}}$ under the map η. Then we have $[\omega_{\boldsymbol{i}}] = [\eta^*(\omega_{\mathcal{I}})]$ in $H^2(Z_{\boldsymbol{i}}; \mathbb{R})$ by (5-5).

Now we have the following diagram which does not necessarily commute because two forms $\eta^* \omega_{\mathcal{I}}$ and $\omega_{\boldsymbol{i}}$ do not necessarily coincide because of taking averages:

$$\begin{tikzcd}
Z_{\boldsymbol{i}} \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\eta"] & \mathbb{R}^N \cong \mathrm{Lie}(\widetilde{\boldsymbol{T}})^* \arrow[d, "L"] \\
Z_{\mathcal{I}} \arrow[r, "\Phi"] & \mathbb{R}^{m_1+\cdots+m_r} \cong \mathrm{Lie}(\boldsymbol{T})^*
\end{tikzcd}$$

Here, the map $L : \mathbb{R}^N \to \mathbb{R}^{m_1+\cdots+m_r}$ is defined as dA^*, where $A : \boldsymbol{T} \to \widetilde{\boldsymbol{T}}$ in (5-3).

But one can see that $L \circ \widetilde{\Phi}$, respectively $\Phi \circ \eta$, is a moment map for $(Z_{\boldsymbol{i}}, \omega_{\boldsymbol{i}}, \boldsymbol{T})$, respectively $(Z_{\boldsymbol{i}}, \eta^* \omega_{\mathcal{I}}, \boldsymbol{T})$. Recall from [Grossberg and Karshon 1994, Theorem 1] that the push-forward of Liouville measure only depends on the cohomology class, so we have that

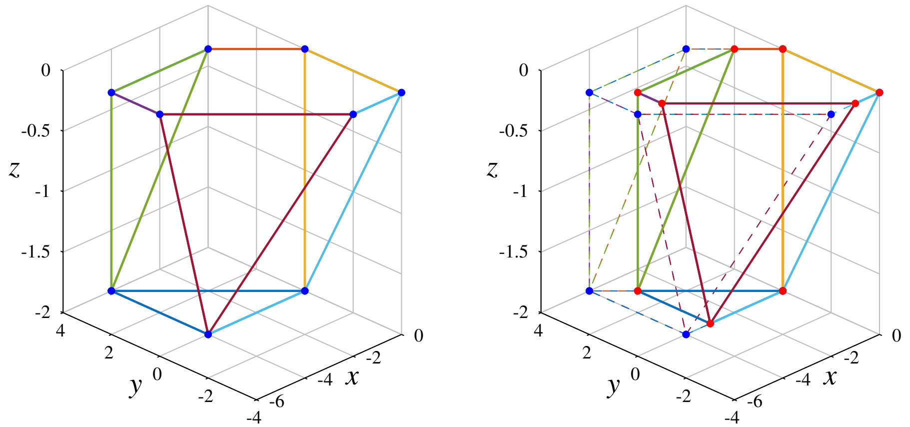
$$(L \circ \widetilde{\Phi})_* \omega_{\boldsymbol{i}}^N = (\Phi \circ \eta)_* (\eta^* \omega_{\mathcal{I}})^N = \Phi_* \omega_{\mathcal{I}}^N.$$

Here the last equality holds since η induces a diffeomorphism between Zariski open dense subsets, and a Zariski closed subset is measure zero. By Theorem 5.6, we have that $\Phi_* \omega_{\mathcal{I}}^N / N! = L_* m_{C(\boldsymbol{i},\boldsymbol{a})}$, so the result follows. □

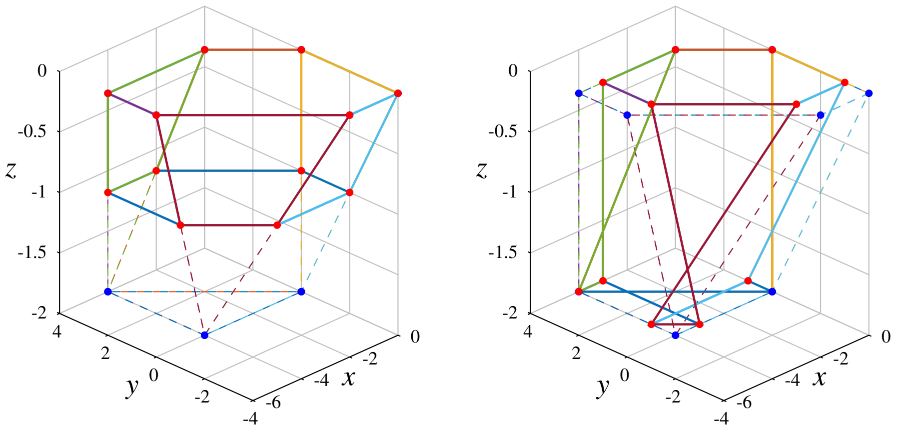
Example 5.7. Let $G = \mathrm{SL}(4)$, $\mathcal{I} = (\{1, 2\}, \{3\})$ and $\boldsymbol{i} = (1, 2, 1, 3)$. The projection map $L = (dA)^* : \mathbb{R}^4 \to \mathbb{R}^3$ is given by the integer matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

as in Example 5.3. In Figure 2 we draw figures for four different pairs of weights $(\lambda_1, \lambda_2) = (2\varpi_1 + 4\varpi_2, 2\varpi_3), (\varpi_1 + 4\varpi_2, 2\varpi_3), (2\varpi_1 + 4\varpi_2, \varpi_3), (2\varpi_1 + 3\varpi_2, 2\varpi_3)$



(1) $(\lambda_1, \lambda_2) = (2\varpi_1 + 4\varpi_2, 2\varpi_3)$. (2) $(\lambda_1, \lambda_2) = (\varpi_1 + 4\varpi_2, 2\varpi_3)$.



(3) $(\lambda_1, \lambda_2) = (2\varpi_1 + 4\varpi_2, \varpi_3)$. (4) $(\lambda_1, \lambda_2) = (2\varpi_1 + 3\varpi_2, 2\varpi_3)$.

Figure 2. The projection images of Grossberg–Karshon twisted cubes.

which determine line bundles $\mathcal{L}_{\mathcal{I},\lambda_1,\lambda_2}$. The polytope in Figure 2(1) has eight facets. When we change an integer vector (λ_1, λ_2) a little bit, some facets move as one can see in the figure. In Figure 2(2)–(4) the red dots represent vertices of the projection for the corresponding integer vector, and the blue dots represent vertices of the projection for $(\lambda_1, \lambda_2) = (2\varpi_1 + 4\varpi_2, 2\varpi_3)$. For pairs $(2\varpi_1 + 4\varpi_2, 2\varpi_3)$, $(\varpi_1 + 4\varpi_2, 2\varpi_3)$, and $(2\varpi_1 + 4\varpi_2, \varpi_3)$, the projections are honest polytopes while the projection for $(2\varpi_1 + 3\varpi_2, 2\varpi_3)$ is not.

Remark 5.8. Note that a Grossberg–Karshon twisted cube is neither closed not convex. When the Grossberg–Karshon twisted cube is a closed convex polytope, then we say it is *untwisted*. In [Lee 2020], an interpretation of untwistedness of Grossberg–Karshon twisted cubes $C(\boldsymbol{i}, \boldsymbol{a})$ using combinatorics of $\boldsymbol{i}$ and $\boldsymbol{a}$ is provided. (Also, see [Harada and Yang 2015; Harada and Lee 2015].) Using the result [Lee 2020, Theorem 1], Grossberg–Karshon twisted cubes appearing in Example 5.7 are all *twisted*. However, their projections can be honest polytopes as we saw in Figure 2. Determining whether the projection of a Grossberg–Karshon twisted cube is an honest polytope is a still open problem.

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NAOKI FUJITA
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
THE UNIVERSITY OF TOKYO
MEGURO-KU
TOKYO
JAPAN
nfujita@ms.u-tokyo.ac.jp

EUNJEONG LEE
CENTER FOR GEOMETRY AND PHYSICS
INSTITUTE FOR BASIC SCIENCE
POHANG
REPUBLIC OF KOREA
eunjeong.lee@ibs.re.kr

DONG YOUP SUH
DEPARTMENT OF MATHEMATICAL SCIENCES
KAIST
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REPUBLIC OF KOREA
dysuh@kaist.ac.kr

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matthias@math.ucla.edu

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Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Wee Teck Gan
Mathematics Department
National University of Singapore
Singapore 119076
matgwt@nus.edu.sg

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

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Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

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