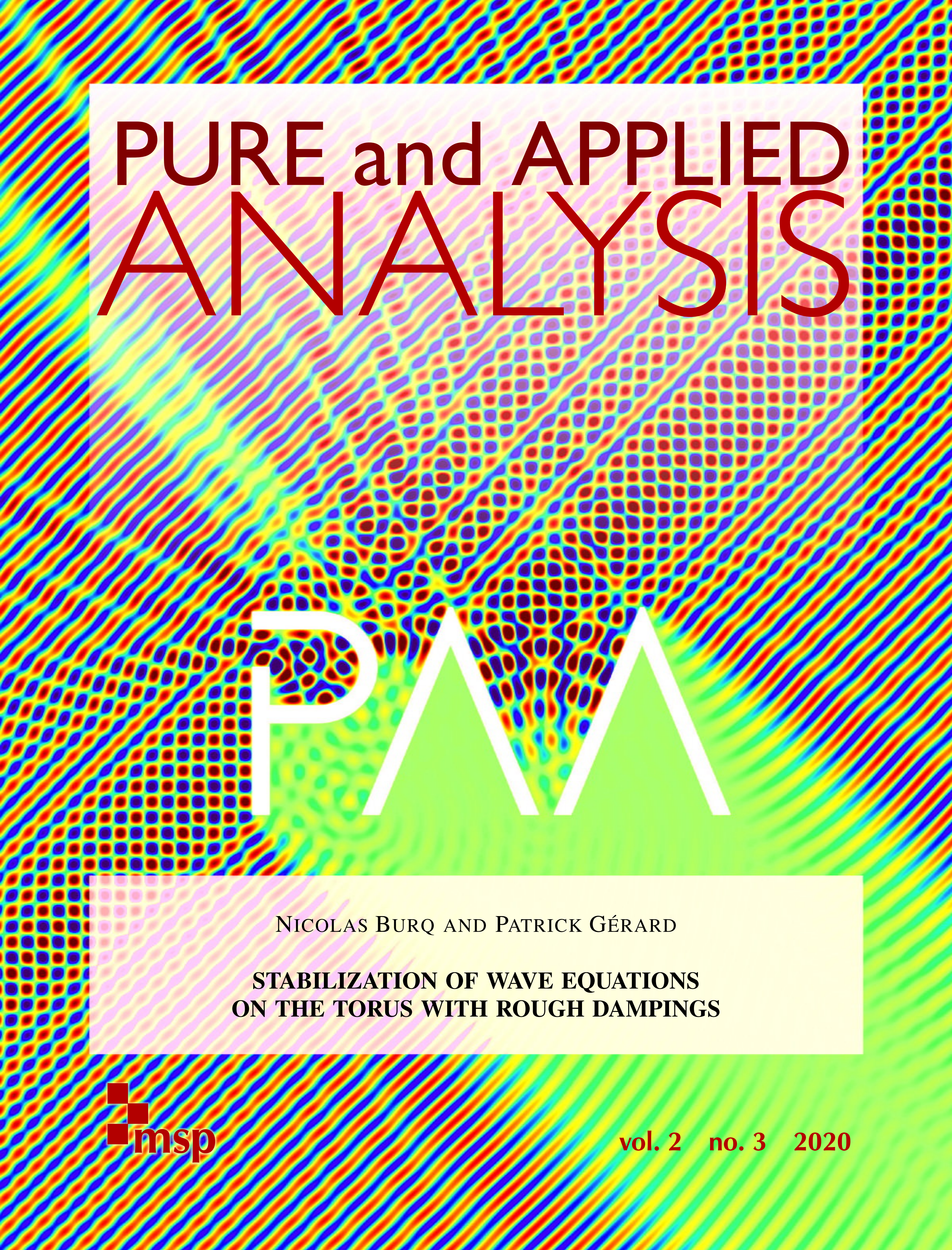

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STABILIZATION OF WAVE EQUATIONS ON THE TORUS WITH ROUGH DAMPINGS

NICOLAS BURQ AND PATRICK GÉRARD

For the damped wave equation on a compact manifold with *continuous* dampings, the geometric control condition is necessary and sufficient for uniform stabilization. On the two-dimensional torus, in the special case where $a(x) = \sum_{j=1}^{N} a_j 1_{x\in R_j}$ (R_j are polygons), we give a very simple necessary and sufficient geometric condition for uniform stabilization. We also propose a natural generalization of the geometric control condition which makes sense for L^∞ dampings. We show that this condition is always necessary for uniform stabilization (for any compact (smooth) manifold and any L^∞ damping), and we prove that it is sufficient in our particular case on $\mathbb{T}^2$ (and for our particular dampings).

Pour l'équation des ondes amortie sur une variété compacte, dans le cas d'un amortissement *continu*, la condition de contrôle géométrique est nécessaire et suffisante pour la stabilisation uniforme. Sur le tore $\mathbb{T}^2$ et dans le cas où $a(x) = \sum_{j=1}^{N} a_j 1_{x\in R_j}$ (R_j sont des polygones), nous exhibons une condition géométrique nécessaire et suffisante très simple. Nous proposons aussi une généralisation naturelle de la condition de contrôle géométrique, pour un amortissement seulement L^∞. Cette généralisation est toujours nécessaire pour la stabilisation uniforme (sur toute variété compacte régulière), et nous démontrons qu'elle est suffisante dans notre cas particulier du tore $\mathbb{T}^2$ (et pour nos fonctions d'amortissement particulières).

1. Notation and main results

Let (M, g) be a (smooth) compact Riemannian manifold endowed with the metric g, Δ_g the Laplace operator on functions on M and for $a \in L^\infty(M)$, let us consider the damped wave (or Klein–Gordon) equation

$$(\partial_t^2 - \Delta + a(x)\,\partial_t + m)u = 0, \quad (u|_{t=0}, \partial_t u|_{t=0}) = (u_0, u_1) \in (H^1 \times L^2)(M), \tag{1-1}$$

where $0 \le m \in L^\infty(M)$. If $a \ge 0$ a.e. it is well known that the energy

$$E_m(u)(t) = \int_M (|\nabla_g u|_g^2 + |\partial_t u|^2 + m|u|^2)\, d\,\mathrm{vol}_g \tag{1-2}$$

is decaying and satisfies

$$E_m(u)(t) = E_m(u)(0) - \int_0^t \int_M 2a(x)|\partial_t u|^2\, d\,\mathrm{vol}_g.$$

We shall say that the *uniform stabilisation* holds for the damping a if one of the following equivalent properties holds (see Appendix B for the equivalence):

MSC2010: 34L20, 35L05, 35Q93, 35S05, 58J40, 93D15.
Keywords: wave equations, control, stabilization, second microlocalization, geometric control.

(1) There exists a rate $f(t)$ such that $\lim_{t\to+\infty} f(t) = 0$ and for any $(u_0, u_1) \in (H^1 \times L^2)(M)$

$$E_m(u)(t) \le f(t) E_m(u)(0).$$

(2) There exists $C, c > 0$ such that for any $(u_0, u_1) \in (H^1 \times L^2)(M)$

$$E_m(u)(t) \le Ce^{-ct} E_m(u)(0).$$

(3) There exists $T > 0$ and $c > 0$ such that for any $(u_0, u_1) \in (H^1 \times L^2)(M)$, if u is the solution to the damped wave equation (1-1), then

$$E_m(u)(0) \le C \int_0^T \int_M 2a(x)|\partial_t u|^2 \, d\mathrm{vol}_g.$$

(4) There exists $T > 0$ and $c > 0$ such that for any $(u_0, u_1) \in (H^1 \times L^2)(M)$, if u is the solution to the undamped wave equation

$$(\partial_t^2 - \Delta + m)u = 0, \quad (u|_{t=0}, \partial_t u|_{t=0}) = (u_0, u_1) \in (H^1 \times L^2)(M), \tag{1-3}$$

then

$$E_m(u)(0) \le C \int_0^T \int_M 2a(x)|\partial_t u|^2 \, d\mathrm{vol}_g.$$

The following result is classical; see [Rauch and Taylor 1974; 1975; Babich and Popov 1981; Babich and Ulin 1981; Ralston 1982; Bardos, Lebeau and Rauch 1992; Burq and Gérard 1997; Lebeau 1996; Koch and Tataru 1995; Sjöstrand 2000; Hitrik 2003.

Theorem 1 [Bardos, Lebeau and Rauch 1992; Burq and Gérard 1997]. *Let $m \ge 0$. Assume that the damping a is continuous. For $\rho_0 = (x_0, \xi_0) \in S^*M$ denote by $\gamma_{\rho_0}(s)$ the geodesic starting from x_0 in (co-)direction ξ_0. Then the damping a stabilizes uniformly the wave equation if and only if the following geometric condition is satisfied*:

$$\exists T, c > 0 \quad \text{such that} \quad \inf_{\rho_0 \in S^*M} \int_0^T a(\gamma_{\rho_0}(s))\, ds \ge c. \tag{GCC}$$

When the damping a is not continuous but merely L^∞, an adaptation of the same techniques gives:

Theorem 2. *Assume that $a \in L^\infty(M)$. Then the strong geometric condition*

$$\begin{aligned} \exists T, c > 0 \quad &\text{such that} \quad \forall \rho_0 \in S^*M, \quad \exists s \in (0, T), \quad \exists \delta > 0 \\ &\text{such that} \quad a \ge c \ \text{ a.e. on } B(\gamma_{\rho_0}(s), \delta) \end{aligned} \tag{SGCC}$$

is ***sufficient*** *for uniform stabilisation, and the weak geometric condition*

$$\exists T > 0 \quad \text{such that} \quad \forall \rho_0 \in S^*M, \quad \exists s \in (0, T) \quad \text{such that} \quad \gamma_{\rho_0}(s) \in \mathrm{supp}(a), \tag{WGCC}$$

where supp(a) *is the support* (*in the distributional sense*) *of a, is* ***necessary*** *for uniform stabilisation.*

Though the question appears to be very natural, until the present work, the only known case in between was essentially an example of [Lebeau 1992, pp. 15–16] (from an idea of J. Rauch) where $M = \mathbb{S}^d$ and a is the characteristic function of the half-sphere (notice however some refinements of (WGCC) in [Humbert, Privat and Trélat 2019; Trélat, Humbert and Privat 2017]). In the case of the half-sphere, uniform stabilisation holds (see [Zhu 2018] for a more detailed proof and a generalization of this result).

Theorem 3 [Lebeau 1992]. *On the d-dimensional sphere*,

$$\mathbb{S}^d = \{x = (x_0, \dots, x_d) \subset \mathbb{R}^{d+1} : \|x\| = 1\},$$

uniform stabilisation holds for the characteristic function of the half-sphere

$$\mathbb{S}^d_+ = \{x = (x_0, \dots, x_d) \subset \mathbb{R}^{d+1} : \|x\| = 1,\ x_0 > 0\}.$$

Remark 1.1. Notice that in this case, all the geodesics enter the interior of the support of a, and hence fulfill the (SGCC) requirements, except the family of geodesics included in the boundary of the support of a, the $(d-1)$-dimensional sphere,

$$\partial\mathbb{S}^d_+ = \{x = (x_0, \dots, x_d) \subset \mathbb{R}^{d+1} : \|x\| = 1,\ x_0 = 0\}.$$

When the manifold is a two-dimensional torus (rational or irrational) and the damping a is a linear combination of characteristic functions of polygons, i.e., there exists N, R_j, $j = 1, \dots, N$, (disjoint and not necessarily vertical) polygons and $0 < a_j$, $j = 1, \dots, N$, such that

$$a(x) = \sum_{j=1}^{N} a_j 1_{x \in R_j}, \tag{1-4}$$

we can state another natural simple geometric condition. Let us endow the torus with an orientation (i.e., we see the torus as a surface in $\mathbb{R}^3$ and define at each point a normal vector $n(x)$). For any initial point x_0 and any norm-1 tangent vector X_0, let γ be the geodesic starting from x_0 in direction X_0 and parametrized by arc length. Let $\nu(\gamma(s))$ be the unique vector normal to γ in the torus and such that $(n(\gamma(s)), \dot\gamma(s), \nu(\gamma(s)))$ is a direct orthonormal frame. By convention, we shall say that ν points to the left of the geodesic (and $-\nu$ to the right).

Assumption 1.2. Assume that the manifold is a two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/A\mathbb{Z} \times B\mathbb{Z}$, $A, B > 0$. Assume that there exists $T > 0$ such that all geodesics (straight lines) of length T either encounter the interior of one of the polygons or follow for some time one of the sides of a polygon R_{j_1} *on the left* and for some time one of the sides of a polygon R_{j_2} (possibly the same) *on the right*.

Our main result is the following:

Theorem 4. *The damping a stabilize uniformly the wave equation if and only if Assumption 1.2 is satisfied.*

Corollary 1.3. *Stabilisation holds for the examples* (a) *and* (d) *of Figure 1*, *but not for* (b), (c) *and* (e).

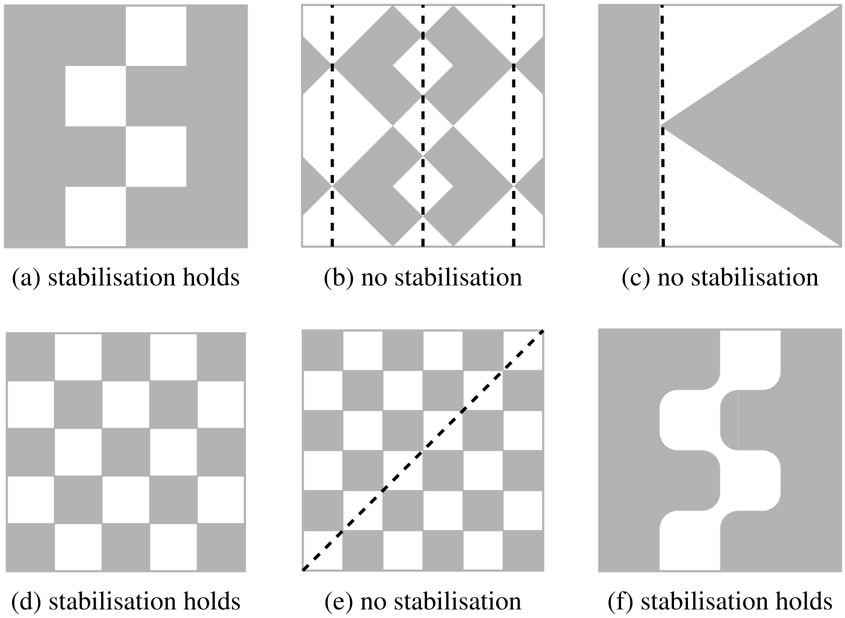

Figure 1. Checkerboards: the damping a is equal to 1 in the gray region, and 0 elsewhere. For all these examples (WGCC) is satisfied but not (SGCC). The dashed lines are geodesics which violate Assumption 1.2.

Remark 1.4. In Assumption 1.2, as soon as the damping is nontrivial (i.e., we have at least one polygon), all nonclosed geodesics will enter the interior of this polygon (because any nonclosed geodesic is dense in the torus). As a consequence, the second part of the assumption has to be checked only for closed geodesics. Actually, closed geodesics corresponding to directions $(\xi, \eta) = (p, q)/\sqrt{p^2 + q^2},\ p \wedge q = 1$, will also enter the polygon as soon as $p^2 + q^2$ is large enough. As a consequence, the second part of Assumption 1.2 has to be checked only for a *finite number* of closed geodesics.

Remark 1.5. As pointed out by a referee, our proof actually gives a sufficient condition for stabilisation in a more general setting where the R_j need not be polygons, but are open subsets, and we assume that all but a finite number of closed geodesics are damped (in the sense that they enter the interior of one of the R_j's) and the remaining closed geodesics satisfy the left/right property on intervals of positive measure (which implies that the boundaries of the open sets R_{j_1} and R_{j_2} have some flat parts). As a consequence, stabilisation holds for Figure 1(f).

Remark 1.6. Stabilisation implies that exact controllability holds for some finite $T > 0$. However our proof relies on a contradiction argument and resolvent estimates. It gives no geometric interpretation for this controllability time. This contradiction argument allows us on tori to avoid a particularly delicate regime at the edge of the uncertainty principle (see Section 3A). Giving a geometric interpretation of the time necessary for control would require dealing with this regime; see [Burq $\geq$ 2020].

The plan of the paper is the following: In Section 2 we give a proof of Theorem 2 using classical methods. In Section 3, we focus on the model case of the checkerboard in Figure 1(a). We first reduce the question of uniform stabilisation to the proof of an observation estimate for high-frequency solutions of Helmholtz equations. We proceed by contradiction and construct good quasimodes, for the study of which we perform a microlocalisation which shows that the only obstruction is the vertical geodesic in the middle of the board. Then we prove a nonconcentration estimate which shows that solutions of Helmholtz equations (quasimodes) cannot concentrate too fast on this trajectory. This is essentially the only point in the proof which is specific to the torus and it relies on the special geometric structure of the torus which was previously used in the context of Schrödinger equations [Burq and Zworski 2004; 2005; 2012; Macià 2010; Anantharaman and Macià 2014; Bourgain, Burq and Zworski 2013] and also for wave equations [Burq and Hitrik 2007; Anantharaman and Léautaud 2014]. Finally, by means of a second microlocalisation with respect to this vertical geodesic, we obtain a contradiction. In Section 4, we show how the general case can be reduced to this model case. Finally, in the last section we introduce a generalized version of (GCC) that makes sense for $a \in L^\infty$ and which is equivalent to Assumption 1.2 in our particular case. We prove that this generalized geometric control condition is always necessary (on any Riemannian manifold and for any damping $0 \le a \in L^\infty$) and we conjecture that it is always sufficient. For the convenience of the reader, we gathered in Appendix A quite a few classical results about the link between resolvent estimates and stabilisation.

The *second microlocalisation* procedure has a well-established history starting with [Laurent 1979; 1985; Kashiwara and Kawai 1980; Sjöstrand 1982; Lebeau 1985] in the analytic context (see also [Bony and Lerner 1989] in the C^∞ framework and [Sjöstrand and Zworski 1999] in the semiclassical setting) and in the framework of defect measures by [Fermanian-Kammerer 2000; Miller 1996; 1997; 2000; Nier 1996; Fermanian-Kammerer and Gérard 2002; 2003; 2004]. Notice that most of these previous works in the framework of measures dealt with lagrangian or involutive submanifolds, and it is worth comparing our contribution with these previous works, in particular [Nier 1996; Anantharaman and Macià 2014]. Here we are interested in the wave equation, while the authors in [Nier 1996; Anantharaman and Macià 2014] were interested in the Schrödinger equation, and (compared to [Anantharaman and Macià 2014]) we are dealing with worse quasimodes ($o(h)$ instead of $o(h^2)$). Another difference is that we perform a second microlocalisation along a symplectic submanifold (namely $\{(x=0, y, \xi=0, \eta) \in T^*\mathbb{T}^2\}$), while they consider an isotropic submanifold $\{x = 0\}$ in [Nier 1996] or $\{(x', x'', \xi'=0, \xi'') \in T^*\mathbb{T}^d\}$ in [Anantharaman and Macià 2014]. An exception is [Fermanian-Kammerer 2005], to which our construction is very close. On the other hand, a feature shared by the present work and [Nier 1996; Anantharaman and Macià 2014] is that in all cases the analysis requires working at the edges of the uncertainty principle and using refinements of some exotic Weyl–Hörmander classes ($S^{1,1}$ in [Nier 1996], $S^{0,0}$ in [Anantharaman and Macià 2014] and $S^{1/2,1/2}$ in the present work); see [Hörmander 1985; Léautaud and Lerner 2017] for related work. Another worthwhile comparison is with [Burq and Hitrik 2007; Anantharaman and Léautaud 2014] on the damped wave equation on the torus when the control domain is arbitrary (in this case (WGCC) is in general not satisfied). However, though both works use some kind of second microlocalisation and deal with the wave equation, in [Burq and Hitrik 2007; Anantharaman and Léautaud

2014] the approaches use Schrödinger equations methods (strong quasimodes) transposed to get wave equations results and consequently lead to much weaker results (polynomial decay vs. exponential decay) under much weaker assumptions (arbitrary open sets).

2. First microlocalisation, proof of Theorem 2

In this section we work on an arbitrary compact manifold M with an arbitrary damping function $a \in L^\infty(M)$ and outline the classical propagation arguments which show that (SGCC) is sufficient for stabilisation, while (WGCC) is necessary. Let us assume (SGCC) holds. According to Proposition A.5, we need to prove (A-5)

$$\begin{aligned} &\exists h_0 > 0 \quad \text{such that} \quad \forall\, 0 < h < h_0, \quad \forall\, (u, f) \in H^2(M) \times L^2(M), \quad (h^2\Delta + 1)u = f, \\ &\qquad \|u\|_{L^2(M)} \leq C\Big(\|a^{\frac{1}{2}} u\|_{L^2} + \frac{1}{h}\|f\|_{L^2}\Big). \end{aligned} \tag{A-5}$$

To prove this estimate we argue by contradiction and obtain sequences $(h_n) \to 0$ and (u_n, f_n) such that

$$(h_n^2\Delta + 1)u_n = f_n, \quad \|u_n\|_{L^2} = 1, \quad \|a^{\frac{1}{2}} u_n\|_{L^2} = o(1)_{n\to+\infty}, \quad \|f_n\|_{L^2} = o(h_n)_{n\to+\infty}.$$

Extracting a subsequence, we can assume that the sequence (u_n) has a semiclassical measure ν on $T^*\mathbb{T}^2$. For $q \in C_0^\infty(T^*M)$, we define $\mathrm{Op}_h(q)$ by the following procedure. Using a partition of unity, we can assume that q is supported in a local chart. Then, in this chart, we define

$$\mathrm{Op}_h(q)(u) = \frac{1}{(2\pi h)^d} \int e^{\frac{i}{h}(x-y)\cdot\xi} q(x, \xi)\zeta(y)u(y)\, dy\, d\xi, \tag{2-1}$$

where $\zeta = 1$ in a neighborhood of the support of q (note that modulo smoothing $O(h^\infty)$ errors, this quantisation does not depend on the choice of the cut-off ζ). Then a semiclassical measure for the sequence (u_n) satisfies

$$\lim_{n\to+\infty} (\mathrm{Op}_{h_n}(q)u_n, u_n)_{L^2(M)} = \langle \nu, q\rangle.$$

In our case, it is supported in the characteristic set

$$\{(X, \Xi) \in S^*M : \|\Xi\|^2 = 1\}.$$

Furthermore, this measure has total mass 1 and is invariant by the bicharacteristic flow:

$$\Xi \cdot \nabla_X \nu = 0.$$

We refer to [Gérard and Leichtnam 1993; Burq 1997; 2002, Section 3] for the definition of semiclassical calculus on manifolds and semiclassical measures and a proof of these results in a very similar context (see also [Zworski 2012]). Let

$$S = \{x \in M : \text{there exists } \delta > 0, c > 0 \text{ such that } a \geq c \text{ on } B(x, \delta)\}.$$

Since $\|a^{1/2}u_n\|_{L^2} = o(1)_{n\to+\infty}$, we get that the measure ν vanishes in a neighborhood of every point $\rho \in S_x^*M$ for all $x \in S$. The assumption (SGCC) ensures that every bicharacteristic contains at least one point in S. Hence ν is identically 0, which contradicts the fact that it has total mass 1!

For the sake of completeness, let us now prove that stabilisation implies (WGCC). We are going to use that for any geodesic there exists a sequence of solutions to the wave equation which concentrates on this geodesic. This can be obtained through geometric optics constructions (see Proposition 5.1) or systematic use of semiclassical measures (see [Gérard 1996] in the special case of constant coefficients). In fact, we shall in Section 5A refine this approach and use *quantitative* estimates for this concentration to show that actually stabilisation implies the stronger condition (GGCC).

Proposition 2.1. *Assume that* (WGCC) *does not hold. Let* $T > 0$. *Consider a geodesic* γ *of length* T *which does not encounter the support of the damping function* a. *Then there exists a sequence* (u_n) *of solutions to the wave equation* (1-3) *which satisfies*

$$\lim_{n\to+\infty} E_m(u_n) = 1, \qquad \lim_{n\to+\infty} \int_0^T \int_M a(x)|\partial_t u|^2(t,x)\,dx\,dt = 0. \tag{2-2}$$

First by compactness, there exists $\delta > 0$ such that

$$\operatorname{dist}(\gamma([-\delta, T+\delta]), \operatorname{supp}(a)) \geq \delta.$$

Then, according to Proposition 5.1, there exists a sequence of approximate solutions (v_n) to the wave equation (with $m = 0$) which concentrate on the geodesic γ and satisfy (5-2) and (5-3). From (5-3), we deduce that

$$\|v_n\|_{L^2(M)} = O(h_n), \tag{2-3}$$

and from the concentration on the geodesic γ (and the δ separation with the support of a)

$$\int_0^T \int_M a(x)|\partial_t v_n|^2(t,x)\,dx\,dt = o(1)_{n\to+\infty} \tag{2-4}$$

uniformly with respect to $t \in [-1, T+1]$. The solution u_n to the wave equation (with m) (1-3) with the same initial data satisfies

$$(\partial_t^2 - \Delta + m)(u_n - v_n) = -m v_n, \quad (u_n - v_n)|_{t=0} = 0, \quad \partial_t(u_n - v_n)|_{t=0} = 0.$$

As a consequence of Duhamel's formula and (2-3),

$$\sup_{t\in[0,T]} \|u_n - v_n\|^2_{H^1(M)} + \|\partial_t u_n - \partial_t v_n\|^2_{L^2(M)} = O(h_n^2).$$

This implies according to (2-4)

$$E_m(u_n) = 1 + O(h_n^2), \qquad \int_0^T \int_M a(x)|\partial_t v_n|^2(t,x)\,dx\,dt = O(h_n^2).$$

3. The model case of a checkerboard

In this section we prove Theorem 4 for the model in Figure 2 on the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/(2\mathbb{Z})^2$. We shall later microlocally reduce the general case to this model.

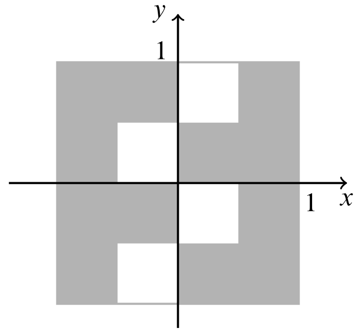

Figure 2. The checkerboard: a microlocal model where the damping a is equal to 1 in the gray region, and 0 elsewhere.

As previously, we prove (A-5) via a contradiction argument and construct the associated semiclassical measure ν. According to the results in Section 2, since the only two bicharacteristics which do not enter the interior of the set where $a = 1$ are

$$\{(x = 0, \xi = 0, \eta = \pm 1)\},$$

we know that ν is supported on the union of these two bicharacteristics.

3A. ***A priori nonconcentration estimate.*** In this section we show that (u_n) cannot concentrate on too small neighbourhoods around $\{x = 0\}$. This is the key (and only) point where we use the particular structure of the torus as a product manifold.

Let us recall that $\|(h_n^2\Delta + 1)u_n\|_{L^2} = o(h_n)$. Define

$$\epsilon(h_n) = \max\left(h_n^{\frac{1}{6}}, \left(\frac{\|(h_n^2\Delta + 1)u_n\|}{h_n}\right)^{\frac{1}{6}}\right), \tag{3-1}$$

so that

$$h_n^{-1}\epsilon^{-6}(h_n)\|(h_n^2\Delta + 1)u_n\|_{L^2} \le 1, \qquad \lim_{n\to+\infty} \epsilon(h_n) = 0. \tag{3-2}$$

The purpose of this section is to prove the following nonconcentration result which is actually related to Kakeya–Nikodym bounds; see [Sogge 2011; Blair and Sogge 2015; Miao, Sogge, Xi and Yang 2016].

Proposition 3.1. *Assume that $\|u_n\|_{L^2} = \mathcal{O}(1)$ and* (3-2) *holds. Then there exists $C > 0$ such that*

$$\forall n \in \mathbb{N}, \quad \|u_n\|_{L^2(\{|x| \le h_n^{1/2}\epsilon^{-2}(h_n)\})} \le C\epsilon^{\frac{1}{2}}(h_n).$$

The proposition follows from the following one-dimensional propagation estimate; see [Burq and Zuily 2015] for related estimates.

Proposition 3.2. *There exist $C > 0$ and h_0 such that for any $0 < h < h_0$, $1 \le \beta \le h^{-1/2}$, and any $(u, f) \in H^2 \times L^2$ a solution of*

$$(h^2(\partial_x^2 + \partial_y^2) + 1)u = f,$$

we have

$$\|u\|_{L^\infty(\{|x|\leq\beta h^{1/2}\};L^2_y)}\leq C\beta^{-\frac{1}{2}}h^{-\frac{1}{4}}(\|u\|_{L^2_{x,y}(\{\beta h^{1/2}\leq|x|\leq 2\beta h^{1/2}\})}+h^{-1}\beta^2\|f\|_{L^2_{x,y}(\{|x|\leq 2\beta h^{1/2}\})}). \tag{3-3}$$

Let us first show that Proposition 3.1 follows from Proposition 3.2. Indeed, choosing $\beta=\epsilon^{-3}(h)$, Hölder's inequality gives

$$\begin{aligned}\|u\|_{L^2(\{|x|\leq h^{1/2}\epsilon^{-2}(h)\})}&\\ &\leq h^{\frac{1}{4}}\epsilon^{-1}(h)\|u\|_{L^\infty(\{|x|\leq h^{1/2}\epsilon^{-3}(h\};L^2_y)}\\ &\leq C\epsilon^{\frac{1}{2}}(h)(\|u\|_{L^2_{x,y}(\{h^{1/2}\epsilon^{-3}(h)\leq|x|\leq 2h^{1/2}\epsilon^{-3}(h)\})}+h^{-1}\epsilon^{-6}(h)\|f\|_{L^2_{x,y}(\{|x|\leq 2\beta h^{1/2}\})})\\ &\leq C\epsilon^{\frac{1}{2}}(h)(\|u\|_{L^2}+h^{-1}\epsilon^{-6}(h)\|f\|_{L^2})\leq 2C\epsilon^{\frac{1}{2}}(h),\end{aligned} \tag{3-4}$$

where in the last inequality we used (3-2).

Now we can prove Proposition 3.2. Denote by v (resp. g) the partial Fourier transform of u (resp. f) with respect to y. For fixed x,

$$\|v(x,\cdot)\|_{L^2_\eta}=(2\pi)^{\frac{1}{2}}\|u(x,\cdot)\|_{L^2_y}.$$

We deduce that (3-3) is equivalent to

$$\begin{aligned}\|v\|_{L^\infty(\{|x|\leq\beta h^{1/2}\};L^2_\eta)}&\\ &\leq C\beta^{-\frac{1}{2}}h^{-\frac{1}{4}}(\|v\|_{L^2(\{\beta h^{1/2}\leq|x|\leq 2\beta h^{1/2}\};L^2_\eta)}+h^{-1}\beta^2\|f\|_{L^2(\{|x|\leq 2\beta h^{1/2}\};L^2_\eta)}).\end{aligned} \tag{3-5}$$

Now, by Minkowski's inequality,

$$\|v\|_{L^\infty_x;L^2_\eta}\leq\|v\|_{L^2_\eta;L^\infty_x}$$

and we deduce that (3-5) is implied by the following one-dimensional result.

Proposition 3.3. *There exist $C>0$ and h_0 such for any $0<h<h_0$, $\eta\in\mathbb{R}$, $1\leq\beta\leq h^{-1/2}$, and any (v,g) a solution of*

$$\left(h^2\frac{d^2}{dx^2}+1-h^2\eta^2\right)v=g,$$

we have

$$\|v\|_{L^\infty(\{|x|\leq\beta h^{1/2}\})}\leq C\beta^{-\frac{1}{2}}h^{-\frac{1}{4}}(\|v\|_{L^2(\{\beta h^{1/2}\leq|x|\leq 2\beta h^{1/2}\})}+h^{-1}\beta^2\|g\|_{L^2(\{|x|\leq 2\beta h^{1/2}\})}). \tag{3-6}$$

We change variables $x=\beta h^{1/2}z$, and it is enough to prove, for solutions of

$$(h\beta^{-2}\,\partial_z^2+1-h^2\eta^2)v=g,$$

that

$$\|v\|_{L^\infty(\{|z|\leq 1\})}\leq C(\|v\|_{L^2(\{1\leq|z|\leq 2\})}+h^{-1}\beta^2\|g\|_{L^2(\{|z|\leq 2\})}). \tag{3-7}$$

Finally, this latter estimate follows (with $\tau=\beta^2h^{-1}(1-h^2\eta^2)$) from the result below, which is a generalization of [Burq and Zuily 2015, Proposition 3.2] (note that taking advantage of the dimension 1, we can replace the L^2 norm in the left of (3.3) of that work by an L^∞ norm).

Lemma 3.4. *There exists $C > 0$ such that, for any $\tau \in \mathbb{R}$ and any solution (v, k) on $(-2, 2)$ of*

$$(\partial_z^2 + \tau)v = k,$$

we have

$$\|v\|_{L^\infty(-1,1)} \leq C\left(\|v\|_{L^2(\{1\leq|z|\leq 2\})} + \frac{1}{\sqrt{1+|\tau|}}\|k\|_{L^1(-2,2)}\right).$$

Let $\chi \in C_0^\infty(-2, 2)$ equal 1 on $(-1, 1)$. Then $u = \chi v$ satisfies

$$(\partial_z^2 + \tau)u = \chi k + 2\partial_z(\chi' v) - \chi'' v. \tag{3-8}$$

We distinguish two regimes.

Elliptic regime: $\tau \leq -1$. Then, multiplying by u and integrating by parts gives

$$\begin{aligned}\|\partial_z u\|^2_{L^2(-2,2)} + |\tau|\,\|u\|^2_{L^2(-2,2)} &= -(\chi k + 2\partial_z(\chi' v) - \chi'' v, u)_{L^2} \\ &= -(\chi k - \chi'' v, u)_{L^2} + 2(\chi' v, \partial_z u)_{L^2},\end{aligned} \tag{3-9}$$

which implies

$$\begin{aligned}&\|\partial_z u\|^2_{L^2(-2,2)} + |\tau|\,\|u\|^2_{L^2(-2,2)} \\ &\qquad \leq C\big(\|k\|_{L^1(-2,2)}\|u\|_{L^\infty} + \|v\|_{L^2(\{1\leq|z|\leq 2\})}(\|u\|_{L^2(\{1\leq|z|\leq 2\})} + \|\partial_z u\|_{L^2(-2,2)})\big),\end{aligned} \tag{3-10}$$

and the one-dimensional Gagliardo–Nirenberg inequality

$$\|u\|_{L^\infty} \leq C\,\|\partial_z u\|^{\frac{1}{2}}_{L^2}\|u\|^{\frac{1}{2}}_{L^2}$$

allows us to conclude in this regime.

Hyperbolic regime: $\tau \geq -1$. Let $\sigma = \sqrt{\tau} \in \mathbb{R}^+ \cup i[0, 1]$. The solution of (3-8) is

$$u(x) = \int_{y=-2}^{x} e^{-i\sigma(x-y)} \int_{z=-2}^{y} e^{i\sigma(y-z)} g(z)\, dz\, dy = \int_{z=-2}^{x} g(z) \int_{y=z}^{x} e^{i\sigma(2y-x-z)}\, dy\, dz,$$

where $g = \chi k - \chi'' v + 2\partial_z(\chi' v) = g_1 + \partial_z g_2$. Since, for $x, z \in [-2, 2]$,

$$\left|\int_{y=z}^{x} e^{i\sigma(2y-x-z)}\, dy\right| \leq \frac{C}{1+|\sigma|},$$

the contribution of g_1 is uniformly bounded by

$$\frac{C}{1+|\tau|}(\|\chi k\|_{L^1(-2,2)} + \|v\|_{L^1(\{1\leq|z|\leq 2\})}).$$

Integrating by parts in the integral involving $\partial_z g_2$, we see that similarly, the contribution of $\partial_z g_2$ is bounded by

$$C\,\|\chi' v\|_{L^1(-2,2)}.$$

3B. ***Second microlocalisation.*** In this section we develop the tools required to understand the concentration properties of our sequence (u_n) on the symplectic submanifold $\{x=0, \xi=0\}$ of the phase space $T^*\mathbb{T}^2$. The construction is very close to the one in [Fermanian-Kammerer 2005].

3B1. *Symbols and operators.* We define S^m to be the class of smooth functions of the variables $(X, \Xi, z, \zeta) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ which have compact supports with respect to the (X, Ξ)-variables and are polyhomogeneous of degree m with respect to the (z, ζ)-variables, with limits in the radial direction

$$\lim_{r\to+\infty} \frac{1}{r^m} a\left(X, \Xi, \frac{(rz, r\zeta)}{\|(z,\zeta)\|}\right) = \tilde{a}\left(X, \Xi, \frac{(z,\zeta)}{\|(z,\zeta)\|}\right).$$

When $m = 0$, via the change of variables

$$(z, \zeta) \mapsto (\tilde{z}, \tilde{\zeta}) = \frac{(z,\zeta)}{\sqrt{1+|z|^2+|\zeta|^2}},$$

such functions are identified with smooth compactly supported functions on $\mathbb{R}^4_{(X,\Xi)} \times \overline{B(0,1)}_{\tilde{z},\tilde{\zeta}}$, where $\overline{B(0,1)}$ denotes the closed unit ball in $\mathbb{R}^2$.

Let $\epsilon(h)$ satisfy

$$\lim_{h\to 0} \epsilon(h) = 0, \quad \epsilon(h) \geq h^{\frac{1}{2}}.$$

In order to perform the second microlocalisation around the submanifold given by the equations $x = 0$, $\xi = 0$, we define, for $a \in S^m$,

$$\mathrm{Op}_h(a) = a\left(x,\ y,\ hD_x,\ hD_y,\ \frac{\epsilon(h)}{h^{\frac{1}{2}}}x,\ \epsilon(h)h^{\frac{1}{2}}D_x\right),$$

where $X = (x, y)$ and $\Xi = (\xi, \eta)$. Notice that this quantification is the usual one [Hörmander 1985, Chapter 18.1], associated to the symbol

$$a\left(x,\ y,\ \xi,\ \eta,\ \frac{\epsilon(h)}{h^{\frac{1}{2}}}x,\ \epsilon(h)h^{\frac{1}{2}}\xi\right).$$

A simple calculation shows that since $\epsilon(h) \geq h^{1/2}$, the latter symbol belongs to the class

$$S((1+\epsilon^2(h)h^{-1}x^2+\epsilon^2(h)h\xi^2)^{\frac{m}{2}}, g)$$

of the Weyl–Hörmander calculus [Hörmander 1985, Chapter 18.5] for the metric

$$g = \frac{\epsilon^2(h)}{h}\frac{dx^2}{1+\epsilon^2(h)h^{-1}x^2+\epsilon^2(h)h\xi^2} + \epsilon^2(h)h\frac{d\xi^2}{1+\epsilon^2(h)h^{-1}x^2+\epsilon^2(h)h\xi^2} + \frac{dy^2}{1+y^2+h^2\eta^2} + h^2\frac{d\eta^2}{1+y^2+h^2\eta^2}. \tag{3-11}$$

As a consequence, we deduce that the operators such defined enjoy good properties and we have a good symbolic calculus; namely for all $a \in S^0$, the operator $\mathrm{Op}(a)$ is bounded on $L^2(\mathbb{R}^2)$ uniformly with respect to h, and

$$\forall a \in S^p,\ b \in S^q,\ ab \in S^{p+q}, \quad \mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(ab) + \epsilon^2(h)r,$$

where $r \in \mathrm{Op}(S^{p+q-1})$, and

$$\forall a \in S^0,\ \ a \geq 0 \quad \Longrightarrow \quad \exists C > 0 \ \text{ such that } \ \mathrm{Re}(\mathrm{Op}(a)) \geq -C\epsilon^2(h) \ \text{ and } \ \|\mathrm{Im}(\mathrm{Op}(a))\| \leq C\epsilon^2(h).$$

3B2. *Definition of the second semiclassical measure.* In this section, we consider a sequence (u_n) of functions on the two-dimensional torus $\mathbb{T}^2$ such that

$$(h_n^2\Delta + 1)u_n = \mathcal{O}(1)_{L^2}. \tag{3-12}$$

We identify u_n with a periodic function on $\mathbb{R}^2$. Now, using the symbolic calculus properties in Section 3B1, and in particular Gårding's inequality and the L^2 boundedness of operators, we can extract a subsequence (still denoted by (u_n)) such that there exists a positive measure $\tilde{\mu}$ on $T^*\mathbb{T}^2 \times \overline{N}$ — $\overline{N}$ denotes the sphere compactification of $N = \mathbb{R}^2_{z,\zeta}$ — such that, for any symbol $a \in S^0$,

$$\lim_{n\to+\infty} (\mathrm{Op}_{h_n}(a)u_n, u_n)_{L^2} = \langle \tilde{\mu}, \tilde{a}\rangle,$$

where the continuous function $\tilde{a}$ on $T^*\mathbb{R}^2 \times \overline{N}$ is naturally defined in the interior by the value of the symbol a and on the sphere at infinity by

$$\tilde{a}(x, y, \xi, \eta, \tilde{z}, \tilde{\zeta}) = \lim_{r\to+\infty} a(x, y, \xi, \eta, r\tilde{z}, r\tilde{\zeta})$$

(which exists because a is polyhomogeneous of degree 0). The measure $\tilde{\mu}$ is of course periodic, and hence defines naturally a measure μ on $T^*\mathbb{T}^2 \times \overline{N}$, and using (3-12), it is easy to see that there is no loss of mass at infinity in the Ξ-variable:

$$\mu(T^*\mathbb{T}^2 \times \overline{N}) = \lim_{n\to+\infty} \|u_n\|^2_{L^2(\mathbb{T}^2)}. \tag{3-13}$$

3B3. *Properties of the second semiclassical measure.* We now turn to the sequence constructed in Section 2 and study refined properties of the second semiclassical measure constructed above, for the choice $\epsilon(h)$ given by (3-1). Notice that compared to (3-12) the sequence considered here satisfies the stronger

$$(h_n^2\Delta + 1)u_n = o(h_n)_{L^2}.$$

Proposition 3.5. *The measure μ satisfies the following properties*:

(1) *Assume only that*

$$(h_n^2\Delta + 1)u_n = O(1)_{L^2}.$$

Then the measure μ has total mass $1 = \|u_n\|^2_{L^2}$ (h_n-oscillation).

(2) *Assume now that*

$$(h_n^2\Delta + 1)u_n = o(h_n)_{L^2}$$

and $\|au_n\|_{L^2} = o(1)$. Then, since the projection of the measure μ on the (x, y, ξ, η)-variables is the measure ν of Section 2 which is invariant by the bicharacteristic flow, we get that the measure μ is supported on the set

$$\{(x, y, \xi, \eta, z, \zeta); x = 0, \xi = 0, \eta = \pm 1\}.$$

(3) *Assume now that*

$$(h_n^2\Delta + 1)u_n = O(h_n\epsilon(h_n))_{L^2}.$$

Then the measure μ is supported on the sphere at infinity in the (z, ζ)-variables.

(4) *Assume now that*

$$(h_n^2\Delta + 1)u_n = O(h_n\epsilon(h_n))_{L^2}, \quad \|1_R u_n\|_{L^2} = o(1),$$

where R is a polygon. Then the measure μ vanishes 2-microlocally at each point of ∂R on the side where the polygon R lies. Namely in our geometry, the measure μ vanishes 2-microlocally on the right on $\{x=0,\ y\in(0,\frac12)\cup(-1,-\frac12)\}$ and 2-microlocally on the left on $\{x=0, y\in(-\frac12,0)\cup(\frac12,1)\}$; more precisely,

$$\begin{aligned}\mu\big(\{(x,y,\xi,\eta,z,\zeta): x=0,\ y\in(0,\tfrac12)\cup(-1,-\tfrac12),\ z>0\}\big)&=0,\\ \mu\big(\{(x,y,\xi,\eta,z,\zeta): x=0,\ y\in(-\tfrac12,0)\cup(\tfrac12,1),\ z<0\}\big)&=0.\end{aligned} \tag{3-14}$$

(5) *According to point* (3) *above, if we identify the sphere at infinity in the (z,ζ)-variables with $\mathbb{S}^1$ by means of the choice of variables $z=r\cos(\theta),\ \zeta=r\sin(\theta),\ r\to+\infty$, the measure μ can be seen as a measure in (x,y,ξ,η,θ)-variables, supported on $x=0,\ \xi=0,\ \eta=\pm1$. In this coordinate system,*

$$(\eta\,\partial_y - \sin^2(\theta)\,\partial_\theta)\mu = 0. \tag{3-15}$$

Remark 3.6. In Proposition 3.5, the only point where we use crucially the particular geometry of the torus (Proposition 3.1) is point (3). For more general geometries, this point is not true. However, for the part on the sphere at infinity of the measure, we can still get an analog of (3-15) for more general geometries, involving the curvature of the surface along the geodesic; see [Burq $\geq$ 2020].

Proof. The proof of point (1) follows from (3-13). To prove point (2), we just remark that the choice of test functions $a(x,\Xi,z,\zeta)=a(X,\Xi)$ shows that the direct image $\pi_*(\mu)$ of μ by the map

$$\pi:(X,\Xi,z,\zeta)\mapsto(X,\Xi)$$

is actually the (first) semiclassical measure ν constructed in Section 2, and consequently, this property follows from Section 2. To prove point (3), we recall that from Proposition 3.1, we have that for any $\chi\in C_0^\infty$, bounded by 1 and supported in $(-A,A)$,

$$\begin{aligned}\|\chi(h_n^{-\frac12}\epsilon(h_n)x)u_n\|_{L^2}^2 &\leq \|u_n\|_{L^2(\{|x|\leq Ah^{1/2}\epsilon^{-1}(h)\}}\\ &\leq \|u_n\|_{L^2(\{|x|\leq h^{1/2}\epsilon^{-2}(h)\}}\leq C\epsilon^{\frac12}(h_n) \quad\Longrightarrow\quad \langle\mu,\chi(z)\rangle=0.\end{aligned} \tag{3-16}$$

To prove point (4), recall from Figure 2 that the damping a is equal to 1 on $(0,\frac12)\times(0,\frac12)$ and that

$$\|au_n\|_{L^2}=\|a^{\frac12}u_n\|_{L^2}=o(1)_{n\to+\infty}.$$

Point (4) will follow from:

Proposition 3.7. *Assume that*

$$\|au_n\|_{L^2}=o(1)_{n\to+\infty},$$

and that the damping is equal to 1 *on $(0,\delta)\times(c,d)$ (resp. $(-\delta,0)\times(c,d)$). Then the measure μ vanishes two-microlocally on the right (resp. on the left) above $T^*\mathbb{T}^2|_{\{0\}\times(c,d)}$:*

$$\begin{aligned}&\mu(\{(x,y,\xi,\eta,z,\zeta): x=0,\ y\in(c,d),\ \eta=\pm1,\ z>0\})=0,\\ (\textit{resp.}\quad &\mu(\{(x,y,\xi,\eta,z,\zeta): x=0,\ y\in(c,d),\ \eta=\pm1,\ z<0\})=0),\end{aligned} \tag{3-17}$$

Let $\psi \in C^\infty(\mathbb{R})$ be supported in $\{1 < r\}$ and equal 1 for $r \geq 2$. Let $\chi \in C_0^\infty(-1,1)$ equal 1 on $\left(-\frac{1}{2}, \frac{1}{2}\right)$, and $\tilde{\chi} \in C_0^\infty(c,d)$ equal 1 on $c+\delta_0, d-\delta_0$, $\delta > 0$. Consider the symbol

$$b(x,y,\xi,\eta,z,\zeta) = \chi\left(\frac{2x}{\delta}\right)\tilde{\chi}(y)\chi(\xi)\chi(\eta-1)\psi\left(\frac{z}{\delta_0|\zeta|}\right)\psi(z^2+\zeta^2).$$

On the other hand, since $\chi(2x/\delta)\tilde{\chi}(y)$ is supported on $\left(-\frac{\delta}{2}, \frac{\delta}{2}\right)_x \times (c,d)_y$ and since $\psi(z/\delta|\zeta|)$ is supported in $z > 0$, we infer that the range of $\mathrm{Op}_{h_n}(b)$ is supported in the domain $\left(0, \frac{\delta}{2}\right)_x \times (c,d)_y$ and consequently

$$\begin{aligned}(\mathrm{Op}_{h_n}(b)u_n, u_n) &= (1_{x\in(0,\frac{\delta}{2})}1_{y\in(c,d)}\,\mathrm{Op}_{h_n}(b)u_n, u_n) = (\mathrm{Op}_{h_n}(b)u_n, 1_{x\in(0,\frac{\delta}{2})}1_{y\in(c,d)}u_n)\\ &= (\mathrm{Op}_{h_n}(b)u_n, 1_{x\in(0,\frac{\delta}{2})}1_{y\in(c,d)}au_n) = o(1)_{n\to+\infty}.\end{aligned} \tag{3-18}$$

This implies

$$\mu(\{(x,y,\xi,\eta,z,\zeta) : x = 0,\ y \in (c+\delta_0, d-\delta_0),\ \eta = 1,\ z \geq 2\delta_0|\zeta|\}) = 0.$$

Taking $\delta_0 > 0$ arbitrarily small, we deduce that on the (z,ζ)-sphere at infinity which contains the support of μ, we have

$$\mu(\{(x,y,\xi,\eta,z,\zeta) : x = 0,\ y \in (c,d),\ \eta = 1,\ z > 0\}) = 0.$$

The case $\eta = -1$ and the other properties in (3-14) follow similarly.

To prove the last property, we write for $q \in S^0$

$$\begin{aligned}&\frac{1}{2ih_n}[h_n^2\Delta + 1, \mathrm{Op}_{h_n}(q)]\\ &\quad= \mathrm{Op}_{h_n}((\xi\,\partial_x + \eta\,\partial_y + \zeta\,\partial_z)q) - i\frac{h_n}{2}\mathrm{Op}_{h_n}(\Delta_{x,y}q) - i\frac{h_n}{2}(\epsilon(h_n)h_n^{-\frac{1}{2}})^2\,\mathrm{Op}_{h_n}(\partial_z^2 q).\end{aligned} \tag{3-19}$$

Since unfolding the bracket shows that, as $n \to \infty$,

$$\frac{1}{2ih_n}([h_n^2\Delta + 1, \mathrm{Op}_{h_n}(q)]u_n, u_n) \to 0,$$

we get

$$o(1)_{n\to\infty} = (\mathrm{Op}_{h_n}((\xi\,\partial_x + \eta\,\partial_y + \zeta\,\partial_z)q)u_n, u_n). \tag{3-20}$$

Let us compute the limit on the sphere at infinity of $(\xi\,\partial_x + \eta\,\partial_y + \zeta\,\partial_z)q$. We denote by $\tilde{q}$ the function q in the (r,θ)-coordinate system. In this system of coordinates, the operator $\zeta\,\partial_z$ is given by

$$-\sin^2(\theta)\,\partial_\theta + r\cos(\theta)\sin(\theta)\,\partial_r.$$

Now we use that, for a polyhomogeneous symbol q of *degree* 0, the main part of q at infinity does not depend on r. As a consequence, the symbol $r\,\partial_r q$ is polyhomogeneous of *degree* -1 (while homogeneity would dictate *degree* 0). Therefore we get, for any polyhomogeneous symbol q of degree 0,

$$\begin{aligned}\zeta\,\partial_z q|_{\mathbb{S}^1} &= \lim_{r\to+\infty}(-\sin^2(\theta)\,\partial_\theta + r\cos(\theta)\sin(\theta)\,\partial_r)\tilde{a}(x,y,\xi,\eta,r,\theta)\\ &= -\sin^2(\theta)\,\partial_\theta \lim_{r\to+\infty}\tilde{q}(x,y,\xi,\eta,r,\theta).\end{aligned} \tag{3-21}$$

Since the measure $\tilde{\mu}$ is supported in $\xi = 0$, (3-15) follows from (3-20). ☐

We can now conclude the contradiction argument, and end the proof of the resolvent estimate (A-5). Notice that the two fixed points for the flow of

$$\dot{\theta} = -\sin^2(\theta)$$

are given by $\theta = 0(\pi)$. We want to show that the measure $\tilde{\mu}$ vanishes identically to get a contradiction with point (1) in Proposition 3.5. For $(x{=}0, \xi{=}0, y_0, \eta_0{=}\pm 1, \theta_0)$ in the support of $\tilde{\mu}$, let us denote by $\phi_s(\theta_0)$ the solution of

$$\frac{d}{ds}\phi_s(\theta_0) = -\sin^2(\phi_s(\theta_0)), \quad \phi_0(\theta_0) = \theta_0,$$

so that $\phi_s(\theta_0) = \operatorname{Arccotan}(s + \operatorname{cotan}(\theta_0))$. From the invariance (3-15) of the measure $\tilde{\mu}$, we deduce that

$$\forall s \in \mathbb{R}, \quad (x = 0,\, y_s = y_0 + s\eta_0 \pmod{2\pi},\, \xi = 0,\, \eta_0,\, \theta_s = \phi_s(\theta_0)) \in \operatorname{supp}(\tilde{\mu}).$$

Consequently, if $\theta_0 \in [0, \pi)$, there exists $s > 0$ such that $y_s \in \left(0, \frac{1}{2}\right) \pmod{2\pi}$ if $\theta_s \in \left[0, \frac{\pi}{2}\right)$, while, if $\theta_0 \in [-\pi, 0)$, there exists $s > 0$ such that $y_s \in \left(-\frac{1}{2}, 0\right) \pmod{2\pi}$ if $\theta_s \in \left[-\pi, -\frac{\pi}{2}\right)$. This is impossible according to (3-14).

4. Back to the general case

Let us work on the torus $\mathbb{T}^2 = \mathbb{R}^2/A\mathbb{Z}\times B\mathbb{Z}$ with $A > 0$, $B > 0$. Since the irrational directions $\Xi = (A\xi, B\eta)$, with $\xi/\eta \notin \mathbb{Q}$, correspond to dense geodesics, and since a is bounded from below on an open set, we deduce that the measure ν defined in Section 2 is supported — in the Ξ-variables — on the set of finitely many rational directions

$$\Xi = (A\xi, B\eta), \quad \text{with } \frac{\xi}{\eta} \in \mathbb{Q},$$

satisfying moreover the elliptic regularity condition, $|\Xi|^2 = 1$, which do not enter the interior of the rectangles. Hence, there exists an isolated direction Ξ_0, so that $(X_0, \Xi_0) \in \operatorname{supp}(\nu)$, which can be written as

$$\Xi_0 = \frac{1}{\sqrt{n^2A^2 + m^2B^2}}(nA, mB), \quad \Xi_0^{\perp} = \frac{1}{\sqrt{n^2A^2 + m^2B^2}}(-mB, nA), \tag{4-1}$$

where the integers n, m may be chosen to have gcd 1. The change of coordinates in $\mathbb{R}^2$,

$$F : (x, y) \longmapsto X = F(x, y) = x\Xi_0^{\perp} + y\Xi_0, \tag{4-2}$$

is orthogonal and hence $-\Delta_X = D_x^2 + D_y^2$.

We have the following simple lemma [Burq and Zworski 2012, Lemma 2.7], which can be deduced from an elementary calculation.

Lemma 4.1. *Suppose that Ξ_0 and F are given by* (4-1) *and* (4-2). *If $u = u(x, y)$ is periodic with respect to $A\mathbb{Z} \times B\mathbb{Z}$ then $F^*u := u \circ F$ satisfies*

$$F^*u(x + k\alpha, y + \ell\beta) = F^*u(x, y - k\gamma), \quad k, \ell \in \mathbb{Z},\ (x, y) \in \mathbb{R}^2, \tag{4-3}$$

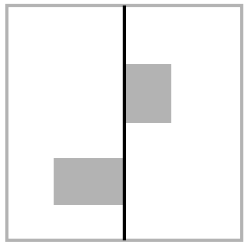

Figure 3. The microlocal model: on the left the rectangle R_1, on the right the rectangle R_2, in the middle the bicharacteristic in the support of μ.

where, for fixed $p, q \in \mathbb{Z}$ *such that* $qn - pm = 1$,

$$\alpha = \frac{AB}{\sqrt{n^2A^2 + m^2B^2}}, \quad \beta = \sqrt{n^2A^2 + m^2B^2}, \quad \gamma = -\frac{pnA^2 + qmB^2}{\sqrt{n^2A^2 + m^2B^2}}.$$

When $B/A = r/s \in \mathbb{Q},\ r, s \in \mathbb{Z} \setminus \{0\}$,

$$F^*u(x + k\tilde{\alpha}, y + \ell\beta) = F^*u(x, y), \quad k, \ell \in \mathbb{Z},\ (x, y) \in \mathbb{R}^2, \tag{4-4}$$

for $\tilde{\alpha} = (n^2s^2 + m^2r^2)\alpha$.

In this new coordinate system, we know that there exists x_0 such that $(x_0, y_0, 0, 1)$ is in the support of the measure $F^*\mu$. By translation invariance, we can assume that $x_0 = 0$. Since

$$(\xi\, \partial_x + \eta\, \partial_y)F^*\mu = 0,$$

we infer that actually the whole line $(x_0{=}0, \mathbb{R} \pmod{2\pi}, 0, 1)$ belongs to the support of $F^*\mu$. If this bicharacteristic curve enters the interior of the support of a (i.e., encounters a point in a neighborhood of which a is bounded away from 0), then by propagation, no point of this bicharacteristic curve lies in the support of μ, which gives a contradiction. On the other hand since Assumption 1.2 is satisfied, we know that there exist two (at least) polygons R_1, R_2 so that the right side of R_1 is $\{0\} \times [\alpha, \beta]$, while the left side of R_2 is $\{0\} \times [\gamma, \delta]$. We may shrink these polygons to rectangles having the same property.

In other words, we are microlocally reduced to the study of the checkerboard in Figure 3. Notice that the change of variables we used in Lemma 4.1 does not keep periodicity with respect to the x-variables but transforms it into some pseudoperiodicity condition (see (4-3)). However, for the study of the checkerboard model in Section 3, we only used periodicity with respect to the y-variables (to prove Proposition 3.1) — which is preserved. The rest of the contradiction argument follows the same lines as in Section 3.

5. Generalized geometric condition

For a general Riemannian manifold and a general damping function $a \in L^\infty(M)$, a natural substitute to (GCC) is the following generalized geometric condition:

$$\exists T, c > 0 \quad \text{such that} \quad \liminf_{\epsilon \to 0} \inf_{\rho_0 \in S^*M} \frac{1}{\mathrm{Vol}(\Gamma_{\rho_0,\epsilon,T})} \int_{\Gamma_{\rho_0,\epsilon,T}} a(x)\, dx \geq c, \tag{GGCC}$$

where $\Gamma_{\rho_0,\epsilon,T}$ is the set of points at distance less than ϵ from the geodesic segment $\{\gamma_{\rho_0}(s) : s \in (0,T)\}$. At first glance, (GGCC) might seem to be a strong condition, difficult to fulfill. We shall prove below that it cannot be relaxed as, on any manifold and for any $a \in L^\infty(M)$, it is a necessary condition for uniform stabilization. On the other hand, we also prove below that in the case of two-dimensional tori it is equivalent to Assumption 1.2. We conjecture that on a general manifold and for general $a \in L^\infty$, uniform stabilization holds if and only if (GGCC) holds. The results in this article show that it is indeed the case on two-dimensional tori, if a satisfies (1-4). For general dampings it is easy to show that (GGCC) implies (WGCC), while the compactness of S^*M shows that it is implied by (SGCC) (δ in (SGCC) can, by compactness, be chosen the same for all $\rho_0 \in S^*M$).

5A. *The generalized geometric condition is necessary for stabilization.*

Theorem 5. *Uniform stabilization implies* (GGCC).

Proof. The proof of this result relies on geometric optics constructions (with complex phases) for the wave equation of [Ralston 1982, Section 2.1] that we recast in our wave equation context.

Proposition 5.1. *Let M be a compact manifold without boundary endowed with a smooth metric g and a smooth density κ. Let*

$$\Delta = \operatorname{div}_\kappa \nabla_g \tag{5-1}$$

be the Laplace operator. Let $\big(t_0, x_0, \tau_0{=}\tfrac{1}{2}, \xi_0\big) \in \operatorname{Char}(\partial_t^2 - \Delta)$, the characteristic manifold. Denote by $\big(t(s){=}t_0 + s, \tau_0{=}\tfrac{1}{2}, \gamma(s), \xi(s)\big)$ the bicharacteristic starting from $\big(t_0, x_0, \tfrac{1}{2}, \xi_0\big)$. Then for any $N > 0$ there exists a family of approximate solutions $v_{h,N}(t,x)$ defined for $0 < h < h_0$ to the wave equation

$$(\partial_t^2 - \Delta)v_{h,N} = \mathcal{O}(h^N)_{L^2(M)}, \quad E(v_{h,N}) = \int_M (|\nabla_g v_{h,N}|^2 + |\partial_t v_{h,N}|^2)\kappa\, dx = 1 + o(h) \tag{5-2}$$

with error terms locally uniformly controlled in time, and which are (locally in time) exponentially localized in $\mathbb{R}_t \times M$ near $(t(s), x(s))$:

$$\begin{gathered} \forall T > 0, \quad \exists C, \alpha > 0 \quad \text{such that} \quad \forall t \in [0,T], \quad \forall x \in M, \\ (|v_{h,N}| + |h\nabla_x v_{h,N}| + |h\, \partial_t v_{h,N}|)(t(s), x) \le C h^{1-\frac{d}{4}} e^{-\alpha \frac{\operatorname{dist}(x, x(s))^2}{h}}. \end{gathered} \tag{5-3}$$

Consequently, if we denote by $\Gamma_T = \gamma([0,T])$ the image of the geodesic in M,

$$\begin{gathered} \forall T > 0, \quad \exists C, \alpha > 0 \quad \text{such that} \quad \forall x \in M, \\ \int_0^T (|\nabla_g v_{h,N}|^2 + |\partial_t v_{h,N}|^2)(t,x)\, dt \le C h^{-\frac{d-1}{2}} e^{-\alpha \frac{\operatorname{dist}(x, \Gamma_T)^2}{h}}. \end{gathered} \tag{5-4}$$

Let us first show how we can deduce Theorem 5 from Proposition 5.1. We are going to test the observation estimates (1-3) on such sequences of solutions.

Let us we assume that (GGCC) does not hold. Fix $T > 0$. Then there exist $\eta_n = (x_n, \xi_n) \in S^*M$, $\epsilon_n \to 0$ such that

$$\lim_{n \to +\infty} \kappa_n = 0, \quad \kappa_n := \frac{1}{\epsilon_n^{d-1}} \int_{\Gamma_{\eta_n,\epsilon_n,T}} a(x)\, dx.$$

Let $t_n = 0$. Let $\rho_n = \big(t_n, \tau_n{=}\frac{1}{2}, x_n, \xi_n\big)$, fix $N = 1$ (we actually need a crude version of Ralston's construction) and let v_h^n be the approximate solution of the wave equation constructed in Proposition 5.1, with initial point ρ_n. We shall use that the family of solutions which depends on two parameters h and the initial point in the cotangent bundle is uniformly controlled with respect to this latter parameter, which will follow from the proof of Proposition 5.1 given below. Since, according to Proposition 5.1, we have

$$\|(v_h^n, \partial_t v_h^n)|_{t=0}\|_{\dot{H}^1\times L^2} = 1 + o(1)_{n\to+\infty},$$

and according to (5-2) and Duhamel's formula, v_h^n is, modulo a $\mathcal{O}(h_n^N)$ error in energy space, equal to the solution to the exact wave equation with the same initial data, to show that uniform stabilization does not hold, it is now enough to show that for a properly chosen sequence $h_n \to 0$

$$\lim_{n\to+\infty}\int_0^T\int_M a(x)|\partial_t v_{h_n,n}|^2\,dx\,dt = 0. \tag{5-5}$$

Extracting a subsequence, we can assume that the sequence of initial points ρ_n converges to $\rho = \big(t_0{=}0, \xi_0{=}\frac{1}{2}, x_0, \xi_0\big)$. The only point we shall use about our approximate solutions is the upper bound (5-4), which implies

$$\begin{aligned}\int_0^T\int_M & a(x)|\partial_t v_{h_n}|^2\,dx\,dt\\ &= \int_0^T\int_M a(x)|\partial_t v_{h_n}|^2\,dx\,dt\\ &\le C\int_{\Gamma_{\rho_n,\epsilon_n,T}} a(x)h_n^{-\frac{d-1}{2}}e^{-\alpha\frac{\operatorname{dist}(x,\Gamma_{\rho_n,T})^2}{h_n}}\,dx + \int_{\Gamma^c_{\rho_n,\epsilon_n,T}} a(x)h_n^{-\frac{d-1}{2}}e^{-\alpha\frac{\operatorname{dist}(x,\Gamma_{\rho_n,T})^2}{h_n}}\,dx. \end{aligned}\tag{5-6}$$

The contribution of the first term is bounded by

$$Ch_n^{-\frac{d-1}{2}}\int_{\Gamma_{\rho_n,\epsilon_n,T}} a(x)\,dx \le \kappa_n\left(\frac{\epsilon_n^2}{h_n}\right)^{\frac{d-1}{2}}.$$

On the other hand, the second term is bounded by

$$\|a\|_{L^\infty}\int_{\Gamma^c_{\rho_n,\epsilon_n,T}} h_n^{-\frac{d-1}{2}}e^{-\alpha\frac{\operatorname{dist}(x,\Gamma_{\rho_n,T})^2}{h_n}}\,dx. \tag{5-7}$$

To estimate this integral we work in (a finite set of) coordinate systems. In such local coordinates, $\Gamma_{\rho_n,T}$ is a finite union of smooth arcs of geodesics (because the geodesic can self-intersect) and it is enough to estimate (5-7) where we replaced $\operatorname{dist}(x, \Gamma_{\rho_n,T})$ by the distance to any such arc. We can change again coordinates such that locally the considered arc of geodesic is

$$\{(y_1 = 0, y' \in \mathbb{R}^{d-1})\},$$

and the distance to the arc γ_n satisfies

$$\exists C > 0 \quad \text{such that} \quad \frac{1}{C}|y'| \le \operatorname{dist}(y, \gamma_n) \le C|y'|.$$

This leads to the estimate (if $\epsilon_n \geq C^2\sqrt{h_n}$)

$$\int_{\mathrm{dist}(x,\gamma)\geq\epsilon_n} h_n^{-\frac{d-1}{2}} e^{-\alpha\frac{\mathrm{dist}(x,\gamma_n)^2}{h_n}}\,dx = \int_{|x'|\geq\frac{\epsilon_n}{C}} h_n^{-\frac{d-1}{2}} e^{-\alpha\frac{|x'|^2}{Ch_n}}\,dx$$
$$= C'\int_{|y'|\geq\frac{\epsilon_n}{C^2\sqrt{h_n}}} e^{-\alpha|y'|^2}\,dy' \leq C'e^{-\alpha\frac{\epsilon_n^2}{C^4h_n}}. \tag{5-8}$$

We now choose

$$h_n = \kappa_n^{\frac{1}{d-1}}\epsilon_n^2 \to 0$$

such that

$$\frac{\epsilon_n^2}{h_n} = \kappa_n^{-\frac{1}{d-1}} \to +\infty, \quad \kappa_n\left(\frac{\epsilon_n^2}{h_n}\right)^{\frac{d-1}{2}} = \kappa_n^{\frac{1}{2}} \to 0. \tag{5-9}$$

This choice implies

$$\int_0^T\int_M a(x)|\partial_t w_{h_n}|^2\,dx\,dt = o(1)_{n\to+\infty},$$

which contradicts (1-3) because the energy of the initial data $(w_{h_n}, \partial_t w_{h_n})$ is constant and nonzero. This completes the proof of Theorem 5. □

Let us now come back to the proof of Proposition 5.1. This is basically done in [Ralston 1982, Section 2.1]. The idea is to define oscillating solutions (phase and symbol) by constructing the germs on the bicharacteristic curve. Let $\rho_0 = \big(t_0, \tau_0{=}\frac{1}{2}, x_0, \xi_0\big)$ be a point in the characteristic variety of the wave equation

$$\mathrm{Char} = \{(t,x,\tau,\xi) \in T^*M : |\tau|^2 = |\xi|^2 = 1\}.$$

Let

$$\Gamma = \big\{t(s), \gamma(s), \tau(s) = \tfrac{1}{2}, \xi(s))\big\}$$

be the bicharacteristic curve issued from ρ_0. For any $T < +\infty$, we can choose systems along the geodesic γ and get an immersion

$$i : (-\epsilon, T+\epsilon)\times B(0,\epsilon) \subset \mathbb{R}\times\mathbb{R}^{d-1} \to M,$$

along which the bicharacteristic takes the form

$$\gamma(s) = \big(t = s,\ x_1 = s,\ x' = 0,\ \tau = \tfrac{1}{2},\ \xi_1 = -\tfrac{1}{2},\ \xi' = 0\big),$$

which allows us to reduce the analysis to $\mathbb{R}^d$. In this coordinate system, (5-1) takes the form

$$\Delta = \frac{1}{\kappa(x)}\sum_{i,j}\frac{\partial}{\partial x_i}g^{i,j}(x)\kappa(x)\frac{\partial}{\partial x_j}.$$

We now write $y = (t,x)$ and seek approximate solutions of the wave equation with the form

$$v_h(t,x) = e^{\frac{i}{h}\psi(t,x)}\sigma(t,x,h), \tag{5-10}$$

where $\sigma(t=0)=\sigma_0\in C_0^\infty(\mathbb{R}^d)$ has sufficiently small compact support near 0. Applying the operator $\partial_t^2-\Delta_x$ we get

$$
\begin{aligned}
(\partial_t^2-\Delta_x)v_h = &-\frac{1}{h^2}\Big((\partial_t\psi)^2-\sum_{1\le k,j\le n} g^{k,j}(x)\,\partial_{x_k}\Psi\,\partial_{x_j}\Psi\Big)\sigma e^{\frac{i}{h}\psi}\\
&+e^{\frac{i}{h}\psi}\frac{i}{h}\Big(2\partial_t\psi\,\partial_t\sigma-2\sum_{1\le k,j\le n} g^{k,j}(x)\,\partial_k\psi\,\partial_j\sigma-\frac{1}{\kappa}\sum_{1\le k,j\le n}\partial_k(g^{k,j}\kappa)(x)\sigma\,\partial_j\psi\Big)\\
&+e^{\frac{i}{h}\psi}(\partial_t^2-\Delta\psi)\sigma.
\end{aligned}
\tag{5-11}
$$

Ralston [1982, Section 2.1] then showed that provided

$$\psi(t(s),x(s))=t(s)-x_1(s)+cste \quad\Longleftrightarrow\quad \partial_{t,x}\psi(t(s),x(s))=(\tau(s),\xi(s)),$$

and choosing

$$\mathrm{Im}\Big(\frac{\partial^2\psi}{\partial x^2}\Big)\Big|_{\gamma(0)}\ge c\mathrm{Id},\quad c>0, \tag{5-12}$$

it is possible to solve both the eikonal equation

$$p=\Big((\partial_t\psi)^2-\sum_{1\le k,j\le n} g^{k,j}(x)\partial_{x_k}\psi\,\partial_{x_j}\psi\Big)=0$$

and the transport equation

$$T=\Big(2\partial_t\psi\,\partial_t\sigma-2\sum_{1\le k,j\le n} g^{k,j}(x)\,\partial_k\psi\,\partial_j\sigma-\frac{1}{\kappa}\sum_{1\le k,j\le n}\partial_k(g^{k,j}\kappa)(x)\sigma\,\partial_j\psi\Big)e^{\frac{i}{h}\psi}+h(\partial_t^2-\Delta\psi)\sigma=0,$$

with

$$\mathrm{Im}\Big(\frac{\partial^2\psi}{\partial x^2}\Big)\Big|_{\gamma(s)}\ge c(s)\mathrm{Id},\quad c(s)>0, \tag{5-13}$$

to arbitrarily large order on the bicharacteristic γ by choosing

$$\sigma=\sum_p h^p\sigma_p.$$

Here by solving to arbitrarily large order, we mean that we can cancel an arbitrarily large number of (t,x) derivatives on γ.

On the torus $\mathbb{T}^d$, these constructions can be performed explicitly and we get

$$\psi(t,x)=t-x_1+i((t-x_1)^2+g(t)|x'|^2)+O(|t-x_1|^3+|x'|^3), \tag{5-14}$$

with g solving

$$2ig'(t)+4g^2(t)=0 \quad\Longleftarrow\quad g(t)=\frac{g(0)}{1-2itg(0)},\ \ g(0):=1.$$

Notice in particular that

$$\mathrm{Re}(g(t))=\frac{1}{1+4t^2}>0, \tag{5-15}$$

and we can choose a symbol

$$\sigma(t, x_1, x') = \frac{\sigma_0(t - x_1, x')}{(1 - 2it)^{\frac{d-1}{2}}} + O(h) + O(|t - x_1| + |x'|).$$

Finally, it remains to cut off the symbol such constructed near the geodesic (taking advantage of (5-13), we see that this truncation will add an exponentially small error), and to normalize by multiplying by

$$ch^{1-\frac{d}{4}}$$

to ensure the normalization of the energy in (5-2) and the error bound (5-3). We leave the details to the reader.

5B. ***Assumption 1.2 and* (GGCC).** On two-dimensional tori and for dampings a satisfying (1-4) we have:

Proposition 5.2. *On a two-dimensional torus* $\mathbb{T}^2$, *if the damping a satisfies* (1-4), *then* (GGCC) *is equivalent to Assumption 1.2.*

Proof. Since Assumption 1.2 implies uniform stabilization (Theorem 4) which in turn implies (GGCC) (Theorem 5), it is enough to show that (GGCC) implies Assumption 1.2.

Let us assume (GGCC). If Assumption 1.2 were not satisfied, then there would, for any $T > 0$, exist a geodesic curve γ of length T which either does not encounter $\mathcal{R} = \bigcup_{j=1}^{N} \overline{R}_j$, or does encounter $\mathcal{R}$ only at corners or only on the left (or only on the right). In the first case, by compactness, the geodesic curve remains at distance ϵ_0 of $\mathcal{R}$, and consequently for $0 < \epsilon < \epsilon_0$ we have

$$\int_{\Gamma_{\rho_0,\epsilon,T}} a(x)\,dx = 0.$$

In the second case (see checkerboard in Figure 1(b)), by compactness, the geodesic curve encounters only a finite number of corners, and consequently ($d = 2$)

$$\int_{\Gamma_{\rho_\sigma,\epsilon,T}} a(x)\,dx = \mathcal{O}(\epsilon^2),$$

with a constant $c > 0$ depending on the angles of the corners, while

$$\mathrm{Vol}(\Gamma_{\rho_0,\epsilon,T}) \sim C\epsilon,$$

which implies that (GGCC) does not hold. In the last case (see the right checkerboard in Figure 1(c) and Figure 4), let us consider the family of geodesics $\gamma_\sigma = \{\gamma_{\rho_\sigma}(s) : s \in (0, T)\}$, $\sigma \in [0, 1)$, parallel on the right to $\gamma_0 = \{\gamma_{\rho_0}(s) : s \in (0, T)\}$ (i.e., if $\rho_0 = (X_0, \Xi_0)$, then $\rho_\sigma = X_0 + \sigma\,\Xi_0^\perp$, where $\Xi_0^\perp$ is the unit vector orthogonal to Ξ_0, pointing on the right of γ_0). Since on the right γ_0 encounters no side of any rectangle R_j, it may encounter only (finitely many) corner points. As a consequence, for any $\sigma > 0$

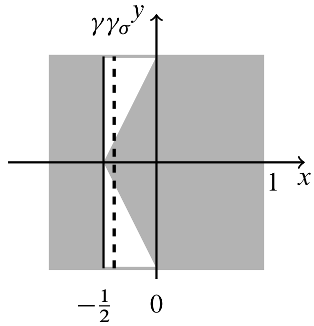

Figure 4. Proof of Proposition 5.2, last case.

sufficiently small, and $0 < \epsilon \ll \sigma$,

$$\int_{\Gamma_{\rho_\sigma,\epsilon,T}} a(x)\,dx \sim c\sigma\epsilon \quad (\epsilon \to 0),$$
$$\mathrm{Vol}(\Gamma_{\rho_\sigma,\epsilon,T}) \sim C\epsilon \quad (\epsilon \to 0).$$

We deduce that

$$\lim_{\epsilon\to 0} \frac{1}{\mathrm{Vol}}(\Gamma_{\rho_\sigma,\epsilon,T}) \int_{\Gamma_{\rho_\sigma,\epsilon,T}} a(x)\,dx = c\sigma,$$

and letting $\sigma \to 0$ shows that (GGCC) does not hold. □

Appendix A: Resolvent estimates and stabilization

In this appendix, we collect a few classical results on resolvent estimates.

A1. ***Resolvent estimates and stabilization.*** It is classical [Gearhart 1978] that stabilization or observability of a self-adjoint evolution system is equivalent to resolvent estimates; see also [Burq and Zworski 2004; Miller 2012; Anantharaman and Léautaud 2014]. For completeness we shall give below a proof (only the fact that resolvent estimates imply stabilization).

Proposition A.1. *Consider a strongly continuous semigroup e^{tA} on a Hilbert space H, with infinitesimal generator A defined on $D(A)$. The following two properties are equivalent*:

(1) *There exist $C, \delta > 0$ such that the resolvent of A, $(A-\lambda)^{-1}$ exists for* $\mathrm{Re}\,\lambda \geq -\delta$ *and satisfies*

$$\exists C > 0 \quad \textit{such that} \quad \forall \lambda \in \mathbb{C}^\delta = \{z \in \mathbb{C} : \mathrm{Re}\, z \geq -\delta\}, \quad \|(A-\lambda)^{-1}\|_{\mathcal{L}(H)} \leq C.$$

(2) *There exist $M, \delta > 0$ such that for any $t > 0$*

$$\|e^{tA}\|_{\mathcal{L}(H)} \leq Me^{-\delta t}.$$

Proof. Let us first prove that (2) implies (1). We start with the resolvent equality (always true for $\mathrm{Re}\,\lambda \gg 0$)

$$(A-\lambda)^{-1} f = -\int_0^{+\infty} e^{t(A-\lambda)} f\,dt,$$

and we deduce that if $\|e^{tA}\| \le Ce^{-\beta t}$, then (1) is satisfied for any $\delta < \beta$. To prove that (1) implies (2), for $u_0 \in D(A)$, and $\chi \in C^\infty(\mathbb{R})$ equal to 0 for $t \le -1$ and to 1 for $t \ge 0$, consider

$$u(t) = \chi(t)e^{t(A-\omega)}u_0.$$

For ω large enough, u belongs to $L^\infty(\mathbb{R}; H)$ because strongly continuous semigroups of operators satisfy

$$\exists C, c > 0 \quad \text{such that} \quad \forall t > 0, \quad \|e^{tA}\| \le Ce^{ct},$$

and u satisfies

$$(\partial_t + \omega - A)u(t) = \chi'(t)e^{t(A-\omega)}u_0 =: v(t).$$

Taking Fourier transforms in the time variable, we get

$$(i\tau + \omega - A)\hat{u}(\tau) = \hat{v}(\tau). \tag{A-1}$$

Since $v(t)$ is supported in $t \in [-1, 0]$, the right-hand side in (A-1) is holomorphic and bounded in any domain

$$\mathbb{C}_\alpha = \{\tau \in \mathbb{C} : \operatorname{Im} \tau \ge \alpha,\ \alpha \in \mathbb{R}\}.$$

From the assumption on the resolvent, we deduce that $\hat{u}$ admits a holomorphic extension to $\{\tau : \operatorname{Im} \tau \le \delta + \omega\}$ which satisfies

$$\|\hat{u}(\tau)\|_H \le C\|\hat{v}(\tau)\|_H.$$

We deduce that

$$\begin{aligned} \|e^{(\omega+\delta)t}u\|_{L^2(\mathbb{R}_t;H))} &= \|\hat{u}(\tau + i(\omega+\delta))\|_{L^2(\mathbb{R}_\tau;H)} \le C\|\hat{v}(\tau + i(\omega+\delta))\|_{L^2(\mathbb{R}_\tau;H)} \\ &\le C\|e^{(\omega+\delta)t}v\|_{L^2(\mathbb{R}_t;H)} \le C'\|u_0\|_H. \end{aligned} \tag{A-2}$$

This implies exponential decay of $e^{tA}u_0$ in the L^2_t norm, with the weight $e^{\delta t}$. Now consider $w(t) := \chi(t-T)e^{tA}u_0$, which satisfies

$$(\partial_t - A)w = \chi'(t-T)e^{tA}u_0, \quad w|_{t=T-1} = 0.$$

From Duhamel formula, we deduce

$$w(T) = \int_{T-1}^{T} e^{(T-s)A}\chi'(t-T)e^{sA}u_0\, ds,$$

and consequently (recall that the semigroup norm is locally bounded in time)

$$\begin{aligned} \|w(T)\|_H &\le \int_{T-1}^{T} \|e^{(T-s)A}\chi'(t-T)e^{sA}u_0\|_H\, ds \\ &\le C \sup_{\sigma\in[0,1]} \|e^{\sigma A}\|_{\mathcal{L}(H)} \int_{T-1}^{T} \|e^{sA}u_0\|_H\, ds \le C'e^{-\delta T}\|e^{\delta s}e^{sA}u_0\|_{L^2(T-1,T);H} \\ &\le C''e^{-\delta T}\|u_0\|_H. \end{aligned}$$

□

A2. *Semigroups for damped wave equations.* The solution to (1-1) is given very classically by

$$\begin{pmatrix} u \\ \partial_t u \end{pmatrix} = e^{tA} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta - m & -a \end{pmatrix}, \tag{A-3}$$

where A is defined on $\mathcal{H} = H^1(M) \times L^2(M)$ with domain $H^2(M) \times H^1(M)$. When $m > 0$, since

$$E(u) = \|u\|_{H^1}^2 + \|\partial_t u\|_{L^2}^2,$$

to study the decay of the energy, we can apply directly the characterization given by Proposition A.1. When $m = 0$, the semigroup e^{tA} is not a contraction semigroup on $H^1 \times L^2$ (because the energy (1-2) does not control the H^1 norm). The main difference from the case $m = 0$ and m nontrivial comes from:

Lemma A.2. *Assume that* $0 \leq m \in L^\infty(M)$ *and* m *is not trivial* $\left(\int_M m(x)\,dx > 0\right)$. *Then the norms*

$$\|u\|_{H^1} = (\|\nabla_x u\|_{L^2}^2 + \|u\|_{L^2}^2)^{\frac{1}{2}}, \quad \|u\| = \sqrt{E(u)} = (\|\nabla_x u\|_{L^2}^2 + \|m^{\frac{1}{2}} u\|_{L^2}^2)^{\frac{1}{2}}$$

are equivalent.

Indeed, as for a classical proof of Poincaré's inequality, we proceed by contradiction to prove the only nontrivial inequality ($\|u\|_{H^1} \leq C\|u\|$), and get a sequence $(u_n) \in H^1(M)$ such that

$$\|u_n\|_{H^1} = 1, \quad \|u_n\| \to_{n \to +\infty} 0.$$

By the weak compactness of the unit ball in H^1 we can extract a subsequence (still denoted by (u_n)) which converges weakly in H^1, and hence because M is compact strongly in L^2, to a limit $u \in H^1$. Since $\|u_n\| \to +\infty$ we get that the sequence actually converges strongly in H^1 and

$$\|u\| = 0 \to \nabla_x u = 0, \quad m^{\frac{1}{2}} u = 0.$$

We deduce that u is constant in M and since $\int_M mu = 0$, we finally get $u = 0$, which contradicts the fact that $\|u_n\|_{H^1} = 1$ (and the strong convergence of u_n to 0).

For $s = 1, 2$, we define $\dot{H}^s = H^s(M)/\mathbb{R}$ to be the quotient space of $H^s(M)$ by the constant functions, endowed with the norm

$$\|\dot{u}\|_{\dot{H}^1} = \|\nabla u\|_{L^2}, \quad \|\dot{u}\|_{\dot{H}^2} = \|\Delta u\|_{L^2}.$$

We define the operator

$$\dot{A} = \begin{pmatrix} 0 & \Pi \\ \dot{\Delta} & -a \end{pmatrix}$$

on $\dot{H}^1 \times L^2$ with domain $\dot{H}^2 \times H^1$, where Π is the canonical projection $H^1 \to \dot{H}^1$ and $\dot{\Delta}$ is defined by

$$\dot{\Delta}\dot{u} = \Delta u$$

(independent of the choice of $u \in \dot{u}$). The operator $\dot{A}$ is maximal dissipative and hence defines a semigroup of contractions on $\dot{\mathcal{H}} = \dot{H}^1 \times L^2$. Indeed for $U = \binom{\dot{u}}{v}$,

$$\mathrm{Re}(\dot{A}U, U)_{\dot{\mathcal{H}}} = \mathrm{Re}(\nabla u, \nabla v)_{L^2} + (\Delta u - av, n)_{L^2} = -(av, v)_{L^2},$$

and

$$
\begin{aligned}
(\dot A-\mathrm{Id})\begin{pmatrix}\dot u\\ v\end{pmatrix}=\begin{pmatrix}\dot f\\ g\end{pmatrix} &\iff \Pi v-\dot u=\dot f,\quad \dot\Delta\dot u-(a+1)v=g\\
&\iff \Pi v-\dot u=\dot f,\quad \Delta v-(1+a)v=g+\Delta f\in H^{-1}(M),
\end{aligned}\tag{A-4}
$$

and we solve this equation by variational theory. Notice that this shows that the resolvent $(\dot A-\mathrm{Id})^{-1}$ is well-defined and continuous from $\dot H^1\times L^2$ to $\dot H^2\times H^1$.

Lemma A.3. *The injection $\dot H^2\times H^1$ to $\dot H^1\times L^2$ is compact.*

This follows from identifying $\dot H^n$ with the kernel of the linear form $u\mapsto\int_M u$.

Corollary A.4. *The operator $(\dot A-\mathrm{Id})^{-1}$ is compact on $\dot{\mathcal H}$.*

On the other hand, it is very easy to show that for $(u_0,u_1)\in H^1\times L^2$

$$
\begin{pmatrix}\Pi&0\\0&\mathrm{Id}\end{pmatrix}e^{tA}=e^{t\dot A}\begin{pmatrix}\Pi&0\\0&\mathrm{Id}\end{pmatrix},
$$

and consequently, stabilization is equivalent to the exponential decay (in norm) of $e^{t\dot A}$ (and consequently, according to Proposition A.1 equivalent to resolvent estimates for $\dot A$).

A3. *Reduction to high-frequency observation estimates.* In this section, we show that for $m\geq 0$, stabilization is equivalent to semiclassical observation estimates; see [Miller 2012].

Proposition A.5. *Assume that $0\leq a\in L^\infty$ is nontrivial $\left(\int_M a>0\right)$. Then stabilization holds for* (1-1) *if and only if*

$$
\begin{aligned}
&\exists h_0>0\quad\textit{such that}\quad\forall\, 0<h<h_0,\quad\forall\,(u,f)\in H^2(M)\times L^2(M),\quad (h^2\Delta+1)u=f,\\
&\qquad\qquad \|u\|_{L^2(M)}\leq C\Big(\|a^{\frac12}u\|_{L^2}+\frac1h\|f\|_{L^2}\Big).
\end{aligned}\tag{A-5}
$$

We prove the proposition for $m=0$. The proof for $m\not\equiv 0$ is similar (slightly simpler as we do not have to work with the operator $\dot A$ but can stick with A). From Proposition A.1, stabilization is equivalent to the fact that the resolvent $(\dot A-\lambda)^{-1}$ is bounded on $\mathbb C^\delta$. Since $\dot A$ is maximal dissipative, its resolvent is defined (and bounded) on any domain $\mathbb C^{-\epsilon}$ ($\epsilon>0$). We deduce that it is equivalent to prove that it is uniformly bounded on $i\mathbb R$ (and consequently by perturbation, on a δ neighborhood of $i\mathbb R$). Since

$$
(\dot A-\lambda)=(1+(1-\lambda)(\dot A-1)^{-1})(\dot A-1),
$$

and $(\dot A-1)^{-1}$ is compact on $\dot{\mathcal H}$ (see Corollary A.4), the operator $(1+(1-\lambda)(\dot A-1)^{-1})$ is Fredholm with index 0 and consequently, $\dot A-\lambda$ is invertible if and only if it is injective. As a consequence, stabilization is equivalent to the following a priori estimates

$$
\begin{aligned}
&\exists C>0\quad\text{such that}\quad\forall\lambda\in\mathbb R,\ U\in\dot H^2\times H^1,\ F\in\dot H^1\times L^2,\\
&\qquad(\dot A-i\lambda)U=F\quad\Longrightarrow\quad\|U\|_{\dot{\mathcal H}}\leq C\|F\|_{\dot{\mathcal H}}.
\end{aligned}\tag{A-6}
$$

A3.1. *High-frequency resolvent estimates imply stabilization.* We argue by contradiction. We assume (A-5) holds and assume that (A-6) does not hold. Then there exist sequences (λ_n), (U_n), (F_n) such that

$$(\dot{A} - i\lambda_n)U_n = F_n, \quad \|U_n\|_{\dot{\mathcal{H}}} > n\|F_n\|_{\dot{\mathcal{H}}}.$$

Since $U_n \neq 0$, we can assume $\|U_n\|_{\dot{\mathcal{H}}} = 1$. Extracting subsequences we can also assume that $\lambda_n \to \lambda \in \mathbb{R} \cup \{\pm\infty\}$ as $n \to \infty$. We write

$$U_n = \begin{pmatrix} \dot{u}_n \\ v_n \end{pmatrix}, \quad F_n = \begin{pmatrix} \dot{f}_n \\ g_n \end{pmatrix},$$

and distinguish according to three cases:

<u>Zero frequency</u>: $\lambda = 0$. In this case, we have

$$\dot{A}U_n = o(1)_{\dot{\mathcal{H}}} \quad \Longleftrightarrow \quad \Pi v_n = o(1)_{\dot{H}^1}, \quad \Delta \dot{u}_n - a v_n = o(1)_{L^2}.$$

We deduce that there exists $c_n \in \mathbb{C}$ such that

$$v_n - c_n = o(1)_{H^1}, \quad \Delta u_n - a c_n = o(1)_{L^2}.$$

But

$$\int_M \Delta u_n = 0 \quad \Longrightarrow \quad c_n \int_M a = o(1) \quad \Longrightarrow \quad c_n = o(1).$$

As a consequence, we get $v_n = o(1)_{L^2}$ and $\Delta u_n = o(1)_{L^2} \Rightarrow \dot{u}_n = o(1)_{\dot{H}^1}$. This contradicts $\|U_n\|_{\dot{\mathcal{H}}} = 1$.

<u>Low frequency</u>: $\lambda \in \mathbb{R}^*$. In this case, we have

$$(\dot{A} - i\lambda)U_n = o(1)_{\dot{\mathcal{H}}} \quad \Longleftrightarrow \quad \Pi v_n - i\lambda \dot{u}_n = o(1)_{\dot{H}^1}, \quad \Delta \dot{u}_n - (i\lambda + a)v_n = o(1)_{L^2}.$$

We deduce

$$\Delta v_n - i\lambda(a + i\lambda)v_n = o(1)_{L^2} + \Delta(o(1)_{\dot{H}^1}) = o(1)_{H^{-1}}.$$

Since (v_n) is bounded in L^2, from this equation, we deduce that Δv_n is bounded in H^{-1} and consequently v_n is bounded in H^1. Extracting another subsequence, we can assume that v_n converges in L^2 to v which satisfies

$$\Delta v + \lambda^2 v - i\lambda a v = 0.$$

Taking the imaginary part of the scalar product with v in L^2 gives (since $\lambda \neq 0$) $\int_M a|v|^2 = 0$, and consequently $av = 0$, which implies that v is an eigenfunction of the Laplace operator. But since the zero set of nontrivial eigenfunctions has Lebesgue measure 0 in M, $av = 0$ implies that $v = 0$ (and consequently $v_n = o(1)_{L^1}$). Now, we have

$$\Delta \dot{u}_n = (i\lambda + a)v_n + o(1)_{L^2} = o(1)_{L^2} \quad \Longrightarrow \quad \dot{u}_n = o(1)_{\dot{H}^1}.$$

This contradicts $\|U_n\|_{\dot{\mathcal{H}}} = 1$.

High frequency: $\lambda_n \to \pm\infty$. We study the case $\lambda_n \to +\infty$; the other case is obtained by considering $\overline{U}_n$. Let $h_n = \lambda_n^{-1}$. Then

$$\begin{aligned}
&(\dot{A} - i\lambda_n)U_n = o(1)_{\dot{\mathcal{H}}} \\
&\quad\iff \quad -i\lambda_n \dot{u}_n + \Pi v_n = o(1)_{\dot{H}^1}, \quad \Delta \dot{u}_n - (i\lambda_n + a)v_n = o(1)_{L^2} \\
&\quad\iff \quad \dot{u}_n = -ih_n \Pi v_n + o(h_n)_{\dot{H}^1}, \quad (h_n^2 \Delta + 1 - ih_n a)v_n = o(h_n)_{L^2} + o(h_n^2)_{H^{-1}}.
\end{aligned} \tag{A-7}$$

To conclude in this regime, we need:

Lemma A.6. *The observation inequality* (A-5) *implies the more general*

$$\begin{aligned}
&\exists h_0 > 0 \quad \text{such that} \quad \forall\, 0<h<h_0, \quad \forall\, (u, f_1, f_2) \in H^2(M) \times L^2(M) \times H^{-1}(M), \quad (h^2\Delta+1)u = f_1 + f_2, \\
&\qquad \|h\nabla_x u\|_{L^2} + \|u\|_{L^2(M)} \le C\left(\|a^{\frac12} u\|_{L^2} + \frac{1}{h}\|f_1\|_{L^2} + \frac{1}{h^2}\|f_2\|_{H^{-1}}\right).
\end{aligned} \tag{A-8}$$

Proof. Let $P_h^{\pm} = h^2\Delta + 1 \pm iha$ be defined on L^2 with domain H^2. Writing

$$P_h^{\pm} = (1 + (2 \pm iha)(h^2\Delta - 1)^{-1})(h^2\Delta - 1),$$

and since $(h^2\Delta - 1)^{-1}$ is compact on L^2, we deduce that $(1 + (2 \pm iha)(h^2\Delta - 1)^{-1})$ is Fredholm with index 0; hence $P_h^{\pm}$ is invertible if and only if it is injective. On the other hand we have

$$h\|a^{\frac12} u\|_{L^2}^2 = \pm \operatorname{Im}(P_h^{\pm} u, u)_{L^2} \le \|P_h^{\pm} u\|_{L^2} \|u\|_{L^2},$$

which combined with (A-5) implies ($(h^2\Delta + 1)u = P_h^{\pm} u \mp ihau$)

$$\begin{aligned}
\|u\|_{L^2}^2 &\le C\left(\|a^{\frac12} u\|_{L^2}^2 + \frac{1}{h^2}(\|P_h^{\pm} u\|_{L^2}^2 + h^2\|au\|_{L^2}^2)\right) \\
&\le \frac{C'}{h}\|P_h^{\pm} u\|_{L^2}\|u\|_{L^2} + \frac{C'}{h^2}\|P_h^{\pm} u\|_{L^2}^2 \quad \Longrightarrow \quad \|u\|_{L^2} \le \frac{C''}{h}\|P_h^{\pm} u\|_{L^2}.
\end{aligned} \tag{A-9}$$

Since

$$\big|\|u\|_{L^2}^2 - \|h\nabla_x u\|_{L^2}^2\big| = |\operatorname{Re}(P_h^{\pm} u, u)_{L^2}| \le \|P_h^{\pm} u\|_{L^2}\|u\|_{L^2}, \tag{A-10}$$

we deduce that $P_h^{\pm}$ is injective and hence bijective from H^2 to L^2 with inverse bounded by C''/h from L^2 to L^2 and by C/h^2 from L^2 to H^1. We now proceed by duality to obtain (A-8). The adjoint of $P_h^{\pm}$ is $P_h^{\mp}$ and is consequently bounded from H^{-1} to L^2 by C/h^2. Using again the identity (A-10) we get

$$P_h^{\pm} u = f_1 + f_2 \quad \Longrightarrow \quad \|h\nabla_x u\|_{L^2} + \|u\|_{L^2(M)} \le \frac{C}{h}\|f_1\|_{L^2} + \frac{C}{h^2}\|f_2\|_{H^{-1}}.$$

Finally

$$(h^2\Delta + 1)u = f_1 + f_2 \quad \Longrightarrow \quad P_h^{+} u = iahu + f_1 + f_2,$$

and we get

$$\begin{aligned}
\|h\nabla_x u\|_{L^2} + \|u\|_{L^2(M)} &\le C\left(\frac{1}{h}\|ihau + f_1\|_{L^2} + \frac{1}{h^2}\|f_2\|_{H^{-1}}\right) \\
&\le C'\left(\|a^{\frac12} u\|_{L^2} + \frac{1}{h}\|f_1\|_{L^2} + \frac{1}{h^2}\|f_2\|_{H^{-1}}\right).
\end{aligned}$$

□

We now come back to our sequence satisfying (A-7). From (A-8), (A-7) implies

$$\|h_n \nabla_x v_n\|_{L^2} + \|v_n\|_{L^2} = o(1)_{n\to+\infty},$$

and in turn

$$\|\nabla_x u_n\|_{L^2} = o(1)_{n\to+\infty}.$$

This contradicts $\|U_n\|_{\dot{\mathcal{H}}} = 1$.

A3.2. *Stabilization implies resolvent estimates.* Consider now $U = \binom{\dot{u}}{v}$, $F = \binom{\dot{f}}{g}$ such that

$$(\dot{A} - i\lambda)U = F \quad \Longleftrightarrow \quad -i\lambda\dot{u} + \Pi v = \dot{f} \quad \text{and} \quad (\Delta v + \lambda^2 - i\lambda a)v = i\lambda g + \Delta f.$$

From (A-5) with $h = \lambda^{-1}$, we get

$$\|v\|_{L^2} + \|h\nabla_x v\|_{L^2} \le C\|g\|_{L^2} + C\|\Delta f\|_{H^{-1}} \le C(\|g\|_{L^2} + C\|\nabla_x f\|_{L^2}),$$

and also

$$\|\nabla_x u\|_{L^2} = h\|\nabla_x(v - f)\| \le C(\|g\|_{L^2} + C\|\nabla_x f\|_{L^2}).$$

Appendix B: Characterization of stabilization

Here we shall prove that the properties (1), (2), (3) and (4) of the Introduction are equivalent. The implication (2) $\Rightarrow$ (1) is trivial. To show (1) $\Rightarrow$ (3) we fix T such that $f(T) \le \frac{1}{2}$. Then since

$$E_m(u)(T) = E_m(u)(0) - \int_0^T\int_M a(x)|\partial_t u|^2(t,x)\,dx\,dt \le \frac{1}{2}E_m(u)(0),$$

we deduce

$$E_m(u)(0) \le 2\int_0^T\int_M a(x)|\partial_t u|^2(t,x)\,dx\,dt,$$

which is (3). Conversely, if (3) is satisfied, we get

$$E_m(u)(T) = E_m(u)(0) - \int_0^T\int_M a(x)|\partial_t u|^2(t,x)\,dx\,dt \le \Big(1 - \frac{1}{C}\Big)E_m(u)(0).$$

Let $\delta = \big(1 - \frac{1}{C}\big) < 1$. Applying the previous estimate between 0 and T, then T and $2T$, etc., we get

$$E_m(u)(nT) \le \delta^n E_m(u)(0),$$

and hence the exponential decay along the discrete sequence of times nT. Finally, writing $nT \le t < (n+1)T$, we get

$$E_m(u)(t) \le E_m(u)(nT) \le \delta^n E_m(u)(0) \le e^{\log(\delta)\left(\frac{t}{T}-1\right)}E_m(u)(0),$$

which is (2). It remains to prove that (3) and (4) are equivalent. We shall actually prove that if (3) holds for some $T > 0$, then (4) holds for the same $T > 0$. Let us fix $T > 0$ and assume that (4) does not hold;

i.e., there exist sequences $(u_0^n, u_1^n) \in H^1 \times L^2$ such that the corresponding solutions to the undamped wave equation (1-3) satisfy

$$E_m(u_n)(0) > n \int_0^T \int_M a(x) |\partial_t u^n|^2(t,x)\, dt\, dx.$$

This implies that u_n is not identically 0 and dividing u_n by $\sqrt{E_m(u_n)(0))}$, we can assume $E_m(u_n)(0) = 1$, and

$$\int_0^T \int_M a(x) |\partial_t u_n|^2(t,x)\, dt\, dx \leq \frac{1}{n}. \tag{B-1}$$

Consider now (v_n) the sequence of solutions to the damped wave equation (1-1), with the same initial data (u_0^n, u_1^n), and $w_n = u_n - v_n$ a solution to

$$(\partial_t^2 - \Delta + a\,\partial_t + m) w_n = -a\,\partial_t u_n, \quad (w_n|_{t=0}, \partial_t w_n|_{t=0}) = (0,0) \in (H^1 \times L^2)(M).$$

From Duhamel's formula and (B-1) we deduce

$$\begin{aligned} \|(w_n, \partial_t w_n)\|_{L^\infty((0,T);H^1(M)\times L^2(M))} &\leq \|a\,\partial_t u_n\|_{L^1(0,T);L^2(M)} \\ &\leq \|a\|_{L^\infty}^{\frac{1}{2}} \|a^{\frac{1}{2}}\,\partial_t u_n\|_{L^1(0,T);L^2(M)} = o(1)_{n\to+\infty}. \end{aligned} \tag{B-2}$$

We deduce

$$E_m(v_n)(0) = 1 + o(1)_{n\to+\infty}$$

and

$$\int_0^T \int_M a(x) |\partial_t v_n|^2(t,x)\, dt\, dx = \|a^{\frac{1}{2}} v_n\|_{L^2((0,T)\times M)} = o(1)_{n\to+\infty},$$

which implies that (3) does not hold. As a consequence, we just proved (3) $\Rightarrow$ (4). The proof of (4) $\Rightarrow$ (3) is similar.

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