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SHOO SETO

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We introduce a generalization of the p-Laplace operator to act on differential forms and generalize an estimate of Gallot and Meyer (1973) for the first nonzero eigenvalue on closed Riemannian manifolds.

1. Introduction

Let (M, g) be an n-dimensional closed Riemannian manifold. Motivated from the variational characterization of the Laplacian eigenvalue problem, we define the L^p-Dirichlet integral on k-forms (introduced in [Scott 1995]) by

$$\mathcal{F}[\alpha] := \int_M \|d\alpha\|^p + \|d^*\alpha\|^p, \quad \alpha \in \Omega^k(M), \tag{1}$$

where d^* is the L^2-adjoint of the exterior derivative d. Note that $\mathcal{F}[\alpha] = 0$ if and only if $\alpha \in \mathcal{H}^k(M)$, that is, the minimum is zero and is attained for harmonic k-forms, i.e., $\alpha \in \ker(d) \cap \ker(d^*)$. For a nonzero infimum we consider the space

$$A_k := \left\{ \alpha \in \mathcal{W}^{1,p}(\Omega^k(M)) \;\middle|\; \fint_M \|\alpha\|^p = 1,\ \int_M \|\alpha\|^{p-2}\langle \alpha, \omega\rangle = 0,\ \omega \in \mathcal{H}^k(M) \right\}, \tag{2}$$

where the space $\mathcal{W}^{1,p}(\Omega^k(M))$ is the $(1, p)$-Sobolev space of differential k-forms defined in [Scott 1995]. See Section 3 for the precise definition. Computing the Euler–Lagrange equation leads us to the defining the following operator:

Definition 1.1 (p-Hodge Laplacian).

$$\Delta_p \alpha := d^*(\|d\alpha\|^{p-2} d\alpha) + d(\|d^*\alpha\|^{p-2} d^*\alpha), \quad \alpha \in \Omega^k(M). \tag{3}$$

When $p = 2$, this becomes the usual Hodge Laplacian. For $p \neq 2$ and $\alpha \in C^\infty(M)$ Δ_p becomes the usual p-Laplacian. The corresponding eigenvalue equation is given by

$$\Delta_p \alpha = \lambda \|\alpha\|^{p-2} \alpha, \quad \alpha \in \Omega^k(M) \tag{4}$$

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and the variational principle tells us that

$$\lambda_1 = \inf\{\mathcal{F}[\alpha] \mid \alpha \in A_k\}.$$

See Section 3 for details. When $p = 2$, there is much work on the spectrum of the Hodge Laplacian acting on differential forms. Among many others, we point out the work of Gallot and Meyer [1973; 1975] who show an estimate of the first eigenvalue using bounds from the Weitzenböck curvature on compact Riemannian manifolds. For manifolds with boundary, among many others, see the works [Kwong 2016; Savo 2009; Raulot and Savo 2011].

For $p \neq 2$, the p-Laplace eigenvalue problem on 0-forms (functions) has attracted much attention. See notes by Lindqvist [2006] for a general reference on the p-Laplace equation. For estimates on the first eigenvalue relating to the curvature, among many other works, see [Matei 2000; Naber and Valtorta 2014; Seto and Wei 2017] for eigenvalue estimates with $\mathrm{Ric} \geq K$, $K \in \mathbb{R}$.

Remark 1.2. There is also a related notion of p-harmonic k-forms which looks at the minimizer in a cohomology class of k-forms with finite L^p-norm, i.e.,

$$\inf_{\alpha \in H^k_d(M)} \int_M \|\alpha\|^p.$$

The critical point of the variation leads to the following definition of p-harmonic, for closed k-forms α, if

$$d^*_p := d^*(\|\alpha\|^{p-2}\alpha) = 0$$

then α is p-harmonic. See [Dung 2017].

In this paper we prove the following lower bound estimate for the first eigenvalue:

Theorem 1.3. *Let M^n be a closed Riemannian manifold with the eigenvalues of the curvature operator bounded below by $H \in \mathbb{R}$ and $p \geq 2$. Then*

$$\lambda_1 \geq \left(\frac{k(n-k)}{2^{(2/p)-1}(C + (p-2)/2)} H \right)^{p/2},$$

where

$$C = \max\left\{ \frac{k}{k+1}, \frac{n-k}{n-k+1} \right\}.$$

Remark 1.4. When $p = 2$, the above recovers the estimate due to Gallot and Meyer [1973] (see also [Gallot and Meyer 1975]), for $1 \leq k \leq \frac{n}{2}$,

$$\lambda_1 \geq k(n-k+1)H.$$

The organization of this paper is as follows. In Section 2 we review some known estimates for differential k-forms. In Section 3 we show that the infimum can be characterized as an eigenvalue problem. In Section 4 we give the main estimate. In

Section 5 we give a brief discussion on boundary conditions for differential forms and possible future directions.

2. Some estimates on $\Omega^k(M)$

We first recall the Weitzenböck curvature

Definition 2.1. Let $p \in M$ and let $\{E_i\}_{i=1}^n$ be an orthonormal frame at p. Then for $\alpha \in \Omega^k(M)$, define the Weitzenböck curvature W_k by

$$W_k(\alpha)(X_1, \ldots, X_k) := \sum (R(E_j, X_i)\alpha)(X_1, \ldots, E_j, \ldots, X_k).$$

Note that on 1-forms, this is simply the Ricci tensor.

If the eigenvalues of the curvature operator are bounded by $H \in \mathbb{R}$, we can show that

$$(W_k(\alpha), \alpha) \geq k(n-k)H\|\alpha\|^2. \tag{5}$$

The Weitzenböck curvature appears in the main tool we use in obtaining our estimate, which is the Bochner–Weitzenböck formula for k-forms:

$$\tfrac{1}{2}\Delta\|\alpha\|^2 = (\Delta\alpha, \alpha) - \|\nabla\alpha\|^2 - (W_k(\alpha), \alpha), \tag{6}$$

where $\Delta := \Delta_2 = dd^* + d^*d$. Note that for exact 1-form $\alpha = df$, since $\nabla df = \operatorname{Hess} f$, the usual Cauchy–Schwarz inequality will give us an estimate on the middle term. For k-forms, we will need the following:

Lemma 2.2 [Gallot and Meyer 1973]. *Let $\alpha \in \Omega^k(M)$, $1 \leq k \leq n-1$. Then*

$$\|\nabla\alpha\|^2 \geq \frac{1}{k+1}\|d\alpha\|^2 + \frac{1}{n-k+1}\|d^*\alpha\|^2. \tag{7}$$

We give a proof for completeness. The proof we give is in the context of conformal Killing forms and can be found in various sources, for instance, [Moroianu and Semmelmann 2003].

Proof. Consider the two linear maps

$$\iota : TM \otimes \Omega^k(M) \to \Omega^{k-1}(M)$$
$$\iota(v, \alpha) = \iota_v\alpha$$

and

$$\wedge : \Omega^1(M) \otimes \Omega^k(M) \to \Omega^{k+1}(M)$$
$$\wedge(\beta, \alpha) = \beta \wedge \alpha.$$

Let ι^* and $\wedge^*$ be their metric adjoint. Then

$$\wedge \circ \iota^*(\alpha) = 0 \quad \text{and} \quad \iota \circ \wedge^*(\alpha) = 0,$$

so that we get the decomposition

$$TM \otimes \Omega^k(M) \simeq \mathrm{im}(\iota^*) \oplus \mathrm{im}(\wedge^*) \oplus Y,$$

where Y is the orthogonal complement. By direct computation, for $\alpha \in \Omega^k(M)$, we have

$$\iota \circ \iota^*(\alpha) = (n-k+1)\alpha \quad \text{and} \quad \wedge \circ \wedge^* (\alpha) = (k+1)\alpha.$$

Viewing $\nabla\alpha \in \Gamma(TM \otimes \Omega^k(M))$, From the decomposition,

$$\nabla\alpha = \iota^*\beta + \wedge^*\gamma + \delta,$$

applying ι, we have

$$\iota\nabla\alpha = (n-k+1)\beta.$$

So the projection operator onto $\mathrm{im}(\iota^*)$ is given by

$$\pi_{\iota^*}\nabla\alpha = \frac{1}{n-k+1}\iota^*\iota\nabla\alpha$$

and similarly

$$\pi_{\wedge^*}\nabla\alpha = \frac{1}{k+1}\wedge^*\wedge\nabla\alpha.$$

Let $T\alpha := \pi_T(\nabla\alpha)$ the projection onto the orthogonal complement space. Since

$$d\alpha = \wedge(\nabla\alpha) \quad \text{and} \quad d^*\alpha = -\iota(\nabla\alpha),$$

we have the decomposition

$$T\alpha(X) = \nabla_X\alpha - \frac{1}{k+1}\iota_X d\alpha + \frac{1}{n-k+1}X^* \wedge d^*\alpha$$

and taking the norm gives us

$$\|\nabla\alpha\|^2 = \|T\alpha\|^2 + \frac{1}{k+1}\|d\alpha\|^2 + \frac{1}{n-k+1}\|d^*\alpha\|^2,$$

which implies (7). □

Remark 2.3. The projection operator T defined above is called the twistor operator and a form $\alpha \in \Omega^k(M)$ is called a conformal Killing form if $T\alpha = 0$.

The following lemma was pointed out by N.T. Dung and gives us a way to control the interior product by using an orthogonal decomposition of forms as the image under an interior product.

Lemma 2.4 [Dung and Sung 2019, Lemma 3.5]. *Let $V \in TM$, $\alpha \in \Omega^{k+1}$, $\beta \in \Omega^k$. Then*

$$|\langle \iota_V\alpha, \beta\rangle| \le \|V\|\|\alpha\|\|\beta\|.$$

3. Variational characterization of the eigenvalue

In this section we will compute the Euler–Lagrange equation of (1) and show that the extremal problem can be reformulated as an eigenvalue problem. Analogous to the 0-form (function) case, we will look at weak solutions lying the $(1, p)$-Sobolev space of differential k-forms first defined by Scott [1995] as

$$\mathcal{W}^{1,p}(\Omega^k(M)) := \left\{\alpha \in W(\Omega^k(M)) \mid \alpha, d\alpha, d^*\alpha \in L^p(\Omega^*(M))\right\},$$

where $W(\Omega^k(M))$ is the classical Sobolev space of k-forms, i.e., α is locally integrable and admits a generalized gradient.

Definition 3.1. We say that λ is an eigenvalue, if there exists a k-form $\alpha \in \mathcal{W}^{1,p}(\Omega^k(M))$ such that

$$\int_M \|d\alpha\|^{p-2}\langle d\alpha, d\beta\rangle + \int_M \|d^*\alpha\|^{p-2}\langle d^*\alpha, d^*\beta\rangle = \lambda \int_M \|\alpha\|^{p-2}\langle \alpha, \beta\rangle,$$

for any $\beta \in C^\infty(\Omega^k(M))$.

We will show the first nonzero eigenvalue λ_1 can be characterized as the infimum of the L^p-Dirichlet energy over the space A_k given in (2).

Proposition 3.2. *For closed manifolds M and $p \geq 2$,*

$$\lambda_1 = \inf\left\{\int_M \|d\alpha\|^p + \|d^*\alpha\|^p \;\middle|\; \alpha \in A_k\right\}.$$

Proof. Let ω be a fixed harmonic form and let $\beta(t) \in A$ for small $t > 0$ such that $\beta(0) = \alpha$. Computing the first variation of (1), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}[\beta(t)]\Big|_{t=0} &= p\int_M \|d\alpha\|^{p-2}\langle d\alpha, d\beta'(0)\rangle + \|d^*\alpha\|^{p-2}\langle d^*\alpha, d^*\beta'(0)\rangle \\ &= p\int_M \langle \Delta_p\alpha, \beta'(0)\rangle.\end{aligned}$$

Next we compute the variation of the constraints so that

$$\frac{d}{dt}\int_M \|\beta\|^p\Big|_{t=0} = p\int_M |\alpha|^{p-2}\langle \alpha, \beta'(0)\rangle$$

and

$$\frac{d}{dt}\int_M \|\beta\|^{p-2}\langle \beta, \omega\rangle\Big|_{t=0} = (p-2)\int_M \|\alpha\|^{p-4}\langle \alpha, \beta'(0)\rangle\langle \alpha, \omega\rangle + \|\alpha\|^{p-2}\langle \beta'(0), \omega\rangle.$$

By the Lagrange multiplier method, there must be some λ and μ such that for $\beta \in \Omega^k(M)$,

$$\int_M \langle \Delta_p\alpha, \beta\rangle = \lambda\int_M \|\alpha\|^{p-2}\langle \alpha, \beta\rangle + \mu\int_M \|\alpha\|^{p-4}\langle \alpha, \beta\rangle\langle \alpha, \omega\rangle + \|\alpha\|^{p-2}\langle \beta, \omega\rangle.$$

Setting $\beta = \omega$, we have

$$0 = \mu \int_M \|\alpha\|^{p-4}\langle \alpha, \omega\rangle^2 + \|\alpha\|^{p-2}\|\omega\|^2$$

so that $\mu = 0$. Therefore,

$$\Delta_p \alpha = \lambda \|\alpha\|^{p-2}\alpha. \qquad \square$$

4. Proof of Theorem 1.3

We will consider the following integral

$$\int_M \langle \Delta_p \alpha, \Delta\alpha\rangle = \int_M \langle \Delta_p \alpha, dd^*\alpha\rangle + \int_M \langle \Delta_p, d^*d\alpha\rangle.$$

Let $\alpha \in \Omega^k(M)$ be an eigenform satisfying (4). Then

$$\begin{aligned} \int_M \langle \Delta_p \alpha, d^*d\alpha\rangle &= \lambda \int_M \|\alpha\|^{p-2}\langle \alpha, d^*d\alpha\rangle \\ &= \lambda \int_M \big\langle d(\|\alpha\|^{p-2}\alpha), d\alpha\big\rangle \\ &= \lambda \int_M \big\langle d(\|\alpha\|^{p-2}) \wedge \alpha, d\alpha\big\rangle + \lambda \int_M \|\alpha\|^{p-2}\|d\alpha\|^2 \end{aligned} \tag{8}$$

and

$$\begin{aligned} \int_M \langle \Delta_p \alpha, dd^*\alpha\rangle &= \lambda \int_M \|\alpha\|^{p-2}\langle \alpha, dd^*\alpha\rangle \\ &= \lambda \int_M \big\langle d^*(\|\alpha\|^{p-2}\alpha), d^*\alpha\big\rangle \\ &= \lambda \int_M \|\alpha\|^{p-2}\|d^*\alpha\|^2 - \lambda \int_M \big\langle \iota_{\nabla\|\alpha\|^{p-2}}\alpha, d^*\alpha\big\rangle. \end{aligned} \tag{9}$$

On the other hand, by using the Bochner–Weitzenböck formula (6) we have

$$\begin{aligned} &\int_M \langle \Delta_p \alpha, \Delta\alpha\rangle \\ &\quad = \lambda \int_M \|\alpha\|^{p-2}\langle \alpha, \Delta\alpha\rangle \\ &\quad = \lambda \int_M \big((p-2)\|\alpha\|^{p-2}|\nabla\|\alpha\||^2 + \|\alpha\|^{p-2}\|\nabla\alpha\|^2 + \|\alpha\|^{p-2}(W_k(\alpha), \alpha)\big). \end{aligned} \tag{10}$$

Combining (8), (9), and (10), we obtain

$$\begin{aligned} &\int_M \big\langle d(\|\alpha\|^{p-2}) \wedge \alpha, d\alpha\big\rangle - \int_M \big\langle \iota_{\nabla\|\alpha\|^{p-2}}\alpha, d^*\alpha\big\rangle + \int_M \|\alpha\|^{p-2}\|d\alpha\|^2 \\ &\qquad + \int_M \|\alpha\|^{p-2}\|d^*\alpha\|^2 \end{aligned} \tag{11}$$

$$(11\ \text{cont.}) \quad = \int_M \big((p-2)\|\alpha\|^{p-2}|\nabla\|\alpha\||^2 + \|\alpha\|^{p-2}\|\nabla\alpha\|^2 + \|\alpha\|^{p-2}(W_k(\alpha),\alpha)\big).$$

Using Lemma 2.4, the first term of (11) can be estimated as

$$\begin{aligned}
\int_M \big\langle d(\|\alpha\|^{p-2})\wedge\alpha, d\alpha\big\rangle &= \int_M \big\langle \alpha, \iota_{\nabla\|\alpha\|^{p-2}}(d\alpha)\big\rangle \\
&\le \int_M \big\|\nabla\|\alpha\|^{p-2}\big\|\,\|d\alpha\|\,\|\alpha\| \\
&= (p-2)\int_M \|\alpha\|^{(p-2)/2}\big\|\nabla\|\alpha\|\big\|\,\|\alpha\|^{(p-2)/2}\|d\alpha\| \\
&\le \frac{(p-2)}{2}\int_M \|\alpha\|^{p-2}\big\|\nabla\|\alpha\|\big\|^2 + \frac{(p-2)}{2}\int_M \|\alpha\|^{p-2}\|d\alpha\|^2
\end{aligned}$$

and similarly for the second term,

$$\begin{aligned}
-\int_M \big\langle \iota_{\nabla\|\alpha\|^{p-2}}\alpha, d^*\alpha\big\rangle &\le \int_M \|\nabla\|\alpha\|^{p-2}\|\alpha\|\,\|d^*\alpha\| \\
&= (p-2)\int_M \|\alpha\|^{(p-2)/2}\big\|\nabla\|\alpha\|\big\|\,\|\alpha\|^{(p-2)/2}\|d^*\alpha\| \\
&\le \frac{(p-2)}{2}\int_M \|\alpha\|^{p-2}\big\|\nabla\|\alpha\|\big\|^2 + \frac{(p-2)}{2}\int_M \|\alpha\|^{p-2}\|d^*\alpha\|^2.
\end{aligned}$$

Applying these estimates to (11), we get

$$\begin{aligned}
&\frac{(p-2)+2}{2}\int_M \|\alpha\|^{p-2}\|d\alpha\|^2 + \frac{(p-2)+2}{2}\int_M \|\alpha\|^{p-2}\|d^*\alpha\|^2 \\
&\ge \int_M \|\alpha\|^{p-2}\|\nabla\alpha\|^2 + \int_M \|\alpha\|^{p-2}(W_k(\alpha),\alpha) \\
&\ge \frac{1}{k+1}\int_M \|\alpha\|^{p-2}\|d\alpha\|^2 + \frac{1}{n-k+1}\int_M \|\alpha\|^{p-2}\|d^*\alpha\|^2 + \int_M \|\alpha\|^{p-2}(W_k(\alpha),\alpha).
\end{aligned}$$

Let

$$C := \max\Big\{\frac{k}{k+1}, \frac{n-k}{n-k+1}\Big\}.$$

Using

$$\int_M \|\alpha\|^{p-2}\|d\alpha\|^2 \le \left(\int_M \|\alpha\|^p\right)^{1-2/p}\left(\int_M \|d\alpha\|^p\right)^{2/p}$$

and

$$\int_M \|\alpha\|^{p-2}\|d^*\alpha\|^2 \le \left(\int_M \|\alpha\|^p\right)^{1-2/p}\left(\int_M \|d^*\alpha\|^p\right)^{2/p},$$

we have

$$\left(C+\frac{(p-2)}{2}\right)\left(\int_M \|\alpha\|^p\right)^{1-2/p}\left[\left(\int_M \|d\alpha\|^p\right)^{2/p}+\left(\int_M \|d^*\alpha\|^p\right)^{2/p}\right] \geq \int_M \|\alpha\|^{p-2}(W_k(\alpha),\alpha).$$

For $p \geq 2$, and using the lower bound of the Weitzenböck curvature (5), we have

$$2^{(2/p)-1}\left(C+\frac{(p-2)}{2}\right)\left(\int_M \|\alpha\|^p\right)^{1-2/p}\left(\int_M \|d\alpha\|^p+\|d^*\alpha\|^p\right)^{2/p} \geq k(n-k)H\int_M \|\alpha\|^p.$$

Using the fact that $\int_M \|d\alpha\|^p+\|d^*\alpha\|^p=\lambda\int_M \|\alpha\|^p$ for eigenform α, we get

$$\lambda^{2/p} \geq \frac{k(n-k)}{2^{(2/p)-1}(C+(p-2)/2)}.$$

5. Boundary conditions

In this section we briefly discuss the situation of a compact manifold M with nonempty smooth boundary ∂M. Let n denote the unit outer normal vector and let $J:\partial M \to M$ be the inclusion. Then $J^*\alpha$ is the restriction of a form to the boundary. Then d and its adjoint d^* are related with an additional boundary term given by

$$\int_M \langle d\alpha,\beta\rangle=\int_M \langle \alpha,d^*\beta\rangle+\int_{\partial M}\langle J^*(\alpha),\iota_n\beta\rangle, \quad \alpha\in\Omega^k(M),\ \beta\in\Omega^{k+1}(M).$$

and the corresponding Green's formula for the p-Laplacian is

$$(\Delta_p\alpha,\beta)=\int_M \|d\alpha\|^{p-2}\langle d\alpha,d\beta\rangle+\int_M \|d^*\alpha\|^{p-2}\langle d^*\alpha,d^*\beta\rangle - \int_{\partial M}\left\langle \iota_n(\|d\alpha\|^{p-2}d\alpha),J^*(\beta)\right\rangle+\int_{\partial M}\left\langle \|d^*\alpha\|^{p-2}J^*(d^*\alpha),\iota_n\beta\right\rangle.$$

The two most common boundary conditions for the classical Laplacian eigenvalue problem are the Dirichlet and Neumann boundary condition. For the Hodge Laplacian, the analogous boundary conditions are the absolute boundary condition

$$\begin{cases}\iota_n\alpha=0,\\ \iota_n\, d\alpha=0, \quad \text{on } \partial M\end{cases}$$

and the relative boundary condition

$$\begin{cases}J^*(\alpha)=0,\\ J^*(d^*\alpha)=0, \quad \text{on } \partial M.\end{cases}$$

The essential feature of the boundary condition is that if α satisfies either of the boundary conditions, then $\Delta_p \alpha = 0$ implies $d\alpha = 0$ and $d^*\alpha = 0$. The boundary terms that will be introduced to (11) are

$$\int_M \langle d(\|\alpha\|^{p-2}) \wedge \alpha, d\alpha \rangle - \int_M \langle \iota_{\nabla \|\alpha\|^{p-2}} \alpha, d^*\alpha \rangle + \int_M \|\alpha\|^{p-2} \|d\alpha\|^2$$
$$+ \int_M \|\alpha\|^{p-2} \|d^*\alpha\|^2 - \int_{\partial M} \|\alpha\|^{p-2} \langle J^*(\alpha), \iota_n(d\alpha) \rangle + \int_{\partial M} \|\alpha\|^{p-2} \langle J^*(d^*\alpha), \iota_n(\alpha) \rangle$$
$$= \int_M \big((p-2) \|\alpha\|^{p-2} |\nabla \|\alpha\||^2 + \|\alpha\|^{p-2} \|\nabla \alpha\|^2 + \|\alpha\|^{p-2} (W_k(\alpha), \alpha) \big).$$

Since the boundary terms will vanish under either of the boundary conditions, we get the same estimate for the boundary value problem as well. It would be interesting to see what the Reilly formula, for instance a generalization of Theorem 3 in [Raulot and Savo 2011] would be in this context, however due to the asymmetry of the weight function in the p-Laplacian, it is not immediate what the appropriate Bochner–Weitzenböck type formula would be for Δ_p.

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SHOO SETO
DEPARTMENT OF MATHEMATICS
CALIFORNIA STATE UNIVERSITY, FULLERTON
FULLERTON, CA
UNITED STATES
shoseto@fullerton.edu

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Los Angeles, CA 90095-1555
matthias@math.ucla.edu

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Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Wee Teck Gan
Mathematics Department
National University of Singapore
Singapore 119076
matgwt@nus.edu.sg

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

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