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FULLY NONLINEAR PARABOLIC DEAD CORE PROBLEMS

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We establish geometric regularity estimates for diffusive models driven by fully nonlinear second-order parabolic operators with measurable coefficients under a strong absorption condition as follows:

$$\mathscr{F}(x,t,Du,D^2u)-\partial_t u=\lambda_0(x,t)u^{\mu}\chi_{\{u>0\}}\quad\text{in }\Omega_T:=\Omega\times(0,T),$$

where $\Omega\subset\mathbb{R}^n$ is a bounded and smooth domain, $0\le\mu<1$ and λ_0 is bounded away from zero and infinity. Such models arise in applied sciences and become mathematically interesting because they permit the formation of dead-core zones, i.e., regions where nonnegative solutions vanish identically. Our main result gives sharp and improved $C^{2/(1-\mu)}$ parabolic regularity estimates along the free boundary $\partial\{u>0\}$. In addition, we derive weak geometric and measure-theoretic properties of solutions and their free boundaries as: nondegeneracy, porosity, uniform positive density and finite speed of propagation. As an application, we prove a Liouville type result for entire solutions and we carry out a blow-up analysis. Finally, we prove the finiteness of parabolic $(n+1)$-Hausdorff measure of the free boundary for a particular class of operators.

1. Introduction

Throughout this article, we are interested in sharp and improved geometric regularity estimates for diffusive models with strong absorption as follows:

$$\textbf{(DCP)}\quad \mathscr{F}(x,t,Du,D^2u)-\partial_t u=\lambda_0(x,t).u^{\mu}\chi_{\{u>0\}}(x,t)\quad\text{in }\Omega_T:=\Omega\times(0,T),$$

with continuous and nonnegative boundary data, where $\Omega\subset\mathbb{R}^n$ is a bounded and smooth domain, $0\le\mu<1$ is the order of reaction, λ_0 is bounded away from zero and infinity and it is known as the *Thiele modulus*. Moreover, $\mathscr{F}:\Omega_T\times\mathbb{R}^n\times\mathrm{Sym}(n)\to\mathbb{R}$ is a fully nonlinear, second-order uniformly elliptic operator with Lipschitz character: there exist constants $\Lambda\ge\lambda>0$ (*ellipticity parameters*) and $\kappa\ge 0$ such that

$$\lambda\|Y\|-\kappa|\varsigma|\le\mathscr{F}(x,t,\xi,X)-\mathscr{F}(x,t,\xi+\varsigma,X+Y)\le\Lambda\|Y\|+\kappa|\varsigma|\tag{1-1}$$

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for every $X, Y \in \mathrm{Sym}(n)$ with $Y \geq 0$, $(x,t,\xi,\varsigma) \in \Omega_T \times \mathbb{R}^n \times \mathbb{R}^n$. It is worth mentioning that $\mathscr{F}$ is assumed to have bounded measurable coefficients. For this reason, bounded viscosity solutions to **(DCP)** have, as the best a priori estimates, a universal Hölder modulus of continuity according to celebrated Krylov–Safonov parabolic estimates; see [Krylov and Safonov 1980] for more details.

Another fundamental aspect of such models is that if $f(u) = \lambda_0 u^\mu \chi_{\{u>0\}}$ is not Lipschitz, then the maximum principle is not applicable. Consequently, nonnegative solutions may create plateau regions, which are known in the literature as *dead-cores* and represent regions where no reaction takes place in the diffusion process from **(DCP)**. Solutions of this class are currently called dead-core solutions and appear in a number of physical-mathematical models; see, for example, [Antontsev et al. 2002; Bandle and Stakgold 1984; Choe and Weiss 2003; da Silva et al. 2018; Guo and Souplet 2005].

The main first result of our manuscript concerns sharp and improved regularity of dead-core solutions along their free boundaries. We refer the reader to Section 2 for the employed notation.

Theorem 1.1 (improved regularity at free boundary points). *Let u be a nonnegative and bounded viscosity solution to* **(DCP)**, *so that $\partial_t u \geq -\mathfrak{c}_0(x,t)u^\mu \chi_{\{u>0\}}$ (in the viscosity sense)*[1] *for $\mathfrak{c}_0$ a nonnegative bounded function and $\mathfrak{K} \Subset \Omega_T$ a compact set. Then there exists a universal constant*[2] *$\mathfrak{C} > 0$ such that for all $(x_0,t_0) \in \partial\{u>0\} \cap \mathfrak{K}$,*

$$u(x,t) \leq \mathfrak{C} \cdot \|u\|_{L^\infty(\Omega_T)} \mathrm{dist}_p((x,t),(x_0,t_0))^{\frac{2}{1-\mu}},$$

for all (x,t) sufficiently close to (x_0,t_0).

We shall also provide how dead-core solutions leave their free boundaries.

Theorem 1.2 (nondegeneracy). *There exists a constant $\mathfrak{C}_0^* = \mathfrak{C}_0^*(n,\lambda,\Lambda,\kappa,\mathfrak{m},\mu) > 0$ such that any viscosity subsolution to* **(DCP)** *satisfies*

$$\sup_{\mathscr{C}_r^-(x_0,t_0)} u(x,t) \geq \mathfrak{C}_0^* \cdot r^{\frac{2}{1-\mu}}, \tag{1-2}$$

for any $(x_0,t_0) \in \overline{\{u>0\}} \cap \Omega_T$ and $\mathscr{C}_r(x_0,t_0) \subset \Omega \times (0,\infty)$.

Finally, our last main result concerns the parabolic Hausdorff measure of the free boundary.

[1] Notice that such an assumption is weaker than those imposed in [Choe and Weiss 2003; Shahgholian 2003; 2008]. It means that solutions can decrease in time direction, but with an appropriate lower bound control.

[2] Throughout this manuscript *universal constants* are those which depend only on dimension and structural parameters of the problem, namely λ, Λ (ellipticity constants of the operator), $\mathfrak{m}$, $\mathfrak{M}$ (bound of λ_0), κ (bounds for the gradient variable of $\mathfrak{F}$), $\mathrm{dist}(\mathfrak{K}, \partial_p\Omega_T)$ and μ.

Theorem 1.3 (Hausdorff measure estimates). *Let u be a viscosity solution to* **(DCP)** *with $\mathscr{F}$ a concave operator. There exists a universal constant $\mathfrak{C} > 0$ such that for all $(x_0, t_0) \in \partial\{u > 0\} \cap \mathscr{C}_{1/2}$,*

$$\mathscr{H}^{n+1}_{\text{par}}(\partial\{u > 0\} \cap \mathscr{C}_\rho(x_0, t_0)) \le \mathfrak{C}\rho^{n+1},$$

for all $\rho \ll 1$, where $\mathscr{H}^{n+1}_{\text{par}}$ is the $(n+1)$-dimensional Hausdorff measure with respect to the parabolic metric.

1A. ***Motivations, state of the art and overview.*** Throughout the last four decades parabolic PDEs with strong absorption conditions have received much attention due to their connections with the modeling of several phenomena in pure and applied sciences (see [Antontsev et al. 2002; Bandle and Stakgold 1984; Díaz 2001; Stakgold 1986]). An illustrative example coming from isothermal, catalytic reaction-diffusion processes is

$$\begin{cases} \Delta u - \partial_t u = u^\mu \chi_{\{u>0\}} & \text{in } \Omega_T, \\ u(x,t) = g(x,t) & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases}$$

where the boundary data satisfy

$$0 < u_0 \in C^0(\overline{\Omega}), \quad g(x,t) = \mathfrak{k} > 0 \quad \text{and} \quad u(x,0) = \mathfrak{k} \quad \text{for all } x \in \partial\Omega.$$

In this context, u represents the concentration of a (gas-liquid) reactant over a diffusing material evolving in time. Hence, the development of dead-core regions occurs precisely when the reactant becomes inactive. Notice that the boundary condition means that the reactant is injected with a fixed isothermal flux on the boundary. From a chemical engineering point of view, to understand the dead-core phenomenon is crucial, since the catalytic material is wasted precisely along the dead-core zone.

Other insights for our study come from the theory of nonlinear geometric free boundary problems (see [Apushkinskaya et al. 2002; Caffarelli et al. 2004; Shahgholian 2003; Teixeira 2016] for some enlightening examples). In this direction, we cite the class of "pseudo" free boundary problems

$$\textbf{(FBP)} \qquad \begin{cases} \max\{\mathfrak{P}[u] - \partial_t u, -u\} = 0 & \text{in } \Omega_T, \\ u(x,t) = g(x,t) & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{on } \overline{\Omega}, \end{cases}$$

where $\mathfrak{P}[u] := \mathscr{F}(x,t,Du,D^2u) - \lambda_0(x,t)u^\mu\chi_{\{u>0\}}(x,t)$ for μ, g, u_0 and $\mathscr{F}$ as before. As a particular application of **(FBP)** to financial markets, we may consider

the operator

$$\mathfrak{P}[\mathscr{V}] = \tfrac{1}{2}(\sigma_1\mathfrak{S}_1, \sigma_2\mathfrak{S}_2)^T \begin{pmatrix} \frac{\partial^2\mathscr{V}}{\partial(\mathfrak{S}_1)^2} & \frac{\partial^2\mathscr{V}}{\partial\mathfrak{S}_1\partial\mathfrak{S}_2} \\ \frac{\partial^2\mathscr{V}}{\partial\mathfrak{S}_2\partial\mathfrak{S}_1} & \frac{\partial^2\mathscr{V}}{\partial(\mathfrak{S}_2)^2} \end{pmatrix} \cdot (\sigma_1\mathfrak{S}_1, \sigma_2\mathfrak{S}_2) + \sum_{i=1}^{2}(r_0 - \sigma_i)\mathfrak{S}_i \frac{\partial\mathscr{V}}{\partial\mathfrak{S}_i} - r_0\mathscr{V}_+^{1-\mu},$$

where $r_0 > 0$ is the interest rate, σ_i is the volatility of the price of the corresponding asset, $\mathscr{V}$ is the American option and $(\mathfrak{S}_1, \mathfrak{S}_2)$ is the price vector of underlying assets. Such nonlinear obstacle problems can be interpreted as the extended model (with zero constraint) in pricing of American options in financial mathematics, which precisely deals with the case $\mu = 0$ (an obstacle type problem). The interested reader may see the references [Blanchet et al. 2005; 2006; Petrosyan and Shahgholian 2007; Shahgholian 2008] for a more complete treatment.

Despite the fact that a number of qualitative and quantitative features for linear models in divergence form like

$$\Delta u - \partial_t u = \lambda_0 u^\mu \chi_{\{u>0\}}(x,t) \quad \text{in } \Omega_T \tag{1-3}$$

have been extensively studied by many authors by using variational approaches in the last four decades (see [Choe and Weiss 2003] for a seminal treatment, see also [Antontsev et al. 2002] and [Díaz 2001] for classical references on this theme), many pivotal issues have not been established for a general model **(DCP)**, until now, due to the rigidity of the structure of such operators. For this reason, the treatment of such free boundary problems in nondivergence form requires the development of new approaches and modern techniques.

We are particularly interested in the smoothness and weak geometric properties around free boundary points of viscosity solutions of models like **(DCP)**. Such issues were our impetus for researching parabolic dead-core problems via a modern, nonvariational and systematic approach based on geometric regularity theory (compare with [Choe and Weiss 2003] for a dead core problem ruled by a heat operator and [da Silva et al. 2018] for its extension to degenerate evolution operators).

Beyond the several applications, the topics treated in this article help to understand general issues in free boundary problems. This fact is illustrated by Theorem 1.1, which shows that the dead-core's analysis brings to light an impressive feature: better regularity estimates (at free boundary points) than those currently available. In effect, in our approach we impose just bounded measurable coefficients for $\mathscr{F}$. Notwithstanding, the modulus of continuity improves upon the expected Hölder regularity coming from the classical Krylov–Safonov regularity estimate (see [Crandall et al. 2000, Section 5; Krylov and Safonov 1980; Wang 1992a, Section 4.4]). Furthermore, even for constant coefficient problems, $\mathscr{F}(Du, D^2u) - \partial_t u = f(x,t,u) \in L^\infty$, our

result is surprising, because in this setting, the $C^{1+\alpha,(1+\alpha)/2}$-estimate is the available regularity; see [Wang 1992b, Section 1.2] and [da Silva and Teixeira 2017, Sections 4 and 6]. Finally, we must compare our estimates with ones coming from Schauder type estimates (see [Tian and Wang 2013] and [Wang 1992b, Section 1.1]). For simplicity, let us suppose that $\mathscr{F}(D^2u) = \Delta u$ and $\lambda_0 > 0$ is constant. Notice that $\lambda_0 u^\mu \in C^{\mu,\mu/2}(\mathscr{C}_1)$. Therefore, the classical Schauder theory implies that $u \in C^{2+\mu,(2+\mu)/2}_{\mathrm{loc}}(\mathscr{C}_1)$. On the other hand, the estimates from Theorem 1.1 tell us that u has $\omega(s) = s^{2/(1-\mu)}$ as modulus of continuity at points on the free boundary. Finally, the main point is that

$$\frac{2}{1-\mu} > 2+\mu$$

for any $\mu \in (0, 1)$. In other words, we obtain an improved decay estimate (at free boundary points for a right-hand side which is not not necessarily Hölder) when compared with classical Schauder estimates. Taking into account the previous statements, our results are new even for linear parabolic problems with nondivergence structure and bounded and merely measurable coefficients.

The insight for the proof of Theorem 1.1 is inspired by techniques from regularity theory of fully nonlinear equations and free boundary problems (see, e.g., [da Silva et al. 2017; da Silva and Teixeira 2017; Shahgholian 2003; 2008; Teixeira 2016]). It consists of a finer geometric decay throughout an iterative process, which is based on the sharp scaling of the equation and maximum principle tools for a limiting caloric profile via a contradiction reasoning. It is worth mentioning that a difficulty in our studies is the absence of a strong maximum principle for $\mathscr{F}$-caloric functions, i.e., $\mathscr{F}(D^2\mathfrak{h}) - \partial_t\mathfrak{h} = 0$ (a viscosity solution to homogeneous problem with constant coefficients). For this very reason, the assumption of control in time variable will play an essential role in our analysis in order to overcome such an obstacle, since in such a limit configuration solutions will become nondecreasing in time direction. This will enable us to apply a strong maximum principle for fully nonlinear equations; see [Da Lio 2004].

Finally, it is worth highlighting that our article extends, as well as generalizes to some extent, the previous seminal results (sharp regularity and weak geometric properties) from [Choe and Weiss 2003] and [Teixeira 2016] by using different approaches and techniques adapted to the general framework of the fully nonlinear parabolic operators (compare also with [Caffarelli et al. 2004] and [Shahgholian 2008]).

The paper is organized as follows: The reader will find the main definitions and assumptions in Section 2. Afterwards, we will present the proofs of the improved regularity and nondegeneracy Theorems 1.1 and 1.2 in Section 3. In Section 4 we put forward a number of consequences of these results. Section 5 treats global analysis results of Liouville and blow-up type. Finally, the Hausdorff estimates in Theorem 1.3 and related results will be delivered in Section 6.

2. Preliminaries and main assumptions

Let us start with some standard parabolic notation. By Ω we shall denote a bounded, open and smooth set in $\mathbb{R}^n$. For $x_0 \in \mathbb{R}^n$ and $r > 0$, we denote by $B_r(x_0)$ the Euclidean open ball with center x_0 and radius r. Also, for a point $(x_0, t_0) \in \Omega \times \mathbb{R}$ and $r > 0$, we consider three kinds of parabolic cylinders:

$$\begin{aligned}
&\mathscr{C}_r(x_0, t_0) := B_r(x_0) \times (t_0 - r^2, t_0 + r^2) && \text{(whole cylinder)}\\
&\mathscr{C}_r^+(x_0, t_0) := B_r(x_0) \times [t_0, t_0 + r^2) && \text{(the upper semi-cylinder)}\\
&\mathscr{C}_r^-(x_0, t_0) := B_r(x_0) \times (t_0 - r^2, t_0] && \text{(the lower semicylinder).}
\end{aligned}$$

Moreover, we will omit the center of the cylinder as $(x_0, t_0) = (0, 0)$.

For a parabolic cylinder $\mathscr{C} = \Omega \times \mathscr{I}$, where $\mathscr{I}$ is a closed interval with endpoints $\mathfrak{a} < \mathfrak{b}$, we define the parabolic boundary by: $\partial_p \mathscr{C} := (\overline{\Omega} \times \{\mathfrak{a}\}) \cup (\partial\Omega \times \mathscr{I})$.

Given (x, t), $(y, s) \in \Omega \times \mathbb{R}$ the parabolic distance (or metric) between (x, t) and (y, s) is given by

$$\operatorname{dist}_p((x, t), (y, s)) := \sqrt{|x - y|^2 + |t - s|}.$$

For $r > 0$ and $\mathscr{O} \subset \mathbb{R}^{n+1}$, we let $\mathscr{N}_r(\mathscr{O}) := \{(x, t) \in \mathbb{R}^{n+1} : \operatorname{dist}_p((x, t), \mathscr{O}) > r\}$ for the parabolic tubular neighborhood of radius r of $\mathscr{O}$.

By $\operatorname{Sym}(n)$ we mean the set of symmetric real matrices of size $n \times n$. If M is a given matrix, we shall use $\|M\|$ to denote any matrix norm.

Throughout this manuscript $\mathscr{F} : \mathscr{C}_1 \times \mathbb{R}^n \times \operatorname{Sym}(n) \to \mathbb{R}$ is a second-order fully nonlinear operator satisfying the structural condition

$$\begin{aligned}
\textbf{(SC)} \qquad \mathscr{P}^-_{\lambda,\Lambda}(M - P) - \kappa|\vec{p} - \vec{q}\,| \le{}& \mathscr{F}(x, t, \vec{p}, M) - \mathscr{F}(x, t, \vec{q}, P)\\
&\le \mathscr{P}^+_{\lambda,\Lambda}(M - P) + \kappa|\vec{p} - \vec{q}\,|,
\end{aligned}$$

for any M, $P \in \operatorname{Sym}(n)$ and $\vec{p}, \vec{q} \in \mathbb{R}^n$, where $\mathscr{P}^{\pm}_{\lambda/n,\Lambda}$ denote the *Pucci's extremal operators*

$$\mathscr{P}^+_{\lambda,\Lambda}(M) := \lambda \cdot \sum_{e_i<0} e_i + \Lambda \cdot \sum_{e_i>0} e_i \quad \text{and} \quad \mathscr{P}^-_{\lambda,\Lambda}(M) := \lambda \cdot \sum_{e_i>0} e_i + \Lambda \cdot \sum_{e_i<0} e_i$$

and $\{e_i : 1 \le i \le n\}$ are the eigenvalues of M.

For a fixed $(x_0, t_0) \in \mathscr{C}_1$, we will measure the oscillation of the coefficients of $\mathscr{F}$ around (x_0, t_0) by the quantity

$$\Theta_{\mathscr{F}}(x_0, t_0, x, t) := \sup_{M \in \operatorname{Sym}(n)\setminus\{0\}} \frac{|\mathscr{F}(x, t, 0, M) - \mathscr{F}(x_0, t_0, 0, M)|}{\|M\|}. \tag{2-1}$$

For notation purposes, we shall often write $\Theta_{\mathscr{F}}(0, 0, x, t) = \Theta_{\mathscr{F}}(x, t)$. Hence, the coefficients of the operator are merely measurable.

In the following definition, we provide the class of solutions that we consider in this work.

Definition 2.1 (viscosity solutions). We say that a function $u \in C^0(\mathscr{C}_1)$ is a viscosity subsolution (resp. supersolution) to

$$\mathscr{F}(x,t,Du(x,t),D^2u(x,t)) - \partial_t u - f(x,t,u) = 0 \quad \text{in } \mathscr{C}_1 \tag{2-2}$$

if for all $\varphi \in C^{2,1}(\mathscr{C}_1)$ whenever $u-\varphi$ has a local minimum (resp. maximum) at $(x_0,t_0) \in \mathscr{C}_1$ then

$$\mathscr{F}(x,t,D\varphi(x,t),D^2\varphi(x,t)) - \partial_t\varphi - f(x,t,\varphi) \geq 0 \quad (resp.\ \leq 0).$$

Finally we say that u is a viscosity solution to (2-2) if it is both a viscosity subsolution and a supersolution.

We recall the existence of a universal constant $\mathfrak{p}_0$, satisfying $\frac{n+2}{2} \leq \mathfrak{p}_0 < n+1$, for which Harnack inequality (resp. Hölder regularity) holds for viscosity solutions with RHS in L^p, provided $p > \mathfrak{p}_0$; see for instance [Crandall et al. 2000, Section 5]. The following compactness result then becomes available:

Proposition 2.2 (compactness of solutions). *Let u be a viscosity solution to*

$$\partial_t u - \mathscr{F}(x,t,Du,D^2u) = \mathfrak{f}(x,t,u) \quad \text{in } \mathscr{C}_r, \tag{2-3}$$

under the assumption $f \in L^p$ with $p > \mathfrak{p}_0$. Then u is locally of class $C^{0,\beta}$ for some $0 < \beta < 1$ and

$$\|u\|_{C^\beta(\mathscr{C}_r)} \leq C(n,\lambda,\Lambda,\kappa) r^{-\beta}(\|u\|_{L^\infty(\mathscr{C}_r)} + r^{2-\frac{n+2}{p}}\|f\|_{L^p(\mathscr{C}_r)}).$$

Another piece of information we need in our approach concerns the stability of the notion of viscosity solutions, i.e., the limit of a sequence of viscosity solutions turns out to be a viscosity solution of the limiting equation. More precisely, we refer to the following lemma, whose proof and general form can be found in [Crandall et al. 2000, Theorem 6.1].

Lemma 2.3 (continuity with respect to the equation). *Let $\mathscr{F}_j, \mathscr{F}$ be normalized (λ,Λ,κ)-operators, $p > \mathfrak{p}_0$, with $f, f_j \in L^p(\mathscr{C}_1) \cap C^0(\mathscr{C}_1)$ and let u_j be viscosity solutions to*

$$\partial_t u_j - \mathscr{F}_j(x,t,Du_j,D^2u_j) = f_j(x,t) \quad \text{in } \mathscr{C}_1$$

for all $j \in \mathbb{N}$. Assume that $u_j \to u$ locally uniformly as $j \to \infty$. Moreover, for all $\mathscr{C}_r(x_0,t_0) \subset \mathscr{C}_1$ and all $\varphi \in C^{2,1}(\mathscr{C}_r(x_0,t_0))$ (test function), assume that

$$g_j(x,t) := \mathscr{F}_j(x,t,D\varphi(x,t),D^2\varphi(x,t)) - f_j(x,t)$$

and

$$g(x,t) := \mathscr{F}(x,t,D\varphi(x,t),D^2\varphi(x,t)) - f(x,t)$$

satisfy

$$\|g - g_j\|_{L^p(\mathscr{C}_r(x_0,t_0))} \to 0 \quad \text{as } j \to \infty. \tag{2-4}$$

Then, u is a viscosity solution to

$$\partial_t u - \mathscr{F}(x, t, Du, D^2u) = f(x, t) \quad \text{in } \mathscr{C}_r(x_0, t_0).$$

Proposition 2.4 (gradient estimates [Crandall et al. 2000, Remark 7.7; Wang 1992b, Section 4.2). *Let u be a viscosity solution to (2-3) with $\mathscr{F}$ a normalized $(\lambda, \Lambda, \kappa)$-operator and $f \in L^p(\mathscr{C}_1)$. If*

$$\lim_{r\to 0+} \sup_{(y,s)\in\mathscr{C}_{\frac{1}{2}}} \left(\fint_{\mathscr{C}_r(y,s)} \Theta^p_{\mathscr{F}}(y, s, x, t) \right)^{\frac{1}{p}} = 0,$$

then $u \in C^{1+\alpha,(1+\alpha)/2}(\overline{\mathscr{C}_{1/2}})$ for some universal $0 < \alpha < 1$. Furthermore, there exists a universal constant $\mathfrak{C} = \mathfrak{C}(n, \lambda, \Lambda, \kappa) > 0$ such that

$$\|Du\|_{L^\infty(\overline{\mathscr{C}_{1/2}})} \le \mathfrak{C}(n, \lambda, \Lambda, \kappa)(\|u\|_{L^\infty(\mathscr{C}_1)} + \|f\|_{L^\infty(\mathscr{C}_1)}).$$

The next result can be proved in a similar way to one in [Crandall et al. 1992, Theorem 8.3].

Theorem 2.5 (comparison principle). *Let u_1 and u_2 be continuous functions in $\overline{\mathscr{C}_1}$ so that*

$$\begin{aligned} \mathscr{F}(x, t, Du_1, D^2u_1) - \partial_t u_1 - \lambda_0(x, t)(u_1)^\mu_+ \\ \le 0 \le \mathscr{F}(x, t, Du_2, D^2u_2) - \partial_t u_2 - \lambda_0(x, t)(u_2)^\mu_+ \quad \text{in } \mathscr{C}_1 \end{aligned} \tag{2-5}$$

in the viscosity sense. If $u_1 \ge u_2$ on $\partial_p\mathscr{C}_1$, then $u_1 \ge u_2$ in $\mathscr{C}_1$.

In the next theorem, we shall state the existence of solutions to the problem

$$\begin{cases} \mathscr{F}(x, t, Du, D^2u) - \partial_t u = \lambda_0(x, t)u^\mu \chi_{\{u>0\}}(x, t) & \text{in } \mathscr{C}_1, \\ u(x, t) = g(x, t) & \text{on} \partial B_1 \times (-1, 1), \\ u(x, 0) = u_0(x) & \text{in} \overline{B_1} \times \{-1\}, \end{cases} \tag{2-6}$$

for continuous functions g and u_0 satisfying the compatibility condition $g(x, 0) = u_0(x)$ for $x \in \partial B_1$. The existence is achieved by the celebrated Perron's method combined with the previous comparison principle, Theorem 2.5.

Theorem 2.6 (existence of dead core solutions). *Suppose that assumption (SC) holds for $\mathscr{F}$, and that λ_0 is continuous. If there exist a viscosity subsolution $u_\sharp$ to (2-6) and a viscosity supersolution $u^\sharp$ to (2-6) such that*

$$u_\sharp = u^\sharp \qquad \text{on } \partial_p\mathscr{C}_1,$$

then there exists a viscosity solution u to problem (2-6). Furthermore, such a u is nonnegative provided the data are nonnegative.

3. Improved regularity estimates and nondegeneracy of solutions

In this section we will prove our main results, Theorem 1.1 and 1.2. We start by deriving the improved regularity estimate in a normalized class of viscosity solutions defined in the unit cylinder, and then we extend the results to general dead-core viscosity solutions.

Definition 3.1. For any fully nonlinear operator $\mathscr{F}$ fulfilling **(SC)** we say that $u \in \mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$ if:

- $\mathscr{F}(x, t, Du, D^2u) - \partial_t u = \lambda_0(x, t)u^\mu \chi_{\{u>0\}}(x, t)$ in $\mathscr{C}_1$ (in the viscosity sense) with $\|\lambda_0 u^\mu\|_{L^\infty(\mathscr{C}_1)} \ll 1$.
- $0 \le u \le 1$, $0 < \mathfrak{m} \le \lambda_0 \le \mathfrak{M}$ in $\mathscr{C}_1$.
- $\partial_t u \ge -\mathfrak{c}_0(x, t)u^\mu \chi_{\{u>0\}}(x, t)$ in $\mathscr{C}_1$ (in the viscosity sense) for a nonnegative and bounded function $\mathfrak{c}_0$.
- $u(0, 0) = 0$.

Hereafter, we shall adopt the notation $\mathscr{S}_{(r,x_0,t_0)}[u] := \sup_{\mathscr{C}_r^-(x_0,t_0)} u(x, t)$.
In the next, we shall define for $u \in \mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$ the set

$$\mathbb{V}[u] := \left\{ j \in \mathbb{N} \cup \{0\};\ \mathscr{S}_{\frac{1}{2^j}}[u] \le 2^{\frac{2}{1-\mu}} \max\left\{1, \frac{1}{\mathfrak{C}_0^*}\right\} \cdot \mathscr{S}_{\frac{1}{2^{j+1}}}[u] \right\},$$

where $\mathfrak{C}_0^* > 0$ is the nondegeneracy constant from Theorem 1.2. Notice that $\mathbb{V}[u]$ is not empty. Indeed, $j = 0 \in \mathbb{V}[u]$ since, in view of Theorem 1.2,

$$\mathscr{S}_{\frac{1}{2}}[u] \ge \mathfrak{C}_0^* \left(\tfrac{1}{2}\right)^{\frac{2}{1-\mu}} \ge \mathfrak{C}_0^* \left(\tfrac{1}{2}\right)^{\frac{2}{1-\mu}} \mathscr{S}_1[u],$$

which implies that

$$\mathscr{S}_1[u] \le 2^{\frac{2}{1-\mu}} \max\left\{1, \frac{1}{\mathfrak{C}_0^*}\right\} \mathscr{S}_{\frac{1}{2}}[u].$$

We now present a key lemma for proving Theorem 1.1, which provides the sharp growth rate for functions on $\mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$.

Lemma 3.2. *There exists a positive constant $\mathfrak{C}_0 = \mathfrak{C}_0(n, \lambda, \Lambda, \mu, \mathfrak{M})$ such that*

$$\mathscr{S}_{\frac{1}{2^{j+1}}}[u] \le \mathfrak{C}_0 \cdot \left(\frac{1}{2^j}\right)^{\frac{2}{1-\mu}} \tag{3-1}$$

for all $u \in \mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$ and $j \in \mathbb{V}[u]$.

Proof. The proof will hold by reductio ad absurdum argument. Let us suppose that the thesis of the lemma fails to hold. Then, for each $k \in \mathbb{N}$ we might find

$u_k \in \mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$ and $j_k \in \mathbb{V}[u_k]$ such that

$$\mathscr{S}_{\frac{1}{2^{j_k+1}}}[u_k] > k\left(\frac{1}{2^{j_k}}\right)^{\frac{2}{1-\mu}}. \tag{3-2}$$

Now, we define the auxiliary function:

$$v_k(x,t) := \frac{u_k\left(\frac{1}{2^{j_k}}x, \frac{1}{2^{2j_k}}t\right)}{\mathscr{S}_{\frac{1}{2^{j_k+1}}}[u_k]} \quad \text{in } \mathscr{C}_1.$$

Thus, v_k fulfills

- $0 \le v_k(x,t) \le \frac{\mathscr{S}_{1/2^{j_k}}[u_k]}{\mathscr{S}_{1/2^{j_k+1}}[u_k]} \le \mathfrak{A} := 2^{2/(1-\mu)} \max\{1, 1/\mathfrak{C}_0^*\}$ in $\mathscr{C}_1^-$ and $v_k(0,0) = 0$.
- $\mathscr{S}_{\frac{1}{2}}[v_k] \ge 1$.
- $\partial_t v_k \ge -\mathfrak{c}_0\left(\frac{1}{2^{j_k}}x, \frac{1}{2^{2j_k}}t\right)\frac{2^{-2j_k}}{\mathscr{S}^{1-\mu}_{1/2^{j_k+1}}[u_k]} v_k^\mu(x,t)$ in $\mathscr{C}_1^-$.
- $\mathscr{F}_k(x,t,Dv_k,D^2v_k) - \partial_t v_k = \frac{1}{2^{2\cdot j_k}}\frac{1}{\mathscr{S}^{1-\mu}_{1/2^{j_k+1}}[u_k]}\lambda_0\left(\frac{1}{2^{j_k}}x, \frac{1}{2^{2\cdot j_k}}t\right)(v_k)_+^\mu(x,t)$ in $\mathscr{C}_1$ in the viscosity sense, where

$$\mathscr{F}_k(x,t,\vec{p},M) := \frac{1}{2^{2j_k}\mathscr{S}_{1/2^{j_k+1}}[u_k]}\mathscr{F}\left(\frac{1}{2^{j_k}}x, \frac{1}{2^{2\cdot j_k}}t, 2^{j_k}\mathscr{S}_{1/2^{j_k+1}}[u_k]\cdot\vec{p}, 2^{2j_k}\mathscr{S}_{1/2^{j_k+1}}[u_k]\cdot M\right).$$

Notice that the operator $\mathscr{F}_k$ fulfills **(SC)**. Moreover,

$$\left\| \frac{1}{2^{2\cdot j_k}}\frac{1}{\mathscr{S}^{1-\mu}_{1/2^{j_k+1}}[u_k]}\lambda_0\left(\frac{1}{2^{j_k}}x, \frac{1}{2^{2\cdot j_k}}t\right)(v_k)_+^\mu(x,t)\right\|_{L^\infty(\mathscr{C}_1)} \le \mathfrak{A}^\mu\cdot\mathfrak{M}\cdot\left(\tfrac{1}{k}\right)^{1-\mu} \to 0 \quad \text{as } k\to\infty.$$

The previous sentences together with standard compactness arguments for fully nonlinear parabolic equations (see Proposition 2.2) imply that, up to a subsequence, $v_k \to v$ locally uniformly in $\overline{\mathscr{C}_{4/5}^-}$ and $\mathscr{F}_k \to \mathscr{F}_0$. Furthermore, by stability results (see Lemma 2.3) we have

- $\mathscr{F}_0(D^2v) - \partial_t v = 0$ in $\mathscr{C}_{4/5}^-$.
- $0 \le v \le \mathfrak{A}$ and $\frac{\partial v}{\partial t} \ge 0$ in $\mathscr{C}_{4/5}^-$.
- $v(0,t) = 0$ for all $t \in \left(-\left(\frac{4}{5}\right)^2, 0\right]$.
- $\mathscr{S}_{1/2}[v] \ge 1$.

Therefore, according to the strong minimum principle (see [Da Lio 2004]) $v \equiv 0$, which contradicts the previous sentence. □

In the next result, we state a version of Theorem 1.1 for the class $\mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$.

Theorem 3.3. *There exists a positive constant* $\mathfrak{C} = \mathfrak{C}(n, \lambda, \Lambda, \mu, \mathfrak{M})$ *such that for all* $u \in \mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$

$$u(x,t) \le \mathfrak{C} \cdot \mathfrak{d}(x,t)^{\frac{2}{1-\mu}} \quad \textit{for all } (x,t) \in \mathscr{C}_{\frac{1}{2}},$$

where

$$\mathfrak{d}(x,t) := \begin{cases} \sup\{r \ge 0;\, \mathscr{C}_r(x,t) \subset \{u > 0\}\} & \textit{for } (x,t) \in \{u > 0\}, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. The proof will be by induction. First of all, we claim that

$$\mathscr{S}_{\frac{1}{2^j}}[u] \le \mathfrak{C}_0 \cdot \left(\frac{1}{2^{j-1}}\right)^{\frac{2}{1-\mu}} \quad \text{for all } j \in \mathbb{N}, \tag{3-3}$$

where $\mathfrak{C}_0$ is the constant coming from Lemma 3.2. Note that if $\mathfrak{C}_0 \ge 1$, which we can suppose without loss of generality, then (3-3) holds for $j = 0$. Suppose now that (3-3) holds for some $j \in \mathbb{N}$. We will verify the $(j+1)$-th step of induction. In fact, if $j \in \mathbb{V}[u]$, the result holds directly by Lemma 3.2. On the other hand, if (3-3) fails, by using the induction hypothesis we obtain

$$\mathscr{S}_{\frac{1}{2^{j+1}}}[u] \le \left(\frac{1}{2}\right)^{\frac{2}{1-\mu}} \cdot \mathscr{S}_{\frac{1}{2^j}}[u] \le \mathfrak{C}_0 \cdot \left(\frac{1}{2}\right)^{\frac{2}{1-\mu}} \left(\frac{1}{2^{j-1}}\right)^{\frac{2}{1-\mu}} = \mathfrak{C}_1 \cdot \left(\frac{1}{2^j}\right)^{\frac{2}{1-\mu}}$$

Therefore, (3-3) holds for all $j \in \mathbb{N}$.

In order to finish the proof for a continuous parameter, for $r \in (0,1)$ let $j \in \mathbb{N}$ be the greatest integer such that $1/2^{j+1} \le r < 1/2^j$. Then,

$$\mathscr{S}_r[u] \le \mathscr{S}_{\frac{1}{2^j}}[u] \le \mathfrak{C}_0 \cdot \left(\frac{1}{2^{j-1}}\right)^{\frac{2}{1-\mu}} \le \mathfrak{C}(n, \lambda, \Lambda, \mu, \mathfrak{M}) \cdot r^{\frac{2}{1-\mu}}.$$

Finally, in order to obtain an estimate for u over the whole cylinder we will use a suitable barrier function from above. Let us define

$$\mathfrak{c} := \left(\frac{\mathfrak{m}(1-\mu)^2}{2[2\Lambda\mu + \kappa(1-\mu)]}\right)^{\frac{1}{1-\mu}}$$

and $\zeta(x,t) := \mathfrak{c} \cdot (|x|^2 + 2n\Lambda \cdot t)^{1/1-\mu}$. Then, we have

$$\mathscr{F}(x,t,D\zeta,D^2\zeta) - \frac{\partial \zeta}{\partial t} - \lambda_0(x,t)\zeta_+^\mu \le 0 \le \mathscr{F}(x,t,Du,D^2u) - \frac{\partial u}{\partial t} - \lambda_0(x,t)u_+^\mu \quad \text{in } \mathscr{C}_1^+.$$

Moreover, we have that $\zeta \ge u$ on $\partial_p \mathscr{C}_1^+$, where we have used $\mathscr{S}_r[u] \le \mathfrak{c} \cdot r^{2/(1-\mu)}$ for the estimate on $\{t = 0\}$. Consequently, the comparison principle for viscosity

solutions in Theorem 2.5 implies that $\zeta \geq u$ in $\mathscr{C}_1^+$. Therefore,

$$\sup_{\mathscr{C}_r} u(x,t) \leq \mathfrak{C}(n,\lambda,\Lambda,\mu,\kappa,\mathfrak{m}) \cdot r^{\frac{2}{1-\mu}}. \qquad \square$$

Remark 3.4. Following the same arguments as in the proof of Theorem 1.1, it is possible to obtain similar regularity estimates for a family of problems with a general (not μ-homogeneous) nonnegative and nonlinear absorption term $\mathfrak{f}: \mathscr{C}_1 \times [0, \|u\|_\infty] \to \mathbb{R}_+$, i.e.,

$$\mathscr{F}(x,t,Du,D^2u) - \partial_t u = \mathfrak{f}(x,t,u) \quad \text{in } \Omega_T,$$

provided $\mathfrak{f}(x,t,r) \leq \mathfrak{C}_* r^\mu$, for all $0 < r \ll 1$ and for some constant $\mathfrak{C}_* > 0$. Some interesting examples include

$$\mathfrak{f}(u) = \begin{cases} \lambda_0(x,t)(e^{u_+^s} - 1) & \text{for } s \geq \mu > 0, \\ \lambda_0(x,t)\ln(u_+^s + 1) & \text{for } s \geq \mu > 0, \\ \lambda_0(x,t)u_+^\mu \ln(u^s + 1) & \text{for } s > 0, \\ \lambda_0(x,t)u_+^\mu/(1+u^s)^m & \text{for } s > 0 \text{ and } 0 < m \leq \mu. \end{cases}$$

We have decided to treat the case $\mathfrak{f}(x,t,u) = \lambda_0(x,t)u_+^\mu(x,t)$ in order to introduce the main novelties in our approach.

Remark 3.5. Notice that Lemma 3.2 ensures that there exists a universal constant $0 < \tau_0 \ll 1$ (small enough) such that if $u \in \mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$ with

$$\|\mathscr{F}(x,t,Du,D^2u) - \partial_t u\|_{L^\infty(\mathscr{C}_1)} \leq \tau_0,$$

then

$$\mathscr{S}_{\frac{1}{2^{j+1}}}[u] \leq \mathfrak{C}_0 \cdot \left(\frac{1}{2^j}\right)^{\frac{2}{1-\mu}}.$$

We now are ready to prove the main result of the article.

Proof of Theorem 1.1. In order to prove Theorem 1.1, we have to reduce the hypothesis presented on it to the framework of Theorem 3.3. We assume without loss of generality that $\mathfrak{K} = \overline{\mathscr{C}_1} \subset \Omega_T$. For $(x,t) \in \{u > 0\} \cap \mathfrak{K}$ let $\mathfrak{d}(x,t)$ the distance comes from Theorem 3.3. For $(x_0,t_0) \in \partial\{u > 0\} \cap \mathfrak{K}$ let us define

$$v(x,t) := \frac{u(x_0 + \mathfrak{R}_0 x, t_0 + \mathfrak{R}_0^2 t)}{\kappa_0} \quad \text{in } \mathscr{C}_1$$

for constants $\kappa_0, \mathfrak{R} > 0$ to be determined universally a posteriori.

From the equation satisfied by u, we easily verify that, in the viscosity sense, v fulfills

$$\mathscr{G}(x,t,Dv,D^2v) - \partial_t v = \hat{\lambda}_0(x,t) \cdot v_+^\mu(x,t), \tag{3-4}$$

for

$$\mathscr{G}(x,t,\vec{p},\mathscr{X}) := \frac{\mathfrak{R}_0^2}{\kappa_0}\mathscr{F}\left(x_0+\mathfrak{R}_0 x, t_0+\mathfrak{R}_0^2, \frac{\kappa_0}{\mathfrak{R}_0}\vec{p}, \frac{\kappa_0}{\mathfrak{R}_0^2}\mathscr{X}\right)$$

and

$$\hat{\lambda}_0(x,t) := \frac{\mathfrak{R}_0^2}{\kappa_0^{1-\mu}}\lambda_0(x_0+\mathfrak{R}_0 x, t_0+\mathfrak{R}_0^2 t).$$

Observe that $\mathscr{G}$ satisfies assumption **(SC)** from Section 2 with the same ellipticity constants λ and Λ, Lipschitz constant κ, and a modulus of continuity $\hat{\omega}(s)=\omega(\mathfrak{R}s)$.

Now, let $\tau_0>0$ be the greatest universal constant, from Remark 3.5 such that Lemma 3.2 holds provided $\|\mathscr{G}(x,t,Dv,D^2u)-\partial_t v\|_{L^\infty(\mathscr{C}_1)}\le\tau_0$. By choosing

$$\kappa_0 := \|u\|_{L^\infty(\Omega_T)} \quad\text{and}\quad 0<\mathfrak{R}<\min\left\{1, \frac{\operatorname{dist}(\mathfrak{K},\partial_p\Omega_T)}{2}, \sqrt{\frac{\tau_0\kappa_0^{1-\mu}}{\mathfrak{M}}}\right\},$$

v fits into the framework of Theorem 3.3. Hence, there exists a constant $\mathfrak{C}=\mathfrak{C}(n,\lambda,\Lambda,\mathfrak{M})$ so that

$$v(x,t)\le\mathfrak{C}\cdot\mathfrak{d}(x,t)^{\frac{2}{1-\mu}},$$

where

$$\mathfrak{d}(x,t) := \begin{cases}\sup\{r\ge 0;\mathscr{C}_r(x,t)\subset\{v>0\}\} & \text{for } (x,t)\in\{v>0\},\\ 0 & \text{otherwise.}\end{cases}$$

By scaling back, we obtain the conclusion of Theorem 1.1. □

This final part is devoted to proving Theorem 1.2 which tells us how dead-core solutions detach their free boundaries. As a byproduct, we shall also give important consequences of such a nondegeneracy property, including uniform Lebesgue density of positive sets $\{u>0\}$, porosity of t-level sets of the free boundary and finite speed propagation of $\{u>0\}$; see Section 4 for more details.

Proof of Theorem 1.2. Firstly, notice that by continuity of viscosity solutions, it is sufficient to show that (1-2) holds for $(x_0,t_0)\in\{u>0\}$. To this end, fix such a point and take $r>0$ so that $\mathscr{C}_r(x_0,t_0)\subset\Omega\times(0,\infty)$. Now, define the comparison function

$$\Psi(x,t) := \mathfrak{c}_1[|x-x_0|^2+\mathfrak{c}_2(t_0-t)]^{\frac{1}{1-\mu}}, \quad (x,t)\in\mathscr{C}_r^-(x_0,t_0),$$

where $\mathfrak{c}_1,\mathfrak{c}_2$ are positive constants satisfying $\mathfrak{c}_2<\frac{\mathfrak{m}(1-\mu)}{2}c_1^{\mu-1}$ and

$$\mathfrak{c}_1 := \left\{\frac{\mathfrak{m}(1-\mu)^2}{4[2\mu\Lambda+(n\Lambda+\kappa)(1-\mu)]}\right\}^{\frac{1}{1-\mu}},$$

where $0 < \mathfrak{m} \le \lambda_0$. Hence, it follows that

$$\mathscr{F}(x,t,D\Psi,D^2\Psi) - \partial_t\psi - \lambda_0(x,t)\psi^\mu \le 0 \quad \text{in } \mathscr{C}_r^-(x_0,t_0).$$

Now, if $\Psi \ge u$ on whole $\partial_p\mathscr{C}_r^-(x_0,t_0)$, then the comparison principle Theorem 2.5 would imply that $u \le \Psi$ in $\mathscr{C}_r^-(x_0,t_0)$. However, this is a contradiction with the fact that $\Psi(x_0,t_0) = 0 < u(x_0,t_0)$. Therefore, there exists a point $(x',t') \in \partial_p\mathscr{C}_r^-(x_0,t_0)$ such that $u(x',t') \ge \Psi(x',t')$. Since $\Psi(x',t') = \mathfrak{c}r^{2/(1-\mu)}$ for $\mathfrak{c}$ independently of (x',t'), we have completed the proof of the theorem. □

4. Applications

Using Theorem 1.1 we are able to prove a similar growth rate for the gradient of dead-core solutions provided that the coefficients of $\mathscr{F}$ are VMO (see Proposition 2.4).

Corollary 4.1 (growth estimates for gradient). *Suppose that the assumptions of Proposition 2.4 are in force. Then a positive constant $\mathfrak{C}_1 = \mathfrak{C}_1(n,\lambda,\Lambda,\mu,\mathfrak{M})$ exists such that for all $u \in \mathfrak{J}(\mathscr{F},\lambda_0,\mu)(\mathscr{C}_1)$:*

$$\|Du(x,t)\| \le \mathfrak{C}_1 \cdot \mathfrak{d}(x,t)^{\frac{1+\mu}{1-\mu}} \quad \textit{for all } \mathscr{C}_{\frac{1}{2}}.$$

Proof. As before, it is enough to prove the estimate

$$\mathscr{S}_{\frac{1}{2^{j+1}}}[\|Du\|] \le \max\left\{\mathfrak{C}_2 \cdot \left(\frac{1}{2^j}\right)^{\frac{1+\mu}{1-\mu}}, \left(\frac{1}{2}\right)^{\frac{1+\mu}{1-\mu}} \mathscr{S}_{\frac{1}{2^j}}[\|Du\|]\right\}, \tag{4-1}$$

for all $j \in \mathbb{N}$ and a constant $\mathfrak{C}_2 = \mathfrak{C}_2(n,\lambda,\Lambda,\mu,\mathfrak{M})$.

Let us suppose that (4-1) does not hold. Then, there exists $u_j \in \mathfrak{J}(\mathscr{F},\lambda_0,\mu)(\mathscr{C}_1)$ such that

$$\mathscr{S}_{\frac{1}{2^{j+1}}}[\|Du_j\|] \ge \max\left\{j\left(\frac{1}{2^j}\right)^{\frac{1+\mu}{1-\mu}}, \left(\frac{1}{2}\right)^{\frac{1+\mu}{1-\mu}} \mathscr{S}_{\frac{1}{2^j}}[\|Du_j\|]\right\}. \tag{4-2}$$

Next, we define the auxiliary normalized and scaled function

$$v_j(x,t) := \frac{2^j u_j\left(\frac{1}{2^j}x, \frac{1}{2^{2j}}t\right)}{\mathscr{S}_{\frac{1}{2^{j+1}}}[\|Du_j\|]}.$$

Notice that using (3-1) and (4-2) we obtain

$$\begin{aligned} 0 \le v_j(x) &\le \frac{2^j \mathfrak{C}(2^{-j})^{\frac{2}{1-\mu}}}{\mathscr{S}_{\frac{1}{2^{j+1}}}[\|Du_j\|]} \\ &\le \frac{\mathfrak{C}_0}{j^{1-\mu}} \quad \text{for } (x,t) \in \mathscr{C}_1. \end{aligned}$$

Furthermore:

- $\mathscr{F}_j(x,t,Dv_j,D^2v_j)-\partial_t v_j=\hat{\lambda_0}(x,t)(v_j)_+^{\mu}(x,t)$ in $\mathscr{C}_1$ in the viscosity sense, where

$$\mathscr{F}_j(x,t,\vec{p},M)$$
$$:=\frac{1}{2^j\mathscr{S}_{\frac{1}{2^{j+1}}}[\|Du_j\|]}\mathscr{F}\left(\frac{1}{2^j}x,\frac{1}{2^{2j}}t,\mathscr{S}_{\frac{1}{2^{j+1}}}[\|Du_j\|]\cdot\vec{p},2^j\mathscr{S}_{\frac{1}{2^{j+1}}}[\|Du_j\|]\cdot M\right)$$

and

$$\hat{\lambda_0}(x,t):=\frac{1}{2^{j(1+\mu)}}\frac{1}{\mathscr{S}^{1-\mu}_{\frac{1}{2^{j+1}}}[\|Du_j\|]}\lambda_0\left(\frac{1}{2^j}x,\frac{1}{2^{2j}}t\right).$$

- $\mathscr{S}_{\frac{1}{2}}[\|Dv_j\|]\geq 1$.

Consequently,

$$\|\hat{\lambda_0}\cdot(v_j)_+^{\mu}\|_{L^\infty(\mathscr{C}_1)}\leq\mathfrak{M}\cdot\mathfrak{C}^{\mu}\cdot\frac{1}{j}.$$

Finally, by using the a priori gradient estimate from Proposition 2.4 we obtain

$$1\leq\mathscr{S}_{\frac{1}{2}}[\|Dv_j\|]\leq\mathfrak{C}(n,\lambda,\Lambda)[\|v_j\|_{L^\infty(\mathscr{C}_1)}+\|\hat{\lambda_0}(v_j)_+^{\mu}\|_{L^\infty(\mathscr{C}_1)}]$$
$$\leq\mathfrak{C}^*\cdot\frac{1}{j}\to 0\quad\text{as } j\to\infty,$$

which is a contradiction. Therefore the proof is ended. □

An interesting piece of information coming from our technique is that by using again the previous iterative geometric argument and supposing better assumptions on $\mathscr{F}$ (to be clarified soon) we will be able to access an improved growth rate for the higher derivatives of u (namely, the temporal derivative and the Hessian of u, respectively) according to Schauder type estimates.

Corollary 4.2 (growth estimates for higher derivatives). *Assume that* (**DCP**) *has locally* $C^{2,1}$ *a priori estimates, i.e., there exists a universal constant* $\mathfrak{C}_\sharp>0$ *such that*

$$\|D^2u\|_{L^\infty(\mathscr{C}_r)},|\partial_t u|_{L^\infty(\mathscr{C}_r)}\leq\mathfrak{C}_\sharp\quad\textit{for all } r\ll 1.$$

Then, there exists a positive constant $\mathfrak{C}_2=\mathfrak{C}_1(n,\lambda,\Lambda,\mu,\mathfrak{M})$ *such that for all* $u\in\mathfrak{J}(\mathscr{F},\lambda_0,\mu)(\mathscr{C}_1)$,

$$\|D^2u(x,t)\|\leq\mathfrak{C}_2\cdot\mathfrak{d}(x,t)^{\frac{2\mu}{1-\mu}}\quad(\textit{resp. } |\partial_t u(x,t)|\leq\mathfrak{C}_2\cdot\mathfrak{d}(x,t)^{\frac{\mu}{1-\mu}})\quad\textit{for all } (x,t)\in\mathscr{C}_{\frac{1}{2}},$$

Remark 4.3. An interesting class of operators for which we can apply Corollary 4.2 is the class of convex (or concave) operators (recall that such a family of operators enjoy local $C^{2+\alpha,(2+\alpha)/2}$ a priori estimates; see [Krylov 1983; Wang 1992b,

Section 4.3]), because under such assumptions on $\mathscr{F}$ it is possible to develop a Schauder type estimate provided the source term enjoys a universal modulus of continuity, suitably integrable at origin (see [Tian and Wang 2013] and [Wang 1992b, Section 1.1] for more details and compare with [da Silva and Teixeira 2017, Section 6] and [da Silva and dos Prazeres 2019, Section 6] for results when such an assumption fails). Finally, in [da Silva and dos Prazeres 2019], Schauder type estimates were proved for flat C^0-viscosity solutions, i.e., solutions with very small oscillation, provided $\mathscr{F}$ is in $C^{1,\tau}(\mathrm{Sym}(n))$ and has Dini continuous coefficients. Therefore, such a family of solutions and operators is an interesting class where Corollary 4.2 holds true.

Remark 4.4 (dead core solutions vs. flat solutions). In view of previous results, we must highlight the relationship between regularity coming from dead-core solutions and that coming from the classical Schauder theory. To this end, let us suppose that u is a flat C^0-viscosity solution to

$$\mathscr{F}(x,t,D^2u)-\partial_t u=\lambda_0(x,t)u_+^{\mu}(x,t)\quad\text{in } \mathscr{C}_1, \tag{4-3}$$

where $0<\mu<1$, $\lambda_0\in C^{0,\mu}(\mathscr{C}_1)$ and $\mathscr{F}$ is subject to the assumptions in, for example, [da Silva and dos Prazeres 2019]; see Remark 4.3. Under such assumptions, the Schauder type estimates from [da Silva and dos Prazeres 2019] ensure that solutions to (4-3) are $C^{2+\mu,(2+\mu/2)}_{\mathrm{loc}}(\mathscr{C}_1)$ (particularly at free boundary points). On the other hand, our main theorem, Theorem 1.1, claims that u is $C^{\varsigma+\alpha,(\varsigma+\alpha)/2}$ at free boundary points, where

$$\varsigma:=\left\lfloor\frac{2}{1-\mu}\right\rfloor\quad\text{and}\quad\alpha:=\frac{2}{1-\mu}-\left\lfloor\frac{2}{1-\mu}\right\rfloor.$$

Nevertheless, notice that for any $0<\mu<1$ we have

$$\varsigma+\alpha=\frac{2}{1-\mu}>2+\mu,$$

which means that dead-core solutions are more regular, along free boundary points, than the best regularity result coming from classical regularity theory in [da Silva and dos Prazeres 2019].

Remark 4.5 (regularity in some problems from geometry). Over the last decades the study of geometric flows has proved to be extremely effective in solving some of the most important problems in topology, differential geometry and geometric analysis. Geometric considerations lead to equations of the form

$$\begin{aligned}\mathscr{F}(x,t,Du,D^2u)-\partial_t u&\\ =\mathfrak{f}\left(x,t,u,\int_{B_1}\mathscr{G}(Du,D^2u)\,dx\right)\chi_{\{u>0\}}&\quad\text{in } \mathscr{M}\subset\mathbb{R}^{n+1},\end{aligned} \tag{4-4}$$

where $\mathscr{F}$ is a convex (concave) operator and $\mathscr{G} \in C^\infty(\mathbb{R}^n \times \mathrm{Sym}(n), \mathbb{R}^m)$ is a vector field (see [Tian and Wang 2013] for more detail on these topics). Such equations appear in many applications of parabolic PDEs in curvature and gradient flows. For this reason, our work has also been motivated by the study of such equations coming from differential geometry and geometric analysis in order to establish high order estimates to solutions along their free boundaries.

Next, we will comment on interior regularity results for general nonlinear curvature and gradient flows (4-4) (at free boundary points); they provide an interesting application in the geometric setting for our main theorem. We consider $\mathscr{M}$ to be a closed manifold without boundary under a volume constraint assumption; thus interior regularity is sufficient. For (4-4), one can to obtain $C^{2,1}$ estimates for solutions via maximum principle

$$\|u\|_{C^{2,1}(\mathscr{C}_r)} \leq C \quad \text{for all } r \ll 1.$$

Furthermore, such an estimate implies that (4-4) is uniformly elliptic; thus the structural condition is satisfied. Notice that such an estimate also implies that the RHS is C^2 in x and bounded and measurable in t. Hence, we fall into the assumptions of Remark 3.4. As a result, Theorem 1.1 can be applied for viscosity solutions to equations of form (4-4).

Finally, this result can be further applied to equations of the form

$$\mathscr{F}(D\mathfrak{h}, D^2\mathfrak{h}) - \partial_t \mathfrak{h} - \mathfrak{h}_+^\mu |\mathfrak{A}|^2 = 0,$$

where $\mathfrak{h}$ is the inwards mean curvature vector of the surface at position x and time t and $|\mathfrak{A}|$ represents the norm of the second fundamental form. This equation is the extended version for models describing the mean curvature hypersurface in the Euclidean space $\mathbb{R}^{n+1}$; see [Sheng and Wang 2010].

Next, we will establish a finer control for dead-core solutions close to free boundary points. In brief, any viscosity solution to **(DCP)** will be "trapped" between the graph of two suitable multiples of $\mathrm{dist}_p(\,\cdot\,, \partial\{u > 0\})^{2/(1-\mu)}$.

Corollary 4.6. *Let u be a nonnegative, bounded viscosity solution to* **(DCP)** *and $\Omega' \Subset \Omega_T$. Given $(x_0, t_0) \in \{u > 0\} \cap \Omega'$, there exists a universal constant $\mathfrak{C}_\sharp > 0$ such that*

$$\mathfrak{C}_\sharp \mathrm{dist}_p((x_0, t_0), \partial\{u > 0\})^{\frac{2}{1-\mu}} \leq u(x_0, t_0) \leq \mathfrak{C}^\sharp \mathrm{dist}_p((x_0, t_0), \partial\{u > 0\})^{\frac{2}{1-\mu}}.$$

Proof. The upper estimate is an immediate consequence of Theorem 1.1. Next, let us suppose that such a $\mathfrak{C}_\sharp > 0$ does not exist. Then there exists a sequence $P_k := (x_k, t_k) \in \{u > 0\} \cap \Omega'$ with

$$d_k := \mathrm{dist}_p(P_k, \partial\{u > 0\} \cap \Omega') \to 0 \quad \text{as } k \to \infty \quad \text{and} \quad u(P_k) \leq k^{-1} d_k^{\frac{2}{1-\mu}}.$$

Now, let us define the auxiliary function $v_k : \mathscr{C}_1 \to \mathbb{R}$ by

$$v_k(y,s) := \frac{u(P_k + (d_k y, d_k^2 s))}{d_k^{\frac{2}{1-\mu}}}.$$

It is easy to verify that:

(1) $\mathscr{F}_k(x,t,Dv_k,D^2v_k) - \partial_t v_k = \widehat{\lambda}_k(x,t)\cdot(v_k)_+^{\mu}$ in $\mathscr{C}_{1/2}$ in the viscosity sense where

$$\mathscr{F}_k(y,s,\vec{p}\,,\mathscr{X}) := d_k^{-\frac{2\mu}{1-\mu}}\mathscr{F}\Big(P_k + (d_k y, d_k^2 s), d_k^{\frac{1+\mu}{1-\mu}}\vec{p}\,, d_k^{\frac{2\mu}{1-\mu}}\mathscr{X}\Big)$$

and

$$\widehat{\lambda}_k(y,s) := \lambda_0(P_k + (d_k y, d_k^2 s)).$$

(2) $u(P_k + (d_k y, d_k^2 s)) \le \sup_{\mathscr{C}^+_{d_k}(\widehat{P}_k)} u(y,s) \le \mathfrak{C} d_k^{2/(1-\mu)}$ according to Theorem 1.1, where $\widehat{P}_k$ is such that $d_k = \operatorname{dist}_p(P_k, \widehat{P}_k)$. Hence, v_k is nonnegative and uniformly bounded.

(3) $v_k(y,s) \le C\cdot d_k^{\alpha} + \frac{1}{k}$ for all $(y,s) \in \mathscr{C}^-_{1/2}$ due to local Hölder regularity of solutions; see [Crandall et al. 2000, Section 5; Krylov and Safonov 1980; Wang 1992a, Section 4.4].

From the nondegeneracy property, Theorem 1.2, and the last sentence we obtain

$$0 < \mathfrak{C}_0\cdot\left(\frac{1}{2}\right)^{\frac{2}{1-\mu}} \le \sup_{\partial_p \mathscr{C}^-_{\frac{1}{2}}} v_k(y,s) \le \sup_{\overline{\mathscr{C}^-_{\frac{1}{2}}}} v_k(y,s) \le C d_k^{\alpha} + \frac{1}{k} \to 0 \quad \text{as } k \to \infty. \tag{4-5}$$

Such a contradiction finishes the proof. □

As an another application, we establish an average control for the μ-power of dead-core solutions. Such an estimate will be useful in order to prove Hausdorff measure estimates.

Corollary 4.7. *Let u be a nonnegative, bounded viscosity solution to* (**DCP**) *and $\Omega' \Subset \Omega_T$. For all $(x_0,t_0) \in \partial\{u>0\} \cap \Omega'$, there exist universal constants $\mathfrak{C}_\sharp > 0$ and*

$$0 < r_0 \ll \min\left\{1, \frac{\operatorname{dist}_p(\Omega', \partial_p \Omega_T)}{2}\right\}$$

such that

$$\fint_{\mathscr{C}^-_r(x_0,t_0)} u^{\mu}(x,t) \ge \mathfrak{C}_\sharp r^{\frac{2\mu}{1-\mu}},$$

for any $r \le r_0$.

Proof. Once more, we will proceed via a contradiction argument. If such a $\mathfrak{C}_* > 0$ does not exist, then there would exist a sequence $P_k := (x_k, t_k) \in \partial\{u > 0\} \cap \Omega'$ such that for any sequence $r_k \to 0+$ as $k \to \infty$ we would have

$$\fint_{\mathscr{C}^-_{r_k}(P_k)} u^\mu(x,t) < k^{-1} r_k^{\frac{2\mu}{1-\mu}}.$$

Now, define the function $v_k : \mathscr{C}^-_1 \to \mathbb{R}$ by

$$v_k(y,s) := \frac{u(P_k + (r_k y, r_k^2 s))}{r_k^{\frac{2}{1-\mu}}}.$$

It is easy to verify

$$\mathscr{G}_k(x,t,Dv_k,D^2v_k) - \partial_t v_k = \lambda_k(x,t)\cdot(v_k)_+^\mu \quad \text{in } \mathscr{C}^-_1$$

in the viscosity sense, where

$$\mathscr{G}_k(y,s,\vec{p},\mathscr{X}) := r_k^{-\frac{2\mu}{1-\mu}} \mathscr{F}(P_k + (r_k y, r_k^2 s), r_k^{\frac{1+\mu}{1-\mu}}\vec{p}, r_k^{\frac{2\mu}{1-\mu}}\mathscr{X})$$

and

$$\lambda_k(y,s) := \lambda_0(P_k + (r_k y, r_k^2 s)).$$

On the one hand, using the contradiction assumption,

$$\fint_{\mathscr{C}^-_{\frac12}(0,0)} v_k^\mu(y,s) = 2^{n+2} \fint_{\mathscr{C}^-_{r_k}(P_k)} \frac{u^\mu(x,t)}{r_k^{\frac{2\mu}{1-\mu}}} < \frac{2^{n+2}}{k} \to 0 \quad \text{as } k \to \infty. \tag{4-6}$$

On the other hand, using Corollary 4.6 we obtain

$$\begin{aligned}\fint_{\mathscr{C}^-_{\frac12}(0,0)} v_k^\mu(y,s) &= \fint_{\mathscr{C}^-_{\frac12}(0,0)} \frac{u^\mu(P_k + (r_k y, r_k^2 s))}{r_k^{\frac{2\mu}{1-\mu}}} \\ &\geq \mathfrak{C}_\sharp^\mu \fint_{\mathscr{C}^-_{\frac12}(0,0)} \left(\frac{\operatorname{dist}_p(P_k + (r_k y, r_k^2 s), \partial\{u>0\})}{r_k}\right)^{\frac{2\mu}{1-\mu}}.\end{aligned}$$

Now, let us denote $d_k(y,s) := \operatorname{dist}_p(P_k + (r_k y, r_k^2 s), \partial\{u > 0\} \cap \Omega')$. Under such a notation we define

$$\mathfrak{D}_k := \{(y,s) \in \mathscr{C}^-_{\frac12}(0,0) \mid d_k(y,s) < a_k r_k\},$$

where

$$a_k := \left(\frac{1}{k}\right)^{\frac{1-\mu}{2\mu}} |\mathscr{C}^-_{\frac12}(0,0)|^{\frac{1-\mu}{2\mu}} (\mathfrak{C}_\sharp^{-\mu})^{\frac{1-\mu}{2\mu}} (2^{\alpha(n+2)})^{\frac{1-\mu}{2\mu}}$$

with $\alpha > 0$ chosen such that

$$\frac{2^{\alpha(n+2)}}{10}|\mathscr{C}^-_{\frac{1}{2}}(0,0)| > 2^{n+2}.$$

Notice that for $k \gg 1$ large enough, $\mathfrak{D}_k \cap \mathscr{C}^-_{r_k}(P_k) \cap \{u > 0\} \neq \varnothing$. Moreover, since $a_k \to 0$ as $k \to \infty$, we have, for $k \gg 1$ large enough, that $|\mathfrak{D}_k^c| \geq \frac{1}{10}|\mathscr{C}^-_{1/2}(0,0)|$. Therefore, we can estimate for $k \gg 1$

$$\begin{aligned}\fint_{\mathscr{C}^-_{\frac{1}{2}}(0,0)} v_k^\mu(y,s) &\geq \frac{\mathfrak{C}_\sharp^\mu}{|\mathscr{C}^-_{\frac{1}{2}}(0,0)|}\int_{\mathfrak{D}_k^c}\left(\frac{d_k(y,s)}{r_k}\right)^{\frac{2\mu}{1-\mu}} \\ &\geq \frac{2^{\alpha(N+2)}}{k}|\mathfrak{D}_k^c| \geq \frac{2^{\alpha(n+2)}}{10k}|\mathscr{C}^-_{\frac{1}{2}}(0,0)| > \frac{2^{n+2}}{k},\end{aligned}$$

which contradicts (4-6). This finishes the proof of the corollary. □

The nondegeneracy property and the growth rate for viscosity solutions to **(DCP)** will lead us to establish some measure-theoretic properties of the free boundary. We start by showing a property of positive density.

Corollary 4.8 (positive Lebesgue density of $\{u > 0\}$). *Let u be as in Theorem 1.1. Then, there exists a positive constant $\zeta = \zeta(n, \lambda, \Lambda, \mathfrak{M}, \|u\|_{L^\infty(\mathscr{C}_1)})$ such that for all $(x_0, t_0) \in \overline{\{u > 0\}}$ and $0 < r < 1$ such that $\mathscr{C}_r(x_0, t_0) \subset \mathscr{C}_{1/2}$, the inclusion*

$$\mathscr{C}_{\zeta r}(x', t') \subset \mathscr{C}_r(x_0, t_0) \cap \{u > 0\},$$

holds for some $(x', t') \in \mathscr{C}^-_r(x_0, t_0)$.

Proof. Let $(x_0, t_0) \in \overline{\{u > 0\}} \cap \overline{\mathscr{C}_{1/2}}$. For r small enough, we have by Theorem 1.2 that there exists $(x', t') \in \mathscr{C}^-_{r/2}(x_0, t_0)$ such that

$$u(x', t') \geq \mathfrak{c} \cdot \left(\frac{r}{2}\right)^{\frac{2}{1-\mu}}. \tag{4-7}$$

Suppose that for all $0 < \mathfrak{d} < 1$ small, there exists a point $(x, t) \in \partial\{u > 0\} \cap \overline{\mathscr{C}_{1/2}}$ satisfying

$$(x', t') \in \mathscr{C}_\mathfrak{d}(x, t) \subset \mathscr{C}_r(x_0, t_0). \tag{4-8}$$

Now, according to (4-7), (4-8) and Theorem 1.1, it follows

$$\mathfrak{c} \cdot \left(\frac{r}{2}\right)^{\frac{2}{1-\mu}} \leq u(x', t') \leq \sup_{\mathscr{C}_\mathfrak{d}(x,t)} u \leq \mathfrak{C}, \mathfrak{d}^{\frac{2}{1-\mu}}.$$

This clearly does not hold for $\mathfrak{d} < \Xi \cdot \frac{r}{2}$, where $\Xi := \left(\frac{\mathfrak{c}}{\mathfrak{C}}\right)^{(1-\mu)/2} < 1$. Hence $\mathscr{C}_{\Xi/4\, r}(x', t') \subset \mathscr{C}_r(x_0, t_0) \cap \{u > 0\}$. This ends the proof of the theorem. □

Remark 4.9. Notice that Corollary 4.8 ensures that the free boundary cannot have Lebesgue points. Consequently, for any compact set $\mathfrak{K} \subset \mathscr{C}_1$, we have

$$\mathscr{L}^{n+1}(\partial\{u > 0\} \cap \mathfrak{K}) = 0.$$

Next, we shall prove, as an easy consequence of the above result, that the free boundary is a porous set. We recall the definition of this notion.

Definition 4.10 (porous set). A set $\mathscr{E} \in \mathbb{R}^n$ is said to be porous with porosity constant $0 < \zeta \leq 1$ if there exists $\mathfrak{R} > 0$ such that for each $x_0 \in \mathscr{E}$ and $0 < \mathfrak{r} < \mathfrak{R}$ there is a point x' so that $B_{\mathfrak{r}\zeta}(x') \subset B_{\mathfrak{r}}(x_0) \setminus \mathscr{E}$.

Observe that a porous set has Hausdorff dimension at most $n - c_0\zeta^n$, where $c_0 = c_0(n) > 0$.

Corollary 4.11 (porosity for t-level of free boundary). *Let u be a viscosity solution to (DCP). For every compact set $\mathscr{K} \subset \mathscr{C}_1$,*

$$\mathscr{H}^{n-\delta}(\partial\{u > 0\} \cap \mathscr{K} \cap \{t = t_0\}) < \infty$$

for a constant $0 < \delta = \delta(n, \lambda, \Lambda, \mu, \mathfrak{M}, \|u\|_{L^\infty(\mathscr{C}_1)}, \operatorname{dist}(\mathscr{K}, \partial\mathscr{C}_1)) \leq 1$.

Proof. The proof is standard (see, for instance, [Choe and Weiss 2003]). However, we quote full details for completeness. Without loss of generality we can suppose that $\mathscr{K} = \overline{\mathscr{C}_{1/2}}$. Let $(z, t_0) \in \partial\{u > 0\} \cap \overline{\mathscr{C}_{1/2}}$; then for $0 < r \ll 1$, according to the nondegeneracy property, there exists $x' \in \partial B_r(z)$ such that

$$u(x', t_0) \geq \mathfrak{C}_0^* \cdot r^{\frac{2}{1-\mu}}.$$

On the other hand, from Theorem 3.3,

$$u(x', t_0) \leq \mathfrak{C} \cdot \mathfrak{d}(x', t_0)^{\frac{2}{1-\mu}}.$$

Consequently,

$$\mathfrak{c} \cdot r^{\frac{2}{1-\mu}} \leq u(x', t_0) \leq \mathfrak{C} \cdot \mathfrak{d}(x', t_0)^{\frac{2}{1-\mu}}.$$

Next, by selecting $\delta = \left(\frac{\mathfrak{c}}{\mathfrak{C}}\right)^{(1-\mu)/2}$, then $\mathfrak{d}(x', t_0) \geq \delta \cdot r$. Therefore

$$B_{\delta \cdot r}(x') \cap B_r(z) \subset \{u(\,\cdot\,, t_0) > 0\}.$$

Now, choose $y \in [z, x']$ such that $|y - x'| = \frac{\delta r}{2}$. Note that for any $y_0 \in B_{(\delta r)/2}(y)$,

$$|y_0 - x'| \leq |y_0 - y| + |y - x'| = \delta r.$$

Moreover, since $|z - x'| = |y - z| + |y - x'|$,

$$|y_0 - z| \leq |y_0 - y| + (|z - x'| - |y - x'|) \leq \frac{\delta r}{2} + \left(r - \frac{\delta r}{2}\right) = r,$$

so we conclude that $B_{(\delta r)/2}(y) \subset B_{\delta r}(x') \cap B_r(z) \subset B_r(z) \setminus (\partial\{u > 0\} \cap \{t = t_0\})$. Therefore, $\partial\{u > 0\} \cap \{t = t_0\} \cap \overline{\mathscr{C}_{1/2}}$ is porous with porosity constant $\frac{\delta}{2}$. □

In contrast to one of the most known properties of heat equations, namely the infinity speed of propagation, fully nonlinear parabolic dead core problems have the property of *finite speed propagation*. Such a property supports the physical soundness of the equation to diffusive models. Moreover, the occurrence of this phenomenon is a consequence of loss of diffusivity of the equation at the level set $u = 0$. The proof will be based on [Choe and Weiss 2003, Corollary 4.4].

Corollary 4.12 (finite speed propagation of $\{u > 0\}$). *There exists a constant* $\mathfrak{c}(n, \lambda, \Lambda, \mu) \geq 1$ *such that, for any solution to* **(DCP)**, *with nonnegative and bounded time derivative, and any* $\mathscr{C}_r^+(x_0, t_0) \subset \Omega \times (0, \infty)$, *the implication*

$$u(\cdot, t_0) = 0 \quad \text{in } B_r(x_0) \Rightarrow u(\cdot, t_0 + \mathfrak{s}^2) = 0 \quad \text{in } B_{\max\{0, r - \mathfrak{cs}\}}(x_0)$$

holds.

Proof. Let us suppose that for $0 < \mathfrak{s}_1 < \frac{r}{\mathfrak{c}}$ there exists a point $x_1 \in B_{r-\mathfrak{cs}_1}(x_0)$ such that $u(x_1, t_0 + \mathfrak{s}_1^2) > 0$. The nondegeneracy property (Theorem 1.2) implies

$$u(x_2, \varsigma) \geq \mathfrak{C}_0^* \mathfrak{s}_1^{\frac{2}{1-\mu}}$$

for some $(x_2, \varsigma) \in \overline{\mathscr{C}_{\mathfrak{s}_1}^-(x_1, t_0 + \mathfrak{s}_1^2)}$. Moreover, since $\frac{\partial u}{\partial t}$ is nonnegative and bounded, we deduce that there exist $0 < \tau(n, \mu) < 1$ and $(x_2, t_0 + \mathfrak{s}_2^2)$ satisfying

$$u(x_2, t_0 + \mathfrak{s}_2^2) > 0, \quad \text{with} \quad 0 \leq \mathfrak{s}_2 \leq (1 - \tau)\mathfrak{s}_1 \quad \text{and} \quad |x_2 - x_1| \leq \mathfrak{s}_1.$$

By iterating the previous reasoning we can obtain a point $(x_k, t_0 + \mathfrak{s}_k^\theta)$ such that

$$u(x_k, t_0 + \mathfrak{s}_k^2) > 0, \quad \text{with } 0 \leq \mathfrak{s}_k \leq (1-\tau)^{k-1}\mathfrak{s}_1 \text{ and } |x_k - x_1| \leq \frac{\mathfrak{s}_1[1 - (1-\tau)^{k-1}]}{\tau}.$$

Finally, up to a subsequence, $x_k \to x_\infty$ as $k \to \infty$; thus we obtain a point $(x_\infty, t_0) \in \overline{\{u > 0\}}$ fulfilling $|x_\infty - x_1| < \mathfrak{s}_1/\tau$. However, this contradicts our assumptions provided $\mathfrak{c} \geq \frac{4}{\tau}$. This contradiction proves the corollary. □

5. Global analysis results

5A. ***Blow-up analysis.*** Throughout this subsection we shall study the blow-up analysis over free boundary points (interior touching points). Historically, such a procedure provides a powerful device in order to study certain one-dimensional profiles in several free boundary problems (see [Apushkinskaya et al. 2002] and [Caffarelli et al. 2004] for some enlightenment). Thus, let u be a solution to **(DCP)**, $\mathfrak{p}_0 := (x_0, t_0) \in \partial\{u > 0\} \cap \mathscr{C}_{1/2}$ and $(\mathfrak{p}_k = (x_k, t_k))_{k \in \mathbb{N}} \in \partial\{u > 0\} \cap \mathscr{C}_{1/2}$ such that

$\mathfrak{p}_k \to \mathfrak{p}_0$. Now, consider, for each $\varepsilon_k \searrow 0$, the *blow-up sequence* $u_{\varepsilon_k} : \mathscr{C}_{1/2} \to \mathbb{R}$ given by

$$u_{\varepsilon_k}^{\mathfrak{p}_k}(x,t) := \frac{u(x_k + \varepsilon_k x, t_k + \varepsilon_k^2 t)}{\varepsilon_k^{\frac{2}{1-\mu}}}.$$

We must stress that this sequence is indeed an ε_k-zoom-in of u (on the free boundary points) rescaled in a suitable way. The next result analyses the "limiting profile" for any blow-up sequence.

Theorem 5.1 (blow-up limit). *Let $\mathfrak{p}_0 = (x_0, t_0) \in \partial\{u > 0\}$ be a free boundary point, $(\mathfrak{p}_k)_{k\in\mathbb{N}} \in \partial\{u > 0\}$ such that $\mathfrak{p}_k \to \mathfrak{p}_0$ and a blow-up sequence $(u_{\varepsilon_k}^{\mathfrak{p}_k})_{k\in\mathbb{N}}$. Then, up to a subsequence*

$$u_{\varepsilon_k}^{\mathfrak{p}_k} \to u_0^{\mathfrak{p}_0} \quad \textit{uniformly in every compact } \mathfrak{K} \subset \mathbb{R}^n \times \mathbb{R}.$$

Furthermore, $u_0^{\mathfrak{p}}$ is a nonnegative viscosity solution to the following global parabolic dead-core problem with constant coefficients:

$$\text{(5-1)}\quad \mathscr{F}(x_0, t_0, D^2 u_0^{\mathfrak{p}_0}(x,t)) - \partial_t u_0^{\mathfrak{p}_0}(x,t) = \lambda_0(x_0,t_0)\cdot (u_0^{\mathfrak{p}_0})_+^{\mu}(x,t) \quad \textit{in } \mathbb{R}^n \times \mathbb{R}.$$

Finally, $(0,0) \in \partial\{u_0^{\mathfrak{p}_0} > 0\}$.

Proof. Note that $u_{\varepsilon_k}^{\mathfrak{p}_k}$ fulfills, in the viscosity sense,

$$\mathscr{F}_{\varepsilon_k}(x,t,Du_{\varepsilon_k}^{\mathfrak{p}_k}, D^2 u_{\varepsilon_k}^{\mathfrak{p}_k}) - \partial_t u_{\varepsilon_k}^{\mathfrak{p}_k} = \widehat{\lambda}_k(x,t)\cdot (u_{\varepsilon_k}^{\mathfrak{p}_k})_+^{\mu} \quad \text{in } \mathscr{C}_{\frac{1}{2\varepsilon_k}}$$

where

$$\mathscr{F}_{\varepsilon_k}(x,t,\vec{p}\,,\mathscr{X}) := \varepsilon_k^{-\frac{2\mu}{1-\mu}} \mathscr{F}(x_k + \varepsilon_k x, t_k + \varepsilon_k^2 t, \varepsilon_k^{\frac{1+\mu}{1-\mu}}\vec{p}\,, \varepsilon_k^{\frac{2\mu}{1-\mu}}\mathscr{X})$$

and

$$\widehat{\lambda}_k(x,t) := \lambda_0(x_k + \varepsilon_k x, t_k + \varepsilon_k^2 t).$$

Observe that $\mathscr{F}_{\varepsilon_k}$ satisfies assumption **(SC)** from Section 2, with the same structural parameters. As a consequence, from Theorem 1.1 we have

$$u_{\varepsilon_k}^{\mathfrak{p}_k}(x,t) \le C(n,\lambda,\Lambda,\mathfrak{M},\mu) \quad \text{for all } (x,t) \in \mathscr{C}_{\frac{1}{2\varepsilon_k}}.$$

Particularly, $u_{\varepsilon_k}^{\mathfrak{p}_k}$ is locally bounded in $\mathscr{C}_{1/(2\varepsilon_k)}$. From universal Hölder regularity, see for instance Proposition 2.2, up to a subsequence, $u_{\varepsilon_k}^{\mathfrak{p}_k} \to u_0^{\mathfrak{p}_0}$ locally uniformly to an entire function. Furthermore, $\partial\{u_0^{\mathfrak{p}_0} > 0\}$ has zero $(n+1)$-Lebesgue measure and by stability results (see Lemma 2.3) $u_0^{\mathfrak{p}_0}$ fulfills

$$\mathscr{F}(x_0,t_0,D^2 u_0^{\mathfrak{p}_0}(x,t)) - \partial_t u_0^{\mathfrak{p}_0}(x,t) = \lambda_0(x_0,t_0)\cdot (u_0^{\mathfrak{p}_0})_+^{\mu}(x,t) \quad \text{in } \mathbb{R}^n \times \mathbb{R}.$$

Obviously, $u_0^{\mathfrak{p}_0}$ is a global nonnegative solution. By nondegeneracy (Theorem 1.2)

$$\sup_{\mathscr{C}_r^-(0,0)} u_0^{\mathfrak{p}_0}(x,t) = \lim_{k\to\infty} \sup_{\mathscr{C}_r^-(0,0)} u_{\varepsilon_k}^{\mathfrak{p}_k}(x,t) \ge \mathfrak{C}_0^* \cdot r^{\frac{2}{1-\mu}},$$

which ensures that $(0,0) \in \partial\{u_0^{\mathfrak{p}_0} > 0\}$. □

From now on, for our purposes, $u_0^{p_0}$ will always denote a limiting function, using the previous reasoning. For this reason, we will label it as the *blow-up solution* at (x_0, t_0).

The next result establishes a quantitative control profile at infinity for a class of entire solutions to the dead-core problem, namely, blow-up solutions.

Theorem 5.2 (behavior of blow-up solutions). *Let u_0 be a blow-up solution at $(0,0) \in \partial\{u > 0\}$. Then, there exist universal constants $\mathfrak{c}_0, \mathfrak{C}_0 > 0$ such that*

$$\begin{aligned} \mathfrak{c}_0 &\le \liminf_{\mathrm{dist}_p((x,t),(0,0))\to\infty} \frac{u_0(x,t)}{\mathrm{dist}_p((x,t),(0,0))^{\frac{2}{1-\mu}}} \\ &\le \limsup_{\mathrm{dist}_p((x,t),(0,0))\to\infty} \frac{u_0(x,t)}{\mathrm{dist}_p((x,t),(0,0))^{\frac{2}{1-\mu}}} \le \mathfrak{C}_0. \end{aligned} \tag{5-2}$$

Proof. Such a lower and upper control at infinity are a consequence of Theorem 1.1 and Theorem 1.2, respectively. □

Remark 5.3. Note that (5-2) implies that blow-up solutions are nontrivial. Moreover, (5-2) says that blow-up solutions satisfy

$$c_0 \cdot r^{\frac{2}{1-\mu}} \le \mathscr{S}_r[u_0] \le C_0 \cdot r^{\frac{2}{1-\mu}},$$

for values of r large enough.

Remark 5.4. In view of Theorem 5.2, the nontrivial space-independent blow-up solution $u = u(t)$ to

$$\mathscr{F}(x_0, t_0, D^2u) - \partial_t u = \lambda_0(x_0, t_0) u_+^{\mu}(x,t) \quad \text{in } \mathbb{R}^n \times \mathbb{R}$$

is given by $u(t) = [-(1-\mu)\lambda_0(x_0,t_0)t]_+^{1/(1-\mu)}$. On the other hand, when $\mathscr{F}(\cdot) = \mathrm{Tr}(\cdot)$, nontrivial time-independent blow-up solutions $u = u(x)$ are of the form

$$u(x) = \{C_{n,\mu} \cdot (x_i)_+^{\frac{2}{1-\mu}}, C_{n,\mu} \cdot (x_i)_-^{\frac{2}{1-\mu}}, C_{n,\mu} \cdot (|x - x_0| - \mathfrak{R}_0)_+^{\frac{2}{1-\mu}}\},$$

for any $i = 1, \dots, n$, where

$$C_{n,\mu} := \left(\frac{\lambda_0 (1-\mu)^2}{2(1+\mu)} \right)^{\frac{1}{1-\mu}}.$$

Notice that the first blow-up type solutions are half-space solutions and the last one is a radial solution with dead core being precisely $B_{\mathfrak{R}_0}(x_0)$; see [da Silva et al. 2017, Section 6] for an analysis about radial solutions of fully nonlinear elliptic dead core problems. Finally, another interesting example of blow-up solutions are those with independent variables, i.e., $u(x,t) = v(x) + w(t)$.

5B. ***A Liouville-type theorem.*** In this section we are concerned with proving a Liouville-type result for global dead-core solutions. In summary, we show that a global viscosity solution must grow faster than $(|x|+|t|^{1/2})^{2/(1-\mu)}$ as $|x|+|t|^{1/2}\to\infty$, unless it is identically zero. The proof will be based on [Teixeira 2016, Theorem 8].

Theorem 5.5. *Let u be an entire viscosity solution to*

$$\mathscr{F}(x,t,D^2u)-\partial_t u(x,t)=\lambda_0(x,t)\cdot u_+^{\mu}(x,t)$$

with $u(0,0)=0$. If $u(x,t)=\mathrm{o}(\mathrm{dist}_p((x,t),(0,0))^{\frac{2}{1-\mu}})$ as $\mathrm{dist}_p((x,t),(0,0))\to\infty$, then $u\equiv 0$.

Proof. For each positive number $r\gg 1$, let us define

$$u_r(x,t):=\frac{u(r\,x,r^2t)}{r^{\frac{2}{1-\mu}}}.$$

Thus, it is easy to check that

$$\mathscr{F}_r(x,t,D^2u_r)-\partial_t u_r=\lambda_0(rx,r^2t)(u_r)_+^{\mu}\quad\text{in }\mathscr{C}_1$$

and $u_r(0,0)=0$, where $\mathscr{F}_r(x,t,\mathscr{X}):=r^{-2\mu/(1-\mu)}\mathscr{F}(rx,r^2t,r^{2\mu/(1-\mu)}\mathscr{X})$. Moreover, we note that $\|u_r\|_{L^\infty(\mathscr{C}_1)}=\mathrm{o}(1)$. In fact, for each $r\in\mathbb{N}$, let $(x_r,t_r)\in\mathbb{R}^n\times\mathbb{R}$ be such that

$$u_r(x_r,t_r)=\sup_{\mathscr{C}_1}u_r(x,t).$$

We must consider two possibilities:

(1) If $\lim_{r\to\infty}\mathrm{dist}_p((rx_r,r^2t_r),(0,0))=\infty$, we get

$$\begin{aligned}u_r(x_r,t_r)&=\mathrm{dist}_p((rx_r,r^2t_r),(0,0))^{-\frac{2}{1-\mu}}u(rx_r,r^2t_r)\mathrm{dist}_p((rx_r,r^2t_r),(0,0))^{\frac{2}{1-\mu}}\\&\le C(n,\lambda,\Lambda,\mu)\cdot o(1)\to 0\quad\text{as } r\to\infty.\end{aligned}$$

(2) On the other hand, if $\lim_{r\to\infty}\mathrm{dist}_p((rx_r,r^2t_r),(0,0))<\infty$, the conclusion is immediate, because u is a continuous function.

Therefore, applying Theorem 1.1 we have

$$u_r(x,t)\le \mathrm{o}(1)\cdot\mathrm{dist}_p((x,t),(0,0))^{\frac{2}{1-\mu}}\quad\text{in }\mathscr{C}_{\frac12}.\tag{5-3}$$

Now, if we assume that there exists $(\hat{x},\hat{t})\in(\mathbb{R}^n\times\mathbb{R})\setminus\{(0,0)\}$ such that $u(\hat{x},\hat{t})>0$, we deduce from (5-3) that

$$\sup_{\mathscr{C}_{\frac12}}\frac{u_r(x,t)}{\mathrm{dist}_p((x,t),(0,0))^{\frac{2}{1-\mu}}}\le\frac{u(\hat{x},\hat{t})}{100\,\mathrm{dist}_p((\hat{x},\hat{t}),(0,0))^{\frac{2}{1-\mu}}},\tag{5-4}$$

provided $r \gg 1$. We now estimate, for $r \gg \max\{2|\hat{x}|, \sqrt{2}|\hat{t}|\}$:

$$\begin{aligned}\frac{u(\hat{x},\hat{t})}{\operatorname{dist}_p((\hat{x},\hat{t}),(0,0))^{\frac{2}{1-\mu}}} &\leq \sup_{\mathscr{C}_{\frac{r}{2}}} \frac{u(x,t)}{\operatorname{dist}_p((x,t),(0,0))^{\frac{2}{1-\mu}}}\\ &\leq \sup_{\mathscr{C}_{\frac{1}{2}}} \frac{u_r(x,t)}{\operatorname{dist}_p((x,t),(0,0))^{\frac{2}{1-\mu}}}\\ &\leq \frac{u(\hat{x},\hat{t})}{100\operatorname{dist}_p((\hat{x},\hat{t}),(0,0))^{\frac{2}{1-\mu}}},\end{aligned}$$

which finally drives us to a contradiction, completing the proof of theorem. □

6. Hausdorff measure estimates

In this section, we will proceed to estimate the parabolic Hausdorff measure (i.e., the Hausdorff measure with respect to the parabolic distance) of the free boundary set of dead-core solutions u. To this end, we need first some preliminary results, which are based on the set of assumptions in Section 2 on the operator $\mathscr{F}$, together with the following additional hypothesis:

(C) (($\mathbb{M}$, $\mathfrak{b}$)-concavity) There exist a constant $\hat{\mathfrak{C}} > 0$ and a bounded symmetric positive definite Lipschitz matrix $\mathbb{M} : \mathscr{C}_1 \to \operatorname{Sym}(n)$, $\mathbb{M} = [m_{ij}]$, so that

$$\mathscr{F}(x, t, \vec{p}, \mathscr{X}) \leq \operatorname{Tr}(\mathbb{M} \cdot \mathscr{X}) + \mathfrak{b}|\vec{p}|,$$

in the viscosity sense, where $\mathfrak{b} \geq 0$ and $0 \leq \mathfrak{c}(x,t) \leq \mathfrak{c}_0^*$. We further assume that there exists a constant $\beta > 0$ such that $m_{ij} \geq \beta$ for all i, j (see [Ricarte et al. 2017, Section 5] for a similar property).

(T) (lower bound for $\partial_t u$) There is a constant $\mathfrak{c}_0 \geq 0$ such that $\partial_t u \geq -\mathfrak{c}_0 u_+^{\mu}$ in the viscosity sense.

Before discussing the main result of this section, let us present some useful notions and preliminary results.

Definition 6.1. Let $\mathscr{A}$ be a subset of a parabolic domain $\mathscr{C}$. We say that $\mathscr{A}$ has the (δ, ζ)-density property if there is $\delta \in (0,1)$ so that there corresponds $\zeta > 0$ with the property

$$\frac{\mathscr{L}^{n+1}(\mathscr{C}_\delta(x,t) \cap \mathscr{A})}{\mathscr{L}^{n+1}(\mathscr{C}_\delta(x,t))} \geq \zeta, \tag{6-1}$$

for all $(x,t) \in \partial A \cap \mathscr{C}$. If (6-1) holds for all δ in $(0,1)$, then we say that $\mathscr{A}$ has uniform density in $\mathscr{C}$ along $\partial\mathscr{A}$.

As a consequence of the above definition, we derive the following facts which will be used in our proof of Hausdorff estimates for the free boundary.

Proposition 6.2. *Let $\mathscr{A} \subset \mathscr{C}$ be open. Then*:

- *If $\mathscr{A}$ has the (δ, ζ)-density property, there is a constant $M = M(n)$ such that*

$$\mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\mathscr{A}) \cap \mathscr{C}_\rho(x_0, t_0)) \le \frac{M(n)}{\zeta}\mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\mathscr{A}) \cap \mathscr{C}_\rho(x_0, t_0) \cap \mathscr{A})$$

  for $(x_0, t_0) \in \partial\mathscr{A} \cap \mathscr{C}$ and $\delta \ll \rho$.

- *If $\mathscr{A}$ has uniform density in $\mathscr{C}$ along $\partial\mathscr{A}$, then $\mathscr{L}^{n+1}(\partial\mathscr{A} \cap \mathscr{C}) = 0$.*

Proof. We prove the first part. Let $(x_0, t_0) \in \partial\mathscr{A} \cap \mathscr{C}$ and $\delta < \rho$. Consider a covering of $\mathscr{N}_\delta(\partial\mathscr{A}) \cap \mathscr{C}_\rho(x_0, t_0)$ by cylinders $\mathscr{C}_\delta(x, t)$ centered at points on $\partial\mathscr{A} \cap \mathscr{C}_\rho(x_0, t_0)$. By [Lieberman 1996, Lemma 7.8], we may extract a countable and disjoint subfamily of cylinders $\{\mathscr{C}_\delta(x_i, t_i)\}_i$ so that $\{\mathscr{C}_{5\delta}(x_i, t_i)\}_i$ covers $\mathscr{N}_\delta(\partial\mathscr{A}) \cap \mathscr{C}_\rho(x_0, t_0)$. Hence:

$$\begin{aligned}\mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\mathscr{A}) \cap \mathscr{C}_\rho(x_0, t_0)) &\le \sum \mathscr{L}^{n+1}(\mathscr{C}_{5\delta}(x_i, t_i)) \\ &\le \frac{M(n)}{\zeta}\mathscr{L}^{n+1}(\mathscr{C}_\delta(x_i, t_i) \cap \mathscr{A}) \\ &\le \frac{M(n)}{\zeta}\mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\mathscr{A}) \cap \mathscr{C}_{2\rho}(x_0, t_0) \cap \mathscr{A}),\end{aligned}$$

where we have used the (δ, ζ)-density property of $\mathscr{A}$ and the fact that $\mathscr{C}_\delta(x_i, t_i) \subset \mathscr{N}_\delta(\partial\mathscr{A}) \cap \mathscr{C}_{2\rho}(x_0, t_0)$ □

We start with the series of preliminary results needed in the proof of Theorem 1.3. The first lemma contains an L^2 estimate on the gradient near free boundary points.

Lemma 6.3. *There exists a constant $C > 0$ such that for all $(x_0, t_0) \in \partial\{u > 0\} \cap \mathscr{C}_{1/2}$ and $\rho < \frac{1}{4}$, the following holds*:

$$\int_{\mathscr{C}_\rho(x_0,t_0)\cap\{0<u<\epsilon^{\frac{2}{1-\mu}}\}} |\nabla u|^2 \le C\epsilon\rho^{n+1}.$$

Proof. Define

$$\Phi(x, t) := u(x, t)\chi_{\{0<u\le\epsilon^{\frac{2}{1-\mu}}\}}(x, t) + \epsilon\chi_{\{u>\epsilon^{\frac{2}{1-\mu}}\}}(x, t). \tag{6-2}$$

Integration by parts gives

$$\begin{aligned}&\int_{\mathscr{C}_\rho(x_0,t_0)} \Phi\cdot m_{ij} D_{ij} u \\ &\quad= \int_{t_0-\rho^2}^{t_0+\rho^2}\left[\frac{1}{\rho}\int_{\partial B_\rho(x_0)} \Phi\cdot m_{ij}\cdot D_j u\cdot(x^i - x_0^i)\, d\mathscr{H}^{n-1} - \int_{B_\rho(x_0)} D_i(\Phi\cdot m_{ij}) D_j u\, dx\right] dt\end{aligned}$$

In view of the assumptions **(C)** and **(T)**, nondegeneracy on average for dead core solutions (Corollary 4.7), the growth rate on gradient, as well as that $\frac{2\mu}{1-\mu} \le \frac{1+\mu}{1-\mu}$ for any $0 < \mu < 1$, we conclude

$$\begin{aligned}\int\limits_{\mathscr{C}_\rho(x_0,t_0)} \mathrm{Tr}(\mathbb{M}D^2u) &\ge \int\limits_{\mathscr{C}_\rho(x_0,t_0)} (\mathfrak{m}-\mathfrak{c}_0)u_+^\mu(x,t) - \mathfrak{b}|Du| \\ &\ge \omega_n\rho^{n+2}[\mathfrak{C}_*(\mathfrak{m}-\mathfrak{c}_0)\rho^{\frac{2\mu}{1-\mu}} - \mathfrak{b}\mathfrak{C}_1\rho^{\frac{1+\mu}{1-\mu}}] \\ &\ge \omega_n\rho^{n+2}\rho^{\frac{2\mu}{1-\mu}}[\mathfrak{C}_*(\mathfrak{m}-\mathfrak{c}_0) - \mathfrak{b}\mathfrak{C}_1] \ge 0.\end{aligned}$$

In particular, we derive

$$\int\limits_{\mathscr{C}_\rho(x_0,t_0)} \Phi\cdot m_{ij}D_{ij}u \ge 0.$$

Hence

$$\int_{\mathscr{C}_\rho(x_0,t_0)\cap\{0<u<\epsilon^{\frac{2}{1-\mu}}\}} m_{ij}D_iu\cdot D_ju \le \frac{1}{\rho}\int_{t_0-\rho^2}^{t_0+\rho^2}\int_{\partial B_\rho(x_0)} \Phi\cdot m_{ij}\cdot D_ju\cdot(x^i-x_0^i)\,d\mathscr{H}^{n-1} - \int_{\mathscr{C}_\rho(x_0,t_0)\cap\{0<u<\epsilon^{\frac{2}{1-\mu}}\}} \Phi D_im_{ij}\cdot D_ju.$$

Using regularity of Du and that $0 < \rho \ll 1$, we conclude the proof of the lemma. □

The above gradient estimate may be applied to get bounds on the Lebesgue measure of the set $\{0 < u < \epsilon^{2/(1-\mu)}\}$ near the free boundary, in terms of the upper bound ϵ. Precisely, we have the next lemma.

Lemma 6.4. *There exists a constant $C > 0$ such that for any $\epsilon > 0$ small enough, any $(x_0, t_0) \in \partial\{u > 0\} \cap \mathscr{C}_{1/2}$ and any ρ small, the estimate*

$$\mathscr{L}^{n+1}(\mathscr{C}_\rho(x_0,t_0)\cap\{0<u<\epsilon^{\frac{2}{1-\mu}}\}) \le C\rho^{n+1}\epsilon,$$

holds.

Proof. From a Vitali covering theorem for parabolic cylinders, see [Lieberman 1996, Lemma 7.8], consider $\{\mathscr{C}_j\}$, a finite covering by cylinders of $\partial\{u > 0\} \cap \mathscr{C}_\rho(x_0, t_0)$, with center at $(x_j, t_j) \in \partial\{u > 0\}$ and radius $C^*\epsilon$, for a constant $C^* > 0$ to be determined a posteriori. Moreover, we require that

$$\bigcup_j \mathscr{C}_j \subset \mathscr{N}_{\frac{1}{4}}(\mathscr{C}_{\frac{1}{2}}) \cap \mathscr{C}_\rho(x_0,t_0)$$

Observe that, from the Heine–Borel lemma there exists a constant $\mathfrak{l} > 0$ (with dimensional dependence) such that

$$\sum_j \chi_{\mathscr{C}_j}(x,t) \le \mathfrak{l}. \tag{6-3}$$

We first prove the estimate

$$\int_{\{0<u<\epsilon^{\frac{2}{1-\mu}}\}\cap\mathscr{C}_j} |\nabla u|^2 \geq C\mathscr{L}^{n+1}(\mathscr{C}_j), \tag{6-4}$$

for some $C > 0$ and C^* to be chosen large enough. Indeed, in view of the nondegeneracy property, there is $(x_j^1, t_j^1) \in \frac{1}{4}\mathscr{C}_j$ such that

$$u(x_j^1, t_j^1) = \sup_{\frac{1}{4}\mathscr{C}_j} u \geq \mathfrak{C}_0^* \cdot \left(\frac{1}{4}C^*\epsilon\right)^{\frac{2}{1-\mu}}.$$

Next, choose $C^* > 0$ large enough so that

$$K := \sup_{\mathscr{N}_{\frac{1}{4}}\left(\mathscr{C}_{\frac{1}{2}}\right)} |\nabla u| \geq \frac{1}{C^*} \quad \text{and} \quad \mathfrak{C}_0^* \cdot (C^*)^{\frac{2}{1-\mu}} > 4^{\frac{2}{1-\mu}}.$$

Next, we choose $\epsilon > 0$ small enough so that if $r_j^1 = \frac{\epsilon}{K}$ and $r_j^2 = \frac{\epsilon}{K}$, then

$$\Phi \geq \frac{3\epsilon}{4} \quad \text{in } \mathscr{C}_j^1 := \mathscr{C}_{r_j^1}(x_j^1, t_j^1) \tag{6-5}$$

and

$$\Phi < \frac{2\epsilon}{3} < \epsilon \quad \text{in } \mathscr{C}_j^2 := \mathscr{C}_{r_j^2}(x_j, t_j), \tag{6-6}$$

where Φ is defined in (6-2). We claim that if $m_j := \fint_{\mathscr{C}_j} \Phi$, then $|\Phi - m_j| > \varsigma\epsilon$ for some $\varsigma > 0$ and for at least one of the cylinders $\mathscr{C}_j^1$ and $\mathscr{C}_j^2$. In fact, if this is not the case, then we can find sequences $(x_k, t_k) \in \mathscr{C}_j^1$ and $(y_k, s_k) \in \mathscr{C}_j^2$ such that

$$\frac{|\Phi(x_k, t_k) - m_j|}{\epsilon} < \frac{1}{k} \quad \text{and} \quad \frac{|\Phi(y_k, s_k) - m_j|}{\epsilon} < \frac{1}{k}.$$

Letting $k \to \infty$, we obtain

$$\frac{|\Phi(x_k, t_k) - \Phi(y_k, s_k)|}{\epsilon} \to 0.$$

This contradicts (6-5) and (6-6). Thus, by Poincaré inequality, we have

$$\varsigma^2\epsilon^2 \leq \fint_{\mathscr{C}_j} |\Phi - m_j| \leq (C^*\epsilon)^2 \fint_{\mathscr{C}_j} |\nabla\Phi|^2$$

and hence, for a universal constant $C_2 > 0$, we conclude that

$$\int_{\{0<u<\epsilon\}\cap\mathscr{C}_j} |\nabla u|^2 \geq C_2\mathscr{L}^{n+1}(\mathscr{C}_j).$$

Moreover, by Corollary 4.6, if $(y,s)\in\{0<u<\epsilon^{2/(1-\mu)}\}\cap\mathscr{C}_\rho(x_0,t_0)$, then

$$\mathfrak{C}_0^*\mathrm{dist}((y,s),\partial\{u>0\})^{\frac{2}{1-\mu}}\le u(y,s)<\epsilon^{\frac{2}{1-\mu}}.$$

Thus,

$$\{0<u<\epsilon^{\frac{2}{1-\mu}}\}\cap\mathscr{C}_\rho(x_0,t_0)\subset\mathscr{N}_{\left(\frac{\epsilon}{\mathfrak{C}_0^*}\right)^{\frac{1-\mu}{2}}}(\partial\{u>0\}\cap\mathscr{C}_{2\rho}(x_0,t_0)).$$

Therefore, by enlarging C^* and diminishing ϵ if necessary, we conclude that

$$\{0<u<\epsilon^{\frac{2}{1-\mu}}\}\cap\mathscr{C}_\rho(x_0,t_0)\subset\bigcup_j 2\mathscr{C}_j\subset\mathscr{C}_{4\rho}.$$

Appealing to Lemma 6.3 and the estimate (6-4), we conclude

$$\begin{aligned}C\epsilon\rho^{n+1}\ge\int\limits_{\{0<u<\epsilon^{2/(1-\mu)}\}\cap\mathscr{C}_j}|\nabla u|^2&\ge\frac{1}{\mathfrak{l}}\sum_j\int\limits_{2\mathscr{C}_j\cap\{0<u<\epsilon^{2/(1-\mu)}\}}|\nabla u|^2\\&\ge\frac{C_2}{\mathfrak{l}}\sum_j\mathscr{L}^{n+1}(\mathscr{C}_j)\\&\ge\frac{C_2}{\mathfrak{l}}\mathscr{L}^{n+1}(\{0<u<\epsilon^{\frac{2}{1-\mu}}\}\cap\mathscr{C}_\rho(x_0,t_0)).\end{aligned}$$

□

Theorem 6.5. *There exists a constant $C>0$ such that*

$$\mathscr{L}^{n+1}(\mathscr{N}_\epsilon(\{u>0\}\cap\mathscr{C}_\rho(x_0,t_0)))\le C\epsilon\rho^{n+1},$$

for $(x_0,t_0)\in\partial\{u>0\}\cap\mathscr{C}_{1/2}$.

Proof. First, observe that

$$[\mathscr{N}_\delta(\partial\{u>0\})\cap\mathscr{C}_\rho(x_0,t_0)\cap\{u>0\}]\subset[\{0<u<\mathfrak{C}\delta^{\frac{2}{1-\mu}}\}\cap\mathscr{C}_\rho(x_0,t_0)]\tag{6-7}$$

for $\mathfrak{C}>0$ as in Theorem 1.1. Indeed, if $(x,t)\in\mathscr{N}_\delta(\partial\{u>0\})\cap\mathscr{C}_\rho(x_0,t_0)\cap\{u>0\}$ and $(y,s)\in\partial\{u>0\}$, then

$$u(x,t)\le\mathfrak{C}(|x-y|+|t-s|^{\frac12})^{\frac{2}{1-\mu}}\le\mathfrak{C}\delta^{\frac{2}{1-\mu}}.$$

By the uniform positive Lebesgue density of the positive set of u (see Corollary 4.8), we have that there exists a constant ζ so that

$$\frac{\mathscr{L}^{n+1}(\mathscr{C}_\delta(x_0,t_0))\cap\{u>0\}}{\mathscr{L}^{n+1}(\mathscr{C}_\delta(x_0,t_0))}\ge\zeta.$$

Hence, the set $\{u>0\}$ has the (δ,ζ)-density property, and then by Proposition 6.2, there is a constant $M>0$

$$\begin{aligned}&\mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\{u>0\})\cap\mathscr{C}_\rho(x_0,t_0))\\&\qquad\le C_2\mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\{u>0\})\cap\mathscr{C}_\rho(x_0,t_0)\cap\{u>0\})+M\delta\rho^{n+1}.\end{aligned}$$

Thus, by appealing to (6-7), we derive

$$\mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\{u>0\})\cap\mathscr{C}_\rho(x_0,t_0))\le C_2\mathscr{L}^{n+1}(\{0<u<\mathfrak{C}\delta^{\frac{2}{1-\mu}}\}\cap\mathscr{C}_\rho(x_0,t_0))+M\delta\rho^{n+1}.$$

From Lemma 6.4, we get for δ small enough that

$$\mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\{u>0\}\cap\mathscr{C}_\rho(x_0,t_0)))\le C\delta\rho^{n+1},$$

for some universal $C>0$. □

Remark 6.6. It will be useful to introduce the notion of *parabolic Hausdorff dimension* for a set $\Sigma_0\subseteq\mathbb{R}^{n+1}$.

$$\mathscr{H}_{\mathrm{par}}(\Sigma_0):=\inf\Big\{0\le s<\infty:\text{for all }\gamma>0\text{ there exists }\{\mathscr{C}_{r_j}(x_j,t_j)\}_{j\ge1}\\ \text{such that }\Sigma_0\subseteq\bigcup_{j\ge1}\mathscr{C}_{r_j}(x_j,t_j)\text{ and }\sum_{j\ge1}r_j^s<\gamma\Big\}.$$

We will finish this section with the proof of the Hausdorff estimate for the free boundary.

Proof of Theorem 1.3. Let $0<\delta<\rho<\frac{1}{4}$, and consider a covering $\mathscr{C}_j$ by cylinders of the set $\partial\{u>0\}\cap\mathscr{C}_\rho(x_0,t_0)$ centered at points in $\partial\{u>0\}\cap\mathscr{C}_\rho(x_0,t_0)$ and with radius δ. Hence,

$$\bigcup_j\mathscr{C}_j\subset\mathscr{N}_\delta(\partial\{u>0\})\cap\mathscr{C}_{\rho+\delta}(x_0,t_0).$$

Thus, we derive

$$\begin{aligned}(6\text{-}8)\qquad \mathscr{H}^{n+1}_{\mathrm{par},\delta}(\partial\{u>0\}\cap\mathscr{C}_\rho(x_0,t_0))&\le C\sum_j\delta^{n+1}=C\sum_j\frac{1}{\delta}\mathscr{L}^{n+1}(\mathscr{C}_j)\\ &\le\frac{C}{\delta}\mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\{u>0\})\cap\mathscr{C}_{\rho+\delta}(x_0,t_0))\\ &\le C(\rho+\delta)^{n+1},\end{aligned}$$

where we have used Theorem 6.5 to obtain the last inequality. Hence, the conclusion is reached by letting $\delta\to0$. □

Remark 6.7. We must highlight that the parabolic Hausdorff dimension and classical Hausdorff dimension have the relationship given by

$$2\mathscr{H}(\Sigma_0)-n\le\mathscr{H}_{\mathrm{par}}(\Sigma_0)\le\mathscr{H}(\Sigma_0)+1.$$

Therefore, $\mathscr{H}((\partial\{u>0\}\cap\mathfrak{K}))\le n+\frac{1}{2}$, for any $\mathfrak{K}\Subset\Omega_T$.

We end this section by providing a t-integral estimate on lower-dimensional Hausdorff measure of the free boundary. This is a direct consequence of the previous arguments.

Lemma 6.8. *Consider the set-valued mapping* $\mathfrak{F}(s) := \partial\{u > 0\} \cap \{t = s\}$. *Hence the mapping* $s \to \mathscr{H}^{n-1}(\mathfrak{F}(s) \cap \mathscr{C}_\rho(x_0, t_0))$, *for* $(x_0, t_0) \in \{u > 0\}$, *is Borel measurable in* $(t_0 - \rho^2, t_0 + \rho^2)$ *and the estimate*

$$\int_{t_0-\rho^2}^{t_0+\rho^2} \mathscr{H}^{n-1}(\mathscr{F}(s) \cap \mathscr{C}_\rho(x_0, t_0))\, ds \le \hat{\mathfrak{C}}\rho^{n+1}, \tag{6-9}$$

holds for a universal constant $\hat{\mathfrak{C}}$ *depending only on dimension.*

Proof. We first show that $\mathfrak{F}$ is upper semicontinuous in $(t_0 - \rho^2, t_0 + \rho^2)$. Indeed, if $\mathscr{U}$ is an open set containing $\mathfrak{F}(s_0) \cap \mathfrak{K}$, $\mathfrak{K} \subset \mathscr{C}_{1/2}$ compact, then $\mathfrak{F}(s) \cap \mathfrak{K} \subset \mathscr{U}$ for all s sufficiently close to s_0. In fact, if this is not the case, then we can build up a sequence $(y_k, s_k) \in \mathfrak{F}(s_k) \cap (\mathfrak{K} \setminus \mathscr{U})$ with $s_k \to s_0$. Passing to a subsequence, we have $(y_k, s_k) \to (y_0, s_0) \in \{u > 0\} \cap (\mathfrak{K} \setminus \mathscr{U})$. This contradicts that $\mathfrak{F}(s_0) \subset \mathscr{U}$. Therefore, the mappings $s \to \mathscr{H}^{n-1}_\delta(\mathfrak{F}(s) \cap \overline{\mathscr{C}_\rho(x_0, t_0)})$ and $s \to \mathscr{H}^{n-1}(\mathfrak{F}(s) \cap \overline{\mathscr{C}_\rho(x_0, t_0)})$ are Borel measurable.

Consider now the covering by cylinders from Lemma 6.4. Hence by (6-8), we obtain

$$\begin{aligned}
\mathfrak{C}\cdot(\rho+\delta)^{n+1} &\ge \frac{1}{\delta}\int_{\bigcup_j \mathscr{C}_j(x_j,t_j)} d\mathscr{L}^{n+1} \\
&= \frac{1}{\delta}\int_{-(\delta+\rho)^2}^{(\delta+\rho)^2} \mathscr{L}^n\left(\bigcup_j \mathscr{C}_j(x_j,t_j)\cap\{t=s\}\right) ds \qquad \text{(by coarea formula)} \\
&\ge \frac{\omega_n}{\mathfrak{l}}\int_{-(\delta+\rho)^2}^{(\delta+\rho)^2} \delta^{n-1}\sum_j \chi_{\{\mathscr{C}_j(x_j,t_j)\cap\{t=s\}\}}\, ds \qquad (\mathfrak{l} \text{ as in (6-3)}) \\
&\ge \frac{\omega_n}{\mathfrak{l}\omega_{n-1}}\int_{-(\delta+\rho)^2}^{(\delta+\rho)^2} \mathscr{H}^{n-1}_\delta(\mathfrak{F}(s)\cap\mathscr{C}_\rho(x_0,t_0))\, ds.
\end{aligned}$$

By letting $\delta \to 0$, we derive the estimate (6-9). □

Remark 6.9. We highlight that Lemma 6.8 implies particularly, from geometric measure theory results (see [Federer 1969, Theorems 4.5.6 and 4.5.11]), that $\chi_{\{u>0\}}(\cdot, t)$ is a function of bounded variation for almost every $t \in (0, T)$. Moreover, for any $\varphi \in C^{0,1}(\Omega : \mathbb{R}^n)$ it follows that

$$\int_{\{u(\cdot,t)>0\}} \operatorname{div}\varphi dx = \int_{\partial_{\mathrm{red}}\{u(\cdot,t)>0\}} \varphi \cdot \vec{\nu}\, d\mathscr{H}^{n-1} \quad \text{for almost every } t \in (0, T),$$

where $\vec{\nu}$ is the normal vector in the measure theoretic sense. Nevertheless, such a previous sentence does not yield any additional information on the singular set of free boundary, because neither the $\chi_{\{u>0\}}(\cdot, t)$ nor the reduced boundary $\partial_{\mathrm{red}}\{u(\cdot, t) > 0\}$ detect the singular set.

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