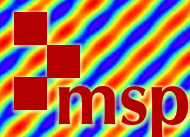


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We prove L^p estimates for the Baouendi–Grushin operator $\Delta_x + |x|^\alpha \Delta_y$ in $L^p(\mathbb{R}^{N+M})$, $1 < p < \infty$, where $x \in \mathbb{R}^N$, $y \in \mathbb{R}^M$. When $p = 2$ more general weights belonging to the reverse Hölder class $B_2(\mathbb{R}^N)$ are allowed.

1. Introduction

We prove L^p estimates for the Baouendi–Grushin operator $L = \Delta_x + |x|^\alpha \Delta_y$ in $L^p(\mathbb{R}^{N+M})$, $1 < p < \infty$, where $x \in \mathbb{R}^N$, $y \in \mathbb{R}^M$; more specifically, we prove the L^p boundedness of the operators $D_{x_i x_j} L^{-1}$, $|x|^\alpha D_{y_i y_j} L^{-1}$ and $|x|^{\frac{\alpha}{2}} D_{x y_i} L^{-1}$. We use these results to characterize the domain of the operator L , denoted by $D_p(L)$, where the solution of the equation $\lambda u - Lu = f$ exists and is unique for any $f \in L^p(\mathbb{R}^{N+M})$ and $\lambda > 0$. In an equivalent way, we describe the domain under which L generates an (analytic and symmetric) semigroup in $L^p(\mathbb{R}^{N+M})$.

If α is an even integer, the operator is hypoelliptic but is not a sublaplacian in the sense of [Folland 1975] and our estimates seem to be known only for $\alpha = 2$ and even $N \geq 2$; see [Koch et al. 2015]. When α is an unrestricted positive real number, many results are known on local regularity of the equation $Lu = f$; see for example [Franchi and Serapioni 1987; Franchi et al. 1994; Franchi and Lanconelli 1984; Garofalo and Vassilev 2007] for the unique continuation property. We refer to [Robinson and Sikora 2008] for heat kernel estimates even in a more general context. However, we are not aware of global regularity results for the second derivatives of u , with the exception of [Wang 2003], where global Hölder regularity is proved for every $\alpha > 0$ and of [Kim 1999], where L^p estimates are proved, when $\alpha = 1$ and $N = 1$, in the half-plane $x > 0$ for the inhomogeneous problem $Lu = f$, $u(0, y) = g(y)$. When $g = 0$ the estimates in [Kim 1999] reduce to ours: even though our results are valid in the whole space, they can be rephrased when $N = 1$ in the half-space $x > 0$ for Dirichlet or Neumann boundary conditions by considering odd and even functions with respect to x , respectively.

We prove L^p estimates through an interpolation theorem in the absence of kernels in homogeneous spaces due to Z. Shen [2005, Theorem 3.1], see also [Auscher and Martell 2007, Theorem 3.14], and weighted mean value inequalities for subsolutions of the elliptic equation $Lu = 0$, with respect to the balls associated with the subelliptic distance defined by the operator, proved in [Franchi and Serapioni 1987; Chanillo and Wheeden 1986]. Some of these results can probably be generalized to the case when $|x|^\alpha$ is replaced by a weight function $\phi(x)$ belonging to the reverse Hölder class $B_p(\mathbb{R}^N)$. This is the

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case when $p = 2$, where the result is not obtained via integration by parts but using maximal inequalities due to P. Auscher and B. Ben Ali [2007] for Schrödinger operators with B_2 potentials. However, local estimates for subsolutions seem to be known only in special cases and they are crucial in our approach when $p \neq 2$. Another restriction comes from the estimates of the mixed derivatives, that is, for the operator $|x|^{\frac{\alpha}{2}} D_{xy_i} L^{-1}$, where our proof relies on scaling. In order to unify our approach and to improve the readability, we consider only the $|x|^\alpha$ case in our L^p estimates. Perturbation arguments from this model case allow us to treat different powers near 0 and ∞ or power-like behavior, but we prefer not to deal with these variants here.

The paper is organized as follows. In Section 2 we define the operator in $L^2(\mathbb{R}^{N+M})$ through a form and prove L^2 estimates via partial Fourier transform and maximal regularity results on Schrödinger operators. In Section 3 we briefly recall the subelliptic distance associated with L and the main geometrical objects needed in L^p estimates. The latter are proved in Section 4, where a separate subsection deals with mixed derivatives.

Notation. We use L^p for $L^p(\mathbb{R}^{N+M})$ and C_c^∞ for $C_c^\infty(\mathbb{R}^{N+M})$. L_c^∞ stands for the space of all bounded measurable functions on $\mathbb{R}^{N+M}$ having compact support. $\mathcal{S}$ is the Schwartz space and $\mathcal{S}'$ the space of tempered distributions. We also write $B(r) := \{x \in \mathbb{R}^N : |x| < r\}$, $B(x_0, r) = x_0 + B(r)$.

2. L^2 estimates

Let $\phi : \mathbb{R}^N \rightarrow [0, +\infty[$ be a nonnegative continuous function and set

$$\mathbb{R}^N \supseteq \Omega_N = \{x \in \mathbb{R}^N : \phi(x) > 0\}, \quad \Omega = \Omega_N \times \mathbb{R}^M.$$

Let L be the operator defined on smooth functions by

$$L = \Delta_x + \phi(x) \Delta_y,$$

where $x \in \mathbb{R}^N$, $y \in \mathbb{R}^M$. Setting

$$a = \left(\begin{array}{c|c} I_N & 0 \\ \hline 0 & \phi I_M \end{array} \right) = (a_{ij})$$

or

$$a_{ij}(x, y) = \begin{cases} 1 & \text{if } i = j \leq N, \\ \phi(x) & \text{if } N+1 \leq i = j \leq N+M \end{cases} \quad (1)$$

and 0 elsewhere, we can write

$$L = \operatorname{div}(a \nabla)$$

and, therefore, L is formally self-adjoint with respect to the Lebesgue measure.

Remark 2.1. L is nondegenerate in the x -direction but degenerates in the y -direction outside Ω . Accordingly, $\nabla_x u$ will denote the distributional gradient (with respect to x) of u in the whole space $\mathbb{R}^{N+M}$ and $\nabla_y u$ only its distributional gradient (with respect to y) in Ω .

We give a formal definition of L through a symmetric form:

Definition 2.2. Consider the sesquilinear form $\mathfrak{a}$ in L^2 defined by

$$\begin{aligned}\mathfrak{a}(u, v) &:= \int_{\mathbb{R}^{N+M}} [\langle \nabla_x u, \nabla_x \bar{v} \rangle + \phi(x) \langle \nabla_y u, \nabla_y \bar{v} \rangle] dx dy, \\ D(\mathfrak{a}) &:= \{u \in L^2 : u \in H_{\text{loc}}^1(\Omega), \nabla_x u, \phi^{\frac{1}{2}} \nabla_y u \in L^2\}.\end{aligned}$$

According to the remark above, we require that the weak gradient $\nabla_y u$ exists only in Ω .

We summarize in the following lemma the main properties of $\mathfrak{a}$. Note that, due to the assumptions on ϕ , $\mathfrak{a}$ is locally uniformly elliptic on Ω .

Lemma 2.3. *The form $\mathfrak{a}$ is densely defined, nonnegative, symmetric and closed in L^2 and the following properties hold:*

- (i) *If Q is an orthogonal matrix in $\mathbb{R}^M$, $y_0 \in \mathbb{R}^M$ and $I_{Q+y_0}u(x, y) = u(x, Qy + y_0)$, then for every $u, v \in D(\mathfrak{a})$, one has $I_{Q+y_0}u, I_{Q+y_0}v \in D(\mathfrak{a})$ and*

$$\mathfrak{a}(I_{Q+y_0}u, I_{Q+y_0}v) = \mathfrak{a}(u, v).$$

- (ii) *If ϕ is homogeneous of degree $\alpha \geq 0$, i.e., $\phi(sx) = s^\alpha \phi(x)$ for $x \in \mathbb{R}^N$, $s > 0$, then defining the dilation*

$$I_s u(x, y) = u(sx, s^{\frac{2+\alpha}{2}} y),$$

for every $u, v \in D(\mathfrak{a})$ one has $I_s u, I_s v \in D(\mathfrak{a})$ and

$$\mathfrak{a}(I_s u, I_s v) = s^{2-N-\frac{2+\alpha}{2}M} \mathfrak{a}(u, v).$$

Proof. Clearly, due to the positivity of ϕ , $\mathfrak{a}$ is a nonnegative symmetric form in L^2 . The closedness of the form follows easily since $\mathfrak{a}$ is locally uniformly elliptic in Ω . The proofs of (i) and (ii) follow by a straightforward computation. $\square$

Let $-L$ be the operator associated with $\mathfrak{a}$, that is,

$$\begin{aligned}D(L) &:= \{u \in D(\mathfrak{a}) : \text{there exists } v \in L^2 \text{ such that } \mathfrak{a}(u, w) = \int_{\mathbb{R}^{N+M}} v \bar{w} dx dy \text{ for all } w \in D(\mathfrak{a})\}, \\ -Lu &:= v.\end{aligned}\tag{2}$$

The basic properties of L are listed below.

Proposition 2.4. *The operator $-L$ defined in (2) is nonnegative and self-adjoint. Moreover:*

- (i) $C_c^\infty \hookrightarrow D(L) \hookrightarrow \{u \in L^2 \cap W_{\text{loc}}^{2,2}(\Omega) : Lu \in L^2\}$ and for every $u \in C_c^\infty$

$$Lu = \Delta_x u + \phi(x) \Delta_y u.$$

- (ii) L generates a contractive analytic semigroup $\{e^{zL} : z \in \mathbb{C}_+\}$ in L^2 .

- (iii) The semigroup $\{e^{tL} : t > 0\}$ is submarkovian; i.e., it is positive and L^∞ -contractive.

- (iv) If Q is an orthogonal matrix in $\mathbb{R}^M$ and $y_0 \in \mathbb{R}^M$, then

$$L = I_{Q+y_0}^{-1} L I_{Q+y_0}, \quad I_{Q+y_0} u(x, y) = u(x, Qy + y_0).$$

(v) If ϕ is homogeneous of degree α , then

$$s^2 L = I_s^{-1} L I_s, \quad I_s u(x, y) = u(sx, s^{\frac{2+\alpha}{2}} y), \quad s > 0.$$

Proof. Part (i) is clear by construction and from interior elliptic regularity (see, however, the proof of Theorem 2.9 for justifying the integration by parts). The generation property of L follows by standard results, see [Ouhabaz 2005, Chapter 1, Section 4]; the positivity of e^{tL} as well its L^∞ -contractivity is a consequence of the Beurling–Deny criteria satisfied by the form $\mathfrak{a}$, see [Ouhabaz 2005, Corollary 2.18], and note that $\mathfrak{a}$ is real; that is, $\mathfrak{a}(u, v) \in \mathbb{R}$ whenever u, v are real functions. Concerning (iv) and (v), let $u \in D(L)$, $v \in D(\mathfrak{a})$ and $s > 0$. Then

$$\begin{aligned} \mathfrak{a}(I_s u, v) &= s^{2-N-\frac{2+\alpha}{2}M} \mathfrak{a}(u, I_{s^{-1}} v) \\ &= -s^{2-N-\frac{2+\alpha}{2}M} \int_{\mathbb{R}^{N+M}} (Lu) I_{s^{-1}} \bar{v} \, dx \, dy = -s^2 \int_{\mathbb{R}^{N+M}} (I_s Lu) \bar{v} \, dx \, dy; \end{aligned}$$

hence $I_s u \in D(L)$ and $L I_s u = s^2 I_s Lu$. The proof for I_{Q+y_0} is similar. $\square$

The following proposition shows that C_c^∞ is dense in $D(L)$ with respect to the graph norm.

Proposition 2.5. C_c^∞ is a core for the operator $(L, D(L))$ and the form $\mathfrak{a}$.

Proof. Since $I - L$ is invertible we have to show that $(I - L)(C_c^\infty)$ is dense in L^2 or, equivalently, that $(I - L)(C_c^\infty)^\perp = \{0\}$. To this aim let $v \in L^2$ such that

$$\int_{\mathbb{R}^{N+M}} (I - L)u \, \bar{v} \, dx \, dy = 0 \quad \text{for all } u \in C_c^\infty.$$

Taking the partial Fourier transform with respect to the y -variable and applying the Fubini and Plancherel theorems we get

$$\int_{\mathbb{R}^{N+M}} [\hat{u}(x, \xi) - \Delta_x \hat{u}(x, \xi) + \phi(x)|\xi|^2 \hat{u}(x, \xi)] \bar{\hat{v}}(x, \xi) \, dx \, d\xi = 0 \quad \text{for all } u \in C_c^\infty.$$

Choosing $u = A(x)B(y) \in C_c^\infty$ we have $\hat{u}(x, \xi) = A(x)\widehat{B}(\xi)$ and

$$\int_{\mathbb{R}^{N+M}} [A(x) - \Delta_x A(x) + \phi(x)|\xi|^2 A(x)] \widehat{B}(\xi) \bar{\hat{v}}(x, \xi) \, dx \, d\xi = 0. \quad (3)$$

Fix $\xi_0 \in \mathbb{R}^M$, $r > 0$ and let

$$w(\xi) = \frac{1}{|B(\xi_0, r)|} \chi_{B(\xi_0, r)} \in L^2(\mathbb{R}^M).$$

Let $(B_n)_n \in C_c^\infty(\mathbb{R}^M)$ be a sequence of test functions such that $B_n \rightarrow \check{w}$ in $L^2(\mathbb{R}^M)$; then $\widehat{B}_n \rightarrow w$ in $L^2(\mathbb{R}^M)$ and taking the limit as $n \rightarrow \infty$ in (3) with $\widehat{B}$ replaced by $\widehat{B}_n$ we obtain

$$\frac{1}{|B(\xi_0, r)|} \int_{B(\xi_0, r)} d\xi \int_{\mathbb{R}^N} [A(x) - \Delta_x A(x) + \phi(x)|\xi|^2 A(x)] \bar{\hat{v}}(x, \xi) \, dx = 0.$$

Letting $r \rightarrow 0$ and using the Lebesgue differentiation theorem, we have for a.e. $\xi_0 \in \mathbb{R}^M$

$$\int_{\mathbb{R}^N} [A(x) - \Delta_x A(x) + \phi(x)|\xi_0|^2 A(x)] \bar{\hat{v}}(x, \xi_0) \, dx = 0,$$

which, since u was arbitrary, is valid for every $A \in C_c^\infty(\mathbb{R}^N)$. The operator $\Delta_x - \phi(\cdot)|\xi|^2$ is a Schrödinger operator in $L^2(\mathbb{R}^N)$ with nonpositive potential $-\phi|\xi|^2 \in L_{\text{loc}}^2(\mathbb{R}^N)$ and $C_c^\infty(\mathbb{R}^N)$ is dense in the domain $D(\Delta_x - \phi(\cdot)|\xi|^2)$ with respect to the graph norm; see [Kato 1972]. The last equation then implies $\hat{v}(\cdot, \xi_0) = 0$ for a.e. $\xi_0 \in \mathbb{R}^M$, which proves the required claim.

Since $D(L)$ is dense in $D(L^{1/2}) = D(\mathfrak{a})$, see [Kato 1966, Theorem VI.2.23], the second statement follows from the first. $\square$

In order to prove the main result of this section we recall the definition of B_p -weights. Let $1 < p \leq \infty$. Then $\omega \in B_p(\mathbb{R}^N)$, where $B_p(\mathbb{R}^N)$ is the class of the reverse Hölder weights of order p , if $\omega \in L_{\text{loc}}^p$, $\omega > 0$ a.e. and there exists a positive constant C such that the inequality

$$\left(\frac{1}{|B|} \int_B \omega^p \right)^{\frac{1}{p}} \leq \frac{C}{|B|} \int_B \omega \quad (4)$$

holds for every ball B . If $p = \infty$, the left-hand side of the inequality above has to be replaced by the essential supremum of ω on B . The smallest positive constant C such that (4) holds is the B_p constant of ω . We recall that powers $|x|^\alpha$ belong to $B_\infty(\mathbb{R}^N)$ whenever $\alpha \geq 0$. This is easily seen first considering balls of radius 1 (and large centers) and then scaling.

Remark 2.6. In the proof of the following result we need the maximal L^2 inequalities for Schrödinger operators $\Delta - V$, $0 \leq V \in B_2(\mathbb{R}^N)$, shown in [Auscher and Ben Ali 2007, Theorem 1.1, Corollary 1.3]. They say that the operator $V(\Delta - V)^{-1}$ is bounded in $L^2(\mathbb{R}^N)$; moreover, the norm of $V(\Delta - V)^{-1}$ is bounded by a constant which depends only on N and the B_2 constant of V . This last fact is not explicitly stated in [Auscher and Ben Ali 2007], even though it follows from the proofs, but can be found in [Carbonaro et al. 2008, Theorem 3.6] in the more general setting of parabolic Schrödinger operators $D_t - \Delta + V$. In fact the norm of $V(\Delta - V)^{-1}$ depends on C in (4) and the constant in the Harnack inequality for the Laplacian in $\mathbb{R}^N$. See also [Shen 1995, Theorem 0.3] where, however, $N \geq 3$ and $V \in B_q$ for some $q \geq \frac{N}{2}$.

Theorem 2.7. Assume that $\phi : \mathbb{R}^N \rightarrow [0, +\infty[$ belongs to $B_2(\mathbb{R}^N)$. Then for every $1 \leq i, j \leq N$, $1 \leq h, k \leq M$, one has

$$\|D_{x_i x_j} u\|_2 + \|\phi D_{y_h y_k} u\|_2 \leq C \|Lu\|_2, \quad u \in D(L).$$

Moreover

$$\|\nabla_x u\|_2 + \|\phi^{\frac{1}{2}} \nabla_y u\|_2 \leq C (\|Lu\|_2 + \|u\|_2), \quad u \in D(L).$$

Proof. By Proposition 2.5 we may assume that $u \in C_c^\infty$. Consider the partial Fourier transform with respect to the y -variable. Let $v(x, \xi) = \hat{u}(x, \xi)$. Then, setting $Lu = f$, we have

$$\Delta_x v(x, \xi) - \phi(x)|\xi|^2 v(x, \xi) = \hat{f}(x, \xi) \in L^2.$$

Observe now that, for every fixed $\xi \in \mathbb{R}^M$, $\Delta_x - \phi(\cdot)|\xi|^2$ is a Schrödinger operator in $\mathbb{R}^N$ with potential $\phi|\xi|^2$. Moreover, since $\phi \in B_2(\mathbb{R}^N)$, it immediately follows that $\phi|\xi|^2$ satisfies the reverse Hölder condition

with the same constant as ϕ . By Remark 2.6 above, we have

$$|\xi|^4 \int_{\mathbb{R}^N} \phi(x)^2 |v(x, \xi)|^2 dx \leq C \int_{\mathbb{R}^N} |\hat{f}(x, \xi)|^2 dx,$$

with a constant C not depending on ξ . Integrating the last inequality over $\mathbb{R}^M$, we get

$$\int_{\mathbb{R}^{N+M}} |\xi|^4 \phi(x)^2 |v(x, \xi)|^2 dx d\xi \leq C \int_{\mathbb{R}^{N+M}} |\hat{f}(x, \xi)|^2 dx d\xi.$$

Since $|\cdot|^2 v(x, \cdot) = \widehat{\Delta_y u}(x, \cdot)$ we get, using Fubini's theorem and the Plancherel theorem in $\mathbb{R}^M$,

$$\int_{\mathbb{R}^{N+M}} \phi(x)^2 |\Delta_y u(x, y)|^2 dx dy \leq C \int_{\mathbb{R}^{N+M}} |f(x, y)|^2 dx dy,$$

which reads as $\|\phi \Delta_y u\|_2 \leq C \|Lu\|_2$; by difference we also get $\|\Delta_x u\|_2 \leq C \|Lu\|_2$.

The Calderón–Zygmund theorem applied separately to each variable implies

$$\begin{aligned} \|D_{x_i x_j} u\|_{L^2(\mathbb{R}^N)}^2 &\leq C(N) \|\Delta_x u\|_{L^2(\mathbb{R}^N)}^2, \quad 1 \leq i, j \leq N, \\ \|D_{y_h y_k} u\|_{L^2(\mathbb{R}^M)}^2 &\leq C(M) \|\Delta_y u\|_{L^2(\mathbb{R}^M)}^2, \quad 1 \leq h, k \leq M. \end{aligned}$$

Integrating the previous inequalities (with the last one multiplied by $\phi(x)^2$) over $\mathbb{R}^M$ and $\mathbb{R}^N$, respectively, we get the first claim.

Concerning the gradient estimates, it is enough to observe that, by interpolation, for every $\epsilon > 0$,

$$\|\nabla_x\|_{L^2(\mathbb{R}^N)} \leq \epsilon \sum_{i,j=1}^N \|D_{x_i x_j} u\|_{L^2(\mathbb{R}^N)} + \frac{C}{\epsilon} \|u\|_{L^2(\mathbb{R}^N)}.$$

The estimates for the first-order derivatives with respect to x immediately follow after integration over $\mathbb{R}^M$ and by using the first part of the theorem. For the gradient with respect to y , we start, analogously, from

$$\|\nabla_y\|_{L^2(\mathbb{R}^M)} \leq \epsilon \sum_{h,k=1}^M \|D_{y_h y_k} u\|_{L^2(\mathbb{R}^M)} + \frac{C}{\epsilon} \|u\|_{L^2(\mathbb{R}^M)}.$$

Choosing $\epsilon = \phi(x)^{1/2}$, the claim follows after the integration over $\mathbb{R}^N$ and by using the first part of the theorem. $\square$

Remark 2.8. If $\phi \in B_N(\mathbb{R}^N)$ and $N \geq 3$, it can be proved along the same lines that

$$\|\phi^{\frac{1}{2}} D_{x_i} u\|_2 \leq C \|Lu\|_2, \quad u \in D(L).$$

In fact, the partial Fourier transform of $D_{x_i y_h} u$ is $-i\xi_h D_{x_i} \hat{u}(x, \xi)$ and $\phi|\xi|^2$ satisfies the $B_N(\mathbb{R}^N)$ reverse Hölder condition with the same constant as ϕ . By [Shen 1995, Theorem 0.8]

$$|\xi|^2 \int_{\mathbb{R}^N} \phi(x) |\nabla_x v(x, \xi)|^2 dx \leq C \int_{\mathbb{R}^N} |\hat{f}(x, \xi)|^2 dx, \quad (5)$$

with a constant C not depending on ξ . Integrating over $\mathbb{R}^M$ and using Plancherel's theorem, we get

$$\int_{\mathbb{R}^{N+M}} \phi(y) |D_{x_i y_h} u(x, y)|^2 dx dy \leq C \int_{\mathbb{R}^{N+M}} |f(x, y)|^2 dx dy.$$

The above result probably holds also for $N = 1, 2$ since the maximal inequality (5) is discussed in [Auscher and Ben Ali 2007] (see the comments after Corollary 1.5); the authors say that their methods give the result for all N but a detailed proof is not given. Since we shall not use this remark in what follows we omit further investigation.

In the following theorem we characterize the domain of the operator L .

Theorem 2.9. *If $\phi \in B_2(\mathbb{R}^N)$ then the domain of the operator L defined in (2) satisfies*

$$D(L) = \{u \in L^2 : \nabla_x u, D_{x_i x_j} u, \phi^{\frac{1}{2}} \nabla_y u, \phi D_{y_h y_k} u \in L^2\}. \quad (6)$$

Proof. Let $\tilde{D}(L)$ be the set defined in the right-hand side of equality (6). Theorem 2.7 then implies $D(L) \subseteq \tilde{D}(L)$. To prove the equality it is then enough to prove that the operator $(L, \tilde{D}(L))$ is dissipative since in this case $I - L : \tilde{D}(L) \rightarrow L^2$ is an injective extension of the resolvent operator $I - L : D(L) \rightarrow L^2$ and so both operators must coincide. Let $u \in \tilde{D}(L)$; then, by the definition, for every compact set $\omega \Subset \Omega$, we have $u, D^2 u \in L^2(\omega)$; hence $u \in H_{\text{loc}}^2(\Omega)$. Moreover a section argument, see for example [Ziemer 1989, Theorem 2.1.4], shows that for a.e. $x \in \Omega_N$, we have $u(x, \cdot) \in H^2(\mathbb{R}^M)$ and

$$\int_{\mathbb{R}^M} u \Delta_y u \, dy = - \int_{\mathbb{R}^M} |\nabla_y u|^2 \, dy \quad \text{for a.e. } x \in \Omega_N.$$

Then multiplying by ϕ , integrating in x and using Fubini's theorem we get

$$\int_{\mathbb{R}^{N+M}} \phi(x) u \Delta_y u \, dx \, dy = - \int_{\mathbb{R}^{N+M}} \phi(x) |\nabla_y u|^2 \, dx \, dy.$$

Analogous reasoning applied to the y -sections shows that

$$\int_{\mathbb{R}^{N+M}} u \Delta_x u \, dx \, dy = - \int_{\mathbb{R}^{N+M}} |\nabla_x u|^2 \, dx \, dy.$$

The last two inequalities imply

$$\int_{\mathbb{R}^{N+M}} u Lu \, dx \, dy = - \int_{\mathbb{R}^{N+M}} (|\nabla_y u|^2 + \phi(x) |\nabla_y u|^2) \, dx \, dy \leq 0,$$

which, since $u \in \tilde{D}(L)$ was arbitrary, implies the dissipativity of $(L, \tilde{D}(L))$. $\square$

The next proposition provides regularity properties of the solution of the resolvent equation with respect to the y -variables.

Proposition 2.10. *Let $u \in D(L)$ be such that $u - Lu = f \in C_c^\infty$. Then for every multiindex α one has $D_y^\alpha u \in D(L)$ and*

$$D_y^\alpha u - L D_y^\alpha u = D_y^\alpha f.$$

Proof. Let $u \in D(L)$ be such that $u - Lu = f \in C_c^\infty$. Then

$$\int_{\mathbb{R}^{N+M}} (uv + \langle \nabla_x u, \nabla_x v \rangle + \phi(x) \langle \nabla_y u, \nabla_y v \rangle) \, dx \, dy = \int_{\mathbb{R}^{N+M}} f v \, dx \, dy \quad \text{for every } v \in D(L). \quad (7)$$

For $h \in \mathbb{R}^M$ let $D_h g(z) := (g(x, y+h) - g(x, y))$ and let us take, in the last equation, $v = D_{-h} D_h u \in D(\mathfrak{a})$. Then, since $D_{-h} = D_h^*$, one has

$$\begin{aligned} \int_{\mathbb{R}^{N+M}} (|D_h u|^2 + |D_h \nabla_x u|^2 \phi(x) + |D_h \nabla_y u|^2) dx dy &= \int_{\mathbb{R}^{N+M}} D_h f D_h u dx dy \\ &\leq \|D_h f\|_2 \|D_h u\|_2 \leq \frac{1}{2} (\|D_h f\|_2^2 + \|D_h u\|_2^2). \end{aligned} \quad (8)$$

In particular for every $\omega \Subset \Omega$ there exists some positive constant $C = C(\omega)$ such that

$$\|D_h \nabla u\|_{L^2(\omega)} \leq C|h| \|\nabla f\|_{L^2(\omega)}$$

for sufficiently small h ; this proves that ∇u is weakly differentiable in ω in the y -variable and that $D_{y_i} u \in H_{\text{loc}}^1(\Omega)$. Moreover, if $e_1, \dots, e_M$ is the standard basis of $\mathbb{R}^M$, $t \neq 0$, and $h = te_i$, then dividing both sides of (8) by t and taking the limit for $t \rightarrow 0$ we obtain

$$\frac{1}{2} \int_{\mathbb{R}^{N+M}} (|D_{y_i} u|^2 + |D_{y_i} \nabla_x u|^2 \phi(x) + |D_{y_i} \nabla_y u|^2) dx dy \leq \int_{\mathbb{R}^{N+M}} |D_{y_i} f|^2 dx dy,$$

which proves that $D_{y_i} u \in D(\mathfrak{a})$. Let us fix now $v \in C_c^\infty$; using (7) with v replaced by $D_{-te_i} v$ we get

$$\begin{aligned} \int_{\mathbb{R}^{N+M}} D_{te_i} f v dx dy &= \int_{\mathbb{R}^{N+M}} f D_{-te_i} v dx dy \\ &= \int_{\mathbb{R}^{N+M}} (u D_{-te_i} v + \langle \nabla_x u, \nabla_x D_{-te_i} v \rangle + \phi(x) \langle \nabla_y u, \nabla_y D_{-te_i} v \rangle) dx dy \\ &= \int_{\mathbb{R}^{N+M}} (D_{te_i} u v + \langle D_{te_i} \nabla_x u, \nabla_x v \rangle + \phi(x) \langle D_{te_i} \nabla_y u, \nabla_y v \rangle) dx dy. \end{aligned}$$

Dividing both sides of the last equation by t and taking the limit for $t \rightarrow 0$ we obtain

$$\int_{\mathbb{R}^{N+M}} D_{y_i} f v dx dy = \int_{\mathbb{R}^{N+M}} (D_{y_i} u v + \langle D_{y_i} \nabla_x u, \nabla_x v \rangle + \phi(x) \langle D_{y_i} \nabla_y u, \nabla_y v \rangle) dx dy.$$

Since by Proposition 2.5 C_c^∞ is a core for $\mathfrak{a}$ and since v is arbitrary in the last equation, we have that $D_{y_i} u \in D(L)$ and $D_{y_i} u - L(D_{y_i} u) = D_{y_i} f$, which is the required claim for $|\alpha| = 1$. An inductive argument easily proves the claim for any multiindex α . Moreover, since $D_y^\alpha u = (I - L)^{-1} D_y^\alpha f$, for some $C = C(\alpha) > 0$ we have

$$\|D_y^\alpha u\|_2 \leq C \|D_y^\alpha f\|_2. \quad \square$$

We end this section by proving a version of Kato's inequality adapted to L , which will be used for proving L^p -estimates.

Proposition 2.11. *Let $u \in D(L)$ and let us define*

$$\text{sign}(u) = \begin{cases} 0 & \text{if } u(x) = 0, \\ u(x)/|u(x)| & \text{if } u(x) \neq 0. \end{cases}$$

Then $|u|$ satisfies the distributional inequality

$$-\mathfrak{a}(|u|, \varphi) \geq \int_{\mathbb{R}^{N+M}} \text{sign}(u) L u \phi dx dy \quad \text{for any } 0 \leq \varphi \in C_c^\infty.$$

Proof. We suppose first that $u \in C_c^\infty$. If

$$u_\epsilon(x) = \sqrt{|u|^2 + \epsilon^2}$$

then $u_\epsilon \geq |u|$ and

$$u_\epsilon(a \nabla u_\epsilon) = u(a \nabla u) \quad (9)$$

(here a is the matrix defined in (1)). Thus (9) implies

$$\begin{aligned} |\nabla_x u_\epsilon| &\leq |u| |u_\epsilon|^{-1} |\nabla_x u| \leq |\nabla_x u|, \\ \phi(x) |\nabla_y u_\epsilon| &\leq |u| |u_\epsilon|^{-1} \phi(x) |\nabla_y u| \leq \phi(x) |\nabla_y u|. \end{aligned} \quad (10)$$

Taking the divergence of (9) we obtain

$$u_\epsilon Lu_\epsilon + |\nabla_x u_\epsilon|^2 + \phi(x) |\nabla_y u_\epsilon|^2 = u Lu + |\nabla_x u|^2 + \phi(x) |\nabla_y u|^2,$$

so by (10)

$$Lu_\epsilon \geq \frac{u}{u_\epsilon} Lu. \quad (11)$$

Integrating by parts the right-hand side of (11), it follows that

$$-\mathfrak{a}(u_\epsilon, \varphi) \geq \int_{\mathbb{R}^{N+M}} \frac{u}{u_\epsilon} Lu \varphi \, dx \, dy \quad \text{for any } 0 \leq \varphi \in C_c^\infty.$$

Letting $\epsilon \rightarrow 0$ we get

$$-\mathfrak{a}(|u|, \varphi) \geq \int_{\mathbb{R}^{N+M}} \text{sign}(u) Lu \varphi \, dx \, dy \quad \text{for any } 0 \leq \varphi \in C_c^\infty.$$

Let now $u \in D(L)$ and let $u_n \in C_c^\infty$ be such that $u_n \rightarrow u$ in $D(L)$. Up to a subsequence, if necessary, we can also suppose that $u_n \rightarrow u$ almost everywhere. Since also $u_n \rightarrow u$ in $D(\mathfrak{a})$ by the last inequalities

$$-\mathfrak{a}(|u_n|, \varphi) \geq \int_{\mathbb{R}^{N+M}} \text{sign}(u_n) Lu_n \varphi \, dx \, dy \quad \text{for any } 0 \leq \varphi \in C_c^\infty;$$

the claim follows letting $n \rightarrow \infty$. □

3. The distance d associated with L

Let $\alpha > 0$ and let

$$L = \Delta_x + |x|^\alpha \Delta_y$$

be the self-adjoint operator defined in Section 2 with $\phi(x) = |x|^\alpha$. In this section we introduce a natural metric d on $\mathbb{R}^{N+M}$ associated with L and which makes the triple $(\mathbb{R}^{N+M}, d, \mathcal{L})$, consisting of $\mathbb{R}^{N+M}$ equipped with the distance d and the Lebesgue measure $\mathcal{L}$, a homogeneous space in the sense of [Coifman and Weiss 1971; 1977].

Definition 3.1. Let $\gamma : [0, T] \rightarrow \mathbb{R}^{N+M}$ be an absolutely continuous curve. We say that γ is a subunit curve if for a.e. $t \in [0, T]$ one has

$$\langle \dot{\gamma}(t), \xi \rangle^2 \leq |\xi_x|^2 + |x|^\alpha |\xi_y|^2 \quad \text{for every } \xi = (\xi_x, \xi_y) \in \mathbb{R}^{N+M}.$$

For every $z_1, z_2 \in \mathbb{R}^{N+M}$ we define

$$\begin{aligned} d(z_1, z_2) &= \inf\{T \in \mathbb{R}^+ : \text{there exists a subunit curve } \gamma : [0, T] \rightarrow \mathbb{R}^{N+M}, \gamma(0) = z_1, \gamma(T) = z_2\} \\ &= \sup\{\psi(z_2) - \psi(z_1) : \psi \in W^{1,\infty}(\mathbb{R}^{N+M}), |\nabla_x \psi|^2 + |x|^\alpha |\nabla_y \psi|^2 \leq 1\}. \end{aligned} \quad (12)$$

We remark that d is a well-defined distance and that any pair of points $z_1, z_2 \in \mathbb{R}^{N+M}$ can be joined by a subunit curve; see [Franchi and Serapioni 1987, Section 2, Example 3.6] and [Franchi and Lanconelli 1984, Definition 2.4]. A proof of the equality in (12) can be found in [Jerison and Sánchez-Calle 1987, Proposition 3.1].

For $z_0 \in \mathbb{R}^{N+M}$, $r > 0$, we write

$$S(z_0, r) := \{z \in \mathbb{R}^{N+M} : d(z_0, z) < r\}$$

to denote the balls of $\mathbb{R}^{N+M}$ with respect to the metric d . In the next proposition we clarify the structure of the metric and define an equivalent system of balls which are explicit and easier to work with. For $z_0 = (x_0, y_0) \in \mathbb{R}^{N+M}$, $r > 0$, let us define the cylindrical set

$$Q(z_0, r) := B(x_0, r) \times B(y_0, r(x_0)), \quad r(x_0) := r(r + |x_0|)^{\frac{\alpha}{2}}. \quad (13)$$

Proposition 3.2. *There exist two positive constants $C_1, C_2 > 0$ such that the distance function d satisfies for every $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in \mathbb{R}^{N+M}$*

$$C_1 F(z_1, z_2) \leq d(z_1, z_2) \leq C_2 F(z_1, z_2),$$

where

$$F(z_1, z_2) = |x_1 - x_2| + \left(\frac{|y_1 - y_2|}{(|x_1| + |x_2|)^{\frac{\alpha}{2}}} \wedge |y_1 - y_2|^{\frac{2}{2+\alpha}} \right).$$

In particular

$$|S(z_0, r)| \simeq \begin{cases} r^{N+M(1+\frac{\alpha}{2})} & \text{if } r \geq |x_0|, \\ r^{N+M} |x_0|^{M\frac{\alpha}{2}} & \text{if } r \leq |x_0|, \end{cases}$$

and the metric balls satisfy the doubling property

$$|S(z_0, sr)| \leq C s^{N+M(1+\frac{\alpha}{2})} |S(z_0, r)| \quad \text{for every } z_0 \in \mathbb{R}^{N+M}, s \geq 1.$$

Furthermore there exists a constant $c > 1$ such that for every $z_0 = (x_0, y_0) \in \mathbb{R}^{N+M}$, $r > 0$,

$$Q(z_0, c^{-1}r) \subseteq S(z_0, r) \subseteq Q(z_0, cr).$$

In particular $|S(z_0, r)| \simeq r^{N+M} (r + |x_0|)^{M\alpha/2}$ and $(\mathbb{R}^{N+M}, d, \mathcal{L})$ is a metric space of homogeneous type.

Proof. The first part of the statement is proved in [Robinson and Sikora 2008, Proposition 5.1, Corollary 5.2] (take in that paper $\delta_1 = \delta'_1 = 0$, $\delta_2 = \delta'_2 = \frac{\alpha}{2}$, $D = D' = N + M(1 + \frac{\alpha}{2})$). A proof of the second part can be found in [Franchi and Serapioni 1987, Proposition 2.7, Example 3.6] and [Franchi et al. 1994, Proposition 1]. $\square$

4. L^p estimates

Let $1 < p < \infty$. In this section we assume that $\phi(x) = |x|^\alpha$, with $\alpha > 0$, and consider therefore the operator

$$L = \Delta_x + |x|^\alpha \Delta_y$$

in L^p with $x \in \mathbb{R}^N$, $y \in \mathbb{R}^M$. Property (iii) of Proposition 2.4 shows that the symmetric semigroup $(e^{tL})_{t \geq 0}$ generated by L in L^2 is submarkovian. Then by a standard result, see for example [Ouhabaz 2005, Chapter 3], it induces a consistent family of strongly continuous semigroups on L^p for any $1 < p < \infty$, still denoted by $(e^{tL})_{t \geq 0}$. Moreover $(e^{tL})_{t \geq 0}$ extends to a contractive holomorphic semigroup on a sector; see [Ouhabaz 2005, Theorem 3.13].

Definition 4.1. For any $p \in (1, \infty)$ we define the sectorial operator $(L, D_p(L))$ as the generator of the extrapolated semigroup $(e^{tL})_{t \geq 0}$ in L^p . We also write $D_2(L) = D(L)$.

$D_p(L) \cap D(L)$ is dense in L^p ; in fact, if $f \in C_c^\infty$, then $e^{tL}f \in D_p(L) \cap D(L)$ and converges to f in L^p . Then $D_p(L) \cap D(L)$, being a dense invariant set, is by construction a core for $(L, D_p(L))$.

Theorem 2.7 holds in the specific situation since $|x|^\alpha \in B_\infty(\mathbb{R}^N)$ and we prove that those estimates extend to $1 < p < \infty$.

We recall that $|x|^\beta$ belongs to $A_t(\mathbb{R}^N)$, the class of Muckenhoupt weights of order $t \geq 1$, whenever $0 \leq \beta < N(t - 1)$. This means that

$$\left(\frac{1}{|B|} \int_B |x|^\beta dx \right) \left(\frac{1}{|B|} \int_B |x|^{\beta(1-t')} dx \right)^{t-1} \leq C$$

for any ball (or cube) B of $\mathbb{R}^N$; see for example [Duoandikoetxea 2001, Chapter 7.3]. However, we need Muckenhoupt weights in $(\mathbb{R}^{N+M}, d, \mathcal{L})$ with respect to the metric defined in Section 3. Since $|x|^\beta$ is independent of y and since the balls S in this space are equivalent to the cylinders $Q(z, r)$ defined in (13), which are products of balls in $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively, one easily verifies that

$$\left(\frac{1}{|S|} \int_S |x|^\beta dx dy \right) \left(\frac{1}{|S|} \int_S |x|^{\beta(1-t')} dx dy \right)^{t-1} \leq C$$

for every ball S (or cylinder) in $(\mathbb{R}^{N+M}, d, \mathcal{L})$.

A theory on these classes of weights in homogeneous spaces is presented for example in [Strömberg and Torchinsky 1989, Chapter I], to which we refer for the proofs of the results needed in what follows. In particular, we recall that Muckenhoupt weights induce doubling measures. The following well-known consequence of the definition is crucial in our approach.

Lemma 4.2. *If $\phi(x, y) = |x|^\beta$, $t \geq 1$, and $\beta < N(t - 1)$, there exists $c > 0$ such that the inequality*

$$\left(\frac{1}{|Q|} \int_Q g \right)^t \leq \frac{c}{\phi(Q)} \int_Q g^t \phi \tag{14}$$

holds for all nonnegative functions g and all cylinders Q in $(\mathbb{R}^{N+M}, d, \mathcal{L})$. Here

$$\phi(Q) = \int_Q \phi.$$

Proof. By Hölder's inequality one has

$$\left(\frac{1}{|Q|} \int_Q g \right)^t = \left(\frac{1}{|Q|} \int_Q g \phi^{\frac{1}{t}} \phi^{-\frac{1}{t}} \right)^t \leq \left(\frac{1}{|Q|} \int_Q g^t \phi \right) \left(\frac{1}{|Q|} \int_Q \phi^{1-t'} \right)^{t-1}$$

and the claim follows from the A_t property of $|x|^\beta$ in $(\mathbb{R}^{N+M}, d, \mathcal{L})$. $\square$

The A_t property of $\phi = |x|^\alpha$, combined with mean value inequalities for Baouendi–Grushin operators, allows us to characterize the domain of the operator. We prove the following result.

Theorem 4.3. *For every $1 \leq i, j \leq N$, $1 \leq h, k \leq M$, the operators $|x|^\alpha D_{y_h y_k} (I - L)^{-1}$, $D_{x_i x_j} (I - L)^{-1}$, originally defined in L^2 , extend to bounded operators in L^p .*

The main tool is the following result due to Shen [2005, Theorem 3.1], which can be considered as a version of the Calderón–Zygmund theorem in the absence of kernels. The original proof, where Euclidean balls are used, can be modified to work also for our space $(\mathbb{R}^{N+M}, d, \mathcal{L})$. Indeed an improved version of Shen's result in more general homogeneous spaces, which covers the cases of our interest, can be found in [Auscher and Martell 2007, Theorem 3.14 and Section V].

Theorem 4.4. *Let $1 \leq p_0 < q_0 \leq \infty$. Suppose that T is a sublinear bounded operator on L^{p_0} . Suppose moreover that there exist $\alpha_2 > \alpha_1 > 1$, $C > 0$, such that*

$$\left(\frac{1}{|Q|} \int_Q |Tf|^{q_0} \right)^{\frac{1}{q_0}} \leq C \left(\frac{1}{|\alpha_1 Q|} \int_{\alpha_1 Q} |Tf|^{p_0} \right)^{\frac{1}{p_0}}$$

for all cylinders Q and for all $f \in C_c^\infty$, with support in $\mathbb{R}^{N+M} \setminus \alpha_2 Q$. Then, for $p_0 \leq p < q_0$, there exists a positive constant C_p such that for all $f \in C_c^\infty$

$$\|Tf\|_p \leq C_p \|f\|_p.$$

We briefly describe our strategy of proof of Theorem 4.3. We first prove the a priori estimates for $p \geq 2$ by applying the above theorem to the operator $T = |x|^\alpha D_{y_h y_k} (I - L)^{-1}$, with $p_0 = 2$, arbitrary $q_0 > 2$ and $\alpha_1 = 3$, $\alpha_2 = 4$. Therefore we have to prove that, if Q is a cylinder and $f \in C_c^\infty$ has support in $\mathbb{R}^{N+M} \setminus 4Q$, then $u = (I - L)^{-1} f$ satisfies

$$\left(\frac{1}{|Q|} \int_Q ||x|^\alpha D_{y_h y_k} u|^{q_0} \right)^{\frac{1}{q_0}} \leq C \left(\frac{1}{|3Q|} \int_{3Q} ||x|^\alpha D_{y_h y_k} u|^2 \right)^{\frac{1}{2}}$$

for some positive C independent of f . Observe that u satisfies in $4Q$ the equation

$$u - Lu = u - \Delta_x u - |x|^\alpha \Delta_y u = 0.$$

Moreover, by Proposition 2.10, the operator L commutes with the second-order derivatives with respect to y and $v = D_{y_h y_k} u$ satisfies the same equation in $4Q$.

To get the a priori estimates in the case $1 < p \leq 2$, we apply Shen's theorem to the adjoint operator T^* .

As a first step we recall a mean value inequality for subsolutions of L , that is, for functions v satisfying the inequality $Lv \geq 0$ in Q , in a weak sense. This means that $\mathfrak{a}(u, \varphi) \leq 0$ for any $0 \leq \varphi \in C_c^\infty(Q)$.

Lemma 4.5 (see [Franchi and Serapioni 1987, Theorem 5.7]). *There exists a positive constant C such that, if v is a local subsolution of L in $4Q$, then*

$$\sup_Q |v| \leq C \left(\frac{1}{|3Q|} \int_{3Q} v^2 \right)^{\frac{1}{2}}.$$

The previous mean value inequality remains true also for $0 < r < \infty$. It follows from the self-improvement of the right-hand side in weak reverse Hölder estimates; see for example [Chanillo and Wheeden 1986, Theorem 4.1]. We give a simple self-contained proof which is a simplification of [Bernicot et al. 2016, Theorem B.1] for the sup-norm. Note that the case $r > 2$ follows from Hölder's inequality.

Lemma 4.6. *For every $0 < r < \infty$, there exists a positive constant C_r such that, if v is local subsolution of L in $4Q$, then*

$$\sup_Q |v| \leq C_r \left(\frac{1}{|3Q|} \int_{3Q} |v|^r \right)^{\frac{1}{r}}.$$

Proof. Let $r < 2$ and $\mathcal{Q}$ be the collection of all cylinders Q' contained in Q . For $\epsilon \in (0, 1)$ let

$$C_r(\epsilon) := \sup_{Q' \in \mathcal{Q}} \frac{\sup_{Q'} |v|}{\left(\frac{1}{|3Q'|} \int_{3Q'} |v|^r \right)^{\frac{1}{r}} + \epsilon} \leq \epsilon^{-1} \sup_Q |v|.$$

Let us fix $Q' \in \mathcal{Q}$ and let Q'' be a cylinder centered at some point of Q' and such that $9Q'' \subseteq 3Q'$. We assume that the radii $r(Q'')$, $r(Q')$ of Q'' and Q' satisfy $c^{-1}r(Q') \leq r(Q'') \leq cr(Q')$ for some fixed constant $c \in (0, 1)$. Applying Lemma 4.5 in Q'' we get

$$\begin{aligned} \sup_{Q''} |v| &\leq C_2 \left(\frac{1}{|3Q''|} \int_{3Q''} v^2 \right)^{\frac{1}{2}} \leq C_2 (\sup_{3Q''} |v|)^{1-\frac{r}{2}} \left(\frac{1}{|3Q''|} \int_{3Q''} |v|^r \right)^{\frac{1}{2}} \\ &\leq C_2 C_r(\epsilon)^{1-\frac{r}{2}} \left[\left(\frac{1}{|9Q''|} \int_{9Q''} |v|^r \right)^{\frac{1}{r}} + \epsilon \right]^{1-\frac{r}{2}} \left(\frac{1}{|3Q''|} \int_{3Q''} |v|^r \right)^{\frac{1}{2}} \\ &\leq C' C_2 C_r(\epsilon)^{1-\frac{r}{2}} \left[\left(\frac{1}{|9Q''|} \int_{9Q''} |v|^r \right)^{\frac{1}{r}} + \epsilon \right] \leq C'' C_2 C_r(\epsilon)^{1-\frac{r}{2}} \left[\left(\frac{1}{|3Q'|} \int_{3Q'} |v|^r \right)^{\frac{1}{r}} + \epsilon \right]. \end{aligned}$$

Since Q'' is arbitrary

$$\sup_{Q'} |v| \leq C'' C_2 C_r(\epsilon)^{1-\frac{r}{2}} \left[\left(\frac{1}{|3Q'|} \int_{3Q'} |v|^r \right)^{\frac{1}{r}} + \epsilon \right].$$

Taking the supremum over $Q' \in \mathcal{Q}$ we get $C_r(\epsilon) \leq (C'' C_2)^{2/r}$ and the thesis follows letting $\epsilon \rightarrow 0$. $\square$

Now we prove that Lemma 4.6 holds if we replace the Lebesgue measure with $|x|^\beta dx$, $\beta > 0$.

Lemma 4.7. Fix $0 < s < \infty$ and v as in Lemma 4.5. Then

$$\sup_Q |v| \leq \left(\frac{C}{\phi(3Q)} \int_{3Q} \phi |v|^s \right)^{\frac{1}{s}},$$

where C depends only on s , p and the B_p constant of $\phi(x) = |x|^\beta$ and

$$\phi(3Q) = \int_{3Q} \phi.$$

Proof. Let $0 < s < \infty$ and Q be a cylinder of $\mathbb{R}^{N+M}$. We fix t as in Lemma 4.2. By using Lemma 4.6 with $r = \frac{s}{t}$ and (14) we obtain

$$\sup_Q |v| \leq C \left(\frac{1}{|3Q|} \int_{3Q} |v|^{\frac{s}{t}} \right)^{\frac{t}{s}} \leq C \left(\frac{1}{\phi(3Q)} \int_{3Q} \phi |v|^s \right)^{\frac{1}{s}}. \quad \square$$

By combining the estimate in Lemma 4.7 and the B_p property we deduce the following.

Corollary 4.8. Let $0 < s < \infty$, $1 < p < \infty$ and v as in Lemma 4.5. Then

$$\left(\frac{1}{|Q|} \int_Q (|x|^\beta |v|^s)^p \right)^{\frac{1}{p}} \leq \frac{C}{|3Q|} \int_{3Q} |x|^\beta |v|^s,$$

where C depends only on s , p and the B_p constant of $|x|^\beta$.

Proof. Using the B_p property of $\phi = |x|^\beta$ and Lemma 4.7 we obtain

$$\left(\frac{1}{|Q|} \int_Q (\phi |v|^s)^p \right)^{\frac{1}{p}} \leq \left(\frac{1}{|Q|} \int_Q \phi^p \right)^{\frac{1}{p}} \sup_Q |v|^s \leq C \left(\frac{1}{|Q|} \int_Q \phi \right) \sup_Q |v|^s \leq \frac{C}{|3Q|} \int_{3Q} \phi |v|^s. \quad \square$$

We can now prove our main result.

Proof of Theorem 4.3. We first consider the operators $|x|^\alpha D_{y_h y_k} (I - L)^{-1}$.

Let us preliminarily treat the case $p \geq 2$. Let us fix $q_0 > 2$ and let Q be a cylinder in $\mathbb{R}^{N+M}$ and $f \in C_c^\infty$ a smooth function with support in $\mathbb{R}^{N+M} \setminus 4Q$. We set

$$T = |x|^\alpha D_{y_h y_k} (I - L)^{-1}, \quad u = (I - L)^{-1} f, \quad v = D_{y_h y_k} (I - L)^{-1} f.$$

By Theorem 2.7, T is bounded on L^2 . Since $f = 0$ in $4Q$, by Proposition 2.10 $v - Lv = 0$ in $4Q$. Combining the last equality with Kato's inequality of Proposition 2.11, we get

$$-\alpha(|v|, \varphi) \geq \int \text{sign } v L v \varphi = \int |v| \varphi \geq 0 \quad \text{for all } 0 \leq \varphi \in C_c^\infty(4Q).$$

It follows that $|v|$ is a local subsolution of L . Note that v is a local solution of $v - Lv = 0$ but we cannot assert that it is a local subsolution of L , that is, $Lv \geq 0$, since its sign is not given. By Corollary 4.8 with $s = 2$ and $\beta = 2\alpha$ we have

$$\left(\frac{1}{|Q|} \int_Q (|x|^{2\alpha} |v|^2)^q \right)^{\frac{1}{q}} \leq \frac{C}{|3Q|} \int_{3Q} |x|^{2\alpha} |v|^2, \quad 1 < q < \infty,$$

or, equivalently,

$$\left(\frac{1}{|Q|} \int_Q (|x|^\alpha |v|)^{2q} \right)^{\frac{1}{2q}} \leq \left(\frac{C}{|3Q|} \int_{3Q} (|x|^\alpha |v|)^2 \right)^{\frac{1}{2}}, \quad 1 < q < \infty.$$

It follows that for $2 \leq q_0 < \infty$

$$\begin{aligned} \left(\frac{1}{|Q|} \int_Q |Tf|^{q_0} \right)^{\frac{1}{q_0}} &= \left(\frac{1}{|Q|} \int_Q (|x|^\alpha |D_{y_h y_k} u|)^{q_0} \right)^{\frac{1}{q_0}} \\ &\leq \left(\frac{C}{|3Q|} \int_{3Q} (|x|^\alpha |D_{y_h y_k} u|)^2 \right)^{\frac{1}{2}} = \left(\frac{C}{|3Q|} \int_{3Q} |Tf|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

By Theorem 4.4, T extends to a bounded operator in L^p for every $2 \leq p < q_0$. Since we can choose q_0 arbitrarily, the case $2 \leq p < \infty$ follows.

To treat the case $p < 2$ we consider the adjoint operator

$$T^* = D_{y_h y_k} (I - L)^{-1} |x|^\alpha,$$

which is bounded in L^2 . By duality, the boundedness of T in L^p for every $1 < p \leq 2$ is equivalent to that of T^* in L^p for every $p \geq 2$. As before we fix $q_0 > 2$ and we prove that T^* satisfies Shen's assumption for every $2 \leq q_0 < \infty$. Let Q be a cylinder in $\mathbb{R}^{N+M}$ and $f \in C_c^\infty$ with support in $\mathbb{R}^{N+M} \setminus 4Q$; set

$$u = (I - L)^{-1} (|x|^\alpha f), \quad v = D_{y_h y_k} u.$$

Then v satisfies $v - Lv = 0$ in $4Q$. By arguing as above, $|v|$ is a local subsolution of L ; hence Lemma 4.5 yields a positive constant C such that

$$\sup_Q |v| \leq C \left(\frac{1}{|3Q|} \int_{3Q} v^2 \right)^{\frac{1}{2}}.$$

It follows that

$$\begin{aligned} \left(\frac{1}{|Q|} \int_Q |T^* f|^{q_0} \right)^{\frac{1}{q_0}} &= \left(\frac{1}{|Q|} \int_Q |D_{y_h y_k} u|^{q_0} \right)^{\frac{1}{q_0}} \\ &\leq \sup_Q |v| \leq C \left(\frac{1}{|3Q|} \int_{3Q} v^2 \right)^{\frac{1}{2}} = \left(\frac{C}{|3Q|} \int_{3Q} |T^* f|^2 \right)^{\frac{1}{2}} \end{aligned}$$

and the proof is complete by Theorem 4.4 applied to T^* .

By difference the operator $\Delta_x (I - L)^{-1}$ is bounded on L^p and, integrating with respect to y the classical Calderón–Zygmund estimates in the x -variables we deduce the L^p boundedness of $D_{x_i x_j} (I - L)^{-1}$. $\square$

Remark 4.9. Theorem 4.3 holds for every $0 \leq \phi \in B_2(\mathbb{R}^N)$ when $1 < p \leq 2$. In fact, the properties of the powers $|x|^\alpha$ stated in Lemma 4.7 and Corollary 4.8 have been used in the above proof only when $p > 2$.

We can now give a partial description of the domain $D_p(L)$, which will be strengthened in Theorem 4.21 after proving L^p -estimates for mixed derivatives.

Proposition 4.10. *Let $p \in (1, \infty)$. Then one has*

$$D_p(L) = \{u \in L^p : \nabla_x u, D_{xx}u, |x|^{\frac{\alpha}{2}} \nabla_y u, |x|^\alpha D_{yy}u \in L^p\}. \quad (15)$$

Moreover

$$\|D_{x_i x_j} u\|_p + \| |x|^\alpha D_{y_h y_k} u \|_p \leq C \|Lu\|_p, \quad u \in D_p(L). \quad (16)$$

Proof. Let $1 < p < \infty$ and let $\tilde{D}_p(L)$ be the set defined in the right-hand side of equality (15).

Let us preliminarily prove that $D_p(L) \subseteq \tilde{D}_p(L)$.

Theorem 4.3 and the consistency of the resolvent operators in L^2 and in L^p imply that

$$\|D_{x_i x_j} u\|_p + \| |x|^\alpha D_{y_h y_k} u \|_p \leq C \|(I - L)u\|_p \quad (17)$$

for any $u \in (I - L)^{-1}(C_c^\infty)$ which is dense in $D_p(L)$ with respect to the graph norm. This implies that (17) extends to $D_p(L)$, proving that u has pure second-order distributional derivatives which satisfy $D_{x_i x_j} u, |x|^\alpha D_{y_h y_k} u \in L^p$ and that

$$\|D_{x_i x_j} u\|_p + \| |x|^\alpha D_{y_h y_k} u \|_p \leq C \|(I - L)u\|_p \leq C(\|u\|_p + \|Lu\|_p), \quad u \in D_p(L). \quad (18)$$

As in Theorem 2.7, an interpolation argument shows that $\nabla_x u, |x|^{\frac{\alpha}{2}} \nabla_y u \in L^p(\mathbb{R}^{N+M})$; i.e., $u \in \tilde{D}_p(L)$.

To get homogeneous estimates, we use Proposition 2.4(v), and apply (18) to $u(x, y) = v(sx, s^{(2+\alpha)/2}y)$, $s > 0$, thus obtaining

$$\|D_{x_i x_j} u\|_p + \| |x|^\alpha D_{y_h y_k} u \|_p \leq C(\|Lv\|_p + s^{-2}\|v\|_p).$$

Letting $s \rightarrow \infty$ we obtain (16).

To prove that $\tilde{D}_p(L) = D(L)$, we proceed as in the proof of Theorem 2.9 and show that the operator $(L, \tilde{D}_p(L))$ is dissipative. Let $u \in \tilde{D}_p(L)$; then the same sectional argument of (6) shows that for a.e. $x \in \Omega_N$, we have $u(x, \cdot) \in W^{2,p}(\mathbb{R}^M)$ and from [Metafune and Spina 2008]

$$\int_{\mathbb{R}^M} u |u|^{p-2} \Delta_y u \, dy = -(p-1) \int_{\mathbb{R}^M} |\nabla_y u|^2 |u|^{p-2} \, dy \quad \text{for a.e. } x \in \Omega_N.$$

Then multiplying by $|x|^\alpha$, integrating in x and using Fubini's theorem we get

$$\int_{\mathbb{R}^{N+M}} |x|^\alpha u |u|^{p-2} \Delta_y u \, dx \, dy = -(p-1) \int_{\mathbb{R}^{N+M}} |x|^\alpha |\nabla_y u|^2 |u|^{p-2} \, dx \, dy.$$

Analogously

$$\int_{\mathbb{R}^{N+M}} u |u|^{p-2} \Delta_x u \, dx \, dy = -(p-1) \int_{\mathbb{R}^{N+M}} |\nabla_x u|^2 |u|^{p-2} \, dx \, dy.$$

The last two inequalities imply

$$\int_{\mathbb{R}^{N+M}} u |u|^{p-2} Lu \, dx \, dy = -(p-1) \int_{\mathbb{R}^{N+M}} (|\nabla_x u|^2 + |x|^\alpha |\nabla_y u|^2) |u|^{p-2} \, dx \, dy \leq 0,$$

which, since $u \in \tilde{D}_p(L)$ is arbitrary, implies the dissipativity of $(L, \tilde{D}_p(L))$. $\square$

Remark 4.11. The previous theorem implies, in particular, the equivalence between the graph norm $\|u\|_p + \|Lu\|_p$ and the norm

$$\|u\|_p + \|\nabla_x u\|_p + \| |x|^{\frac{\alpha}{2}} \nabla_y u \|_p + \|D_{xx} u\|_p + \| |x|^\alpha D_{yy} u \|_p,$$

since this last clearly dominates $\|Lu\|_p$.

The following proposition shows that C_c^∞ also is a core for $(L, D_p(L))$.

Proposition 4.12. *For any $p \in (1, \infty)$, C_c^∞ is a core for the operator $(L, D_p(L))$.*

Proof. Let $u \in D_p(L)$; we preliminarily approximate u with functions in $D_p(L)$ having compact support in $\mathbb{R}^{N+M}$. Let $\eta \in C_c^\infty(\mathbb{R}^N)$ be a smooth function such that $\chi_{B_1} \leq \eta \leq \chi_{B_2}$ and, for every $n \in \mathbb{N}$, $x \in \mathbb{R}^N$, define $\eta_n(x) = \eta(\frac{x}{n})$. Set $u_n = \eta_n u$. Now u_n has, by construction, compact support in x and, using the characterization in (15), one can easily recognize that $u_n \in D_p(L)$. Lebesgue's theorem immediately implies that u_n , $|x|^{\alpha/2} \nabla_y u_n$, $|x|^\alpha D_{yy} u_n$ tend to u , $|x|^{\alpha/2} \nabla_y u$, $|x|^\alpha D_{yy} u$ in L^p , respectively. Concerning the x -gradient, we have

$$\begin{aligned} \|\nabla_x(\eta_n u) - \nabla_x(u)\|_p^p &\leq \int_{\mathbb{R}^{N+M}} |\eta_n - 1|^p |\nabla_x u|^p dx dy + \int_{\mathbb{R}^{N+M}} |\nabla_x \eta_n|^p |u|^p dx dy \\ &\leq \int_{\mathbb{R}^{N+M}} |\eta_n - 1|^p |\nabla u|^p dx dy + Cn^{-p} \int_{\{n \leq |x| \leq 2n\}} |u|^p dx dy, \end{aligned}$$

which tends to 0 by dominated convergence. Similarly $D_{xx} u_n$ tends to $D_{xx} u$ in L^p . By Proposition 4.10 and Remark 4.11, this proves that u_n tends to u in $D_p(L)$. Using a similar argument with η replaced by an analogous cut-off function $\eta \in C_c^\infty(\mathbb{R}^M)$, we can approximate u with functions in $D_p(L)$ having compact support also in the y -variable.

Let us suppose first that $p < 2$ and let $u \in D_p(L)$ (the case $p = 2$ is already proved in Proposition 2.5). From the first part of the proof, we can suppose u has compact support. Let $\eta \in C_c^\infty$ such that $\eta = 1$ on the support of u and, using Proposition 2.5, let $(u_n)_{n \in \mathbb{N}}$ be a sequence of C_c^∞ functions such that $u_n \rightarrow u$ in $D_2(L)$ as $n \rightarrow \infty$. This implies

$$\|L(\eta u_n - u)\|_p + \|\eta u_n - u\|_p \leq C[\|L(\eta u_n - u)\|_2 + \|\eta u_n - u\|_2] \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where C depends on the measure of the support of u . This proves the claim for $p < 2$.

The proof of the case $p > 2$ can be carried out by slightly adapting the arguments used in the proof of Proposition 2.5. We equivalently show that $(I - L)(C_c^\infty)$ is dense in L^p and to this aim let $v \in L^{p'}(\mathbb{R}^{N+M})$ such that

$$\int_{\mathbb{R}^{N+M}} (I - L)u \bar{v} dx dy = 0 \quad \text{for all } u \in C_c^\infty.$$

Since $1 < p' < 2$, the partial Fourier transform of $v(x, \cdot) \in L^{p'}(\mathbb{R}^M)$, with respect to the y -variable, exists as a function in $L^p(\mathbb{R}^M)$ for a.e. $x \in \mathbb{R}^N$. Therefore, taking in the last equality the Fourier transform with respect to the y -variable and applying the Fubini and Plancherel theorems, we get

$$\int_{\mathbb{R}^{N+M}} [\hat{u}(x, \xi) - \Delta_x \hat{u}(x, \xi) + |x|^\alpha |\xi|^2 \hat{u}(x, \xi)] \bar{\hat{v}}(x, \xi) dx d\xi = 0 \quad \text{for all } u \in C_c^\infty.$$

By [Semenov 1977, Theorem 1.1], since the potential is nonnegative and in L^p_{loc} , we know $C_c^\infty(\mathbb{R}^N)$ is a core for $\Delta_x - \phi(\cdot)|\xi|^2$ in L^p ; proceeding then as in the proof of Proposition 2.5 we conclude that $\hat{v}(\cdot, \xi) = 0$ for a.e. $\xi \in \mathbb{R}^M$ and the proof follows. $\square$

Mixed derivatives. By using classical covering results and Rellich inequalities we obtain here L^p estimates for the mixed second-order derivatives. To simplify the notation we write x for any of the variables x_i , $i = 1, \dots, N$, and y for any of the variables y_h , $h = 1, \dots, M$.

Theorem 4.13. *For every $u \in D_p(L)$*

$$\| |x|^{\frac{\alpha}{2}} D_{xy} u \|_p \leq C \|Lu\|_p.$$

We need a Rellich-type inequality.

Lemma 4.14. *Let $p \neq N, \frac{N}{2}$. There exist a positive constant C such that for $u \in C_c^\infty$ satisfying $u(0, y) = 0$, $\nabla_x u(0, y) = 0$ for all $y \in \mathbb{R}^M$, we have*

$$\left\| \frac{u}{|x|^2} \right\|_p \leq C \|Lu\|_p.$$

Proof. Let $u \in C_c^\infty$ such that $u(0, y) = 0$, $\nabla_x u(0, y) = 0$ so that $\|u/|x|^2\|_p < \infty$. Then by [Metafune et al. 2019, Theorem 4.2; 2015, Theorem 3.3]

$$\int_{\mathbb{R}^N} \left| \frac{u}{|x|^2} \right|^p dx \leq C \int_{\mathbb{R}^N} |\Delta_x u|^p dx.$$

Integrating the previous inequality over $\mathbb{R}^M$ and using Theorem 4.3,

$$\left\| \frac{u}{|x|^2} \right\|_p \leq C \|\Delta_x u\|_p \leq C(\|Lu\|_p + \| |x|^\alpha \Delta_y u \|_p) \leq C \|Lu\|_p. \quad \square$$

Remark 4.15. The above Rellich inequality uses Theorem 4.3 to replace Δ_x with the operator L . However, even its version in dimension 1 (that is, for D_{xx} rather than L) is not obvious and probably cannot be obtained by integration by parts; see, e.g., [Metafune et al. 2019], where it is shown that Rellich inequalities can be proved for the Laplacian in $L^p(\mathbb{R}^N)$ when $p < \frac{N}{2}$, a condition which is never verified in dimension 1.

We first prove mixed derivatives estimates assuming that both u and $\nabla_x u$ vanish for $x = 0$. We need the following covering result.

Proposition 4.16 [Cupini and Fornaro 2004, Proposition 6.1]. *For every $0 \leq k < \frac{1}{2}$ there exists a natural number $\zeta = \zeta(N, k)$ with the following property. Given $\mathcal{F} = \{x + B(\rho(x))\}_{x \in \mathbb{R}^N}$, where $\rho : \mathbb{R}^N \rightarrow \mathbb{R}_+$ is a Lipschitz continuous function with Lipschitz constant k , there exist a countable subcovering $\{x_n + B(\rho(x_n))\}_{n \in \mathbb{N}}$ of $\mathbb{R}^N$ such that at most ζ among the double balls $\{x_n + B(2\rho(x_n))\}_{n \in \mathbb{N}}$ overlap.*

The following lemma is an immediate consequence of the classical Calderón–Zygmund inequalities for the Laplacian.

Lemma 4.17. *Let $1 < p < \infty$. Then for every $a \in \mathbb{R}$, $u \in C_c^\infty$, one has*

$$\|D_{xx}u\|_p + \|a^2 D_{yy}u\|_p + \|a D_{xy}u\|_p \leq C \|\Delta_x u + a^2 \Delta_y u\|_p.$$

Proof. It is sufficient to apply the Calderón–Zygmund inequalities to $v(x, y) = u(x, ay)$. $\square$

Lemma 4.18. *Let $p \neq \frac{N}{2}$, N . One has*

$$\|D_{xx}u\|_p + \||x|^\alpha D_{yy}u\|_p + \||x|^{\frac{\alpha}{2}} D_{xy}u\|_p \leq C \|Lu\|_p$$

for every $u \in C_c^\infty$ such that $u(0, y) = 0$, $\nabla_x u(0, y) = 0$ for all $y \in \mathbb{R}^M$.

Proof. We fix $x_0 \in \mathbb{R}^N$ and choose $\vartheta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \vartheta \leq 1$, $\vartheta(x) = 1$ for $x \in B(0, 1)$ and $\vartheta(x) = 0$ for $x \in \mathbb{R}^N \setminus B(0, 2)$. Moreover, we set $\vartheta_\rho(x) = \vartheta((x - x_0)/\rho)$, where $\rho = \frac{1}{4}|x_0|$. We apply Lemma 4.17 to the function $\vartheta_\rho u$ and obtain

$$\|D_{xx}(\vartheta_\rho u)\|_p + \||x_0|^\alpha D_{yy}(\vartheta_\rho u)\|_p + \||x_0|^{\frac{\alpha}{2}} D_{xy}(\vartheta_\rho u)\|_p \leq C \|\Delta_x(\vartheta_\rho u) + |x_0|^\alpha \Delta_y(\vartheta_\rho u)\|_p.$$

By the classical interpolation inequalities for the gradient we get, for every $\eta > 0$,

$$\begin{aligned} & \|D_{xx}u\|_{L^p(B(x_0, \rho))} + \||x_0|^\alpha D_{yy}u\|_{L^p(B(x_0, \rho))} + \||x_0|^{\frac{\alpha}{2}} D_{xy}u\|_{L^p(B(x_0, \rho))} \\ & \leq C \left(\|\Delta_x u + |x_0|^\alpha \Delta_y u\|_{L^p(B(x_0, 2\rho))} + \frac{1}{\rho} \|\nabla_x u\|_{L^p(B(x_0, 2\rho))} + \frac{1}{\rho^2} \|u\|_{L^p(B(x_0, 2\rho))} + \frac{|x_0|^{\frac{\alpha}{2}}}{\rho} \|\nabla_y u\|_{L^p(B(x_0, 2\rho))} \right) \\ & \leq C \left(\|\Delta_x u + |x_0|^\alpha \Delta_y u\|_{L^p(B(x_0, 2\rho))} + \eta \|\Delta_x u\|_{L^p(B(x_0, 2\rho))} + \eta \||x_0|^\alpha \Delta_y u\|_{L^p(B(x_0, 2\rho))} + \frac{1}{\eta \rho^2} \|u\|_{L^p(B(x_0, 2\rho))} \right). \end{aligned}$$

Since

$$\rho = \frac{1}{4}|x_0|, \quad \frac{1}{2}|x_0| \leq |x| \leq \frac{3}{2}|x_0|, \quad x \in B(x_0, 2\rho),$$

we get

$$\begin{aligned} & \|D_{xx}u\|_{L^p(B(x_0, \rho))} + \||x|^\alpha D_{yy}u\|_{L^p(B(x_0, \rho))} + \||x|^{\frac{\alpha}{2}} D_{xy}u\|_{L^p(B(x_0, \rho))} \\ & \leq C \left(\|\Delta_x u + |x|^\alpha \Delta_y u\|_{L^p(B(x_0, 2\rho))} + \eta \|\Delta_x u\|_{L^p(B(x_0, 2\rho))} \right. \\ & \quad \left. + \eta \||x|^\alpha \Delta_y u\|_{L^p(B(x_0, 2\rho))} + \frac{1}{\eta} \left\| \frac{u}{|x|^2} \right\|_{L^p(B(x_0, 2\rho))} \right). \quad (19) \end{aligned}$$

Let $\{B(x_n, \rho(x_n))\}$ be a countable covering of $\mathbb{R}^N$ as in Proposition 4.16 such that at most ζ among the double balls $\{B(x_n, 2\rho(x_n))\}$ overlap. Writing (19) with x_n instead of x_0 and summing over n it follows that

$$\begin{aligned} & \|D_{xx}u\|_p + \||x|^\alpha D_{yy}u\|_p + \||x|^{\frac{\alpha}{2}} D_{xy}u\|_p \\ & \leq C \left(\|\Delta_x u + |x|^\alpha \Delta_y u\|_p + \eta \|\Delta_x u\|_p + \eta \||x|^\alpha \Delta_y u\|_p + \frac{1}{\eta} \left\| \frac{u}{|x|^2} \right\|_p \right). \end{aligned}$$

By choosing η small enough we get

$$\|D_{xx}u\|_p + \||x|^\alpha D_{yy}u\|_p + \||x|^{\frac{\alpha}{2}} D_{xy}u\|_p \leq C \left(\|\Delta_x u + |x|^\alpha \Delta_y u\|_p + \left\| \frac{u}{|x|^2} \right\|_p \right).$$

and the claim follows from Lemma 4.14. $\square$

Next we prove mixed derivatives estimates assuming that either u or $\nabla_x u$ vanishes for $x = 0$.

Lemma 4.19. *If $p \neq \frac{2N}{2-\alpha}, \frac{N}{2}, N$, then*

$$\| |x|^{\frac{\alpha}{2}} D_{xy} u \|_p \leq C \|Lu\|_p$$

for every $u \in C_c^\infty$ such that $u(0, y) = 0$ or $u_x(0, y) = 0$ for all $y \in \mathbb{R}^M$.

Proof. Let $u \in C_c^\infty$ such that $u(0, y) = 0$ and let $v(x, y) = \frac{1}{\lambda} u(\lambda x, y)$. Then $v(0, y) = 0$ and $\nabla_x v(0, y) = \nabla_x u(0, y)$. This implies that $w = u - v$ satisfies $w(0, x) = w_x(0, x) = 0$. Moreover

$$\| |x|^{\frac{\alpha}{2}} D_{xy} v \|_p = \lambda^{-\frac{\alpha}{2} - \frac{N}{p}} \| |x|^{\frac{\alpha}{2}} D_{xy} u \|_p$$

and, applying Theorem 4.3,

$$\begin{aligned} \|Lv\|_p &\leq \|v_{xx}\|_p + \| |x|^\alpha \Delta_y v \|_p \\ &= \lambda^{1 - \frac{N}{p}} \|u_{xx}\|_p + \lambda^{-\alpha - 1 - \frac{N}{p}} \| |x|^\alpha \Delta_y u \|_p \leq C(\lambda) \|Lu\|_p. \end{aligned}$$

Applying Lemma 4.18 to w we then have

$$\begin{aligned} \| |x|^{\frac{\alpha}{2}} D_{xy} u \|_p &\leq \| |x|^{\frac{\alpha}{2}} D_{xy} w \|_p + \| |x|^{\frac{\alpha}{2}} D_{xy} v \|_p \leq C(\|Lw\|_p + \| |x|^{\frac{\alpha}{2}} D_{xy} v \|_p) \\ &\leq C(\|Lu\|_p + \|Lv\|_p + \| |x|^{\frac{\alpha}{2}} D_{xy} v \|_p) \leq C(\lambda) \|Lu\|_p + C \| |x|^{\frac{\alpha}{2}} D_{xy} v \|_p \\ &= C(\lambda) \|Lu\|_p + C \lambda^{-\frac{\alpha}{2} - \frac{N}{p}} \| |x|^{\frac{\alpha}{2}} D_{xy} u \|_p. \end{aligned}$$

The claim then follows by choosing λ large enough such that $C \lambda^{-\alpha/2 - N/p} \leq \frac{1}{2}$.

Assume now $u_x(0, y) = 0$ and let $v(x, y) = u(\lambda x, y)$. Then $u(0, y) = v(0, y)$ and $v_x(0, y) = \lambda u_x(0, y) = 0$. Moreover

$$\| |x|^{\frac{\alpha}{2}} D_{xy} v \|_p = \lambda^{1 - \frac{\alpha}{2} - \frac{N}{p}} \| |x|^{\frac{\alpha}{2}} D_{xy}^2 u \|_p.$$

It follows that $w = u - v$ satisfies $w(0, x) = w_x(0, x) = 0$. Hence an analogous argument yields

$$\| |x|^{\frac{\alpha}{2}} D_{xy} u \|_p \leq C(\lambda) \|Lu\|_p + C \lambda^{1 - \frac{\alpha}{2} - \frac{N}{p}} \| |x|^{\frac{\alpha}{2}} D_{xy}^2 u \|_p.$$

Choosing λ large enough or small enough according to $1 - \frac{\alpha}{2} - \frac{N}{p} > 0$ or $1 - \frac{\alpha}{2} - \frac{N}{p} < 0$ we get the claim for $1 - \frac{\alpha}{2} - \frac{N}{p} \neq 0$ or, equivalently, $p \neq \frac{2N}{2-\alpha}$. $\square$

Proof of Theorem 4.13. Let us suppose, preliminarily, $p \neq \frac{2N}{2-\alpha}$, $p \neq \frac{N}{2}$, $p \neq N$ and let $u \in C_c^\infty$. We introduce the operators

$$Pu(x, y) = \frac{u(x, y) + u(-x, y)}{2}, \quad Qu(x, y) = \frac{u(x, y) - u(-x, y)}{2}$$

Observe that

$$u = Pu + Qu, \quad Qu(0, y) = 0, \quad \nabla_x (Pu)(0, y) = 0, \quad y \in \mathbb{R}^M,$$

P and Q commute with the second-order derivatives and $\|P(Lu)\|_p + \|Q(Lu)\|_p$ is equivalent to $\|Lu\|_p$. Moreover

$$L(Pu) = P(Lu), \quad L(Qu) = Q(Lu).$$

We can therefore apply the results in Lemma 4.19 to Pu and Qu . For the mixed second-order derivatives we get

$$\begin{aligned} \| |x|^{\frac{\alpha}{2}} D_{xy} u \|_p &\leq \| |x|^{\frac{\alpha}{2}} P(D_{xy} u) \|_p + \| |x|^{\frac{\alpha}{2}} Q(D_{xy} u) \|_p = \| |x|^{\frac{\alpha}{2}} D_{xy}(Pu) \|_p + \| |x|^{\frac{\alpha}{2}} Q_{xy}(Qu) \|_p \\ &\leq C(\|L(Pu)\|_p + \|L(Qu)\|_p) = C(\|P(Lu)\|_p + \|Q(Lu)\|_p) \leq C\|Lu\|_p. \end{aligned}$$

By density the proof extends to $u \in D_p(L)$.

Suppose now $p = \frac{2N}{2-\alpha}$. Observe that, by the previous part of the proof, the operator $|x|^{\alpha/2} D_{xy}(I - L)^{-1}$ is bounded in L^p for some $p_1 < \frac{2N}{2-\alpha} < p_2$, with $p_1, p_2 \neq \frac{N}{2}, N$. The Riesz–Thorin interpolation theorem then yields the boundedness of $|x|^{\alpha/2} D_{xy}(I - L)^{-1}$ also for $p = \frac{2N}{2-\alpha}$; the same scaling argument used in the proof of Proposition 4.10 then proves the required claim. We can argue similarly for $p = N$, $p = \frac{N}{2}$. $\square$

As a corollary we improve gradient estimates near $x = 0$ showing that $|x|^{\alpha/2-1} \nabla_y u \in L^p$, $u \in D_p(L)$, when $(\frac{\alpha}{2} - 1)p + N > 0$. This last condition is necessary for the above integrability, since otherwise the weight $|x|^{\alpha/2-1}$ is not locally p -summable near $x = 0$. We also recall that $|x|^{\alpha/2} \nabla_y u \in L^p$ by Proposition 4.10.

Corollary 4.20. *Let $(\frac{\alpha}{2} - 1)p + N > 0$. Then for every $u \in D_p(L)$*

$$\| |x|^{\frac{\alpha}{2}-1} \nabla_y u \|_p \leq C\|Lu\|_p.$$

Proof. By density we may assume that $u \in C_c^\infty$. By the Hardy inequality, see for example [Metafune et al. 2015, Proposition 8.1],

$$\int_{\mathbb{R}^N} |x|^{(\frac{\alpha}{2}-1)p} |\nabla_y u|^p dx \leq \left(\frac{p}{(\frac{\alpha}{2}-1)p + N} \right)^p \int_{\mathbb{R}^N} |x|^{\frac{\alpha}{2}p} |D_{xy} u|^p dx.$$

Integrating over $\mathbb{R}^M$ and using Theorem 4.13, the claim follows. $\square$

We can now strengthen Proposition 4.10.

Theorem 4.21. *Let $p \in (1, \infty)$. Then one has*

$$D_p(L) = \{u \in L^p : \nabla_x u, D_{xx} u \in L^p, |x|^{\frac{\alpha}{2}} \nabla_y u, |x|^{\frac{\alpha}{2}} D_{xy} u, |x|^\alpha D_{yy} u \in L^p\}.$$

In particular the graph norm $\|u\|_p + \|Lu\|_p$ is equivalent to

$$\|u\|_p + \|\nabla_x u\|_p + \| |x|^{\frac{\alpha}{2}} \nabla_y u \|_p + \|D_{xx} u\|_p + \| |x|^\alpha D_{yy} u \|_p + \| |x|^{\frac{\alpha}{2}} D_{xy} u \|_p.$$

Moreover, if $(\frac{\alpha}{2} - 1)p + N > 0$, then also $|x|^{\frac{\alpha}{2}-1} \nabla_y u \in L^p$.

Proof. The proof follows from Proposition 4.10, Theorem 4.13 and Corollary 4.20. $\square$

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