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Semiclassical approximation of the magnetic Schrödinger operator on a strip: dynamics and spectrum

Mouez Dimassi

In the semiclassical regime (i.e., $\epsilon \searrow 0$), we study the effect of a slowly varying potential $V(\epsilon t, \epsilon z)$ on the magnetic Schrödinger operator $P = D_x^2 + (D_z + \mu x)^2$ on a strip $[-a, a] \times \mathbb{R}_z$. The potential $V(t, z)$ is assumed to be smooth. We derive the semiclassical dynamics and we describe the asymptotic structure of the spectrum and the resonances of the operator $P + V(\epsilon t, \epsilon z)$ for ϵ small enough. All our results depend on the eigenvalues corresponding to $D_x^2 + (\mu x + k)^2$ on $L^2([-a, a])$ with Dirichlet boundary condition.

1. Introduction

The quantum dynamics of an electron in a strip subject to an uniform magnetic field and an external slowly varying potential is governed by the Schrödinger operator

$$H(\epsilon) := P + V(\epsilon t, \epsilon z) = D_x^2 + (D_z + \mu x)^2 + V(\epsilon t, \epsilon z), \qquad D_\nu = \frac{1}{i}\partial_\nu, \quad \epsilon, \mu > 0,$$

where μ is proportional to the strength of the magnetic field and ϵ is a small parameter. The potential V is assumed to be smooth and real valued.

The operator

$$P = D_x^2 + (D_z + \mu x)^2,$$

is defined on $\{u \in H^2(C_a);\ u|_{\partial C_a} = 0\}$, where $H^2(C_a)$ denotes the second order Sobolev space on a strip $C_a := \{(x, z) \in \mathbb{R}^2;\ -a \le x \le a\}$. The Fourier transformation with respect to z reduces the spectral problem of P to an analysis of the (k depending) eigenvalues $E_0(k), E_1(k), \ldots$ of the Sturm-Liouville operator

$$P(k) = -\partial_x^2 + (k + \mu x)^2,$$

on the interval $[-a, a]$ with Dirichlet boundary condition at $-a$ and a.

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In this paper, we are interested in the asymptotic solutions of the time-dependent Schrödinger equation

$$D_t u = H(\epsilon)u, \quad u|_{t=0} = u_\epsilon(x, z), \tag{1-1}$$

as $\epsilon \searrow 0$. In particular we derive the semiclassical dynamics and we describe the asymptotic structure of the spectrum and the resonances of the operator $H(\epsilon)$ for ϵ small enough.

The hydrogen atom in a homogeneous magnetic field is a model of quantum chaos. See for example [Viehweger et al. 1990]. The spectral properties of $H(\epsilon)$ on $\mathbb{R}^2$ have been intensively studied in the last twenty years. In the case of perturbations, the Landau levels $\lambda_n(\mu) = \mu(2n+1)$ become accumulation points of the eigenvalues of $H(\epsilon)$ and the asymptotics of the function counting the number of the eigenvalues lying in a neighborhood of $\lambda_n(\mu)$ have been examined by many authors in different aspects. For recent results, the reader may consult [Gérard and Łaba 2002; Ivrii 2018; Fournais and Helffer 2010].

The spectrum of P on a bounded domain $\Omega \subset \mathbb{R}^2$ were considered by many others. In particular the asymptotic behavior of the bottom of the spectrum of P as μ tends to infinity has been treated for different geometry of Ω (see [Fournais and Helffer 2010]). In the case where Ω is the semiinfinite plane or the disk, the WKB approximations of the energies and the eigenfunctions are obtained in [Spehner et al. 1998; Bonnaillie-Noël et al. 2016].

S. De Bièvre and J. F. Pulé [1999] studied the perturbed operator $H(1)$ on the half plane with Dirichlet boundary condition. They showed that the spectrum of $H(1)$ is purely absolutely continuous in a spectral interval of size $\gamma\mu$ (for some $\gamma < 1$) between the Landau levels of the operator P. A similar problem has been considered in [Briet et al. 2008; 2009; Bony et al. 2009] for $H(1)$ on a strip C_a. Moreover the behavior of the spectral shift function near the thresholds $E_i(0)$ was studied in [Briet et al. 2008].

In this work, by the WKB method we construct nontrivial asymptotic solutions of (1-1) (see Theorem 3.1). From the eikonal equation, we derive the classical effective Hamiltonian corresponding to (1-1). In particular we show that the equations of motion in the z-direction are given by $\dot{z} = -\partial_k E_l(k)$, $\dot{k} = \partial_z V(s, z)$. These WKB approximate solutions fail at the so called turning points. In such neighborhoods, where the semiclassical approximation fails, we use the semiclassical Airy equation to describe the solution of (1-1). Next the connection of the two solutions in the matching regions leads to the Bohr–Sommerfeld quantization conditions. In Section 5 we use these quantization conditions to determine asymptotically the eigenvalues and the resonances of $H(\epsilon)$ for ϵ small enough. Particular attention will be paid to the asymptotic behavior of the spectrum near the thresholds of P.

The paper is organized as follows: Section 2 is devoted to the study of the operator $P(k)$ on the interval $[-a, a]$. In Section 3, we construct the approximate solutions of (1-1). In Section 4, we study the concept of a turning point t_i for equations of the form (1-1). We describe also the asymptotic behavior as $\epsilon \searrow 0$ of solutions in a neighborhood of t_i, and we derive the Bohr–Sommerfeld quantization conditions.

2. The unperturbed Hamiltonian

Consider the $2D$ Schrödinger operator with constant magnetic field in the strip C_a:

$$P = D_x^2 + (D_z + \mu x)^2.$$

The operator P is unitarily equivalent to

$$\mathcal{F} P \mathcal{F}^* = \int_{\mathbb{R}}^{\oplus} P(k)\, dk, \tag{2-1}$$

where $\mathcal{F}$ is the partial Fourier transform with respect to z,

$$(\mathcal{F}u)(x, k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-izk} u(x, z)\, dz,$$

and

$$P(k) = D_x^2 + (k + \mu x)^2,$$

is the operator defined on $\mathcal{H}_a := \{u \in H^2([-a, a]);\ u(-a) = u(a) = 0\}$. We begin with a general result on such operators.

Theorem 2.1. *The operator $P(k)$ has a simple discrete spectrum i.e., $\sigma(P(k)) = \bigcup_{j=1}^{\infty} \{E_j(k)\}$ with $E_1(k) < E_2(k) < E_3(k) < \cdots$. Moreover, for every j, $E_j(k)$ is an even real analytic function in k, with the following properties*:

$$k E_j'(k) > 0, \quad k \neq 0 \qquad \textit{and} \qquad E_j'(0) = 0, \quad E_j''(0) > 0, \tag{2-2}$$

$$E_j(k) = E_j(0) + \sum_{i=1}^{\infty} a_{j,i} k^{2i}, \quad (k \to 0), \quad a_{j,1} > 0, \tag{2-3}$$

$$E_j(k) = k^2 - 2a\mu k + \nu_j (2\mu k)^{\frac{2}{3}} + \mathcal{O}(1), \quad k \to +\infty, \tag{2-4}$$

where $0 < \nu_1 < \nu_2 < \cdots < \nu_j < \cdots$ are the eigenvalues of the operator $M = D_x^2 + x$ on $\mathbb{R}^+$. Here $E_j(0)$ are the eigenvalues of the operator $D_x^2 + \mu^2 x^2$. In particular, $E_j(0) \sim (2j-1)\mu$ for strong magnetic field (i.e., μ large enough), and $E_j(0) \sim (j\pi)^2/a^2 + (\mu^2/a^2)\big(\frac{1}{3} - 1/(2\pi^2 j^2)\big)$ for weak magnetic field (i.e., $\mu \ll 1$). The normalized eigenfunctions $\Psi_j(\,\cdot\,, k)$ corresponding to $E_j(k)$ can be *chosen real-valued analytic with respect to k satisfying*:

$$\text{for all } p \in \mathbb{N}, \text{ there exists } C_p \text{ such that } \int_{-a}^{a} \big(\partial_k^p \Psi(x,k)\big)^2\, dx \le C_p. \tag{2-5}$$

Proof. From the Sturm–Liouville theory (see for instance [Marchenko 1986]), it is well known that $P(k)$ has a simple discrete spectrum: $E_1(k) < E_2(k) < \cdots$. The change of variable $x \mapsto -x$ shows that $E_l(k) = E_l(-k)$. Since the eigenvalues are simple, ordinary perturbation theory shows that $E_l(k)$ (and the corresponding eigenfunction) are analytic functions in k (see [Kato 1966; Reed and Simon 1978]). The estimate (2-2) is proved by [Geĭler and Senatorov 1997] in a more general setting (see [Geĭler and Senatorov 1997, Theorem 2]). Formula (2-3) follows from the fact that $E_j(k)$ is an even real analytic function with $E_j''(0) > 0$. The asymptotic behavior of $E_j(0)$ for μ small enough (resp. large enough) follows from the perturbation theory (resp. semiclassical analysis).

To prove (2-4) it suffices to study the operator[1] $D_x^2 + 2\mu xk + k^2$. Replacing x by $t = \mu(x+a)$ and rescaling $t \mapsto t/\lambda$ (with $\lambda = (2\mu k)^{\frac{1}{3}}$) we transform $\widetilde{H}(k)$ into

$$\lambda^2(D_t^2 + t) - 2a\mu k + 2k^2 : L^2([0, 2\lambda\mu a]) \to L^2([0, 2\lambda\mu a]),$$

which yields (2-4) since[2] $\lambda \to +\infty$ as $k \to +\infty$.

The only point remaining concerns the estimate (2-5). Let $\Psi_n(\,\cdot\,, k)$ be the normalized real-valued[3] analytic function corresponding to $E_n(k)$. Since Ψ_n is real and $\|\Psi_n(\,\cdot\,, k)\| = 1$, it follows that

$$\frac{\partial}{\partial k} \int_{-a}^{a} \Psi_n(x,k)^2\, dx = 0 = 2\int_{-a}^{a} \Psi_n(x,k) \frac{\partial}{\partial k} \Psi_n(x,k)\, dx. \tag{2-6}$$

Put $\widehat{P}(k) = D_x^2 + 2xk + x^2$, and let Γ_n be a simple closed contour around $E_n(k) - k^2$ such that $\operatorname{dist}\big(\Gamma_n, \sigma(\widehat{P}(k))\big) \ge C > 0$ uniformly on k. Let $\Pi_n(k)$ be the orthogonal projection onto $\Psi_n(\,\cdot\,, k)$:

$$\Pi_n(k) = \frac{1}{2\pi i} \int_{\Gamma_n} (\widehat{P}(k) - z)^{-1}\, dz = \langle \,\cdot\,, \Psi_n(\,\cdot\,, k)\rangle \Psi_n(x,k). \tag{2-7}$$

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[1] By the min-max principle the spectrum of $D_x^2 + 2kx + k^2$ and $P(k)$ differ by a constant for k large enough.

[2] The eigenvalues of the Airy equation, $(D_t^2 + (t - \nu_j))u(t) = 0$, on $L^2([0, \mu])$ with Dirichlet condition $u(0) = u(\mu) = 0$ are the solutions of the equation

$$\mathrm{Ai}(-\nu_j) = \mathrm{Bi}(-\nu_j) \frac{\mathrm{Ai}(-\nu_j + \mu)}{\mathrm{Bi}(-\nu_j + \mu)}. \tag{E}$$

Here $\mathrm{Ai}(x)$ is the Airy function and $\mathrm{Bi}(x) = \mathrm{Ai}(e^{2\pi i/3}x)$. Since the right hand side of (E) tends to zero as μ tends to $+\infty$, $-\nu_j$ are approximated by the zeros of the Airy function.

[3] Since $D_x^2 + (x+k)^2 = \overline{D_x^2 + (x+k)^2}$, $\Psi_n(x,k)$ can be chosen real-valued.

From (2-6) we deduce that $\Pi_n(k)\partial_k\Psi_n(x,k) = 0$. Combining this with the fact that $\Pi_n(k)\Psi_n(x,k) = \Psi_n(x,k)$ and using (2-7), we get

$$\begin{aligned}\partial_k\Psi_n(x,k) &= \partial_k\Pi_n(k)\Psi_n(x,k)\\ &= \frac{1}{2\pi i}\int_{\Gamma_n}(\widehat{P}(k)-z)^{-1}2x(\widehat{P}(k)-z)^{-1}\,dz\,\Psi_n(x,k),\end{aligned}\tag{2-8}$$

which yields

$$\|\partial_k\Psi_n(\,\cdot\,,k)\| = \mathcal{O}(1)\|\Psi_n(\,\cdot\,,k)\| = \mathcal{O}(1).$$

We now proceed by induction using (2-8). □

3. The perturbed Hamiltonian

For the simplicity of the notation we take $\mu = 1$. As stated in the introduction, we consider the time-dependent Schrödinger equation with perturbed potentials:

$$\big[D_t - H(\epsilon)\big]u = 0, \quad u = u(t,x,z,\epsilon). \tag{3-1}$$

With the change of variables

$$s = \epsilon t \quad \text{(adiabatic scale)} \qquad \text{and} \qquad y = \epsilon z \quad \text{(long spacial scale)},$$

Equation (3-1) becomes

$$\big[\epsilon D_s - \widehat{H}(\epsilon)\big]v = 0, \quad v = v(s,x,y,\epsilon), \tag{3-2}$$

where

$$\widehat{H}(\epsilon) := D_x^2 + (\epsilon D_y + x)^2 + V(s,y).$$

Now if $\widehat{H}(\epsilon)$ is regarded as an ϵ-pseudodifferential operator on (s, y) with operator-valued symbol, one looks for a local solution of the form

$$v(s,x,y,\epsilon) = e^{i\phi(s,y)/\epsilon}m(s,x,y,\epsilon), \tag{3-3}$$

$$m(s,x,y;\epsilon) = m_0(s,x,y) + \epsilon m_1(s,x,y) + \cdots. \tag{3-4}$$

Substituting (3-3) into (3-2) and collecting terms which are the same order in ϵ, we get

$$\begin{aligned}&e^{-i\phi(s,y)/\epsilon}\big[\epsilon D_s - \widehat{H}(\epsilon)\big]v\\ &\quad= \big[\partial_s\phi - P(\phi_y'(s,y)) - V(s,y)\big]m + \big[\epsilon D_s - \partial_k P(\phi_y'(s,y))\epsilon D_y + i\epsilon(\Delta\phi)\big]m\\ &\qquad\qquad + \epsilon^2\Delta m\\ &\quad= c_0(s,x,y) + \epsilon c_1(s,x,y) + \cdots + \epsilon^{N+2}c_{N+2}(s,x,y,\epsilon),\end{aligned}\tag{3-5}$$

with

$$c_0(s,x,y) = \big[\phi'_s - P(\phi'_y) - V(s,y)\big]m_0, \tag{3-6}$$

$$c_1(s,x,y) = Km_0 + \big[\phi'_s - P(\phi'_y) - V(s,y)\big]m_1, \tag{3-7}$$

and for $j = 2, 3, \dots, N+2$,

$$c_j(s,x,y) = Km_{j-1} + \Delta_y m_{j-2} + \big[\phi'_s - P(\phi'_y) - V(s,y)\big]m_j. \tag{3-8}$$

Here

$$K = i\big[\partial_k P(\phi'_y)\partial_y + \phi''_{yy} - \partial_s\big], \tag{3-9}$$

and

$$\phi'_y = \partial_y \phi(s,y), \quad \phi''_{yy} = \partial^2_{yy}\phi(s,y), \quad \partial_k P(k) = 2(k+x). \tag{3-10}$$

Notice that, when ϕ is real-valued, (3-3) is the standard ansatz of geometric optics. In the construction of geometric optics solutions one requires that

$$c_0(s,x,y) = 0, \tag{3-11}$$

$$c_j(s,x,y) = 0, \quad j = 1, 2, \cdots. \tag{3-12}$$

Eikonal equation and semiclassical dynamics. From now on we fix l, and we let $\Psi_l(\cdot\,,k)$ be the normalized eigenfunction corresponding to $E_l(k)$:

$$P(k)\Psi_l(\cdot\,,k) = E_l(k)\Psi_l(\cdot\,,k), \qquad \int_{-a}^{a} \Psi_l(x,k)^2\,dx = 1. \tag{3-13}$$

By Theorem 2.1, the function $k \to \Psi_l(\cdot\,,k)$ can be chosen real analytic.

Equations (3-11) and (3-6) tell us that for all s, y, $m_0(s,\ \cdot\,,y)$ is an eigenfunction of $P(\phi'_y)$ with eigenvalue $\partial_s\phi - V(s,y)$. Hence, we can satisfy (3-11) by choosing

$$\phi'_s = E_l(\phi'_y) + V(s,y), \quad \text{(eikonal equation)} \tag{3-14}$$

and setting

$$m_0(s,x,y) = f_0(s,y)\Psi_l(x,\phi'_y). \tag{3-15}$$

Since the eikonal equation is derived from the "effective Hamiltonian"

$$G(s,\sigma,y,k) = \sigma - E_l(k) - V(s,y),$$

we see that equation of motion in the y-direction are

$$\dot{s} = 1, \quad \dot{\sigma} = \partial_s V(s,y), \quad \dot{y} = -\partial_k E_l(k), \quad \dot{k} = \partial_y V(s,y). \tag{3-16}$$

By applying the classical "method of characteristics," one can solve the eikonal equation (3-14) at least for small s. From now on we assume that ϕ is constructed.

Propagation of the amplitude. For simplicity we ignore the dependence of the following operators and functions on ϕ'_y and we write P, $\partial_k P$, E_l, $\partial_k E_l$, Ψ and $f_0\Psi_l$ instead of $P(\phi'_y)$, $\partial_k P(\phi'_y)$, $E_l(\phi'_y)$, $\partial_k E_l(\phi'_y)$, $\Psi_l(x, \phi'_y)$ and $f_0(s, y)\Psi_l(x, \phi'_y)$.

By the Fredholm alternative in $L^2([-a, a])$, we can solve (3-12) for $j = 1$ if and only if the first term of the right hand side of (3-7) is orthogonal to

$$\ker\big[P - (\phi'_s - V(y))\big] = \ker\big[P - E_l\big] = \mathrm{Vect}(\Psi_l),$$

where we have used (3-14) and (3-15). In view of (3-7) and (3-8) this is equivalent to

$$\big\langle[\partial_k P\partial_y + \phi''_y - \partial_s](f_0\Psi_l),\ \Psi_l\big\rangle = 0.$$

We conclude from (2-6) that $\langle\partial_s\Psi_l, \Psi_l\rangle = 0$, hence that

$$\langle\partial_k P\Psi_l, \Psi_l\rangle\partial_y f_0 - \partial_s f_0 + \big[\langle\partial_k P\partial_y\Psi_l, \Psi_l\rangle + \phi''_y\big]f_0 = 0. \tag{3-17}$$

Taking the derivative with respect to k in (3-13),

$$\big[P(k) - E_l(k)\big]\partial_k\Psi(\,\cdot\,, k) = \big[\partial_k E_l(k) - \partial_k P(k)\big]\Psi_l(\,\cdot\,, k), \tag{3-18}$$

and taking the inner product with Ψ_l and using again (3-13) we get

$$\partial_k E_l(k) = \langle(\partial_k P)\Psi_l, \Psi_l\rangle = 2\int_{-a}^{a}(x + k)\Psi_l(x, k)^2\,dx. \tag{3-19}$$

Next, taking the derivative with respect to y of $\partial_k E_l = \partial_k E_l(\phi'_y)$, we obtain

$$\partial_y \cdot \partial_k E_l = 2\phi''_y + 2\langle(\partial_k P)\partial_y(\Psi_l), \Psi_l\rangle. \tag{3-20}$$

Substituting (3-19) and (3-20) into the left hand side of (3-17), we get the transport equation for f_0:

$$\partial_k E_l\partial_y f_0 - \partial_s f_0 + \tfrac{1}{2}\big[\partial_y \cdot \partial_k E_l\big]f_0 = 0. \tag{3-21}$$

Assuming that $\phi(s, y)$ is selected and let $U_s : y = y(0) \to y(s)$ be the flow on the configuration space $\mathbb{R}_y$ corresponding to

$$\dot y(s) = -\partial_k E_l\big(\phi'_y(s, y(s))\big).$$

It is well known that

$$-\partial_y \cdot \partial_k E_l\big(\phi'_y(s, y(s))\big) = \frac{d}{ds}\log\left|\frac{\partial y(s)}{\partial y}\right|.$$

Along $y(s)$ the differential equation (3-21) takes the form

$$\frac{d}{ds}\big[f_0(s, y(s))\big] + \left[\frac{1}{2}\frac{d}{ds}\log\left|\frac{\partial y(s)}{\partial y}\right|\right]f_0 = 0, \tag{3-22}$$

which yields a kind of energy conservation

$$\frac{d}{ds}\left[|f_0|^2\left|\frac{\partial y(s)}{\partial y}\right|\right]=0. \tag{3-23}$$

Thus, $\int_{\mathbb{R}}|f_0(s,y)|^2\,dy$ does not depend on s and consequently

$$\int_{-a}^{a}\int_{\mathbb{R}_y}|u(s,x,y,\epsilon)|^2\,dx\,dy=\int_{-a}^{a}\int_{\mathbb{R}_y}|u(0,x,y,\epsilon)|^2\,dx\,dy+\mathcal{O}(\epsilon).$$

We now derive the transport equation for $m_1(s,x,y)$. Like (3-7), Equation (3-8) can be solved for $j=2$ if and only

$$\big\langle i\big[\partial_k P\partial_y+\phi''_y-\partial_s\big]m_1+\Delta_y m_0,\ \Psi_l\big\rangle=0. \tag{3-24}$$

Writing

$$m_1(s,x,y)=f_1(s,y)\Psi_l(x,\phi'_y)+m_1^{\perp}(s,x,y) \tag{3-25}$$

with

$$\langle\Psi_l(\,\cdot\,,\phi'_y),\ m_1^{\perp}\rangle=0.$$

According to (3-7) and (3-25), the term $m_1^{\perp}$ is given by

$$m_1^{\perp}=-\big[\phi'_s-P(\phi'_y)-V(s,y)\big]^{-1}(Km_0). \tag{3-26}$$

Inserting (3-25) in (3-24) and using (3-26) we see that $f_1(s,y)$ satisfies an inhomogeneous version of the transport equation (4-3):

$$\partial_k E_l\partial_y f_1-\partial_s f_1+\tfrac{1}{2}[\partial_y\cdot\partial_k E_l]f_1=-\big\langle[\partial_k P\partial_y+\phi''_y-\partial_s]m_1^{\perp}+i\Delta_y m_0,\ \Psi_l\big\rangle. \tag{3-27}$$

We repeat this process (by solving the transport equation with a right-hand side) and get explicitly all the terms m_j (at least for s small). This gives a solution of (3-1) modulo $\mathcal{O}(\epsilon^{\infty})$. Consequently, we have proved:

Theorem 3.1. *Given $N\in\mathbb{N}$, $\phi\in C^{\infty}(\mathbb{R})$ and $f\in C_0^{\infty}(\mathbb{R})$. There exists $T,\epsilon_0>0$ and an approximate solution*

$$v(s,x,y;\epsilon)=m_0(s,x,y)+\epsilon m_1(s,x,y)+\cdots+\epsilon^N m_N(s,x,y)$$

such that for all $|s|<T$ and $\epsilon\in\,]0,\epsilon_0[$ we have:

$$\|v(0,x,y;\epsilon)-e^{i\phi(y)/\epsilon}\Psi_l(x,\phi'(y))\|=\mathcal{O}(\epsilon),$$
$$\|(\epsilon D_s-\widehat{H}(\epsilon))v\|=\mathcal{O}_N(\epsilon^N),$$

with

$$m_0(s,x,y)=e^{i\phi(s,y)/\epsilon}f_0(s,y)\Psi_l(x,\partial_y\phi(s,y)),$$

where $\phi(s,y)$ and $f_0(s,y)$ are solutions of (3-14) *and* (3-21) *respectively with initial condition $f_0(0,y)=f(y)$ and $\phi(0,y)=\phi(y)$.*

However, as is well known, if we try to construct WKB-solutions globally (that is in some large given region), ϕ may develop singularities at "caustic" and the transport equations then become undefined. The consideration of these difficulties, beginning with [Keller 1958; Maslov and Fedoriuk 1981], lead to the development of the theory of Fourier integral operators as given by Hörmander [1971]. Since here the problem is reduced to study a one-dimensional Hamiltonian in the y-direction, we will use in the next section the standard semiclassical techniques based on the Airy function.

4. Quantization conditions

Recall that $H(\epsilon)$ and $\widehat{H}(\epsilon)$ have the same spectrum, since they are unitarily equivalent by a change of variable (see (3-1) and (3-2)). Hence, in this section we will be concerned with the spectrum of the operator $\widehat{H}(\epsilon)$. From now on we assume that V is time independent (i.e., $V(y) := V(t, y)$).

Fix an energy e, and consider the stationary equation

$$(\widehat{H}(\epsilon) - e)w = 0, \quad w = e^{i\phi(y)/\epsilon}\big(m_0(x, y) + \epsilon m_1(x, y) + \cdots\big). \tag{4-1}$$

Clearly, f is a solution of (4-1) if and only if $v(x, y, s, \epsilon) = e^{ise/\epsilon}w$ is a solution of (3-2). In particular, the eikonal and transport equations corresponding to (4-1) are

$$e = E_l(\phi'(y)) + V(y), \tag{4-2}$$

$$\partial_y f_0 + \frac{1}{2}\left[\frac{\partial_y \cdot \partial_k E_l}{\partial_k E_l}\right] f_0 = 0. \tag{4-3}$$

Let $\Sigma_e^l = \{(y, k) \in \mathbb{R} \times \mathbb{C},\ E_l(k) + V(y) = e\}$ be the isoenergy curve. Recalling that $k = 0$ is the only critical point of $k \mapsto E_l(k)$. Assume that

$$V'(y) \neq 0 \text{ on the set of turning points } \Gamma_e^l := \big\{y \in \mathbb{R};\ V(y) = e - E_l(0)\big\}. \tag{4-4}$$

Thus, Γ_e^l is a discrete set: $\Gamma_e^l = \{\cdots < y_{-1} < y_0 < y_1 < \cdots\}$. Each finite interval; $[y_j, y_{j+1}]$ is covered by a closed finite branch γ_e of Σ_e, which consists of two regular branches γ_+, γ_-:

$$\gamma_+ : k = \tau(y), \quad \gamma_- : k = -\tau(y).$$

(*Classical allowed region*). Consider an interval $[y_j, y_{j+1}]$ covered by a real closed branch γ_e of Σ_e. The construction of Section 3 and (4-3) give us two solutions $w, \overline{w}$ of (4-1) such that

$$w = e^{i\phi(y)/\epsilon}(m_0 + \epsilon m_1 + \cdots), \quad \text{with } \phi(y) = \int^y \tau(t)\,dt, \tag{4-5}$$

and

$$m_0(x, y) = C_0 \frac{1}{|\partial_k E_l(\phi'_y(y))|^{\frac{1}{2}}} \Psi_l(x, \phi'_y(y)).$$

(*Classical forbidden region*). In the regions $]y_{j-1}, y_j[$ and $]y_{j+1}, y_{j+2}[$, which are classically forbidden, we can also construct a solution of the form (4-5). But now, since in Σ_e^l the number k is complex, the phases $\phi(y)$ are purely imaginary. We denote the corresponding solution by g_1. From g_1 we can construct a linearly independent other solution g_2 by the change $k \to -k$, (we recall that $E_l(k)$ is an even function). The solutions g_1 and g_2 in this regions are decreasing and increasing exponential functions. Notice that, the turning points separate the projections of the real and complex branches of the isoenergy curve to the y-axis. As indicated above, in vicinities of the turning points y_j the semiclassical approximations w, $\overline{w}$, g_1 and g_2 are note defined, since $\phi'_y(y_j) = 0$ and $\partial_k E_l(\phi'_y(y_j)) = 0$. To describe the solutions near y_j we use the standard semiclassical Airy equation. More precisely, near the turning point y_j we replace the variable y by the new one $\tilde{y} = \epsilon^{-\frac{2}{3}}(y - y_j)$, and we consider instead of (4-1) the equation

$$\big[D_x^2 + (\epsilon^{\frac{1}{3}} D_{\tilde{y}} + x)^2 + V(y_j + \epsilon^{\frac{2}{3}} \tilde{y}) - e\big] w(x, \tilde{y}; \epsilon) = 0, \tag{4-6}$$

with

$$w(x, \tilde{y}; \epsilon) = \sum_{l \geq 0} \epsilon^{l/\epsilon} m_l(x, \tilde{y}). \tag{4-7}$$

Expanding the operator in the left hand side of (4-6) in powers of $\epsilon^{\frac{1}{3}}$ and substituting into (4-7), we obtain

$$\big[D_x^2 + x^2 + V(y_j) - e\big] m_0(x, \tilde{y}) = 0, \tag{4-8}$$

$$\big[D_x^2 + x^2 + V(y_j) - e\big] m_1(x, \tilde{y}) = -2x D_z m_0(x, \tilde{y}), \tag{4-9}$$

$$\begin{aligned} \big[D_x^2 + x^2 + V(y_j) - e\big] m_2(x, \tilde{y}) = {} & -2x D_{\tilde{y}} m_1(x, \tilde{y}) \\ & - \big[D_{\tilde{y}}^2 + V'(y_j)\tilde{y}\big] m_0(x, \tilde{y}). \end{aligned} \tag{4-10}$$

Since $E_l(0) + V(y_j) = e$, it follows from (4-8) that

$$m_0(x, \tilde{y}) = N(\tilde{y}) \Psi_l(x, 0). \tag{4-11}$$

Notice that $x \mapsto \psi_l(x, 0)^2$ is an even function, hence the right hand side of (4-9) is orthogonal to $\Psi_l(x, 0)$. We conclude from the Fredholm alternative that (4-9) is always soluble and its solution is given by

$$m_1(x, \tilde{y}) = B(\tilde{y}) \Psi_l(x, 0) + i N'(\tilde{y}) \big[D_x^2 + x^2 + V(y_j) - e\big]^{-1} (2x \Psi_l(x, 0)).$$

Next, applying (3-18) to $k=0$, and recalling that $\partial_k P(0) = 2x$ and $\partial_k E_l(0) = 0$, we deduce that

$$\left[D_x^2 + x^2 + V(y_j) - e\right]^{-1}(2x\Psi_l(x,0)) = -\partial_k \Psi_l(x,0).$$

Consequently,

$$m_1(x,\tilde{y}) = B(\tilde{y})\Psi_l(x,0) - iN'(\tilde{y})\partial_k\Psi_l(x,0). \tag{4-12}$$

The right hand side of (4-10) can be written as

$$2ixB'(\tilde{y})\Psi_l(x,0) + 2ixN''(\tilde{y})\partial_k\Psi_l(x,0) \\ + \left[-N''(\tilde{y}) + V'(y_j)\tilde{y}N(\tilde{y})\right]\Psi_l(x,0). \tag{4-13}$$

Combining this with fact that $x\Psi_l(x,0)$ is orthogonal to $\Psi_l(x,0)$, we deduce that Equation (4-10) has a solution if and only if

$$-N''(\tilde{y}) + V'(y_j)\tilde{y}N(\tilde{y}) - 2N''(\tilde{y})\int_{-a}^{a} x\Psi_l(x,0)\partial_k\Psi_l(x,0)\,dx = 0. \tag{4-14}$$

On the other hand, it follows from (3-19) that

$$\kappa_l^{-1} := \tfrac{1}{2}\partial_k^2 E_l(0) = 1 + \int_{-a}^{a} 2x\Psi_l(x,0)\partial_k\Psi_l(x,0)\,dx, \tag{4-15}$$

which together with (4-14) yields the following Airy equation for $N(\tilde{y})$:

$$-N''(\tilde{y}) + \eta_l\tilde{y}N(\tilde{y}) = 0, \quad \eta_l := \kappa_l V'(y_j). \tag{4-16}$$

$$m_0(x,\tilde{y}) = \left[C_3 \operatorname{Ai}(\eta_l^{\frac{1}{3}}\tilde{y}) + C_4 \operatorname{Ai}(\eta_l^{\frac{1}{3}} e^{i2\pi/3}\tilde{y})\right]\Psi_l(x,0). \tag{4-17}$$

Thus, the leading term of the series (4-7) is given by (4-11) where for $N(\tilde{y})$ we can choose an arbitrary solution of (4-16). All the remaining terms of (4-7) can easily be constructed. This gives a solution near the turning points.

It now remains to construct a global approximate solution to Equation (4-1). For the wave function to be square-integrable, we must take only the exponentially decaying solutions $(g_{1,j}, g_{1,j+1})$ in the two classically forbidden regions $]y_{j-1}, y_j[$ and $]y_{j+1}, y_{j+2}[$. These must then connect properly through the turning points y_j to the classically allowed region. Let us fix a solution in the allowed region $]y_j, y_{j+1}[$:

$$v(x,y;\epsilon) = (C_1 e^{i\phi(y)/\epsilon} + C_2 e^{-i\phi(y)/\epsilon})\left(\frac{1}{|\partial_k E_l(\phi_y'(y))|^{\frac{1}{2}}}\Psi_l(x,\phi_y'(y)) + \mathcal{O}(\epsilon)\right).$$

Next, it is possible to find a condition under which $v(x,y;\epsilon)$ satisfies the following property: the continuation of v through the turning points y_j and y_{j+1} by means of solutions of the form (4-17) leads to solutions of the form $g_{1,j}$, $g_{1,j+1}$.

This condition, called the “Bohr–Sommerfeld quantization condition,” has the following form:

$$\int_{\gamma_e} k\,dy = \pi(2n + \operatorname{ind}(\gamma_e)/2)\epsilon + \sum_{n\geq 2} \omega_n \epsilon^n, \tag{4-18}$$

where (ω_n) is some sequence of 1-form and $\operatorname{ind}(\gamma_e)$ is the Maslov index of γ_e (see [Maslov and Fedoriuk 1981]).

This condition, which can be considered as a condition on the spectral parameter e, plays a crucial role in calculation of eigenvalues or resonances (see the next section).

Remark. Consider the ϵ-pseudodifferential operator

$$H^l_{\text{eff}}(\epsilon) = E_l(\epsilon D_y) + V(y) - e.$$

The equations (4-2) and (4-3) are exactly the eikonal and transport equations in the construction of asymptotic solutions to $H^l_{\text{eff}}(\epsilon)u = \mathcal{O}(\epsilon^2)\|u\|^2$. The operator $H^l_{\text{eff}}(\epsilon)$ is called[4] the effective Hamiltonian of order $\mathcal{O}(\epsilon^2)$ corresponding to $\widehat{H}(\epsilon)$ near e. By using the Feshbach method (or Grushin problem) see [Dimassi and Sjöstrand 1999], we can construct an effective Hamiltonian of any order $\mathcal{O}(\epsilon^N)$ corresponding to $\widehat{H}(\epsilon)$ (see [Dimassi 1993; Martinez 1991a]).

5. Asymptotic behavior of the spectrum

According to (2-1), (2-4) and (2-3), we have

$$\sigma(P) = \sigma_{\text{ac}}(P) = \bigcup_{j=1} \bigcup_{k\in\mathbb{R}} E_j(k) = [E_1(0), +\infty[,$$

and $E_j(0)$, $j = 1, 2, \dots$ are thresholds in $\sigma(P)$.

It is known that the spectrum of the perturbed operator $\widehat{H}(\epsilon)$ depends on the asymptotic behavior of V at infinity. Here we distinguish two definite types of these asymptotics:

$$V(y) \to +\infty, \quad y \to \infty, \tag{A}$$

$$V(y) \to 0, \qquad y \to \infty. \tag{B}$$

We refer to the remark on page 211 for other type of asymptotics.

Case (A). If $V(y) \to +\infty$ as $y \to \infty$, the spectrum of $\widehat{H}(\epsilon)$ is simple and purely discrete. For instance, let us assume:

[4] An effective Hamiltonian of order $\mathcal{O}(\epsilon^N)$ is a Hamiltonian that acts in a reduced space (here $L^2(\mathbb{R}_y)$ instead of $L^2([-a, a] \times \mathbb{R}_y)$) and only describes a part of the spectrum of the true Hamiltonian $H(\epsilon)$ modulo an error term $\mathcal{O}(\epsilon^N)$.

Assumption A. A nondegenerate minimum occurs at $y = 0$ and $V(y)$ is strictly increasing when $y > 0$ and strictly decreasing when $y < 0$.

Then the discrete spectrum of $\widehat{H}(\epsilon)$ is included in $[E_1(0) + V(0), +\infty[$.

For $e \in]E_1(0) + V(0), E_2(0) + V(0)[$, the isoenergy curve Σ_e^1 has a nontrivial real branch γ_e^1 which is oval. The projection of γ_e^1 on the y-axis is $[y_1, y_2]$, where $\{y_1, y_2\}$ are the unique turning points of Σ_e^1. The quantization condition (4-18) leads to the asymptotic description of the allowed discrete spectrum near e. The corresponding eigenfunction are localized asymptotically in $[y_1, y_2]$. For $l \geq 2$, the real branches of the isoenergy curve Σ_e^l are absent.

Fix $e \in]E_N(0)+V(0), E_{N+1}(0)+V(0)[$. For $l \in \{1, 2, \dots, N\}$ the real branch γ_e^l of the isoenergy curve Σ_e^l is oval. The projection of γ_e^l on the y-axis is $[y_{1,l}, y_{2,l}]$, where $\{y_{1,l}, y_{2,l}\}$ are the unique turning points of Σ_e^l. The quantization condition for each curve γ_e^l,

$$\int_{\gamma_e^l} k\,dy = \pi(2n + \operatorname{ind}(\gamma_e^l)/2)\epsilon + \sum_{n\geq 2} \omega_n^l \epsilon^n, \tag{5-1}$$

gives a set of eigenvalues $e_n^l(\epsilon) \sim \sum_j a_{j,n,l}\epsilon^j$. The corresponding eigenfunction are concentrated in $[y_{1,l}, y_{2,l}]$. For $l \geq N + 1$, the real branches of the isoenergy curve Σ_e^l are absent.

Now let us treat the general case (i.e., without the monotonicity assumption). Fix $e \in I :=]E_1(0) + \inf_{x\in\mathbb{R}} V(x), +\infty[$. We recall that the spectrum of $\widehat{H}(\epsilon)$ is discrete and included in $\bar{I}$. The isoenergy curve Σ_e^l has a finite real branches $\gamma_e^{l,j}$, $j = 1, \dots, N(l)$. Under the assumption (4-4), the curve $\gamma_e^{l,j}$ is oval. The quantization condition (4-18) for each curve $\gamma_e^{l,j}$, $j = 1, \dots, N(l)$ gives the description of the asymptotic behavior with respect to ϵ of the corresponding eigenvalue. If all these asymptotic series are different they give the description of the asymptotic properties of the eigenvalues. However, if some of these series are equal (this happens for example if V is even) we have to take into account the interaction effects to describe their splitting. The interaction between two series corresponding to different real branches of the isoenergy curve can be estimated in terms of tunneling through all the intervals separating the asymptotic supports of the eigenfunctions and covered by the complex branches of the isoenergy curve. This interaction leads to exponentially small displacement of the eigenvalues. Thus, the spectrum of $\widehat{H}(\epsilon)$ can be described essentially as the simple unification of the contributions of the separate γ_e^l.

Finally, let us study the bottom of the spectrum (i.e., $e = E_1(0) + \inf_{y\in\mathbb{R}} V(y)$). Without any loss of generality we may assume that $\inf_{y\in\mathbb{R}} V(y) = V(0)$ with $V'(0) = 0$, $V''(0) > 0$ and $V(y) > 0$ for all $y \neq 0$. In this case, the isoenergy curve Σ_e^1 is reduced to a single point $X_0 = (0, 0)$ and $\Sigma_e^l = \varnothing$ for $l \geq 2$. Therefore, the approximate solutions of the Equation (4-1) are localized in $\{y = 0\}$. In the Appendix, we follow the standard construction of approximate solutions near a nondegenerate minimum of the semiclassical Schrödinger operator (see [Dimassi and Sjöstrand 1999, Chapter 3]). More precisely, denote $\kappa_m := \sqrt{E_1''(0)V''(0)}\big(m + \frac{1}{2}\big)$, $m \in \mathbb{N}$. We have:

Theorem 5.1. *Fix C_0 in $]\kappa_N, \kappa_{N+1}[$. For ϵ small enough the operator $\widehat{H}(\epsilon)$ has exactly N eigenvalues $e_1(\epsilon), \dots, e_N(\epsilon)$ in $]-\infty, E_1(0) + V(0) + C_0\epsilon[$. Moreover, for $j \in \{1, 2, \dots, N\}$, the following asymptotics hold*:

$$e_j(\epsilon) = E_1(0) + V(0) + \kappa_j\epsilon + \sum_{l=2}^{\infty} c_{j,l}\epsilon^l.$$

Case (B). Assume that V tends to zero at infinity. By the Weyl criterion the essential spectrum of $\widehat{H}(\epsilon)$ and P are the same, and coincide with $[E_1(0), +\infty[$. In $]-\infty, E_1(0)[$ we have a discrete spectrum caused by the potential V. This part of the spectrum (except near $E_1(0)$) can be studied as above. If $e \to E_1(0)$, the isoenergy curve becomes infinite. In particular, the total number of eigenvalues near $E_1(0)$ can be infinite. In this case, the asymptotic behavior of eigenvalues near $E_1(0)$ is the same as for the operator $-(E_1''(0)/2)\epsilon^2 \frac{\partial^2}{\partial y^2} + V(y)$ on $L^2(\mathbb{R})$.

To investigate the effect of V on the continuous spectrum of $\widehat{H}(\epsilon)$, it is natural to study the resonances. One can treat the resonances by two different ways. If the potential V is analytic in some neighborhood of the real axis, one can consider the resonances as the eigenvalues of the spectrally deformed Hamiltonian [Hislop and Sigal 1996]. If the analytical continuation of V is impossible, the resonances can be considered as a poles of the meromorphic continuation of the kernel of the resolvent $(\widehat{H}(\epsilon) - z)^{-1}$ on some weighted L^2 space (see [Helffer and Martinez 1987]).

Let us assume that V satisfies the assumption A with $v_0 := -V(0) > 0$. The structure of the isoenergy curve Σ_e^l depends on the correlation of v_0 and $E_{j+1}(0) - E_j(0)$. For simplicity let us assume that

$$v_0 < \min\big(E_2(0) - E_1(0), E_3(0) - E_2(0)\big).$$

If $e \in]E_1(0), E_2(0) + V(0)[$, then the isoenergy curve Σ_e^1 is unbounded, and its projection on the y-axis is $\mathbb{R}$. The real branches of the isoenergy curve Σ_e^l are absent for $l \geq 2$. Thus, the quantization conditions (4-18) lose their meaning and for each e there are two eigenfunctions of the continuous spectrum which correspond to two separated unbounded parts γ_- and γ_+ of Σ_e^1 (see Section 4).

If $e = E_2(0) + V(0)$, then Σ_e^1 is unbounded and $\Sigma_e^2 = \{(0, 0)\}$. Real branches of the isoenergy curve Σ_e^l are absent for $l \geq 3$, so the contribution of these branches to the spectrum is empty asymptotically. As indicated above the real branch of Σ_e^1

are two separated unbounded curves γ_- and γ_+. Thus the set Σ_e^1 is nontrapping[5] and then the operator K_1 does not produce resonances near e. The operator K_2 has a discrete spectrum near e which is resonances of the operator $\widehat{H}(\epsilon)$. More precisely, let

$$K_2 = \tfrac{1}{2}\big(E_2''(0)\epsilon^2 D_y^2 + V''(0)y^2\big),$$

and let $\zeta_1\epsilon < \zeta_2\epsilon \cdots < \zeta_j\epsilon < \cdots$ be the eigenvalues of K_2. As in the proof of Theorem 5.1 we have:

Theorem 5.2. *Fix C_0 in $]\zeta_N, \zeta_{N+1}[$. For ϵ small enough the operator $\widehat{H}(\epsilon)$ has exactly N resonances $z_1(\epsilon), \dots, z_N(\epsilon)$ in the disk*

$$D(e, C_0\epsilon) := \{z \in \mathbb{C};\ |z-e| < C_0\epsilon\}.$$

Moreover, for $j \in \{1, 2, \dots, N\}$, the following asymptotics hold:

$$z_j(\epsilon) = E_2(0) + V(0) + \kappa_j\epsilon + \sum_{l=2}^{\infty} \epsilon^l d_{j,l}. \tag{5-2}$$

Formula (5-2) shows that $\Im z_j(\epsilon) = \mathcal{O}(\epsilon^\infty)$. Assuming that V is analytic in a complex conic neighborhood of the real axis we can show as in [Martinez 1991b] that $\Im z_j(\epsilon) = \mathcal{O}(e^{-C/\epsilon})$ for some positive constant C. We cannot exclude the existence of embedded eigenvalues as the case $\mu = 0$ (i.e., $\widehat{H}(\epsilon) = D_x^2 + \epsilon^2 D_y^2 + V(y)$) shows. So in this paper we make no distinction between real eigenvalues and resonances.

Next, fix $e \in]E_2(0) + V(0), E_2(0)[$. The isoenergy curve Σ_e^1 is unbounded, the real branch γ_e of Σ_e^2 is oval and the real branches of Σ_e^l are absent for $l \geq 3$. Again the branch Σ_e^1 does not produce resonances near e, and the quantization condition (4-18) corresponding to γ_e gives the asymptotic expansion in powers of ϵ of the real part of some resonances $z_j(\epsilon)$. The study of the resonances of $H(\epsilon)$ near $E_2(0)$ is related to the study of the operator $-(E_2''(0)/2)\epsilon^2 \frac{\partial^2}{\partial y^2} + V(y)$ on $L^2(\mathbb{R})$.

Remark. (1) Our results hold for more general potential V. In particular, one can consider the case where V depends on the variable x (i.e., $V = V(x, \epsilon t, \epsilon z)$). In this case the results depend on the eigenvalues $G_l(s, k, y)$ corresponding to $D_x^2 + (\mu x + k)^2 + V(x, s, y)$ on $L^2([-a, a])$ with Dirichlet boundary condition.

(2) The asymptotic behavior of the spectrum of $\widehat{H}(\epsilon)$ in the case where V is periodic will be treated elsewhere. The case of the plane was considered in [Brüning et al. 2002] for strong magnetic field (μ is large enough).

[5] Σ_e^1 is nontrapping for the classical Hamiltonian $p(y; k) = E_1(k) + V(y)$ if for all $(y, k) \in \Sigma_e^1$, $|\exp(tH_p(y, k)| \to \infty$ when $t \to \infty$.

(3) If $V(y) - y$ tends to zero at infinity then the spectrum of $\widehat{H}(\epsilon)$ covers the real axis. In this case to study the resonances we can use the analytic deformation. More precisely, set $W(y) = V(y) - y$, and suppose that W admits a holomorphic extension into the domain $\Gamma_\delta := \{z \in \mathbb{C};\ |\Im z| < \delta\}$ for some $\delta > 0$. We also assume that $W(z)$ tends to zero uniformly on $z \in \Gamma_\delta$.

For real θ the operator $\widehat{H}(\epsilon)$ is unitarily equivalent to

$$\widehat{H}_\theta(\epsilon) := D_x^2 + (\epsilon D_y + x)^2 + y + \theta + W(y+\theta).$$

The above assumption on W implies that $\widehat{H}_\theta(\epsilon)_{\theta\in\Gamma_\delta}$ is an analytic family of type-A in the sense of Kato [1966]. Fix $\theta = -i\nu$ with $\nu > 0$. According to the Weyl criterion, we have

$$\sigma_{\text{ess}}(\widehat{H}_{-i\nu}(\epsilon)) = \sigma(\widehat{H}_{-i\nu}(\epsilon) - W(y - i\delta)) = \mathbb{R} - i\nu.$$

Thus on the upper half plane $\{z \in \mathbb{C};\ \Im z > -\nu\}$, the operator $\widehat{H}_{-i\nu}(\epsilon)$ has discrete eigenvalues of finite multiplicities. These eigenvalues are the resonances of $\widehat{H}(\epsilon)$.

Appendix: Sketch of proof of Theorem 5.1

Recalling that $\Sigma_e^1 = \{(0,0)\}$ and $\Sigma_e^l = \varnothing$ for $l \neq 1$. Thus the approximate solutions w of (4-1) are localized in $y = 0$. Therefore, we want to find

$$w = e^{i\phi(y)/\epsilon}\big(m_0(x,y) + \epsilon m_1(x,y) + \cdots\big), \quad \text{and} \quad e = e_0 + \epsilon e_1 + \epsilon^2 e_2 + \cdots$$

solutions of (4-1) near $y = 0$, with $\Re(i\phi(y)) > 0$ for $y \neq 0$ and $\Re(i\phi(0)) = 0$.

From (3-21), (3-27), (4-2) and (4-3) we have:

$$E_l(\phi'(y)) + V(y) = e_0, \tag{A-1}$$

$$(\mathcal{L} - ie_1) f_0 = 0, \tag{A-2}$$

$$(\mathcal{L} - ie_1) f_1 = -\big\langle [\partial_k P \partial_y + \phi_y''] m_1^\perp + i\Delta_y m_0, \Psi_l \big\rangle - ie_2 f_0, \tag{A-3}$$

and for[6] $j = 2, 3, \ldots$

$$(\mathcal{L} - ie_1) f_j = F(y) - ie_j f_0. \tag{A-4}$$

Here

$$\mathcal{L} := \partial_k E_l(\phi'(y))\partial_y + \tfrac{1}{2}\phi''(y)\partial_k^2 E_l(\phi'(y)). \tag{A-5}$$

Since $k \to E_1(k)$ is an even real analytic function, the same is true for $E_1(ik)$, and it follows from (2-3) that

$$E_1(ik) - E_1(0) = -\frac{E_1''(0)}{2}k^2 + \mathcal{O}(k^4). \tag{A-6}$$

[6] F is a function given by the preceding equations depending on $m_0, \ldots, m_{j-1}$ and $m_j^\perp$.

Combining this with the fact that

$$V(y) = V(0) + \frac{V''(0)}{2} y^2 + \mathcal{O}(y^3), \tag{A-7}$$

we deduce that there exists a unique real valued function $\phi_0(y)$ such that $y\phi_0(y) > 0$ for $y \neq 0$ and

$$E_l(i\phi_0'(y)) + V(y) = e_0 = E_l(0) + V(0), \quad \phi'(0) = 0, \tag{A-8}$$

for y small enough. From (A-6) and (A-7), we have

$$\phi_0(y) = \frac{1}{2}\sqrt{\frac{V''(0)}{E_1''(0)}} y^2 + \mathcal{O}(y^3). \tag{A-9}$$

Next, replacing ϕ by $i\phi_0$ in (A-5) and using (A-8), we obtain

$$\mathcal{L} = i(yc_0(y)\partial_y + c_1(y)), \tag{A-10}$$

where $c_0(y)$ and $c_1(y)$ are real valued function with $c_0(0) = \sqrt{E_1''(0)V''(0)}$ and $c_1(0) = \frac{1}{2}\sqrt{E_1''(0)V''(0)}$.

We now turn to Equation (A-2). We look for a solution of (A-2) in the form $f_0(y) = y^m g_0(y)$ with $g_0(0) = 1$. An easy computation shows that

$$(mc_0(y) + c_1(y) - e_1)g_0(y) + yg_0'(y) = 0.$$

Hence $mc_0(0) + c_1(0) - e_1 = 0$, since $g_0(0) = 1$. This gives the allowed values of e_1:

$$e_1 = mc_0(0) + c_1(0) = \sqrt{E_1''(0)V''(0)}\bigl(m + \tfrac{1}{2}\bigr) =: \kappa_m, \quad m \in \mathbb{N}. \tag{A-11}$$

In this case $g_0(y)$ is uniquely determined by

$$(yc_0(y)\partial_y + mc_0(y) + c_1(y) - \lambda_1)g_0(y) = 0, \quad g_0(0) = 1. \tag{A-12}$$

From now on we fix $e_1 = mc_0(0) + c_1(0)$ and $f_0(y) = y^m g_0(y)$ with $g_0(0) = 1$. Let us solve Equation (A-3) for the unknown (e_2, f_1). Applying Taylor's formula to the first term of the right hand side of (A-3) and using (A-10), we see that (e_2, f_1) is a solution of the following equation

$$\bigl(yc_0(y)\partial_y + c_1(y) - e_1\bigr)f_1 = \sum_{j=0}^{m-1} \gamma_j y^j + y^m(k(y) + e_2 g_0(y)). \tag{A-13}$$

Put $f_1(y) = \sum_{j=0}^{m-1} \nu_j y^j + y^m g_1(y)$. Using again Taylor's formula for $c_0(y)$ and $c_1(y)$ at $y = 0$, and equating the coefficients of y^j in both sides of (A-13) we get

$$(c_1(0) - e_1)\nu_0 = \gamma_0, \tag{A-14}$$

and by induction for $j \in \{1, \dots, m-1\}$,

$$\big(jc_0(0) + c_1(0) - e_1\big)\nu_j + F(\nu_0, \dots, \nu_{j-1}) = \gamma_j. \tag{A-15}$$

By (A-11), $jc_0(0) + c_1(0) - e_1 \neq 0$ for $j = 0, 1, 2, \dots, m-1$. Thus, the equations (A-14) and (A-15) uniquely determine ν_j. This gives the polynomial $p_{m-1}(y) := \sum_{j=0}^{m-1} \nu_j y^j$. Similarly, comparing the coefficient of y^m on both sides of (A-13), we see that e_2 is given by

$$e_2 = -k(0) + \sum_{j=1}^{m-1} \frac{(m-j)}{j!} c_0^{(j)}(0)\nu_{m-j} + \frac{1}{m!}\partial_y^m (c_1 p_{m-1})(0),$$

and thus, g_1 satisfies

$$(yc_0(y)\partial_y + mc_0(y) + c_1(y) - e_1)g_1(y) = r(y) + k(y) + e_2 g_0(y). \tag{A-16}$$

Here $r(y)$ only depends on $c_0(y)$, $c_1(y)$ and $p_{m-1}(y)$ with $r(0) + k(0) + e_2 = 0$. Combining this with (A-11) and (A-15) we see that $g_1(y)$ is uniquely determined by adding $g_1(0) = 0$.

We can now proceed analogously to construct (e_j, f_{j-1}) for $j = 2, 3, \dots$. This yields Theorem 5.1.

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