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# On $p$ -adic vanishing cycles of log smooth families

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In this paper, we will prove that the sheaf of  $p$ -adic vanishing cycles on a regular log smooth family is generated by Milnor symbols, assuming that the base dvr contains a primitive  $p$ -th root of unity. Our result generalizes the surjectivity results of Bloch and Kato (*Inst. Hautes Études Sci. Publ. Math.* **63** (1986), 107–152) and Hyodo (*Invent. Math.* **91**:3 (1988), 543–557) to a regular log smooth case.

## 1. Introduction

Let  $K$  be a henselian discrete valuation field of mixed characteristic  $(0, p)$ , with residue field  $k$ . Let  $O_K$  be the ring of integers in  $K$ , and let  $X$  be a regular scheme which is flat of finite type over  $\text{Spec}(O_K)$ . We consider cartesian squares of schemes

$$\begin{array}{ccccc}
 X_K & \hookrightarrow & X & \longleftarrow & X_k \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{Spec}(K) & \hookrightarrow & \text{Spec}(O_K) & \longleftarrow & \text{Spec}(k)
 \end{array}$$

The Kummer short exact sequence of étale sheaves on  $X_K$

$$0 \rightarrow \mu_{p^n} \rightarrow \mathcal{O}_{X_K}^\times \xrightarrow{\times p^n} \mathcal{O}_{X_K}^\times \rightarrow 0$$

yields a long exact sequence of étale sheaves on  $X_k$

$$0 \rightarrow i^* j_* \mu_{p^n} \rightarrow i^* j_* \mathcal{O}_{X_K}^\times \xrightarrow{\times p^n} i^* j_* \mathcal{O}_{X_K}^\times \xrightarrow{\delta} i^* R^1 j_* \mu_{p^n} \rightarrow i^* R^1 j_* \mathcal{O}_{X_K}^\times \rightarrow \dots$$

Since  $X$  is regular, we have  $i^* R^1 j_* \mathcal{O}_{X_K}^\times = 0$  and the connecting map  $\delta$  in this sequence induces an isomorphism

$$i^* R^1 j_* \mu_{p^n} \cong \text{Coker}\left(i^* j_* \mathcal{O}_{X_K}^\times \xrightarrow{\times p^n} i^* j_* \mathcal{O}_{X_K}^\times\right).$$

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A motivation for this note is to extend this fundamental fact to higher cohomological degrees. More precisely, we are concerned with the surjectivity of a geometric version of Tate's norm residue homomorphism

$$\varrho_{X,n}^q : \mathcal{K}_q^M / p^n \rightarrow i^* R^q j_* \mu_{p^n}^{\otimes q},$$

where  $\mathcal{K}_q^M$  denotes a Milnor  $K$ -sheaf defined as a quotient of  $(i^* j_* \mathcal{O}_{X_K}^\times)^{\otimes q}$  and  $\mathcal{K}_q^M / p^n$  denotes the cokernel of the multiplication by  $p^n$  on  $\mathcal{K}_q^M$ , see [Section 4](#) below. The sheaf  $i^* R^q j_* \mu_{p^n}^{\otimes q}$  on the right hand side is the, so called, *sheaf of  $p$ -adic vanishing cycles*, which is an étale sheaf of arithmetic and geometric interest. Bloch and Kato [[1986](#)] proved that the map  $\varrho_{X,n}^q$  is surjective in the case where  $X$  is smooth over  $\mathrm{Spec}(O_K)$ . Later, Hyodo [[1988](#)] extended this surjectivity to the case where  $X$  is a semistable family over  $\mathrm{Spec}(O_K)$ . These surjectivity facts play a fundamental role in a construction of  $p$ -adic period maps in the  $p$ -adic Hodge theory, see [[Kurihara 1987](#); [Kato 1987](#); [1994](#); [Tsuji 1999](#); [2000](#); [Yamashita and Yasuda 2014](#)].

To state our main results more precisely, we introduce the following generalized situation with log poles. Let  $D$  be a normal crossing divisor on  $X$  which is flat over  $\mathrm{Spec}(O_K)$ , and let

$$\psi : U := X - (X_k \cup D) \hookrightarrow X$$

be the natural open immersion. We then have a version of symbol map with log poles

$$\varrho_{(X,D),n}^q : \mathcal{K}_q^M / p^n \rightarrow M_n^q := i^* R^q \psi_* \mu_{p^n}^{\otimes q},$$

where  $\mathcal{K}_q^M$  is again a Milnor  $K$ -sheaf defined as a quotient of  $(i^* \psi_* \mathcal{O}_U^\times)^{\otimes q}$ , see [Section 4](#) below. Now we state a main result of this paper, where quasilog smoothness is a generalization of log-smoothness (see [Definition 5.2](#), [Example 5.3](#)):

**Theorem 1.1** ([Theorem 5.4](#)). *If  $(X, D)$  is quasilog smooth over  $\mathrm{Spec}(O_K)$  and  $K$  contains a primitive  $p$ -th root of unity, then  $\varrho_{(X,D),n}^q$  is surjective for any  $q \geq 2$ .*

Tsuji [[2000](#)] proves an isomorphism in the derived category of étale sheaves on  $X_k$ :

$$\mathcal{S}_n(q)_{(X,D)} \cong \tau_{\leq q} i^* R\psi_* \mu_{p^n}^{\otimes q} \quad (1.2)$$

for quasilog smooth  $(X, D)$  assuming  $0 \leq q \leq p - 2$ , where  $\mathcal{S}_n(q)_{(X,D)}$  denotes a log syntomic complex. His strategy is to show that both hand sides in (1.2) are invariant under log blow-ups, and then to reduce his assertion to the case that  $X$  is smooth over  $\mathrm{Spec}(O_K)$  (and  $D = \emptyset$ ). This last case is due to Kurihara [[1987](#)]. We will prove [Theorem 1.1](#) using his arguments on log blow-ups, which is the first key ingredient of our results. We have to note that [Theorem 1.1](#) does not follow from (1.2). Indeed, it is not clear that the  $q$ -th cohomology sheaf of  $\mathcal{S}_n(q)_{(X,D)}$  is generated by symbols, which is rather a consequence of (1.2) and [Theorem 1.1](#).

To continue the outline of our proof of [Theorem 1.1](#), we introduce a subsheaf  $\mathcal{U}^1 \mathcal{K}_q^M$  of  $\mathcal{K}_q^M$ , which is the subsheaf generated by the image of

$$i^*(1 + \mathcal{I})^\times \otimes (i^* \psi_* \mathcal{O}_U^\times)^{\otimes (q-1)},$$

where  $\mathcal{I}$  denotes the ideal sheaf of  $\mathcal{O}_X$  defining the reduced part  $Y := (X_k)_{\text{red}}$  of  $X_k$ , and  $(1 + \mathcal{I})^\times$  means the kernel of the map  $\mathcal{O}_X^\times \rightarrow i_* \mathcal{O}_Y^\times$ . We will further introduce a multi-index descending filtration on  $\mathcal{U}^1 \mathcal{K}_q^M$ , where the multi-indexes are assigned to irreducible components of  $X_k$ , see [Definition 4.2](#). To investigate the map  $\varrho_{(X,D),n}^q$ , we will need to control the behavior of the sheaf

$$\mathcal{U}^1 M_1^q := \text{Im}(\mathcal{U}^1 \mathcal{K}_q^M \rightarrow M_1^q)$$

and a certain absolute logarithmic differential sheaf  $\widetilde{\omega}_{Y,\log}^q$  under log blow-ups (see [Lemma 5.10](#) below), which corresponds to a key computation by Hyodo [[1988](#), Lemma (3.5)] in the semistable family case. Our second key ingredient is the computations on the multigraded quotients of the induced filtration on  $\mathcal{U}^1 M_1^q$ , which will be carried out by ideas of Kato, who introduced a new Cartier operator on the absolute differential modules with log poles, see Sections 3–4. By this computation on multigraded quotients, the behavior of  $\mathcal{U}^1 M_1^q$  and  $\widetilde{\omega}_{Y,\log}^q$  will be calculated by standard facts on the vanishing of the higher direct image of the structure sheaf under log blow-ups. We would like to mention also that the idea of our computation on multigraded quotients of  $\mathcal{U}^1 \mathcal{K}_q^M$  has been used in a recent joint paper of the first author with Rülling [[Rülling and Saito 2018](#)].

Throughout this paper, we will work with the [Setting 1.4](#) stated below.

**Definition 1.3.** Let  $k$  be a field.

- (1) A *normal crossing variety* over  $k$  is a pure-dimensional scheme  $Y$  which is separated of finite type over  $k$  and everywhere étale locally isomorphic to

$$\text{Spec}(k[T_1, \dots, T_N]/(T_1 \cdots T_a)) \quad \text{for some } 1 \leq a \leq N = \dim(Y) + 1.$$

- (2) An *admissible divisor* on a normal crossing variety  $Y$  is a reduced effective Cartier divisor  $E$  such that the immersion  $E \hookrightarrow Y$  is everywhere étale locally isomorphic to

$$\begin{aligned} \text{Spec}(k[T_1, \dots, T_N]/(T_1 \cdots T_a, T_{a+1} \cdots T_{a+b})) \\ \hookrightarrow \text{Spec}(k[T_1, \dots, T_N]/(T_1 \cdots T_a)) \end{aligned}$$

for some  $a, b \geq 1$  with  $a + b \leq N$ .

Let  $K$  be a henselian discrete valuation field of characteristic 0 whose residue field  $k$  has characteristic  $p > 0$ . Let  $O_K$  be the integer ring of  $K$ . Unless mentioned otherwise, we do *not* assume that  $k$  is perfect. Put  $B := \text{Spec}(O_K)$  and  $s := \text{Spec}(k)$ .

**Setting 1.4.**  $X$  is a regular scheme of finite type over  $B$ , and  $D$  is a reduced divisor on  $X$  which is flat over  $B$  ( $D$  may be empty). We put  $Y := (X \times_B s)_{\text{red}}$  and  $U := X \setminus (Y \cup D)$ , and assume the following two conditions:

- The divisor  $Y \cup D$  has normal crossings on  $X$ .
- $Y$  is a normal crossing variety over  $s$ , and  $(D \times_B s)_{\text{red}}$  is an admissible divisor on  $Y$ .

When  $k$  is perfect, the first condition implies the second condition.

## 2. Absolute differential modules with log poles

Let the notation be as in [Setting 1.4](#), and let  $i$  and  $\psi$  be as follows:

$$Y \xhookrightarrow{i} X \xleftarrow{\psi} U = X \setminus (Y \cup D).$$

Put

$$\mathcal{L} := \psi_* \mathcal{O}_U^\times \cap \mathcal{O}_X \subset \psi_* \mathcal{O}_U,$$

which we regard as a sheaf of commutative monoids by the multiplication of functions. Let

$$\alpha : i^* \mathcal{L} \rightarrow \mathcal{O}_Y$$

be the natural map of étale sheaves, where  $i^*$  denotes the topological inverse image of étale sheaves. In this section, we study the following étale sheaves.

**Definition 2.1.** (1) Let  $\Omega_{Y/\mathbb{Z}}^1$  be the usual absolute Kähler differential sheaf on  $Y_{\text{ét}}$ . We define the étale sheaf  $\tilde{\omega}_Y^1$  on  $Y$  as the quotient sheaf of

$$\Omega_{Y/\mathbb{Z}}^1 \oplus (\mathcal{O}_Y \otimes_{\mathbb{Z}} i^* \psi_* \mathcal{O}_U^\times)$$

divided by the  $\mathcal{O}_Y$ -submodule generated by local sections of the form

$$(d\alpha(x), 0) - (0, \alpha(x) \otimes x) \quad \text{with } x \in i^* \mathcal{L}.$$

There is a logarithmic differential map

$$d \log : i^* \psi_* \mathcal{O}_U^\times \rightarrow \tilde{\omega}_Y^1, \quad x \mapsto (0, 1 \otimes x).$$

Put  $\tilde{\omega}_Y^0 := \mathcal{O}_Y$  and  $\tilde{\omega}_Y^q := \bigwedge_{\mathcal{O}_Y}^q \tilde{\omega}_Y^1$  for  $q \geq 2$ .

- (2) We define  $\tilde{\mathcal{L}}_Y^q$  as the kernel of  $d : \tilde{\omega}_Y^q \rightarrow \tilde{\omega}_Y^{q+1}$ , and  $\tilde{\mathcal{B}}_Y^q$  as the image of  $d : \tilde{\omega}_Y^{q-1} \rightarrow \tilde{\omega}_Y^q$ , respectively, and put

$$\tilde{\omega}_{Y, \log}^q := \text{Im}(d \log : (i^* \psi_* \mathcal{O}_U^\times)^{\otimes q} \rightarrow \tilde{\omega}_Y^q).$$

**Remark 2.2.** The natural map  $\mathcal{L} \rightarrow \mathcal{O}_X$  gives a log structure on  $X$  in the sense of [\[Kato 1989\]](#). In terms of log schemes, the sheaf  $\tilde{\omega}_Y^1$  means the differential module

$\omega_{(Y,L)/\mathbb{Z}}^1$  defined in [loc. cit., (1.7)], where  $L$  denotes the inverse image log structure of  $\mathcal{L}$  onto  $Y$  [loc. cit., (1.4)].

**Theorem 2.3.** (1) *The sheaf  $\tilde{\omega}_Y^q$  is locally free over  $\mathcal{O}_Y$ .*

(2) *There is a unique isomorphism*

$$C^{-1} : \tilde{\omega}_Y^q \xrightarrow{\cong} \mathcal{H}^q(\tilde{\omega}_Y) = \tilde{\mathcal{L}}_Y^q / \tilde{\mathcal{B}}_Y^q$$

*sending a local section  $x \cdot d \log(y_1) \wedge \cdots \wedge d \log(y_q)$  with  $x \in \mathcal{O}_Y$  and each  $y_i \in i^* \psi_* \mathcal{O}_U^\times$ , to  $x^p \cdot d \log(y_1) \wedge \cdots \wedge d \log(y_q) + \tilde{\mathcal{B}}_Y^q$ .*

(3) *There is a short exact sequence on  $Y_{\text{ét}}$*

$$0 \longrightarrow \tilde{\omega}_{Y,\log}^q \longrightarrow \tilde{\mathcal{L}}_Y^q \xrightarrow{1-C^{-1}} \mathcal{H}^q(\tilde{\omega}_Y) \longrightarrow 0.$$

*Proof.* We first reduce the problem to the case that  $X$  is a regular semistable family over  $B = \text{Spec}(O_K)$ . Note that we may work étale locally. Indeed, once we prove (2) étale locally, then the isomorphisms  $C^{-1}$  patch together automatically by the uniqueness. Assume the following three conditions (see [Setting 1.4](#)):

- *$X$  is affine, and  $Y = \text{Spec}(k[T_1, \dots, T_d]/(T_1 \cdots T_a))$  for  $1 \leq \exists a \leq d = \dim(X)$ .*
- *The irreducible components of  $D$  are regular and principal.*
- *There exists a regular sequence  $t_1, \dots, t_d$  of prime elements of  $\Gamma(X, \mathcal{O}_X)$  such that  $t_\lambda$  lifts  $T_\lambda \in \Gamma(Y, \mathcal{O}_Y)$  for  $1 \leq \forall \lambda \leq d$  and such that  $t_{a+1}, \dots, t_{a+b}$  are uniformizers of the irreducible components of  $D$  for  $0 \leq \exists b \leq d - a$ .*

Let  $\pi$  be a prime element of  $O_K$ . We have

$$\pi = u t_1^{e_1} t_2^{e_2} \cdots t_a^{e_a} \tag{2.4}$$

for some  $u \in \Gamma(X, \mathcal{O}_X^\times)$  and some  $e_1, \dots, e_a \geq 1$ . Put

$$X' := \text{Spec}(O_K[S_1, \dots, S_d]/(S_1 \cdots S_a - \pi)),$$

$$Y' := (X')_s = \text{Spec}(k[S_1, \dots, S_d]/(S_1 \cdots S_a)),$$

$$U' := \text{Spec}(K[S_1, \dots, S_d, S_{a+1}^{-1}, \dots, S_{a+b}^{-1}]/(S_1 \cdots S_a - \pi)).$$

Let  $\psi'$  be the open immersion  $U' \hookrightarrow X'$  and  $i'$  the closed immersion  $Y' \hookrightarrow X'$ , and let  $\beta$  be the isomorphism of schemes

$$\beta : Y \xrightarrow{\cong} Y', \quad S_\lambda \mapsto T_\lambda \quad (1 \leq \lambda \leq d).$$

Put  $\mathcal{K} := \text{Ker}(\mathcal{O}_X^\times \rightarrow \mathcal{O}_Y^\times)$  and  $\mathcal{K}' := \text{Ker}(\mathcal{O}_{X'}^\times \rightarrow \mathcal{O}_{Y'}^\times)$ . By (2.4),  $\psi_* \mathcal{O}_U^\times / \mathcal{O}_X^\times$  is a free abelian sheaf generated by  $t_1, \dots, t_{a+b}$ . Similarly  $\psi'_* \mathcal{O}_{U'}^\times / \mathcal{O}_{X'}^\times$  is a free abelian sheaf generated by  $S_1, \dots, S_{a+b}$ . Hence there is an isomorphism of sheaves on  $Y_{\text{ét}}$

$$\beta^*(i'^* \psi'_* \mathcal{O}_{U'}^\times / \mathcal{K}') \xrightarrow{\cong} i^* \psi_* \mathcal{O}_U^\times / \mathcal{K}$$

that extends the isomorphism  $\beta^* \mathcal{O}_{Y'}^\times \xrightarrow{\cong} \mathcal{O}_Y^\times$  and sends  $S_\lambda \mapsto t_\lambda$  for  $1 \leq \lambda \leq a + b$ . By this isomorphism we see that  $\beta^* \tilde{\omega}_Y^q \cong \tilde{\omega}_Y^q$ . Thus we are reduced to the case that  $X$  is a regular semistable family over  $B$ .

We assume that  $X$  is a regular semistable family over  $B$  in what follows. Let  $\omega_Y^q$  be the cokernel of the map

$$\tilde{\omega}_Y^{q-1} \rightarrow \tilde{\omega}_Y^q, \quad x \mapsto d \log(\pi) \wedge x.$$

We have a short exact sequence of complexes

$$0 \longrightarrow \omega_Y^\bullet[-1] \xrightarrow{d \log(\pi) \wedge} \tilde{\omega}_Y^\bullet \longrightarrow \omega_Y^\bullet \longrightarrow 0 \tag{2.5}$$

and a short exact sequence of the  $q$ -th cohomology sheaves for any  $q \geq 0$

$$0 \longrightarrow \mathcal{H}^{q-1}(\omega_Y^\bullet) \xrightarrow{d \log(\pi) \wedge} \mathcal{H}^q(\tilde{\omega}_Y^\bullet) \longrightarrow \mathcal{H}^q(\omega_Y^\bullet) \longrightarrow 0, \tag{2.6}$$

see [Tsuji 2000, Lemma A.7] with  $m = 0$ . We recall here the following facts due to Tsuji [2000, Theorems A.3 and A.4] (see [Kato 1989, Proposition (3.10), Theorem (4.12)(1)]):

**Fact 2.7.** (1)  $\omega_Y^q$  is a locally free  $\mathcal{O}_Y$ -module, and there is a unique isomorphism

$$C^{-1} : \omega_Y^q \xrightarrow{\cong} \mathcal{H}^q(\omega_Y^\bullet)$$

sending a local section of the form  $x \cdot d \log(y_1) \wedge \cdots \wedge d \log(y_q)$  with  $x \in \mathcal{O}_Y$  and each  $y_i \in i^* \psi_* \mathcal{O}_U^\times$ , to  $x^p \cdot d \log(y_1) \wedge \cdots \wedge d \log(y_q) + d \omega_Y^{q-1}$ .

(2) Let  $V$  be an open subset of  $Y$  which is smooth over  $k$  and for which  $D \times_X V$  is empty. Then the short exact sequence (2.5) splits on  $V$  as complexes. Consequently the exact sequence (2.6) splits on  $V$ , i.e., we have

$$\mathcal{H}^q(\tilde{\omega}_V^\bullet) \xrightarrow{\cong} \mathcal{H}^q(\Omega_V^\bullet) \oplus \mathcal{H}^{q-1}(\Omega_V^\bullet).$$

(3) There is a short exact sequence on  $Y_{\text{ét}}$

$$0 \longrightarrow \omega_{Y, \log}^q \longrightarrow \mathcal{X}_Y^q \xrightarrow{1-C^{-1}} \mathcal{H}^q(\omega_Y^\bullet) \longrightarrow 0,$$

where  $\omega_{Y, \log}^q$  is defined as  $\text{Im}(d \log : (i^* \psi_* \mathcal{O}_U^\times)^{\otimes q} \rightarrow \omega_Y^q)$ .

Theorem 2.3(1) follows from (2.5) and Fact 2.7(1). We prove Theorem 2.3(2). Let  $V$  be a dense open subset of  $Y$  which is smooth over  $k$  and for which  $D \times_X V$  is empty. Let  $\sigma$  be the open immersion  $V \hookrightarrow Y$ . We first show that the canonical adjunction map

$$\mathcal{H}^q(\tilde{\omega}_Y^\bullet) \rightarrow \sigma_* \sigma^* \mathcal{H}^q(\tilde{\omega}_Y^\bullet) \cong \sigma_* (\mathcal{H}^q(\Omega_V^\bullet) \oplus \mathcal{H}^{q-1}(\Omega_V^\bullet)) \tag{2.8}$$

is injective. Indeed by (2.6) there is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{H}^{q-1}(\omega_Y^\bullet) & \xrightarrow{d \log(\pi)^\wedge} & \mathcal{H}^q(\tilde{\omega}_Y^\bullet) & \longrightarrow & \mathcal{H}^q(\omega_Y^\bullet) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \sigma_* \mathcal{H}^{q-1}(\Omega_Y^\bullet) & \xrightarrow{d \log(\pi)^\wedge} & \sigma_* \mathcal{H}^q(\tilde{\omega}_Y^\bullet) & \longrightarrow & \sigma_* \mathcal{H}^q(\Omega_Y^\bullet) \longrightarrow 0
 \end{array}$$

(2.8)

where the vertical arrows are adjunction maps. The left and right vertical arrows are injective by Fact 2.7(1). Hence the map (2.8) is injective. We define the map  $C^{-1} : \tilde{\omega}_Y^q \rightarrow \mathcal{H}^q(\tilde{\omega}_Y^\bullet)$  as follows. Using differential symbols, we easily see that the image of the composite map

$$\tilde{\omega}_Y^q \longrightarrow \sigma_*(\Omega_Y^q \oplus \Omega_Y^{q-1}) \xrightarrow{C^{-1}} \sigma_*(\mathcal{H}^q(\Omega_Y^\bullet) \oplus \mathcal{H}^{q-1}(\Omega_Y^\bullet)),$$

is contained in  $\mathcal{H}^q(\tilde{\omega}_Y^\bullet)$ . We thus obtain the map  $C^{-1} : \tilde{\omega}_Y^q \rightarrow \mathcal{H}^q(\tilde{\omega}_Y^\bullet)$ . By the construction,  $C^{-1}$  is described by the local assignment as in Theorem 2.3(2), which implies the uniqueness of  $C^{-1}$ . Moreover it is bijective by the following commutative diagram with exact rows and Fact 2.7(1):

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \omega_Y^{q-1} & \xrightarrow{d \log(\pi)^\wedge} & \tilde{\omega}_Y^q & \longrightarrow & \omega_Y^q \longrightarrow 0 \\
 & & \downarrow C^{-1} \wr & & \downarrow C^{-1} & & \downarrow C^{-1} \wr \\
 0 & \longrightarrow & \mathcal{H}^{q-1}(\omega_Y^\bullet) & \xrightarrow{d \log(\pi)^\wedge} & \mathcal{H}^q(\tilde{\omega}_Y^\bullet) & \longrightarrow & \mathcal{H}^q(\omega_Y^\bullet) \longrightarrow 0
 \end{array}$$

This completes the proof of Theorem 2.3(2).

We prove Theorem 2.3(3). By the local presentation of  $C^{-1}$ , it is easy to see that the map  $1 - C^{-1} : \tilde{\mathcal{L}}_Y^q \rightarrow \mathcal{H}^q(\tilde{\omega}_Y^\bullet)$  is surjective and that its kernel contains  $\tilde{\omega}_{Y, \log}^q$ . Put

$$L := \text{Ker}(1 - C^{-1} : \tilde{\mathcal{L}}_Y^q \rightarrow \mathcal{H}^q(\tilde{\omega}_Y^\bullet)).$$

We show that the natural inclusion  $\tilde{\omega}_{Y, \log}^q \hookrightarrow L$  is surjective. By (2.5), (2.6) and Fact 2.7(1), there is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \tilde{\mathcal{L}}_Y^{q-1} & \xrightarrow{d \log(\pi)^\wedge} & \tilde{\mathcal{L}}_Y^q & \longrightarrow & \mathcal{L}_Y^q \longrightarrow 0 \\
 & & \downarrow 1 - C^{-1} & & \downarrow 1 - C^{-1} & & \downarrow 1 - C^{-1} \\
 0 & \longrightarrow & \mathcal{H}^{q-1}(\omega_Y^\bullet) & \xrightarrow{d \log(\pi)^\wedge} & \mathcal{H}^q(\tilde{\omega}_Y^\bullet) & \longrightarrow & \mathcal{H}^q(\omega_Y^\bullet) \longrightarrow 0
 \end{array}$$

By this diagram and [Fact 2.7\(3\)](#), the lower row of the following commutative diagram of complexes is exact:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \omega_{Y,\log}^{q-1} & \xrightarrow{d \log(\pi) \wedge} & \tilde{\omega}_{Y,\log}^q & \longrightarrow & \omega_{Y,\log}^q \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \parallel \\
 0 & \longrightarrow & \omega_{Y,\log}^{q-1} & \xrightarrow{d \log(\pi) \wedge} & L & \longrightarrow & \omega_{Y,\log}^q \longrightarrow 0
 \end{array}$$

Hence the middle vertical arrow is surjective, and we obtain [Theorem 2.3\(3\)](#).  $\square$

### 3. Another Cartier isomorphism

Let the notation be as in [Setting 1.4](#). Let  $i$  and  $\psi$  be as follows:

$$Y \hookrightarrow X \xleftarrow{\psi} U = X \setminus (Y \cup D).$$

Let  $\{Y_\lambda\}_{\lambda \in \Lambda}$  be the irreducible components of  $Y$ . For  $\lambda \in \Lambda$ , let  $\mathcal{I}_\lambda \subset \mathcal{O}_X$  be the defining ideal of  $Y_\lambda$ . For  $\mathbf{m} = (m_\lambda)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda$ , put

$$\mathcal{I}^{(\mathbf{m})} := \prod_{\lambda \in \Lambda} \mathcal{I}_\lambda^{m_\lambda} \subset \mathcal{O}_X,$$

where  $\mathbb{N}$  denotes the set of natural numbers  $\{0, 1, 2, \dots\}$ .

**Definition 3.1.** (1) We endow  $\mathbb{N}^\Lambda$  with a semiorder as follows. For  $\mathbf{m} = (m_\lambda)_{\lambda \in \Lambda}$  and  $\mathbf{n} = (n_\lambda)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda$ , we say that  $\mathbf{m} \leq \mathbf{n}$  if  $m_\lambda \leq n_\lambda$  for all  $\lambda \in \Lambda$ .

(2) We put  $\mathbf{0} := (0)_{\lambda \in \Lambda}$  and  $\mathbf{1} := (1)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda$ .

(3) For  $\mathbf{m}, \mathbf{l} \in \mathbb{N}^\Lambda$ , we define a sheaf  $\omega_{\mathbf{m},\mathbf{l}}^q$  ( $q \geq 0$ ) on  $Y_{\text{ét}}$  as

$$\omega_{\mathbf{m},\mathbf{l}}^q := \mathcal{I}^{(\mathbf{m})} / \mathcal{I}^{(\mathbf{m}+\mathbf{l})} \otimes_{\mathcal{O}_X} \tilde{\omega}_Y^q,$$

which is a locally free module over  $\mathcal{O}_X / \mathcal{I}^{(\mathbf{l})}$  if  $\mathbf{l} \leq \mathbf{1}$ . We define a map  $d_{\mathbf{m}}^q : \mathcal{I}^{(\mathbf{m})} \otimes_{\mathcal{O}_X} \tilde{\omega}_Y^q \rightarrow \mathcal{I}^{(\mathbf{m})} \otimes_{\mathcal{O}_X} \tilde{\omega}_Y^{q+1}$  by the local assignment

$$\prod_{\lambda \in \Lambda} \pi_\lambda^{m_\lambda} \otimes \omega \mapsto \prod_{\lambda \in \Lambda} \pi_\lambda^{m_\lambda} \otimes \left( d\omega + \sum_{\lambda \in \Lambda} m_\lambda \cdot d \log(\pi_\lambda) \wedge \omega \right) \quad (\omega \in \tilde{\omega}_Y^q),$$

where  $\pi_\lambda \in \mathcal{O}_X$  denotes a local uniformizer of  $Y_\lambda$  for each  $\lambda \in \Lambda$ . This map does not depend on the choice of local uniformizers  $\{\pi_\lambda\}_{\lambda \in \Lambda}$ . One can easily check that  $d_{\mathbf{m}}^{q+1} \circ d_{\mathbf{m}}^q = 0$  and that  $d_{\mathbf{n}}^q$  is compatible with  $d_{\mathbf{m}}^q$  for  $\mathbf{n} \geq \mathbf{m}$ . Hence  $d_{\mathbf{m}}^q$  induces a differential operator

$$d = d_{\mathbf{m},\mathbf{l}}^q : \omega_{\mathbf{m},\mathbf{l}}^q \rightarrow \omega_{\mathbf{m},\mathbf{l}}^{q+1}.$$

Using this  $d$ , we regard  $\omega_{\mathbf{m},\mathbf{l}}^\bullet = (\omega_{\mathbf{m},\mathbf{l}}^\bullet, d)$  as a complex.

- (4) We define  $\mathcal{Z}_{\mathfrak{m},\mathfrak{l}}^q$  (resp.  $\mathcal{B}_{\mathfrak{m},\mathfrak{l}}^q$ ) as the kernel of  $d : \omega_{\mathfrak{m},\mathfrak{l}}^q \rightarrow \omega_{\mathfrak{m},\mathfrak{l}}^{q+1}$  (resp. the image of  $d : \omega_{\mathfrak{m},\mathfrak{l}}^{q-1} \rightarrow \omega_{\mathfrak{m},\mathfrak{l}}^q$ ), which are étale subsheaves of  $\omega_{\mathfrak{m},\mathfrak{l}}^q$ .

The following result is due to Kato:

**Theorem 3.2.** *Let  $\mathfrak{m} = (m_\lambda)_{\lambda \in \Lambda}$  and  $\mathfrak{l} = (\ell_\lambda)_{\lambda \in \Lambda}$  be elements of  $\mathbb{N}^\Lambda$  with  $\mathbf{0} \leq \mathfrak{l} \leq \mathbf{1}$ . Let  $\mathfrak{m}' \in \mathbb{N}^\Lambda$  be the smallest element that satisfies  $p \cdot \mathfrak{m}' \geq \mathfrak{m}$ , and define  $\mathfrak{l}' = (\ell'_\lambda)_{\lambda \in \Lambda}$  by*

$$\ell'_\lambda := \begin{cases} 1 & \text{if } \ell_\lambda = 1 \text{ and } p \mid m_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Then there is an isomorphism

$$C^{-1} : \omega_{\mathfrak{m}',\mathfrak{l}'}^q \xrightarrow{\cong} \mathcal{H}^q(\omega_{\mathfrak{m},\mathfrak{l}}^\bullet) = \mathcal{Z}_{\mathfrak{m},\mathfrak{l}}^q / \mathcal{B}_{\mathfrak{m},\mathfrak{l}}^q, \quad x \otimes \omega \mapsto x^p \otimes \omega,$$

where  $x$  (resp.  $\omega$ ) denotes a local section of  $\mathcal{S}^{(\mathfrak{m})}$  (resp.  $\tilde{\omega}_{Y,\log}^q$ ).

We first state an immediate consequence of this theorem. For  $\mu \in \Lambda$ , put  $\tilde{\omega}_{Y_\mu}^q := \mathcal{O}_{Y_\mu} \otimes_{\mathcal{O}_Y} \tilde{\omega}_Y^q$  and define  $\mathfrak{d}_\mu = (\delta_{\mu,\lambda})_{\lambda \in \Lambda}$  by

$$\delta_{\mu,\lambda} := \begin{cases} 1 & (\lambda = \mu), \\ 0 & (\lambda \neq \mu). \end{cases}$$

Then we have  $\omega_{\mathfrak{m},\mathfrak{d}_\mu}^q = \mathcal{S}^{(\mathfrak{m})} \otimes_{\mathcal{O}_X} \tilde{\omega}_{Y_\mu}^q$ , and [Theorem 3.2](#) implies the following:

- Corollary 3.3.** (1) *The complex  $\omega_{\mathfrak{m},\mathfrak{d}_\mu}^\bullet$  is acyclic (i.e., exact) if  $p \nmid m_\mu$ .*  
(2) *If  $p \mid m_\mu$ , then we have an isomorphism*

$$C^{-1} : \omega_{\mathfrak{m}',\mathfrak{d}_\mu}^q \xrightarrow{\cong} \mathcal{H}^q(\omega_{\mathfrak{m},\mathfrak{d}_\mu}^\bullet),$$

where  $\mathfrak{m}' \in \mathbb{N}^\Lambda$  is the smallest element satisfying  $p \cdot \mathfrak{m}' \geq \mathfrak{m}$ .

*Proof of [Theorem 3.2](#).* Let  $\pi_\lambda \in \mathcal{O}_X$  be a local uniformizer of  $Y_\lambda$  for each  $\lambda \in \Lambda$ . If  $p$  divides  $m_\lambda$  for any  $\lambda \in \Lambda$ , then the map  $d : \omega_{\mathfrak{m},\mathfrak{l}}^q \rightarrow \omega_{\mathfrak{m},\mathfrak{l}}^{q+1}$  sends

$$\prod_{\lambda \in \Lambda} \pi_\lambda^{m_\lambda} \otimes \omega \quad \text{to} \quad \prod_{\lambda \in \Lambda} \pi_\lambda^{m_\lambda} \otimes d\omega,$$

and the assertion follows from [Theorem 2.3\(2\)](#).

We prove the general case. Take a sequence of elements of  $\mathbb{N}^\Lambda$

$$\mathfrak{m} = \mathfrak{m}_0 \leq \mathfrak{m}_1 \leq \mathfrak{m}_2 \leq \cdots \leq \mathfrak{m}_t = p \cdot \mathfrak{m}'$$

such that

$$\sum_{\lambda \in \Lambda} m_{i+1,\lambda} - \sum_{\lambda \in \Lambda} m_{i,\lambda} = 1 \quad \text{for } 0 \leq i < t,$$

where  $\mathfrak{m}_i = (m_{i,\lambda})_{\lambda \in \Lambda}$  and  $\mathfrak{m}_{i+1} = (m_{i+1,\lambda})_{\lambda \in \Lambda}$ . For  $0 \leq i \leq t$ , define  $\mathfrak{l}_i = (\ell_{i,\lambda})_{\lambda \in \Lambda}$  by

$$\ell_{i,\lambda} := \begin{cases} 1 & \text{if } \ell_\lambda = 1 \text{ and } m_{i,\lambda} = m_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

We have inclusions of complexes

$$\omega_{m,t}^\bullet = \omega_{m_0,l_0}^\bullet \supset \omega_{m_1,l_1}^\bullet \supset \omega_{m_2,l_2}^\bullet \supset \cdots \supset \omega_{m_t,l_t}^\bullet = \omega_{p \cdot m',l'}^\bullet,$$

and exact sequences ( $0 \leq i < t$ )

$$0 \rightarrow \omega_{m_{i+1},l_{i+1}}^\bullet \rightarrow \omega_{m_i,l_i}^\bullet \rightarrow \omega_{m_i,\partial_\mu}^\bullet / \mathcal{S}^{(l_i)} \omega_{m_i,\partial_\mu}^\bullet \rightarrow 0,$$

where  $\mu = \mu(i)$  is the unique element of  $\Lambda$  such that  $m_{i+1,\mu} > m_{i,\mu}$  and  $\partial_\mu$  is as we defined before [Corollary 3.3](#). It is enough to show the following two assertions:

(A) *The complex  $\omega_{m_i,\partial_\mu}^\bullet / \mathcal{S}^{(l_i)} \omega_{m_i,\partial_\mu}^\bullet$  is acyclic for  $0 \leq i < t$ .*

(B) *There is an isomorphism  $C^{-1} : \omega_{m',l'}^q \xrightarrow{\cong} \mathcal{H}^q(\omega_{p \cdot m',l'}^\bullet)$ .*

The assertion (B) follows from the proved case of the theorem. Because  $\mathcal{S}^{(l_i)}$  is locally free over  $\mathcal{O}_X$  for any  $0 \leq i < t$ , the assertion (A) is reduced to the following:

**Lemma 3.4.** *Let  $\mu \in \Lambda$  and assume  $p \nmid m_\mu$ . Then the complex  $\omega_{m,\partial_\mu}^\bullet$  is acyclic.*

We prove this lemma in what follows. Let  $\tilde{\omega}_{Y_\mu}^q$  be as before [Corollary 3.3](#). Note that  $\tilde{\omega}_{Y_\mu}^q$  is generated by  $\Omega_{Y_\mu}^q$  (usual Kähler  $q$ -forms) and  $q$ -forms of the form  $d \log(\pi_\lambda) \wedge \eta$  with  $\lambda \in \Lambda$  and  $\eta \in \Omega_{Y_\mu}^{q-1}$ . For  $q \geq 1$ , there is a residue homomorphism

$$\text{Res}^q : \tilde{\omega}_{Y_\mu}^q \rightarrow \tilde{\omega}_{Y_\mu}^{q-1}$$

characterized by the following two properties:

(1) *For  $\omega \in \Omega_{Y_\mu}^q$ ,  $\text{Res}^q(\omega)$  is zero.*

(2) *For  $\eta \in \Omega_{Y_\mu}^{q-1}$ , we have*

$$\text{Res}^q(d \log(\pi_\lambda) \wedge \eta) = \begin{cases} \eta & (\lambda = \mu), \\ 0 & (\lambda \neq \mu). \end{cases}$$

Since  $\omega_{m,\partial_\mu}^q = \mathcal{S}^{(m)} \otimes_{\mathcal{O}_X} \tilde{\omega}_{Y_\mu}^q$ , we define a residue homomorphism

$$\text{Res}^q : \omega_{m,\partial_\mu}^q \rightarrow \omega_{m,\partial_\mu}^{q-1}$$

by  $\text{Res}^q(a \otimes \omega) := a \otimes \text{Res}^q(\omega)$  for  $a \in \mathcal{S}^{(m)}$  and  $\omega \in \tilde{\omega}_{Y_\mu}^q$ . We show that

$$d \text{Res}^q(x) + \text{Res}^{q+1}(dx) = m_\mu \cdot x \quad \text{for any } x \in \omega_{m,\partial_\mu}^q, \tag{3.5}$$

which implies that  $\omega_{m,\partial_\mu}^\bullet$  is acyclic if  $p \nmid m_\mu$ . Put

$$\xi^m := \prod_{\lambda \in \Lambda} \pi_\lambda^{m_\lambda} \in \prod_{\lambda \in \Lambda} \mathcal{S}_\lambda^{m_\lambda} = \mathcal{S}^{(m)}.$$

For  $x = \xi^m \otimes \omega$  with  $\omega \in \Omega_{Y_\mu}^q$ , we have  $\text{Res}^q(x) = 0$  and

$$\begin{aligned} \text{Res}^{q+1}(dx) &= \text{Res}^{q+1}(\xi^m \otimes (d\omega + \sum_{\lambda \in \Lambda} m_\lambda \cdot d \log(\pi_\lambda) \wedge \omega)) \\ &= \xi^m \otimes m_\mu \cdot \omega = m_\mu \cdot x. \end{aligned}$$

For  $x = \xi^m \otimes d \log(\pi_\nu) \wedge \eta$  with  $\eta \in \Omega_{Y_\mu}^{q-1}$  and  $\nu \neq \mu$ , we have  $\text{Res}^q(x) = 0$  and

$$\begin{aligned} \text{Res}^{q+1}(dx) &= \text{Res}^{q+1}(\xi^m \otimes (-d \log(\pi_\nu) \wedge d\eta + \sum_{\lambda \in \Lambda} m_\lambda \cdot d \log(\pi_\lambda) \wedge d \log(\pi_\nu) \wedge \eta)) \\ &= \xi^m \otimes m_\mu \cdot d \log(\pi_\nu) \wedge \eta = m_\mu \cdot x. \end{aligned}$$

Finally for  $x = \xi^m \otimes d \log(\pi_\mu) \wedge \eta$  with  $\eta \in \Omega_{Y_\mu}^{q-1}$ , we have

$$d \text{Res}^q(x) = d(\xi^m \otimes \eta) = \xi^m \otimes (d\eta + \sum_{\lambda \in \Lambda} m_\lambda \cdot d \log(\pi_\lambda) \wedge \eta)$$

and

$$\begin{aligned} \text{Res}^{q+1}(dx) &= \text{Res}^{q+1}(\xi^m \otimes (-d \log(\pi_\mu) \wedge d\eta + \sum_{\lambda \in \Lambda} m_\lambda \cdot d \log(\pi_\lambda) \wedge d \log(\pi_\mu) \wedge \eta)) \\ &= \xi^m \otimes (-d\eta - \sum_{\lambda \neq \mu} m_\lambda \cdot d \log(\pi_\lambda) \wedge \eta) = -d \text{Res}^q(x) + m_\mu \cdot x. \end{aligned}$$

Thus we obtain (3.5), Lemma 3.4 and Theorem 3.2.  $\square$

**Corollary 3.6.**  $\mathcal{L}_{m,1}^q$  is generated by local sections of the following forms:

- (1)  $\prod_{\lambda \in \Lambda} \pi_\lambda^{m_\lambda} \otimes (d\eta + \sum_{\lambda \in \Lambda} m_\lambda \cdot d \log(\pi_\lambda) \wedge \eta)$  with  $\eta \in \widetilde{\omega}_Y^{q-1}$ , where  $\pi_\lambda \in \mathcal{O}_X$  is a local uniformizer of  $Y_\lambda$  for each  $\lambda \in \Lambda$ .
- (2)  $x^p \otimes \omega$  with  $x \in \mathcal{I}^{(m')}$  and  $\omega \in \widetilde{\omega}_{Y, \log}^q$ , where  $m' \in \mathbb{N}^\Lambda$  is the smallest element satisfying  $p \cdot m' \geq m$ .

#### 4. Structure of the sheaf $\mathcal{U}^1 M_1^q$

Let the notation be as in Setting 1.4. Let  $i$  and  $\psi$  be as follows:

$$Y \xrightarrow{i} X \xleftarrow{\psi} U = X \setminus (Y \cup D).$$

For  $q \geq 0$  and  $n \geq 1$ , we define étale sheaves  $M_n^q$  and  $\mathcal{K}_q^M$  on  $Y$  as

$$M_n^q := i^* R^q \psi_* \mu_{p^n}^{\otimes q} \quad \text{and} \quad \mathcal{K}_q^M := \begin{cases} \mathbb{Z} & (q = 0), \\ i^* \psi_* \mathcal{O}_U^\times & (q = 1), \\ (i^* \psi_* \mathcal{O}_U^\times)^{\otimes q} / J_q & (q \geq 2). \end{cases}$$

Here  $J_q$  for  $q \geq 2$  denotes the subsheaf of  $(i^* \psi_* \mathcal{O}_U^\times)^{\otimes q}$  generated by local sections  $x_1 \otimes \cdots \otimes x_q$  such that  $x_r + x_{r'} = 0$  or 1 for some  $1 \leq r < r' \leq q$ . There is a homomorphism of étale sheaves (see [Bloch and Kato 1986, 1.2])

$$\varrho_{(X,D),n}^q : \mathcal{K}_q^M / p^n \rightarrow M_n^q, \quad (4.1)$$

which is a geometric version of Tate's norm residue map. For  $x_1, \dots, x_q \in i^* \psi_* \mathcal{O}_U^\times$ , we denote the image of  $\{x_1, x_2, \dots, x_q\} \in \mathcal{K}_q^M$  under (4.1) again by  $\{x_1, x_2, \dots, x_q\}$ .

**Definition 4.2.** (1) We define  $\mathcal{U}^0\mathcal{K}_q^M$  as the full-sheaf  $\mathcal{K}_q^M$ , and  $\mathcal{U}^1\mathcal{K}_q^M$  as the subsheaf generated locally by symbols of the form

$$\{1 + x, y_1, \dots, y_{q-1}\} \quad \text{with } x \in i^*\mathcal{I} \text{ and } y_1, \dots, y_{q-1} \in i^*\psi_*\mathcal{O}_U^\times,$$

where  $\mathcal{I} \subset \mathcal{O}_X$  denotes the defining ideal of  $Y$ . We define  $\mathcal{U}^0M_n^q$  and  $\mathcal{U}^1M_n^q$  as the image of  $\mathcal{U}^0\mathcal{K}_q^M$  and  $\mathcal{U}^1\mathcal{K}_q^M$  under the map (4.1), respectively.

(2) For  $\mathfrak{m} = (m_\lambda)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda$  with  $\mathfrak{m} \geq \mathbf{1}$ , we define  $\mathcal{U}^{(\mathfrak{m})}\mathcal{K}_q^M$  as the subsheaf generated locally by symbols of the form

$$\{1 + x, y_1, \dots, y_{q-1}\} \quad \text{with } x \in i^*\mathcal{I}^{(\mathfrak{m})} \text{ and } y_1, \dots, y_{q-1} \in i^*\psi_*\mathcal{O}_U^\times,$$

where  $\mathcal{I}^{(\mathfrak{m})}$  is as we defined in the previous section. We define  $\mathcal{U}^{(\mathfrak{m})}M_n^q$  as the image of  $\mathcal{U}^{(\mathfrak{m})}\mathcal{K}_q^M$  under the map (4.1).

(3) We define  $\mathfrak{e} = (e_\lambda)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda$  as follows. For  $\lambda \in \Lambda$ , let  $e_\lambda$  be the absolute ramification index of the discrete valuation ring  $\mathcal{O}_{X, \eta_\lambda}$ , where  $\eta_\lambda$  denotes the generic point of  $Y_\lambda$ . We put  $e'_\lambda := pe_\lambda/(p-1)$  for  $\lambda \in \Lambda$  and  $\mathfrak{e}' := (e'_\lambda)_{\lambda \in \Lambda} \in \mathbb{Q}^\Lambda$ .

(4) For  $\mathfrak{m} = (m_\lambda)_{\lambda \in \Lambda}$  and  $\mathfrak{n} = (n_\lambda)_{\lambda \in \Lambda} \in \mathbb{Q}^\Lambda$ , we say that  $\mathfrak{m} < \mathfrak{n}$  (resp.  $\mathfrak{m} \leq \mathfrak{n}$ ) if  $m_\lambda < n_\lambda$  (resp.  $m_\lambda \leq n_\lambda$ ) for any  $\lambda \in \Lambda$ .

The following lemma is straight-forward, and left to the reader:

**Lemma 4.3.** (1) We have  $\mathcal{U}^1M_n^q = \mathcal{U}^{(\mathbf{1})}M_n^q$ , where  $\mathbf{1}$  denotes  $(1)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda$ .

(2)  $M_n^q$ ,  $\mathcal{U}^0M_n^q$  and  $\mathcal{U}^1M_n^q$  are contravariantly functorial in the pair  $(X, D)$ .

The main result of this section is the following, which is also due to Kato:

**Theorem 4.4.** Let  $\mathfrak{m}$  and  $\mathfrak{l}$  be elements of  $\mathbb{N}^\Lambda$ .

(1) Assume  $\mathbf{1} \leq \mathfrak{m} < \mathfrak{e}' + \mathbf{1}$  and  $\mathbf{0} \leq \mathfrak{l} \leq \mathbf{1}$ . For each  $\lambda \in \Lambda$ , assume  $\ell_\lambda = 0$  if  $m_\lambda \geq e'_\lambda$ . Then there is an isomorphism

$$\omega_{\mathfrak{m}, \mathfrak{l}}^{q-1} / \mathcal{L}_{\mathfrak{m}, \mathfrak{l}}^{q-1} \xrightarrow{\cong} \mathcal{U}^{(\mathfrak{m})}M_1^q / \mathcal{U}^{(\mathfrak{m}+\mathfrak{l})}M_1^q$$

given by the local assignment

$$x \otimes d \log(y_1) \wedge \cdots \wedge d \log(y_{q-1}) \mapsto \{1 + \tilde{x}, y_1, \dots, y_{q-1}\} + \mathcal{U}^{(\mathfrak{m}+\mathfrak{l})}M_1^q$$

for  $x \in i^*(\mathcal{I}^{(\mathfrak{m})} / \mathcal{I}^{(\mathfrak{m}+\mathfrak{l})})$  and  $y_1, \dots, y_{q-1} \in i^*\psi_*\mathcal{O}_U^\times$ , where  $\tilde{x}$  is a lift of  $x$  to  $i^*\mathcal{I}^{(\mathfrak{m})}$ .

(2) If  $\mathfrak{m} \geq \mathfrak{e}'$ , then  $\mathcal{U}^{(\mathfrak{m})}M_1^q$  is zero.

Theorem 4.4 describes the structure of the sheaf  $\mathcal{U}^1M_1^q$  as follows.

**Corollary 4.5.** Take a sequence of elements of  $\mathbb{N}^\Lambda$

$$\mathbf{1} = \mathfrak{m}_0 \leq \mathfrak{m}_1 \leq \mathfrak{m}_2 \leq \cdots \leq \mathfrak{m}_i \leq \cdots \leq \mathfrak{m}_t$$

satisfying the following conditions:

- (a)  $l_i := m_{i+1} - m_i$  satisfies  $l_i \leq 1$  for any  $i \leq t-1$ .  
 (b)  $e' \leq m_t < e' + 1$ . (Note that such  $m_t$  is unique in  $\mathbb{N}^\Lambda$ .)  
 (c) For any  $(i, \lambda)$  with  $i \leq t-1$  and  $m_{i,\lambda} \geq e'_\lambda$ , the  $\lambda$ -component of  $l_i$  is zero.

We then have

$$\mathcal{U}^{(m_i)}M_1^q / \mathcal{U}^{(m_{i+1})}M_1^q \cong \omega_{m_i, l_i}^{q-1} / \mathcal{F}_{m_i, l_i}^{q-1} \quad \text{for } 0 \leq i \leq t-1, \quad \mathcal{U}^{(m_t)}M_1^q = 0,$$

by [Theorem 4.4](#)(1) and (2), respectively.

To prove [Theorem 4.4](#), we need the following lemma:

- Lemma 4.6.** (1) For  $m, n \in \mathbb{N}^\Lambda$ , we have  $\{\mathcal{U}^{(m)}\mathcal{K}_q^M, \mathcal{U}^{(n)}\mathcal{K}_q^M\} \subset \mathcal{U}^{(m+n)}\mathcal{K}_{q+q'}^M$ .  
 (2) There is a surjective homomorphism

$$i^*\mathcal{O}_X \otimes (i^*\psi_*\mathcal{O}_U^\times)^{\otimes r} \rightarrow \tilde{\omega}_Y^r, \quad x \otimes y_1 \otimes \cdots \otimes y_r \mapsto \bar{x} \cdot d \log(y_1) \wedge \cdots \wedge d \log(y_r),$$

where for  $x \in i^*\mathcal{O}_X$ ,  $\bar{x}$  denotes its residue class in  $\mathcal{O}_Y$ . The kernel of this map is generated by local sections of the following forms:

- $x \otimes y_1 \otimes \cdots \otimes y_r$  with  $x \in i^*\mathcal{I}$  or  $y_s \in i^*(1 + \mathcal{I})$  for some  $1 \leq s \leq r$ ,
- $x \otimes y_1 \otimes \cdots \otimes y_r$  with  $y_s = y_{s'}$  for some  $1 \leq s < s' \leq r$ ,
- $\sum_{s=1}^m (x_s \otimes x_s \otimes y_1 \otimes \cdots \otimes y_{r-1}) - \sum_{t=1}^\ell (x'_t \otimes x'_t \otimes y_1 \otimes \cdots \otimes y_{m-1})$  such that all  $x_s$  and  $x'_t$  belong to  $i^*(\mathcal{O}_X \cap \psi_*\mathcal{O}_U^\times)$  and such that the sums  $\sum_{s=1}^m x_s$  and  $\sum_{t=1}^\ell x'_t$  taken in  $i^*\mathcal{O}_X$  satisfy  $\sum_{s=1}^m x_s \equiv \sum_{t=1}^\ell x'_t \pmod{i^*\mathcal{I}}$ .

*Proof.* (1) This follows from the same argument as in [[Bloch and Kato 1986](#), Lemma 4.1].

(2) Let  $z$  be a point on  $Y$ . Put  $A := \mathcal{O}_{X, \bar{z}}^{\text{sh}}$ ,  $I := \mathcal{I}_{\bar{z}}$  and  $L := (\psi_*\mathcal{O}_U^\times)_{\bar{z}}$ . Let  $A[L]$  be the free  $A$ -module over the set  $L$ . There is a surjective  $A$ -homomorphism  $A[L] \rightarrow (\tilde{\omega}_Y^1)_{\bar{z}}$  sending  $a[b]$  to  $\bar{a} \cdot d \log(b)$ . Its kernel is the  $A$ -submodule generated by elements of the following forms:

- (i)  $a[b]$  with  $a \in I$  or  $b \in 1 + I$ ,  
 (ii)  $[b \cdot b'] - [b] - [b']$  with  $b, b' \in L$ ,  
 (iii)  $\sum_{s=1}^m a_s[a_s] - \sum_{t=1}^\ell a'_t[a'_t]$  ( $a_s, a'_t \in A \cap L$ ) with  $\sum_{s=1}^m a_s \equiv \sum_{t=1}^\ell a'_t \pmod{I}$ .

The claim follows from this fact. The details are straight-forward and left to the reader.  $\square$

*Proof of [Theorem 4.4](#).* (1) By [Lemma 4.6](#), the local assignment in the theorem gives a well-defined surjective homomorphism of sheaves

$$\rho_{m, l} : \omega_{m, l}^{q-1} \longrightarrow \mathcal{U}^{(m)}M_1^q / \mathcal{U}^{(m+l)}M_1^q.$$

We prove  $\rho_{m,l}(\mathcal{Z}_{m,l}^{q-1}) = 0$  assuming  $m < \epsilon'$ , locally on  $Y$ . We may assume that  $Y_\lambda$ 's are principal on  $X$ . Fix uniformizers  $\pi_\lambda \in i^*\mathcal{O}_X$  of  $Y_\lambda$  ( $\lambda \in \Lambda$ ) and put

$$\xi^m := \prod_{\lambda \in \Lambda} \pi_\lambda^{m_\lambda} \in i^*\mathcal{I}^{(m)}.$$

It is enough to show that local sections of  $\mathcal{Z}_{m,l}^{q-1}$  of the forms (1) and (2) of [Corollary 3.6](#) map to zero under  $\rho_{m,l}$ . For  $y_1 \in i^*\mathcal{O}_X^\times$  and  $y_2, \dots, y_{q-1} \in i^*\psi_*\mathcal{O}_U^\times$ , we have

$$\begin{aligned} \{1 + \xi^m y_1, y_1, y_2, \dots, y_{q-1}\} + \sum_{\lambda \in \Lambda} m_\lambda \cdot \{1 + \xi^m y_1, \pi_\lambda, y_2, \dots, y_{q-1}\} \\ = \{1 + \xi^m y_1, \xi^m y_1, y_2, \dots, y_{q-1}\} \\ = -\{1 + \xi^m y_1, -1, y_2, \dots, y_{q-1}\} \in \mathcal{U}^{m+l}\mathcal{K}_q^M. \end{aligned}$$

Hence  $\rho_{m,l}(\omega) = 0$  for

$$\begin{aligned} \omega = \xi^m \otimes (d\eta + \sum_{\lambda \in \Lambda} m_\lambda \cdot d \log(\pi_\lambda) \wedge \eta) \in \mathcal{Z}_{m,l}^{q-1} \\ \text{with } \eta = y_1 \cdot d \log(y_2) \wedge \dots \wedge d \log(y_{q-1}) \in \widetilde{\omega}_Y^{q-2}. \end{aligned}$$

Next let  $m' \in \mathbb{N}^\Lambda$  be the smallest element that satisfies  $p \cdot m' \geq m$ . For  $x \in i^*\mathcal{I}^{(m')}$  and  $y_1, \dots, y_{q-1} \in i^*\psi_*\mathcal{O}_U^\times$ , we have

$$\{1 + x^p, y_1, \dots, y_{q-1}\} - p \cdot \{1 + x, y_1, \dots, y_{q-1}\} \in \mathcal{U}^{m'+\epsilon}\mathcal{K}_q^M \subset \mathcal{U}^{m+l}\mathcal{K}_q^M,$$

where we have used the assumption  $m < \epsilon'$  to verify  $m' + \epsilon \geq m + l$ . Hence  $\rho_{m,l}(\omega) = 0$  for

$$\omega = x^p \otimes d \log(y_1) \wedge \dots \wedge d \log(y_{q-1}) \in \mathcal{Z}_{m,l}^{q-1}.$$

Thus we obtain  $\rho_{m,l}(\mathcal{Z}_{m,l}^{q-1}) = 0$  for  $m < \epsilon'$ .

We prove  $\rho_{m,l}(\mathcal{Z}_{m,l}^{q-1}) = 0$  for  $m < \epsilon' + 1$ . Define  $n = (n_\lambda)_{\lambda \in \Lambda}$  and  $l' = (l'_\lambda)_{\lambda \in \Lambda}$  as

$$n_\lambda := \begin{cases} m_\lambda & \text{if } m_\lambda < \epsilon'_\lambda, \\ m_\lambda - 1 & \text{if } m_\lambda \geq \epsilon'_\lambda, \end{cases} \quad \text{and} \quad l'_\lambda := \begin{cases} \ell_\lambda & \text{if } m_\lambda < \epsilon'_\lambda, \\ 1 & \text{if } m_\lambda \geq \epsilon'_\lambda. \end{cases}$$

We have  $n \leq m$ ,  $\mathbf{1} \leq n < \epsilon'$ ,  $\mathbf{0} \leq l' \leq \mathbf{1}$  and  $n + l' = m + l$  by the assumptions on  $m$  and  $l$ , and there is a commutative diagram

$$\begin{array}{ccc} \omega_{m,l}^{q-1} & \hookrightarrow & \omega_{n,l'}^{q-1} \\ \rho_{m,l} \downarrow & & \downarrow \rho_{n,l'} \\ \mathcal{U}^{(m)}M_1^q / \mathcal{U}^{(m+l)}M_1^q & \hookrightarrow & \mathcal{U}^{(n)}M_1^q / \mathcal{U}^{(n+l')}M_1^q \end{array}$$

where the top horizontal arrow maps  $\mathcal{L}_{m,l}^{q-1}$  into  $\mathcal{L}_{n,l'}^{q-1}$ . Hence  $\rho_{m,l}(\mathcal{L}_{m,l}^{q-1})$  is zero by the previous case and the injectivity of the bottom horizontal arrow.

It remains to prove the injectivity of the induced map

$$\bar{\rho}_{m,l} : \omega_{m,l}^{q-1} / \mathcal{L}_{m,l}^{q-1} \rightarrow \mathcal{U}^{(m)} M_1^q / \mathcal{U}^{(m+l)} M_1^q.$$

Since  $\omega_{m,l}^{q-1} / \mathcal{L}_{m,l}^{q-1}$  is a subsheaf of  $\omega_{m,l}^q$ , the canonical adjunction map

$$\omega_{m,l}^{q-1} / \mathcal{L}_{m,l}^{q-1} \rightarrow \bigoplus_{y \in Y^0} i_{y*} i_y^*(\omega_{m,l}^{q-1} / \mathcal{L}_{m,l}^{q-1})$$

is injective by [Theorem 2.3\(1\)](#), where for  $y \in Y^0$ ,  $i_y$  denotes the natural map  $y \hookrightarrow Y$ . Hence we may replace  $X$  with  $\text{Spec}(\mathcal{O}_{X, y_\mu})$  ( $\mu \in \Lambda$ ), where  $y_\mu$  denotes the generic point of  $Y_\mu$ . By the definition of  $d : \omega_{m,l}^{q-1} \rightarrow \omega_{m,l}^q$ , we have

$$\omega_{m,l}^{q-1} / \mathcal{L}_{m,l}^{q-1} \cong \begin{cases} \Omega_{y_\mu}^{q-1} & \text{if } p \nmid m_\mu, \ell_\mu = 1 \text{ and } m_\mu < e'_\mu, \\ d\Omega_{y_\mu}^{q-1} \oplus d\Omega_{y_\mu}^{q-2} & \text{if } p \mid m_\mu, \ell_\mu = 1 \text{ and } m_\mu < e'_\mu, \\ 0 & \text{otherwise,} \end{cases}$$

and the assertion follows from [\[Bloch and Kato 1986, Corollary 1.4.1\(ii\)–\(iv\)\]](#).

(2) For  $m \in \mathbb{N}^\Lambda$  with  $m \geq e'$ ,  $1 + \mathcal{I}^{(m)}$  is contained in  $(1 + \mathcal{I}^{(m-e)})^p$ . The assertion follows from this fact.  $\square$

## 5. Surjectivity of the symbol map

Let  $O_K$  be as in [Section 1](#), and let  $\pi$  be a prime element of  $O_K$ .

**Definition 5.1.** For an injective morphism of monoids

$$h : \mathbb{N} \rightarrow \mathbb{N}^d, \quad 1 \mapsto (e_\lambda)_{1 \leq \lambda \leq d},$$

we define a scheme  $X^h$  and a divisor  $D^h$  on  $X^h$  as

$$\begin{aligned} X^h &:= \text{Spec}(O_K[T_1, \dots, T_d] / (\prod_{\lambda \text{ with } e_\lambda \geq 1} T_\lambda^{e_\lambda} - \pi)) \\ D^h &:= \{\prod_{\lambda \text{ with } e_\lambda = 0} T_\lambda = 0\} \subset X^h. \end{aligned}$$

Put  $Y^h := (X^h)_{s,\text{red}}$ . We define a scheme  $\mathcal{Y}^h$  as  $\text{Spec}(k[T_1, T_2, \dots, T_d])$  and denote the natural closed immersion  $Y^h \hookrightarrow \mathcal{Y}^h$  by  $\iota^h$ .

Let  $(X, D)$  be as in [Setting 1.4](#). We introduce the following terminology:

**Definition 5.2.** We say that  $(X, D)$  is *quasilog smooth* over  $B = \text{Spec}(O_K)$ , if it is, everywhere étale locally on  $X$ , isomorphic to  $(X^h, D^h)$  for some injective morphism of monoids  $h : \mathbb{N} \rightarrow \mathbb{N}^d$  and a prime element  $\pi \in O_K$ , where  $d$  denotes  $\dim(X)$ .

**Example 5.3.** Let  $(X, D)$  be a pair as in [Setting 1.4](#).

- (1) Let  $\mathcal{M}$  be the log structure on  $X$  associated with  $D$ , and let  $\mathcal{N}$  be the log structure on  $B = \text{Spec}(O_K)$  associated with the closed point  $s \in B$ . If the canonical morphism  $(X, \mathcal{M}) \rightarrow (B, \mathcal{N})$  of log schemes is smooth in the sense of [Kato 1989, (3.3)], then the pair  $(X, D)$  is quasilog smooth over  $\text{Spec}(O_K)$  in our sense. Note also that the converse is not necessarily true.
- (2) As a consequence of (1), a pair  $(X, D)$  as in [Setting 1.4](#) is quasilog smooth over  $\text{Spec}(O_K)$ , if the multiplicities of the irreducible components of  $X_s$  are prime to  $p$ .

**Theorem 5.4.** *Assume that  $K$  contains a primitive  $p$ -th root of unity  $\zeta_p$ , and that  $(X, D)$  is quasilog smooth over  $B$ . Then the symbol map in (4.1),*

$$\varrho_{(X,D),1}^q : \mathcal{K}_q^M / p^n \rightarrow M_n^q,$$

is surjective, and there is an isomorphism

$$M_1^q / \mathcal{U}^1 M_1^q \xrightarrow{\cong} \tilde{\omega}_{Y,\log}^q$$

fitting into a commutative diagram

$$\begin{array}{ccc}
 & \mathcal{K}_q^M / p & \\
 \overline{\varrho_{(X,D),n}^q} \swarrow & & \searrow d \log \\
 M_1^q / \mathcal{U}^1 M_1^q & \xrightarrow{\cong} & \tilde{\omega}_{Y,\log}^q
 \end{array} \tag{5.5}$$

Our proof of [Theorem 5.4](#) will be complete in the next section. In this section, we reduce the theorem to [Lemma 5.10](#) below. Let  $y$  be a generic point of  $Y$ , and let  $i_y$  be the natural map  $y \hookrightarrow Y$ . The strict henselian local ring  $\mathcal{O}_{X,\bar{y}}^{\text{sh}}$  is a discrete valuation ring by the regularity of  $X$ . Hence there is an isomorphism

$$i_y^*(M_1^q / \mathcal{U}^1 M_1^q) \xrightarrow{\cong} \Omega_{y,\log}^q \oplus \Omega_{y,\log}^{q-1} = i_y^* \tilde{\omega}_{Y,\log}^q \tag{5.6}$$

[[Bloch and Kato 1986](#), Lemma 5.3]. Since, by [Theorem 2.3\(1\)](#),  $\tilde{\omega}_Y^q$  is locally free over  $\mathcal{O}_Y$ , the adjunction map

$$\tilde{\omega}_{Y,\log}^q \rightarrow \bigoplus_{y \in Y^0} i_{y*} i_y^* \tilde{\omega}_{Y,\log}^q$$

is injective, and the isomorphism (5.6) induces a surjective map

$$\mathcal{U}^0 M_1^q / \mathcal{U}^1 M_1^q \twoheadrightarrow \tilde{\omega}_{Y,\log}^q$$

(see [[Sato 2013](#), Lemma 2.3]). This map fits into the diagram (5.5) with  $M_1^q / \mathcal{U}^1 M_1^q$  replaced by  $\mathcal{U}^0 M_1^q / \mathcal{U}^1 M_1^q$ . In what follows, put

$$M^q := M_1^q, \quad N^q := \text{Ker}(\mathcal{U}^0 M^q / \mathcal{U}^1 M^q \rightarrow \tilde{\omega}_{Y,\log}^q) \quad \text{and} \quad L^q := M^q / \mathcal{U}^0 M^q.$$

We have show that  $N^q = 0$  and  $L^q = 0$ . Note also that once we show  $L^q = 0$ , we will have shown that  $M_n^q = \mathcal{U}^0 M_n^q$  for all  $n \geq 1$  by a standard argument as in [Bloch and Kato 1986, Corollary 6.1.1]. We first prove that  $N^q = L^q = 0$  in a simple case:

**Lemma 5.7.** *Assume that  $(X, D)$  is quasilog smooth over  $B$  and that the underlying scheme  $X$  is smooth over  $B$ . Then we have  $N^q = 0$  and  $L^q = 0$  (i.e., Theorem 5.4 holds for  $(X, D)$ ).*

*Proof.* When  $D = \emptyset$ , the assertion follows from a theorem of Bloch and Kato [1986, Theorem 1.4]. We proceed with the proof of the lemma by using induction on the number of the irreducible components of  $D$ . Since the problem is étale local on  $X$  and  $X$  is smooth over  $B$  by assumption, we may suppose that  $X = \text{Spec}(O_K[T_2, T_3, \dots, T_d])$  and that  $D = \{T_2 T_3 \cdots T_r = 0\} \subset X$ , where  $2 \leq r \leq d = \dim(X)$ .

Fix an irreducible component  $V$  of  $D$ , which is also smooth over  $B$ . Put  $D' := D - V$  as an effective Cartier divisor, and let  $E$  be the pullback of  $D'$  onto  $V$ . It is easy to see that  $E$  is a simple normal crossing divisor on  $V$  and the pair  $(V, E)$  is quasilog smooth over  $B$  and that the underlying scheme  $V$  is smooth over  $B$ . Recall that  $Y := X_{s,\text{red}} (= X_s)$  and  $U := X \setminus (Y \cup D)$ . Now put  $Z := V_{s,\text{red}} (= V_s)$ ,  $U' := X \setminus (Y \cup D')$  and  $W := V \setminus (Z \cup E)$ , and consider a commutative diagram of schemes

$$\begin{array}{ccccc}
 Z & \xrightarrow{i'} & V & \xleftarrow{\theta} & W \\
 \downarrow \iota & & \downarrow & & \downarrow \alpha \\
 Y & \xrightarrow{i} & X & \xleftarrow{\psi'} & U' & \xleftarrow{\beta} & U \\
 & & & \swarrow \psi & & & \\
 & & & & & & 
 \end{array}$$

We then have a commutative diagram of étale sheaves on  $Y$  whose upper row is a complex and whose lower row is exact

$$\begin{array}{ccccc}
 \mathcal{K}_{q,(X,D')}^M/p & \longrightarrow & \mathcal{K}_q^M/p & \longrightarrow & \iota_* \mathcal{K}_{q-1,(V,E)}^M/p \\
 \varrho_1^q \downarrow & & \varrho_1^q \downarrow & & \varrho_1^{q-1} \downarrow \\
 M_{(X,D')}^q & \longrightarrow & M^q & \longrightarrow & \iota_* M_{(V,E)}^{q-1}
 \end{array} \tag{5.8}$$

Here we put  $M_{(X,D')}^q := i^* R^q \psi'_* \mu_p^{\otimes q}$  and  $M_{(V,E)}^{q-1} := i'^* R^{q-1} \theta_* \mu_p^{\otimes (q-1)}$ , and the sheaves  $\mathcal{K}_{q,(X,D')}^M$  and  $\mathcal{K}_{q-1,(V,E)}^M$  are Milnor  $K$ -sheaves defined as quotients of  $(i^* \psi'_* \mathcal{O}_{U'}^{\otimes q})^{\otimes q}$  and  $(i'^* \theta_* \mathcal{O}_W^{\otimes (q-1)})^{\otimes (q-1)}$ , respectively. The right arrow in the upper row is a boundary map of Milnor  $K$ -sheaves, which one can check to be surjective. The lower row is obtained by applying  $i^* R^q \psi'_*$  to the Gysin distinguished triangle on  $(U')_{\text{ét}}$ ,

$$\alpha_* \mu_p^{\otimes (q-1)}[-2] \rightarrow \mu_p^{\otimes q} \rightarrow R\beta_* \mu_p^{\otimes q} \rightarrow \alpha_* \mu_p^{\otimes (q-1)}[-1].$$

Now the left and the right vertical arrows in (5.8) are surjective by the induction hypothesis, and we obtain the surjectivity of  $\mathcal{Q}_{(X,D),1}^q$  by a simple diagram chase, that is,  $L^q = 0$ .

To prove that  $N^q = 0$ , we consider the following commutative diagram of sheaves on  $Y_{\text{ét}}$ , whose middle row is exact and whose other rows and columns are complexes:

$$\begin{array}{ccccccc}
 \mathcal{U}^1 M_{(X,D')}^q & \longrightarrow & \mathcal{U}^1 M^q & \xrightarrow{r_1} & \iota_* \mathcal{U}^1 M_{(V,E)}^{q-1} & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 M_{(X,D')}^q & \longrightarrow & M^q & \xrightarrow{r_2} & \iota_* M_{(V,E)}^{q-1} & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \tilde{\omega}_{Y',\log}^q & \longrightarrow & \tilde{\omega}_{Y,\log}^q & \xrightarrow{r_3} & \iota_* \tilde{\omega}_{Z,\log}^{q-1} \longrightarrow 0
 \end{array}$$

where  $\tilde{\omega}_{Y',\log}^q$  (resp.  $\tilde{\omega}_{Z,\log}^{q-1}$ ) denotes the logarithmic differential sheaf defined for the pair  $(X, D')$  (resp.  $(V, E)$ ); the arrow  $r_1$  denotes the map induced by  $r_2$ , and  $r_3$  denotes a residue map of logarithmic differential sheaves. By the smoothness of  $X$  over  $B$ , we have

$$\tilde{\omega}_{Y,\log}^q \cong \Omega_Y^q(\log D_Y)_{\log} \oplus \Omega_Y^{q-1}(\log D_Y)_{\log} \quad (D_Y := D \cap Y)$$

and similar presentations for  $\tilde{\omega}_{Y',\log}^q$  and  $\tilde{\omega}_{Z,\log}^{q-1}$ . By this fact and a simple variant of the purity of logarithmic differential sheaves [Shiho 2007, Theorem 3.2], we see that the bottom row is exact. On the other hand, one can easily check that the sequence

$$(0 \longrightarrow ) \mathcal{U}^m M_{(X,D')}^q \longrightarrow \mathcal{U}^m M^q \xrightarrow{r_1} \iota_* \mathcal{U}^m M_{(V,E)}^{q-1} \longrightarrow 0$$

is exact for  $1 \leq m \leq e' := pe/(p - 1)$  by Theorem 4.4 and descending induction on  $m$ . Hence the top row in the above diagram is also exact. Now the left and the right columns are exact by the induction hypothesis, and the middle column is exact as well by a simple diagram chase, which shows that  $N^q = 0$ .  $\square$

We now reduce the general case of Theorem 5.4 to Lemma 5.10 below. Fix an arbitrary point  $x \in Y$ . We show the stalks  $(N^q)_{\bar{x}}$  and  $(L^q)_{\bar{x}}$  are zero by induction on  $c := \text{codim}_Y(x)$ . If  $c = 0$ , then  $(N^q)_{\bar{x}} = (L^q)_{\bar{x}} = 0$  by (5.6). In what follows, assume  $c \geq 1$  and the following induction hypothesis:

- (1)  $(N^q)_{\bar{y}} = (L^q)_{\bar{y}} = 0$  for any  $q \geq 0$  and any  $y \in Y$  of codimension  $\leq c - 1$ .

Since the problem is étale local, we may assume  $(X, D) = (X^h, D^h)$  for an injective morphism  $h : \mathbb{N} \hookrightarrow \mathbb{N}^d$  of monoids ( $d = \dim(X)$ ). Sorting the components of  $\mathbb{N}^d$  if necessarily, we assume the following two conditions:

- (2) *The first component of  $h(1) \in \mathbb{N}^d$  is nonzero.*  
(3) *The composite map*

$$Y \hookrightarrow \mathcal{Y} = \mathrm{Spec}(k[T_1, \dots, T_d]) \longrightarrow \mathrm{Spec}(k[T_{c+2}, \dots, T_d])$$

*sends  $x$  to the generic point of  $\mathrm{Spec}(k[T_{c+2}, \dots, T_d])$  [Tsuji 2000, Lemma 5.3].*

Following the idea of Tsuji [2000, proof of Theorem 5.1], we decompose  $h : \mathbb{N} \rightarrow \mathbb{N}^d$  into a sequence of morphisms of monoids

$$h : \mathbb{N} \xrightarrow{h^0} \mathbb{N}^d \xrightarrow{\kappa^1} \mathbb{N}^d \xrightarrow{\kappa^2} \dots \xrightarrow{\kappa^r} \mathbb{N}^d$$

which satisfies the following two conditions:

- (4)  $h^0(1) = (e, \overbrace{0, \dots, 0}^{c\text{-copies}}, *, \dots, *)$  for some  $e \neq 0$  (see (2)).  
(5) For  $1 \leq t \leq d$ , let  $\epsilon_t \in \mathbb{N}^d$  be the element whose  $t$ -th component is 1 and whose other components are 0. Then for  $1 \leq v \leq r$ ,  $\kappa^v$  sends  $\epsilon_t$  ( $1 \leq t \leq d$ ) to

$$\begin{cases} \epsilon_t & (t \neq m), \\ \epsilon_m + \epsilon_n & (t = m), \end{cases} \quad \text{for some } m \neq n \text{ with } 1 \leq m \leq c+1, 1 \leq n \leq c+1.$$

Put  $h^v := \kappa^v \kappa^{v-1} \dots \kappa^1 h^0$  and  ${}^v X := X^{h^v}$ , and let  $f^v$  be the morphism induced by  $\kappa^v$ :

$$f^v : {}^v X \rightarrow {}^{v-1} X.$$

We further fix some notation. Put  ${}^v Y := Y^{h^v} = ({}^v X)_{s, \mathrm{red}}$ , and let  $x^v \in {}^v Y$  be the image of  $x \in Y$  under the composite

$$Y = {}^r Y \xrightarrow{g^r} {}^{r-1} Y \xrightarrow{g^{r-1}} \dots \xrightarrow{g^{v+1}} {}^v Y,$$

where  $g^v : {}^v Y \rightarrow {}^{v-1} Y$  denotes the morphism induced by  $f^v$ . For  $0 \leq v \leq r$ , let  $\sigma^v$  be the composite map

$${}^v Y \xrightarrow{\iota^v} \mathcal{Y}^{h^v} = \mathrm{Spec}(k[T_1, \dots, T_d]) \rightarrow \mathrm{Spec}(k[T_{c+2}, \dots, T_d]),$$

where  $\iota^v$  denotes  $\iota^{h^v}$ . Since  $\sigma^v = \sigma^{v-1} g^v$  by (5), the point  $\sigma^v(x^v)$  is the generic point of  $\mathrm{Spec}(k[T_{c+2}, \dots, T_d])$  for any  $0 \leq v \leq r$  by (3). This implies the following:

- (6) *For any  $0 \leq v \leq r$ ,  $x^v$  has codimension  $c$  on  ${}^v Y$ . Consequently,  $x^v$  is a closed point of  $(g^v)^{-1}(x^{v-1})$ .*

We also need the following fact (see [Tsuji 2000, Lemmas 3.2 and 3.4]):

(7) For  $1 \leq \nu \leq r$ ,  $f^\nu$  factors as

$${}^\nu X \hookrightarrow {}^\nu \overline{X} \xrightarrow{f^\nu} {}^{\nu-1} X,$$

where the left arrow is an open immersion and  $f^\nu$  is the blow-up at the closed subscheme  $\{T_m = T_n = 0\} \subset {}^{\nu-1} X$ . The fibers of  $f^\nu$  have dimension at most one.

Put

$${}^\nu D := D^{h_\nu} \quad \text{and} \quad {}^\nu U := {}^\nu X \setminus ({}^\nu Y \cup {}^\nu D),$$

and define the sheaves  ${}^\nu M^q$ ,  ${}^\nu L^q$  and  ${}^\nu N^q$  on  ${}^\nu Y_{\text{ét}}$  for the diagram

$${}^\nu Y \hookrightarrow {}^\nu X \longleftarrow {}^\nu U$$

in the same way as for  $M^q$ ,  $L^q$  and  $N^q$  on  $Y_{\text{ét}}$ , respectively. In what follows, we prove

$$({}^\nu L^q)_{\overline{x}^\nu} = ({}^\nu N^q)_{\overline{x}^\nu} = 0$$

by induction on  $0 \leq \nu \leq r$ . We first note:

**Lemma 5.9.**  $({}_0 L^q)_{\overline{x}^0} = 0$  and  $({}_0 N^q)_{\overline{x}^0} = 0$ .

*Proof.* By the assumption (4), the pair  $({}^0 X, {}^0 D)$  is, étale locally around  $x^0$ , isomorphic to a quasilog smooth pair  $(X', D')$  over a henselian discrete valuation ring  $A'$  of mixed characteristic such that  $X'$  is smooth over  $A'$ , see the first isomorphism on page 559 of [Tsuji 2000]. Hence the assertion follows from Lemma 5.7.  $\square$

Assume  $\nu \geq 1$  and the following induction hypothesis:

$$(8) \quad ({}_{\nu-1} L^q)_{\overline{x}^{\nu-1}} = 0 \quad \text{and} \quad ({}_{\nu-1} N^q)_{\overline{x}^{\nu-1}} = 0.$$

We change the notation slightly and put

$$\left\{ \begin{array}{l} X := \text{Spec}(\mathcal{O}_{\nu-1 X, \overline{x}^{\nu-1}}^{\text{sh}}), \\ Y := X_{s, \text{red}}, \\ D := \text{Spec}(\mathcal{O}_{\nu-1 D, \overline{x}^{\nu-1}}^{\text{sh}}), \\ U := X \setminus (Y \cup D), \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} X' := \overline{\nu X} \times_{\nu-1 X} \text{Spec}(\mathcal{O}_{\nu-1 X, \overline{x}^{\nu-1}}^{\text{sh}}), \\ Y' := (X')_{s, \text{red}}, \\ D' := \overline{\nu D} \times_{\overline{\nu X}} X', \\ U' := X' \setminus (Y' \cup D'), \end{array} \right.$$

for simplicity. Here  $\overline{\nu D}$  denotes the closure of  $(f^\nu)^{-1}({}^\nu D)_{\text{red}} \setminus (\overline{\nu X})_{s, \text{red}} \subset \overline{\nu X}$ . Note that  ${}^\nu D = \overline{\nu D} \times_{\overline{\nu X}} {}^\nu X$ . Let  $i' : Y' \hookrightarrow X'$  and  $\psi' : U' \hookrightarrow X'$  be the canonical closed and open immersions, respectively. We define étale sheaves  $M^q$ ,  $L^q$  and  $N^q$  on  $Y'$  as

$$M^q := i'^* R^q \psi'_* \mu_p^{\otimes q}, \quad N^q := \text{Ker}(\mathcal{U}^0 M^q / \mathcal{U}^1 M^q \rightarrow \widetilde{\omega}_{Y', \log}^q), \quad L^q := M^q / \mathcal{U}^0 M^q.$$

In view of (6), once we prove  $N^q$  and  $L^q$  are zero, we will finish the induction on  $\nu$  and  $c$ . We will prove the following lemma in the next section:

**Lemma 5.10** (cf. [Hyodo 1988, Lemma (3.5)]). *For any  $t \geq 0$ , we have*

$$\Gamma(Y, \tilde{\omega}_{Y, \log}^t) \cong \Gamma(Y', \tilde{\omega}_{Y', \log}^t), \quad H^1(Y', \mathcal{U}^1 M^t) = H^1(Y', \tilde{\omega}_{Y', \log}^t) = 0.$$

We prove here that  $N^q$  and  $L^q$  are zero admitting this lemma. Noting that  $\mu_p \cong \mathbb{Z}/p\mathbb{Z}$  on  $U'$  by the assumption on  $K$ , we compute the Leray spectral sequence

$$E_2^{a,b} = H^a(Y', M^b) \implies H^{a+b}(U', \mu_p^{\otimes a}) \cong H^{a+b}(U, \mu_p^{\otimes a}),$$

where we have used the proper base-change theorem [SGA 4<sub>3</sub> 1973, XII.5.2] for the identification  $H^a(Y', M^b) \cong H^a(X', R^b \psi'_* \mu_p^{\otimes a})$  and also used the fact that  $\bar{f}^v$  induces an isomorphism  $U' \cong U$ . Since  $\text{cd}_p(Y') \leq 1$  by (7), this spectral sequence yields a short exact sequence

$$0 \rightarrow H^1(Y', M^{q-1}) \rightarrow H^q(U, \mu_p^{\otimes q}) \rightarrow \Gamma(Y', M^q) \rightarrow 0.$$

Because  $L^t$  and  $N^t$  are skyscraper sheaves on  $Y'$  for any  $t \geq 0$  by the induction hypothesis (1) for  $X'$ , both  $H^1(Y', M^{q-1})$  and  $H^1(Y', \mathcal{U}^0 M^q)$  are zero by Lemma 5.10. Hence there is a commutative diagram whose lower row is exact

$$\begin{array}{ccccccc} \mathcal{U}^0 H^q(U, \mu_p^{\otimes q}) & \xlongequal{\quad} & H^q(U, \mu_p^{\otimes q}) & & & & \\ \downarrow & & \wr \downarrow & & & & \\ 0 & \longrightarrow & \Gamma(Y', \mathcal{U}^0 M^q) & \longrightarrow & \Gamma(Y', M^q) & \longrightarrow & \Gamma(Y', L^q) \longrightarrow 0 \end{array}$$

where  $\mathcal{U}^\bullet H^q(U, \mu_p^{\otimes q})$  ( $\bullet = 0, 1$ ) denotes the filtration on the stalk of the sheaf of  $p$ -adic vanishing cycles on  $Y$  (see Definition 4.2(1) and Lemma 4.3(2)), and the upper equality follows from the induction hypothesis (8). This diagram shows that the skyscraper sheaf  $L^q$  is zero. We next show that  $N^q$  is zero. Put

$$\text{gr}_{\mathcal{U}}^0 M^q := \mathcal{U}^0 M^q / \mathcal{U}^1 M^q = M^q / \mathcal{U}^1 M^q.$$

Since  $N^q$  is skyscraper by (1), there is an exact sequence

$$0 \rightarrow \Gamma(Y', N^q) \rightarrow \Gamma(Y', \text{gr}_{\mathcal{U}}^0 M^q) \xrightarrow{\alpha} \Gamma(Y', \tilde{\omega}_{Y', \log}^q) \rightarrow 0$$

and a commutative diagram with exact rows (see Lemma 4.3(2))

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{U}^1 H^q(U, \mu_p^{\otimes q}) & \longrightarrow & H^q(U, \mu_p^{\otimes q}) & \longrightarrow & \Gamma(Y, \tilde{\omega}_{Y, \log}^q) \longrightarrow 0 \\ & & \downarrow & & \wr \downarrow & & \beta \downarrow \\ 0 & \longrightarrow & \Gamma(Y', \mathcal{U}^1 M^q) & \longrightarrow & \Gamma(Y', M^q) & \longrightarrow & \Gamma(Y', \text{gr}_{\mathcal{U}}^0 M^q) \longrightarrow 0 \end{array}$$

Here the upper row is exact by the induction hypothesis (8), and the lower row is exact by Lemma 5.10. The arrow  $\beta$  denotes the map induced by the left square, and the middle vertical arrow is bijective by the proof of the vanishing of  $L^q$ . Now

this diagram shows that  $\alpha$  is bijective, because  $\beta$  is surjective and  $\alpha\beta$  is bijective by [Lemma 5.10](#). Hence the skyscraper sheaf  $N^q$  is zero. Thus the induction on  $v$  and  $c$  is complete and we obtain [Theorem 5.4](#), assuming [Lemma 5.10](#).

### 6. Proof of [Lemma 5.10](#)

In this section we prove [Lemma 5.10](#) to finish the proof of [Theorem 5.4](#). Let the notation be as in [Setting 1.4](#). Assume that

$$X = \text{Spec}(O_K[T_1, \dots, T_d]/(T_1^{e_1} \cdots T_a^{e_a} - \pi)) \quad (e_1, \dots, e_a \geq 1, d = \dim(X)),$$

$$D = \{T_{a+1} \cdots T_d = 0\} \subset X,$$

where  $\pi$  is a prime element of  $O_K$  and  $D$  is empty if  $a = d$ . Let

$$f : X' \rightarrow X$$

be the blow-up at the regular closed subscheme  $\{T_b = T_c = 0\} \subset X$  with  $1 \leq b < c \leq d$ . Put  $Y' := (X')_{s,\text{red}}$ . We define a reduced normal crossing divisor  $D'$  on  $X'$  as

$$D' := \overline{f^{-1}(D)_{\text{red}} \setminus Y'} \subset X'$$

and define the sheaves  $\tilde{\omega}_{Y',\log}^q$  and  $M_{1,X'}^q$  on  $(Y')_{\text{ét}}$  in the same way as for  $\tilde{\omega}_{Y,\log}^q$  and  $M_1^q$  on  $Y_{\text{ét}}$ . Let

$$g : Y' \rightarrow Y$$

be the morphism induced by  $f$ . [Lemma 5.10](#) follows from:

**Lemma 6.1.** *Let  $D(Y_{\text{ét}})$  be the derived category of complexes of étale sheaves on  $Y$ .*

- (1)  $\tilde{\omega}_{Y,\log}^q \xrightarrow{\cong} Rg_*\tilde{\omega}_{Y',\log}^q$  in  $D(Y_{\text{ét}})$  for any  $q \geq 0$ .
- (2)  $R^1g_*(\mathcal{L}^1M_{1,X'}^q) = 0$  for any  $q \geq 0$ .

We introduce some notation that will be useful throughout the proof of [Lemma 6.1](#). For  $\lambda \in \Lambda := \{1, 2, \dots, a\}$ , let  $Y_\lambda$  be the closed subset  $\{T_\lambda = 0\} \subset X$  endowed with the reduced subscheme structure, which is an irreducible component of  $Y$ . Let  $\{Y'_\lambda\}_{\lambda \in \Lambda'}$  be the irreducible components of  $Y'$ . We have

$$\Lambda' = \begin{cases} \Lambda \cup \{\infty\} & \text{if } b \leq a, \\ \Lambda & \text{if } a < b, \end{cases}$$

where  $Y'_\lambda$  for  $\lambda \in \Lambda$  means the strict transform of  $Y_\lambda$  and  $Y'_\infty$  is the exceptional fiber of  $f$ . For  $\mathfrak{m} = (m_\lambda)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda$ , we define  $f^*\mathfrak{m} = (n_\lambda)_{\lambda \in \Lambda'} \in \mathbb{N}^{\Lambda'}$  as

$$n_\lambda = \begin{cases} m_\lambda & \text{if } \lambda \in \Lambda, \\ m_b & \text{if } b \leq a < c \text{ and } \lambda = \infty, \\ m_b + m_c & \text{if } c \leq a \text{ and } \lambda = \infty. \end{cases}$$

For  $\lambda \in \Lambda'$ , let  $\mathcal{I}'_\lambda \subset \mathcal{O}_{X'}$  be the defining ideal of  $Y'_\lambda$ . For  $\mathfrak{n} \in \mathbb{N}^{\Lambda'}$ , we define  $\mathcal{I}'_{X'}^{(\mathfrak{n})} \subset \mathcal{O}_{X'}$  in the same way as for  $\mathcal{I}^{(\bullet)} \subset \mathcal{O}_X$ , see [Section 3](#). We have

$$f^* \mathcal{I}^{(\mathfrak{m})} = \mathcal{I}'_{X'}^{(f^* \mathfrak{m})}. \quad (6.2)$$

We will often write  $\mathcal{I} \subset \mathcal{O}_X$  and  $\mathcal{I}' \subset \mathcal{O}_{X'}$  for the defining ideals of  $Y$  and  $Y'$ , respectively.

**Sublemma 6.3.** (1) *We have  $g^* \widetilde{\omega}_Y^q \xrightarrow{\sim} \widetilde{\omega}_{Y'}^q$ , for any  $q \geq 0$ , where  $g^*$  denotes the inverse image of coherent sheaves.*

(2) *Assume that  $c \leq a$ , and let  $\mathfrak{n} = (n_\lambda)_{\lambda \in \Lambda'} \in \mathbb{N}^{\Lambda'}$  be an arbitrary element with  $n_b + n_c = n_\infty + 1$ . Then we have  $Rg_* (\mathcal{I}'_{X'}^{(\mathfrak{n})} / \mathcal{I}'_{X'}^{(\mathfrak{n} + \mathfrak{d}_\infty)}) = 0$  in  $D(Y_{\text{ét}})$ . (See the definitions before [Corollary 3.3](#) for  $\mathfrak{d}_\infty \in \mathbb{N}^{\Lambda'}$ .)*

(3) *We have  $\mathcal{O}_Y \xrightarrow{\sim} Rg_* \mathcal{O}_{Y'}$  in  $D(Y_{\text{ét}})$ .*

*Proof of Sublemma 6.3.* (1) Let  $(X, \mathcal{L})$  and  $(X', \mathcal{L}')$  be the log schemes associated with the pairs  $(X, D)$  and  $(X', D')$ , respectively (see [Remark 2.2](#)), and let  $L$  (resp.  $L'$ ) be the inverse image log structure onto  $Y$  (resp. onto  $Y'$ ). The morphism  $(Y', L') \rightarrow (Y, L)$  of log schemes induced by  $f$  is étale in the sense of [\[Kato 1989, \(3.3\)\]](#) by [\[loc. cit., Theorem \(3.5\)\]](#). Hence the assertion follows from [\[loc. cit., Proposition 3.12\]](#).

(2) When  $c \leq a$ ,  $Y'_\infty$  is isomorphic to the trivial  $\mathbb{P}^1$ -bundle over  $Y_b \cap Y_c$ , and  $Y'_\infty \cap Y'_b$  and  $Y'_\infty \cap Y'_c$  are relative hyperplane sections of  $Y'_\infty$  over  $Y_b \cap Y_c$ . Since  $(\mathcal{I}'_\infty)^{n_\infty} / (\mathcal{I}'_\infty)^{n_\infty + 1} \cong \mathcal{O}(n_\infty)$  on  $Y'_\infty$ , we have

$$\mathcal{I}'_{X'}^{(\mathfrak{n})} / \mathcal{I}'_{X'}^{(\mathfrak{n} + \mathfrak{d}_\infty)} \cong (\mathcal{I}'_b)^{n_b} \otimes_{\mathcal{O}_{X'}} (\mathcal{I}'_c)^{n_c} \otimes_{\mathcal{O}_{X'}} ((\mathcal{I}'_\infty)^{n_\infty} / (\mathcal{I}'_\infty)^{n_\infty + 1}) \cong \mathcal{O}(-1)$$

on  $Y'_\infty$ , where we have used the fact that  $\mathcal{I}'_\lambda$  is principal for  $\lambda \neq b, c, \infty$  and the assumption that  $n_b + n_c = n_\infty + 1$ . Hence we obtain the assertion by a standard fact on the cohomology of projective lines [\[Hartshorne 1977, Chapter III Theorem 5.1\]](#).

(3) Noting that  $f : X' \rightarrow X$  is the blow-up along a closed subscheme of  $X$  defined by a regular sequence and that  $\mathcal{I} = T_0 T_1 \cdots T_a \mathcal{O}_X$  is a free  $\mathcal{O}_X$ -module (of rank 1), we have

$$\mathcal{O}_X \xrightarrow{\sim} Rf_* \mathcal{O}_{X'} \quad \text{and} \quad \mathcal{I} \xrightarrow{\sim} Rf_* f^* \mathcal{I} \quad \text{in } D(X_{\text{ét}})$$

by the theorem on formal functions [\[Hartshorne 1977, Chapter III Theorem 11.1\]](#) and the cohomology of projective lines. Our task is to check

$$Rf_* (\mathcal{I}' / f^* \mathcal{I}) = 0 \quad \text{in } D(X_{\text{ét}}). \quad (6.4)$$

Note that  $\mathcal{I}' = \mathcal{I}'_{X'}^{(\mathbf{1})}$  and that  $f^* \mathcal{I} = \mathcal{I}'_{X'}^{(f^* \mathbf{1})}$  by (6.2). If  $a < c$ , then we have  $f^* \mathbf{1} = \mathbf{1}$  in  $\mathbb{N}^{\Lambda'}$  and the assertion (6.4) is obvious. If  $c \leq a$ , then we have

$$f^* \mathbf{1} = \mathbf{1} + \mathfrak{d}_\infty \quad \text{in } \mathbb{N}^{\Lambda'} \quad (6.5)$$

and the assertion follows from [Sublemma 6.3\(2\)](#).  $\square$

*Proof of Lemma 6.1.* (1) By [Theorem 2.3\(2\)](#) and (3), it is enough to show that

$$(a) \quad \tilde{\omega}_Y^q \xrightarrow{\cong} Rg_*\tilde{\omega}_{Y'}^q, \quad \text{and} \quad (b) \quad \tilde{\mathcal{L}}_Y^q \xrightarrow{\cong} Rg_*\tilde{\mathcal{L}}_{Y'}^q \quad \text{in } D(Y_{\text{ét}}).$$

We first show (a). Since  $\tilde{\omega}_Y^q$  is locally free over  $\mathcal{O}_Y$  by [Theorem 2.3\(1\)](#), we have

$$Rg_*\tilde{\omega}_Y^q \cong Rg_*g^*\tilde{\omega}_Y^q \cong Rg_*(\mathcal{O}_{Y'} \otimes_{\mathcal{O}_{Y'}} g^*\tilde{\omega}_Y^q) \cong (Rg_*\mathcal{O}_{Y'}) \otimes_{\mathcal{O}_Y} \tilde{\omega}_Y^q \cong \tilde{\omega}_Y^q$$

by the projection formula and [Sublemma 6.3\(3\)](#). We next show (b). The case  $q = 0$  follows from (a) and the isomorphism  $\tilde{\mathcal{L}}_Y^0 = (\mathcal{O}_Y)^p \cong \mathcal{O}_Y$ . We proceed with the proof by using induction on  $q$ . There is a commutative diagram with distinguished rows in  $D(Y_{\text{ét}})$ :

$$\begin{array}{ccccccc} \tilde{\mathcal{L}}_Y^q & \longrightarrow & \tilde{\omega}_Y^q & \xrightarrow{d} & \tilde{\mathcal{B}}_Y^{q+1} & \longrightarrow & \tilde{\mathcal{L}}_Y^q[1] \\ \downarrow & & \downarrow \wr & & \downarrow & & \downarrow \\ Rg_*\tilde{\mathcal{L}}_{Y'}^q & \longrightarrow & Rg_*\tilde{\omega}_{Y'}^q & \xrightarrow{d} & Rg_*\tilde{\mathcal{B}}_{Y'}^{q+1} & \longrightarrow & Rg_*\tilde{\mathcal{L}}_{Y'}^q[1] \end{array}$$

By [Theorem 2.3\(2\)](#), there is another commutative diagram with distinguished rows in  $D(Y_{\text{ét}})$ :

$$\begin{array}{ccccccc} \tilde{\mathcal{B}}_Y^{q+1} & \longrightarrow & \tilde{\mathcal{L}}_Y^{q+1} & \xrightarrow{C} & \tilde{\omega}_Y^{q+1} & \longrightarrow & \tilde{\mathcal{B}}_Y^{q+1}[1] \\ \downarrow & & \downarrow & & \downarrow \wr & & \downarrow \\ Rg_*\tilde{\mathcal{B}}_{Y'}^{q+1} & \longrightarrow & Rg_*\tilde{\mathcal{L}}_{Y'}^{q+1} & \xrightarrow{C} & Rg_*\tilde{\omega}_{Y'}^{q+1} & \longrightarrow & Rg_*\tilde{\mathcal{B}}_{Y'}^{q+1}[1] \end{array}$$

where  $C$  denotes the inverse of the isomorphism  $C^{-1}$ . The induction on  $q$  works by these diagrams.

(2) We first show

$$R^1g_*(\mathcal{W}^{(f^*1)}M_{1,X'}^q) = 0. \quad (6.6)$$

Take a sequence of elements of  $\mathbb{N}^A$

$$\mathbf{1} = \mathbf{m}_0 \leq \mathbf{m}_1 \leq \mathbf{m}_2 \leq \cdots \leq \mathbf{m}_i \leq \cdots \leq \mathbf{m}_t$$

satisfying the following conditions:

- (i) If  $i \leq t - 1$ , then  $\mathfrak{l}_i := \mathbf{m}_{i+1} - \mathbf{m}_i$  agrees with  $\mathfrak{d}_\mu$  for some  $\mu = \mu(i) \in \Lambda$ .
- (ii) We have  $\mathbf{m}_i \geq \mathfrak{e}'$  (in  $\mathbb{Q}^A$ ) if and only if  $i = t$ .

See the definitions before [Corollary 3.3](#) for  $\mathfrak{d}_\mu$ , and see [Definition 4.2\(3\)](#) for  $\mathfrak{e}' \in \mathbb{Q}^A$ . We define  $\mathfrak{e}' \in \mathbb{Q}^A$  for  $X'$  in the same way as we defined  $\mathfrak{e}'$  for  $X$ . For  $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^A$

with  $\mathbf{0} \leq n' \leq \mathbf{1}$ , we define the sheaf  $\omega_{n,n',X'}^{q-1}$  on  $Y'$  in the same way as for  $\omega_{*,*}^{q-1}$  on  $Y$ . By the choice of the above  $m_i$ 's and the fact that  $f^*e' = \mathfrak{E}'$ , we see that

- (i') For  $i \leq t-1$ , we have  $f^*l_i = f^*m_{i+1} - f^*m_i \in \mathbb{N}^{A'}$  and  $\mathbf{0} \leq f^*l_i \leq \mathbf{1}$ .
- (ii') We have  $f^*m_i \geq \mathfrak{E}'$  (in  $\mathbb{Q}^{A'}$ ) if and only if  $i = t$ .

Hence by [Theorem 4.4](#) for  $X'$ , we have

$$\begin{aligned} \mathcal{U}^{(f^*m_i)} M_{1,X'}^q / \mathcal{U}^{(f^*m_{i+1})} M_{1,X'}^q &\cong \omega_{f^*m_i, f^*l_i, X'}^{q-1} / \mathcal{Z}_{f^*m_i, f^*l_i, X'}^{q-1} \quad \text{for } 0 \leq i \leq t-1, \\ \mathcal{U}^{(f^*m_t)} M_{1,X'}^q &= 0, \end{aligned}$$

and we are reduced to showing that

$$R^1 g_* (\omega_{f^*m, f^*l, X'}^{q-1} / \mathcal{Z}_{f^*m, f^*l, X'}^{q-1}) = 0 \quad (6.7)$$

for  $m, l \in \mathbb{N}^A$  with  $m \geq 1$  and  $l = \mathfrak{d}_\mu$  for some  $\mu \in \Lambda$ . We prove (6.7). By (6.2), there is a short exact sequence on  $Y'_{\text{ét}}$

$$0 \rightarrow f^* \mathcal{S}^{(m+l)} \otimes_{\mathcal{O}_{X'}} \widetilde{\omega}_{Y'}^{q-1} \rightarrow f^* \mathcal{S}^{(m)} \otimes_{\mathcal{O}_{X'}} \widetilde{\omega}_{Y'}^{q-1} \rightarrow \omega_{f^*m, f^*l, X'}^{q-1} \rightarrow 0.$$

Since  $\mathcal{S}^{(m+l)}$  and  $\mathcal{S}^{(m)}$  are locally free  $\mathcal{O}_X$ -modules, we obtain

$$\omega_{m,l}^{q-1} \xrightarrow{\cong} Rg_* \omega_{f^*m, f^*l, X'}^{q-1}$$

by applying  $Rf_*$  to the above short exact sequence and the projection formula together with the claim (a) in the proof of [Lemma 6.1\(1\)](#). In particular we obtain  $R^1 g_* \omega_{f^*m, f^*l, X'}^{q-1} = 0$  and (6.7), because  $R^1 g_*$  is right exact for  $p$ -torsion sheaves for the reason of the dimension of the fibers of  $g$  [[SGA 4<sub>3</sub> 1973](#), X.5.2, XII.5.2]. Thus we obtain (6.6).

By (6.6), it remains to check

$$R^1 g_* (\mathcal{U}^1 M_{1,X'}^q / \mathcal{U}^{(f^*1)} M_{1,X'}^q) = 0 \quad (6.8)$$

If  $a < c$ , then we have  $f^*1 = \mathbf{1}$  in  $\mathbb{N}^{A'}$  and the assertion is obvious. As for the case  $c \leq a$ , we have

$$\mathcal{U}^1 M_{1,X'}^q / \mathcal{U}^{(f^*1)} M_{1,X'}^q \cong \omega_{\mathbf{1}, \mathfrak{d}_\infty, X'}^{q-1} / \mathcal{Z}_{\mathbf{1}, \mathfrak{d}_\infty, X'}^{q-1}$$

by (6.5) and [Theorem 4.4](#). We have

$$Rg_* \omega_{\mathbf{1}, \mathfrak{d}_\infty, X'}^{q-1} \cong Rg_* (\mathcal{S}_{X'}^{(\mathbf{1})} / \mathcal{S}_{X'}^{(f^*1)} \otimes_{\mathcal{O}_{Y'}} g^* \widetilde{\omega}_Y^{q-1}) \cong Rg_* (\mathcal{S}_{X'}^{(\mathbf{1})} / \mathcal{S}_{X'}^{(f^*1)}) \otimes_{\mathcal{O}_Y} \widetilde{\omega}_Y^{q-1} = 0$$

by (6.4). Thus we obtain  $R^1 g_* \omega_{\mathbf{1}, \mathfrak{d}_\infty, X'}^{q-1} = 0$  and (6.8), noting that  $R^1 g_*$  is right exact for  $p$ -torsion sheaves. This completes the proof of [Lemmas 6.1](#) and [5.10](#), and [Theorem 5.4](#).  $\square$

**Remark 6.9.** By [Lemma 5.10](#) and the proof of [Theorem 5.4](#) (see the last diagram of [Section 5](#)), we have  $\mathcal{U}^1 M_1^q \cong g_*(\mathcal{U}^1 M_{1,X'}^q)$  as sheaves on  $Y_{\text{ét}}$  for any  $q \geq 0$  (under the notation of this section). Consequently, we have

$$\mathcal{U}^1 M_1^q \xrightarrow{\cong} Rg_*(\mathcal{U}^1 M_{1,X'}^q) \quad \text{in } D(Y_{\text{ét}})$$

by [Lemma 6.1\(2\)](#) and the fact that  $R^s g_*(\mathcal{U}^1 M_{1,X'}^q) = 0$  for  $s \geq 2$ , see [[SGA 4<sub>3</sub> 1973](#), X.5.2, XII.5.2].

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