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**VALUE DISTRIBUTION PROPERTIES FOR THE GAUSS MAPS
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We study the value distribution theory for the immersed harmonic surfaces and K -QC harmonic surfaces. We first investigate the value distribution properties for the generalized Gauss map Φ of an immersed harmonic surface, similar to the result of Fujimoto and Ru in the minimal surfaces case. After building a relation between Φ and the classical Gauss map n for the K -QC harmonic surfaces, we derive that, for a complete harmonic and K -quasiconformal surface immersed in \mathbb{R}^3 , if its unit normal n omits seven directions in S^2 and any three of which are not contained in a plane in \mathbb{R}^3 , then the surface must be flat. In the last section, under an additional condition, we give an estimate of the Gauss curvature for the K -QC harmonic surfaces, generalizing the result of the minimal surfaces in the case that the unit normal n omits a neighborhood of some fixed direction.

1. Introduction

Since R. Osserman and S. S. Chern [Chern 1965; Chern and Osserman 1967; Osserman 1964] initiated the study of the value distribution properties for the Gauss map of complete minimal surfaces immersed in \mathbb{R}^n , it has grown into a very rich theory due to the works of F. Xavier [1981], H. Fujimoto [1993], Osserman and Ru [1997] and M. Ru [1991] etc. On the other hand, as early as the late 60s, T. K. Milnor [1967; 1968] started to consider whether the theory carries over in an interesting way to the larger class of harmonically immersed surfaces. Many similar results have been obtained (see [Alarcón and López 2013; Connor et al. 2015; 2018; Dioos and Sakaki 2019; Jensen and Rigoli 1988; Kalaj 2013; Milnor 1979; 1980; 1983]). In particular, it was first observed by Milnor [1983] that, instead of the Gauss map n , the map Φ (in this paper, we call it the *generalized* Gauss map; see the next section for the definition) carries the same value distribution properties as the Gauss map n in the minimal surface case. The observation is that if the induced metric ds^2 on M (from the standard metric in \mathbb{R}^3) is complete, then

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the conformal metric $\|\Phi\|^2$ (which is called *the associated Klotz metric*) is also complete, so it allows us to study the value distribution properties for Φ . In this paper, we first study the value distribution properties for the map Φ . In particular, we obtain a result which is similar to Fujimoto and Ru’s result in the minimal surface case (see Theorem 3.3). We then use the results we obtained to study the value distribution property for the Gauss map n of harmonically immersed surfaces by comparing n with Φ (see Theorem 4.7). In the last section, under an additional condition, we give an estimate of the Gauss curvature for the K -QC harmonic surfaces, generalizing the result of the minimal surfaces in the case that the unit normal n omits a neighborhood of some fixed direction.

2. Immersed harmonic surfaces

We study the maps $X : M \rightarrow \mathbb{R}^n$, with $n \geq 3$, where M is a complex Riemann surface, and X is a regular and immersed map. The surface X is called *an immersed harmonic surface* if X is harmonic. Under a local coordinate $z = u + \sqrt{-1}v$ for the Riemann surface M , it is well known that X is harmonic if and only if

$$\Delta X = 4\partial^2 X / \partial z \partial \bar{z} \equiv 0$$

where $\frac{\partial}{\partial z} = \frac{1}{2}(\partial/\partial u - \sqrt{-1}\partial/\partial v)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial/\partial u + \sqrt{-1}\partial/\partial v)$. Thus X is harmonic if and only if

$$(1) \quad \phi := \frac{\partial X}{\partial z} = (\phi_1, \dots, \phi_n)$$

is holomorphic, where $\phi_i = \frac{\partial}{\partial z} X^i$ for $i = 1, \dots, n$ when we write $X = (X^1, \dots, X^n)$. Note that although ϕ_i are only locally defined, the holomorphic one-forms $\Phi_i := \phi_i dz$ are globally defined on M . Thus the map $\Phi := [\Phi_1 : \dots : \Phi_n] : M \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$ is well-defined and holomorphic. We call it the *generalized Gauss map* of the harmonic surface X .

Let ds^2 be the metric on M induced through X from the standard inner product on \mathbb{R}^n . In terms of local coordinate (u, v) , the first fundamental form of ds^2 is given by

$$(2) \quad ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

with

$$E = X_u \cdot X_u, \quad F = X_u \cdot X_v, \quad G = X_v \cdot X_v.$$

Let $z = u + \sqrt{-1}v$. Then we have

$$X_u = \phi + \bar{\phi}, \quad X_v = \sqrt{-1}(\phi - \bar{\phi}).$$

Utilizing the complex local coordinate (z, \bar{z}) , we can rewrite (2) as

$$(3) \quad ds^2 = h dz^2 + 2\|\phi\|^2 |dz|^2 + \overline{h} d\bar{z}^2,$$

where

$$(4) \quad h = \phi \cdot \phi = \frac{E - G - 2\sqrt{-1}F}{4}, \quad \|\phi\|^2 = \phi \cdot \bar{\phi} = \frac{E + G}{4}.$$

We call the quadratic differential $\eta := h dz^2$ the *Hopf differential*. It is clear that

$$(5) \quad |h| < \|\phi\|^2.$$

We call the metric $\Gamma := \frac{1}{2} \|\phi\|^2 |dz|^2$ the *associated conformal metric* of ds^2 , also known as the *Klotz metric*. From (3) and (5), if ds^2 is complete, then the associated Klotz metric $\|\phi\|^2 |dz|^2$ is also complete (see also Lemma 1 in [Milnor 1976]). The immersion X is said to be *weakly complete* if the associated Klotz metric $\|\phi\|^2 |dz|^2$ is complete.

Denote by $\mathfrak{K}(I)$ the intrinsic curvature (Gauss curvature) of the induced metric ds^2 above and $\mathfrak{K}(\Gamma)$ the Gauss curvature with respect to the Klotz metric $\Gamma := \frac{1}{2} \|\phi\|^2 |dz|^2$. By Lemma 1 in [Milnor 1980], there exists a positive function $\mu \leq 1$ such that

$$(6) \quad \mathfrak{K}(\Gamma) \leq \mu \mathfrak{K}(I).$$

For any choice of a unit normal vector field \mathbf{n} , one has an associated second fundamental form

$$II(\mathbf{n}) = L du^2 + 2M dudv + N dv^2$$

with $\Delta X \cdot \mathbf{n} = L + N \equiv 0$. Thus $\det(II(\mathbf{n})) = -(L^2 + M^2) \leq 0$. It follows that $\mathfrak{K}(I) \leq 0$ since

$$(7) \quad \mathfrak{K}(I) = \frac{\sum_{j=1}^{n-2} \det(II(\mathbf{n}_j))}{EG - F^2}$$

for any choices (\mathbf{n}_j) of $n - 2$ mutually orthogonal unit normal vector fields.

3. Value distribution properties of the generalized Gauss map

Let

$$X = (X^1, \dots, X^n) : M \rightarrow \mathbb{R}^n$$

be an immersed harmonic surface with the induced metric, where M is a Riemann surface. Let $\Phi = [\Phi_1 : \dots : \Phi_n] : M \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$, where $\Phi_i := \left(\frac{\partial}{\partial z} X^i\right) dz$, be the generalized Gauss map. In this section, we study the value distribution properties for the generalized Gauss map Φ . We begin with the following lemma.

Lemma 3.1. *Let $X = (X^1, \dots, X^n) : M \rightarrow \mathbb{R}^n$ be an immersed harmonic surface with the induced metric, where M is a Riemann surface. Let*

$$\Phi = [\Phi_1 : \dots : \Phi_n] : M \rightarrow \mathbb{P}^{n-1}(\mathbb{C}),$$

where $\Phi_i := \left(\frac{\partial}{\partial z} X^i\right) dz$, be the generalized Gauss map. If Φ is constant, then $X(M)$ lies in a 2-plane.

Proof. As above, denote by $\mathfrak{K}(\Gamma)$ the Gauss curvature with respect to the Klotz metric $\Gamma := \frac{1}{2} \|\phi\|^2 |dz|^2$. Then

$$\mathfrak{K}(\Gamma) = -\frac{\|\phi'\|^2 \|\phi\|^2 - (\phi' \cdot \bar{\phi})(\bar{\phi}' \cdot \phi)}{(\|\phi\|^2)^3}.$$

Since Φ is constant, we conclude that ϕ is constant, so $\mathfrak{K}(\Gamma) \equiv 0$. From (6), we get

$$0 = \mathfrak{K}(\Gamma) \leq \mu \mathfrak{K}(I),$$

where $\mu > 0$. Using the fact that $\mathfrak{K}(I) \leq 0$, we get $\mathfrak{K}(I) \equiv 0$. Now, from (7) and $\det(II(n_j)) \leq 0$ for all j , we conclude that $II(n_j) \equiv 0$ for all j , so $X(M)$ lies in a 2-plane. \square

We recall the following result due to Milnor [1983] (see also Theorem 2.1 in [Jensen and Rigoli 1988]).

Theorem 3.2 [Milnor 1983, Theorem 3]. *Let $X = (X^1, \dots, X^n) : M \rightarrow \mathbb{R}^n$ be a complete immersed harmonic surface with the induced metric, where M is a Riemann surface. Let $\Phi = [\Phi_1 : \dots : \Phi_n] : M \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$, where $\Phi_i := \left(\frac{\partial}{\partial z} X^i\right) dz$ for $i = 1, \dots, n$, be the generalized Gauss map. Then either $X(M)$ is a 2-plane or else $\Phi(M)$ comes arbitrarily close to every hyperplane $\sum_{k=1}^n a_k w_k = 0$ in $\mathbb{P}^{n-1}(\mathbb{C})$.*

We note that Theorem 3.2 corresponds to Chern’s theorem [1965] in the theory of minimal surfaces. It is known that Chern’s result has been extended to a much sharper result by Fujimoto [1990] and Ru [1991] for the Gauss maps of minimal surfaces. In the following, we have the result corresponding to the result of Fujimoto and Ru.

Theorem 3.3. *Let M be an open Riemann surface and*

$$X = (X^1, \dots, X^n) : M \rightarrow \mathbb{R}^n$$

be a harmonic immersion. Let $\Phi : M \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$ be the generalized Gauss map. Assume that X is weakly complete with respect to the reduced metric. If Φ omits more than $\frac{1}{2}n(n + 1)$ hyperplanes in $\mathbb{P}^{n-1}(\mathbb{C})$ in general position, then $X(M)$ lies in a 2-plane.

Remark 3.4. As we noted above, if the induced metric ds^2 is complete, then the associated Klotz metric $\|\phi\|^2 |dz|^2$ is also complete (we say that X is weakly complete in this case).

In order to prove Theorem 3.3, we need to introduce some basic concepts and the following auxiliary results.

Let $H = \{[z_0 : z_1 : \dots : z_k] \mid a_0 z_0 + \dots + a_k z_k = 0\}$ be a hyperplane in $\mathbb{P}^k(\mathbb{C})$; here $\mathbf{a} = (a_0, \dots, a_k) \in \mathbb{C}^{k+1} \setminus \{\mathbf{0}\}$ is called *the normal vector* associated to H .

Hyperplanes H_1, \dots, H_q are said to be *in m -subgeneral position* (with $m \geq k$) if and only if for every injective map $\mu : \{0, 1, \dots, m\} \rightarrow \{1, \dots, q\}$, the linear span of those corresponding normal vectors $\mathbf{a}_{\mu(0)}, \dots, \mathbf{a}_{\mu(m)}$ is \mathbb{C}^{k+1} . When $m = k$, then we just say the H_1, \dots, H_q are in general position in $\mathbb{P}^k(\mathbb{C})$. It is clear that if the hyperplanes H_1, \dots, H_q in $\mathbb{P}^m(\mathbb{C})$ are in general position, then, for $k \leq m$ and regarding $\mathbb{P}^k(\mathbb{C}) \subset \mathbb{P}^m(\mathbb{C})$, the restricted hyperplanes $H_1 \cap \mathbb{P}^k(\mathbb{C}), \dots, H_q \cap \mathbb{P}^k(\mathbb{C})$ are in m -subgeneral position.

We need the following lemma.

Lemma 3.5 [Chen 1987; Nochka 1983]. *Let $\{H_j\}_{j=1}^q$ be a set of hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in m -subgeneral position. Then there exist a function $\varpi : J = \{1, \dots, q\} \rightarrow \mathbb{R}$ and a number $\theta > 0$ with the following properties:*

- $0 < \varpi(j) \leq 1$ for all $j \in J$.
- $q - 2m + k - 1 = \theta(\sum_{j=1}^q \varpi(j) - k - 1)$.
- $1 \leq \frac{m+1}{k+1} \leq \theta \leq \frac{2m-k+1}{k+1}$.

We call $\varpi(j)$ the Nochka weight associated to the hyperplane $H_j (1 \leq j \leq q)$.

Next, we recall the definition of the *derived curves*. Let F be a nondegenerate holomorphic map of Δ_R into $\mathbb{P}^k(\mathbb{C})$ (i.e., $F(\Delta_R)$ is not contained in any proper subspaces of $\mathbb{P}^k(\mathbb{C})$), where $\Delta_R := \{z \mid |z| < R\} \subset \mathbb{C}$ and $0 < R \leq \infty$. Take a reduced representation $\tilde{F} = (f_0, f_1, \dots, f_k)$ of F , i.e., $\tilde{F} : \Delta_R \rightarrow \mathbb{C}^{k+1} \setminus \{\mathbf{0}\}$ and $\mathbb{P}(\tilde{F}) = F$, where \mathbb{P} is the natural projection. Let $\|\tilde{F}\| = (\sum_{j=0}^k |f_j|^2)^{\frac{1}{2}}$. Take the s -th derivative $\tilde{F}^{(s)} = (f_0^{(s)}, f_1^{(s)}, \dots, f_k^{(s)})$ and define

$$(8) \quad \tilde{F}_s = \tilde{F}^{(0)} \wedge \tilde{F}^{(1)} \wedge \dots \wedge \tilde{F}^{(s)} : \Delta_R \rightarrow \bigwedge_{s+1} \mathbb{C}^{k+1}$$

for each $0 \leq s \leq k$. Obviously, $\tilde{F}_{k+1} \equiv 0$. Let $F_s = \mathbb{P}(\tilde{F}_s)$. We call the map F_s the *s -th derived curve of F* .

For holomorphic functions f_0, f_1, \dots, f_k , we denote

$$W(f_0, f_1, \dots, f_k) := \det(f_j^{(s)}, 0 \leq j, s \leq k).$$

Let $\{e_0, e_1, \dots, e_k\}$ be the standard basis of \mathbb{C}^{k+1} . Then we can write, for $0 \leq s \leq k$,

$$\tilde{F}_s = \sum_{0 \leq i_0 < \dots < i_s \leq k} W(f_{i_0}, \dots, f_{i_s}) e_{i_0} \wedge \dots \wedge e_{i_s}.$$

Hence,

$$|\tilde{F}_s|^2 := \sum_{0 \leq i_0 < \dots < i_s \leq k} |W(f_{i_0}, \dots, f_{i_s})|^2.$$

From above, it is easy to see that if F is nondegenerate, then $\tilde{F}_s \neq 0$ for any $0 \leq s \leq k - 1$.

For a hyperplane H_j in $\mathbb{P}^k(\mathbb{C})$ with the normal vector $\mathbf{a}_j = (a_{j0}, \dots, a_{jk})$, we define, for $0 \leq s \leq k$ and $1 \leq j \leq q$,

$$(9) \quad |\tilde{F}_s(H_j)|^2 = \sum_{0 \leq i_1 < \dots < i_s \leq k} \left| \sum_{t \neq i_1, \dots, i_s} a_{jt} W(f_t, f_{i_1}, \dots, f_{i_s}) \right|^2.$$

Notice that

$$|\tilde{F}(H_j)| = |\tilde{F}_0(H_j)| = |a_{j0}f_0 + a_{j1}f_1 + \dots + a_{jk}f_k|.$$

From (9), we see that $\tilde{F}_s(H_j) \equiv 0$ if and only if it holds for all i_1, \dots, i_s that

$$\sum_{t \neq i_1, \dots, i_s} a_{jt} W(f_t, f_{i_1}, \dots, f_{i_s}) \equiv 0.$$

Thus if F is nondegenerate, then $\tilde{F}_s(H_j) \neq 0$ for all $0 \leq s \leq k - 1$ and $1 \leq j \leq q$. Indeed, if $\tilde{F}_s(H_j) \equiv 0$ for some s and j , then

$$W(\tilde{F}(H_j), f_{i_1}, \dots, f_{i_s}) \equiv 0$$

for all i_1, \dots, i_s . This implies that $\tilde{F}(H_j), f_{i_1}, \dots, f_{i_s}$ are linearly independent, which contradicts the nondegeneracy of f .

The following result is due to Ru [1991, Main Lemma]. It plays an important role in the proof of our theorem.

Lemma 3.6 [Ru 1991, Main Lemma]. *Let $F : \Delta_R \rightarrow \mathbb{P}^k(\mathbb{C})$ be a nondegenerate holomorphic map with its reduced representation \tilde{F} . Let $\{H_j\}_{j=1}^q$ be a set of hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in m -subgeneral position, and $\varpi(j)$ be their associated Nochka weights. If $q > 2m - k + 1$ and*

$$N > \frac{2qk(k+2)}{\sum_{j=1}^q \varpi(j) - (k+1)},$$

then there exists a positive constant C such that

$$|\tilde{F}|^\chi \frac{|\tilde{F}_k|^{1+\frac{2q}{N}} \prod_{j=1}^q (\prod_{s=0}^{k-1} |\tilde{F}_s(H_j)|)^{\frac{4}{N}}}{\prod_{j=1}^q |\tilde{F}(H_j)|^{\varpi(j)}} \leq C \left(\frac{2R}{R^2 - |z|^2} \right)^{\frac{1}{2}k(k+1) + \frac{2q}{N} \sum_{s=0}^k s^2},$$

where $\chi = \sum_{j=1}^q \varpi(j) - (k+1) - \frac{2q}{N}(k^2 + 2k - 1)$.

We also need the following lemma.

Lemma 3.7 [Fujimoto 1993, Lemma 1.6.7]. *Let $d\sigma^2$ be a conformal flat metric on an open Riemann surface M . Then for each point $p \in M$, there exists a local diffeomorphism Ψ of a disk $\Delta_R = \{w \in \mathbb{C} \mid |w| < R\}$ ($0 < R \leq \infty$) onto an open neighborhood of p with $\Psi(0) = p$ such that Ψ is a local isometry (i.e., the pullback $\Psi^*(d\sigma^2)$ is equal to the standard Euclidean metric ds_E^2 on Δ_R),*

and there exists a point a_0 with $|a_0| = 1$, such that the Ψ -image Γ_{a_0} of the line $L_{a_0} = \{w = a_0 t : 0 < t < R\}$ is divergent in M .

Now we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. Assume that the holomorphic map $\Phi : M \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$ omits the hyperplanes H_1, \dots, H_q , which are in general position in $\mathbb{P}^{n-1}(\mathbb{C})$ with $q > \frac{1}{2}n(n+1)$. From Lemma 3.1, it suffices to prove that Φ is constant. By taking the universal cover of M if necessary, one can assume that M is simply connected. It follows from the uniformization theorem that M is conformally equivalent to \mathbb{C} or the unit disk Δ . By Nochka's result [1983] (see also [Chen 1987]) about the Cartan conjecture, we know that Φ is constant when M is conformally equivalent to \mathbb{C} . So the result holds in this case.

Therefore it suffices to consider the case where M is the unit disc Δ . Assume that Φ is not constant. We want to derive a contradiction. From the assumption that Φ is not constant, there exists k ($1 \leq k \leq n-1$) such that the image of Φ is contained in $\mathbb{P}^k(\mathbb{C}) \subset \mathbb{P}^{n-1}(\mathbb{C})$, but not in any subspace whose dimension is lower than k . In other words, $\Phi : \Delta \rightarrow \mathbb{P}^k(\mathbb{C})$ is a nondegenerate map. Let

$$\tilde{\Phi} = (\phi_0, \phi_1, \dots, \phi_k),$$

then, by the assumption, the metric $\|\tilde{\Phi}(z)\|^2 |dz|^2$ is complete. Let $\tilde{H}_j := H_j \cap \mathbb{P}^k(\mathbb{C})$, $1 \leq j \leq q$. Then these hyperplanes are in $(n-1)$ -subgeneral position in $\mathbb{P}^k(\mathbb{C})$. One may assume that \tilde{H}_j is given by

$$\tilde{H}_j : a_{j0}z_0 + a_{j1}z_1 + \dots + a_{jk}z_k = 0 \quad (1 \leq j \leq q).$$

Since $\Phi : \Delta \rightarrow \mathbb{P}^k(\mathbb{C})$ is nondegenerate, from the discussion above, $\tilde{\Phi}_k(z) \neq 0$ and none of the $\tilde{\Phi}_s(\tilde{H}_j)$, $0 \leq s \leq k$, $1 \leq j \leq q$, vanishes identically. Thus, by (9) for each $\tilde{\Phi}_s(\tilde{H}_j)$, there exist i_1, i_2, \dots, i_s such that

$$(10) \quad \psi_{js} := \sum_{t \neq i_1, \dots, i_s} a_{jt} W(\phi_t, \phi_{i_1}, \dots, \phi_{i_s})$$

does not vanish identically. Note that every ψ_{js} is a holomorphic function and has only isolated zeros.

Let $\varpi(j)$ be the Nochka weight associated to the hyperplane $\{\tilde{H}_j\}$ for $1 \leq j \leq q$. By Lemma 3.5, one has

$$q - 2(n-1) + k - 1 = \theta \left(\sum_{j=1}^q \varpi(j) - k - 1 \right),$$

and

$$\theta \leq \frac{2(n-1) - k + 1}{k + 1}.$$

Since $q > \frac{1}{2}n(n+1)$, and noticing that $n(n+1)/2 \geq (k+1)(n-k/2-1) + n$

always holds for any $0 \leq k \leq n-1$, it is easy to verify that

$$\frac{2(q-2(n-1)+k-1)}{k(2(n-1)-k+1)} = \frac{2(q-2n+k+1)}{2kn-k^2-k} > 1,$$

for all $1 \leq k \leq n-1$. Hence

$$\begin{aligned} \frac{2(\sum_{j=1}^q \varpi(j) - k - 1)}{k(k+1)} &= \frac{2(q-2(n-1)+k-1)}{\theta k(k+1)} \\ &\geq \frac{2(q-2(n-1)+k-1)}{k(2(n-1)-k+1)} > 1, \end{aligned}$$

which yields that

$$\sum_{j=1}^q \varpi(j) - k - 1 - \frac{1}{2}k(k+1) > 0.$$

Let

$$\begin{aligned} \chi &:= \sum_{j=1}^q \varpi(j) - (k+1) - \frac{2q}{N}(k^2 + 2k - 1), \\ \lambda_0 &:= \frac{1}{\chi} \left(\frac{1}{2}k(k+1) + \frac{2q}{N} \sum_{s=0}^k s^2 \right). \end{aligned}$$

Choose some N such that

$$\frac{\sum_{j=1}^q \varpi(j) - k - 1 - \frac{k}{2}(k+1)}{\frac{2}{q} + \sum_{s=0}^k (k-s)^2 + k^2 + 2k - 1} < \frac{2q}{N} < \frac{\sum_{j=1}^q \varpi(j) - k - 1 - \frac{k}{2}(k+1)}{\sum_{s=0}^k (k-s)^2 + k^2 + 2k - 1},$$

which implies that

$$0 < \lambda_0 < 1, \quad \frac{4}{N\chi(1-\lambda_0)} > 1.$$

We define a new metric

$$(11) \quad d\sigma^2 = \left(\frac{\prod_{j=1}^q |\tilde{\Phi}(\tilde{H}_j)|^{\varpi(j)}}{|\tilde{\Phi}_k|^{1+\frac{2q}{N}} \prod_{j=1}^q (\prod_{s=0}^{k-1} |\psi_{js}|)^{\frac{4}{N}}} \right)^{\frac{2}{(1-\lambda_0)\chi}} |dz|^2$$

on the set $M' := \Delta \setminus \{p \in \Delta \mid \text{either } \tilde{\Phi}_k = 0 \text{ or } \prod_{j=1}^q \prod_{s=0}^{k-1} |\psi_{js}| = 0\}$.

Notice that $d\sigma^2$ is a flat metric on M' . Fix a point $p_0 \in M'$; by Lemma 3.7 there exists a local diffeomorphism Ψ of a disk $\Delta_R = \{w \in \mathbb{C} : |w| < R\}$ ($0 < R \leq \infty$) onto an open neighborhood of p_0 with $\Psi(0) = p_0$ such that Ψ is a local isometry. Furthermore, there exists a point a_0 with $|a_0| = 1$ and the Ψ -image Γ_{a_0} of the line $L_{a_0} = \{w = a_0 t : 0 < t < R\}$ is divergent in M' . Again, by Nochka's result [1983] (see also [Chen 1987]), we know that $R < \infty$ since Φ is nonconstant. We

claim that the Ψ -image Γ_{a_0} actually is divergent to the boundary of Δ . To this end, we assume the contrary, that the curve Γ_{a_0} is divergent to a point z_0 which either satisfies $\tilde{\Phi}_k(z_0) = 0$ or $\psi_{js}(z_0) = 0$ for some s and j . Thus, we have

$$\liminf_{z \rightarrow z_0} |\tilde{\Phi}_k|^{(N+2q)\delta_0/2} \prod_{1 \leq j \leq q, 1 \leq s \leq k-1} |\psi_{js}|^{2\delta_0} \cdot v > 0,$$

where

$$v = \left(\frac{\prod_{j=1}^q |\tilde{\Phi}(\tilde{H}_j)|^{\varpi(j)}}{|\tilde{\Phi}_k|^{1+\frac{2q}{N}} \prod_{j=1}^q (\prod_{s=0}^{k-1} |\psi_{js}|)^{\frac{4}{N}}} \right)^{\frac{2}{(1-\lambda_0)\chi}}$$

and

$$\delta_0 = \frac{4}{N\chi(1-\lambda_0)} > 1.$$

Thus,

$$\begin{aligned} R &= \int_{L_{a_0}} \Psi^* d\sigma = \int_{\Gamma_{a_0}} d\sigma \\ &= \int_{\Gamma_{a_0}} \left(\frac{\prod_{j=1}^q |\tilde{\Phi}(\tilde{H}_j)|^{\varpi(j)}}{|\tilde{\Phi}_k|^{1+\frac{2q}{N}} \prod_{j=1}^q (\prod_{s=0}^{k-1} |\psi_{js}|)^{\frac{4}{N}}} \right)^{\frac{1}{(1-\lambda_0)\chi}} |dz| \\ &\geq c \int_{\Gamma_{a_0}} \frac{1}{|z-z_0|^{\delta_0}} |dz| = \infty, \end{aligned}$$

which yields a contradiction.

Therefore $\Gamma_{a_0} = \Psi(L_{a_0})$ is divergent to the boundary of Δ . To compute the length of Γ_{a_0} with respect to the Klotz metric $\|\tilde{\Phi}\|^2 |dz|^2$ where $\tilde{\Phi} = (\phi_0, \dots, \phi_k)$, we introduce the following functions defined on $\{w \mid |w| < R\}$:

$$f_s(w) := \phi_s(\Psi(w)) \quad (0 \leq s \leq k),$$

and $F(w) := (f_0(w), f_1(w), \dots, f_k(w))$. For $1 \leq j \leq q$, $0 \leq s \leq k$, we define

$$F(H_j) := a_{j0} f_0 + \dots + a_{jk} f_k, \quad F_k := W(f_0, f_1, \dots, f_k)$$

and

$$\varphi_{js} := \sum_{t \neq i_1, \dots, i_s} a_{jt} W(f_t, f_{i_1}, \dots, f_{i_s}),$$

where (i_1, \dots, i_s) is the index in the definition of ψ_{js} in (10). Noticing the fact that, for $0 \leq s \leq k$,

$$F_s(w) = (F \wedge F' \wedge \dots \wedge F^{(s)})(w) = (\tilde{\Phi} \wedge \dots \wedge \tilde{\Phi}^{(s)})(z) \left(\frac{dz}{dw} \right)^{s(s+1)/2},$$

we have, from (11), that

$$\begin{aligned}
 \Psi^* d\sigma &= \Psi^* \left(\frac{\prod_{j=1}^q |\tilde{\Phi}(\tilde{H}_j)|^{\varpi(j)}}{|\tilde{\Phi}_k|^{1+\frac{2q}{N}} \prod_{j=1}^q (\prod_{s=0}^{k-1} |\psi_{js}|)^{\frac{4}{N}}} \right)^{\frac{1}{(1-\lambda_0)\chi}} \left| \frac{dz}{dw} \right| |dw| \\
 &= \left(\frac{\prod_{j=1}^q |F(\tilde{H}_j)|^{\varpi(j)}}{|F_k|^{1+\frac{2q}{N}} \prod_{j=1}^q (\prod_{s=0}^{k-1} |\varphi_{js}|)^{\frac{4}{N}}} \right)^{\frac{1}{(1-\lambda_0)\chi}} \\
 &\quad \times \left| \frac{dz}{dw} \right|^{\frac{(1+\frac{2q}{N})\frac{k(k+1)}{2} + \frac{2q}{N} \sum_{s=0}^{k-1} s(s+1)}{(1-\lambda_0)\chi} + 1} |dw| \\
 &= \left(\frac{\prod_{j=1}^q |F(\tilde{H}_j)|^{\varpi(j)}}{|F_k|^{1+\frac{2q}{N}} \prod_{j=1}^q (\prod_{s=0}^{k-1} |\varphi_{js}|)^{\frac{4}{N}}} \right)^{\frac{1}{(1-\lambda_0)\chi}} \left| \frac{dz}{dw} \right|^{\frac{\lambda_0}{1-\lambda_0} + 1} |dw| \\
 &= \left(\frac{\prod_{j=1}^q |F(\tilde{H}_j)|^{\varpi(j)}}{|F_k|^{1+\frac{2q}{N}} \prod_{j=1}^q (\prod_{s=0}^{k-1} |\varphi_{js}|)^{\frac{4}{N}}} \right)^{\frac{1}{(1-\lambda_0)\chi}} \left| \frac{dz}{dw} \right|^{\frac{1}{1-\lambda_0}} |dw|.
 \end{aligned}$$

Using the isometry property of Ψ , i.e., $|dw| = \Psi^* d\sigma$, we get

$$(12) \quad \left| \frac{dw}{dz} \right| = \left(\frac{\prod_{j=1}^q |F(\tilde{H}_j)|^{\varpi(j)}}{|F_k|^{1+\frac{2q}{N}} \prod_{j=1}^q (\prod_{s=0}^{k-1} |\varphi_{js}|)^{\frac{4}{N}}} \right)^{\frac{1}{\chi}}.$$

Now, denote by $l(\Gamma_{a_0})$ the length of the curve Γ_{a_0} with respect to the Klotz metric $\|\tilde{\Phi}\|^2 |dz|^2$; then from (12),

$$\begin{aligned}
 l(\Gamma_{a_0}) &= \int_{\Gamma_{a_0}} \|\tilde{\Phi}\| |dz| = \int_{L_{a_0}} \Psi^*(\|\tilde{\Phi}\| |dz|) \\
 &= \int_{L_{a_0}} \|\tilde{\Phi}(\Psi(w))\| \left| \frac{dz}{dw} \right| |dw| \\
 &= \int_{L_{a_0}} \|F\| \left(\frac{|F_k|^{1+\frac{2q}{N}} \prod_{j=1}^q (\prod_{s=0}^{k-1} |\varphi_{js}|)^{\frac{4}{N}}}{\prod_{j=1}^q |F(\tilde{H}_j)|^{\varpi(j)}} \right)^{\frac{1}{\chi}} |dw| \\
 &\leq \int_{L_{a_0}} \left(\frac{\|F\|^\chi |F_k|^{1+\frac{2q}{N}} \prod_{j=1}^q (\prod_{s=0}^{k-1} |F_s(\tilde{H}_j)|)^{\frac{4}{N}}}{\prod_{j=1}^q |F(\tilde{H}_j)|^{\varpi(j)}} \right)^{\frac{1}{\chi}} |dw|.
 \end{aligned}$$

In the above inequality, we use the fact that $|\varphi_{js}| \leq |F_s(\tilde{H}_j)|$ for all $0 \leq s \leq k$, $1 \leq j \leq q$. By Lemma 3.6, it can be concluded that, noticing that $0 < \lambda_0 < 1$,

$$l(\Gamma_{a_0}) \leq C \int_0^R \left(\frac{2R}{R^2 - |w|^2} \right)^{\lambda_0} |dw| < \infty,$$

which contradicts the completeness of the Klotz metric $\|\tilde{\Phi}\|^2 |dz|^2$. Thus Φ is a constant map. □

4. Value distribution properties for the Gauss map of the harmonic and QC-harmonic surface immersed in \mathbb{R}^3

In this section, we study the value distribution properties for the Gauss map (i.e., its unit normal) of harmonic and K -quasiconformal surfaces immersed in \mathbb{R}^3 . Our method is to compare the Gauss map of the surface (i.e., its unit-normal) with the generalized Gauss map Φ and apply the results obtained in the previous section. The classical Bernstein theorem says that a minimal graph over a plane is planar. W. H. Meeks III and H. Rosenberg [2005] showed that a complete embedded minimal surface over a plane is either a plane or a helicoid. It is known that the classical Bernstein theorem still holds for the K -quasiconformal harmonic graph, but fails if we only assume the graph is harmonic. So in order to get the desired value distribution properties for the Gauss map of harmonic surfaces immersed in \mathbb{R}^3 , the additional condition that X be K -quasiconformal seems necessary.

Let M be an open Riemann surface and

$$X = (X^1, X^2, X^3) : M \rightarrow \mathbb{R}^3$$

be a harmonic immersion. Write

$$(13) \quad \|\nabla X\|^2 = E + G = 4\|\phi\|^2,$$

where $\|\nabla X\|^2$ is the Hilbert–Schmidt norm defined by

$$\|\nabla X\|^2 := \|\partial X / \partial u\|^2 + \|\partial X / \partial v\|^2.$$

Also the Jacobian of X is given by

$$(14) \quad J_X = \|X_u \times X_v\| = \sqrt{EG - F^2} = 2\sqrt{\|\phi\|^4 - |h|^2},$$

where ϕ is given by (1), and h and $\|\phi\|$ are defined by (4).

Let \mathbf{n} be a normal vector on a harmonic surface $X : M \rightarrow \mathbb{R}^3$, and \mathbf{b} be a unit vector in \mathbb{R}^3 . The following proposition aims to give a relation between the intersection of \mathbf{n} and \mathbf{b} and the projective distance of the generalized Gauss map Φ to a hyperplane with the normal \mathbf{b} . Instead of the proof which is originally from Lemma 1.1 in [Osserman 1964] we use the mixed product to deal with it.

Proposition 4.1. *Let $X : M \rightarrow \mathbb{R}^3$ be a harmonic surface. Then every normal vector makes an angle α with the unit vector \mathbf{b} at a given point if and only if*

$$(15) \quad \frac{|\mathbf{b} \cdot \phi|^2}{\|\phi\|^2} \frac{\|\phi\|^4}{\|\phi\|^4 - |h|^2} - \frac{(\mathbf{b} \cdot \phi)^2 \bar{h} + (\mathbf{b} \cdot \bar{\phi})^2 h}{2\|\phi\|^2} \frac{\|\phi\|^2}{\|\phi\|^4 - |h|^2} = \frac{1}{2} \sin^2 \alpha,$$

where ϕ is given by (1), and h and $\|\phi\|$ are defined by (4).

Proof. The unit normal vector is given by $\mathbf{n} = (X_u \times X_v)/J_X$. For a unit vector \mathbf{b} , we denote by $|A|$ the mixed product of the three vectors \mathbf{b} , X_u and X_v , where A is a matrix determined by three row vectors \mathbf{b} , X_u and X_v . Then it follows from determinant expansion of $|AA^T|$ in its first row that

$$\begin{aligned} |\mathbf{n} \cdot \mathbf{b}|^2 &= \frac{|X_u \times X_v \cdot \mathbf{b}|^2}{EG - F^2} = \frac{|AA^T|}{EG - F^2} \\ &= 1 - \frac{(\mathbf{b} \cdot X_u)^2 G + (\mathbf{b} \cdot X_v)^2 E - 2(\mathbf{b} \cdot X_u)(\mathbf{b} \cdot X_v)F}{EG - F^2}. \end{aligned}$$

Utilizing the relations

$$X_u = \phi + \bar{\phi} \quad \text{and} \quad X_v = \sqrt{-1}(\phi - \bar{\phi}),$$

we have, from the above relation, that

$$|\mathbf{n} \cdot \mathbf{b}|^2 = 1 - \frac{(\mathbf{b} \cdot \phi)^2 (G - E - 2iF) + (\mathbf{b} \cdot \bar{\phi})^2 (G - E + 2iF)}{EG - F^2} - \frac{2|\mathbf{b} \cdot \phi|^2 (E + G)}{EG - F^2}.$$

By the relations (4), (13) and (14), the above equality is equivalent to

$$(16) \quad |\mathbf{n} \cdot \mathbf{b}|^2 = 1 + \frac{(\mathbf{b} \cdot \phi)^2 \bar{h} + (\mathbf{b} \cdot \bar{\phi})^2 h}{\|\phi\|^2} \frac{\|\phi\|^2}{\|\phi\|^4 - |h|^2} - \frac{2|\mathbf{b} \cdot \phi|^2}{\|\phi\|^2} \frac{\|\phi\|^4}{\|\phi\|^4 - |h|^2}.$$

Since \mathbf{n} and \mathbf{b} are two unit vectors, $|\mathbf{n} \cdot \mathbf{b}| = |\cos \alpha|$. Hence, the normal vector \mathbf{n} makes an angle α with the unit vector \mathbf{b} if and only if the equality (15) holds. \square

Remark 4.2. If $X : M \rightarrow \mathbb{R}^3$ is a minimal surface with a conformal minimal immersion X , then we have $h \equiv 0$. In this case, Proposition 4.1 implies that

$$\frac{|\phi \cdot \mathbf{b}|^2}{\|\phi\|^2} = \frac{1}{2} \sin^2 \alpha.$$

This relation shows that in the case of minimal surfaces, a normal vector \mathbf{n} makes an angle of at least α with a given vector \mathbf{b} if and only if its generalized Gauss map Φ has a positive projective distance to a hyperplane H with the unit normal \mathbf{b} . If we take \mathbf{b} to be the x^3 -axis, then our proposition is Lemma 1.1 in [Osserman 1964] in the \mathbb{R}^3 case.

Remark 4.3. By the fact that $|h| < \|\phi\|^2$ for a harmonic surface, the relation (15) gives us the following inequality

$$\frac{1 - |\mathbf{n} \cdot \mathbf{b}|^2}{2} = \frac{1}{2} \sin^2 \alpha \geq \frac{|\mathbf{b} \cdot \phi|^2}{\|\phi\|^2} \frac{\|\phi\|^2}{\|\phi\|^2 + |h|} \geq \frac{1}{2} \frac{|\mathbf{b} \cdot \phi|^2}{\|\phi\|^2},$$

which shows that the inequality $|\mathbf{b} \cdot \phi|^2 / \|\phi\|^2 \geq \epsilon > 0$ implies that $|\mathbf{n} \cdot \mathbf{b}| \leq \eta < 1$, that is, it will force normals to avoid some neighborhood of the unoriented direction determined by the vector \mathbf{b} . Conversely, it is also true that

$$\frac{|\mathbf{b} \cdot \phi|^2}{\|\phi\|^2} \geq \frac{1 - |\mathbf{n} \cdot \mathbf{b}|^2}{2} \frac{\|\phi\|^2 - |h|}{\|\phi\|^2}.$$

However, the result that $|\mathbf{b} \cdot \phi|^2 / \|\phi\|^2$ has a positive lower bound cannot be derived from this inverse inequality under the condition that $|\mathbf{n} \cdot \mathbf{b}| \leq \eta < 1$, that is, the condition that normals avoid some neighborhood of the unoriented direction determined by a vector \mathbf{b} does not imply that $|\mathbf{b} \cdot \phi|^2 / \|\phi\|^2 \geq \epsilon > 0$. The harmonic rotational horn with the generalized Gauss map $\Phi = [dz : \sqrt{-1}dz : 1/zdz]$, which is given by A. Alarcón and F. J. López [2013], is such a counterexample. To verify it, one can choose the unit vector $\mathbf{b} = 1/\sqrt{u^2 + v^2}(v, u, 0)$, then $\mathbf{n} \cdot \mathbf{b} = 0$ and $|\mathbf{b} \cdot \phi|^2 / \|\phi\|^2 \rightarrow 0$ as $|z| \rightarrow 0$.

From Remark 4.3, we see that Theorem 3.2 would not tell any useful information about the unit-normal \mathbf{n} . Indeed the Bernstein type theorem fails for some harmonic immersed surfaces. In order to derive some useful consequence from the previous results about Φ , we need to derive the lower bound for $|\mathbf{b} \cdot \phi|^2 / \|\phi\|^2$ when $|\mathbf{n} \cdot \mathbf{b}| \leq \eta < 1$. This requires the assumption that X is K -quasiconformal.

An immersion $X = (X^1, X^2, X^3) : M \rightarrow \mathbb{R}^3$ is called K -quasiconformal if it satisfies the inequality

$$(17) \quad \|\nabla X\|^2 \leq \left(K + \frac{1}{K}\right) J_X,$$

which is equivalent to

$$(18) \quad \|\phi\|^2 \leq \frac{K^2 + 1}{2K} \sqrt{\|\phi\|^4 - |h|^2}.$$

Note that we adopt the definition of quasiconformality given by D. Kalaj [2013] (see also [Alarcón and López 2013]). If $K = 1$, then the above inequality can be changed into the two relations

$$(19) \quad |X_u| = |X_v| \quad \text{and} \quad X_u \cdot X_v = 0,$$

where we say that X is an isothermal parametrization (isothermal coordinate) of the surface M . If a Riemann surface M admits a K -quasiconformal harmonic

immersion X into \mathbb{R}^3 , we call such a surface a K -quasiconformal harmonic surface, and say its immersion is a K -quasiconformal harmonic immersion.

Lemma 4.4. *Let M be a surface with a harmonic immersion X . Then X is K -quasiconformal in the sense of the definition given by (17) if and only if its Hopf differential η and conformal metric $\|\Phi\|$ satisfy*

$$(20) \quad |\eta| \leq Q_K \|\Phi\|^2,$$

where $Q_K = (K^2 - 1)/(K^2 + 1)$. Particularly, if $K = 1$ then X is a conformal immersion. Furthermore, the metric ds^2 and its associated conformal metric $\|\Phi\|^2$ satisfy the inequality

$$(21) \quad 2(1 - Q_K)\|\Phi\|^2 \leq ds^2 \leq 2(1 + Q_K)\|\Phi\|^2.$$

The inequalities of the above relations hold simultaneously if and only if X is a conformal immersion.

Proof. The inequality (18) implies that

$$(22) \quad |h| \leq Q_K \|\phi\|^2,$$

i.e., $|\eta| \leq Q_K \|\Phi\|^2$ since $\eta := h dz^2$. Conversely, the inequality (20) (i.e., inequality (22)) implies that

$$\|\nabla X\|^2 \leq \left(K + \frac{1}{K}\right) J_X.$$

Therefore, X is K -QC in the sense of definition given by D. Kalaj. Finally, it is easy to see that the inequality (20) (i.e., inequality (22)) implies both the upper and lower bound of (21) from the first fundamental form (3). \square

Under the assumption that X is K -quasiconformal, we give an estimate of the quantity

$$\frac{\frac{1}{2}(1 - |\mathbf{n} \cdot \mathbf{b}|^2)}{|\phi \cdot \mathbf{b}|^2 / \|\phi\|^2}$$

in the quasiconformal distortion constant K for a K -quasiconformal harmonic surface in \mathbb{R}^3 .

Lemma 4.5. *Let M be a K -quasiconformal harmonic surface in \mathbb{R}^3 . Then for every unit normal vector \mathbf{n} and every unit vector \mathbf{b} at the same point p on M ,*

$$(23) \quad \frac{K^2 + 1}{2K^2} \frac{|\phi \cdot \mathbf{b}|^2}{\|\phi\|^2} \leq \frac{1 - |\mathbf{n} \cdot \mathbf{b}|^2}{2} \leq \frac{K^2 + 1}{2} \frac{|\phi \cdot \mathbf{b}|^2}{\|\phi\|^2}.$$

In particular, when $K = 1$ we have

$$(24) \quad \frac{1 - |\mathbf{n} \cdot \mathbf{b}|^2}{2} = \frac{|\phi \cdot \mathbf{b}|^2}{\|\phi\|^2}.$$

Proof. From the relations (16), (18) and (22), we have

$$\begin{aligned} \frac{1 - |\mathbf{n} \cdot \mathbf{b}|^2}{2} &= \frac{|\mathbf{b} \cdot \phi|^2}{\|\phi\|^2} \frac{\|\phi\|^4}{\|\phi\|^4 - |h|^2} - \frac{(\mathbf{b} \cdot \phi)^2 \bar{h} + (\mathbf{b} \cdot \bar{\phi})^2 h}{2\|\phi\|^2} \frac{\|\phi\|^2}{\|\phi\|^4 - |h|^2} \\ &\leq \left(\frac{\|\phi\|^4}{\|\phi\|^4 - |h|^2} + \frac{\|\phi\|^2 |h|}{\|\phi\|^4 - |h|^2} \right) \frac{|\mathbf{b} \cdot \phi|^2}{\|\phi\|^2} \\ &\leq (1 + Q_K) \frac{\|\phi\|^4}{\|\phi\|^4 - |h|^2} \frac{|\mathbf{b} \cdot \phi|^2}{\|\phi\|^2} \\ &\leq \frac{(1 + Q_K)(K + 1/K)^2}{4} \frac{|\mathbf{b} \cdot \phi|^2}{\|\phi\|^2} = \frac{K^2 + 1}{2} \frac{|\mathbf{b} \cdot \phi|^2}{\|\phi\|^2}. \end{aligned}$$

Hence we complete the proof of the right-hand side of the relation (23).

Similarly, we have

$$\begin{aligned} \frac{1 - |\mathbf{n} \cdot \mathbf{b}|^2}{2} &\geq \left(\frac{\|\phi\|^4}{\|\phi\|^4 - |h|^2} - \frac{\|\phi\|^2 |h|}{\|\phi\|^4 - |h|^2} \right) \frac{|\mathbf{b} \cdot \phi|^2}{\|\phi\|^2} \\ &\geq \left(\frac{\|\phi\|^2}{\|\phi\|^2 + |h|} \right) \frac{|\mathbf{b} \cdot \phi|^2}{\|\phi\|^2} \geq \frac{K^2 + 1}{2K^2} \frac{|\mathbf{b} \cdot \phi|^2}{\|\phi\|^2}. \end{aligned}$$

Thus the proof of Lemma 4.5 is finished. □

Combining Lemma 4.5 with Theorem 3.2, we get the following theorem.

Theorem 4.6. *Let $X : M \rightarrow \mathbb{R}^3$ be a complete harmonic and K -quasiconformal immersion with the induced metric, where M is an open Riemann surface, and let \mathbf{n} be the unit normal of M . If its Gauss map (i.e., the normal \mathbf{n}) omits a neighborhood of a direction in S^2 , then X must be a plane.*

The above theorem recovers the classical Bernstein theorem for graphs of harmonic and K -quasiconformal surfaces defined on \mathbb{R}^2 . It is known that the Bernstein theorem fails for graphs of harmonic surfaces on \mathbb{R}^2 without the K -quasiconformal assumption. Thus the additional assumption of K -quasiconformality seems necessary and reasonable in order to study the value distribution for its Gauss map.

From Lemma 4.5, we see that $|\mathbf{n} \cdot \mathbf{b}| = 1$ if and only if $\phi \cdot \mathbf{b} = 0$. For any unit-vector $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$, it corresponds to a hyperplane

$$H_{\mathbf{b}} = \{b_1 w_1 + b_2 w_2 + b_3 w_3\} \subset \mathbb{P}^2(\mathbb{C}).$$

Notice that $H_{\mathbf{b}_j}, 1 \leq j \leq q$, are in general position if and only if any three of among $\mathbf{b}_1, \dots, \mathbf{b}_q$ are not contained in a plane in \mathbb{R}^3 . Thus Lemma 4.5 and Theorem 3.3 imply the following interesting result about the value distribution of the Gauss map \mathbf{n} which extends Theorem 4.6.

Theorem 4.7. *Let $X : M \rightarrow \mathbb{R}^3$ be a complete harmonic and K -quasiconformal immersion with the induced metric, where M is an open Riemann surface, and let \mathbf{n}*

be the unit normal of M . If its Gauss map (i.e., the normal \mathbf{n}) omits seven directions, say unit vectors $\mathbf{b}_1, \dots, \mathbf{b}_7$ in S^2 , and any three of which are not contained in a plane in \mathbb{R}^3 , then X must be a plane.

5. Estimate of Gauss curvature for K -QC harmonic surfaces

In this section, for a K -quasiconformal harmonic immersion $X : M \rightarrow \mathbb{R}^3$, we will study its Gauss curvature estimate if the Gauss map (not the generalized Gauss map) omits a neighborhood of some direction. For the result in the minimal surface case, see [Osserman and Ru 1997].

We first derive the expression for the Gauss curvature \mathfrak{K} of a harmonic immersed surface in terms of its generalized Gauss map Φ . Let $X = (X^1, X^2, X^3)$ be its harmonic immersion. Take a local coordinate $z = u + \sqrt{-1}v$. It follows from $X_u = \phi + \bar{\phi}$ and $X_v = \sqrt{-1}(\phi - \bar{\phi})$ that

$$X_{uu} = \phi' + \bar{\phi}', \quad X_{uv} = \sqrt{-1}(\phi' - \bar{\phi}'), \quad X_{vv} = -(\phi' + \bar{\phi}')$$

and

$$(25) \quad \mathbf{n} = \frac{X_u \times X_v}{\sqrt{EG - F^2}} = \frac{\sqrt{-1}(\bar{\phi} \times \phi)}{\sqrt{\|\phi\|^4 - |h|^2}}$$

which imply that the Gauss curvature can be expressed by

$$\mathfrak{K} = \frac{LN - M^2}{EG - F^2} = -4 \frac{|X_u \times X_v \cdot \bar{\phi}'|^2}{(EG - F^2)^2} = -\frac{|\bar{\phi} \times \phi \cdot \bar{\phi}'|^2}{(\|\phi\|^4 - |h|^2)^2}.$$

Furthermore, it can be rewritten by an expansion of the determinant as

$$(26) \quad \mathfrak{K} = -\frac{4}{(\sqrt{EG - F^2})^3} \left\{ \sqrt{EG - F^2} \|\phi'\|^2 + \frac{4\bar{h}(\phi' \cdot \phi)(\bar{\phi}' \cdot \phi)}{\sqrt{EG - F^2}} + \frac{4h(\phi' \cdot \bar{\phi})(\bar{\phi}' \cdot \bar{\phi})}{\sqrt{EG - F^2}} - \frac{4\|\phi\|^2[(\phi' \cdot \bar{\phi})(\bar{\phi}' \cdot \phi) + (\phi' \cdot \phi)(\bar{\phi}' \cdot \bar{\phi})]}{\sqrt{EG - F^2}} \right\}.$$

In particular, if X is a conformal harmonic immersion, that is, $h = 0$, the above relation can be simplified as

$$(27) \quad \mathfrak{K} = -\frac{\Delta \log \lambda}{\lambda^2} = -\frac{\|\phi'\|^2 \|\phi\|^2 - (\phi' \cdot \bar{\phi})(\bar{\phi}' \cdot \phi)}{(\|\phi\|^2)^3} = -\frac{\sum_{i < j \leq 3} |\phi_i \phi'_j - \phi'_i \phi_j|^2}{\|\phi\|^6},$$

where $\lambda = \sqrt{2}\|\phi\|$.

It will be useful to introduce the meromorphic functions $\psi_k(z)$ defined by

$$(28) \quad \psi_k(z) = \frac{\phi_k(z)}{\phi_3(z)}, \quad k = 1, 2, 3.$$

Then

$$\psi'_k = \frac{\phi_3\phi'_k - \phi'_3\phi_k}{\phi_3^2}$$

and

$$\psi_j\psi'_k - \psi'_j\psi_k = \frac{\phi_j\phi'_k - \phi'_j\phi_k}{\phi_3^2}, \quad j, k = 1, 2, 3.$$

Hence, we can rewrite (27) as

$$(29) \quad \mathfrak{K} = -\frac{|\psi_1\psi'_2 - \psi'_1\psi_2|^2 + \sum_{j=1}^2 |\psi'_j|^2}{|\phi_3|^2(1 + \sum_{j=1}^2 |\psi_j|^2)^3}.$$

Lemma 5.1. *Let M be a K -quasiconformal harmonic surface from the unit disk $\{z \mid |z| < 1\}$ into R^3 whose unit normal makes an angle of at least $\alpha > 0$ with the x^3 -axis at every point of the surface. If p is the point of M corresponding to $z = 0$, then the distance d from p to the boundary of M satisfies*

$$(30) \quad d \leq \frac{2K}{\sqrt{K^2 + 1}} |\csc \alpha| |\phi_3(0)|.$$

Proof. After choosing \mathfrak{b} to be a x^3 -axis, we have $|\phi \cdot \mathfrak{b}| = |\phi_3|$. If a normal \mathfrak{n} makes at least an angle α with the x^3 -axis, then $|\mathfrak{n} \cdot \mathfrak{b}| \leq |\cos \alpha|$. Thus the right-hand side of relation (23) becomes

$$(31) \quad \frac{|\phi_3|^2}{\|\phi\|^2} \geq \frac{1}{K^2 + 1} \sin^2 \alpha.$$

Let c be an arbitrary curve going from $z = 0$ to the boundary $|z| = 1$. Thus by (21) and (31), the length of the image γ on M of the curve c is given by

$$\begin{aligned} d_\gamma &= \int_\gamma ds \leq \sqrt{2(1 + Q_K)} \int_c \|\phi\| |dz| \\ &\leq \frac{2K}{\sqrt{K^2 + 1}} |\csc \alpha| \int_c |\phi_3| |dz|. \end{aligned}$$

Set $F(z) = \int_0^z \phi_3(\zeta) d\zeta$. The inequality (31) shows that $F'(z) \neq 0$. Then $\zeta = F(z)$ will have an inverse $z = G(\zeta)$ in some disk with center $\zeta = 0$. Let R be the largest radius such that G is holomorphic, then by Liouville's theorem we have that R is finite. Hence, there exists a point ζ_0 on the circle $|\zeta| = R$ such that G cannot be extended

to a neighborhood of ζ_0 . Let L be the line segment from $\zeta = 0$ to ζ_0 and $\Gamma = G(L)$. Thus $G(R\zeta)$ satisfies Schwarz's lemma, which shows $|G'(0)| \leq 1/R$. Thus,

$$\begin{aligned} d &= \inf_c \int_c ds \leq \frac{2K}{\sqrt{K^2 + 1}} |\csc \alpha| \int_c |\phi_3| |dz| \\ &= \frac{2K}{\sqrt{K^2 + 1}} |\csc \alpha| \int_L |d\zeta| = \frac{2K}{\sqrt{K^2 + 1}} |\csc \alpha| R \leq \frac{2K |\csc \alpha|}{\sqrt{K^2 + 1} |G'(0)|} \\ &= \frac{2K}{\sqrt{K^2 + 1}} |\csc \alpha| |F'(0)| = \frac{2K}{\sqrt{K^2 + 1}} |\csc \alpha| |\phi_3(0)|. \quad \square \end{aligned}$$

Theorem 5.2. *Let M be a K -quasiconformal harmonic surface in \mathbb{R}^3 . Suppose that its unit normal makes an angle of at least $\alpha > 0$ with some fixed direction at every point of M . In addition, we assume that $|(\phi' \cdot \phi)(\overline{\phi'} \cdot \phi)| / \|\phi\|^4 \leq N_K$, where N_K is a constant. If $d(p)$ is the distance of p to the boundary of M , then the Gauss curvature $\mathfrak{K}(p)$ of M at p satisfies the inequality*

$$(32) \quad |\mathfrak{K}(p)| d(p)^2 \leq \frac{4K^2 \csc^2 \alpha}{(K^2 + 1)(1 - Q_K^2)^{3/2}} \left[2(K^2 + 1) \csc^2 \alpha - 2 + \frac{(K^2 - 1)N_K}{K} \right].$$

Proof. After a rotation we assume that the normals make an angle of at least α with the x^3 -axis. Let \tilde{M} be the universal covering surface of M under a universal covering transformation $z(\zeta)$. Suppose that the point $\zeta = 0$ in \tilde{M} corresponds to p in M . For a conformal metric ds_z on M , we have $ds_z = \rho(z)|dz| = \rho(z(\zeta))|dz/d\zeta||d\zeta|$, which implies that

$$\mathfrak{K}_{\tilde{M}}(\zeta) = -\frac{\Delta_\zeta \log(\rho(z(\zeta))|dz/d\zeta|)}{(\rho(z(\zeta))|dz/d\zeta|)^2} = -\frac{\Delta_z \log \rho}{\rho^2} \circ z(\zeta) = \mathfrak{K}_M(z(\zeta)).$$

If we form the functions $\psi_k(\zeta)$ by (28), then from the relation (31), we have

$$(33) \quad \sum_{k=1}^2 |\psi_k(\zeta)|^2 \leq (K^2 + 1) \csc^2 \alpha - 1.$$

Thus we have bounded holomorphic functions ψ_k , $k = 1, 2$, on \tilde{M} . If \tilde{M} is the entire ζ -plane, Liouville's theorem says that ψ_k , $k = 1, 2$ are constants. Then the relation (29) shows that the Gauss curvature $\mathfrak{K} \equiv 0$. Thus, the relation (32) holds automatically. Now we need to consider the case that \tilde{M} is the unit disk $|\zeta| < 1$. For convenience, we adopt the following notation:

$$(34) \quad C_k = |\psi_k(0)|, \quad D_k = |\psi'_k(0)|, \quad M_k = \sup_{|w| < 1} |\psi_k(w)|.$$

From the application of the Schwarz-pick lemma to the function ψ_k/M_k , we obtain

$$(35) \quad D_k \leq M_k \left(1 - \frac{C_k^2}{M_k^2} \right) = M_k \eta_k,$$

where

$$(36) \quad \eta_k = 1 - \frac{C_k^2}{M_k^2}.$$

Utilizing the relation (29) at the point $w = 0$ in \tilde{M} , we next give an estimate of Gauss curvature \mathfrak{K} at a given point $p \in M$. By (33), we get

$$(37) \quad C_k^2 \leq M_k^2 \leq (K^2 + 1) \csc^2 \alpha - 1,$$

and

$$(38) \quad D_k^2 \leq ((K^2 + 1) \csc^2 \alpha - 1) \eta_k^2.$$

By the Cauchy–Schwarz inequality, we have

$$|\psi_1(0)\psi_2'(0) - \psi_1'(0)\psi_2(0)|^2 = (C_1D_2 + C_2D_1)^2 \leq \sum_{j=1}^2 C_j^2 \sum_{k=1}^2 D_k^2.$$

Thus, the combination of (37)–(38) and the above inequality yields

$$\begin{aligned} |\psi_1(0)\psi_2'(0) - \psi_2'(0)\psi_1(0)|^2 + \sum_{j=1}^2 |\psi_j'(0)|^2 &\leq \sum_{k=1}^2 D_k^2 \left(1 + \sum_{j=1}^2 C_j^2\right) \\ &\leq ((K^2 + 1) \csc^2 \alpha - 1) \left(1 + \sum_{j=1}^2 C_j^2\right) \sum_{j=1}^2 \eta_j^2. \end{aligned}$$

Hence,

$$\frac{\|\phi\|^2 \|\phi'\|^2 - (\phi' \cdot \bar{\phi})(\bar{\phi}' \cdot \phi)}{\|\phi\|^6} \leq \frac{((K^2 + 1) \csc^2 \alpha - 1) \sum_{j=1}^2 \eta_j^2}{|\phi_3(0)|^2 (1 + \sum_{j=1}^2 C_j^2)^2}.$$

Furthermore, it follows from (26) and Lemma 5.1 that

$$\begin{aligned} |\mathfrak{K}(0)| &\leq \frac{1}{(\sqrt{1-Q_K^2})^3} \left[\frac{\|\phi\|^2 \|\phi'\|^2 - (\phi' \cdot \bar{\phi})(\bar{\phi}' \cdot \phi)}{\|\phi\|^6} + \frac{K^2 - 1}{K} \frac{|(\phi' \cdot \phi)(\bar{\phi}' \cdot \phi)|}{\|\phi\|^6} \right] \\ &\leq \frac{1}{(\sqrt{1-Q_K^2})^3} \left[\frac{2(K^2 + 1) \csc^2 \alpha - 2}{|\phi_3(0)|^2} + \frac{(K^2 - 1)}{K} \frac{|(\phi' \cdot \phi)(\bar{\phi}' \cdot \phi)|}{\|\phi\|^6} \right] \\ &\leq \frac{1}{(\sqrt{1-Q_K^2})^3 |\phi_3(0)|^2} \left[2(K^2 + 1) \csc^2 \alpha - 2 + \frac{(K^2 - 1)N_K}{K} \right] \\ &\leq \frac{4K^2 \csc^2 \alpha}{(K^2 + 1)(\sqrt{1-Q_K^2})^3 d^2(0)} \left[2(K^2 + 1) \csc^2 \alpha - 2 + \frac{(K^2 - 1)N_K}{K} \right]. \quad \square \end{aligned}$$

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
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