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SCATTERING RESONANCES ON TRUNCATED CONES

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We consider the problem of finding the resonances of the Laplacian on truncated Riemannian cones. In a similar fashion to Cheeger and Taylor, we construct the resolvent and scattering matrix for the Laplacian on cones and truncated cones. Following Stefanov, we show that the resonances on the truncated cone are distributed asymptotically as $Ar^n + o(r^n)$, where A is an explicit coefficient. We also conclude that the Laplacian on a nontruncated cone has no resonances.

1. Introduction

In this note, we consider the resonances on truncated Riemannian cones and establish a Weyl-type formula for their distribution. To fix notation, we let (Y, h) be a compact $(n-1)$-dimensional Riemannian manifold (with or without boundary) and let $C(Y)$ denote the cone over Y. In other words, $C(Y)$ is diffeomorphic to the product $(0, \infty)_r \times Y$ and is equipped with the incomplete Riemannian metric $g = dr^2 + r^2 h$. We refer the reader to the foundational works [Cheeger and Taylor 1982a; 1982b] for more details on the geometric set-up. We also introduce the *truncated* Riemannian cone $C_a(Y)$ formed by introducing a boundary at $r = a$; i.e., $C_a(Y)$ is diffeomorphic to $[a, \infty)_r \times Y$ and equipped with the same metric.

The (negative-definite) Laplacian on $C(Y)$ (or $C_a(Y)$ with a choice of boundary conditions) has the form

$$\partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_h,$$

where Δ_h denotes the Laplacian of (Y, h). Its resolvent $R(\lambda)$ is given by

$$R(\lambda) = (\Delta + \lambda^2)^{-1}.$$

We consider the *cutoff resolvent* $\chi R(\lambda)\chi$, where χ is a (fixed) smooth compactly supported function on $C(Y)$ (or $C_a(Y)$). One consequence of the resolvent formula of Theorem 2.1 is that the cutoff resolvent extends meromorphically to the logarithmic cover of $\mathbb{C} \setminus \{0\}$.

More precisely, we identify elements λ of the logarithmic cover of $\mathbb{C} \setminus \{0\}$ by a magnitude $|\lambda|$ and a phase $\arg \lambda \in \mathbb{R}$. We identify the "physical half-plane" as those λ with $\arg \lambda \in (0, \pi)$. These λ correspond to the resolvent set $\mathbb{C} \setminus [0, \infty)$ via the map $\lambda \mapsto |\lambda|^2 e^{2i \arg \lambda}$. The cutoff resolvent then extends to be meromorphic as a function of λ on this logarithmic cover.

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The poles of the cutoff resolvent consist of possibly finitely many L^2-eigenvalues lying in the upper half-plane (which do not appear with Dirichlet boundary conditions) and poles lying on other sheets of the cover. The latter poles are called the *resonances* of Δ.

The main theorem of this paper counts the most physically relevant resonances for the truncated cone. In particular, we count those resonances λ nearest to the physical half-plane, i.e., those with $\arg\lambda \in \left(-\frac{\pi}{2}, 0\right)$ and $\arg\lambda \in \left(\pi, \frac{3\pi}{2}\right)$. The resonances on other "sheets" of the cover remain more mysterious and are related to the zeros of Hankel functions near the real axis. We consider the resonance counting function on these sheets, defined by

$$N(r) = \#\{\lambda : \lambda \text{ is a resonance and } |\lambda| \le r\}.$$

The following theorem provides an asymptotic formula for $N(r)$.

Theorem 1.1. *Suppose either that the set of periodic geodesics of* (Y, h) *has Liouville measure zero or that* $Y = \mathbb{S}^{n-1}$ *equipped with a constant rescaling of the standard metric. Consider the truncated cone* $C_1(Y)$ *equipped with the Dirichlet Laplacian and let* $N(r)$ *denote its resonance counting function on the neighboring sheets as above. We then have, as* $r \to \infty$,

$$N(r) = A_n \operatorname{Vol}(Y, h) r^n + o(r^n),$$

where A_n *is an explicit constant* (*defined below in* (7)) *and* $\operatorname{Vol}(Y, h)$ *denotes the volume of the Riemannian manifold* (Y, h).

The constant $A_n \operatorname{Vol}(Y, h)$ in Theorem 1.1 is the same constant as computed in [Stefanov 2006] for the resonance counting function on the domain exterior to a ball in $\mathbb{R}^n$. When $Y = \mathbb{S}^{n-1}$ is equipped with its standard metric, the truncated cone $C_1(Y)$ can be thought of as the exterior of the unit ball in Euclidean space. Theorem 1.1 recovers Stefanov's result. (When $Y = \mathbb{S}^{n-1}$, n odd, is equipped with its standard metric, the cutoff resolvent in fact continues to the complex plane; this can be seen in the resolvent formulae below.)

We also state the following theorem, which is known to the community but does not seem to be in the literature.

Theorem 1.2. *If* (Y, h) *is a compact Riemannian manifold* (*with or without boundary*) *then the cone* $C(Y)$ *has no resonances.*

In fact, Theorem 2.1 below shows that λ is a resonance of the truncated cone $C_1(Y)$ if and only if λ/a is a resonance of the truncated cone $C_a(Y)$. Sending a to 0 then pushes all resonances out to infinity and provides evidence for Theorem 1.2.

The proof of Theorem 1.1 has two main steps. We first separate variables and obtain an explicit resolvent formula in Theorem 2.1 to characterize the resonances as zeros of a Hankel function. In Section 3 we consider the asymptotic distribution of the zeros of each Hankel function appearing in the resolvent formula. The hypothesis on the link (Y, h) is used to control the error terms when synthesizing the result. Theorem 1.2 is an immediate corollary of the resolvent formula in Theorem 2.1.

The proof of Theorem 1.1 follows an argument of [Stefanov 2006] very closely. Stefanov established a Weyl-type law for the distribution of resonances for the exterior of a ball in odd-dimensional Euclidean space. The main contribution of this paper is the observation that, after some natural modifications, the core of Stefanov's argument applies to the setting of cones. Borthwick [2010; 2012] and Borthwick and Philipp [2014] showed that a similar approach works in the asymptotically hyperbolic setting.

We further remark that we have specialized to the Dirichlet Laplacian in Theorem 1.1 only for simplicity. For Neumann or Robin boundary conditions, the resolvent formula of Theorem 2.1 has an analogous expression. The resonance counting problem then involves counting zeros of $H_\nu^{(2)'} + C\nu H_\nu^{(2)}$, which can be handled with similar arguments.

2. Resolvent construction

In this section we write down an explicit formula (via separation of variables) for the resolvent and then show that the cut-off resolvent has a meromorphic continuation to the logarithmic cover Λ of the complex plane. The construction is essentially contained in the [Cheeger and Taylor 1982a; 1982b], but the resolvent is not explicitly written there.

Suppose ϕ_j form an orthonormal family of eigenfunctions for $-\Delta_h$ with corresponding eigenvalues μ_j^2. We decompose $L^2(C(Y))$ into a direct sum in terms of the eigenspaces of $-\Delta_h$, i.e.,

$$L^2(C_a(Y); \mathbb{C}) = \bigoplus_{j=0}^{\infty} L^2((a,\infty); E_j), \qquad f(r,y) = \sum_{j=0}^{\infty} f_j(r)\phi_j(y),$$

where the first space is defined with respect to the volume form induced by the metric and the latter spaces can be identified (via the identification $f(r)\phi_j(y) \mapsto f(r)$) with the space $L^2((a,\infty); \mathbb{C})$ equipped with the volume form $r^{n-1}\,dr$.

For $\arg\lambda \in (0,\pi)$, the resolvent $R(\lambda)$ splits as a direct sum of operators $R_j(\lambda)$ acting on $L^2((a,\infty), E_j)$, with measure $r^{n-1}\,dr$.

$$R(\lambda)\left(\sum_{j=1}^{\infty} f_j(r)\phi_j(y)\right) = \bigoplus_{j=1}^{\infty} (R_j(\lambda) f_j)\phi_j(y).$$

In this section, we prove the following explicit formula for the j-th piece of the resolvent. For the cone $C(Y)$ (i.e., for $a = 0$), we use the Friedrichs extension of the Laplacian to guarantee self-adjointness (though in high enough dimension the Laplacian is essentially self-adjoint):

Theorem 2.1. *The piece of the resolvent corresponding to the j-th eigenvalue has the following explicit expression on the truncated cone $C_a(Y)$ or the cone $C(Y)$ ($a = 0$):*

$$(R_j(\lambda) f)(r) = \int_a^\infty K_{a,j}(r,\tilde{r}) f(\tilde{r})\tilde{r}^{n-1}\,d\tilde{r},$$

where $K_{a,j}(r,\tilde{r})$ is given by

$$K_{a,j}(r,\tilde{r}) = \frac{\pi}{2i}(\tilde{r}r)^{-(n-2)/2} \begin{cases} H_{\nu_j}^{(1)}(\lambda\tilde{r}) J_{\nu_j}(\lambda r) - (J_{\nu_j}(\lambda a)/H_{\nu_j}^{(1)}(\lambda a)) H_{\nu_j}^{(1)}(\lambda\tilde{r}) H_{\nu_j}^{(1)}(\lambda r), & r < \tilde{r}, \\ J_{\nu_j}(\lambda\tilde{r}) H_{\nu_j}^{(1)}(\lambda r) - (J_{\nu_j}(\lambda a)/H_{\nu_j}^{(1)}(\lambda a)) H_{\nu_j}^{(1)}(\lambda\tilde{r}) H_{\nu_j}^{(1)}(\lambda r), & r > \tilde{r}. \end{cases}$$

Here J_ν are the standard Bessel functions of the first kind and $H_\nu^{(1)}$ are the Hankel functions of the first kind. The second term in both expressions should be interpreted as 0 *when* $a = 0$.

Proof. After separating variables, we may assume that $f = f_j(r)\phi_j(y)$. We construct the resolvent for $\Im\lambda > 0$ and then meromorphically continue the expression.

Writing $u = u_j(r)\phi_j(y)$, the equation $(\Delta + \lambda^2)u = f$ induces the following differential equation for u_j:

$$\partial_r^2 u_j + \frac{n-1}{r}\partial_r u_j - \frac{\mu_j^2}{r^2}u_j + \lambda^2 u_j = f_j. \tag{1}$$

We solve this equation by showing it is equivalent to a Bessel equation.

Changing variables to $\rho = \lambda r$ and writing $\tilde{u}(\rho) = u(\rho/\lambda)$ yields

$$\partial_\rho^2 \tilde{u} + \frac{n-1}{\rho}\partial_\rho \tilde{u} + \left(1 - \frac{\mu_j^2}{\rho^2}\right)\tilde{u} = \frac{1}{\lambda^2}\tilde{f}(\rho).$$

Writing $v = \rho^{(n-2)/2}\tilde{u}$, we obtain a Bessel equation for v:

$$v'' + \frac{1}{\rho}v' + \left(1 - \frac{\nu_j^2}{\rho^2}\right)v = g(\rho), \tag{2}$$

where

$$\nu_j^2 = \mu_j^2 + ((n-2)/2)^2 \quad \text{and} \quad g(\rho) = \frac{\rho^{(n-2)/2}}{\lambda^2}\tilde{f}(\rho).$$

We now proceed by the standard ODE technique of variation of parameters. One basis for the space of solutions of the homogeneous version of this Bessel equation is $\{J_{\nu_j}(\rho), H_{\nu_j}^{(1)}(\rho)\}$, where J_ν is the Bessel function of the first kind and $H_\nu^{(1)}$ is the Hankel function of the first kind. We thus may use the following basis for the space of solutions of the homogeneous equation:

$$w_1(r) = r^{-(n-2)/2}J_{\nu_j}(\lambda r), \quad w_2(r) = r^{-(n-2)/2}H_{\nu_j}^{(1)}(\lambda r). \tag{3}$$

For $\Im\lambda > 0$, $R_j(\lambda)f_j$ must lie in $L^2((a,\infty), r^{n-1}\,dr)$. If f_j is compactly supported, this means that $u_j = R_j(\lambda)f_j$ must be a multiple of $r^{-(n-2)/2}H_{\nu_j}^{(1)}(\lambda r)$ near infinity. When $a > 0$, u_j must satisfy the boundary condition at $r = a$. When $a = 0$, the choice of the Friedrichs extension requires that both u_j and u_j' lie in the weighted L^2 space near 0 and so u_j must be a multiple of $r^{-(n-2)/2}J_{\nu_j}(\lambda r)$ near $r = 0$ as any nonzero multiple of w_2 will not have this property.

We may thus write

$$u_j(r) = \left(\int_r^\infty \frac{w_2(\tilde{r})f_j(\tilde{r})}{W(w_1,w_2)(\tilde{r})}\,d\tilde{r}\right)w_1(r) + \left(C + \int_a^r \frac{w_1(\tilde{r})f_j(\tilde{r})}{W(w_1,w_2)(\tilde{r})}\,d\tilde{r}\right)w_2(r),$$

where C is a yet-to-be-determined constant, the functions w_1 and w_2 are as in (3), and $W(w_1, w_2)$ is their Wronskian. The Wronskian W can be easily computed in terms of the Wronskian of the Bessel and Hankel functions and is

$$W(w_1,w_2)(r) = r^{-(n-1)}\frac{2i}{\pi}.$$

We now turn our attention to the boundary condition. For $a = 0$, the requirement that the solution and its derivative live in L^2 forces $C = 0$, yielding the result. For $a \neq 0$, we require that $u_j(a) = 0$; i.e.,

$$\left(\frac{\pi}{2i}\int_a^\infty H^{(1)}_{\nu_j}(\lambda\tilde{r})\tilde{r}^{n/2} f(\tilde{r})\,d\tilde{r}\right) a^{-(n-2)/2} J_{\nu_j}(\lambda a) + C a^{-(n-2)/2} H^{(1)}_{\nu_j}(\lambda a) = 0,$$

and so we must have

$$C = -\frac{\pi}{2i}\frac{J_{\nu_j}(\lambda a)}{H^{(1)}_{\nu_j}(\lambda a)}\int_a^\infty H^{(1)}_{\nu_j}(\lambda\tilde{r})\tilde{r}^{n/2} f(x)\,dx,$$

finishing the proof. □

We now claim that $\chi R(\lambda)\chi$ has a meromorphic continuation:

Lemma 2.2. *Given a fixed $\chi \in C_c^\infty(\mathbb{R}_+ \times Y)$, $\chi R(\lambda)\chi$ meromorphically continues from*

$$\{\lambda \in \mathbb{C} : \Im\lambda > 0\}$$

to the logarithmic cover Λ of the complex plane.

Proof. We first prove the statement for the full cone; the statement for the truncated cone will follow by an appeal to the analytic Fredholm theorem.

Fix $\chi \in C_c^\infty((0, \infty))$ and regard $\chi(r)$ as a compactly supported smooth function on $C(Y)$. We let $R(\lambda)$ denote the resolvent on the nontruncated cone (i.e., $a = 0$) and $K(\lambda; r, y, \tilde{r}, \tilde{y})$ denote its integral kernel. In order to show that $\chi R(\lambda)\chi$ meromorphically continues, it suffices to show that for any $f, g \in L^2(C(Y))$, the function

$$\lambda \mapsto \langle \chi R(\lambda)\chi f, g\rangle$$

meromorphically continues to Λ.

Fix two such functions $f, g \in L^2(C(Y))$ and let $f_j(r)$ and $g_j(r)$ denote their coefficients in the expansion in terms of eigenfunctions of Δ_h, i.e.,

$$f(r, y) = \sum_{j=0}^\infty f_j(r)\phi_j(y).$$

We observe that because f and g are square-integrable, the sum and the integral commute; i.e.,

$$\|f\|^2_{L^2(C(Y))} = \int_0^\infty \sum_{j=0}^\infty |f_j(r)|^2 r^{n-1}\,dr = \sum_{j=0}^\infty \int_0^\infty |f_j(r)|^2 r^{n-1}\,dr.$$

From Theorem 2.1, we may write

$$\langle \chi R(\lambda)\chi f, g\rangle = \sum_{j=0}^\infty \left(\int_0^\infty\int_0^r (\tilde{r}r)^{-(n-2)/2}\chi(r)\chi(\tilde{r})f_j(\tilde{r})g_j(r)J_{\nu_j}(\lambda\tilde{r})H^{(1)}_{\nu_j}(\lambda r)\tilde{r}^{n-1}r^{n-1}\,d\tilde{r}\,dr \right.$$
$$\left. + \int_0^\infty\int_r^\infty (\tilde{r}r)^{-(n-2)/2}\chi(r)\chi(\tilde{r})f_j(\tilde{r})g_j(r)J_{\nu_j}(\lambda r)H^{(1)}_{\nu_j}(\lambda\tilde{r})\tilde{r}^{n-1}r^{n-1}\,d\tilde{r}\,dr\right), \quad (4)$$

where J_ν and $H_\nu^{(1)}$ are as above. Because each term in (4) meromorphically continues to the Riemann surface Λ, it suffices to show that the partial sums of the series converge locally (in λ) uniformly (in j).

By the asymptotic expansions of Bessel functions for large order, we know [DLMF 2018, 10.19] that, locally in $\lambda \in \Lambda$, and for $r \in \operatorname{supp} \chi$,

$$J_\nu(\lambda r) = \frac{1}{\sqrt{2\pi\nu}}\left(\frac{e\lambda r}{2\nu}\right)^{\nu} + o\left(\frac{1}{\sqrt{\nu}}\left(\frac{e\lambda r}{2\nu}\right)^{\nu}\right),$$

$$H_\nu^{(1)}(\lambda r) = \frac{1}{i}\sqrt{\frac{2}{\pi\nu}}\left(\frac{e\lambda r}{2\nu}\right)^{-\nu} + o\left(\frac{1}{\sqrt{\nu}}\left(\frac{e\lambda r}{2\nu}\right)^{-\nu}\right),$$

as $\nu \to \infty$ through the positive reals. In particular, for j large enough, each term in (4) can be bounded by

$$C\int_0^\infty\int_0^r \frac{1}{\pi\nu_j}\chi(r)\chi(\tilde{r}) f_j(\tilde{r}) g_j(r)\left[\left(\frac{\tilde{r}}{r}\right)^{\nu_j}(1+o(1))\right](\tilde{r}r)^{n/2}\,d\tilde{r}\,dr$$
$$+ C\int_0^\infty\int_r^\infty \frac{1}{\pi\nu_j}\chi(r)\chi(\tilde{r}) f_j(\tilde{r}) g_j(r)\left[\left(\frac{r}{\tilde{r}}\right)^{\nu_j}(1+o(1))\right](\tilde{r}r)^{n/2}\,d\tilde{r}\,dr.$$

Observe that in the first integral, $\tilde{r}/r$ is bounded by 1, while $r/\tilde{r}$ is bounded by 1 in the second.

Because χ is compactly supported, we may therefore bound each term (for j large enough) by

$$\frac{C_\chi}{\nu_j}\|f_j\|_{L^2}\|g_j\|_{L^2}.$$

This sequence is absolutely summable, so the partial sums of the series in (4) converge locally uniformly. This establishes that the cut-off resolvent on the full cone ($a = 0$) meromorphically extends to the logarithmic cover Λ of the complex plane.

We now proceed to the case of the truncated cone ($a > 0$). We proceed by an appeal to the analytic Fredholm theorem.

Fix $\chi_0, \chi_\infty \in C^\infty((a,\infty))$ so that $\chi_0(r)$ is supported near $r = a$, $\chi_\infty(r)$ is identically zero near $r = a$, and $\chi_0 + \chi_\infty = 1$. We let $R_\infty(\lambda)$ denote the resolvent on the nontruncated cone and $R_0(\lambda)$ denote the resolvent on a compact manifold with boundary into which the support of χ_0 embeds isometrically. We define the parametrix

$$Q(\lambda) = \tilde{\chi}_0 R_0(\lambda)\chi_0 + \tilde{\chi}_\infty R_\infty(\lambda)\chi_\infty,$$

where $\tilde{\chi}$ have similar support properties and are identically 1 on the support of their counterparts. Applying $\Delta + \lambda^2$ yields a remainder of the form $I + \sum[\Delta, \tilde{\chi}_i]R_i(\lambda)\chi_i$. Both terms are compact and the operator is invertible for large $\Im\lambda$ by Neumann series, so applying $R_a(\lambda)$ to both sides and inverting the remainder shows that it has a meromorphic continuation. □

3. Proof of Theorem 1.1

By the formula for the resolvent in Theorem 2.1, the resonances of $R_a(\lambda)$ correspond to those λ for which $H_{\nu_j}^{(1)}(\lambda a) = 0$ for some j. For simplicity we will discuss only the case $a = 1$ as the other cases can be found by rescaling. As mentioned in the introduction, we consider only those resonances nearest to the upper half-plane, i.e., those with

$$-\tfrac{\pi}{2} < \arg\lambda < 0 \quad \text{or} \quad \pi < \arg\lambda < \tfrac{3\pi}{2}. \tag{5}$$

Because ν_j is real, we may relate the zeros of $H^{(1)}_{\nu_j}(\lambda)$ in the region given by (5) to zeros of $H^{(2)}_{\nu_j}(\lambda)$ in the quadrant $0 < \arg\lambda < \frac{\pi}{2}$ via analytic continuation formulae. Indeed, it is well known [DLMF 2018, 10.11.5, 10.11.9] that

$$\begin{aligned} H^{(1)}_\nu(ze^{\pi}) &= -e^{-\nu\pi\imath} H^{(2)}_\nu(z),\\ H^{(1)}_\nu(\bar z) &= \overline{H^{(2)}_\nu(z)}. \end{aligned} \tag{6}$$

The first of these equations identifies zeros of $H^{(1)}_\nu$ in $\pi < \arg\lambda < \frac{3\pi}{2}$ to zeros of $H^{(2)}_\nu$ in the first quadrant; the second equation does the same for zeros of $H^{(1)}_\nu$ with $-\frac{\pi}{2} < \arg\lambda < 0$. In particular, each zero of $H^{(2)}_\nu$ with $0 \le \arg\lambda \le \frac{\pi}{2}$ corresponds to exactly two resonances.

For large enough ν, the zeros of the Hankel function $H^{(2)}_\nu$ in the first quadrant lie near the boundary of (a scaling of) an "eye-like" domain $K \subset \mathbb{C}$. The domain K is symmetric about the real axis and is bounded by the following curve and its conjugate:

$$z = \pm(t\coth t - t^2)^{1/2} + i(t^2 - t\tanh t)^{1/2}, \quad 0 \le t \le t_0,$$

where t_0 is the positive root of $t = \coth t$. We refer to the piece of the boundary of K lying in the upper half-plane by ∂K_+.

The constant A_n above is given by

$$A_n = \frac{2(n-1)\operatorname{Vol}(B_{n-1})}{n(2\pi)^n} \int_{\partial K_+} \frac{|1-z^2|^{1/2}}{|z|^{n+1}}\, d|z|, \tag{7}$$

where B_{n-1} is the $(n-1)$-dimensional unit ball. Observe that, up to a factor of the volume of the unit sphere (which is replaced by the volume of Y in the theorem statement), the constant A_n is the same constant computed in [Stefanov 2006].

We use below two different parametrizations of the piece of ∂K_+ lying the in the quadrant $0 \le \arg z \le \frac{\pi}{2}$. The first parametrization is by the argument of z, i.e., by the map

$$\left[0, \tfrac{\pi}{2}\right] \to \partial K_+, \quad \theta = \arg z \mapsto z = z(\theta).$$

For the second parametrization, we introduce the function ρ, defined by

$$\rho(z) = \tfrac{2}{3}\zeta^{3/2} = \log\frac{1+\sqrt{1-z^2}}{z} - \sqrt{1-z^2}, \quad |\arg z| < \pi, \tag{8}$$

where (following [Stefanov 2006, Section 4; Olver 1974, Chapter 10]) the branches of the functions above are chosen so that ζ is real when z is. Another characterization is that the principal branches are chosen when $0 < z < 1$ and continuity is demanded elsewhere.

The boundary ∂K is the vanishing set of $\Re\rho$. This yields a parametrization of the part of ∂K_+ lying in $0 \le \arg z \le \frac{\pi}{2}$:

$$\left[0, \tfrac{\pi}{2}\right] \to \partial K_+, \quad t \mapsto \rho^{-1}(-it) = z.$$

The transition between the two parametrizations is given by

$$\frac{dt}{d\theta} = \frac{dt}{dz}\frac{dz}{d\theta} = (i\rho'(z))(iz) = \sqrt{1-z^2}.$$

The function ζ defined in (8) is the solution of the ODE

$$\left(\frac{d\zeta}{dz}\right)^2 = \frac{1-z^2}{\zeta z^2}$$

that is infinitely differentiable on the positive real axis (including at $z = 1$). As is implicit in (8), it can be analytically continued to the complex plane with a branch cut along the negative real axis.

Because the resonances correspond to zeros of $H^{(2)}_{\nu_j}$, we must also consider the asymptotic distribution of the ν_j. In what follows, we consider only the case when the periodic geodesics of (Y, h) have measure zero.[1] The eigenvalues μ_j^2 of Δ_h obey Weyl's law:

$$\begin{aligned} N_h(\mu) &= \#\{\mu_j : \mu_j \le \mu \text{ with multiplicity}\} \\ &= \frac{\operatorname{Vol} B_{n-1}}{(2\pi)^{n-1}} \operatorname{Vol}(Y, h)\mu^{n-1} + R(\mu). \end{aligned}$$

Here $\operatorname{Vol}(B_{n-1})$ denotes the volume of the unit ball in $\mathbb{R}^{n-1}$ and $\operatorname{Vol}(Y, h)$ is the volume of Y equipped with the metric h. In general, $R(\mu) = O(\mu^{n-2})$, but if we now impose the dynamical hypothesis (that the set of periodic geodesics of (Y, h) has Liouville measure zero), then a theorem of [Duistermaat and Guillemin 1975] (in the boundaryless case) and [Ivrii 1980; 1982] (in the boundary case) shows that

$$R(\lambda) = o(\mu^{n-2}).$$

The nonperiodicity assumption then allows us to count eigenvalues on intervals of length 1:

$$\begin{aligned} N_h(\mu, \mu+1) &= \#\{\mu_j : \mu \le \mu_j \le \mu + 1 \text{ with multiplicity}\} \\ &= (n-1)\frac{\operatorname{Vol}(B_{n-1})}{(2\pi)^{n-1}} \operatorname{Vol}(Y, h)\mu^{n-2} + o(\mu^{n-2}). \end{aligned}$$

As $\nu_j^2 = \mu_j^2 + (n-2)^2/4$, the same counting formula holds for ν_j; i.e.,

$$\begin{aligned} N_\nu(\rho, \rho+1) &= \#\{\nu_j : \rho \le \nu_j \le \rho + 1 \text{ with multiplicity}\} \\ &= (n-1)\frac{\operatorname{Vol}(B_{n-1})}{(2\pi)^{n-1}} \operatorname{Vol}(Y, h)\rho^{n-2} + o(\rho^{n-2}). \end{aligned} \tag{9}$$

We now turn our attention to the zeros of the Hankel function $H^{(2)}_\nu(z)$ with $\arg z \in \left[0, \frac{\pi}{2}\right]$. An argument from [Watson 1944, pages 511–513] is easily adapted to give a precise count of the number of zeros of

[1] When (Y, h) is a sphere, the analysis is simplified slightly. In that case, one replaces the use of the Weyl formula with explicit formulae for the eigenvalues μ_j^2 and their multiplicities.

$H_\nu^{(2)}$ in this sector. Indeed, that argument shows that the number of zeros is given by the closest integer to $\frac{1}{2}\nu - \frac{1}{4}$ (when $\nu - \frac{1}{2}$ is an integer, there is a zero on the imaginary axis and so rounds up).

As $\nu \to \infty$ through positive real values, we have an asymptotic expansion [DLMF 2018, 10.20.6] relating the Hankel function to the Airy function

$$H_\nu^{(2)}(\nu z) \sim 2e^{i\pi/3}\left(\frac{4\zeta}{1-z^2}\right)^{1/4}\left(\frac{\mathrm{Ai}(e^{-2\pi i/3}\nu^{2/3}\zeta)}{\nu^{1/3}}\sum_{k=0}^{\infty}\frac{A_k(\zeta)}{\nu^{2k}} + \frac{\mathrm{Ai}'(e^{-2\pi i/3}\nu^{2/3}\zeta)}{\nu^{5/3}}\sum_{k=0}^{\infty}\frac{B_k(\zeta)}{\nu^{2k}}\right). \quad (10)$$

Here A_k and B_k are real and infinitely differentiable for $\zeta \in \mathbb{R}$. This expansion is uniform in $|\arg z| \le \pi - \delta$ for fixed $\delta > 0$. In particular, for large enough ν, the zeros of the Hankel function are well-approximated by zeros of the Airy function and we may identify each zero $h_{\nu,k}$ of the Hankel function $H_\nu^{(2)}$ with a zero of the Airy function $\mathrm{Ai}(-z)$.

Let a_k denote the k-th zero of the Airy function $\mathrm{Ai}(-z)$; all a_k are positive and

$$a_k = \left[\tfrac{3}{2}\left(k\pi - \tfrac{\pi}{4}\right)\right]^{2/3} + O(k^{-4/3}).$$

We now define $\lambda_{\nu,k}$ and $\tilde{\lambda}_{\nu,k}$ via the Airy zeros and their leading approximations:

$$\lambda_{\nu,k} = \nu\zeta^{-1}(\nu^{-2/3}e^{-i\pi/3}a_k) = \nu\rho^{-1}\left(-i\tfrac{2}{3}a_k^{3/2}\nu^{-1}\right),$$
$$\tilde{\lambda}_{\nu,k} = \nu\rho^{-1}\left(-i\left(k-\tfrac{1}{4}\right)\pi\nu^{-1}\right),$$

where $k = 1, \dots, \left\lfloor \frac{1}{2}\nu + \frac{1}{4}\right\rfloor$. By the Hankel expansion (10), $|h_{\nu,k} - \lambda_{\nu,k}| \le C/\nu$ for large enough ν, while $|h_{\nu,k} - \tilde{\lambda}_{\nu,k}| \le C/\nu$ for large enough ν and k. As we have identified $\left\lfloor \frac{1}{2}\nu + \frac{1}{4}\right\rfloor$ approximate zeros, we can conclude that these account for all $h_{\nu,k}$.

We now divide our attention into those zeros with small argument and those with large argument. We introduce the auxiliary counting function

$$N(r, \theta_1, \theta_2) = \#\{\sigma : \sigma \text{ is a resonance with } |\sigma| \le r,\ \arg\sigma \in [\theta_1, \theta_2]\}.$$

We first address those with small argument. Fix $\epsilon > 0$ and consider those zeros with $|z| < r$ and $\arg z \in [0, \epsilon]$. We need count those $\lambda_{\nu,k}$ with $\arg\lambda_{\nu,k} \in [0, \epsilon]$ and $|\lambda_{\nu,k}| \le r$. As $|\lambda_{\nu,k}|$ is comparable to ν, we can over-count these zeros by counting all $\lambda_{\nu,k}$ with argument in $[0, \epsilon]$ and $\nu \le Cr$.

Because $|\rho| \le C\epsilon^{3/2}$ for those $\lambda_{\nu,k}$ with $\arg\lambda_{\nu,k} \in [0, \epsilon]$, we must only count those a_k with $a_k \le C\nu^{2/3}\epsilon$. The leading order asymptotic [DLMF 2018, 9.9.6] for the zeros of the Airy function shows that this number is $O(\nu\epsilon^{3/2})$.

We now count those resonances with argument in $[0, \epsilon]$. Putting together the asymptotic for ν_j in (9) with the previous two paragraphs, we have (with $m(\nu_j)$ denoting the multiplicity of ν_j)

$$\begin{aligned} N(r, 0, \epsilon) &= \sum_{j=1}^{\infty} m(\nu_j)\#\{h_{\nu_j,k} : |h_{\nu_j,k}| \le r, \arg h_{\nu_j,k} \in [0, \epsilon]\} \\ &\le \sum_{j=1}^{Cr} m(\nu_j)C\nu_j\epsilon^{3/2} \le C\epsilon^{3/2}\sum_{\rho=0}^{Cr}\sum_{\nu_j \in [\rho,\rho+1]} m(\nu_j)\rho \le C\epsilon^{3/2}r^n. \end{aligned} \quad (11)$$

We now consider those resonances with argument in $\left[\epsilon, \frac{\pi}{2}\right]$. For large enough ν, the approximations $\tilde{\lambda}_{\nu,k}$ are valid for these resonances. We count those approximate resonances with $\nu_j \in [\rho, \rho+1)$ and $\arg \lambda_{\nu,k} \in [\theta, \theta+\Delta\theta]$. We start by introducing, for fixed ν, the number Δk_ν of $\tilde{\lambda}_{\nu,k}$ with argument lying in $[\theta, \theta+\Delta\theta]$. Observe that the definition of $\tilde{\lambda}_{\nu,k}$ relates Δk_ν with Δt by

$$\Delta k_\nu = \frac{\nu}{\pi}\Delta t + O(1),$$

where Δt denotes the change in t corresponding to $\Delta\theta$ in the parametrizations above. Note that Δt is *independent* of the choice of ν. We can then write

$$\#\{\tilde{\lambda}_{\nu,k} : \nu_j \in [\rho, \rho+1), \arg \tilde{\lambda}_{\nu,k} \in [\theta, \theta+\Delta\theta]\} = \sum_{\rho \le \nu_j \le \rho+1} m(\nu_j)\Delta k_\nu = \sum_{\rho \le \nu_j < \rho+1} m(\nu_j)\left(\frac{\nu_j}{\pi}\Delta t + O(1)\right).$$

By the definition of the approximate zeros $\tilde{\lambda}_{\nu,k}$, we can estimate their size $|\tilde{\lambda}_{\nu,k}|$ in terms of $|z(\theta)|$, provided that $\arg \tilde{\lambda}_{\nu,k} \in [\theta, \theta+\Delta\theta]$, yielding

$$|\tilde{\lambda}_{\nu,k}| = \nu(|z(\theta)| + O(\Delta\theta)).$$

In particular, if $\nu_j|z(\theta)| \ge r$ but $|\lambda_{\nu,k}| \le r$, then

$$\nu_j \in \left[\frac{r}{|z(\theta)|}(1 - c\Delta\theta), \frac{r}{|z(\theta)|}\right].$$

We may thus rewrite our counting function as

$$\begin{aligned}\#\{\tilde{\lambda}_{\nu,k} : |\tilde{\lambda}_{\nu,k}| \le r, \arg \tilde{\lambda}_{\nu,k} \in [\theta, \theta+\Delta\theta]\} &= \sum_{\substack{|\tilde{\lambda}_{\nu,k}| \le r \\ \arg \tilde{\lambda}_{\nu,k} \in [\theta,\theta+\Delta\theta]}} m(\nu_j) \\ &= \sum_{\substack{\nu_j|z(\theta)| \le r \\ \arg \tilde{\lambda}_{j,k} \in [\theta,\theta+\Delta\theta]}} m(\nu_j) + \sum_{\substack{\nu_j \in [(r/|z(\theta)|)(1-c\Delta\theta), r/|z(\theta)|] \\ \arg \tilde{\lambda}_{\nu,k} \in [\theta,\theta+\Delta\theta]}} m(\nu_j).\end{aligned}$$

By our improved Weyl's law (9), the second term is $O(r^{n-2})$.

We now focus our attention on the first term (here $\lfloor \cdot \rfloor$ denotes the "floor" function):

$$\begin{aligned}\sum_{\substack{\nu_j|z(\theta)| \le r \\ \arg \tilde{\lambda}_{j,k} \in [\theta,\theta+\Delta\theta]}} m(\nu_j) &= \sum_{\rho=0}^{\lfloor r/|z| - 1 \rfloor} \sum_{\nu_j \in [\rho,\rho+1)} \sum_{\arg \tilde{\lambda}_{\nu,k} \in [\theta,\theta+\Delta\theta]} m(\nu_j) + \sum_{\nu_j \in [\lfloor r/z \rfloor, r/z]} \sum_{\arg \tilde{\lambda}_{\nu,k} \in [\theta,\theta+\Delta\theta]} m(\nu_j) \\ &= \sum_{\rho=0}^{\lfloor r/|z| - 1 \rfloor} \sum_{\nu_j \in [\rho,\rho+1)} m(\nu_j)\Delta k_\nu + \sum_{\nu_j \in [\lfloor r/z \rfloor, r/z]} \sum_{\arg \tilde{\lambda}_{\nu,k} \in [\theta,\theta+\Delta\theta]} m(\nu_j).\end{aligned}$$

Again by Weyl's law, we observe that the second term is $O(r^{n-2})$. By relating Δt and Δk_ν we can rewrite the first term:

$$\sum_{\rho=0}^{\lfloor r/|z| - 1 \rfloor} \sum_{\nu_j \in [\rho,\rho+1)} m(\nu_j)\Delta k_\nu = \sum_{\rho=0}^{\lfloor r/|z| - 1 \rfloor} \sum_{\nu_j \in [\rho,\rho+1)} m(\nu_j)\frac{\nu_j}{\pi}\Delta t + \sum_{\nu_j \le \lfloor r/|z| \rfloor} m(\nu_j)O(1).$$

By Weyl's law (9), the second term is $O(r^{n-1})$, so we again consider the first term.

As Δt is independent of ν_j, we may use Weyl's law as well on the first term:

$$\sum_{\rho=0}^{\lfloor r/|z|-1\rfloor} \sum_{\nu_j\in[\rho,\rho+1)} m(\nu_j)\frac{\nu_j}{\pi}\Delta t = \sum_{\rho=0}^{\lfloor r/|z|-1\rfloor}\left[\frac{n-1}{2^{n-1}\pi^n}\operatorname{Vol}(B_{n-1})\operatorname{Vol}(Y,h)\rho^{n-1}\Delta t + O(\rho^{n-2}) + o(\rho^{n-1})\Delta t\right]$$
$$= \frac{2(n-1)}{(2\pi)^n}\operatorname{Vol}(B_{n-1})\operatorname{Vol}(Y,h)\Delta t \sum_{\rho=0}^{\lfloor r/|z|-1\rfloor}\rho^{n-1} + O(r^{n-1}) + o(r^n)\Delta t$$
$$= \frac{2(n-1)}{(2\pi)^n n}\operatorname{Vol}(B_{n-1})\operatorname{Vol}(Y,h)\frac{1}{n}\left(\frac{r}{|z(\theta)|}\right)^n\Delta t + O(r^{n-1}) + o(r^n)\Delta t.$$

We finally introduce a Riemann sum in t to understand this main term:

$$\#\{\tilde\lambda_{\nu,k} : |\tilde\lambda_{\nu,k}| \le r,\ \arg\tilde\lambda_{\nu,k}\in[\epsilon,\tfrac{\pi}{2}]\}$$
$$= \int_{t^{-1}(\epsilon)}^{\pi/2}\left(\frac{2(n-1)\operatorname{Vol}(B_{n-1})}{(2\pi)^n n}\operatorname{Vol}(Y,h)\right)\frac{r^n}{|z(\theta)|^n}\,dt + O(r^{n-1}) + o(r^n)$$
$$= \frac{(n-1)\operatorname{Vol}(B_{n-1})}{(2\pi)^n n}\operatorname{Vol}(Y,h)r^n\int_{\partial K_+}\frac{1}{|z(\theta)|^n}\,dt + O(\epsilon r^n) + o(r^n)$$
$$= \left(\frac{(n-1)\operatorname{Vol}(B_{n-1})}{(2\pi)^n n}\operatorname{Vol}(Y,h)\int_{\partial K_+}\frac{|1-z^2|^{1/2}}{|z|^{n+1}}\,d|z|\right)r^n + O(\epsilon r^n) + o(r^n)$$
$$= A_n\operatorname{Vol}(Y,h)r^n + O(\epsilon r^n) + o(r^n). \tag{12}$$

Here the prefactor of 2 disappeared because the first integral parametrizes only half of ∂K_+. It reappears in the statement of Theorem 1.1 because each zero there corresponds to two resonances (one on each sheet). We further observe that the constant $A_n\operatorname{Vol}(Y,h)$ agrees with the leading term found in the Euclidean case found in [Stefanov 2006].

Sending ϵ to 0 establishes the theorem for the approximate zeros $\lambda_{\nu,k}$. Because each $\lambda_{\nu,k}$ is in a C/ν neighborhood of a zero $h_{\nu,k}$, this finishes the proof of the theorem.

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Cover image: The figure shows the outgoing scattered field produced by scattering a plane wave, coming from the northwest, off of the (stylized) letters P A A. The total field satisfies the homogeneous Dirichlet condition on the boundary of the letters. It is based on a numerical computation by Mike O'Neil of the Courant Institute.

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