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RESONANT SPACES FOR VOLUME-PRESERVING ANOSOV FLOWS

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We consider Anosov flows on closed 3-manifolds preserving a volume form Ω. Following Dyatlov and Zworski (*Invent. Math.* **210**:1 (2017), 211–229) we study spaces of invariant distributions with values in the bundle of exterior forms whose wavefront set is contained in the dual of the unstable bundle. Our first result computes the dimension of these spaces in terms of the first Betti number of the manifold, the cohomology class $[\iota_X \Omega]$ (where X is the infinitesimal generator of the flow) and the helicity. These dimensions coincide with the Pollicott–Ruelle resonance multiplicities under the assumption of *semisimplicity*. We prove various results regarding semisimplicity on 1-forms, including an example showing that it may fail for time changes of hyperbolic geodesic flows. We also study non-null-homologous deformations of contact Anosov flows, and we show that there is always a splitting Pollicott–Ruelle resonance on 1-forms and that semisimplicity persists in this instance. These results have consequences for the order of vanishing at zero of the Ruelle zeta function. Finally our analysis also incorporates a flat unitary twist in the resonant spaces and in the Ruelle zeta function.

1. Introduction

We study resonant spaces of invariant distributions with values in the bundle of exterior forms for volume-preserving Anosov flows on 3-manifolds. One of the main motivations for looking at these spaces is that when a natural restriction is placed on the wave front set of the distributions, their dimensions are related to the Pollicott–Ruelle resonance multiplicities, which in turn determine the order of vanishing at zero of the Ruelle zeta function. For the case of contact Anosov flows this analysis was carried out in [Dyatlov and Zworski 2017] and here we show that the transition from "contact" to "volume-preserving" presents some new features, making the overall picture more involved, partially due to the nonsmoothness of the stable plus unstable bundle.

Let (M, Ω) be a closed 3-manifold equipped with a volume form Ω and let φ_t be a volume-preserving Anosov flow with infinitesimal generator X. If we write the Anosov splitting as $TM = \mathbb{R}X \oplus E_s \oplus E_u$, then we define the spaces E_0^*, E_s^* and E_u^* as the duals of $\mathbb{R}X$, E_u and E_s respectively. In particular, this means that for each $x \in M$, $E_u^*(x)$ is the annihilator of $\mathbb{R}X(x) \oplus E_u(x)$ and $E_u^* \subset T^*M$, a closed conic subset. We denote by $\mathcal{D}'_{E_u^*}(M; \Omega^k)$ the space of distributions with values in the bundle of exterior k-forms and with wave front set contained in E_u^* (see Section 2 for background on these notions). The resonant spaces that we are interested in are

$$\mathrm{Res}_k(0) := \{u \in \mathcal{D}'_{E_u^*}(M; \Omega^k) : \iota_X u = 0,\ \iota_X\, du = 0\}.$$

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The dimensions of the spaces can be considered as *geometric multiplicities*. We note that [Dang and Riviere 2017] studies generalised resonant spaces of forms (at zero) for arbitrary Anosov flows and these have a good cohomology theory (see Remark 2.2 for more details and definitions) but in principle these generalised resonant forms are not in the kernel of ι_X and might only be in the kernel of some power of the Lie derivative.

Our first result computes the dimension of these geometric spaces in terms of the first Betti number $b_1(M)$ of the manifold M and two natural characteristics of the flow that we now recall.

Since X preserves the volume form Ω, its Lie derivative $\mathcal{L}_X\Omega$ is equal to 0. Hence the 2-form $\omega := \iota_X\Omega$ must be closed.

Definition 1.1. We say that X is *null-homologous* if the cohomology class $[\omega]$ is equal to 0, i.e., ω is exact. For a null-homologous X, its *helicity* is the number

$$\mathcal{H}(X) := \int_M \tau(X)\,\Omega,$$

where τ is any 1-form such that $d\tau = \omega$.

It is easy to check that this definition is independent of the choice of primitive τ. The helicity (also referred to as the *asymptotic Hopf invariant*) measures how much in average field lines wrap and coil around one another. We refer to [Arnold and Khesin 1998] for a complete account of this concept as well as its interpretation as an average self-linking number.

We can now state our first result:

Theorem 1.2. *Let (M, Ω) be a closed 3-manifold with volume form Ω and let φ_t be a volume-preserving Anosov flow. Then*:

(1) $\dim \mathrm{Res}_0(0) = \dim \mathrm{Res}_2(0) = 1$.

(2) *If* $[\omega] \neq 0$, *then* $\dim \mathrm{Res}_1(0) = b_1(M) - 1$.

(3) *If* $[\omega] = 0$, *then*

$$\dim \mathrm{Res}_1(0) = \begin{cases} b_1(M) & \text{if } \mathcal{H}(X) \neq 0, \\ b_1(M)+1 & \text{if } \mathcal{H}(X) = 0. \end{cases}$$

This result generalises [Dyatlov and Zworski 2017, Proposition 3.1] as a contact Anosov flow fits into $[\omega] = 0$ and $\mathcal{H}(X) \neq 0$, since in that case we can take τ to be the contact 1-form and $\tau(X) = 1$. In Section 5 we give some examples to illustrate the various cases in Theorem 1.2, but we should point out right away that we do not know of any example of a volume-preserving Anosov flow with zero helicity.

We note that all the notions involved in Theorem 1.2 are invariant under time changes. Namely, if f is a positive smooth function, the flow of fX is also Anosov and with the *same* E_u^*. Hence the resonant spaces $\mathrm{Res}_k(0)$ are the same for all such flows. Also the notion of being null-homologous or having nonzero helicity is unaffected by time changes.

As mentioned before, the dimensions of $\mathrm{Res}_k(0)$ are important since they are related to the Pollicott–Ruelle resonance multiplicities $m_k(0)$. In general $m_k(0) \geq \dim \mathrm{Res}_k(0)$, and equality holds under the following condition (see Lemma 2.1):

Definition 1.3. X or φ_t is said to be k-semisimple if given $u \in \mathcal{D}'_{E_u^*}(M; \Omega^k)$ with $\iota_X u = 0$ and $\iota_X du \in \mathrm{Res}_k(0)$, then $u \in \mathrm{Res}_k(0)$, i.e., $\iota_X du = 0$.

Semisimplicity for $k = 0, 2$ will be easy to establish, but 1-semisimplicity does not always hold. In the case of contact Anosov flows, 1-semisimplicity was proved in [Dyatlov and Zworski 2017, Lemma 3.5]. For general volume-preserving Anosov flows the bundle $E_u \oplus E_s$ is only Hölder continuous [Foulon and Hasselblatt 2003] and thus the 1-form adapted to the flow, defined to be zero on $E_u \oplus E_s$ and 1 on the generator X, is only Hölder continuous. As a consequence the computations done in [Dyatlov and Zworski 2017, Lemma 3.5] are no longer viable due to this lack of smoothness.

Our next two results show that the picture for volume-preserving Anosov flow is rather more subtle. Let $\mathcal{X}_\Omega$ denote the set of vector fields that preserve Ω and let $\mathcal{X}_\Omega^0 \subset \mathcal{X}_\Omega$ denote those which are null-homologous.

Theorem 1.4. *Let (M, Ω) be a closed 3-manifold with volume form Ω. Consider a smooth 1-parameter family X_ε of volume-preserving Anosov vector fields with X_0 1-semisimple*:

(1) *If $X_\varepsilon \in \mathcal{X}_\Omega^0$ for every ε and $\mathcal{H}(X_0) \neq 0$, then X_ε is 1-semisimple for all ε sufficiently small.*

(2) *If X_0 is not null-homologous, then X_ε is 1-semisimple for all ε sufficiently small.*

For any hyperbolic surface, there is a time change of the geodesic flow which is not 1-semisimple.

Consider now a contact Anosov flow X with contact form α on a closed 3-manifold M. In particular, by Theorem 1.4 we know that 1-semisimplicity persists in $\mathcal{X}_\Omega^0$ and near X, where $\Omega = -\alpha \wedge d\alpha$. The next theorem gives us a local picture for what happens near X and away from $\mathcal{X}_\Omega^0$.

Theorem 1.5. *Consider $Y \in \mathcal{X}_\Omega \setminus \mathcal{X}_\Omega^0$. Then for sufficiently small ε, the flow $X_\varepsilon = X + \varepsilon Y$ is 1-semisimple. Moreover, there is a splitting Pollicott–Ruelle resonance $-i\lambda_\varepsilon = O(\varepsilon^2)$ of $-i\mathcal{L}_{X_\varepsilon}$ acting on $\Omega^1 \cap \ker \iota_{X_\varepsilon}$ with $\lambda_\varepsilon < 0$ for $\varepsilon \neq 0$, with Pollicott–Ruelle multiplicity 1 (see Figure 1).*

1A. *Ruelle zeta function.* We denote the set of primitive closed orbits of X by $\mathcal{G}_0$ (i.e., the ones that are not powers of a closed orbit in M); the period of $\gamma \in \mathcal{G}_0$ is denoted by l_γ. The Ruelle zeta function is defined as

$$\zeta(s) := \prod_{\gamma \in \mathcal{G}_0} (1 - e^{-s l_\gamma}). \tag{1-1}$$

The infinite product converges for $\mathrm{Re}\, s \gg 1$ and its meromorphic continuation to all $\mathbb{C}$ was first established in [Giulietti et al. 2013] in full generality and subsequently in [Dyatlov and Zworski 2016], where a microlocal approach was employed; see [Pollicott 2013] for a survey of dynamical zeta functions. Moreover, it was shown in [Dyatlov and Zworski 2016] that there is a factorisation (assuming that E_s and E_u are orientable)

$$\zeta(s) = \frac{\zeta_1(s)}{\zeta_0(s)\zeta_2(s)}, \tag{1-2}$$

where $\zeta_k(s)$ is an entire function with the order of vanishing at each $s \in \mathbb{C}$ equal to $m_k(is)$ for $k = 0, 1, 2$. Here $m_k(\lambda)$ is the Pollicott–Ruelle resonance multiplicity (see Section 2 for more details). Hence the order of vanishing of ζ at $s = 0$ is determined by $m(0) := m_1(0) - m_0(0) - m_2(0)$. Using this and Theorem 1.2 we derive the following:

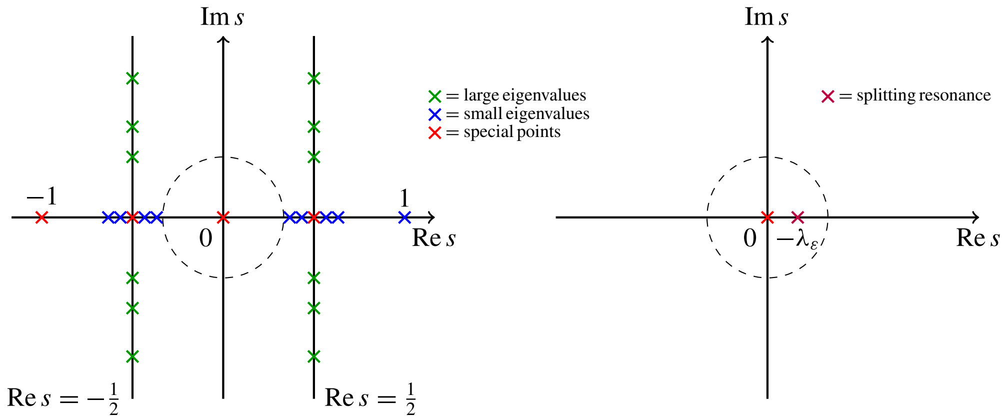

Figure 1. Left: resonance spectrum of $\mathcal{L}_X$ acting on $\Omega^1(S\Sigma)$ for a closed hyperbolic surface Σ. According to [Guillarmou et al. 2018; Dyatlov et al. 2015] and Remark 8.3 below, the green crosses correspond to (large) eigenvalues $\mu \geq \frac{1}{4}$ of $-\Delta_\Sigma$, the blue ones correspond to (small) eigenvalues $\mu \leq \frac{1}{4}$ and the red ones are "special". Right: resonance spectrum of $\mathcal{L}_{X_\varepsilon}$ acting on $\Omega^1(S\Sigma)$ and the splitting resonance, according to Theorem 1.5. We remark that the resonances in the rest of this paper will often be given by $\lambda = is$, i.e., obtained by a rotation of $\frac{\pi}{2}$ from this picture.

Corollary 1.6. *Let (M, Ω) be a closed 3-manifold with a volume-preserving Anosov flow φ_t whose stable and unstable bundles are orientable. Then*

$$s^{n(M,X)}\zeta(s)$$

is holomorphic close to zero, where

$$\begin{aligned} n(M, X) &= 3 - b_1(M) \quad \textit{if } [\omega] \neq 0, \\ n(M, X) &= 2 - b_1(M) \quad \textit{if } [\omega] = 0 \textit{ and } \mathcal{H}(X) \neq 0, \\ n(M, X) &= 1 - b_1(M) \quad \textit{if } [\omega] = 0 \textit{ and } \mathcal{H}(X) = 0. \end{aligned}$$

Moreover, if φ_t is 1-semisimple, then $s^{n(M,X)}\zeta(s)|_{s=0} \neq 0$.

The Ruelle zeta function for the suspension of a hyperbolic toral automorphism $A \in \mathrm{SL}(2, \mathbb{Z})$ is equal to

$$\zeta(s) = \frac{(e^{-s} - \lambda)(e^{-s} - 1/\lambda)}{(e^{-s} - 1)^2},$$

where λ and $1/\lambda$ are eigenvalues of A. This has a pole of order 2 at $s = 0$, which of course matches the computation in Corollary 1.6 since $b_1(M) = 1$. However, the corollary asserts that *any* other volume-preserving non-null-homologous Anosov flow on M will have ζ with the same behaviour at $s = 0$ since 1-semisimplicity holds trivially given that $\mathrm{Res}_1(0)$ is zero-dimensional. An interesting class of Anosov flows with $[\omega] \neq 0$ is given in [Bonatti and Langevin 1994]. These examples have a transverse torus, but they are not conjugate to suspensions. We do not know if they are 1-semisimple.

Magnetic flows are also examples to which the previous corollary applies. They are null-homologous (see Section 5), but they are generically *not* contact (see [Dairbekov and Paternain 2005]); hence they were not covered by the main result in [Dyatlov and Zworski 2017]. In this setting, magnetic flows can be described by a vector field of the form $X + (\lambda \circ \pi)V$, where X is the geodesic vector field, V is the vertical vector field of the circle fibration $\pi : S\Sigma \to \Sigma$, and $\lambda \in C^\infty(\Sigma)$ (here M is equal to $S\Sigma$, the unit circle bundle of the orientable surface Σ). They are volume-preserving since X and V preserve the canonical volume form. Suppose the geodesic flow is Anosov. Thanks to item (1) in Theorem 1.4, if λ is small enough, the magnetic flows remain Anosov and 1-semisimple and hence the order of vanishing of the zeta function at zero is the same as for Anosov geodesic flows, namely $-\chi(\Sigma)$.

The last statement in Theorem 1.4 and Theorem 1.5 have consequences for the zeta function. The failure of 1-semisimplicity means that $m_1(0) \geq b_1(M) + 1$, and hence the order of vanishing at zero of the zeta function is *strictly bigger* than that of the geodesic flow case. Hence time changes can a priori produce alterations in the properties of ζ near zero. Similarly the cohomology class $[\omega]$ can also produce alterations. For the particular construction of Theorem 1.4 we do not know the precise order of vanishing at zero.

Corollary 1.7. *The order of vanishing of the zeta function $\zeta_{X_\varepsilon}(s)$ of the flow X_ε from Theorem 1.5 at zero, for $\varepsilon \neq 0$, is equal to $b_1(M) - 3$. Moreover, for the time change fX of the geodesic flow on the hyperbolic surface constructed in Theorem 1.4, the order of vanishing is greater than or equal to $-\chi(\Sigma) + 1$.*

1B. *Flat unitary twists.* It is possible (and natural) to introduce a unitary twist in the discussion above. Consider (M, Ω) a closed 3-manifold with volume form Ω and X a volume-preserving Anosov vector field. Let $\mathcal{E}$ be a Hermitian vector bundle over M, equipped with a unitary connection A. We consider $\mathcal{D}'_{E_u^*}(M; \Omega^k \otimes \mathcal{E})$ the space of distributions with values in the bundle of $\mathcal{E}$-valued exterior k-forms and with wave front set contained in E_u^*. We replace the exterior differential d by the covariant derivative d_A (induced by the connection A) acting on $\mathcal{E}$-valued differential forms. Thus we can define resonant spaces

$$\mathrm{Res}_{k,A}(0) := \{u \in \mathcal{D}'_{E_u^*}(M; \Omega^k \otimes \mathcal{E}) : \iota_X u = 0,\ \iota_X\, d_A u = 0\}.$$

We shall compute the dimensions of these spaces in analogy to Theorem 1.2 under the assumption that A is *flat and unitary*, i.e., $d_A^2 = 0$ and d_A is compatible with the Hermitian inner product on $\mathcal{E}$. Recall that flat unitary connections are in 1-1 correspondence with representations of $\pi_1(M)$ into the unitary group. Under this condition, one can define twisted Betti numbers $b_i(M, \mathcal{E})$ in the standard way (we note that these numbers may depend on A). The upshot is a theorem similar to Theorem 1.2 where the Betti numbers $b_i(M)$ are replaced by $b_i(M, \mathcal{E})$; see Theorem 4.1 for the full statement. With this information in hand we can study a *twisted Ruelle zeta function*,

$$\zeta_A(s) := \prod_{\gamma \in \mathcal{G}_0} \det(\mathrm{Id} - \alpha_\gamma e^{-s l_\gamma}). \tag{1-3}$$

Here, given a point x_0 on $\gamma \in \mathcal{G}_0$, we denote by α_γ the parallel transport map (i.e., an element of the holonomy group) along the loop determined by γ. It is easy to check that the product is independent of the choice of x_0 on γ, as this amounts to conjugating α_γ by a linear map. Note that if $\mathcal{E} = M \times \mathbb{C}$ and $d_A = d$, the expression in (1-3) reduces to that in (1-1). If the connection A is flat, we recover the definition of the twisted Ruelle zeta function considered in [Fried 1986]; it was also studied in [Adachi 1988; Adachi and Sunada 1987], where functions of this type were called *L-functions* in analogy with number theory. Fried conjectured that the coefficient at zero of ζ_A for an acyclic connection (i.e., one that has vanishing Betti numbers) is related to the analytic torsion, but proved it only for hyperbolic manifolds. For recent progress on this conjecture and more information, see [Dang et al. 2020; Shen 2018; Zworski 2018].

The notion of semisimplicity extends naturally to the twisted case (just replace d by d_A in Definition 1.3). In that case we will say a flow φ_t or X is 1-semisimple with respect to d_A. Putting everything together we shall derive the following corollary:

Corollary 1.8. *Let (M, Ω) be a closed 3-manifold with a volume-preserving Anosov flow φ_t whose stable and unstable bundles are orientable. Let $\mathcal{E}$ be a Hermitian vector bundle equipped with a unitary flat connection A. Then*

$$s^{n(M,X,A)}\zeta_A(s)$$

is holomorphic close to zero, where

$$\begin{aligned} n(M, X, A) &= 3b_0(M, \mathcal{E}) - b_1(M, \mathcal{E}) && \text{if } [\omega] \neq 0, \\ n(M, X, A) &= 2b_0(M, \mathcal{E}) - b_1(M, \mathcal{E}) && \text{if } [\omega] = 0 \text{ and } \mathcal{H}(X) \neq 0, \\ n(M, X, A) &= b_0(M, \mathcal{E}) - b_1(M, \mathcal{E}) && \text{if } [\omega] = 0 \text{ and } \mathcal{H}(X) = 0. \end{aligned}$$

Moreover, if X is 1-semisimple with respect to d_A, then $s^{n(M,X,A)}\zeta_A(s)|_{s=0} \neq 0$.

A particular instance of the corollary arises when we consider A to be the pullback of a flat connection on a surface Σ. In this case it is easy to check that (see Lemma 2.9)

$$2b_0(M, \mathcal{E}) - b_1(M, \mathcal{E}) = \operatorname{rank}(\mathcal{E})\, \chi(\Sigma).$$

Thus:

Corollary 1.9. *Let $\mathcal{E}$ be a Hermitian vector bundle over an oriented closed Riemannian surface (Σ, g), equipped with a unitary flat connection A. We consider $M = S\Sigma$ with footpoint map π and any Anosov flow, 1-semisimple with respect to d_{π^*A}, null-homologous with nonzero helicity, preserving the volume form of $S\Sigma$. We consider the pullback bundle $\pi^*\mathcal{E}$ with the pullback connection π^*A. Then in a neighbourhood of zero we have $s^{\operatorname{rank}(\mathcal{E})\cdot\chi(\Sigma)} \cdot \zeta_{\pi^*A}(s)$ holomorphic such that*

$$s^{\operatorname{rank}(\mathcal{E})\cdot\chi(\Sigma)} \cdot \zeta_{\pi^*A}(s)|_{s=0} \neq 0.$$

We remark that Corollary 1.9 applies in particular to contact flows, since for those 1-semisimplicity holds with respect to any flat and unitary d_A.

This paper is organised as follows. Section 2 gives preliminary information, recalls the Pollicott–Ruelle resonances and proves some necessary lemmas. In Section 3 we recall the factorisation of the twisted zeta function in terms of some traces of operators on $\mathcal{E}$-valued k-forms. In Section 4, we compute the dimension of the resonant spaces $\operatorname{Res}_{k,A}(0)$ and obtain Theorem 1.2 as a particular case. Corollary 1.8 is also proved in this section. Section 5 gives examples and develops material needed for the study of time changes. Section 6 discusses perturbations and proves the main result needed for items (1) and (2) in Theorem 1.4. Theorem 1.5 is proved in Section 7. Finally, Section 8 exhibits a time change of the geodesic flow of a hyperbolic surface for which 1-semisimplicity fails, thus completing the proof of Theorem 1.4.

2. Preliminary results

In this section we review the necessary tools to prove the results stated in the Introduction. In particular, we recall the Pollicott–Ruelle resonances and put forward some preparatory lemmas.

2A. ***Microlocal analysis.*** Here we outline the microlocal tools necessary for our proofs. For more information on distribution spaces and properties of wavefront sets see [Grigis and Sjöstrand 1994, Chapter 7] or [Hörmander 1983, Chapters VI, VIII] and for more about pseudodifferential operators see [Grigis and Sjöstrand 1994, Chapter 3] or [Hörmander 1985, Chapter XVIII].

Let M be a closed manifold and $\mathcal{E}$ a smooth complex vector bundle. We consider the space of infinitely differentiable smooth sections and the space of distributional sections, respectively,

$$C^\infty(M;\mathcal{E}) \quad \text{and} \quad \mathcal{D}'(M;\mathcal{E}).$$

We recall the notion of the *wavefront set* of a distribution, which keeps track of the directional singularities. Given $u \in \mathcal{D}'(\mathbb{R}^n)$, we have $(x,\xi) \notin \mathrm{WF}(u) \subset T^*\mathbb{R}^n \setminus 0 = \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ if there exists $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi(x) \neq 0$ and an open conical neighbourhood U of ξ such that

$$|\widehat{\varphi u}(\eta)| = O(\langle \eta \rangle^{-\infty})$$

for $\eta \in U$. Here we let $\langle \eta \rangle = (1+|\eta|^2)^{1/2}$ and by $O(\langle \eta \rangle^{-\infty})$ we mean an expression bounded by $C_N\langle \eta \rangle^{-N}$ for every N. A vector-valued distribution $u \in \mathcal{D}'(\mathbb{R}^n;\mathbb{R}^m)$ for some $m \in \mathbb{N}$ may be identified with a vector $u = (u_1,\dots,u_m)$ with $u_i \in \mathcal{D}'(\mathbb{R}^n)$. Then

$$\mathrm{WF}(u) := \bigcup_{i=1}^m \mathrm{WF}(u_i).$$

It is standard that these definitions are coordinate invariant, so for $u \in \mathcal{D}'(M;\mathcal{E})$ we have

$$\mathrm{WF}(u) \subset T^*M \setminus 0.$$

It is moreover true that for any pseudodifferential operator A we have

$$\mathrm{WF}(Au) \subset \mathrm{WF}(A) \cap \mathrm{WF}(u) \subset \mathrm{WF}(u),$$

a fact that will be used later on. Then, we introduce for a closed conic set $\Gamma \subset T^*M \setminus 0$ the space

$$\mathcal{D}'_\Gamma(M;\mathcal{E}) = \{u \in \mathcal{D}'(M;\mathcal{E}) \mid \mathrm{WF}(u) \subset \Gamma\}.$$

Note that by the above relation on wavefront sets, the spaces $\mathcal{D}'_\Gamma(M;\mathcal{E})$ are invariant under the action of pseudodifferential operators.

2B. ***Pollicott–Ruelle resonances.*** Let us now quickly recall the microlocal approach to Pollicott–Ruelle resonances, as in [Dyatlov and Zworski 2017]. Let M be a compact smooth manifold without boundary and X be a smooth vector field. We assume that the flow φ_t of X is Anosov, i.e., that there is a splitting of the tangent space

$$T_xM = \mathbb{R}X(x) \oplus E_u(x) \oplus E_s(x)$$

for each $x \in M$, where $E_u(x)$ and $E_s(x)$ depend continuously on x and are invariant under the flow and, moreover, that for some constants $C, \nu > 0$ and a fixed metric on M

$$|d\varphi_t(x) \cdot v| \leq Ce^{-\nu|t|} \cdot |v|, \qquad \begin{cases} t \geq 0, & v \in E_s(x), \\ t \leq 0, & v \in E_u(x). \end{cases}$$

We call $E_s(x)$ the *stable* bundle or direction and $E_u(x)$ the *unstable* bundle or direction. It is a well-known fact that the geodesic flow on the unit tangent bundle $M = SN$ for N with negative sectional curvature is Anosov.

Let us define the spaces $E_0^*(x)$, $E_u^*(x)$, $E_s^*(x)$ as the duals of $E_0(x) := \mathbb{R}X(x)$, $E_s(x)$, $E_u(x)$ respectively. Explicitly, $E_u^*(x)$ is the annihilator of $\mathbb{R}X(x) \oplus E_u(x)$, $E_s^*(x)$ is the annihilator of $\mathbb{R}X(x) \oplus E_s(x)$ and $E_0^*(x)$ is the annihilator of $E_s(x) \oplus E_u(x)$. The continuous vector bundle $E_u^* := \bigcup_{x \in M} E_u^*(x) \subset T^*M$ is a closed conic subset.

Let us consider a complex vector bundle $\mathcal{E}$ over M, equipped with a connection A (which defines the covariant derivative d_A) and a smooth potential Φ (section of the endomorphism bundle of $\mathcal{E}$). This defines a first-order operator

$$P = -i\iota_X d_A + \Phi \tag{2-1}$$

acting on sections of $\mathcal{E}$, denoted by $C^\infty(M; \mathcal{E})$. Later on we will dispense with Φ, but for the moment it can be included without trouble.

For $\lambda \in \mathbb{C}$ with sufficiently large $\operatorname{Im} \lambda > C_0 > 0$, we have the integral

$$R(\lambda) := i \int_0^\infty e^{i\lambda t} e^{-itP}\, dt : L^2(M; \mathcal{E}) \to L^2(M; \mathcal{E}) \tag{2-2}$$

converges and defines a bounded operator, holomorphic in λ and, moreover, $R(\lambda) = (P - \lambda)^{-1}$ on L^2. The propagator e^{itP} is defined by solving the appropriate first-order PDE and the constant C_0 depends on P.

In [Faure and Sjöstrand 2011] (see also [Dyatlov and Zworski 2016]) it is proved that the operator $R(\lambda)$ has a meromorphic extension to the entire complex plane

$$R(\lambda) : C^\infty(M; \mathcal{E}) \to \mathcal{D}'(M; \mathcal{E}) \tag{2-3}$$

for $\lambda \in \mathbb{C}$ and the poles of this continuation are the *Pollicott–Ruelle resonances*.

We proceed to define the multiplicity of a Pollicott–Ruelle resonance λ_0. By definition, there is a Laurent expansion of $R(\lambda)$ at λ_0 (see [Dyatlov and Zworski 2019, Appendix C])

$$R(\lambda) = R_H(\lambda) - \sum_{j=1}^{J(\lambda_0)} \frac{(P - \lambda_0)^{j-1}\Pi}{(\lambda - \lambda_0)^j}, \qquad \Pi,\ R_H(\lambda) : \mathcal{D}'_{E_u^*}(M; \mathcal{E}) \to \mathcal{D}'_{E_u^*}(M; \mathcal{E}) \tag{2-4}$$

where $R_H(\lambda)$ is the holomorphic part at λ_0 and $\Pi = \Pi_{\lambda_0}$ is a finite-rank projector given by

$$\Pi_{\lambda_0} = \frac{1}{2\pi i} \oint_{\lambda_0} (\lambda - P)^{-1} \, d\lambda. \tag{2-5}$$

Here, the integral is along a small closed loop around λ_0 and it can be easily checked that $\Pi_{\lambda_0}^2 = \Pi_{\lambda_0}$, $[\Pi_{\lambda_0}, P] = 0$. The fact that $R_H(\lambda)$ and Π can be extended to continuous operators on $\mathcal{D}'_{E_u^*}$ follows from the restrictions on the wave front sets given in [Dyatlov and Zworski 2016, Proposition 3.3] and [Grigis and Sjöstrand 1994, Theorem 7.8]. The *Pollicott–Ruelle multiplicity* of λ_0, denoted by $m_P(\lambda_0)$, is defined as the dimension of the range of Π_{λ_0}.

By applying $P - \lambda$ to (2-4), we obtain $(P - \lambda_0)^{J(\lambda_0)} \Pi_{\lambda_0} = 0$ and so $\operatorname{ran} \Pi_{\lambda_0} \subset \ker(P - \lambda_0)^{J(\lambda_0)}$. The elements of $\operatorname{ran} \Pi_{\lambda_0}$ are called *generalised resonant states* and we will define, for $j \in \mathbb{N}$,

$$\operatorname{Res}_P^{(j)}(\lambda_0) = \{u \in \mathcal{D}'_{E_u^*}(M; \mathcal{E}) : (P - \lambda_0)^j u = 0\}. \tag{2-6}$$

We also write

$$\operatorname{Res}_P(\lambda_0) = \{u \in \mathcal{D}'_{E_u^*}(M; \mathcal{E}) : (P - \lambda_0)^{J(\lambda_0)} u = 0\}.$$

In fact, we may show that $\operatorname{Res}_P(\lambda_0)$ is equal to the range of Π_{λ_0} and we may think of $J(\lambda_0)$ as the size of the largest Jordan block.

Lemma 2.1. *Let $u \in \mathcal{D}'_{E_u^*}(M; \mathcal{E})$ be such that $(P - \lambda_0)^{j_0} u = 0$ with $j_0 \in \mathbb{N}_0$ the minimal such number. Then $j_0 \leq J(\lambda_0)$, $\Pi_{\lambda_0} u = u$ and* $\ker(P - \lambda_0)^{J(\lambda_0)} = \operatorname{ran} \Pi_{\lambda_0}$.

Proof. Assume that $j_0 > J(\lambda_0)$ for the sake of contradiction. Since Sobolev spaces filter out $\mathcal{D}'(M; \mathcal{E})$, there is an $s > 0$ such that $u \in H^{-s}(M; \mathcal{E})$. Recalling the definition of the anisotropic space $\mathcal{H}_{rG}(M; \mathcal{E})$ for $r > 0$ (see (6-1) below), we get

$$\mathcal{D}'_{E_u^*}(M; \mathcal{E}) \cap H^{-r}(M; \mathcal{E}) \subset \mathcal{H}_{rG}(M; \mathcal{E})$$

since $\mathcal{H}_{rG}$ is microlocally equivalent to H^{-r} near E_u^*. Therefore $u \in \mathcal{H}_{rG}(M; \mathcal{E})$ for $r > s$ and by Lemma 6.1 below $(P - \lambda)^{-1} : \mathcal{H}_{rG}(M; \mathcal{E}) \to \mathcal{H}_{rG}(M; \mathcal{E})$ is meromorphic near λ_0 for $r \gg s$.

Let us set $v := (P - \lambda_0)^{j_0 - 1} u$. Then $(P - \lambda)^{-1} v = (\lambda_0 - \lambda)^{-1} v$ and by applying (2-5) to v we get $\Pi_{\lambda_0} v = v$. Note that (2-4) also implies $(P - \lambda_0)^{J(\lambda_0)} \Pi_{\lambda_0} t = 0$ for all $t \in \mathcal{H}_{rG}$. But all this implies

$$(P - \lambda_0)^{j_0 - 1} u = \Pi_{\lambda_0} (P - \lambda_0)^{j_0 - 1} u = (P - \lambda_0)^{j_0 - 1} \Pi_{\lambda_0} u = 0. \tag{2-7}$$

This contradicts the minimality of j_0 and proves the first claim.

For the second claim, take some $u \in \operatorname{Res}_P^{(j_0)}(\lambda_0)$ and use induction on j_0. Note that the first two equalities of (2-7) show $\Pi_{\lambda_0} u = u$ for $j_0 = 1$ and more generally that

$$(P - \lambda_0)^{j_0 - 1} (\Pi_{\lambda_0} u - u) = 0.$$

The fact that Π_{λ_0} is a projector and the induction hypothesis show $\Pi_{\lambda_0} u = u$, proving the claim.

Lastly, if $u \in \operatorname{ran} \Pi_{\lambda_0}$ then $\Pi_{\lambda_0} u = u$ and so $(P - \lambda_0)^{J(\lambda_0)} u = 0$ by (2-4), which together with the previous paragraph shows $\ker (P - \lambda_0)^{J(\lambda_0)} \cap \mathcal{D}'_{E_u^*}(M; \mathcal{E}) = \operatorname{ran} \Pi_{\lambda_0}$. ☐

Remark 2.2. Generalised resonant spaces of forms (at zero) have a good cohomology theory; see [Dang and Riviere 2017, Theorem 2.1]. We emphasise that here we study resonant spaces at zero with $j = 1$ in (2-6) and such that the elements are in the kernel of ι_X, as well as conditions under which there are no Jordan blocks.

2C. *Coresonant states.* Here we study the connection between the semisimplicity and a suitable pairing between resonant and coresonant states. We start off with a lemma relating the adjoint of the spectral projector and the spectral projector of the adjoint.

Lemma 2.3. *Let P be a first-order differential operator acting on sections of $\mathcal{E}$ with principal symbol $-i\sigma(X) \times \mathrm{Id}_{\mathcal{E}}$ and consider the adjoint operator P^*. Denote the spectral projector of P at $\lambda_0 \in \mathbb{C}$ by Π_{λ_0} and of P^* by Π'_{λ_0}. Also, denote the resolvent by $R_P(\lambda) = (P - \lambda)^{-1}$. Then*[1]

$$R_P(\lambda)^* = -R_{-P^*}(-\bar{\lambda}) \quad \textit{and} \quad \Pi^*_\lambda = \Pi'_{-\bar{\lambda}}.$$

Proof. Firstly note that for $\operatorname{Im} \lambda \gg 1$ and all $u, v \in L^2(M; \mathcal{E})$, by (2-2) we have the identity

$$\langle R_P(\lambda)u, v\rangle_{L^2} = \langle u, -R_{-P^*}(-\bar{\lambda})v\rangle_{L^2}. \tag{2-8}$$

Then by analytic continuation we have the equality in (2-8) for any $u, v \in C^\infty$ for all $\lambda \in \mathbb{C}$. Moreover, by continuity and the mapping properties of $R_P(\lambda) : \mathcal{D}'_{E_u^*}(M; \mathcal{E}) \to \mathcal{D}'_{E_u^*}(M; \mathcal{E})$ and $R_{-P^*}(-\bar{\lambda}) : \mathcal{D}'_{E_s^*}(M; \mathcal{E}) \to \mathcal{D}'_{E_s^*}(M; \mathcal{E})$ outside the poles, we have (2-8) for all $u \in \mathcal{D}'_{E_u^*}$ and $v \in \mathcal{D}'_{E_s^*}$. This proves the first claim. Now let $u \in \mathcal{D}'_{E_u^*}(M; \mathcal{E})$ and $v \in \mathcal{D}'_{E_s^*}(M; \mathcal{E})$. We may write

$$\langle \Pi_{\lambda_0} u, v\rangle = -\frac{1}{2\pi i} \oint_{\lambda_0} \langle R_P(\lambda)u, v\rangle \, d\lambda = \frac{1}{2\pi i} \oint_{\lambda_0} \langle u, R_{-P^*}(-\bar{\lambda})v\rangle \, d\lambda = \langle u, \Pi'_{-\bar{\lambda}_0} v\rangle.$$

This proves $\Pi^*_{\lambda_0} = \Pi'_{-\bar{\lambda}_0}$. ☐

We proceed to define the *coresonant states*. Given an operator P as in Lemma 2.3 and a resonance $\lambda_0 \in \mathbb{C}$, the space of coresonant states at λ_0 is $\mathrm{Res}_{-P^*}(-\bar{\lambda}_0) \subset \mathcal{D}'_{E_s^*}(M; \mathcal{E})$. By the wavefront set conditions, notice that we may multiply resonances and coresonances in the scalar case, or form inner products; see, e.g., [Grigis and Sjöstrand 1994, Proposition 7.6]. We are now ready to reinterpret the semisimplicity in terms of the pairing

$$\mathrm{Res}_P(\lambda_0) \times \mathrm{Res}_{-P^*}(-\bar{\lambda}_0) \to \mathbb{C}, \; (u, v) \mapsto \langle u, v\rangle_{L^2}. \tag{2-9}$$

Observe that the pairing (2-9) is nondegenerate: we have $\langle u, v\rangle = 0$ for all $v \in \mathrm{Res}_{-P^*}(-\bar{\lambda}_0)$ if and only if $\langle u, \Pi'_{-\bar{\lambda}_0}\varphi\rangle = 0$ for all $\varphi \in C^\infty(M; \mathcal{E})$. Then by Lemma 2.3 and since $\Pi_{\lambda_0} u = u$, this holds if and only if $u \equiv 0$; by an analogous argument for the other entry, we obtain the nondegeneracy. In particular, $m_P(\lambda_0) = m_{-P^*}(-\bar{\lambda}_0)$ and also $J(\lambda_0) = J'(-\bar{\lambda}_0)$. Here $J'(\mu)$ denotes the size of the largest Jordan block of $-P^*$ at μ.

[1]Here we interpret $-R_{-P^*}(-\bar{\lambda}) : C^\infty(M; \mathcal{E}) \to \mathcal{D}'(M; \mathcal{E})$ as the operator obtained by meromorphic continuation, but with respect to the flow generated by $-X$.

Lemma 2.4. *Assume P satisfies the assumptions of Lemma 2.3. Then we have that the semisimplicity for P at λ_0 holds if and only if the semisimplicity for $-P^*$ at $-\bar{\lambda}_0$ holds. Moreover, P is semisimple at λ_0 if and only if the pairing*

$$\operatorname{Res}^{(1)}_P(\lambda_0) \times \operatorname{Res}^{(1)}_{-P^*}(-\bar{\lambda}_0) \to \mathbb{C}, \quad (u, v) \mapsto \langle u, v\rangle_{L^2}. \tag{2-10}$$

is nondegenerate.

Proof. For the first claim, simply note that by the previous paragraph we have $J(\lambda_0) = J'(-\bar{\lambda}_0)$.

For the second claim, assume first that the pairing (2-10) is nondegenerate. Assume we have $u, u' \in \mathcal{D}'_{E_u^*}(M; \mathcal{E})$, with $(P - \lambda_0)u = u'$ where $u' \in \operatorname{Res}^{(1)}_P(\lambda_0)$. We want to show $u' = 0$. We have, for any $v \in \operatorname{Res}^{(1)}_{-P^*}(-\bar{\lambda}_0)$,

$$\langle u', v\rangle = \langle (P - \lambda_0)u, v\rangle = \langle u, (P^* - \bar{\lambda}_0)v\rangle = 0.$$

Now nondegeneracy implies $u' = 0$.

Assume next the semisimplicity holds for P at λ_0 and let $u \in \operatorname{Res}^{(1)}_P(\lambda_0)$ satisfy $\langle u, v\rangle = 0$ for all $v \in \operatorname{Res}^{(1)}_{-P^*}(-\bar{\lambda}_0)$. Then we have, for all $\varphi \in C^\infty(M; \mathcal{E})$,

$$\langle u, \varphi\rangle = \langle \Pi_{\lambda_0} u, \varphi\rangle = \langle u, \Pi'_{-\bar{\lambda}_0}\varphi\rangle = 0.$$

Here we used Lemma 2.3 and the assumption. Thus $u \equiv 0$. The fact that $-P^*$ is semisimple at $-\bar{\lambda}_0$ and an analogous argument for the other entry proves the nondegeneracy and finishes the proof. ☐

2D. *Further preparatory results.* We start by quoting an important technical result; see [Dyatlov and Zworski 2017, Lemma 2.3].

Lemma 2.5. *Suppose there exist a smooth volume form on M and a smooth inner product on the fibres of $\mathcal{E}$ for which $P^* = P$ on $L^2(M; \mathcal{E})$. Suppose that $u \in \mathcal{D}'_{E_u^*}(M; \mathcal{E})$ satisfies*[2]

$$Pu \in C^\infty(M; \mathcal{E}), \quad \operatorname{Im}\langle Pu, u\rangle_{L^2} \geq 0.$$

Then $u \in C^\infty(M; \mathcal{E})$. In particular, the conclusion of the lemma holds for u a resonant state with the eigenvalue $\lambda \in \mathbb{R}$—just swap P with $P - \lambda$.

We also need a simple regularity result analogous to [Dyatlov and Zworski 2017, Lemma 2.1]. We give it here for completeness

Lemma 2.6. *Assume d_A is flat and let $\Gamma \subset T^*M \setminus 0$ be a closed conic set. Assume that $u \in \mathcal{D}'_\Gamma(M; \Omega^k \otimes \mathcal{E})$ and $d_A u \in C^\infty(M; \Omega^{k+1} \otimes \mathcal{E})$. Then there exists $v \in C^\infty(M; \Omega^k \otimes \mathcal{E})$ and $w \in \mathcal{D}'_\Gamma(M; \Omega^{k-1} \otimes \mathcal{E})$ such that $u = v + d_A w$.*

Proof. The proof follows formally by replacing d with d_A and δ with d_A^* in the proof [Dyatlov and Zworski 2017, Lemma 2.1]. ☐

[2] The inner product in this paper is complex conjugate in the second variable.

2E. ***Cohomology in a flat bundle.*** Given a manifold M of dimension n and a Hermitian vector bundle $\mathcal{E}$ with a flat connection A, we may consider the complex given by

$$0 \xrightarrow{d_A} C^\infty(M;\mathcal{E}) \xrightarrow{d_A} C^\infty(M;\Omega^1\otimes\mathcal{E}) \xrightarrow{d_A} \cdots \xrightarrow{d_A} C^\infty(M;\Omega^n\otimes\mathcal{E}) \xrightarrow{d_A} 0. \tag{2-11}$$

Here we extend, as usual, the action of d_A to vector-valued differential forms by asking that the Leibnitz rule holds. The homology of this complex will be denoted by $H^k_A(M;\mathcal{E})$ for $k=0,\dots,n$. Consider now Σ an oriented Riemannian surface and let $\mathcal{E}$ be a Hermitian vector bundle over Σ equipped with a unitary, flat connection A. We can pull back the bundle $\mathcal{E}$ to the unit sphere bundle $\pi : S\Sigma\to\Sigma$ to obtain $\pi^*\mathcal{E}$, equipped with a unitary, flat connection π^*A.

Lemma 2.7. *Assume Σ has genus $g\neq 1$. Then the following map is an isomorphism*:

$$\pi^* : H^1_A(\Sigma;\mathcal{E}) \to H^1_{\pi^*A}(S\Sigma;\pi^*\mathcal{E}). \tag{2-12}$$

Proof. There is a vertical vector field V that generates the rotation in the fibres of $S\Sigma$. We first check π^* is injective, so assume $\pi^*\theta = d_{\pi^*A}F$, where $\theta\in C^\infty(\Sigma;\Omega^1\otimes\mathcal{E})$ is d_A-closed and $F\in C^\infty(S\Sigma;\mathcal{E})$. This implies $\iota_V d_{\pi^*A}F=0$. Note that if $x\in\Sigma$, there is a small ball B with $x\in B$, over which $\mathcal{E}$ is trivial. Thus $\iota_V d_{\pi^*A}F=0$ implies $VF=0$ (since $\iota_V\pi^*A=0$) and so $F=\pi^*f$ locally; this is easily seen to extend to $F=\pi^*f$ globally for some $f\in C^\infty(\Sigma;\mathcal{E})$. This implies $\pi^*(d_Af-\theta)=0$ and so $d_Af=\theta$.

For surjectivity, take $u\in C^\infty(S\Sigma;\Omega^1\otimes\pi^*\mathcal{E})$ with $d_{\pi^*A}u=0$. We want to prove there are v and F such that $u=\pi^*v+d_{\pi^*A}F$, where v is d_A-closed. This implies

$$\iota_V u = \iota_V d_{\pi^*A}F. \tag{2-13}$$

If we solve (2-13), then $w=u-d_{\pi^*A}F$ satisfies $d_{\pi^*A}w=0$ and $\iota_Vw=0$. By going to local trivialisations where $A=0$, a computation implies $w=\pi^*v$ for some 1-form v locally. Again, by uniqueness this may be easily extended to some global $v\in C^\infty(\Sigma;\mathcal{E})$ with $d_Av=0$. We now focus on (2-13) and finding such F.

To this end, we introduce the pushforward map $\pi_*: C^\infty(S\Sigma;\Omega^1\otimes\pi^*\mathcal{E})\to C^\infty(\Sigma;\mathcal{E})$ by integrating along the fibres

$$\pi_* : \alpha(x,v)\mapsto \beta(x)=\int_{S_x\Sigma}\alpha. \tag{2-14}$$

One can show that the pushforward is well-defined and that it intertwines d_A and d_{π^*A}; after going to a trivialisation where $A=0$, this reduces to showing commutation with d, which follows from [Bott and Tu 1982, Proposition 6.14.1]. Thus π_* descends to cohomology; i.e., we have $\pi_*: H^1_{\pi^*A}(S\Sigma;\mathcal{E})\to H^0_A(\Sigma;\mathcal{E})$.

Now observe that (2-13) can be solved if and only if $\pi_*u=0$. We introduce the section $s\in C^\infty(\Sigma;\mathcal{E})$ with $s(x)=\pi_*u$. Note that $d_As=0$. Moreover, we have for K the Gaussian curvature of Σ:

$$\int_{S\Sigma}\langle u,\pi^*(sK\,d\,\mathrm{vol}_\Sigma)\rangle=\int_\Sigma\langle\pi_*u,sK\,d\,\mathrm{vol}_\Sigma\rangle=\int_\Sigma\|s\|^2K\,d\,\mathrm{vol}_\Sigma=\|s\|^2 2\pi\chi(\Sigma). \tag{2-15}$$

Here we used that $\|s\|^2$ is constant, since s is parallel and A is unitary, and we applied Gauss–Bonnet theorem. In the first equality we use a generalisation of [Bott and Tu 1982, Proposition 6.15]. We use the convention that $\langle s\alpha, s'\beta\rangle=\langle s,s'\rangle_{\mathcal{E}}\alpha\wedge\bar\beta$, where α and β are forms of complementary degree and s, s' are sections.

On the other hand, we have $\pi^*(K\, d\,\mathrm{vol}_\Sigma) = -d\psi$, where ψ is the connection 1-form on $S\Sigma$. Therefore we have the pointwise identity, as $d_{\pi^*A} u = 0$ and $d_A s = 0$,

$$\langle u, \pi^*(sK\, d\,\mathrm{vol}_\Sigma)\rangle = d\langle u, (\pi^* s)\psi\rangle.$$

So by Stokes' theorem we obtain that the first integral in (2-15) is zero. Since $g \neq 1$, we have $\chi(\Sigma) \neq 0$ and so $s = 0$. Therefore $\pi_* u = 0$, which concludes the proof. □

Remark 2.8. Alternatively, we could have proved Lemma 2.7 more abstractly using a version of the Gysin sequence for twisted de Rham complexes; see [Bott and Tu 1982, p. 177] for more details.

We now compute the Euler characteristic of the twisted de Rham complex. This shows that, although the twisted Betti numbers, i.e., dimensions of $H^k_A(M; \mathcal{E})$ can jump by changing A, the Euler characteristic is independent of the choice of flat connection. We could not find an appropriate reference for this result.

Lemma 2.9. *The Euler characteristic of the chain complex* (2-11), *denoted by* $\chi_A(M; \mathcal{E})$, *is equal to*

$$\chi_A(M; \mathcal{E}) = \mathrm{rank}(\mathcal{E})\chi(M).$$

Proof. A way to prove this is given by an application of the Atiyah–Singer index theorem; we sketch the proof here. It starts by noting that, as with the usual nontwisted forms, we have

$$d_A + d_A^* : C^\infty(M; \Omega^{\mathrm{odd}} \otimes \mathcal{E}) \to C^\infty(M; \Omega^{\mathrm{even}} \otimes \mathcal{E}). \tag{2-16}$$

Here $\Omega^{\mathrm{even}} = \bigoplus_i \Omega^{2i}$ and $\Omega^{\mathrm{odd}} = \bigoplus_i \Omega^{2i+1}$ are the bundles of even and odd differential forms, respectively. Let us introduce the twisted Hodge laplacian, $\Delta_A = d_A^* d_A + d_A d_A^*$. By Hodge theory, we have $H^k_A(M; \mathcal{E}) \cong \ker \Delta_A|_{\Omega^k \otimes \mathcal{E}}$. Therefore, we also have $\mathrm{ind}(d_A + d_A^*) = \chi_A(M; \mathcal{E})$, where by ind we denote the index of an operator.

By the Atiyah–Singer index theorem,

$$\begin{aligned}
\mathrm{ind}(d_A + d_A^*) &= \int_{T^*M} \mathrm{ch}(\mathrm{d}(d_A + d_A^*))\mathcal{T}(TM) \\
&= \int_{T^*M} \mathrm{ch}(\mathcal{E})\,\mathrm{ch}(\mathrm{d}(d + d^*))\mathcal{T}(TM) \\
&= \mathrm{rank}(\mathcal{E}) \int_{T^*M} \mathrm{ch}(\mathrm{d}(d + d^*))\mathcal{T}(TM) = \mathrm{rank}(\mathcal{E})\chi(M).
\end{aligned} \tag{2-17}$$

Here, $\mathcal{T}$ denotes the Todd class and ch denotes the Chern character.[3] The letter d denotes the *difference bundle*. Since $(\mathcal{E}, A)$ is flat by assumption, we have $\mathrm{ch}(\mathcal{E}) = \mathrm{rank}(\mathcal{E})$. The transition to the second line is justified since the principal symbol of $d_A + d_A^*$ is equal to $\sigma(d + d^*) \otimes \mathrm{Id}_{\mathcal{E}}$, so that

$$\mathrm{d}(d_A + d_A^*) = \mathrm{d}(\sigma(d + d^*) \otimes \mathrm{Id}) = [G_1 \otimes \mathcal{E}] - [G_2 \otimes \mathcal{E}] = ([G_1] - [G_2]) \cdot [\mathcal{E}] \in K^{\mathrm{comp}}(T^*M).$$

Here G_1 and G_2 are certain vector bundles over a one-point compactification of T^*M and K^{comp} denotes the suitable K-theory. Since ch is multiplicative over the K-theory, we get the product of characters. The

[3] More explicitly, these are given for a vector bundle V over M with curvature two-form Ω and $w = -\Omega/(2\pi i)$, by $\mathrm{ch}(V) = \mathrm{tr}\exp w$ and $\mathcal{T}(V) = \det(w/(1 - \exp(-w)))$. Here we apply the Taylor series at zero to forms.

last equality follows from the Atiyah–Singer index theorem for the operator $d + d^* : \Omega^{\text{odd}} \to \Omega^{\text{even}}$ and the nontwisted Hodge theory. □

3. Meromorphic continuation of $\zeta_A(s)$

We devote this section to showing meromorphic continuation of $\zeta_A(s)$ given by (1-1) for an arbitrary (possibly nonflat, nonunitary) A. We note that the meromorphic continuation of the Ruelle zeta function was first established in [Giulietti et al. 2013] and later in [Dyatlov and Zworski 2016], and that here we follow the latter microlocal approach. Let (M, g) be a compact Riemannian manifold and $\mathcal{E}$ a Hermitian vector bundle over M equipped with a connection A and an endomorphism-valued function Φ. Also assume M admits an Anosov flow φ_t with generator X. We consider the first-order operator $P = -i\iota_X d_A + \Phi$.

Let us denote by $\alpha_{x,t}$ the parallel transport (with respect to P) in the fibres of $\mathcal{E}$ along integral curves of φ_t:

$$\alpha_{x,t} : \mathcal{E}(x) \to \mathcal{E}(\varphi_t(x)). \tag{3-1}$$

Recall now that the propagator e^{-itP} is the one-parameter family of operators, defined by solving the first-order PDE in (t, x) for $u \in C^\infty(M; \mathcal{E})$

$$\Big(\frac{\partial}{\partial t} + iP\Big)(e^{-itP}u) = 0. \tag{3-2}$$

Then the solution $u(t, x) = (e^{-itP}u)(t, x) \in C^\infty(\mathbb{R} \times M; \mathcal{E})$ (we pull back $\mathcal{E}$ to $\mathbb{R} \times M$) and we have

$$(e^{-itP}u)(t, x) = u(t, x) = \alpha_{\varphi_{-t}x,t}\, u(\varphi_{-t}x). \tag{3-3}$$

This follows by a computation in local coordinates. In fact, in a local coordinate system $U \ni x$ over which $\mathcal{E}|_U \cong U \times \mathbb{C}^m$ is trivial and for small t, we have

$$(\partial_t + A(\partial_t) + i\Phi(\varphi_t x))\alpha_{x,t} = 0. \tag{3-4}$$

We write A for the matrix of 1-forms associated to $d_A = d + A$ and identify $\alpha_{x,t}$ with a matrix. Then we may compute, using the chain rule,

$$\begin{aligned}
\partial_t u(t, x) &= -\big(A(X(x)) + i\Phi(x)\big)\alpha_{\varphi_{-t}x,t}u(\varphi_{-t}x) - (X\alpha)_{\varphi_{-t}x,t}u(\varphi_{-t}x) - \alpha_{\varphi_{-t}x,t}Xu(\varphi_{-t}x)\\
&= -iP(\alpha_{\varphi_{-t}x,t}u)(t, x) + X(\alpha_{\varphi_{-t}x,t}u)(t, x) - (X\alpha_{\varphi_{-t}x,t})u(\varphi_{-t}x) - \alpha_{\varphi_{-t}x,t}Xu(\varphi_{-t}x)\\
&= -iPu(t, x).
\end{aligned}$$

Here we used (3-4) in the first equality, the definition of P in the second and the chain rule in the last one. We thus obtain (3-3) for small t and by iteration we obtain it for all t. As a consequence, we obtain for any $f \in C^\infty(M)$ and $u \in C^\infty(M; \mathcal{E})$

$$e^{-itP}(fu) = f \circ \varphi_{-t} \cdot e^{-itP}u. \tag{3-5}$$

Denote by $\mathcal{P}_{x,t}$ the linearised Poincaré map for any time t and point $x \in M$:

$$\mathcal{P}_{x,t} = (d\varphi_t(x))^{-T} : \Omega^1_0(x) \to \Omega^1_0(\varphi_t x),$$

where, for $x \in M$ and $k \in \mathbb{N}$, we define the subbundle of differential forms in the kernel of ι_X by

$$\Omega_0^k = \Omega^k \cap \ker \iota_X.$$

We write $-T$ for the inverse transpose. Note $\mathcal{L}_X$ acts on sections of Ω_0^k for any k. Also, we have that φ_t^* is a one-parameter family of maps acting on Ω_0^k, for any k, such that we may write $(\varphi_t)^* = e^{t\mathcal{L}_X}$. So we obtain that, by the definition of φ_{-t}^* for any η a smooth k-form (see (3-3))

$$\textstyle\bigwedge^k \mathcal{P}_{x,t}(\eta(x)) = e^{-t\mathcal{L}_X}\eta(\varphi_t x). \tag{3-6}$$

Here $\bigwedge^k \mathcal{P}_{x,t}$ is the exterior product of maps acting on Ω_0^k. Given a closed orbit γ with period T, we consider a point $x_0 \in \gamma$ and define

$$\operatorname{tr}\alpha_\gamma := \operatorname{tr}\alpha_{x_0,T}.$$

Since the maps $\alpha_{\varphi_t x_0,T}$ are conjugate for varying t, the trace is independent of γ. Similarly, we define

$$\det(\mathrm{Id} - \mathcal{P}_\gamma) := \det(\mathrm{Id} - \mathcal{P}_{x_0,T}).$$

In what follows, for technical purposes we assume that we have a constant $\beta \in \mathbb{N}$ such that

$$|\det(\mathrm{Id} - \mathcal{P}_\gamma)| = (-1)^\beta \det(\mathrm{Id} - \mathcal{P}_\gamma). \tag{3-7}$$

This happens in particular if E_s and E_u are orientable, where $\beta = \dim E_s$. This assumption may be removed by using a suitable twist with an orientation bundle; see [Dyatlov and Guillarmou 2016; Dyatlov and Zworski 2016; Giulietti et al. 2013] for details.

We will denote by $\gamma^\#$ a general primitive periodic orbit, and if γ is an arbitrary periodic orbit, then $l_\gamma^\#$ will denote the period of the primitive periodic orbit corresponding to γ.

Theorem 3.1. *Define for* $\operatorname{Re} s \gg 1$

$$F_P(s) := \sum_{\gamma \in \mathcal{G}} \frac{e^{-sl_\gamma} l_\gamma^\# \operatorname{tr}\alpha_\gamma}{|\det(\mathrm{Id} - \mathcal{P}_\gamma)|}, \tag{3-8}$$

where the sum is over all periodic trajectories. Then $F_P(s)$ extends meromorphically to all $s \in \mathbb{C}$. The poles of $F_P(s)$ are precisely $s \in \mathbb{C}$, where is a Pollicott–Ruelle resonance of P. Moreover, the poles are simple with residues equal to the Pollicott–Ruelle multiplicity $m_P(is)$.

Proof. We give only a sketch of the proof here, as it follows from [Dyatlov and Zworski 2016]. The sum (3-8) converges by [loc. cit., Lemma 2.2] and as $\|\alpha_\gamma\| \le Ce^{Cl_\gamma}$ for some $C > 0$. Observe that by (3-3), we have that the Schwartz kernel K of the propagator e^{-itP}, as a distribution $K(t, y, x) \in \mathcal{D}'(\mathbb{R} \times M \times M)$, satisfies $\mathrm{WF}(K) \subset N^*S$, where $S = \{(t, \varphi_t(x), x) : x \in M,\ t \in \mathbb{R}\}$ and N^*S denotes the conormal bundle of S. Therefore, Guillemin's trace formula [loc. cit., Appendix B] applies to give, for $t > 0$,

$$\operatorname{tr}^\flat e^{-itP}|_{C^\infty(M;\mathcal{E})} = \sum_{\gamma \in \mathcal{G}} \frac{l_\gamma^\# \operatorname{tr}\alpha_\gamma \delta(t - l_\gamma)}{|\det(\mathrm{Id} - \mathcal{P}_\gamma)|}.$$

All that is left to do is to note that the remainder of the proof in [loc. cit., Section 4] is not sensitive to changing φ_{-t}^* to a general propagator e^{-itP} for P as above. This completes the proof.

Alternatively, the whole statement follows from more general work [Dyatlov and Guillarmou 2016, Theorem 4] on open systems. □

We now prove the meromorphic extension of the zeta function using the meromorphic continuation of the trace above.

Proposition 3.2. *The zeta function $\zeta_A(s)$ is given by*

$$\zeta_A(s) = \prod_{\gamma^\#} \det\left(\mathrm{Id} - \alpha_{\gamma^\#} e^{-sl_\gamma^\#}\right) \tag{3-9}$$

for large $\operatorname{Re} s$ *and holomorphic in that region. Moreover, it has a meromorphic extension to the whole of* $\mathbb{C}$ *and the poles and zeros of the extension are determined by Pollicott–Ruelle resonances of* $P = -i\iota_X d_A + \Phi$ *acting on differential forms with values in* $\mathcal{E}$.

Proof. We follow the now standard procedure of writing $\log \zeta_A$ as an alternating sum of traces of maps between bundles of differential forms with values in a vector bundle; see [Dyatlov and Zworski 2016, equation (2.5)], originally due to [Ruelle 1976]. We write for large $\operatorname{Re} s$

$$\begin{aligned}
\log \zeta_A(s) &= \sum_{\gamma^\#} \log\det(\mathrm{Id} - \alpha_{\gamma^\#} e^{-sl_\gamma^\#}) = \sum_{\gamma^\#} \operatorname{tr}\log(\mathrm{Id} - \alpha_{\gamma^\#} e^{-sl_\gamma^\#}) \\
&= -\sum_{\gamma^\#, j} \frac{\operatorname{tr}(\alpha_{\gamma^\#}^j) e^{-jsl_\gamma^\#}}{j} = -\sum_\gamma \operatorname{tr}(\alpha_\gamma) e^{-sl_\gamma} \frac{l_\gamma^\#}{l_\gamma} \\
&= \sum_{k=0}^{n-1} (-1)^{k+\beta+1} \sum_\gamma \frac{\operatorname{tr}\left(\bigwedge^k \mathcal{P}_\gamma\right) \operatorname{tr}(\alpha_\gamma) e^{-sl_\gamma}}{|\det(\mathrm{Id} - \mathcal{P}_\gamma)|} \frac{l_\gamma^\#}{l_\gamma} = \sum_{k=0}^{n-1} (-1)^{k+\beta} g_k(s).
\end{aligned} \tag{3-10}$$

We used the formula $\log\det(\mathrm{Id} + A) = \operatorname{tr}\log(\mathrm{Id} + A)$, which works for $\|A\|$ small enough, the fact that there is a $C > 0$ such that $\|\alpha_\gamma\| \leq Ce^{Cl_\gamma}$ and [Dyatlov and Zworski 2016, Lemma 2.2]. The function g_k is defined as

$$g_k(s) = -\sum_\gamma \frac{\operatorname{tr}\left(\bigwedge^k \mathcal{P}_\gamma\right) \operatorname{tr}(\alpha_\gamma) e^{-sl_\gamma}}{|\det(\mathrm{Id} - \mathcal{P}_\gamma)|} \frac{l_\gamma^\#}{l_\gamma}.$$

Also, we used the identity

$$\det(\mathrm{Id} - \mathcal{P}_\gamma) = \sum_{k=0}^{n-1} (-1)^k \operatorname{tr}\left(\bigwedge^k \mathcal{P}_\gamma\right),$$

which comes from linear algebra. Introduce then

$$F_k(s) := -g_k'(s) = -\sum_\gamma \frac{\operatorname{tr}\left(\bigwedge^k \mathcal{P}_\gamma\right) \operatorname{tr}(\alpha_\gamma) e^{-sl_\gamma} l_\gamma^\#}{|\det(\mathrm{Id} - \mathcal{P}_\gamma)|}. \tag{3-11}$$

This is reminiscent of (3-8). In fact, consider the vector bundle $\mathcal{E}_k := \Omega_0^k \otimes \mathcal{E}$. We extend the action of P on $\mathcal{E}$ to the action on $\mathcal{E}_k$ by the Leibnitz rule and denote the associated first-order differential operator by P_k. We have, for $w \in C^\infty(M; \Omega_0^k)$ and $s \in C^\infty(M; \mathcal{E})$,

$$P_k(s \otimes w) = (-i\iota_X d_A + \Phi)(s \otimes w) = Ps \otimes w + s \otimes (-i\mathcal{L}_X w). \tag{3-12}$$

Then we observe that, by using (3-12),

$$(\partial_t + iP_k)(e^{-itP} s \otimes e^{-t\mathcal{L}_X} w) = 0. \tag{3-13}$$

Introduce the parallel transport $\beta_{k,x,t} : \mathcal{E}_k(x) \to \mathcal{E}_k(\varphi_t x)$ along the fibres of $\mathcal{E}_k$. Then by (3-3), (3-6) and (3-13)

$$\begin{aligned}\beta_{k,x,t}(s(x) \otimes w(x)) &= e^{-itP_k}(s \otimes w)(\varphi_t x)\\ &= e^{-itP} s(\varphi_t x) \otimes e^{-t\mathcal{L}_X} w(\varphi_t x) = \alpha_{x,t}(s(x)) \otimes \textstyle\bigwedge^k \mathcal{P}_{x,t}(w(x)).\end{aligned} \tag{3-14}$$

We claim that for $k = 0, 1, \ldots, n-1$

$$F_{P_k}(s) = F_k(s).$$

To see this, observe that along a periodic orbit γ of period l_γ by (3-14) we have

$$\operatorname{tr}(\beta_{k,\gamma}) = \operatorname{tr}\big(\alpha_\gamma \otimes \textstyle\bigwedge^k \mathcal{P}_\gamma\big) = \operatorname{tr}(\alpha_\gamma) \cdot \operatorname{tr}\big(\textstyle\bigwedge^k \mathcal{P}_\gamma\big).$$

Here we write $\beta_{k,\gamma} = \beta_{k,x_0,l_\gamma}$, where x_0 is any point on γ. The trace $\operatorname{tr}\beta_{k,\gamma}$ is independent of x_0. This proves the claim.

By Theorem 3.1 and an elementary argument, for each k there exists a holomorphic function $\zeta_{k,A}(s)$ such that

$$\frac{\zeta_{k,A}'}{\zeta_{k,A}} = -F_k(s) = g_k'(s).$$

Thus by (3-10) we obtain the factorisation

$$\zeta_A(s) = \prod_{k=0}^{n-1} \zeta_{k,A}^{(-1)^{k+\beta}}(s). \tag{3-15}$$

By Theorem 3.1, $s \in \mathbb{C}$ is a zero of $\zeta_{k,A}(s)$ precisely when is is a Pollicott–Ruelle resonance of P_k and the multiplicity of the zero is equal to the Pollicott–Ruelle multiplicity at is. □

For convenience we restate the factorisation above for 3-manifolds.

Corollary 3.3. *Consider a closed* 3*-manifold* (M, g) *with an Anosov flow* X. *Let* $\mathcal{E}$ *be a vector bundle over* M *equipped with a connection* A *and a potential* Φ. *Then, assuming* E_s *is orientable, we have the factorisation, where* $\zeta_{k,A}$ *is entire for* $k = 0, 1, 2$,

$$\zeta_A(s) = \frac{\zeta_{1,A}(s)}{\zeta_{0,A}(s)\zeta_{2,A}(s)}. \tag{3-16}$$

Moreover, the order of zero at a point s *of* $\zeta_A(s)$ *is equal to*

$$m_{P_1}(is) - m_{P_0}(is) - m_{P_2}(is), \tag{3-17}$$

where $m_{P_k}(is)$ *denotes the Pollicott–Ruelle resonance multiplicity at* is *of the operators* $P_k = -i\iota_X d_A + \Phi$ *acting on sections of the vector bundle* $\mathcal{E}_k = \Omega_0^k(M) \otimes \mathcal{E}$ *for* $k = 0, 1, 2$.

4. Resonant spaces

In this section we prove:

Theorem 4.1. *Let* (M, Ω) *be a closed* 3*-manifold with volume form* Ω *and let* φ_t *be a volume-preserving Anosov flow. Let* $\mathcal{E}$ *be a Hermitian vector bundle equipped with a unitary flat connection* A. *Then*:

(1) $\dim \mathrm{Res}_{0,A}(0) = \dim \mathrm{Res}_{2,A}(0) = b_0(M, \mathcal{E})$.

(2) *If* $[\omega] \neq 0$, *then* $\dim \mathrm{Res}_{1,A}(0) = b_1(M, \mathcal{E}) - b_0(M, \mathcal{E})$.

(3) *If* $[\omega] = 0$, *then*

$$\dim \mathrm{Res}_{1,A}(0) = \begin{cases} b_1(M, \mathcal{E}) & \text{if } \mathcal{H}(X) \neq 0, \\ b_1(M, \mathcal{E}) + b_0(M, \mathcal{E}) & \text{if } \mathcal{H}(X) = 0. \end{cases}$$

Moreover, k*-semisimplicity holds for* $k = 0, 2$.

In particular, as a consequence we obtain:

Proof of Theorem 1.2. This is a direct consequence of Theorem 4.1 applied to trivial bundle $\mathcal{E} = M \times \mathbb{C}$ and the trivial connection $d_A = d$. □

We break down the proof of Theorem 4.1 into the following subsections.

4A. ***Smooth invariant 1-forms.*** We first show that smooth resonant 1-forms are zero. The idea is that an invariant 1-form decays along the stable direction in the future and in the unstable direction in the past and so must vanish. This first subsection is quite general and holds in any dimension for any unitary connection A and Hermitian matrix field Φ. Recall that $\Omega_0^k = \Omega^k \cap \ker \iota_X$.

Lemma 4.2. *We have*

$$\mathrm{Res}_{1,A,\Phi}(0) \cap C^\infty(M; \Omega_0^1 \otimes \mathcal{E}) = \{0\}. \tag{4-1}$$

Proof. We start by proving the following formula, which holds for any $u \in C^\infty(M; \Omega^k \otimes \mathcal{E})$:

$$\alpha_{x,t}(u_x(\xi^k)) = e^{-t(\iota_X d_A + i\Phi)} u_{\varphi_t x}\big(\big(\textstyle\bigwedge^k d\varphi_t\big)\xi^k\big). \tag{4-2}$$

Here $\xi^k \in \Lambda_x^k M$ is a k-vector and x is any point in M. The definitions of $\alpha_{x,t}$ are given in (3-1) and (3-3).

Note firstly that it suffices to prove the claim above for $u = s \otimes w$, where w is a k-form and s is a section of $\mathcal{E}$, since we can write u as a sum of such terms near x and a term which is zero close to x. But this follows from (3-14) and by the definition of the map $\mathcal{P}_{x,t}$.

If $u \in \mathrm{Res}_{1,A,\Phi}(0) \cap C^\infty(M; \Omega_0^1 \otimes \mathcal{E})$ we must have $(-i\iota_X d_A + \Phi)u = 0$ and $\iota_X u = 0$. This further implies $e^{-t(\iota_X d_A + i\Phi)} u = u$, since $(\partial_t + \iota_X d_A + i\Phi)u = 0$. Then by (4-2) for $k = 1$ and $\xi \in E_s(x)$

$$|u_x(\xi)| = |\alpha_{x,t} u_x(\xi)| = |u_{\varphi_t x}(d\varphi_t \xi)| \lesssim |d\varphi_t \xi|_g \lesssim e^{-\lambda t}, \quad t > 0. \tag{4-3}$$

Here we used that $\alpha_{x,t}$ is a unitary isomorphism[4] the Anosov property of X and that $t > 0$ in the last inequality. By taking the limit $t \to \infty$, we get u is zero in the direction of E_s. Similarly, we get that u is zero in the direction of E_u, so u is zero. □

Remark 4.3. The above method shows that for an arbitrary smooth k-form $u \in \mathrm{Res}_{k,A,\Phi}(0)$, we have $u|_{\bigwedge^k E_u} = 0$ and $u|_{\bigwedge^k E_s} = 0$, and more generally one could compare rates of contraction and expansion to obtain vanishing on larger subspaces. Other components can be nonzero, as can be seen, e.g., below from the computation for $\mathrm{Res}_{2,A}(0)$ for A flat.

4B. $\mathrm{Res}_{0,A}(0)$ *and* $\mathrm{Res}_{2,A}(0)$.

Recall that $\omega = i_X\Omega$ and assume from now on that A is flat.

Lemma 4.4. *We have*

$$\mathrm{Res}_{0,A}(0) = \{s \in C^\infty(M; \mathcal{E}) : d_A s = 0\} = H^0_A(M, \mathcal{E}), \tag{4-4}$$

$$\mathrm{Res}_{2,A}(0) = \{s\,\omega : s \in C^\infty(M; \mathcal{E}),\ d_A s = 0\}. \tag{4-5}$$

Moreover, k-semisimplicity holds for $k = 0, 2$.

Proof. We distinguish the cases $k = 0$ or 2.

Case $k = 0$: If $s \in \mathrm{Res}_{0,A}(0)$, then $s \in C^\infty(M; \mathcal{E})$ by Lemma 2.5. Since A is flat, $d_A^2 s = 0$ and therefore $d_A s \in \mathrm{Res}_{1,A}(0) \cap C^\infty(M, \Omega^1_0 \otimes \mathcal{E})$ and by Lemma 4.2 we have $d_A s = 0$. So in this case we get a bijection with the parallel sections of $\mathcal{E}$.

For semisimplicity, consider $s \in \mathcal{D}'_{E_u^*}(M; \mathcal{E})$ with $\iota_X d_A s =: v \in \mathrm{Res}_{0,A}(0)$. Then $v \in C^\infty(M; \mathcal{E})$ by Lemma 2.5 and v is parallel by the previous paragraph. For $u \in C^\infty(M; \mathcal{E})$ parallel, since d_A is unitary, we have

$$\int_M \langle \iota_X d_A s, u\rangle_{\mathcal{E}}\, \Omega = \int_M X\langle s, u\rangle_{\mathcal{E}}\, \Omega = 0. \tag{4-6}$$

By picking $u = v$, we get $v = 0$ and so $s \in \mathrm{Res}_{0,A}(0)$.

Case $k = 2$: For $u \in \mathrm{Res}_{2,A}(0)$, we may write $u = s\omega$ for some distributional section $s \in \mathcal{D}'_{E_u^*}(M; \mathcal{E})$. Then $\iota_X d_A u = 0$ implies $\iota_X d_A s = 0$, as $\mathcal{L}_X \Omega = d\omega = 0$. By the analysis of $\mathrm{Res}_{0,A}(0)$, we immediately get that s is parallel.

For semisimplicity, assume $\iota_X d_A u = v \in \mathrm{Res}_{2,A}(0)$ with $u \in \mathcal{D}'_{E_u^*}(M; \Omega^2_0 \otimes \mathcal{E})$. So $u = s\omega$ for some $s \in \mathcal{D}'_{E_u^*}(M; \mathcal{E})$ and $v = s'\omega$ with s' smooth and parallel. Therefore $s' = \iota_X\, d_A s \in \mathrm{Res}_{0,A}(0)$ and by semisimplicity in the $k = 0$ case, we obtain $s' = 0$. □

Remark 4.5. In the proof of Lemma 4.4, the fact that $J(0) = 1$ in the case $k = 0$ also holds for A nonflat and unitary. To see this, consider the spectral theoretic inequality, which holds for $\varphi \in C^\infty(M; \mathcal{E})$,

$$\|(P - \lambda)\varphi\|_{L^2} \cdot \|\varphi\|_{L^2} \geq |\operatorname{Im}\langle (P - \lambda)\varphi, \varphi\rangle_{L^2}| = |\operatorname{Im}\lambda|\,\|\varphi\|^2_{L^2}. \tag{4-7}$$

[4]This can be shown as follows. Fix $x \in M$ and take two parallel sections u_1 and u_2 of $\mathcal{E}$ along the orbit $\{\varphi_t x : t \in \mathbb{R}\}$, solving locally in some trivialisation $(\partial_t + A(\partial_t) + i\Phi)u_j = 0$ for $j = 1, 2$. Then $\partial_t \langle u_1, u_2\rangle_{\mathcal{E}(\varphi_t x)} = \langle (\partial_t + A(\partial_t))u_1, u_2\rangle + \langle u_1, (\partial_t + A(\partial_t))u_2\rangle = -i\langle \Phi u_1, u_2\rangle + i\langle u_1, \Phi u_2\rangle = 0$, as d_A is unitary and Φ is Hermitian. Therefore the parallel transport preserves inner products and $\alpha_{x,t}$ is unitary.

Here we used that $P = P^*$ on L^2. Therefore $\|R(\lambda)\|_{L^2\to L^2} \le 1/|\operatorname{Im}\lambda|$ for $\operatorname{Im}\lambda > 0$, which implies $J(0) = 1$.

4C. $\mathbf{Res}_{1,A}(0)$. Recall that $H^0_A(M;\mathcal{E})$ is the space of parallel sections (i.e., smooth sections s of $\mathcal{E}$ such that $d_A s = 0$). We start with a solvability result along the lines of [Dyatlov and Zworski 2017, Proposition 3.3.].

Proposition 4.6. *Assume X preserves a smooth volume form Ω and A is unitary and flat. Let $f \in C^\infty(M;\mathcal{E})$ and assume $\int_M \langle f, s\rangle_{\mathcal{E}}\Omega = 0$ for all $s \in C^\infty(M;\mathcal{E})$ parallel. Then there exists $u \in \mathcal{D}'_{E_u^*}(M;\mathcal{E})$ such that $\iota_X d_A u = f$.*

Proof. Let us set $P = -i\iota_X\, d_A$. By Lemma 4.4 we have the 0-semisimplicity and so $J(0) = 1$. Thus by (2-4) near zero, where $\Pi = \Pi_0$,

$$R(\lambda) = R_H(\lambda) - \frac{\Pi}{\lambda}.$$

Therefore, by applying $P - \lambda$ to this equation we obtain close to zero

$$(P-\lambda)R_H(\lambda) + \Pi_0 = \mathrm{Id}. \tag{4-8}$$

We introduce $u := -iR_H(0)f$, which lies in $\mathcal{D}'_{E_u^*}(M;\mathcal{E})$ by the mapping properties of $R_H(\lambda)$ in (2-4). Then, assuming $\Pi_0 f = 0$ we have by (4-8), evaluated at $\lambda = 0$,

$$f = f - \Pi_0 f = PR_H(0)f = (iP)(-iR_H(0)f) = \iota_X d_A u.$$

Now we prove that $\Pi_0 f = 0$. By Lemmas 2.1 and 4.4, we get

$$\operatorname{ran}(\Pi_0) = \ker(P|_{\mathcal{D}'_{E_u^*}(M;\mathcal{E})}) = \operatorname{Res}_{0,A}(0) = H^0_A(M;\mathcal{E}).$$

Since X is volume-preserving and A is unitary, we have $P^* = P$. Therefore $\operatorname{ran}\Pi_0' = H^0_A(M;\mathcal{E})$ analogously, where Π_0' denotes the spectral projector of $-P$ with respect to the flow $-X$. Now Lemma 2.3 gives $\Pi_0^* = \Pi_0'$ and so for any $g \in C^\infty(M;\mathcal{E})$

$$\langle \Pi_0 f, g\rangle_{L^2} = \langle f, \Pi_0^* g\rangle_{L^2} = 0.$$

Thus $\Pi_0 f = 0$, which concludes the proof. □

We proceed with:

Lemma 4.7. *There is a linear map $T : \operatorname{Res}_{1,A}(0) \to H^0_A(M;\mathcal{E})$ such that $d_A u = T(u)\omega$, where $u \in \operatorname{Res}_{1,A}(0)$. The map T satisfies the following*:

(1) *If $[\omega] \neq 0$ or $\mathcal{H}(X) \neq 0$, then T is trivial.*

(2) *If $\mathcal{H}(X) = 0$, then T is surjective.*

Proof. Let $u \in \operatorname{Res}_{1,A}(0)$. Since A is flat, $d_A^2 = 0$ and hence $d_A u \in \operatorname{Res}_{2,A}(0)$ and so $d_A u = s\omega$ with s parallel and smooth, by Lemma 4.4. If we set $T(u) = s$, this defines a linear map such that $d_A u = T(u)\omega$.

Next note that given parallel sections $p, q \in H^0_A(M; \mathcal{E})$, the inner product $\langle q, p\rangle_{\mathcal{E}}$ is a constant function on M. By Lemma 2.6 there is a smooth v such that $d_A u = d_A v$. We write

$$d\langle T(u), v\rangle_{\mathcal{E}} = \langle T(u), d_A v\rangle_{\mathcal{E}} = \|T(u)\|^2 \omega$$

and observe that the left-hand side is exact. Hence we must have $T \equiv 0$ if $[\omega] \neq 0$.

If $[\omega] = 0$, we set $\omega = d\tau$ and thus

$$d_A(u - T(u)\tau) = 0.$$

Using Lemma 2.6, we can write $u - T(u)\tau = \eta + d_A F$, where η is a smooth 1-form with $d_A\eta = 0$ and $F \in \mathcal{D}'_{E_u^*}(M; \mathcal{E})$. Contracting with X and taking (pointwise) the inner product with $T(u)$ we derive

$$-\|T(u)\|^2 \tau(X) = \varphi(X) + X\langle T(u), F\rangle_{\mathcal{E}}, \tag{4-9}$$

where φ is the smooth, closed 1-form $\varphi := \langle T(u), \eta\rangle$. But note that

$$\int_M \varphi(X)\Omega = \int_M \varphi \wedge d\tau = -\int_M d(\varphi \wedge \tau) = 0.$$

Hence integrating (4-9) yields

$$-\|T(u)\|^2 \mathcal{H}(X) = 0$$

and therefore $T \equiv 0$ if $\mathcal{H}(X) \neq 0$, thus showing item (1) in the lemma.

To show item (2) assume $\mathcal{H}(X) = 0$ and let s be a parallel section. We shall show that there is $u \in \mathrm{Res}_{1,A}(0)$ with $T(u) = s$. Note that for any parallel section p

$$\int_M \langle s\tau(X), p\rangle_{\mathcal{E}}\, \Omega = \langle s, p\rangle_{\mathcal{E}}\, \mathcal{H}(X) = 0.$$

By Proposition 4.6 there is an $F \in \mathcal{D}'_{E_u^*}(M; \mathcal{E})$ such that $\iota_X d_A F = s\tau(X)$ and hence $u := s\tau - d_A F \in \mathrm{Res}_{1,A}(0)$ and $T(u) = s$ as desired. ☐

Lemma 4.8. *There is an injection*

$$\ker T \hookrightarrow H^1_A(M; \mathcal{E}). \tag{4-10}$$

The injection can be described as follows: Let $u \in \ker T$. Then there exists $F \in \mathcal{D}'_{E_u^*}(M; \mathcal{E})$ such that

$$u - d_A F \in C^\infty(M; \mathcal{E} \otimes \Omega^1) \tag{4-11}$$

and also $d_A(u - d_A F) = 0$. The injection map is given by

$$S : u \in \ker T \mapsto [u - d_A F] \in H^1_A(M; \mathcal{E}). \tag{4-12}$$

An element $[\eta] \in H^1_A(M; \mathcal{E})$ is in the image of S if and only if

$$\int_M \langle p, \eta(X)\rangle_{\mathcal{E}}\, \Omega = 0$$

for any parallel section p.

Proof. Let $u \in \ker T$, so that $d_A u = 0$. By Lemma 2.6 there is $F \in \mathcal{D}'_{E_u^*}(M;\mathcal{E})$ such that $u - d_A F \in C^\infty(M;\Omega^1 \otimes \mathcal{E})$. We claim that the class $[u - d_A F] \in H^1_A(M;\mathcal{E})$ is independent of our choice of F. Suppose there is a G such that $u - d_A G$ is smooth and d_A-closed. Then $d_A(F-G) \in C^\infty(M;\Omega^1 \otimes \mathcal{E})$, so by Lemma 2.6 (or ellipticity), $F - G$ is smooth and thus $u - d_A F$ and $u - d_A G$ belong to the same class.

For injectivity, we assume that $u - d_A F$ is exact; so without loss of generality assume $u = d_A F$. Then $\iota_X u = 0$ implies $d_A F(X) = 0$, so by Lemma 4.4 we have F smooth and parallel, so $u = 0$.

If $[\eta]$ is in the image of S, then $\eta = u - d_A F$ for some $F \in \mathcal{D}'_{E_u^*}(M;\mathcal{E})$. Contracting with X, we see that $\eta(X) = -d_A F(X)$ and hence $\langle p, \eta(X)\rangle_{\mathcal{E}} = -X\langle p, F\rangle_{\mathcal{E}}$. Integrating gives

$$\int_M \langle p, \eta(X)\rangle_{\mathcal{E}}\, \Omega = 0.$$

Conversely, if the last integral is zero for all p, Proposition 4.6 gives $F \in \mathcal{D}'_{E_u^*}(M;\mathcal{E})$ such that $-\eta(X) = d_A F(X)$ and $u := \eta + d_A F \in \ker T$ and $Su = [\eta]$. □

And finally we can compute the rank of S in terms of whether X is null-homologous or not.

Lemma 4.9. *We have*:

(1) $\dim S(\ker T) = b_1(M,\mathcal{E})$ *if* $[\omega] = 0$.

(2) $\dim S(\ker T) = b_1(M,\mathcal{E}) - b_0(M,\mathcal{E})$ *if* $[\omega] \neq 0$.

Proof. If X is null-homologous, we write $\omega = d\tau$. We use Lemma 4.8 to show that S is surjective. Consider $\eta \in H^1_A(M;\mathcal{E})$ and $p \in H^0_A(M;\mathcal{E})$. Since the 1-form $\varphi := \langle p, \eta\rangle$ is closed we have

$$\int_M \varphi(X)\,\Omega = \int_M \varphi \wedge d\tau = -\int_M d(\varphi \wedge \tau) = 0$$

and item (1) follows.

Suppose now $[\omega] \neq 0$. We define a map $W : H^1_A(M,\mathcal{E}) \to (H^0_A(M,\mathcal{E}))^*$ by

$$W([\eta])(p) := \int_M \langle p, \eta(X)\rangle_{\mathcal{E}}\, \Omega.$$

By Lemma 4.8 the image of S coincides with the kernel of W. Thus, to prove item (2) it suffices to show that W is surjective. By Poincaré duality there is a closed 1-form φ such that

$$\int_M \varphi \wedge \omega \neq 0.$$

If p and q are parallel sections we compute

$$W([q\varphi])(p) = \langle p, q\rangle_{\mathcal{E}} \int_M \varphi(X)\,\Omega = \langle p, q\rangle_{\mathcal{E}} \int_M \varphi \wedge \omega$$

and hence W is onto. □

We are now in shape to put the ingredients together and prove:

Proof of Theorem 4.1. The theorem follows directly after applying Lemmas 4.7 and 4.9. □

Putting together the material from this section and Section 3 we obtain:

Proof of Corollary 1.8. The order of vanishing of $\zeta(s)$ is equal to $m_1(0)-m_0(0)-m_2(0)$ by Corollary 3.3. By Theorem 4.1 we have that $m_0(0)=m_2(0)=b_0(M,\mathcal{E})$ and $m_1(0)\geq \dim \mathrm{Res}_{1,A}(0)$, which concludes the proof. □

Moreover, we obtain:

Proof of Corollary 1.6. This is a direct consequence of Corollary 1.8 applied to the case $\mathcal{E}=M\times\mathbb{C}$ and the trivial connection $d_A=d$. □

5. Examples

In this section we consider a few noncontact examples of Anosov flows on the unit tangent bundle of a surface. They illustrate the various cases in Theorem 1.2 and give specific deformations for Theorem 1.5.

5A. ***Structural equations.*** As a general reference for structural equations, see [Singer and Thorpe 1967, Chapter 7]. For this section assume (Σ, g) is a compact oriented negatively curved surface. Let X be the geodesic vector field on the unit sphere bundle $S\Sigma$. Denote by $\pi: S\Sigma\to\Sigma$ the footpoint projection. Then, there are 1-forms α, β and ψ on $S\Sigma$ defined by, for $\xi\in T^*_{(x,v)}S\Sigma$,

$$\begin{aligned}\alpha_{(x,v)}(\xi)&=\langle v, d\pi(\xi)\rangle_x,\\ \beta_{(x,v)}(\xi)&=\langle d\pi(\xi), iv\rangle_x,\\ \psi_{(x,v)}(\xi)&=\langle\mathcal{K}(\xi), iv\rangle_x.\end{aligned}\tag{5-1}$$

The 1-form α is called the contact form. From the defining equation one obtains $\iota_X\alpha=0$ and $\iota_X\,d\alpha=0$, and $\Omega=-\alpha\wedge d\alpha$ is a volume form. Also, here $\mathcal{K}: TT\Sigma\to T\Sigma$ is the *connection map*, i.e., the projection along the horizontal subbundle, and ψ is called the *connection* 1*-form*. The expression iv denotes the vector v rotated by an angle of $\frac{\pi}{2}$ (we fix an orientation). Explicitly,

$$\mathcal{K}_{(x,v)}(\xi):=\frac{DZ}{dt}(0)\in T_x\Sigma,\tag{5-2}$$

where $(\gamma(t), Z(t))$ is an arbitrary local curve in $T\Sigma$ with the initial data $(\gamma(0), Z(0))=(x,v)$ and $(\dot\gamma(0),\dot Z(0))=\xi$; $\frac{D}{dt}$ denotes the Levi-Civita derivative along the curve. One can then show that $\{\alpha,\beta,\psi\}$ form a coframe on $S\Sigma$ such that the following *structural equations* (see [Singer and Thorpe 1967, p. 188]) hold:

$$\begin{aligned}d\alpha&=\psi\wedge\beta,\\ d\beta&=-\psi\wedge\alpha,\\ d\psi&=-K\alpha\wedge\beta.\end{aligned}\tag{5-3}$$

From this, we deduce the following properties

$$\iota_X\beta=\iota_X\psi=0,\quad \iota_X\,d\beta=\psi,\quad \iota_X\,d\psi=-K\beta.\tag{5-4}$$

Furthermore, there is a natural choice of metric on $S\Sigma$, called the *Sasaki metric*. It is defined by the splitting

$$T_{(x,v)}S\Sigma=\mathbb{H}(x,v)\oplus\mathbb{V}(x,v)=\ker(\mathcal{K}(x,v)|_{S\Sigma})\oplus\ker(d\pi(x,v))$$

into *horizontal* and *vertical* subspaces, respectively. Then the new metric is defined as

$$\langle\langle \xi, \eta \rangle\rangle := \langle \mathcal{K}(\xi), \mathcal{K}(\eta) \rangle + \langle d\pi(\xi), d\pi(\eta) \rangle. \tag{5-5}$$

It follows after some checking from relations (5-3) and the definitions that $\{\alpha, \beta, \psi\}$ is an orthonormal coframe for $T^*S\Sigma$ with respect to the Sasaki metric. This also yields an orthonormal dual frame $\{X, H, V\}$. We record the structural equations (5-3) for these vector fields:

$$\begin{aligned} [H, V] &= X, \\ [V, X] &= H, \\ [X, H] &= KV. \end{aligned} \tag{5-6}$$

Here V is the generator of rotations in the vertical fibres.

We now use the Hodge star operator $*$ with respect to the Sasaki metric on $S\Sigma$ to write $\mathcal{L}_X^* = - * \mathcal{L}_X *$ on 1-forms. We also have an extra structure given by

$$\alpha \wedge Ju = *u \tag{5-7}$$

for u a section of Ω_0^1. Here $J : \Omega_0^1 \to \Omega_0^1$ is the (dual) almost-complex structure associated to the symplectic form $d\alpha$ on $\ker \alpha = \operatorname{span}\{V, H\}$ and is given by

$$J(u_2\beta + u_3\psi) = u_3\beta - u_2\psi, \quad J^2 = -\mathrm{Id}.$$

Therefore $(\mathcal{L}_X^*)^k u = 0$ for some $k \in \mathbb{N}$ is equivalent to $\mathcal{L}_X^k Ju = 0$ and we obtain

$$\operatorname{Res}_{-i\mathcal{L}_X^*, \Omega_0^1}(0) = J^{-1} \operatorname{Res}_{i\mathcal{L}_X, \Omega_0^1}(0). \tag{5-8}$$

In the next section we use this relation together with time changes to derive an explicit expression for coresonant states at zero.

5B. ***Time-reversal and resonant spaces.*** Here we consider the action under pullback of the time-reversal map $R : S\Sigma \to S\Sigma$, given by $R(x, v) = (x, -v)$. We first collect the information on this action on the orthonormal frames and coframes given in (5-3) and (5-6).

Proposition 5.1. *We have $R^*\alpha = -\alpha$, $R^*\beta = -\beta$ and $R^*\psi = \psi$. Similarly, we have $R^*X = -X$, $R^*H = -H$ and $R^*V = V$.*

Proof. We consider the coframe case first. Simply observe that

$$R^*\alpha_{(x,v)}(\xi) = \langle -v, d\pi\, dR\xi \rangle_x = -\alpha_{(x,v)}(\xi)$$

so $R^*\alpha = -\alpha$. Similarly

$$R^*\beta_{(x,v)}(\xi) = \langle -iv, d\pi\, dR\xi \rangle_x = -\beta_{(x,v)}(\xi)$$

so $R^*\beta = -\beta$. Finally, recall that $\mathcal{K}(\xi) = \frac{DZ}{dt}(0)$, where $c(t) = (\gamma(t), Z(t))$ is any curve in $T\Sigma$ with $\dot{c}(0) = \xi$. Therefore

$$\mathcal{K}(dR\xi) = -\frac{DZ}{dt}(0) = -\mathcal{K}(\xi)$$

since $\tilde{c}(t) = (\gamma(t), -Z(t))$ is the curve adapted to $-dR\xi$. Now we easily see that $R^*\psi = \psi$ from the definition.

The frame case follows from the coframe case, since contractions commute with pullbacks. □

Now note that in any unit sphere bundle SN over an Anosov manifold (N, g_1), the pullback by R swaps the stable and unstable bundles. More precisely, we have

$$R^* E^X_{u,s} = E^{R^*X}_{u,s} = E^{-X}_{u,s} = E_{s,u}, \quad R^* E_0 = E_0.$$

The upper index denotes the vector field with respect to which we are taking the stable/unstable bundles. This follows from the fact that R intertwines the flows of X and $-X$. Thus we also have

$$R^* E^*_{u,s} = E^*_{s,u}, \quad R^* E^*_0 = E^*_0.$$

The upshot is of course that R^* is an isomorphism between resonant and coresonant spaces, i.e., the ones with the wavefront set in E^*_u and in E^*_s.

Proposition 5.2. *The pairing* (2-9) *between resonant and coresonant states is equivalent to the pairing on*

$$\operatorname{Res}_{-i\mathcal{L}_X,\Omega^1_0}(0) \times \operatorname{Res}_{-i\mathcal{L}_X,\Omega^1_0}(0), \quad (u, v) := \int_{S\Sigma} u \wedge \alpha \wedge R^*\bar{v}. \tag{5-9}$$

The pairing (5-9) *is Hermitian* (*i.e.*, *conjugate symmetric*).

Proof. We first claim that

$$\operatorname{Res}_{-i\mathcal{L}^*_X,\Omega^1_0}(0) = J^{-1} R^* \operatorname{Res}_{-i\mathcal{L}_X,\Omega^1_0}(0). \tag{5-10}$$

This is obtained from (5-8) and by observing that $v \in \operatorname{Res}_{i\mathcal{L}_X,\Omega^1_0}(0)$ if and only if $R^*v \in \operatorname{Res}_{-i\mathcal{L}_X,\Omega^1_0}(0)$, since R^* commutes with ι_X and d, and as R^* swaps E^*_u and E^*_s by the discussion above. Thus by another application of (5-7), we obtain (5-9). For the symmetry part, observe that R is orientation-preserving and

$$(u, v) = \int_{SM} u \wedge \alpha \wedge R^*\bar{v} = -\int_{SM} R^*u \wedge \alpha \wedge \bar{v} = \overline{(v, u)}.$$ □

5C. *Magnetic flows.* These flows are determined by a smooth function $\lambda \in C^\infty(\Sigma)$. The relevant vector field is $X_\lambda := X + \lambda V$. A calculation using the structure equations shows

$$\iota_{X_\lambda}\Omega = -d\alpha + \lambda\alpha \wedge \beta = -d\alpha + \lambda\pi^*\sigma,$$

where σ is the area form of g. If Σ has negative Euler characteristic, then $K\sigma$ generates $H^2(\Sigma)$ and thus there is a constant c and a 1-form γ such that

$$\lambda\sigma = cK\sigma + d\gamma.$$

Therefore

$$\iota_{X_\lambda}\Omega = -d\alpha + \lambda\pi^*\sigma = d(-\alpha - c\psi + \pi^*\gamma),$$

and hence $X_\lambda \in \mathcal{X}^0_\Omega$. If X is Anosov and λ is small, X_λ remains Anosov. In general these flows are *not* contact; see [Dairbekov and Paternain 2005].

5D. ***Explicit flows with*** $[\omega]\neq 0$**.** In this subsection, we construct explicit volume-preserving non-null-homologous Anosov flows that are close to the geodesic flow on a compact oriented negatively curved surface (Σ, g). Let $\theta\neq 0$ be a *harmonic* 1-form on Σ. At the level of $S\Sigma$ this can be seen in terms of two equations

$$\begin{aligned} X(\theta)+HV(\theta)&=0,\\ H(\theta)-XV(\theta)&=0. \end{aligned}\tag{5-11}$$

This first is zero divergence, the second is $d\theta=0$. To check these equations one can argue as follows. We will use that $d\pi_{(x,v)}(X(x,v))=v$ and $d\pi_{(x,v)}(H(x,v))=iv$. Given θ, we consider $\pi^*\theta$ and note (using the standard formula for d applied to $\pi^*\theta$)

$$d(\pi^*\theta)(X,H)=X\pi^*\theta(H)-H(\pi^*\theta(X))-\pi^*\theta([X,H]).$$

By the structural equations, the term $[X,H]$ is purely vertical; hence it is killed by $\pi^*\theta$. Now one can check that $\pi^*\theta(H)(x,v)=\theta(iv)=V(\theta)=-(*\theta)(v)$ and $\pi^*\theta(X)=\theta(v)$. Finally since

$$d(\pi^*\theta)(X,H)=\pi^*\,d\theta(X,H)=d\theta(d\pi(X),d\pi(H))=d\theta(v,iv),$$

one obtains that $d\theta=0$ if and only if $H(\theta)-XV(\theta)=0$. The form θ has zero divergence if and only if $*\theta$ is closed so the first equation also follows.

We consider the vector field $Y:=\theta X+V(\theta)H$. This vector field is dual to the 1-form on $S\Sigma$ given by $\pi^*\theta=\theta\alpha+V(\theta)\beta$. This form is closed as well as $\varphi:=-V(\theta)\alpha+\theta\beta$ which is the pullback $\pi^*(*\theta)$. We can easily check that $\varphi(Y)=0$ and $\pi^*\theta(Y)=[\theta]^2+[V(\theta)]^2$.

The flows we wish to consider are of the form $X_\varepsilon=X+\varepsilon Y$, where X is the Anosov geodesic vector field and ε is small so that it remains Anosov. Using the above we observe:

- X_ε preserves the volume form $\Omega=\alpha\wedge\beta\wedge\psi$. This is thanks to the fact that θ has zero divergence.
- $[\iota_{X_\varepsilon}\Omega]\neq 0$ for $\varepsilon\neq 0$. This is because $\pi^*\theta(Y)=[\theta]^2+[V(\theta)]^2\geq 0$, and hence if θ is not trivial,

$$\int_{S\Sigma}\pi^*\theta(X_\varepsilon)\,\Omega=\varepsilon\int_{S\Sigma}\pi^*\theta(Y)\,\Omega\neq 0.\tag{5-12}$$

What we will prove in the coming sections is that X_ε has a splitting resonance for 1-forms near zero, and the semisimplicity does not break down.

6. Perturbations

In this section we study the behaviour of the Pollicott–Ruelle multiplicities under small deformations and start with the proof of Theorem 1.4.

6A. ***Uniform anisotropic Sobolev spaces.*** We start by laying out the necessary tools to study perturbations of Anosov flows and associated anisotropic Sobolev spaces. We will follow the recent approach of [Guedes Bonthonneau 2020], where a uniform weight function that works in a neighbourhood of the initial vector field is constructed. For brevity, we will only outline the necessary details. We refer the reader to [Faure and Sjöstrand 2011] for more details in the case of a fixed vector field, and to [Dang et al. 2020] for an alternative construction of a weight function that works for perturbed vector fields. The use of anisotropic spaces in hyperbolic dynamics has its origins in the works of many authors; see [Baladi 2005; Baladi and Tsujii 2007; Blank et al. 2002; Butterley and Liverani 2007; Liverani 2004; Gouëzel and Liverani 2006].

Let M be compact and X_0 an Anosov vector field. By [Guedes Bonthonneau 2020, Section 2], there exists a 0-homogeneous weight function $m \in C^\infty(T^*M \setminus 0)$ that applies to all flows with generators $\|X - X_0\|_{C^1} < \eta$, for some $\eta > 0$, in a sense to be explained. It satisfies, for all such X,

$$m = 1 \quad \text{near } E_u^*, \qquad m = -1 \quad \text{near } E_s^*, \qquad X_* m \leq 0.$$

Here X_* is the symplectic lift of X to T^*M. We set $G(x,\xi) \sim m(x,\xi)\log(1+|\xi|)$ for all $|\xi|$ large. The anisotropic Sobolev spaces are defined as, for $r \in \mathbb{R}$,

$$\mathcal{H}_{h,rG} = \operatorname{Op}_h(e^{-rG})L^2(M). \tag{6-1}$$

Here $h > 0$ and Op_h denotes a semiclassical quantisation on M; we write $\operatorname{Op} := \operatorname{Op}_1$. We will write $\mathcal{H}_{rG} = \operatorname{Op}(e^{-rG})L^2(M)$. Frequently we consider a smooth vector bundle $\mathcal{E}$ over M and in that case we consider the corresponding spaces $\mathcal{H}_{h,rG} = \operatorname{Op}_h(e^{-rG\times \mathrm{Id}_{\mathcal{E}}})L^2(M;\mathcal{E})$. We will write

$$\mathcal{H}_{h,rG+k\log\langle\xi\rangle} = \operatorname{Op}_h(e^{-rG})H^k(M;\mathcal{E}).$$

We will use the special notation $\mathcal{H}_{rG,k} := \mathcal{H}_{1,rG+k\log\langle\xi\rangle} = \operatorname{Op}(e^{-rG})H^k(M;\mathcal{E})$. We remark that the spaces $\mathcal{H}_{h,rG}$ for varying h are all the same as sets, equipped with a family of distinct, but equivalent norms.

Let X_ε be a smooth family of Anosov vector fields on M. Consider also a smooth family of differential operators P_ε with principal symbol $\sigma(X_\varepsilon) \times \mathrm{Id}_{\mathcal{E}}$. We will consider any $Q \in \Psi^{-\infty}(M;\mathcal{E})$ compactly microsupported, self-adjoint operator, elliptic in the neighbourhood of the zero section in T^*M. Introduce now the spaces

$$\mathcal{D}^{\varepsilon}_{h,rG} := \{u \in \mathcal{H}_{h,rG} : P_\varepsilon u \in \mathcal{H}_{h,rG}\}$$

and equip them with the norm $\|u\|^2_{\mathcal{D}^\varepsilon_{h,rG}} = \|u\|^2_{\mathcal{H}_{h,rG}} + \|hP_\varepsilon u\|^2_{\mathcal{H}_{h,rG}}$. Completely analogously with $\mathcal{H}_{rG}$, we introduce $\mathcal{D}^\varepsilon_{rG}$, and also $\mathcal{D}^\varepsilon_{rG,k}$ for an integer k.

Then [Guedes Bonthonneau 2020, Lemma 9] states:

Lemma 6.1. *There exists an $\varepsilon_0 > 0$ such that the following holds. Given any $s_0 > 0$, $k \in \mathbb{Z}$ and $r > r(s_0) + |k|$, there is $h_k > 0$ such that for $0 < h < h_k$, $\operatorname{Im} s > -s_0$, $|\operatorname{Re} s| < h^{-1/2}$ and $|\varepsilon| < \varepsilon_0$,*

$$P_\varepsilon - h^{-1}Q - s : \mathcal{D}^\varepsilon_{h,rG+k\log\langle\xi\rangle} \to \mathcal{H}_{h,rG+k\log\langle\xi\rangle}$$

is invertible and the inverse is bounded as $O(1)$ independently of ε.

Here $r(s)$ is a nonincreasing function of $\operatorname{Im} s$, so that $r(s) > r_{P_\varepsilon}(\operatorname{Im} s)$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Also, here $r_{P_\varepsilon}(s_0)$ represents a certain threshold (see [Guedes Bonthonneau 2020, p. 4]) depending on P_ε such that for r bigger than this quantity the resolvent $(P_\varepsilon - h^{-1}Q - s)^{-1} : \mathcal{H}_{h,rG} \to \mathcal{H}_{h,rG}$ is holomorphic and $(P_\varepsilon - s)^{-1} : \mathcal{H}_{rG} \to \mathcal{H}_{rG}$ admits a meromorphic extension to $\operatorname{Im} s > -s_0$ and $|\operatorname{Re} s| \leq h^{-1/2}$.

6B. ***Pollicott–Ruelle multiplicities are locally constant.*** In this section we prove, using the construction of anisotropic Sobolev spaces in the previous section, that in some fixed bounded region, the sums of multiplicities of resonances are locally constant. Observe that under the assumptions in Lemma 6.1, we have the factorisation property

$$(P_\varepsilon - s)(P_\varepsilon - h^{-1}Q - s)^{-1} = \mathrm{Id} + h^{-1}Q(P_\varepsilon - h^{-1}Q - s)^{-1}. \tag{6-2}$$

This holds for s in $\Omega_{h,s_0} := \{s : \operatorname{Im} s > -s_0,\ |\operatorname{Re} s| < h^{-1/2}\}$. We introduce the notation

$$D(\varepsilon, s) := h^{-1}Q(P_\varepsilon - h^{-1}Q - s)^{-1}.$$

Since Q is smoothing, we have that $D(\varepsilon, s)$ is of trace class, and, moreover, since for any $\varepsilon, \varepsilon'$

$$D(\varepsilon, s) - D(\varepsilon', s) = h^{-1}Q(P_{\varepsilon'} - h^{-1}Q - s)^{-1}(P_{\varepsilon'} - P_\varepsilon)(P_\varepsilon - h^{-1}Q - s)^{-1},$$

we have that $\varepsilon \mapsto D(\varepsilon, s)$ is continuous with values in holomorphic maps from Ω_{h,s_0+1} to $\mathcal{L}(\mathcal{H}_{rG}, \mathcal{H}_{rG})$. Here $\mathcal{L}(A, B)$ denotes the space of bounded operators from A to B, with the operator norm.

Then $P_\varepsilon - s : \mathcal{D}^\varepsilon_{rG} \to \mathcal{H}_{rG}$ are an analytic family of Fredholm operators for $\operatorname{Im} s > -s_0$. Consider now a resonance s_1 of $P = P_0$, and a simple closed curve γ around s_1 containing no resonances on itself or in its interior except s_1, such that $\gamma \subset \Omega_{h,s_0}$. The fact that $D(\varepsilon, s)$ is continuous allows us to say that for ε small, a neighbourhood of γ still contains no resonances of P_ε. Introduce the family of projectors

$$\Pi_\varepsilon := \frac{1}{2\pi i}\oint_\gamma (s - P_\varepsilon)^{-1}\, ds.$$

Our first aim is to prove:

Lemma 6.2. *The ranks of* Π_ε *are locally constant*; *i.e.*, *there is an* $\varepsilon_1 > 0$ *such that* rank Π_ε *is constant for* $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$.

Proof. We first claim that, for ε small enough,

$$\frac{1}{2\pi i}\operatorname{tr}\oint_\gamma \partial_s(\mathrm{Id} + D(\varepsilon, s))^{-1}(\mathrm{Id} + D(\varepsilon, s))\, ds = -\operatorname{rank}\Pi_\varepsilon. \tag{6-3}$$

The left-hand side is well-defined by the generalised argument principle [Dyatlov and Zworski 2019, Theorem C.11], since the contour integral is a finite-rank operator. To prove the equality in (6-3), we apply the residue theorem for meromorphic families of operators. Use (6-2) to obtain the left-hand side of (6-3) is equal to

$$\begin{aligned}&\frac{1}{2\pi i}\operatorname{tr}\oint_\gamma \big((s - P_\varepsilon)^{-1} + (P_\varepsilon - h^{-1}Q - s)(P_\varepsilon - s)^{-2}\big)(P_\varepsilon - s)(P_\varepsilon - h^{-1}Q - s)^{-1}\, ds\\ &\quad= -\frac{1}{2\pi i}\operatorname{tr}\oint_\gamma (P_\varepsilon - h^{-1}Q - s)^{-1}\, ds + \frac{1}{2\pi i}\operatorname{tr}\oint_\gamma (P_\varepsilon - h^{-1}Q - s)(P_\varepsilon - s)^{-1}(P_\varepsilon - h^{-1}Q - s)^{-1}\, ds.\end{aligned}$$

The first integrand in the second line above vanishes, since $(P_\varepsilon - h^{-1}Q - s)^{-1}$ is holomorphic; the second one is equal to $-\operatorname{tr}\Pi_\varepsilon = -\operatorname{rank}\Pi_\varepsilon$, by the cyclicity of traces. This shows (6-3).

Now recall by Jacobi's formula that we have

$$\frac{1}{2\pi i}\operatorname{tr}\oint_\gamma \partial_s(\mathrm{Id}+D(\varepsilon,s))^{-1}(\mathrm{Id}+D(\varepsilon,s))\,ds = -\frac{1}{2\pi i}\oint_\gamma \operatorname{tr}((\mathrm{Id}+D(\varepsilon,s))^{-1}\partial_s D(\varepsilon,s))\,ds$$
$$= -\frac{1}{2\pi i}\oint_\gamma \frac{\partial_s \det(\mathrm{Id}+D(\varepsilon,s))}{\det(\mathrm{Id}+D(\varepsilon,s))}\,ds.$$

Here we used integration by parts, and that $\partial_s D(\varepsilon,s)$ is a smoothing operator to commute trace and integration. In particular, the continuity of $\varepsilon\mapsto D(\varepsilon,s)$ as above and so that of the Fredholm determinant $\varepsilon\mapsto\det(\mathrm{Id}+D(\varepsilon,s))$ and its derivative $\varepsilon\mapsto\partial_s\det(\mathrm{Id}+D(\varepsilon,s))$ imply that for ε small enough the integrand changes by a small margin, and since the integral is integer-valued, we obtain the claim.[5] □

Note that a priori projections Π_ε are continuous only as functions of ε with values in $\mathcal{L}(\mathcal{H}_{rG,1},\mathcal{H}_{rG})$ and $\mathcal{L}(\mathcal{H}_{rG},\mathcal{H}_{rG,-1})$ if the resolvents $(P_\varepsilon - s)^{-1}$ are. The maps $\Pi_\varepsilon : \operatorname{ran}\Pi_0\to\operatorname{ran}\Pi_\varepsilon$ are isomorphisms for small ε by Lemma 6.2. We will show $\varepsilon\mapsto\Pi_\varepsilon\in\mathcal{L}(\mathcal{H}_{rG},\mathcal{H}_{rG})$ is continuous; we follow the argument in [Chaubet and Dang 2019, Appendix A]. Pick a basis $\varphi^j\in\mathcal{H}_{rG,1}$, $j=1,\dots,k=\operatorname{rank}\Pi_0$, of $\operatorname{ran}\Pi_0$, and define $\varphi^j_\varepsilon := \Pi_\varepsilon\varphi^j$; then $\varepsilon\mapsto\varphi^j_\varepsilon\in\mathcal{H}_{rG}$ is continuous. Define also $\tilde\varphi^j_\varepsilon = \Pi_0\Pi_\varepsilon\varphi^j$ and note $\varepsilon\mapsto\tilde\varphi^j_\varepsilon\in\mathcal{H}_{rG}$ is also continuous. Let ν^j_ε be the dual basis in $\operatorname{ran}\Pi_0$ of $\tilde\varphi^j_\varepsilon$; then $\varepsilon\mapsto\nu^j_\varepsilon\in(\operatorname{ran}\Pi_0)'$ is continuous. Here the prime denotes the dual. Finally, let $l^j_\varepsilon := \nu^j_\varepsilon\circ\Pi_0\circ\Pi_\varepsilon$, continuous as a map $\varepsilon\mapsto l^j_\varepsilon\in\mathcal{H}'_{rG}$. Then we may write

$$\Pi_\varepsilon = \sum_{j=1}^k \varphi^j_\varepsilon\otimes l^j_\varepsilon.$$

By construction, this map is continuous $\mathcal{H}_{rG}\to\mathcal{H}_{rG}$ for $r > r(s_0)+1$.

One may further bootstrap this argument as in [Chaubet and Dang 2019] to reobtain [Guedes Bonthonneau 2020, Lemma 10]:

Lemma 6.3. *For $r > r(s_0)+k+1$ and ε small enough, $\varepsilon\mapsto\Pi_\varepsilon$ is a C^k family of bounded operators on $\mathcal{H}_{rG}$.*

We are now in good shape to prove some of the basic perturbation statements from the Introduction.

Proof of Theorem 1.4(1) *and* (2). If $X_0\in\mathcal{X}^0_\Omega$ has nonzero helicity, then for ε small enough, $\mathcal{H}(X_\varepsilon)\neq 0$ and we may assume by Lemma 6.2 that $m_{1,X_\varepsilon}(0)\le m_{1,X_0}(0)=b_1(M)$. Thus by Theorem 1.2, we have $\dim\operatorname{Res}_{-i\mathcal{L}_{X_\varepsilon},\Omega^1_0}(0)=b_1(M)=m_{1,X_\varepsilon}(0)$, so that X_ε is 1-semisimple, which proves (1). The proof of (2) is completely analogous to the proof above and we omit it. □

7. Proof of Theorem 1.5

In this section we discuss what happens with semisimplicity if we perturb an arbitrary contact Anosov flow. For this purpose, consider M, a closed orientable 3-manifold, and a contact Anosov flow X on M. This implies there is a contact 1-form α such that $\Omega=-\alpha\wedge d\alpha$ is a volume form, $\alpha(X)=1$ and $\iota_X\,d\alpha=0$.

[5] Alternatively, one may apply the generalised Rouché's theorem [Dyatlov and Zworski 2019, Theorem C.12] to conclude that the sums of *null multiplicities* (in the sense of Gohberg–Sigal theory; see [Dyatlov and Zworski 2019, Appendix C]) over the resonances in the interior of γ of operators $\mathrm{Id}+D(\varepsilon,s)$ for small enough ε are constant. By (6-3), we know that these sums of null multiplicities are equal to $\operatorname{rank}\Pi_\varepsilon$, which proves the claim.

We consider a frame $\{X_1, X_2\}$ of $\ker\alpha$ (such a frame exists since M is parallelizable) such that $d\alpha(X_1, X_2) = -1$. The dual coframe $\{\alpha, \alpha_1, \alpha_2\}$ to $\{X, X_1, X_2\}$ satisfies

$$d\alpha = \alpha_2 \wedge \alpha_1, \quad \Omega = -\alpha \wedge d\alpha = \alpha \wedge \alpha_1 \wedge \alpha_2.$$

Next, consider a Riemannian metric g on M making $\{X, X_1, X_2\}$ an orthonormal frame. Observe that $\Omega^1 = \mathbb{R}\alpha \oplus \Omega_0^1$ and for any $u = u_1\alpha_1 + u_2\alpha_2 \in \mathcal{D}'(M; \Omega_0^1)$, we have for the action of the Hodge star $*$ of g

$$*u = u_1\alpha_2 \wedge \alpha + u_2\alpha \wedge \alpha_1 = \alpha \wedge (u_2\alpha_1 - u_1\alpha_2). \tag{7-1}$$

We introduce the complex structure $J : \Omega_0^1 \to \Omega_0^1$ given by

$$Ju := u_2\alpha_1 - u_1\alpha_2,$$

so that $*u = \alpha \wedge Ju$. In particular, we have $\mathcal{L}_X^* u = - * \mathcal{L}_X * u = 0$ if and only if

$$\mathcal{L}_X Ju = 0. \tag{7-2}$$

Let $Y \in \mathcal{X}_\Omega$. Since Y preserves Ω we may consider the winding cycle map associated to Y:

$$W_Y : H^1(M) \to \mathbb{C}, \quad W_Y(\theta) := \int_M \theta(Y)\,\Omega.$$

Clearly Y is null-homologous if and only if $W_Y \equiv 0$. The next lemma characterises the property of Y being null-homologous in terms of a distinguished resonant state of X. Let Π denote the spectral projector at zero of $-i\mathcal{L}_X$ acting on Ω^1 (see (2-5)). Set

$$u := \Pi\mathcal{L}_Y\alpha \in \operatorname{Res}_{-i\mathcal{L}_X, \Omega^1}(0).$$

Lemma 7.1. *We have $\iota_X u = 0$. Let θ be a (real) smooth closed 1-form and let $\psi \in \mathcal{D}'_{E_s^*}(M)$ be such that $v := (J)^{-1}(\theta + d\psi) \in \operatorname{Res}_{-i\mathcal{L}_X^*, \Omega_0^1}(0)$. Then*

$$\langle u, v\rangle_{L^2} = -W_Y(\theta).$$

In particular, Y is null-homologous if and only if $u = 0$.

Proof. We may write for some $a, a_1, a_2 \in C^\infty(M)$

$$Y = aX + a_1X_1 + a_2X_2$$

and a calculation shows

$$\mathcal{L}_Y\alpha = (\iota_Y d + d\iota_Y)\alpha = a_1\iota_{X_1}\,d\alpha + a_2\iota_{X_2}\,d\alpha + da. \tag{7-3}$$

Therefore, we have

$$\iota_X u = \Pi\iota_X\mathcal{L}_Y\alpha = \Pi Xa = X\Pi a = 0. \tag{7-4}$$

In the previous equation we used that Πa is constant by Theorem 1.2 and that Π commutes with X. Next we compute, using that $*v = \alpha \wedge (\theta + d\psi)$,

$$\begin{aligned} \langle \mathcal{L}_Y \alpha, v \rangle_{L^2} &= \int_M (a_1 \iota_{X_1} d\alpha + a_2 \iota_{X_2} d\alpha + da) \wedge \alpha \wedge (\theta + d\psi) \\ &= -\int_M (a_1 \iota_{X_1} + a_2 \iota_{X_2})(\theta + d\psi)\, \Omega \\ &= -\int_M \iota_Y (\theta + d\psi)\, \Omega = -\int_M \iota_Y \theta\, \Omega = -W_Y(\theta). \end{aligned} \tag{7-5}$$

Here we used the graded commutation rule for contractions, integration by parts and the following facts: $\theta + d\psi$ is closed, $\iota_X(\theta + d\psi) = 0$ and Y is volume-preserving. By Lemma 2.3 it follows that $\Pi^* v = v$. By this and the computation in (7-5), it follows that

$$\langle u, v \rangle_{L^2} = \langle \Pi \mathcal{L}_Y \alpha, v \rangle_{L^2} = \langle \mathcal{L}_Y \alpha, v \rangle_{L^2} = -W_Y(\theta)$$

as desired. Clearly, the relation $\langle u, v \rangle = -W_Y(\theta)$ implies that if $u = 0$, then Y is null-homologous. If Y is null-homologous, then $\langle u, v \rangle = 0$ for all v. Since 1-semisimplicity holds for X, Lemma 2.4 implies $u = 0$ and the lemma is proved. ☐

The next lemma provides important information about the pairing between resonant and coresonant states in the contact case.

Lemma 7.2. *Let θ be a smooth closed 1-form on M. Let $\varphi \in \mathcal{D}'_{E_u^*}(M)$ and $\psi \in \mathcal{D}'_{E_s^*}(M)$ be such that*

$$\begin{aligned} u &= \theta + d\varphi \in \operatorname{Res}_{-i\mathcal{L}_X, \Omega_0^1}(0), \\ v &= (J)^{-1}(\theta + d\psi) \in \operatorname{Res}_{-i\mathcal{L}_X^*, \Omega_0^1}(0). \end{aligned} \tag{7-6}$$

Then

$$\operatorname{Re}\langle u, v \rangle_{L^2} = \operatorname{Re} \int_M (\theta + d\varphi) \wedge \alpha \wedge (\bar\theta + d\bar\psi) \le 0$$

with equality if and only if θ is exact, or in other words $u = v = 0$.

Proof. By (7-6) we have $\iota_X u = 0$ and $\iota_X v = 0$, so $X\varphi = X\psi = -\theta(X)$. We have the chain of equalities

$$\begin{aligned} \operatorname{Re}\langle u, v \rangle_{L^2} &= -\int_M \operatorname{Re}(\theta \wedge \bar\theta) \wedge \alpha - \operatorname{Re} \int_M \varphi\, d\alpha \wedge \bar\theta \\ &= \operatorname{Re} \int_M \varphi \bar\theta(X)\, \Omega = -\operatorname{Re}\langle \varphi, X\varphi \rangle_{L^2} = \operatorname{Im}\langle -iX\varphi, \varphi \rangle_{L^2}. \end{aligned} \tag{7-7}$$

Here we used $X\varphi = -\theta(X)$, $\operatorname{Re}(\theta \wedge \bar\theta) = 0$ and integration by parts.

Assume now $\operatorname{Re}\langle u, v \rangle_{L^2} \ge 0$. By the computation in (7-7), Lemma 2.5 implies $\varphi \in C^\infty(M)$, so $u \in C^\infty(M; \Omega_0^1)$ and Lemma 4.2 implies $u \equiv 0$ and θ exact, so also $v \equiv 0$. ☐

7A. ***Constructing the splitting resonance.*** Let $Y \in \mathcal{X}_\Omega$ such that Y is *not* null-homologous and consider a perturbation of X

$$X_\varepsilon = X + \varepsilon Y.$$

Consider a simple closed curve γ around zero, so that no resonances of $-i\mathcal{L}_{X_\varepsilon}$ on $\Omega^1(M)$ cross the curve γ for small enough values of the parameter ε. Consider the family of projectors given by

$$\Pi_\varepsilon := \Pi_{\mathcal{L}_{X_\varepsilon}} = \frac{1}{2\pi i}\oint_\gamma (\lambda + i\mathcal{L}_{X_\varepsilon})^{-1}\,d\lambda. \tag{7-8}$$

By Lemma 6.3, the Π_ε are C^k in ε in suitable topologies. More precisely, we have $\varepsilon \mapsto \Pi_\varepsilon \in \mathcal{L}(\mathcal{H}_{rG}, \mathcal{H}_{rG})$ is C^k for $r > r(0) + k + 1$ (i.e., r large enough).

We will construct the splitting resonant state "by hand". For that purpose, consider

$$t_\varepsilon = \mathcal{L}_{X_\varepsilon}\Pi_\varepsilon\alpha = \varepsilon\Pi_\varepsilon\mathcal{L}_Y\alpha.$$

Here we used that Π_ε commutes with ι_{X_ε} and d, which follows since the integral defining Π_ε does so. Our candidate for the splitting resonance is

$$u_\varepsilon := \Pi_\varepsilon\mathcal{L}_Y\alpha.$$

Firstly, we note that $\iota_{X_\varepsilon}u_\varepsilon = 0$, which follows from

$$\iota_{X_\varepsilon}t_\varepsilon = \mathcal{L}_{X_\varepsilon}\Pi_\varepsilon(1 + \varepsilon\alpha(Y)) = 0.$$

This is because

$$\Pi_\varepsilon f = \frac{1}{\operatorname{vol}(M)}\int_M f\,\Omega$$

is constant, which follows from Theorem 1.2. We also understand that Π_ε acts on forms of any degree, and is given by the expression (7-8). This implies directly that $\iota_{X_\varepsilon}u_\varepsilon = 0$ for $\varepsilon \neq 0$, and then by continuity we have $\iota_{X_\varepsilon}u_\varepsilon = 0$ for all ε.

Fix now $\varepsilon \neq 0$. Then either exactly one resonance "splits" by Lemma 6.2 and Theorem 1.2, so we must have $\mathcal{L}_{X_\varepsilon}t_\varepsilon = \mu_\varepsilon t_\varepsilon$ for some $\mu_\varepsilon \neq 0$ and thus $\mathcal{L}_{X_\varepsilon}u_\varepsilon = \mu_\varepsilon u_\varepsilon$, or a resonant state does not split, in which case $\mathcal{L}_{X_\varepsilon}t_\varepsilon = 0$ and so $\mathcal{L}_{X_\varepsilon}u_\varepsilon = 0$. Also, we clearly have $\mathcal{L}_X u_0 = 0$. Therefore, there exists a function λ_ε such that for each small enough ε

$$\mathcal{L}_{X_\varepsilon}u_\varepsilon = \lambda_\varepsilon u_\varepsilon. \tag{7-9}$$

Hence we may write

$$\lambda_\varepsilon = \frac{\langle \mathcal{L}_{X_\varepsilon}u_\varepsilon, u^*\rangle}{\langle u_\varepsilon, u^*\rangle},$$

where u^* is a coresonant 1-form at zero such that $\langle u_0, u^*\rangle \neq 0$. Such a 1-form exists by Lemma 2.4. Therefore, for ε small enough and by continuity the above expression makes sense, so we conclude that λ_ε is in C^2 for ε in an interval around zero. Note that $\lambda_0 = 0$ and that by Lemma 7.1, $u_0 \neq 0$ since Y is not null-homologous.

7B. ***Proving that $\lambda_\varepsilon \neq 0$.*** We dedicate this subsection to proving that $\lambda_\varepsilon \neq 0$ for $\varepsilon \neq 0$ and we achieve this by looking at the second-order derivatives of λ_ε in ε. Recall we have a C^2 family of resonant 1-forms $u_\varepsilon = \Pi_\varepsilon \mathcal{L}_Y \alpha$ corresponding to resonances $-i\lambda_\varepsilon$ for the flow $X + \varepsilon Y$ such that

$$\begin{aligned} \iota_{X+\varepsilon Y}\, du_\varepsilon &= \lambda_\varepsilon u_\varepsilon, \\ \iota_{X+\varepsilon Y} u_\varepsilon &= 0. \end{aligned} \tag{7-10}$$

We will denote u_0 by u and λ_0 by λ, and we apply the same principle to the derivatives of λ and u at zero. We want to linearise (7-10) by taking derivatives in ε.

First linearisation of (7-10): We take the first derivative of (7-10) to get

$$\begin{aligned} \iota_Y\, du_\varepsilon + \iota_{X+\varepsilon Y}\, d\dot{u}_\varepsilon &= \dot{\lambda}_\varepsilon u_\varepsilon + \lambda_\varepsilon \dot{u}_\varepsilon, \\ \iota_Y u_\varepsilon + \iota_{X+\varepsilon Y} \dot{u}_\varepsilon &= 0. \end{aligned} \tag{7-11}$$

Evaluating (7-11) at $\varepsilon = 0$, we get the system

$$\begin{aligned} \iota_Y\, du + \iota_X\, d\dot{u} &= \dot{\lambda} u, \\ \iota_Y u + \iota_X \dot{u} &= 0. \end{aligned} \tag{7-12}$$

This further simplifies, since u is a resonant state at zero, so by Lemma 4.7 we have $du = 0$. By (7-1) we may write $*u^* = \alpha \wedge w$, where $w = Ju^*$ and we have $\mathcal{L}_X w = 0$ and $\iota_X w = 0$. Much as before, since $w \in \mathcal{D}'_{E_s^*}(M; \Omega_0^1)$ we have $dw = 0$. Therefore, by taking the inner product with u^* in (7-12), we get

$$\begin{aligned} \dot{\lambda}\langle u, u^*\rangle = \langle \iota_X\, d\dot{u}, u^*\rangle &= \int_M \iota_X\, d\dot{u} \wedge \alpha \wedge w \\ &= -\int_M d\dot{u} \wedge w = -\int_M \dot{u} \wedge dw = 0. \end{aligned}$$

This implies $\dot{\lambda} = 0$.

Second linearisation of (7-10): By taking the ε derivative of (7-11) we get

$$\begin{aligned} 2\iota_Y\, d\dot{u}_\varepsilon + \iota_{X+\varepsilon Y}\, d\ddot{u}_\varepsilon &= \ddot{\lambda}_\varepsilon u_\varepsilon + 2\dot{\lambda}_\varepsilon \dot{u}_\varepsilon + \lambda_\varepsilon \ddot{u}_\varepsilon, \\ 2\iota_Y \dot{u}_\varepsilon + \iota_{X+\varepsilon Y} \ddot{u}_\varepsilon &= 0. \end{aligned} \tag{7-13}$$

We evaluate (7-13) at $\varepsilon = 0$ to get

$$\begin{aligned} 2\iota_Y\, d\dot{u} + \iota_X\, d\ddot{u} &= \ddot{\lambda} u, \\ 2\iota_Y \dot{u} + \iota_X \ddot{u} &= 0. \end{aligned} \tag{7-14}$$

Consider the same coresonant state u^* as above. Pairing (7-14) with u^* yields

$$\ddot{\lambda}\langle u, u^*\rangle = 2\int_M \iota_Y\, d\dot{u} \wedge \alpha \wedge w + \int_M \iota_X\, d\ddot{u} \wedge \alpha \wedge w. \tag{7-15}$$

Now the second integral above is equal to $-\int_M d\ddot{u} \wedge w = 0$, by integration by parts.

The first integral is a bit trickier and it is equal to

$$\begin{aligned}\int_M \iota_Y\, d\dot{u}\wedge\alpha\wedge w &= \int_M (a_1\iota_{X_1}+a_2\iota_{X_2})\, d\dot{u}\wedge\alpha\wedge w\\ &= \int_M (a_1\iota_{X_1}+a_2\iota_{X_2}) w\, d\dot{u}\wedge\alpha = \int_M w(Y)\, d\dot{u}\wedge\alpha.\end{aligned} \tag{7-16}$$

Here we used that $\iota_X\, d\dot{u}=0$ by the first linearisation analysis and $\iota_X w=0$. Note that $\iota_X\, d\dot{u}=0$ also implies that $d\dot{u}\wedge\alpha$ is X-invariant, so the integral $\int_M w(Y)\, d\dot{u}\wedge\alpha$ may be interpreted as "some winding cycle".

Observe that $\mathrm{WF}(d\dot{u})\subset\mathrm{WF}(\dot{u})\subset E_u^*$. This follows by differentiating Π_ε at zero to deduce

$$\dot{\Pi}_0=\frac{1}{2\pi i}\oint_\gamma (\lambda+i\mathcal{L}_X)^{-1}(-i\mathcal{L}_Y)(\lambda+i\mathcal{L}_X)^{-1}\, d\lambda = i(R_H(0)\mathcal{L}_Y\Pi_0+\Pi_0\mathcal{L}_Y R_H(0)).$$

At this point, we recall that $(-i\mathcal{L}_X-\lambda)^{-1}=R_H(\lambda)-\Pi_0/\lambda$. Since Π_0 and $R_H(0)$ extend to maps $\mathcal{D}'_{E_u^*}(M;\Omega^1)\to\mathcal{D}'_{E_u^*}(M;\Omega^1)$, we have that $\dot{u}=\dot{\Pi}_0\mathcal{L}_Y\alpha\in\mathcal{D}'_{E_u^*}(M;\Omega^1)$.

By Theorem 1.2 it follows that $d\dot{u}\wedge\alpha=c\Omega$ for some constant c. In fact, we have

$$\begin{aligned}c\,\mathrm{vol}(M) &= \int_M d\dot{u}\wedge\alpha = \int_M \dot{u}\wedge d\alpha = -\int_M \dot{u}(X)\,\Omega\\ &= \int_M u(Y)\,\Omega = W_Y(u).\end{aligned} \tag{7-17}$$

In these lines we used the second equation of (7-12) and $\iota_X u=0$. Combining (7-17), (7-15) and (7-16) we have

$$\ddot{\lambda}\langle u,u^*\rangle = 2c\int_M w(Y)\,\Omega = 2cW_Y(w)=\frac{2W_Y(u)W_Y(w)}{\mathrm{vol}(M)}. \tag{7-18}$$

Next we choose a special u^*. Namely, if we write $u=\theta+d\varphi$ for some (real) smooth closed 1-form θ and $\varphi\in\mathcal{D}'_{E_u^*}(M)$, then we choose $u^*=v$ as in Lemma 7.2. This ensures that $\langle u,u^*\rangle<0$ and, moreover, by Lemma 7.1 we have

$$\langle u,u^*\rangle = -W_Y(\theta)<0.$$

Hence (7-18) simplifies to

$$\ddot{\lambda}=\frac{-2W_Y(\theta)}{\mathrm{vol}(M)}<0.$$

By the symmetry of the Pollicott–Ruelle resonance spectrum, we have that λ_ε is real, since otherwise we would contradict Lemma 6.2. We conclude by Taylor's theorem

$$\lambda_\varepsilon=\varepsilon^2\left(-\frac{W_Y(\theta)}{\mathrm{vol}(M)}+O(\varepsilon)\right).$$

In particular λ_ε is negative (so nonzero) for sufficiently small $\varepsilon\neq 0$. Therefore, the resonance $-i\lambda_\varepsilon$ of $-i\mathcal{L}_{X_\varepsilon}$ splits to the upper half-plane and 0 is a strict local maximum for λ_ε. This completes the proof of Theorem 1.5.

We conclude this section with:

Proof of the first part of Corollary 1.7. By Corollary 3.3, the order of vanishing of the Ruelle zeta function at zero is equal to $m_1(0) - m_0(0) - m_2(0)$. By Theorem 1.2, we know $m_2(0) = m_0(0) = 1$ and by Theorem 1.5 and Lemma 6.2 we have $m_1(0) = b_1(M) - 1$ for small enough nonzero ε. □

8. Time changes

In this section we consider the transformation $X \mapsto \widetilde{X} = fX$, where X is an Anosov vector field and $f > 0$ a positive smooth function and call such a transformation a *time change*. By [de la Llave et al. 1986, Lemma 2.1], we have that $\widetilde{X}$ is also Anosov and, moreover, its stable and unstable bundles $\widetilde{E}^s$ and $\widetilde{E}^u$ are given by

$$\widetilde{E}^s = \{Z + \theta(Z)X : Z \in E^s\}. \tag{8-1}$$

Here the continuous 1-form θ is given by solving $\mathcal{L}_X(f^{-1}\theta) = f^{-2}\,df$. Therefore, we notice that $\widetilde{E}_u^* = (\widetilde{E}^s \oplus \mathbb{R}\widetilde{X})^* = E_u^*$ and $\widetilde{E}_s^* = (\widetilde{E}^u \oplus \mathbb{R}\widetilde{X})^* = E_s^*$, where we used (8-1). This means that the resonant states associated to the flow fX lie in suitable spaces $\mathcal{D}'_{E_u^*}$, which will be very convenient.

We begin by recasting Lemma 2.4 to the case of 1-forms and consider a time change.

Proposition 8.1. *Let X be an Anosov flow on a manifold M and let $f > 0$ be a positive smooth function. Then $\mathcal{L}_{fX}$ acting on Ω_0^1 is semisimple at zero if and only if the pairing*

$$\operatorname{Res}^{(1)}_{-i\mathcal{L}_X,\Omega_0^1}(0) \times \operatorname{Res}^{(1)}_{-i\mathcal{L}_X^*,\Omega_0^1}(0) \to \mathbb{C},\ (u,v) \mapsto \left\langle \frac{u}{f}, v \right\rangle_{L^2(M;\Omega^1)} \tag{8-2}$$

is nondegenerate.

Proof. Let us determine the appropriate resonant spaces of $\mathcal{L}_{fX}$ and $\mathcal{L}_{fX}^*$ at zero. Note first that $\ker \mathcal{L}_{fX} = \ker \mathcal{L}_X$ on $\mathcal{D}'_{E_u^*}(M;\Omega_0^1)$, since time changes preserve the E_u^* set. Next, we compute $\mathcal{L}_{fX}^* = \mathcal{L}_X^*(f\,\cdot)$ on Ω_0^1, with respect to a fixed smooth inner product (e.g., given by a metric). Therefore, we have

$$\operatorname{Res}^{(1)}_{-i\mathcal{L}_{fX}^*,\Omega_0^1}(0) = \frac{1}{f}\operatorname{Res}^{(1)}_{-i\mathcal{L}_X^*,\Omega_0^1}(0).$$

Thus the nondegeneracy of the pairing between resonances and coresonances is equivalent to the nondegeneracy of (8-2) and applying Lemma 2.4 finishes the proof. □

8A. ***Time changes of the geodesic flow on a hyperbolic surface.*** The aim of this subsection is to explicitly specify the equations for 1-forms in the kernel of $\mathcal{L}_X$ on the unit sphere bundle $M = S\Sigma$ of a closed hyperbolic surface Σ. We start by considering the case of general variable curvature and use the orthonormal frame $\{\alpha, \beta, \psi\}$ constructed in Section 5A.

Let $u \in \mathcal{D}'(M;\Omega_0^1)$. Then $u = b\beta + f\psi$ for some $b, f \in \mathcal{D}'(M)$ and we have

$$du = \alpha \wedge (X(b) - fK) + \beta \wedge \psi(H(f) - V(b)) + \alpha \wedge \psi(b + X(f)).$$

Therefore, $du = 0$ implies

$$\begin{aligned} X(b) &= Kf, \\ X(f) &= -b, \\ H(f) &= V(b). \end{aligned} \tag{8-3}$$

The first two equations come from $\iota_X\, du = 0$. The third is an additional one, which we know holds if $u \in \mathcal{D}'_{E_u^*}(M;\Omega_0^1)$ and $\iota_X\, du = 0$; it can be explained as an additional horocyclic invariance (see [Guillarmou and Faure 2018] and below).

Now we specialise to $K = -1$, i.e., the case of hyperbolic surfaces. Then in the $\{\beta, \psi\}$ coframe spanning Ω_0^1, the operator $\mathcal{L}_X$ may be written as

$$\mathcal{L}_X = X \times \mathrm{Id} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the first two equations in (8-3) then read

$$(X-1)(b-f)=0,$$
$$(X+1)(b+f)=0.$$

Thus $f = -b$ as there are no resonances with positive imaginary part, since X is volume-preserving.[6] The third equation in (8-3) now gives $U_- b = 0$, where $U_- = H + V$ is the horocyclic vector field spanning E_u. Now we may also write, where the adjoint is with respect to the Sasaki metric on $S\Sigma$,

$$\mathcal{L}_X^* = -X \times \mathrm{Id} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore $\mathcal{L}_X^* v = 0$, where $v = b'\beta + f'\psi$ for some $b', f' \in \mathcal{D}'_{E_s^*}(M)$, is the same as

$$(-X+1)(b'+f')=0,$$
$$(-X-1)(b'-f')=0.$$

Since we are looking at the vector field $-X$, no resonance with positive imaginary part gives $f' = -b'$ and so $(X+1)b' = 0$. The third equation in (8-3) then reads $U_+ b' = 0$, where $U_+ = H - V$ spans the E_s bundle.

Therefore, we have

$$\begin{aligned} \mathrm{Res}^{(1)}_{-i\mathcal{L}_X}(0) &= \{b(\beta-\psi) \in \mathcal{D}'(M) : (X-1)b = 0,\ (H+V)b = 0\}, \\ \mathrm{Res}^{(1)}_{-i\mathcal{L}_X^*}(0) &= \{b(\beta-\psi) \in \mathcal{D}'(M) : (X+1)b = 0,\ (H-V)b = 0\}. \end{aligned} \tag{8-4}$$

Note that we may drop the wavefront set conditions, since they follow from the equations being satisfied. We remark that since we know $-i\mathcal{L}_X$ at 0 is semisimple by [Dyatlov and Zworski 2017], then so is $-iX$ at $-i$ by the correspondence (8-4) and $\dim \mathrm{Res}_{-iX}(-i) = b_1(M)$. Alternatively, we may use [Guillarmou et al. 2018, Theorem 1] to deduce semisimplicity even at the special point $-i$ for hyperbolic surfaces.

Proposition 8.2. *Let $f \in C^\infty(M)$ and $f > 0$. Semisimplicity for $-i\mathcal{L}_{fX}$ at zero acting on Ω_0^1 is equivalent to the nondegeneracy of the pairing*

$$\mathrm{Res}^{(1)}_{-iX}(-i) \times \mathrm{Res}^{(1)}_{iX}(-i), \quad (b_1, b_2) \mapsto \left\langle \frac{b_1}{f}, b_2 \right\rangle_{L^2(M)}. \tag{8-5}$$

[6] This can be seen from (2-2), since $e^{-itP} = \varphi_{-t}^*$ is an isometric isomorphism on $L^2(M)$ and so the integral defining the resolvent converges for $\mathrm{Im}\,\lambda > 0$.

Proof. The proof is based on the correspondence (8-4) and Proposition 8.1. Then for $b_1(\beta-\psi)\in \mathrm{Res}^{(1)}_{-i\mathcal{L}_{fX}}(0)$ and $(b_2/f)(\beta-\psi)\in \mathrm{Res}^{(1)}_{-i\mathcal{L}^*_{fX}}(0)$, we have

$$\left\langle b_1(\beta-\psi),\, \frac{b_2}{f(\beta-\psi)}\right\rangle_{L^2(M;\Omega^1)} = 2\left\langle b_1,\, \frac{b_2}{f}\right\rangle_{L^2(M)}.$$

This proves that the pairing (8-5) is equivalent to the pairing (8-2), which finishes the proof. □

In the next sections, we would like to find out more about the pairing (8-5), similar to [Dyatlov et al. 2015; Guillarmou et al. 2018], where a pairing formula for generic resonances is proved.

Remark 8.3. Using the decomposition $u=a\alpha+b\beta+f\psi$, by (8-3) it may be seen that $(\mathcal{L}_X+s)u=0$ is equivalent to $(X+1+s)(b+f)=0$, $(X-1+s)(b-f)=0$ and $(X+s)a=0$. This enables us to determine the resonance spectrum of $\mathcal{L}_X$ on 1-forms from the resonance spectrum of X on functions, using the works of [Dyatlov et al. 2015; Guillarmou et al. 2018]. In particular, for $\operatorname{Re} s>-1$ we obtain $b+f=0$, which suffices to determine the spectrum on the left in Figure 1. The small and large eigenvalues in this figure are in the sense of [Ballmann et al. 2016].

8B. ***Reduction to distributions on the boundary.*** We follow the notation from [Dyatlov et al. 2015, Section 3]. We consider the hyperboloid model

$$\mathbb{H}^2=\{x=(x_0,x_1,x_2)=(x_0,x')\in\mathbb{R}^3:\langle x,x\rangle_{\mathcal{M}}=x_0^2-x_1^2-x_2^2=1,\ x_0>0\}$$

of hyperbolic geometry, equipped with the Riemannian metric $-\langle\cdot,\cdot\rangle_{\mathcal{M}}$, restricted to $T\mathbb{H}^2$. Here $\langle\cdot,\cdot\rangle_{\mathcal{M}}$ is called the Lorentzian metric. We also consider the action the isometry group $G=\mathrm{PSO}(1,2)$ of $\mathbb{H}^2$, consisting of matrices preserving the Lorentzian metric, orientation and the sign of x_0. This action extends to an action on the unit sphere bundle $S\mathbb{H}^2$, since G consists of isometries and in fact $G\ni\gamma\mapsto\gamma\cdot(1,0,0,0,1,0)\in S\mathbb{H}^2$ is a diffeomorphism. We also have explicitly

$$S\mathbb{H}^2=\{(x,\xi)\in\mathbb{H}^2:x,\xi\in\mathbb{R}^3,\ \langle\xi,\xi\rangle_{\mathcal{M}}=-1,\ \langle x,\xi\rangle_{\mathcal{M}}=0\}. \tag{8-6}$$

We will write φ_t for the geodesic flow on $S\mathbb{H}^2$ and X for the geodesic vector field. In the identification (8-6), we may write

$$X=\xi\cdot\partial_x+x\cdot\partial_\xi.$$

Therefore the geodesic flow on $S\mathbb{H}^2$ may be explicitly written as

$$\varphi_t(x,\xi)=(x\cosh t+\xi\sinh t,\, x\sinh t+\xi\cosh t). \tag{8-7}$$

We may compactify $\mathbb{H}^2$ to the closed unit ball $\bar{B}^2$ by embedding it with the map $\psi_0(x)=x'/(x_0+1)$ and we call S^1 bounding B^2 the *boundary at infinity*. Note that to a point $\nu\in S^1$ we may associate a ray $\{(s,s\nu):s>0\}$, which is asymptotic to the hyperboloid ray $\{(\sqrt{1+s^2},s\nu):s>0\}$. The action of G extends to an action on the boundary at infinity S^1 as follows. Let $\gamma\in G$ and $\nu\in S^1$. Then the matrix action on $\mathbb{R}^3$

$$\gamma\cdot(1,\nu)=N_\gamma(\nu)(1,L_\gamma(\nu)) \tag{8-8}$$

defines an action of $\gamma\in G$ on S^1 via L_γ. It also defines the multiplicative map $N_\gamma:S^1\to\mathbb{R}_+$.

Denote by $\pi : S\mathbb{H}^2 \to \mathbb{H}^2$ the footpoint projection. We will consider the mappings

$$B_\pm(x,\xi) : S\mathbb{H}^2 \to S^1, \quad B_\pm(x,\xi) = \lim_{t\to\pm\infty} \pi(\varphi_t(x,\xi)). \tag{8-9}$$

The limit in (8-9) is interpreted as the point of intersection of the geodesic starting at x and with tangent vector ξ with the boundary at infinity. We introduce also

$$\Phi_\pm : S\mathbb{H}^2 \to \mathbb{R}_+, \quad \Phi_\pm(x,\xi) := x_0 \pm \xi_0 > 0. \tag{8-10}$$

In fact, then we can write for any $(x,\xi) \in S\mathbb{H}^2$

$$x \pm \xi = \Phi_\pm(x,\xi)(1, B_\pm(x,\xi)). \tag{8-11}$$

The maps $B_\pm$ and $\Phi_\pm$ have nice interactions with the geodesic vector field X and the horocyclic vector fields $U_\pm$, defined in Section 8A. By this we mean that

$$dB_\pm \cdot X = 0, \quad U_\pm B_\pm = 0. \tag{8-12}$$

The first equation holds since $B_\pm$ is constant along X and the second one since $B_\pm$ is constant along horospheres. We also have

$$X\Phi_\pm = \pm\Phi_\pm, \quad U_\pm\Phi_\pm = 0. \tag{8-13}$$

Here, the first equation follows from $\Phi_\pm(\varphi_t(x,\xi)) = e^{\pm t}\Phi_\pm(x,\xi)$, which is true by (8-7). The second one also follows from a computation. Finally, since $\langle x+\xi, x-\xi\rangle_{\mathcal{M}} = 2$ and by (8-11), for $(x,\xi) \in S\mathbb{H}^2$, we have

$$\Phi_+(x,\xi)\Phi_-(x,\xi)(1 - B_+(x,\xi)\cdot B_-(x,\xi)) = 2. \tag{8-14}$$

The maps $\Phi_\pm$ and $B_\pm$ are G-equivariant in the following sense. We have

$$B_\pm(\gamma\cdot(x,\xi)) = L_\gamma(B_\pm(x,\xi)), \quad \Phi_\pm(\gamma\cdot(x,\xi)) = N_\gamma(B_\pm(x,\xi))\Phi_\pm(x,\xi). \tag{8-15}$$

Now the Jacobian of the map $L_\gamma : S^1 \to S^1$ may be computed explicitly and is given by

$$\langle dL_\gamma(\nu)\cdot\zeta_1, dL_\gamma(\nu)\cdot\zeta_2\rangle_{\mathbb{R}^2} = N_\gamma(\nu)^{-2}\langle\zeta_1,\zeta_2\rangle_{\mathbb{R}^2}, \quad \zeta_1,\zeta_2 \in T_\nu S^1. \tag{8-16}$$

Consider $\Sigma = \Gamma\backslash\mathbb{H}^2$ a compact hyperbolic surface, where $\Gamma \subset \mathrm{PSO}(1,2)$ is a discrete subgroup. Then we may identify the unit sphere bundle as $S\Sigma = \Gamma\backslash S\mathbb{H}^2$. We introduce the space of boundary distributions as

$$\mathrm{Bd}^0(\lambda) = \{w \in \mathcal{D}'(S^1) \mid L_\gamma^* w(\nu) = N_\gamma^{-\lambda}(\nu)w(\nu),\ \gamma\in\Gamma,\ \nu\in S^1\}. \tag{8-17}$$

The generator X of the geodesic flow descends to $S\Sigma$ and we define the *first band resonant states* by

$$\mathrm{Res}^0_X(\lambda) = \{u \in \mathcal{D}'_{E_u^*}(S\Sigma) \mid (X+\lambda)u = 0,\ U_-u = 0\}.$$

We similarly introduce the first band coresonant states via (see Section 2C)

$$\mathrm{Res}^0_{X^*}(\lambda) = \{u \in \mathcal{D}'_{E_s^*}(S\Sigma) \mid (X-\bar{\lambda})u = 0,\ U_+u = 0\}.$$

Then we have the correspondence, valid for all $\lambda \in \mathbb{C}$ proved in [Dyatlov et al. 2015, Lemma 5.6], which we prove here for completeness. Note that by $\Phi_\pm^\lambda$ for $\lambda \in \mathbb{C}$ we simply mean the exponentiation of the function $\Phi_\pm > 0$ by the exponent λ.

Lemma 8.4. *Let $\pi_\Gamma : S\mathbb{H}^2 \to S\Sigma$ be the natural projection. Then*

$$\pi_\Gamma^* \operatorname{Res}_X^0(\lambda) = \Phi_-^\lambda B_-^* \operatorname{Bd}^0(\lambda). \tag{8-18}$$

Similarly we have, for the space of coresonant states,

$$\pi_\Gamma^* \operatorname{Res}_{X^*}^0(\lambda) = \Phi_+^{\bar\lambda} B_+^* \operatorname{Bd}^0(\bar\lambda). \tag{8-19}$$

We also have $\overline{\operatorname{Bd}^0(\lambda)} = \operatorname{Bd}^0(\bar\lambda)$.

Proof. Let $w \in \operatorname{Bd}^0(\lambda)$ and put $v = \Phi_-^\lambda B_-^* w \in \mathcal{D}'(S\mathbb{H}^2)$ (the pullback of distributions under submersions is well-defined; see [Grigis and Sjöstrand 1994, Corollary 7.9]). We use now the invariance properties $\Phi_\pm$ and $B_\pm$ given by (8-15) to prove v is Γ-invariant. For $\gamma \in \Gamma$ we have

$$\gamma^* v = (\gamma^*\Phi_-)^\lambda \gamma^* B_-^* w = B_-^*(N_\gamma)^\lambda \Phi_-^\lambda B_-^* L_\gamma^* w = \Phi_-^\lambda B_-^* w = v.$$

Thus v is Γ-invariant and descends to $\mathcal{D}'(SM)$.

Now using (8-12) and (8-13), we obtain directly that $(X+\lambda)v = 0$ and $U_- v = 0$. This proves $\Phi_-^\lambda B_-^* \operatorname{Bd}^0(\lambda) \subset \pi_\Gamma^* \operatorname{Res}_X^0(\lambda)$ (the wavefront set condition on v follows from [Grigis and Sjöstrand 1994, Chapter 7]). The other direction follows by reversing the steps above and noting that a function (distribution) invariant by X and U_- is immediately a pullback by B_-. The statement about coresonant states follows similarly. □

We now introduce the set of coordinates $(\nu_-, \nu_+, s) \in (S^1 \times S^1)_\Delta \times \mathbb{R}$ on $S\mathbb{H}^2$, yielding a diffeomorphism $F : (S^1 \times S^1)_\Delta \times \mathbb{R} \to S\mathbb{H}^2$, and given by identification

$$(\nu_-, \nu_+, s) = \left(B_-(x,\xi),\, B_+(x,\xi),\, \frac{1}{2} \log \frac{\Phi_+(x,\xi)}{\Phi_-(x,\xi)} \right). \tag{8-20}$$

Here $(S_1 \times S_1)_\Delta$ denotes the torus $S^1 \times S^1$ without the diagonal Δ. The coordinates (8-20) can be interpreted as (ν_-, ν_+) parametrises the geodesic γ starting at ν_- and ending at ν_+ and s is the parameter on this geodesic such that $\gamma(-s)$ is the point on γ closest to $e_0 = (1,0,0)$ (or 0 in the disk model). The geodesic flow in these coordinates is simply $\varphi_t : (\nu_-, \nu_+, s) \mapsto (\nu_-, \nu_+, s+t)$.

The coordinates (8-20) enable us to write a product of distributions in resonant and coresonant spaces more explicitly, but we first require an explicit computation of the Jacobian of the change of coordinates $(x,\xi) \to (\nu_-, \nu_+, s)$.

Lemma 8.5. *For the coordinate system introduced in* (8-20), *we have the equality*

$$F^*(dx\,d\xi) = \frac{2\,d\nu_-\,d\nu_+\,ds}{|\nu_- - \nu_+|^2}. \tag{8-21}$$

Proof. This is the content of [Nicholls 1989, Theorem 8.1.1]. □

Remark 8.6. The Jacobian popping up in Lemma 8.5 is well known and the current in (8-21) is called the *Liouville current*.

We now prove that the invariant distributions formed as products of resonant and coresonant states have a very nice form in the coordinates (8-20).

Proposition 8.7. *Let $w_1 \in \mathrm{Bd}^0(\lambda)$ and $w_2 \in \mathrm{Bd}^0(\bar{\lambda})$, and consider the invariant distributions $v_1 = \Phi_-^{\lambda} B_-^* w_1$ and $v_2 = \Phi_+^{\bar{\lambda}} B_+^* w_2$ constructed in Lemma 8.4. Then the product distribution in (ν_-, ν_+, s) coordinates takes the form*[7]

$$F^*((v_1\bar{v}_2)(x,\xi)\,dx\,d\xi) = 2^{2\lambda+1}\frac{w_1(\nu_-)\bar{w}_2(\nu_+)}{|\nu_- - \nu_+|^{2(\lambda+1)}}\,d\nu_-\,d\nu_+\,ds. \tag{8-22}$$

In particular, for $\lambda = -1$ the product $F^*(v_1\bar{v}_2)$ extends to a distribution on $S^1 \times S^1 \times \mathbb{R}$.

Proof. By definition, we have the following expression for the product $v_1\bar{v}_2$:

$$(v_1\bar{v}_2)(x,\xi) = (\Phi_-(x,\xi)\Phi_+(x,\xi))^{\lambda} B_-^* w_1(x,\xi) B_+^* \bar{w}_2(x,\xi). \tag{8-23}$$

Now changing the coordinates to (ν_-, ν_+, s) given in (8-20) and by using the identity (8-14) we get

$$F^*(v_1\bar{v}_2)(\nu_-,\nu_+,s) = 2^{\lambda}(1-\nu_-\cdot\nu_+)^{-\lambda} w_1(\nu_-)\bar{w}_2(\nu_+) = 2^{2\lambda}\frac{w_1(\nu_-)\bar{w}_2(\nu_+)}{|\nu_- - \nu_+|^{2\lambda}}. \tag{8-24}$$

Using the Jacobian computation in Lemma 8.5, we establish (8-22).

In the special case $\lambda = -1$, using (8-22) we may write

$$F^*(v_1\bar{v}_2(x,\xi)\,dx\,d\xi) = \tfrac{1}{2} w_1(\nu_-)\bar{w}_2(\nu_+)\,ds\,d\nu_-\,d\nu_+. \tag{8-25}$$

In particular, for $\lambda = -1$ the distribution $F^*(v_1\bar{v}_2)$ extends to a distribution on the space $S^1 \times S^1 \times \mathbb{R}$. □

Remark 8.8. The distributions in (8-14) are called distributions of Patterson–Sullivan type. See [Anantharaman and Zelditch 2007] for more details, where the particular case of $\lambda = -\frac{1}{2} + ir_j$ is studied, in connection to eigenvalues of Δ on Σ with eigenvalue $\frac{1}{4} + r_j^2$. Note however there is an extra factor of $|\nu_- - \nu_+|^2$ compared to (8-24), obtained by changing coordinates according to (8-20).

8C. *Construction of a time change that is not semisimple on 1-forms.* Here we construct a smooth, positive function on the unit sphere bundle $S\Sigma$ of a compact hyperbolic surface $\Sigma = \Gamma\backslash\mathbb{H}^2$ such that under a time change of the geodesic flow, the action of the Lie derivative on resonant 1-forms at zero is not semisimple. We establish a few auxiliary lemmas first. We denote by $\pi_\Gamma : \mathbb{H}^2 \to \Gamma\backslash\mathbb{H}^2$ the associated projection.

Lemma 8.9. *Let $w \in \mathrm{Bd}^0(-1)$. Then $w(\nu)\,d\nu$ is Γ-invariant and we have*

$$\int_{S^1} w(\nu)\,d\nu = 0.$$

[7] Formally, by (8-22) we mean an equality in the sense of 0-currents. More explicitly, we mean an equality in the sense of distributions $\langle 2^{2\lambda+1} w_1(\nu_-)\bar{w}_2(\nu_+)/|\nu_- - \nu_+|^{2(\lambda+1)}, f\rangle_{(S^1\times S^1)_\Delta\times\mathbb{R}} = \langle v_1\bar{v}_2, f\circ F^{-1}\rangle_{S\mathbb{H}^2}$.

Proof. For the first claim, recall that by (8-16) we have $L_\gamma^* \, d\nu = N_\gamma^{-1}(\nu)\, d\nu$ for any $\gamma \in G$. Therefore, by (8-17) we have also $L_\gamma^*(w\, d\nu) = w\, d\nu$ for any $\gamma \in \Gamma$, which gives the required property.

The second property is a direct consequence of the works [Dyatlov et al. 2015; Guillarmou et al. 2018] on pairings. Note that [Dyatlov et al. 2015, Lemma 5.11] proves a pairing formula, which for $\lambda = -1$ gives

$$\langle \pi_* v_1, \pi_* v_2 \rangle_\Sigma = 0 \tag{8-26}$$

for all v_1 resonance states at -1 and v_2 coresonant states at -1. Here π_* maps first band resonant and coresonant states at -1 to eigenfunctions of Δ on Σ at zero by [Dyatlov et al. 2015, Lemma 5.8], so $\pi_* v_1$ and $\pi_* v_2$ are constants. Using the time-reversal map R from Section 5B we may identify resonant and coresonant states; i.e., we have $R^* : \mathrm{Res}_X^0(-1) \to \mathrm{Res}_{X^*}^0(-1)$ is an isomorphism. Moreover, we claim that $\pi_* R^* v = \pi_* v$ for any $v \in \mathrm{Res}_X^0(-1)$. For this recall the connection 1-form ψ on $S\Sigma$ (dual to the vertical fibre), and observe that $\pi_* v = \pi_*(v\psi)$. Then for any 2-form θ on Σ

$$\langle \pi_*(R^* v\psi), \theta \rangle_\Sigma = \int_{S\Sigma} R^* v\psi \wedge \pi^*\theta = \langle \pi_*(v\psi), \theta \rangle_\Sigma.$$

Here we used $R^*\psi = \psi$ and $\pi \circ R = \pi$. By applying (8-26) to $v_2 = R^* v_1$, we obtain that π_* is zero on both resonant and coresonant states.

Alternatively, this follows directly from the proof of [Guillarmou et al. 2018, Theorem 1] (more precisely, see p. 19 of that work and the start of discussion of the $\lambda_0 = -n$ case). □

Next we prove an auxiliary lemma that relies on the dynamics of the action of Γ on S^1.

Lemma 8.10. *Let $w \in \mathrm{Bd}^0(-1)$ and let $(\nu_-, \nu_+) \in S^1 \times S^1$ with $\nu_- \neq \nu_+$. Then there exists a $\varphi \in C^\infty(S^1)$, such that*:

(1) $\varphi \geq 0$.

(2) $\varphi(\nu_+) \neq 0$.

(3) *φ vanishes in a neighbourhood of ν_-.*

(4) $\langle w, \varphi \rangle_{S^1} = 0$.

Proof. We denote by $B_\varepsilon(A)$ the ε-neighbourhood of a set A. Let $\varphi_\varepsilon \in C^\infty(S^1)$ be a nonnegative function with $\varphi_\varepsilon = 1$ outside $B_\varepsilon(\nu_-)$ and $\varphi_\varepsilon = 0$ in $B_{\varepsilon/2}(\nu_-)$; assume also $0 \leq \varphi_\varepsilon \leq 1$. Here $\varepsilon > 0$ is a small enough positive number. If $\langle w, \varphi_\varepsilon \rangle = 0$ for some ε, we are done by setting $\varphi = \varphi_\varepsilon$. If not, then we may assume $\langle w, \varphi_\varepsilon \rangle > 0$ for every $\varepsilon > 0$. Assume $\langle w, \varphi_\varepsilon \rangle > 0$ and $\langle w, \varphi_\delta \rangle < 0$ for some $\varepsilon, \delta > 0$. Then if we take $s = -\langle w, \varphi_\varepsilon \rangle / \langle w, \varphi_\delta \rangle > 0$, we have $\langle w, \varphi_\varepsilon + s\varphi_\delta \rangle = 0$ and so we are done by setting $\varphi = \varphi_\varepsilon + s\varphi_\delta$.

Next, we may without loss of generality assume $\langle w, \varphi_\varepsilon \rangle > 0$ for all $\varepsilon > 0$ small enough. By Lemma 8.9 we have $\langle w, 1 \rangle = 0$, which implies $\langle w, 1 - \varphi_\varepsilon \rangle < 0$. The invariance of $w(\nu)\, d\nu$ under the action of Γ following from Lemma 8.9 then yields that for any $\psi \in C^\infty(S^1)$

$$\langle w, \psi \rangle = \int_{S^1} L_\gamma^*(w(\nu)\, d\nu)\psi = \int_{S^1} w(\nu)\psi \circ L_{\gamma^{-1}}(\nu)\, d\nu = \langle w, \psi \circ L_{\gamma^{-1}} \rangle. \tag{8-27}$$

Now use that since $\Gamma \cong \pi_1(M)$ has $2g \geq 4$ generators, it is not elementary by [Katok 1992, Theorem 2.4.3]. Therefore, by Exercise 2.13 of that work we have that Γ contains infinitely many hyperbolic elements (fixing exactly two elements of S^1), no two of which have a common fixed points.

So take $\gamma \in \Gamma$ hyperbolic such that ν_-, ν_+ are not in the set of fixed points of γ, which we denote by $\{p_1, p_2\}$. Assume without loss of generality p_1 is an attractor and p_2 is a repeller.

By (8-27) for $\psi = 1-\varphi_\varepsilon$, we get $\langle w, 1-\varphi_\varepsilon\rangle = \langle w, (1-\varphi_\varepsilon)\circ L_{\gamma^{-1}}\rangle < 0$. Since $\operatorname{supp}((1-\varphi_\varepsilon)\circ L_{\gamma^{-1}}) = L_\gamma(B_\varepsilon(\nu_-))$, we have that for $n \geq N_0$ large enough, $\varphi_{\varepsilon,n} := (1-\varphi_\varepsilon)\circ L_{\gamma^{-n}}$ has support arbitrarily close to p_1, so disjoint from ν_- and ν_+. Therefore, for $s = -\langle w, \varphi_\varepsilon\rangle / \langle w, \varphi_{\varepsilon,n}\rangle > 0$, we have

$$\langle w, \varphi_\varepsilon + s\varphi_{\varepsilon,n}\rangle = 0.$$

Then $\varphi = \varphi_\varepsilon + s\varphi_{\varepsilon,n}$ does the job. □

With this in hand, we can prove the following claim:

Theorem 8.11. *Let $\Sigma = \Gamma\backslash\mathbb{H}^2$ be a closed hyperbolic surface. Fix $w_2 \in \operatorname{Bd}^0(-1)$ and let $v_2 \in \operatorname{Res}^0_{X^*}(-1)$ be the corresponding coresonant state, according to Lemma 8.4. Then there exists an $f \in C^\infty(S\Sigma)$ with $f > 0$ such that*

$$\int_{S\Sigma} f v_1 \bar{v}_2 \, dx\, d\xi = 0 \tag{8-28}$$

for all $v_1 \in \operatorname{Res}^0_X(-1)$. In other words, semisimplicity of the Lie derivative $\mathcal{L}_{-iX/f}$ acting on resonant 1-forms at zero fails.

Proof. We divide the construction of f into several steps.

Step 1: First, fix $(x_0, \xi_0) \in S\mathbb{H}^2$. Denote the corresponding coordinates of (x_0, ξ_0) by $(\nu_{0-}, \nu_{0+}, s_0)$, according to (8-20). By Lemma 8.10, there is a nonnegative $\varphi_+ \in C^\infty(S^1)$, nonvanishing at ν_{0+}, vanishing near ν_{0-} and in the kernel of w_2. Now let $\varphi_- \in C^\infty(S^1)$ be such that $\varphi_- \geq 0$, $\varphi_-(\nu_{0-}) \neq 0$ and $\operatorname{supp}(\varphi_+) \cap \operatorname{supp}(\varphi_-) = \varnothing$. Also, let $\psi \in C_0^\infty(\mathbb{R})$ be such that $\psi(s_0) \neq 0$ and $\psi \geq 0$. Set $\chi(\nu_-, \nu_+, s) := \varphi_+(\nu_+)\varphi_-(\nu_-)\psi(s)$. Take any $w_1 \in \operatorname{Bd}^0(-1)$ and denote the corresponding element of $\operatorname{Res}^0_X(-1)$ by v_1. Then by the computation in Proposition 8.7 for $\lambda = -1$, we have $F^*\pi_\Gamma^*(v_1\bar{v}_2\, dx\, d\xi) = \frac{1}{2} w_1(\nu_-)\bar{w}_2(\nu_+)\, d\nu_-\, d\nu_+\, ds$ and

$$\begin{aligned} \langle \pi_\Gamma^*(v_1\bar{v}_2\, dx\, d\xi), F_*\chi\rangle_{S\mathbb{H}^2} &= \tfrac{1}{2}\langle w_1(\nu_-)\bar{w}_2(\nu_+)\, d\nu_-\, d\nu_+\, ds, \chi\rangle_{(S^1\times S^1)_\Delta\times\mathbb{R}} \\ &= \tfrac{1}{2}\langle w_1, \varphi_-\rangle\langle \bar{w}_2, \varphi_+\rangle\langle ds, \psi\rangle = 0 \end{aligned} \tag{8-29}$$

since $\langle w_2, \varphi_+\rangle = 0$ by the construction. We will denote the χ above by $\chi_{(x_0,\xi_0)}$ and by $U_{(x_0,\xi_0)}$ a neighbourhood of (x_0, ξ_0) where $F_*\chi_{(x_0,\xi_0)} > 0$. Note that χ is a function in $C_0^\infty((S^1\times S^1)_\Delta \times \mathbb{R})$, by the condition on disjoint supports of φ_- and φ_+ in the construction, and as $\psi \in C_0^\infty(\mathbb{R})$. Therefore we have $F_*\chi$ a function in $C_0^\infty(S\mathbb{H}^2)$.

Step 2: Denote by $\mathcal{D} \subset \mathbb{H}^2$ a compact fundamental domain for Σ. Then $S\mathcal{D}$ is a fundamental domain for $S\Sigma$. By compactness, we have an $N > 0$ and $(x_i, \xi_i) \in S\mathbb{H}^2$ for $i = 1, 2, \dots, N$ such that

$$S\mathcal{D} \subset \bigcup_{(x_i,\xi_i)} U_{(x_i,\xi_i)}.$$

Define then

$$F_*\chi(x,\xi) := \sum_{i=1}^{N} F_*\chi_{(x_i,\xi_i)}(x,\xi) \in C_0^\infty(S\mathbb{H}^2).$$

By the construction, we have

$$\langle \pi_\Gamma^*(v_1\bar{v}_2\,dx\,d\xi),\, F_*\chi\rangle_{S\mathbb{H}^2} = \frac{1}{2}\sum_{i=1}^{N}\langle w_1(\nu_-)\bar{w}_2(\nu_+)\,d\nu_-\,d\nu_+\,ds,\, \chi_{(x_i,\xi_i)}\rangle_{(S^1\times S^1)_\Delta\times\mathbb{R}} = 0. \tag{8-30}$$

Step 3: We introduce the pushforward map $\pi_* : C_0^\infty(S\mathbb{H}^2) \to C^\infty(S\Sigma)$ by defining for any $\eta \in C_0^\infty(S\mathbb{H}^2)$

$$\pi_*\eta(x,\xi) := \sum_{\gamma\in\Gamma} \eta(\gamma\cdot(x_0,\xi_0)) \in C^\infty(S\Sigma). \tag{8-31}$$

Here $(x_0,\xi_0) \in \pi_\Gamma^{-1}(x,\xi) \subset S\mathbb{H}^2$ is an arbitrary point in the fibre and the definition of π_* is independent of any choices. Note that the only accumulation points of orbits of Γ are on the boundary at infinity S^1, so the pushforward is well-defined and sequentially continuous. Note also that π_* is dual to π_Γ^* in the sense of distributions.

Then we observe that $f(x,\xi) := \pi_* F_*\chi(x,\xi) \in C^\infty(S\Sigma)$ satisfies the required properties. Firstly,

$$\langle v_1\bar{v}_2\,dx\,d\xi,\, f\rangle_{S\Sigma} = \langle \pi_\Gamma^*(v_1\bar{v}_2\,dx\,d\xi),\, F_*\chi\rangle_{S\mathbb{H}^2} = 0 \tag{8-32}$$

by (8-30) from Step 2 and duality of π_* with π_Γ^*. Secondly, we have $f > 0$. To see this, let $(x,\xi) \in S\Sigma$ and denote a lift to $S\mathbb{H}^2$ by (x_0,ξ_0). Then there exists $\gamma' \in \Gamma$ with $\gamma'\cdot(x_0,\xi_0) \in \mathcal{D}$. Therefore, there is an $i \in \{1,2,\dots,N\}$ with $\gamma'\cdot(x_0,\xi_0) \in U_{(x_i,\xi_i)}$ and so $F_*\chi_{(x_i,\xi_i)}(\gamma'\cdot(x_0,\xi_0)) > 0$. Hence

$$f(x,\xi) = \sum_{\gamma\in\Gamma} F_*\chi(\gamma\cdot(x_0,\xi_0)) \ge \sum_{i=1}^{N} F_*\chi_{(x_i,\xi_i)}(\gamma'\cdot(x_0,\xi_0)) \ge F_*\chi_{(x_i,\chi_i)}(\gamma'\cdot(x_0,\xi_0)) > 0.$$

This proves the first claim. The final claim now follows directly from the correspondence in (8-4) and Proposition 8.1. □

Remark 8.12. One may see the element in the kernel of $\mathcal{L}_{X/f}^2$ and not in the kernel of $\mathcal{L}_{X/f}$ constructed in Theorem 8.11 more explicitly. Namely, one such element is given by the formula

$$u' = -iR_H(0)(fu).$$

Here $u \in \mathrm{Res}_X^0(-1)$ is an element such that $\int_{S\Sigma} fuv\,dx\,d\xi = 0$ for all $v \in \mathrm{Res}_{X^*}^0(-1)$ and $R_H(\lambda)$ is the holomorphic part at zero of $(-i\mathcal{L}_X - \lambda)^{-1}$ on 1-forms. The conclusion follows as $\Pi_0(fu) = 0$ and $-iR_H(0)$ is an inverse to $\mathcal{L}_X$ on $\ker \Pi_0 \cap \mathcal{D}'_{E_u^*}(M;\Omega^1)$.

Theorem 8.11 completes the proof of Theorem 1.4. We conclude this section with the following:

Proof of the second part of Corollary 1.7. By Theorem 1.4 there is a time change fX on the unit sphere bundle $S\Sigma$ of a closed hyperbolic surface Σ with $\ker \mathcal{L}_{fX}^2 \neq \ker \mathcal{L}_{fX}$ on $\Omega_0^1(S\Sigma)$. By Theorem 1.2, for the flow fX we have $m_0(0) = m_2(0) = 1$ and $\dim \mathrm{Res}_1(0) = b_1(\Sigma)$, so that $m_1(0) \ge b_1(\Sigma) + 1$. The claim then follows by applying Corollary 3.3. □

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Cover image: The figure shows the outgoing scattered field produced by scattering a plane wave, coming from the northwest, off of the (stylized) letters P A A. The total field satisfies the homogeneous Dirichlet condition on the boundary of the letters. It is based on a numerical computation by Mike O'Neil of the Courant Institute.

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