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RESONANCES AND VISCOSITY LIMIT FOR THE WIGNER–VON NEUMANN-TYPE HAMILTONIAN

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The resonances for the Wigner–von Neumann-type Hamiltonian are defined by the periodic complex distortion in the Fourier space. Also, following Zworski, we characterize resonances as the limit points of discrete eigenvalues of the Hamiltonian with a quadratic complex-absorbing potential in the viscosity-type limit.

1. Introduction

We consider the one-dimensional Schrödinger operator

$$P = -\frac{d^2}{dx^2} + V(x) \quad \text{on } L^2(\mathbb{R})$$

and its resonances, where $V(x)$ is an oscillatory and slowly decaying potential. A typical example is

$$P = -\frac{d^2}{dx^2} + a\frac{\sin 2x}{x} \quad \text{on } L^2(\mathbb{R}),$$

where $a \in \mathbb{R}$. We note that P is not dilation-analytic in this case since the potential is exponentially growing in the complex direction. More generally, we consider the following class of potentials.

Assumption A. The potential $V(x)$ has the form

$$V(x) = \sum_{j=1}^{J} s_j(x) W_j(x),$$

where $J \in \mathbb{N}$, $s_j \in C(\mathbb{R}; \mathbb{R})$ are periodic functions with period π whose Fourier series converge absolutely, and $W_j \in C^\infty(\mathbb{R}; \mathbb{R})$ have analytic continuations to the region $\{z = x + iy \mid |x| > R_0, |y| < K|x|\}$ for some $R_0 > 0$ and $K > 0$ with the bound $|W_j(z)| \le C|z|^{-\mu}$ for some $\mu > 0$ in this region; see Figure 1, left.

We note that $V(x) = a(\sin 2x)/x$ satisfies Assumption A for any $K > 0$. We also note that dilation-analytic potentials satisfy Assumption A by setting $s_j(x) = 1$. We first show that resonances can be defined for this class of potentials. We write the set of threshold by $\mathcal{T} = \{n^2 \mid n \in \mathbb{N} \cup \{0\}\}$ (see Remark 2.2 for the necessity of $\mathcal{T}$). The resolvent on the upper half-plane is denoted by $R_+(z) = (z - P)^{-1}$, $\operatorname{Im} z > 0$.

Theorem 1.1. *Under Assumption A, there exists a complex neighborhood $\Omega \subset \mathbb{C}$ of $[0, \infty) \setminus \mathcal{T}$ such that the following holds: for any $f, g \in L^2_{\mathrm{comp}}(\mathbb{R})$, the matrix element $(f, R_+(z)g)$ has a meromorphic continuation to Ω.*

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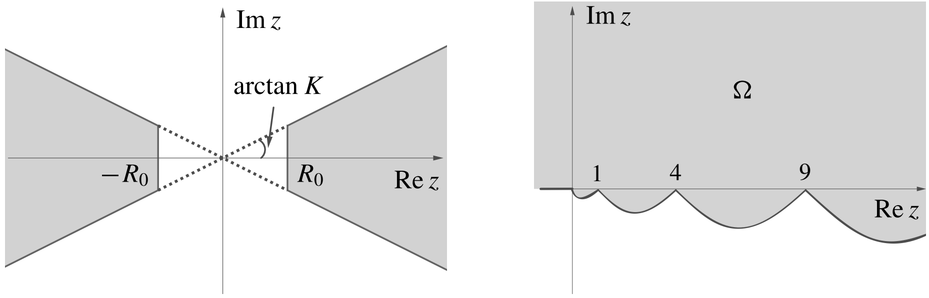

Figure 1. Left: the domain of analyticity of W_j from Assumption A. Right: the domain Ω in Theorem 1.1 and Theorem 1.6.

Remark 1.2. The neighborhoods Ω in Theorems 1.1 and 1.6 are given explicitly in Sections 2 and 3; see also Figure 1, right.

Remark 1.3. Unfortunately, the original Wigner–von Neumann potential [von Neumann and Wigner 1929], see also [Reed and Simon 1978, Section XIII.13],

$$V(x) = (1+g(x)^2)^{-2}(-32\sin x)(g(x)^3\cos x - 3g(x)^2\sin^3 x + g(x)\cos x + \sin^3 x),$$

where $g(x) = 2x - \sin 2x$, does not seem to satisfy Assumption A. In fact, the argument principle implies that if $\nu > \frac{1}{2}$ and $\ell \gg 1$ with $\ell \in \mathbb{Z}$, then $g(z) \pm i$ have two zeros in the region

$$\bigl\{z \in \mathbb{C} \bigm| \bigl(\ell - \tfrac{1}{2}\bigr)\pi \le \operatorname{Re} z \le \bigl(\ell + \tfrac{1}{2}\bigr)\pi,\ -\nu\log\ell \le \operatorname{Im} z \le \nu\log\ell\bigr\}.$$

Thus another method is needed to study the complex resonances for the original Wigner–von Neumann Hamiltonian.

Following the standard theory of resonances, the complex resonances are defined using this meromorphic continuation.

Definition 1.4. Let $R_+(z)$ be the meromorphic continuation of the resolvent for P as in Theorem 1.1. A complex number $z \in \Omega$ is called a resonance if z is a pole of $(f, R_+(z)g)$ for some $f, g \in L^2_{\mathrm{comp}}(\mathbb{R})$ and the multiplicity m_z is defined as the maximal number m such that there exist $f_1, \dots, f_m, g_1, \dots, g_m \in L^2_{\mathrm{comp}}(\mathbb{R})$ with

$$\det\biggl(\frac{1}{2\pi i}\oint_{C(z)} (f_i, R_+(\zeta)g_j)\,d\zeta\biggr)_{i,j=1}^{m} \neq 0,$$

where $C(z)$ is a small circle around z. The set of resonances is denoted by $\operatorname{Res}(P)$.

Remark 1.5. $\operatorname{Res}(P)$ is discrete in Ω and $m_z < \infty$ for any $z \in \Omega$ (see Remark 2.3).

We prove Theorem 1.1 by introducing the periodic complex distortion in the Fourier space (see Section 2 for the definition and the underlying idea).

We now introduce the complex dissipative potential

$$P_\varepsilon = -\frac{d^2}{dx^2} + V(x) - i\varepsilon x^2, \quad \varepsilon > 0.$$

We easily see that $P_\varepsilon,\ \varepsilon > 0$, has purely discrete spectrum on $L^2(\mathbb{R})$. Zworski [2018] proved that the set of resonances can be characterized as limit points of the eigenvalues of P_ε as $\varepsilon \to 0$, namely $\lim_{\varepsilon\to 0} \sigma_d(P_\varepsilon) = \mathrm{Res}(P)$ for compactly supported potentials employing the dilation-analytic method. Zworski [2018] also proposed a problem of finding a potential $V(x)$ such that the limit set of $\sigma_d(P_\varepsilon)$ when $\varepsilon \to 0$ is not discrete, and suggested $V(x) = (\sin x)/x$ as a candidate for such a $V(x)$. Our next result disproves this conjecture (away from the thresholds).

Theorem 1.6. *Under Assumption A, there exists a complex neighborhood* $\Omega \subset \mathbb{C}$ *of* $[0,\infty) \setminus \mathcal{T}$ *such that* $\lim_{\varepsilon\to 0} \sigma_d(P_\varepsilon) = \mathrm{Res}(P)$ *in* Ω *including multiplicities. In particular,* $\lim_{\varepsilon\to 0} \sigma_d(P_\varepsilon)$ *is discrete in* Ω. *More precisely, for any* $z \in \Omega$ *there exists* $\rho_0 > 0$ *such that for any* $0 < \rho < \rho_0$ *there exists* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$

$$\#\sigma_d(P_\varepsilon) \cap B(z,\rho) = m_z,$$

where $B(z,\rho) = \{w \in \mathbb{C} \mid |w - z| \le \rho\}$.

Wigner–von Neumann-type Hamiltonians have been investigated by many authors. See for instance [Behncke 1991; 1994; Cruz-Sampedro et al. 2002; Devinatz et al. 1991; Froese and Herbst 1982; Hinton et al. 1991; Klaus 1991; Lukic 2013; Rejto and Taboada 1997; Richard et al. 2016]. To our knowledge, the definition of the complex resonances based on the complex distortion for Schrödinger operators with oscillatory and slowly decaying potentials is new. The complex distortion in the momentum variables is studied in [Cycon 1985; Sigal 1984] for radially symmetric dilation-analytic or sufficiently smooth exponentially decaying potentials. In [Nakamura 1990], this method is extended to the not necessarily radially symmetric case. See the references in that work for related earlier works on the complex distortion.

Stefanov [2005] studied the approximation of resonances by the fixed complex-absorbing potential method in the semiclassical limit. Similar methods are used in generalized geometric settings in [Nonnenmacher and Zworski 2009; 2015; Vasy 2013]. As mentioned above, Theorem 1.6 was proved by Zworski [2018] for compactly supported potentials. This was extended to more general dilation-analytic potentials in [Xiong 2020]. Analogous results were proved for Pollicott–Ruelle resonances in [Dyatlov and Zworski 2015] (see also [Dang and Riviere 2017; Drouot 2017]), and for 0th-order pseudodifferential operators in [Galkowski and Zworski 2019]. For the numerical results and original approach in physical chemistry, see the references in [Stefanov 2005; Zworski 2018].

This paper is organized as follows. In Section 2, we present the proofs of the theorems for the model case $V(x) = a(\sin 2x)/x$, which contain all the essential ideas for the general case. In Section 3, we present technical arguments which complete the proofs for the general case.

2. The model case

In this section, we explain the general ideas for the proofs and give the full proofs for the model case $V(x) = a(\sin 2x)/x,\ a \in \mathbb{R}$.

2A. ***Periodic distortion in the Fourier space.*** The main idea of Theorem 1.1 is as follows: We note the standard dilation-analytic method for the complex resonances does not apply to our potentials. On the other hand, it is known that if we set

$$A' = \tfrac{1}{2}(x \cdot D' + D' \cdot x), \quad D'u(x) = \tfrac{1}{2\pi i}(u(x+\pi) - u(x-\pi)),$$

then we can construct a Mourre theory with this conjugate operator; see [Nakamura 2014]. We can use this operator as the generator of complex distortion to define the resonances for our model. Actually, in the Fourier space, A' is a differential operator

$$\widetilde{A}' = \tfrac{1}{2\pi}((i\partial_\xi) \cdot \sin(\pi\xi) + \sin(\pi\xi) \cdot (i\partial_\xi)),$$

and this generates a periodic complex distortion in the Fourier space; see [Nakamura 1990] for Hunziker-type local distortion in the Fourier space.

Thus we introduce the periodic distortion in the Fourier space

$$\Phi_\theta(\xi) = \xi + \theta\sin(\pi\xi), \quad U_\theta f(\xi) = \Phi'_\theta(\xi)^{\frac{1}{2}} f(\Phi_\theta(\xi)),$$

where $\theta \in (-\pi^{-1}, \pi^{-1})$. In the Fourier space, P has the form $\widetilde{P} = \xi^2 + \widetilde{V}$, where $\widetilde{V} = (2\pi)^{-1/2}\widehat{V}*$ is a convolution operator and $\widehat{V}$ is the Fourier transform $\widehat{V}(\xi) = (2\pi)^{-1/2}\int V(x)e^{-ix\xi}\,dx$. Hence we have

$$\widetilde{P}_\theta := U_\theta \widetilde{P} U_\theta^{-1} = (\xi + \theta\sin(\pi\xi))^2 + \widetilde{V}_\theta, \quad \widetilde{V}_\theta = U_\theta \widetilde{V} U_\theta^{-1}.$$

Lemma 2.1. *Let* $V(x) = a(\sin 2x)/x$ *for* $a \in \mathbb{R}$. *Then* $\widetilde{V}_\theta = (\Phi'_\theta)^{1/2}\widetilde{V}(\Phi'_\theta)^{1/2}$, *where* $(\Phi'_\theta)^{1/2}$ *is a multiplication operator by* $\Phi'_\theta(\xi)^{1/2}$, *and* $\widetilde{V} = (a/2)\chi_{[-2,2]}*$, *where* $\chi_{[-2,2]}$ *denotes the characteristic function of* $[-2, 2]$. *In particular,* $\widetilde{V}_\theta$ *is analytic with respect to* θ *and* ξ^2*-compact for* $\theta \in \mathbb{C} \setminus ((-\infty, -\pi^{-1}] \cup [\pi^{-1}, \infty))$.

Proof. By direct computation, we immediately have $\widetilde{V} = (a/2)\chi_{[-2,2]}*$. Thus we have, for $\theta \in (-\pi^{-1}, \pi^{-1})$,

$$\begin{aligned}\widetilde{V}_\theta f(\xi) &= U_\theta \widetilde{V} U_\theta^{-1} f(\xi) \\ &= \int_{\mathbb{R}} \Phi'_\theta(\xi)^{\frac{1}{2}} \frac{a}{2}\chi_{[-2,2]}(\Phi_\theta(\xi) - \eta)(\Phi_\theta^{-1})'(\eta)^{\frac{1}{2}} f(\Phi_\theta^{-1}(\eta))\,d\eta \\ &= \int_{\mathbb{R}} \Phi'_\theta(\xi)^{\frac{1}{2}} \frac{a}{2}\chi_{[-2,2]}(\Phi_\theta(\xi) - \Phi_\theta(\eta))\Phi'_\theta(\eta)^{\frac{1}{2}} f(\eta)\,d\eta.\end{aligned}$$

On the other hand, we note

$$\frac{d}{d\xi}(\Phi_\theta(\xi) - \Phi_\theta(\eta)) = 1 + \theta\pi\cos(\pi\xi) > 0$$

for $\theta \in (-\pi^{-1}, \pi^{-1})$. Moreover, we have

$$\Phi_\theta(\eta \pm 2) - \Phi_\theta(\eta) = \pm 2 + \theta(\sin(\pi(\eta \pm 2)) - \sin(\pi\eta)) = \pm 2.$$

These imply that $-2 \le \Phi_\theta(\xi) - \Phi_\theta(\eta) \le 2$ if and only if $-2 \le \xi - \eta \le 2$. Thus we have

$$\begin{aligned}\widetilde{V}_\theta f(\xi) &= \int_{\mathbb{R}} \Phi'_\theta(\xi)^{\frac{1}{2}} \frac{a}{2}\chi_{[-2,2]}(\xi - \eta)\Phi'_\theta(\eta)^{\frac{1}{2}} f(\eta)\,d\eta \\ &= (\Phi'_\theta)^{\frac{1}{2}}\widetilde{V}(\Phi'_\theta)^{\frac{1}{2}} f(\xi).\end{aligned}$$

The second part of Lemma 2.1 follows from the first part. We note that $(\Phi'_\theta)^{\frac{1}{2}}$ is well-defined for $\theta \in \mathbb{C} \setminus ((-\infty, -\pi^{-1}] \cup [\pi^{-1}, \infty))$ since $\Phi'_\theta(\xi) = 1 + \theta\pi\cos(\pi\xi) \neq 0$ and $\mathbb{C} \setminus ((-\infty, -\pi^{-1}] \cup [\pi^{-1}, \infty))$ is simply connected. ☐

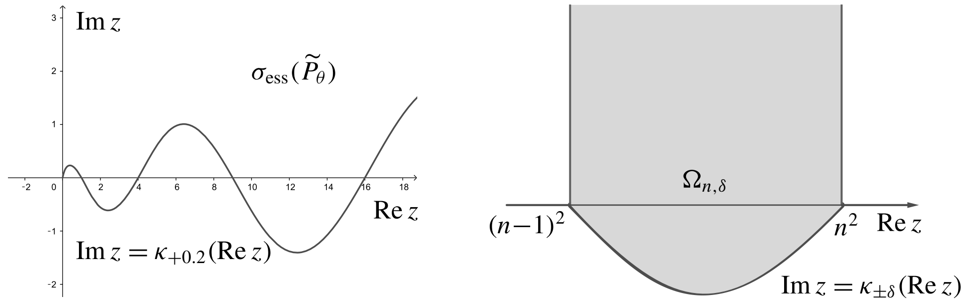

Figure 2. Left: $\sigma_{\mathrm{ess}}(\widetilde{P}_\theta)$ for $\theta = 0.2i$. Right: the region $\Omega_{n,\delta}$.

2B. ***Definition of resonances.*** In Sections 2B and 2C, we assume that $V(x) = a(\sin 2x)/x$ for $a \in \mathbb{R}$. The modifications needed for the general case are explained in Section 3.

By Lemma 2.1 we learn that $\widetilde{P}_\theta$ is analytic with respect to θ in the sense of Kato, and the essential spectrum of $\widetilde{P}_\theta$ is given by

$$\sigma_{\mathrm{ess}}(\widetilde{P}_\theta) = \{(\xi + \theta \sin(\pi\xi))^2 \mid \xi \in \mathbb{R}\};$$

see Figure 2, left.

Remark 2.2. We note that, for complex θ,

$$\sigma_{\mathrm{ess}}(\widetilde{P}_\theta) \cap [0, \infty) = \{n^2 \mid n \in \mathbb{N} \cup \{0\}\}.$$

Thus $\mathcal{T} = \{n^2 \mid n \in \mathbb{N} \cup \{0\}\} \subset [0, \infty)$ is considered as the set of thresholds with respect to our periodic complex distortion in the Fourier space and is analogous to the set of threshold $\{0\} \subset [0, \infty)$ in the case of the usual complex scaling. In addition to the usual threshold 0, the set $\mathcal{T}$ contains energy n^2, $n \in \mathbb{N}$, at which corresponding plane waves $e^{\pm inx}$ are half-harmonics, i.e., the waves of half-multiple frequencies of the oscillating part of the potential.

We fix $n \in \mathbb{N}$, and for the energy interval $((n-1)^2, n^2)$ we take $\theta = (-1)^n i\delta = \pm i\delta$. We easily see that for $0 < \delta < \pi^{-1}$ the essential spectrum of $\widetilde{P}_{\pm i\delta}$ is the graph of a function $\kappa_{\pm\delta} : [0, \infty) \to \mathbb{R}$ in $\mathbb{R}^2 \cong \mathbb{C}$. Namely, we may define $\kappa_{\pm\delta}(x)$, $x = \operatorname{Re} z \geq 0$, by the relation

$$\sigma_{\mathrm{ess}}(\widetilde{P}_{\pm i\delta}) = \{z \in \mathbb{C} \mid \operatorname{Im} z = \kappa_{\pm\delta}(\operatorname{Re} z),\ \operatorname{Re} z \geq 0\}.$$

More explicitly, if $x = \xi^2 - \delta^2 \sin^2(\pi\xi)$ for $\xi \in \mathbb{R}$, then $\kappa_{\pm\delta}(x) = \pm 2\delta\xi \sin(\pi\xi)$. A important fact is that $\kappa_{(-1)^n\delta}(x) < 0$ for $x \in ((n-1)^2, n^2)$.

We set $\delta_0 = \pi^{-1}$ and take any $0 < \delta < \delta_0$. We also set

$$\Omega_{n,\delta} = \{z = x + iy \mid (n-1)^2 < x < n^2,\ y > \kappa_{(-1)^n\delta}(x)\};$$

see Figure 2, right. Note that $\Omega_{n,\delta} \subset \Omega_{n,\delta'}$ if $0 < \delta < \delta' < \delta_0$.

Proof of Theorem 1.1 for the model case. We fix $n \in \mathbb{N}$ and $\delta > 0$ as above, and we write $\mathcal{A} = L^2_{\mathrm{comp}}(\mathbb{R})$. We first note that $U_\theta \hat{f}$ $(f \in \mathcal{A})$ has an analytic continuation for complex θ. We denote the resolvent

$R_+(z)$ on the Fourier space by $\widetilde{R}_+(z)$. For $f, g \in \mathcal{A}$, we have

$$(\hat{f}, \widetilde{R}_+(z)\hat{g}) = (U_\theta \hat{f}, U_\theta \widetilde{R}_+(z) U_\theta^{-1} U_\theta \hat{g}) = (U_{\bar{\theta}} \hat{f}, (z - \widetilde{P}_\theta)^{-1} U_\theta \hat{g}),$$

where $\theta \in \mathbb{R}$ and $\operatorname{Im} z > 0$. The right-hand side is analytic with respect to θ by Lemma 2.1, where θ ranges over a complex neighborhood of $\{(-1)^n i\delta \mid 0 \le \delta < \delta_0\}$. This in turn implies that the left-hand side has a meromorphic continuation to Ω_{n,δ_0} with respect to z. Thus Theorem 1.1 is proved for $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_{n,\delta_0}$. □

Remark 2.3. We set

$$\Pi_z^\theta = \frac{1}{2\pi i} \oint_{C(z)} (\zeta - \widetilde{P}_\theta)^{-1}\, d\zeta$$

to be the spectral projection for $\widetilde{P}_\theta$. Then we have

$$\frac{1}{2\pi i} \oint_{C(z)} (f, R_+(\zeta) g)\, d\zeta = \frac{1}{2\pi i} \oint_{C(z)} (U_{\bar{\theta}} \hat{f}, (\zeta - \widetilde{P}_\theta)^{-1} U_\theta \hat{g})\, d\zeta = (U_{\bar{\theta}} \hat{f}, \Pi_z^\theta U_\theta \hat{g}).$$

We note that $\{U_\theta \hat{f} \mid f \in \mathcal{A}\}$ is dense in L^2, which is proved by an argument similar to [Hunziker 1986, Theorem 3]. This implies that $m_z = \operatorname{rank}[\Pi_z^\theta]$. Namely, the resonances coincide with the discrete eigenvalues of $\widetilde{P}_\theta$ including multiplicities. In particular, $\operatorname{Res}(P)$ is discrete and $m_z < \infty$ for any $z \in \Omega$.

2C. *Viscosity limit.* As in [Zworski 2018], the essential ingredient of the proof of Theorem 1.6 is the resolvent estimate of the distorted operator which is uniform with respect to ε in the case of $V = 0$. We prove this by employing the semiclassical analysis in the Fourier space with the semiclassical parameter $h = \sqrt{\varepsilon}$. Since we work in the Fourier space, the term $-i\varepsilon x^2 = i\varepsilon \partial_\xi^2$ is the usual viscosity term (multiplied by i) and the viscosity limit corresponds to the semiclassical limit.

For notational simplicity, we set $P_0 = P$, $\widetilde{P}_0 = \widetilde{P}$ and $\widetilde{P}_{0,\theta} = \widetilde{P}_\theta$. In the Fourier space, P_ε, $\varepsilon \ge 0$, has the form

$$\widetilde{P}_\varepsilon = \xi^2 + \widetilde{V} + i\varepsilon \partial_\xi^2.$$

Hence the distorted operator $\widetilde{P}_{\varepsilon,\theta} = U_\theta \widetilde{P}_\varepsilon U_\theta^{-1}$ is given by

$$\widetilde{P}_{\varepsilon,\theta} = (\xi + \theta \sin(\pi\xi))^2 + \widetilde{V}_\theta - i\varepsilon D_\xi (1 + \pi\theta \cos(\pi\xi))^{-2} D_\xi - i\varepsilon r_\theta(\xi),$$

where $r_\theta(\xi) = -\Phi_\theta'(\xi)^{-1/2} \partial_\xi (\Phi_\theta'(\xi)^{-1} \partial_\xi (\Phi_\theta'(\xi)^{-1/2}))$ is a function which is analytic with respect to θ and bounded with respect to ξ. Since $\widetilde{P}_{\varepsilon,\theta}$ has a compact resolvent, $\widetilde{P}_{\varepsilon,\theta}$, $\varepsilon > 0$, has purely discrete spectrum. Moreover, for fixed $\varepsilon > 0$, $\widetilde{P}_{\varepsilon,\theta}$ is analytic with respect to θ in the sense of Kato. These imply that the eigenvalues of $\widetilde{P}_{\varepsilon,\theta}$ coincide with those of $\widetilde{P}_\varepsilon$ including multiplicities by the same argument as in Remark 2.3. Thus it is enough to show that the eigenvalues of $\widetilde{P}_{\varepsilon,\theta}$ converge to those of $\widetilde{P}_\theta$ as $\varepsilon \to +0$.

Proof of Theorem 1.6 for the model case. We first prove the resolvent estimate (2-1) for the distorted free Hamiltonian

$$\widetilde{Q}_{\varepsilon,\theta} = (\xi + \theta \sin(\pi\xi))^2 - i\varepsilon D_\xi (1 + \pi\theta \cos(\pi\xi))^{-2} D_\xi - i\varepsilon r_\theta(\xi), \quad \varepsilon \ge 0.$$

In the following, we fix $n \in \mathbb{N}$, and set $\theta = (-1)^n i\delta = \pm i\delta$, $0 < \delta < \delta_0$, as in Section 2B.

We set $h=\sqrt{\varepsilon}$ and view $\widetilde{Q}_{\varepsilon,\theta}$ as an h-pseudodifferential operator in the Fourier space. Recall that $\Omega_{n,\delta}$ is defined in Section 2B; see Figure 2, right. We easily see that the numerical range of the h-principal symbol of $\widetilde{Q}_{\varepsilon,\theta}$, i.e.,

$$\{(\xi+\theta\sin(\pi\xi))^2-i(1+\pi\theta\cos(\pi\xi))^{-2}x^2 \mid x,\xi\in\mathbb{R}\},$$

is disjoint from $\Omega_{n,\delta}$ for small $\delta>0$. For instance, this is true for $0<\delta\le\delta_1$, where $\delta_1=(\sqrt{2}-1)\pi^{-1}$. The constant δ_1 comes from requiring

$$\sup_{x\ge 0}\left|\frac{d}{dx}\kappa_{\pm\delta}(x)\right|=\left|\frac{d}{dx}\kappa_{\pm\delta}(0)\right|=\frac{2\pi\delta}{1-\pi^2\delta^2}$$

is less than or equal to the minimal value $\frac{1}{2}(1/(\pi\delta)-\pi\delta)$ with respect to $\xi\in\mathbb{R}$ of the absolute value of the slope of the half-line $\{-i(1\pm\pi\delta i\cos(\pi\xi))^{-2}x^2 \mid x\in\mathbb{R}\}$ in the complex plane. For simplicity, we consider $0<\delta<\delta_1$ and do not pursue the optimal δ. Now we fix $0<\delta<\delta_1$ and $z\in\Omega_{n,\delta}$. Then there exists $\rho_0>0$ such that there is no resonance in $B(z,\rho_0)\Subset\Omega_{n,\delta}$ possibly expect for z, where $B(z,\rho)$ denotes the ball of radius ρ with the center at z. In the following, we fix $0<\rho<\rho_0$, and let $w\in B_z=B(z,\rho)$. By the standard semiclassical calculus we learn $(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}$ exists and

$$\|(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}\|_{L^2\to L^2}\le C \tag{2-1}$$

for $w\in B_z$ and for sufficiently small $\varepsilon>0$. We note that it also holds for $\varepsilon=0$.

We next employ the perturbation argument. Since $(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}$ exists, we have

$$\widetilde{P}_{\varepsilon,\theta}-w=(1+\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1})(\widetilde{Q}_{\varepsilon,\theta}-w).$$

By Lemma 2.1 and the boundedness of $(\xi^2+i)(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}$, we learn $\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}$ is compact for $\varepsilon\ge 0$. Thus the analytic Fredholm theory can be applied. We have

$$\begin{aligned}(w-\widetilde{P}_{\varepsilon,\theta})^{-1}&=(\partial_w(\widetilde{P}_{\varepsilon,\theta}-w))(\widetilde{P}_{\varepsilon,\theta}-w)^{-1}\\&=(\partial_w\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1})(1+\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1})^{-1}\\&\qquad+(1+\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1})(w-\widetilde{Q}_{\varepsilon,\theta})^{-1}(1+\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1})^{-1}.\end{aligned}$$

The Gohberg–Sigal factorization [1971, Theorem 3.1] applied to $1+\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}$, Cauchy's theorem and the cyclicity of the trace imply that

$$\operatorname{tr}\oint_{\partial B_z}(1+\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1})(w-\widetilde{Q}_{\varepsilon,\theta})^{-1}(1+\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1})^{-1}\,dw=0.$$

Thus the number of the eigenvalues of $P_{\varepsilon,\theta}$, $\varepsilon\ge 0$, in B_z is given by

$$\operatorname{tr}\frac{1}{2\pi i}\oint_{\partial B_z}(w-\widetilde{P}_{\varepsilon,\theta})^{-1}\,dw=\operatorname{tr}\frac{1}{2\pi i}\oint_{\partial B_z}(\partial_w\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1})(1+\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1})^{-1}\,dw.$$

Note that the right-hand side of this equality is the number of zeros of $1+\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}$ in B_z in the sense of [Gohberg and Sigal 1971, Theorem 2.1]. Thus the operator-valued Rouché theorem [Gohberg and Sigal 1971, Theorem 2.2] implies that in order to prove Theorem 1.6, it suffices to show

$$\|((1+\widetilde{V}_\theta(\widetilde{Q}_{0,\theta}-w)^{-1})-(1+\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}))(1+\widetilde{V}_\theta(\widetilde{Q}_{0,\theta}-w)^{-1})^{-1}\|_{L^2\to L^2}<1$$

for $w\in\partial B_z$ and small $\varepsilon>0$. Since $(1+\widetilde{V}_\theta(\widetilde{Q}_{0,\theta}-w)^{-1})^{-1}$ exists and independent of $\varepsilon>0$ for $w\in\partial B_z$, the above estimate holds if we show

$$\lim_{\varepsilon\to 0}\|\widetilde{V}_\theta(\widetilde{Q}_{0,\theta}-w)^{-1}-\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}\|_{L^2\to L^2}=0 \tag{2-2}$$

uniformly for $w\in\partial B_z$.

Let $\gamma>0$. We claim that we can decompose $\widetilde{V}_\theta=\widetilde{V}_{\theta,1}+\widetilde{V}_{\theta,2}$, where $\widetilde{V}_{\theta,1}$ is a smoothing pseudodifferential operator in the Fourier space and $\|\widetilde{V}_{\theta,2}\|_{L^2\to L^2}<\gamma$. To see this, we take the decomposition

$$\widetilde{V}_\theta=(\Phi'_\theta)^{\frac12}\widetilde{V}(\Phi'_\theta)^{\frac12}=(\Phi'_\theta)^{\frac12}\widetilde{V}_{1,R}(\Phi'_\theta)^{\frac12}+(\Phi'_\theta)^{\frac12}\widetilde{V}_{2,R}(\Phi'_\theta)^{\frac12}=\widetilde{V}_{\theta,1}+\widetilde{V}_{\theta,2}$$

for large $R>0$, where $\widetilde{V}_{j,R}$ is the Fourier multiplier on the Fourier space by $V_{j,R}$, $\chi\in C_c^\infty(\mathbb{R})$ such that $\chi=1$ near $x=0$, and

$$a\frac{\sin 2x}{x}=a\frac{\sin 2x}{x}\chi\Big(\frac{x}{R}\Big)+a\frac{\sin 2x}{x}\Big(1-\chi\Big(\frac{x}{R}\Big)\Big)=V_{1,R}+V_{2,R}.$$

Then the claimed properties are easily verified.

Since $\|(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}\|_{L^2\to L^2}\le C$ for small $\varepsilon\ge 0$ and $w\in B_z$, we have

$$\|\widetilde{V}_{\theta,2}(\widetilde{Q}_{0,\theta}-w)^{-1}-\widetilde{V}_{\theta,2}(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}\|_{L^2\to L^2}\le 2C\gamma,$$

where C is independent of γ. By the resolvent equation, we also learn

$$\begin{aligned}&\widetilde{V}_{\theta,1}(\widetilde{Q}_{0,\theta}-w)^{-1}-\widetilde{V}_{\theta,1}(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}\\&\qquad=-i\varepsilon\widetilde{V}_{\theta,1}(\widetilde{Q}_{0,\theta}-w)^{-1}(D_\xi(1+\pi\theta\cos(\pi\xi))^{-2}D_\xi+r_\theta(\xi))(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}.\end{aligned}$$

Since $\widetilde{V}_{\theta,1}$ is a smoothing pseudodifferential operator and $(\widetilde{Q}_{0,\theta}-w)^{-1}$ is also a pseudodifferential operator with a bounded symbol, $\widetilde{V}_{\theta,1}(\widetilde{Q}_{0,\theta}-w)^{-1}D_\xi^2$ is L^2-bounded. Thus we have

$$\|\widetilde{V}_{\theta,1}(\widetilde{Q}_{0,\theta}-w)^{-1}-\widetilde{V}_{\theta,1}(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}\|_{L^2\to L^2}\le C_\gamma\varepsilon,$$

with some (γ-dependent) constant $C_\gamma>0$. If ε is so small that $\varepsilon\le(C/C_\gamma)\gamma$, we have

$$\|\widetilde{V}_\theta(\widetilde{Q}_{0,\theta}-w)^{-1}-\widetilde{V}_\theta(\widetilde{Q}_{\varepsilon,\theta}-w)^{-1}\|_{L^2\to L^2}\le 2C\gamma+C_\gamma\varepsilon\le 3C\gamma$$

and thus (2-2) is proved since $\gamma>0$ may be arbitrary small. Thus Theorem 1.6 is proved for $\Omega=\bigcup_{n\in\mathbb{N}}\Omega_{n,\delta_1}$. □

3. The general case

3A. *Analyticity of $\widetilde{V}_\theta$.* We recall that $\widetilde{V}_\theta$ was defined in Section 2A.

Lemma 3.1. *Under Assumption A, $\widetilde{V}_\theta$ is analytic with respect to θ and ξ^2-compact for θ in some complex neighborhood of $\{i\delta\mid -K\pi^{-1}<\delta<K\pi^{-1}\}$, where K is the constant in Assumption A.*

Proof of Lemma 3.1. For real θ, the integral kernel $\widetilde{V}_\theta(\xi,\eta)$ of $\widetilde{V}_\theta$ is given by

$$\widetilde{V}_\theta(\xi,\eta)=\frac{1}{\sqrt{2\pi}}\Phi_\theta'(\xi)^{\frac{1}{2}}\widehat{V}(\Phi_\theta(\xi)-\Phi_\theta(\eta))\Phi_\theta'(\eta)^{\frac{1}{2}},\quad \xi,\eta\in\mathbb{R}.$$

We first consider the case of $V\in C_c^\infty(\mathbb{R};\mathbb{R})$. Then the Paley–Wiener estimate implies that $\widetilde{V}_\theta(\xi,\eta)$ is analytic with respect to $\theta\in\mathbb{C}$ and has the off-diagonal decay bounds

$$|\partial_\xi^\alpha\partial_\eta^\beta\widetilde{V}_\theta(\xi,\eta)|\le C_{\alpha,\beta,N}\langle\xi-\eta\rangle^{-N},\quad \xi,\eta\in\mathbb{R},$$

for any α, β and N, where $C_{\alpha,\beta,N}$ is independent of θ when $\theta\in\mathbb{C}$ ranges over a bounded set. We also recall the formula, see, e.g., [Zworski 2012, Section 8.1],

$$\widetilde{V}_\theta=b^{\mathrm{w}}(\xi,D_\xi;\theta),\quad b(\xi,x;\theta)=\int_{\mathbb{R}}\widetilde{V}_\theta\left(\xi+\frac{\eta}{2},\xi-\frac{\eta}{2}\right)e^{-i\langle\eta,x\rangle}\,d\eta,$$

where b^{w} denotes the Weyl quantization

$$b^{\mathrm{w}}(\xi,D_\xi;\theta)f(\xi)=\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}b\left(\frac{\xi+\eta}{2},x;\theta\right)e^{i\langle\xi-\eta,x\rangle}f(\eta)\,d\eta\,dx.$$

In fact, the integral kernel of $b^{\mathrm{w}}(\xi,D_\xi;\theta)$ is

$$\frac{1}{2\pi}\int_{\mathbb{R}}b\left(\frac{\xi+\eta}{2},x;\theta\right)e^{i\langle\xi-\eta,x\rangle}\,dx$$

and this coincides with $\widetilde{V}_\theta(\xi,\eta)$ by simple computations. These imply that $\widetilde{V}_\theta$ is a pseudodifferential operator in the Fourier space with a symbol rapidly decaying with respect to x (that is,

$$|\partial_\xi^\alpha\partial_x^\beta b(\xi,x;\theta)|\le C_{\alpha,\beta,N}\langle x\rangle^{-N},\quad \xi,x\in\mathbb{R},$$

for any α, β and N, where $C_{\alpha,\beta,N}$ is independent of θ when $\theta\in\mathbb{C}$ ranges over a bounded set) and analytic with respect to θ. Thus Lemma 3.1 is proved in this case.

We next consider the case of $V(x)=s(x)W(x)$, where $s(x)$ and $W(x)$ satisfy the condition in Assumption A; see Figure 1, left. We first estimate the Fourier transform of $W(x)$. By the deformation of the integral (see Figure 3, left), we have

$$\widehat{W}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{C_{\pm,\tau}}W(z)e^{-iz\xi}\,dz,\quad \pm\xi>0,$$

where

$$C_{\pm,\tau}=(e^{\pm i\tau}(-\infty,0]-2R_0)\cup[-2R_0,2R_0]\cup(2R_0+e^{\mp i\tau}[0,\infty)),$$

$0<\tau<\arctan K$, and R_0 is that in Assumption A. This expression shows that $\widehat{W}(\xi)$ has an analytic continuation to

$$S_\tau=\{z\in\mathbb{C}^*\mid-\tau<\arg z<\tau\}\cup\{z\in\mathbb{C}^*\mid-\tau<\arg z-\pi<\tau\};$$

see Figure 3, right. We see that $\widehat{W}(\xi)$ decays rapidly in S_τ when $|\xi|\to\infty$ thanks to the smoothness of W. For small $\xi\in S_\tau$, we have $|\widehat{W}(\xi)|\le C|\xi|^{-1/(1+\mu)}$, where $\mu>0$ is the constant in Assumption A. To see

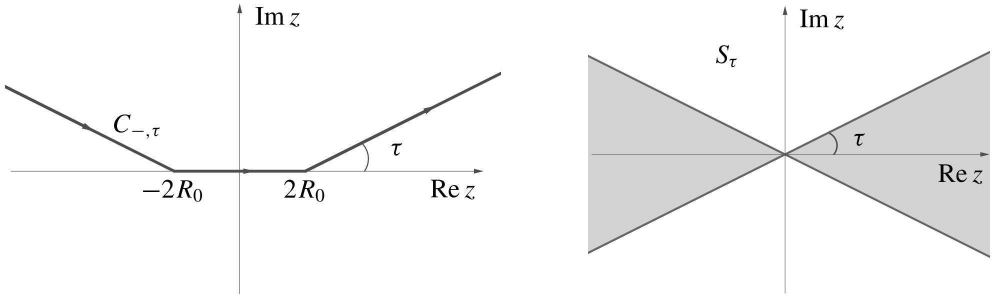

Figure 3. Left: the curve $C_{-,\tau}$, and $C_{+,\tau}$ is its reflection with respect to the real axis. Right: the domain S_τ.

this, we take $C_{\pm,\tau'}$ for $0 < \tau < \tau' < \arctan K$ and estimate

$$|\widehat{W}(\xi)| \le C \int_0^\infty e^{-cx|\xi|} \langle x \rangle^{-\mu}\, dx = C|\xi|^{-1} \int_0^\infty e^{-c|x|} \langle x/|\xi| \rangle^{-\mu}\, dx.$$

We divide the integral into $\int_0^\varepsilon + \int_\varepsilon^\infty$ and we obtain the bound

$$\frac{\varepsilon}{|\xi|} + \frac{1}{|\xi|} \langle \varepsilon/|\xi| \rangle^{-\mu}.$$

Taking $\varepsilon = |\xi|^{\mu/(1+\mu)}$, we have $|\widehat{W}(\xi)| \le C|\xi|^{-1/(1+\mu)}$.

We next claim that the Fourier transform $\widehat{V}(\xi)$ has an analytic continuation to the region $T_\tau = \bigcup_{k\in\mathbb{Z}} T_{\tau,k}$, where (see Figure 4)

$$T_{\tau,k} = \{z \in \mathbb{C} \setminus \{0, 2\} \mid -\tau < \arg z < \tau,\ -\tau < \arg(2 - z) < \tau\} + 2k,$$

and the estimate

$$\sum_{k\in\mathbb{Z}} \sup_{\xi\in T_{\tau,k}} |\xi - 2k|^{\frac{1}{1+\mu}} |\xi - 2k - 2|^{\frac{1}{1+\mu}} |\widehat{V}(\xi)| < \infty \tag{3-1}$$

holds. To see this, we first denote the Fourier transform of s by $\hat{s}(\xi) = \sqrt{2\pi} \sum_{k\in\mathbb{Z}} a_k \delta(\xi - 2k)$. Then we have

$$\widehat{V}(\xi) = \sum_{k\in\mathbb{Z}} a_k \widehat{W}(\xi - 2k).$$

By Assumption A, we have $\sum_{k\in\mathbb{Z}} |a_k| < \infty$. The estimates on $\widehat{W}(\xi)$ above show

$$\sum_{k\in\mathbb{Z}} \sup_{\xi\in T_{\tau,k}} |\xi - 2k|^{\frac{1}{1+\mu}} |\xi - 2k - 2|^{\frac{1}{1+\mu}} |\widehat{W}(\xi)| < \infty.$$

Then the estimate (3-1) follows from Young's inequality in $\ell^1(\mathbb{Z})$ applied to sequences $\{a_k\}_{k\in\mathbb{Z}}$ and

$$\Big\{ \sup_{\xi\in T_{\tau,k}} |\xi - 2k|^{\frac{1}{1+\mu}} |\xi - 2k - 2|^{\frac{1}{1+\mu}} |\widehat{W}(\xi)| \Big\}_{k\in\mathbb{Z}}.$$

By (3-1), we have $|\widetilde{V}_\theta(\xi, \eta)| \le g(\xi - \eta)$ for some integrable function g. This is also true for $(\partial/\partial\theta)\widetilde{V}_\theta(\xi, \eta)$ by Cauchy's formula with respect to θ. Thus Young's inequality implies that the

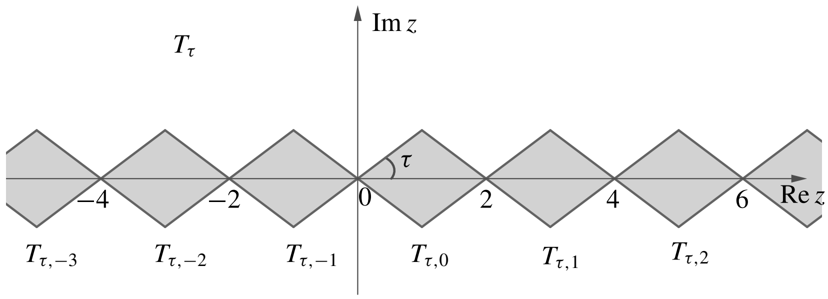

Figure 4. The domains T_τ and $T_{\tau,k}$.

operator $\widetilde{V}_\theta$ with integral kernel $\widetilde{V}_\theta(\xi,\eta)$ is L^2-bounded and analytic with respect to θ. We note that if θ is purely imaginary, we have

$$|\operatorname{Im}(\Phi_\theta(\xi)-\Phi_\theta(\eta))| \le \pi|\theta|\,|\operatorname{Re}(\Phi_\theta(\xi)-\Phi_\theta(\eta))-2k|,$$

with any $k\in\mathbb{Z}$, in particular k such that $|\xi-\eta-2k|\le 1$. Thus θ may be taken from a complex neighborhood of $\{i\delta \mid -\pi^{-1}\tan\tau<\delta<\pi^{-1}\tan\tau\}$. Since $0<\tau<\arctan K$ is arbitrary, $\widetilde{V}_\theta$ is analytic for θ as claimed in Lemma 3.1.

To see ξ^2-compactness, we approximate V by C_c^∞ functions. Take $\chi\in C_c^\infty(\mathbb{R})$ such that $\chi=1$ near $x=0$. We take the decomposition $V(x)=V_{1,R}+V_{2,R}$, where $R>0$,

$$V_{1,R}=\chi\Big(\frac{x}{R}\Big)W(x)\sum_{|k|\le R}a_k e^{2ikx},$$

$$V_{2,R}=W(x)\sum_{|k|>R}a_k e^{2ikx}+\Big(1-\chi\Big(\frac{x}{R}\Big)\Big)W(x)\sum_{|k|\le R}a_k e^{2ikx}.$$

We also denote the corresponding distorted operators on the Fourier space by $\widetilde{V}_{\theta,1,R}$ and $\widetilde{V}_{\theta,2,R}$. Since $V_{1,R}\in C_c^\infty$, we know $\widetilde{V}_{\theta,1,R}$ is ξ^2-compact. We also see that $\lim_{R\to\infty}\|\widetilde{V}_{\theta,2,R}\|_{L^2\to L^2}=0$ by the estimate for $V=s(x)W(x)$ as above. This completes the proof of Lemma 3.1. □

3B. ***Proofs of theorems for the general case.*** Although we set $\delta_0=\pi^{-1}$ for the model case in Section 2, we set $\delta_0=\min\{\pi^{-1},K\pi^{-1}\}$ for the general case in this subsection in view of Lemma 3.1. Similarly we set $\delta_1=\min\{(\sqrt{2}-1)\pi^{-1},K\pi^{-1}\}$ in this subsection. Then all the statements in Sections 2B and 2C remain true for these δ_0 and δ_1.

Proof of Theorem 1.1 for the general case. The proof is exactly the same as that for the model case in Section 2 if we replace Lemma 2.1 by Lemma 3.1. □

Proof of Theorem 1.6 for the general case. The proof is almost the same as that for the model case in Section 2. The only necessary change is the following: In the claim that we can take the decomposition $\widetilde{V}_\theta=\widetilde{V}_{\theta,1}+\widetilde{V}_{\theta,2}$, where $\widetilde{V}_{\theta,1}$ is a smoothing pseudodifferential operator in the Fourier space and $\|\widetilde{V}_{\theta,2}\|_{L^2\to L^2}<\gamma$, we set $\widetilde{V}_{\theta,1}=\widetilde{V}_{\theta,1,R}$ and $\widetilde{V}_{\theta,2}=\widetilde{V}_{\theta,2,R}$ for large $R>0$, where $\widetilde{V}_{\theta,j,R}$ was defined in the ξ^2-compactness part of the proof of Lemma 3.1. □

Remark 3.2. In the case of $V = a(\sin 2x)/x + V_0$, $V_0 \in C_c^\infty(\mathbb{R};\mathbb{R})$, Lemma 2.1 and the proof of Lemma 3.1 show that Lemma 3.1 holds for $\theta \in \mathbb{C} \setminus ((-\infty, -\pi^{-1}] \cup [\pi^{-1}, \infty))$. Thus the set of resonances $\mathrm{Res}_n(P)$ is defined in $\mathbb{C} \setminus (0, \infty)$ for any $n \in \mathbb{N}$ including multiplicities by the meromorphic continuation of $(f, R_+(z)g)$ from $\{z \mid 0 < \arg z < \pi\}$ to

$$\{z \mid 0 < \arg z < \pi\} \cup \{z \mid \arg z = 0, (n-1)^2 < |z| < n^2\} \cup \{z \mid -2\pi < \arg z < 0\}.$$

This poses the problem of whether $\mathrm{Res}_n(P) \neq \mathrm{Res}_{n'}(P)$ when $n \neq n'$.

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Cover image: The figure shows the outgoing scattered field produced by scattering a plane wave, coming from the northwest, off of the (stylized) letters P A A. The total field satisfies the homogeneous Dirichlet condition on the boundary of the letters. It is based on a numerical computation by Mike O'Neil of the Courant Institute.

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