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WILLMORE TYPE INEQUALITY USING MONOTONICITY FORMULAS

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Simon type monotonicity formulas for the Willmore functional $\int |H|^2$ in the hyperbolic space $\mathbb{H}^n$ and $\mathbb{S}^n$ are obtained. These formulas give a lower bound of $\int_\Sigma |H|^2$ where Σ^2 is any closed surface in $\mathbb{H}^n$.

1. Introduction

We study the Willmore functional

$$\int_\Sigma |\boldsymbol{H}|^2$$

for a 2-surface Σ^2 in the standard hyperbolic n-space $\mathbb{H}^n$ of constant sectional curvature -1 and standard n-sphere $\mathbb{S}^n$. Here $\boldsymbol{H}$ is the mean curvature vector of Σ in the respective ambient spaces. Willmore established the following classic result in the Euclidean space $\mathbb{R}^3$:

Theorem 1 (See, for example, [Willmore 1971]). *Given any closed smooth 2-surface $\Sigma^2 \subset \mathbb{R}^3$,*

$$\int_\Sigma |\boldsymbol{H}|^2 \geqslant 16\pi, \tag{1}$$

with equality occurring if and only if Σ is a standard sphere of any radius.

There are many proofs of the inequality (1). For instance, one can invoke classical differential geometry methods (see, for example, [Kühnel 2002, Theorem 4.46]), and Simon [1993] (see also [Gilbarg and Trudinger 1977, Equation (16.31)]) obtained the inequality using a monotonicity formula.

By a stereographic projection from

$$\mathbb{S}^3 \backslash \{0, 0, 0, 1\} \subset \mathbb{R}^4$$

to $\mathbb{R}^3 \times \{0\}$, one obtains an analogous result for $\mathbb{S}^3$ from the result for the Euclidean 3-space.

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Theorem 2 (See the introduction in [Marques and Neves 2014]). *Given any closed smooth surface* $\Sigma^2 \subset \mathbb{S}^3$,

$$\frac{1}{4}\int_\Sigma |\boldsymbol{H}|^2 \geqslant 4\pi - |\Sigma|,$$

with equality if and only if Σ *is a geodesic sphere in* $\mathbb{S}^3$.

Chen [1974] observed the conformal invariance properties of the Willmore functional, so his proof worked well for both the hyperbolic n-space and the n-sphere.

Theorem 3 [Chen 1974]. *Given any closed smooth* 2*-surface* $\Sigma^2 \subset \mathbb{H}^n$,

$$\frac{1}{4}\int_\Sigma |\boldsymbol{H}|^2 \geqslant 4\pi + |\Sigma| \tag{2}$$

with equality occurring if and only if Σ *lies in a* $\mathbb{H}^3$ *subspace as a geodesic sphere.*

We are concerned with the hyperbolic case mainly, and with notation which we will clarify later, we have the following result.

Theorem 4. *Given any closed* 2*-surface* $\Sigma^2 \subset \mathbb{H}^n$, *if* $o \in \Sigma$ *is a point of multiplicity* $k \geqslant 1$,

$$|\Sigma| + 4k\pi = -\int_\Sigma \left|\frac{1}{w}X^\perp + \frac{1}{2}\boldsymbol{H}\right|^2 + \frac{1}{4}\int_\Sigma |\boldsymbol{H}|^2. \tag{3}$$

This easily implies Chen's result [1974] as a corollary. Also, based on a similar idea, we prove the following theorem for a surface $\Sigma^2 \subset \mathbb{S}^n$.

Theorem 5. *Given any closed* $\Sigma^2 \subset \mathbb{S}^n$ *and* $0 < \sigma < \rho < \pi$, *if* $o \in \Sigma$ *is a point of multiplicity* $k \geqslant 1$, *then*

$$-\frac{2}{w(\rho)}\int_{\Sigma_\rho} \cos r + 4k\pi - |\Sigma_\rho| = \frac{1}{w(\rho)}\int_{\Sigma_\rho} X^\perp \cdot \boldsymbol{H} - \int_{\Sigma_\rho}\left|\frac{1}{w}X^\perp + \frac{1}{2}\boldsymbol{H}\right|^2 + \frac{1}{4}\int_{\Sigma_\rho}|\boldsymbol{H}|^2.$$

Since our proofs are based on monotonicity formulas, one can follow the same philosophy as in [Hoffman and Spruck 1974] and generalize Theorems 4 and 5 to general Riemannian manifolds with upper sectional curvature bounds. However, for clarity of exposition, we deal only with the two special cases $\mathbb{H}^n$ and $\mathbb{S}^n$.

2. Preliminaries

First, we recall a basic comparison theorem of sectional curvature.

Lemma 1 (See Theorem 27 of [Petersen 1998, Chapter 5]). *Assume that* (M^n, g) *satisfies* $\sec \leqslant K$ *and that the metric* g *written in geodesic polar coordinates centered at* $x \in M$ *is* $dr^2 + g_r$, *then*

$$\mathrm{Hess}_M\, r \geqslant \frac{\mathrm{sn}_K'(r)}{\mathrm{sn}_K(r)} g_r.$$

Here, sn_K *is defined to be*

$$\mathrm{sn}_K(r) = \frac{1}{\sqrt{K}}\sin(\sqrt{K}r) \qquad \text{if } K > 0;$$
$$\mathrm{sn}_K(r) = r \qquad \text{if } K = 0;$$
$$\mathrm{sn}_K(r) = \frac{1}{\sqrt{-K}}\sinh(\sqrt{-K}r) \qquad \text{if } K < 0.$$

If $K > 0$, *the estimate on* $\mathrm{Hess}_M r$ *is only valid with* $r < \pi/\sqrt{K}$.

Suppose that the Levi-Civita connection on (M, g) is ∇, then $r(y) = \mathrm{dist}_M(x, y)$ gives a vector field ∇r. We define X to be

$$X = \mathrm{sn}_K(r)\nabla r.$$

Given any 2-surface Σ in (M, g), we are concerned with an estimate of the quantity $\mathrm{div}_\Sigma X$.

Lemma 2. *Given any* $x \in \Sigma^2 \subset M^n$ *with* $\sec \leqslant K$,

$$\mathrm{div}_\Sigma X \geqslant 2\,\mathrm{sn}'_K(r). \tag{4}$$

Proof. Let $\{e_i\}$, $i = 1, 2$, be a chosen orthonormal frame spanning $T\Sigma$; we use the convention of summation over repeated indices, and apply Lemma 1:

$$\begin{aligned}
\mathrm{div}_\Sigma X &= \langle \nabla_{e_i} X, e_i\rangle \\
&= \langle \nabla_{e_i}(\mathrm{sn}_K(r)\nabla r), e_i\rangle \\
&= [\mathrm{sn}_K(r)\langle \nabla_{e_i}\nabla r, e_i\rangle + \mathrm{sn}'_K(r)|\nabla e_i r|^2] \\
&= [\mathrm{sn}_K(r)\,\mathrm{Hess}_M r(e_i, e_i) + \mathrm{sn}'_K(r)|\nabla e_i r|^2] \\
&\geqslant \mathrm{sn}'_K(r)[g_r(e_i, e_i) + |\nabla e_i r|^2] \\
&= \mathrm{sn}'_K(r) g(e_i, e_i) \\
&= 2\,\mathrm{sn}'_K(r).
\end{aligned}$$

□

Although one can obtain the same estimate by Jacobi fields or exponential maps similar to that of [Hoffman and Spruck 1974], using Lemma 1 is much more direct and convenient in our settings.

3. Willmore type inequalities

Our main result, Theorem 4, is a finer version of the Willmore type inequality (2). In its proof we establish the monotonicity formula (6).

Let $\phi(r)^{-1} = w(r) = \int_0^r \sinh t\,\mathrm{d}t$. Note that $w(r)$ is a constant multiple of the volume of a $\mathbb{H}^2$-geodesic ball. We see that $\phi'(r) = -\sinh r/w^2$.

Proof. Let ∇ and ∇^Σ, respectively, be the connections on $\mathbb{H}^n$ and Σ. Given any number $\sigma > 0$, we define a cutoff version of ϕ as $\phi_\sigma(r) = \phi(\max\{\sigma, r\})$.

Let $r(x) = \mathrm{dist}_{\mathbb{H}^n}(o, x)$ be the geodesic distance from o to any point $x \in \mathbb{H}^n$ and $\Sigma_\rho = \{x \in \Sigma : r(x) < \rho\}$. We choose the Lipschitz vector field $Y(x) = (\phi_\sigma(r) - \phi(\rho))_+ X$ where $0 < \sigma < \rho < \infty$. Let $V(x) = \cosh r(x) = \mathrm{sn}_1'(r)$; we see that $\nabla V = X$. V is called a static potential in the general relativity literature; see Chruściel and Herzlich [2003]. We often write ϕ instead of $\phi(r)$ and similarly for other quantities.

We make use of the first variation formula,

$$\int_\Sigma \mathrm{div}_\Sigma\, Y = -\int_\Sigma \langle Y, \boldsymbol{H}\rangle, \tag{5}$$

where $\boldsymbol{H}$ is the mean curvature vector.

We calculate $\mathrm{div}_\Sigma\, Y$ first. By Lemma 2,

$$\mathrm{div}_\Sigma\, Y = 2(\phi_\sigma - \phi(\rho))V + \sinh r |\nabla^\Sigma r|^2((\phi_\sigma - \phi(\rho))_+)'.$$

Note that $|\nabla^\perp r|$, the length of ∇r along the normal direction of Σ, is $|\nabla^\perp r|^2 = 1 - |\nabla^\Sigma r|^2$. Integrating $\mathrm{div}_\Sigma\, Y$ over Σ then gives

$$\begin{aligned}\int_\Sigma \mathrm{div}_\Sigma\, Y &= -2\phi(\rho)\int_{\Sigma_\rho} V + 2\phi(\sigma)\int_{\Sigma_\sigma} V + 2\int_{\Sigma_\rho\setminus\Sigma_\sigma}\phi V \\ &\qquad + \int_{\Sigma_\rho\setminus\Sigma_\sigma}\phi'(r)\sinh r(1 - |\nabla^\perp r|^2) \\ &= -2\phi(\rho)\int_{\Sigma_\rho} V + 2\phi(\sigma)\int_{\Sigma_\sigma} V \\ &\qquad + |\Sigma_\rho\setminus\Sigma_\sigma| - \int_{\Sigma_\rho\setminus\Sigma_\sigma}\phi'(r)\sinh r|\nabla^\perp r|^2, \end{aligned} \tag{6}$$

where we have used a consequence of a simple calculus,

$$2\phi V + \phi' \sinh r = 1. \tag{7}$$

Since $\boldsymbol{H}$ is a vector normal to Σ,

$$-\int_\Sigma \langle Y, \boldsymbol{H}\rangle = -\int_{\Sigma_\rho} (\phi_\sigma(r) - \phi(\rho))_+ \sinh r \nabla r \cdot \boldsymbol{H} \tag{8}$$

$$\begin{aligned} &= \phi(\rho)\int_{\Sigma_\rho} \sinh r \nabla^\perp r \cdot \boldsymbol{H} - \phi(\sigma)\int_{\Sigma_\sigma} \sinh r \nabla^\perp r \cdot \boldsymbol{H} \\ &\qquad - \int_{\Sigma_\rho\setminus\Sigma_\sigma} \phi \sinh r \nabla^\perp r \cdot \boldsymbol{H}. \end{aligned} \tag{9}$$

We then have

$$-\phi \sinh r \nabla^{\perp} r \cdot \boldsymbol{H} + \phi' \sinh r |\nabla^{\perp} r|^2 = -\frac{1}{w} X^{\perp} \cdot \boldsymbol{H} - \frac{|X^{\perp}|^2}{w^2} = -\left|\frac{1}{w} X^{\perp} + \frac{1}{2}\boldsymbol{H}\right|^2 + \frac{1}{4}|\boldsymbol{H}|^2, \tag{10}$$

where $X^{\perp} = \sinh r \nabla^{\perp} r$ is the normal component to Σ of the vector field X.

To collect (6), (9) and (10), one has

$$\begin{aligned} -2\phi(\rho) \int_{\Sigma_\rho} V + 2\phi(\sigma) \int_{\Sigma_\sigma} V + |\Sigma_\rho \setminus \Sigma_\sigma| \\ = \phi(\rho) \int_{\Sigma_\rho} \sinh r \nabla^{\perp} r \cdot \boldsymbol{H} - \phi(\sigma) \int_{\Sigma_\sigma} \sinh r \nabla^{\perp} r \cdot \boldsymbol{H} \\ - \int_{\Sigma_\rho \setminus \Sigma_\sigma} \left|\frac{1}{w} X^{\perp} + \frac{1}{2}\boldsymbol{H}\right|^2 + \frac{1}{4} \int_{\Sigma_\rho \setminus \Sigma_\sigma} |\boldsymbol{H}|^2. \end{aligned} \tag{11}$$

Then (11) is the monotonicity formula we desired. Since Σ is closed, letting $\rho \to +\infty$ and $\sigma \to 0$, the above simplifies as

$$|\Sigma| + 2 \lim_{\sigma \to 0} \phi(\sigma) \int_{\Sigma_\sigma} V = -\int_{\Sigma} \left|\frac{1}{w} X^{\perp} + \frac{1}{2}\boldsymbol{H}\right|^2 + \frac{1}{4} \int_{\Sigma} |\boldsymbol{H}|^2.$$

Since Σ_σ is locally Euclidean with multiplicity k at the point $o \in \Sigma$ and $V(0) = 1$, the fact that the limit

$$\lim_{\sigma \to 0} \phi(\sigma) \int_{\Sigma_\sigma} V = \lim_{\sigma \to 0} \frac{\pi \sigma^2}{\int_0^\sigma \sinh t \, \mathrm{d}t} \cdot \left(\frac{1}{\pi \sigma^2} \int_{\Sigma_\sigma} V \right) = 2\pi \lim_{\sigma \to 0} \frac{|\Sigma_\sigma|}{\pi \sigma^2} = 2k\pi \tag{12}$$

exists will finally give (3). □

Remark 1. For Σ in an ambient space with upper sectional curvature bound $\sec \leqslant -1$, the proof is verbatim. In the proof, we have only used

$$\operatorname{div}_{\Sigma} X = 2 \operatorname{sn}'_K(r)$$

in the inequality (4). With the sectional curvature bound, we use $\operatorname{div}_{\Sigma} X \geqslant 2 \operatorname{sn}'_K(r)$.

If Σ has multiplicity greater than 1 somewhere, (2) is similar to the proof of Li and Yau [1982] when a point in Σ is covered multiple times by the immersion. In fact, we have the following interesting result analogous to [Li and Yau 1982, Theorem 6]:

Corollary 1. *Give any closed 2-surface* $\Sigma^2 \subset \mathbb{H}^n$,

$$\frac{1}{4} \int_{\Sigma} |\boldsymbol{H}|^2 < |\Sigma| + 8\pi, \tag{13}$$

then Σ *is embedded in* $\mathbb{H}^n$.

Proof. If Σ is of multiplicity at least 2 somewhere x is in Σ, then by (12),

$$\frac{1}{4}\int_\Sigma |\boldsymbol{H}|^2 \geqslant |\Sigma| + 8\pi,$$

but this contradicts (13). Hence, Σ has to be embedded. □

When $\Sigma^2 \subset \mathbb{H}^n$ is a surface with boundary, one should apply the first variation formula with boundary,

$$\int_\Sigma \operatorname{div}_\Sigma Y = -\int_\Sigma \langle Y, \boldsymbol{H}\rangle + \int_{\partial\Sigma} \langle Y, \eta\rangle,$$

where η is the outward pointing normal of $\partial\Sigma$ to Σ. We similarly prove the following:

Corollary 2. *Suppose that Σ is a 2-surface with nonempty boundary. If $o \in \Sigma$ is an interior point of Σ, then*

$$|\Sigma| + 4\pi \leqslant \int_{\partial\Sigma} \left\langle \frac{1}{w} X, \eta \right\rangle - \int_\Sigma \left| \frac{1}{w} X^\perp + \frac{1}{2}\boldsymbol{H} \right|^2 + \frac{1}{4}\int_\Sigma |\boldsymbol{H}|^2;$$

and if $o \in \partial\Sigma$ is a boundary point of Σ, then

$$|\Sigma| + 2\pi \leqslant \int_{\partial\Sigma} \left\langle \frac{1}{w} X, \eta \right\rangle - \int_\Sigma \left| \frac{1}{w} X^\perp + \frac{1}{2}\boldsymbol{H} \right|^2 + \frac{1}{4}\int_\Sigma |\boldsymbol{H}|^2.$$

We now give an estimate of $\frac{1}{4}\int|\boldsymbol{H}|^2$ in the same fashion as [Gilbarg and Trudinger 1977, Equation (16.31)] and arrive at the following:

Theorem 6. *Given any closed 2-surface in $\mathbb{H}^n$ and a real number $\rho > 0$, then*

$$4\pi + |\Sigma_\rho| \leqslant \frac{1}{w(\rho)}\int_{\Sigma_\rho} V + \frac{1}{4}\int_\Sigma |\boldsymbol{H}|^2. \tag{14}$$

Proof. The proof is as before, with some modifications. From (5), (9), (6) and (7), and also using the shorthand

$$X = \sinh r \nabla r,$$

we have

$$\begin{aligned} -2\phi(\rho)\int_{\Sigma_\rho} V + 2\phi(\sigma)\int_{\Sigma_\sigma} V + |\Sigma_\rho \backslash \Sigma_\sigma| \\ = -\int_{\Sigma_\rho} (\phi_\sigma - \phi(\rho))_+ X\cdot\boldsymbol{H} - \int_{\Sigma_\rho\backslash\Sigma_\sigma} \frac{|X^\perp|^2}{w^2}. \end{aligned} \tag{15}$$

For any $x \in \Sigma_\rho \backslash \Sigma_\sigma$, we claim

$$-(\phi_\sigma - \phi(\rho))_+ X\cdot\boldsymbol{H} - \frac{|X^\perp|^2}{w^2} \leqslant \frac{1}{4}|\boldsymbol{H}|^2. \tag{16}$$

Then from (15) and (16), we have

$$-2\phi(\rho)\int_{\Sigma_\rho} V + 2\phi(\sigma)\int_{\Sigma_\sigma} V + |\Sigma_\rho \setminus \Sigma_\sigma| = -\int_{\Sigma_\sigma} (\phi_\sigma - \phi(\rho))_+ X\cdot \boldsymbol{H} + \frac{1}{4}\int_{\Sigma_\rho\setminus\Sigma_\sigma} |\boldsymbol{H}|^2 \leqslant -\int_{\Sigma_\sigma} (\phi_\sigma - \phi(\rho))_+ X\cdot \boldsymbol{H} + \frac{1}{4}\int_{\Sigma} |\boldsymbol{H}|^2.$$

By letting $\sigma \to 0$, we obtain (14) immediately.

Now we turn to the proof of the claimed estimate (16). Indeed, when $X\cdot \boldsymbol{H} \geqslant 0$, the inequality is trivial. When $X\cdot \boldsymbol{H} \leqslant 0$, much as in (10), we have

$$-(\phi_\sigma - \phi(\rho))_+ X\cdot\boldsymbol{H} - \frac{|X^\perp|^2}{w^2} = -\left|\frac{1}{w}X^\perp + \frac{1}{2}\boldsymbol{H}\right|^2 + \frac{1}{4}\boldsymbol{H}^2 + \phi(\rho)X\cdot\boldsymbol{H} \leqslant \frac{1}{4}|\boldsymbol{H}|^2. \quad \square$$

Now we turn to the case of surfaces in an n-sphere $\mathbb{S}^n$.

Proof of Theorem 5. Let $r(x) = \mathrm{dist}_{\mathbb{S}^n}(x, o)$, $X = \sin r \nabla r$, $\phi(r)^{-1} = w(r) = \int_0^r \sin t\,\mathrm{d}t$. We proceed as in the proof of Theorem 4, noting the relation

$$2\phi\cos r + \phi' \sin r = -1. \qquad \square$$

As in the proof of Theorem 6, we have:

Theorem 7. *Given any closed 2-surface in $\mathbb{S}^n$ and a real number $0 < \rho < \pi$, then*

$$4\pi - |\Sigma_\rho| \leqslant \frac{1}{w(\rho)}\int_{\Sigma_\rho} \cos r + \frac{1}{4}\int_\Sigma |\boldsymbol{H}|^2.$$

Remark 2. The reason that we do not take a limit $\rho \to \infty$ is the existence of conjugate points in $\mathbb{S}^n$.

4. Equality case of Theorem 3

Now we turn to discuss the equality case of Theorem 3 based on (3). We recall some basics of the hyperboloid model of the hyperbolic space $\mathbb{H}^n$. The readers can find relevant materials in [Petersen 1998, Chapter 3]. $\mathbb{H}^n$ can be realized as a pseudosphere in Minkowski space, i.e.,

$$\mathbb{H}^n = \{x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{1,n} : \langle x, x\rangle = -1, x_0 > 0\},$$

where the bilinear form $\langle\cdot,\cdot\rangle$ is defined to be

$$\langle x, y\rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n.$$

The notation $\langle\cdot,\cdot\rangle$ will be used without confusion. By differentiating the relation $\langle x, x\rangle = -1$, we find that the tangent space $T_x\mathbb{H}^n$ at $x \in \mathbb{H}^n$ is $\{y \in \mathbb{R}^{1,n} : \langle x, y\rangle = 0\}$.

By restricting $\langle \cdot, \cdot \rangle$ to $T_x\mathbb{H}^n$, we get the standard Riemannian metric $\langle \cdot, \cdot \rangle_{T_x\mathbb{H}^n}$ on $\mathbb{H}^n$. The geodesic passing through $x \in \mathbb{H}^n$ with unit velocity $z \in T_x\mathbb{H}^n$ is

$$x(\rho) = x\cosh\rho + z\sinh\rho, \quad \rho \in (-\infty, \infty). \tag{17}$$

If we know the endpoints $x \neq y$ of a geodesic segment, we can solve $\rho = \rho(x, y)$ and $z := z(x, y) \in T_x\mathbb{H}^n$:

$$\langle x, y\rangle = -\cosh\rho, z = \sinh^{-1}\rho(y + x\langle x, y\rangle). \tag{18}$$

Theorem 8. *The equality holds in* (2) *of Theorem 3 if and only if* Σ *lies in a* $\mathbb{H}^3$ *subspace as a geodesic* 2*-sphere.*

Proof. One easily verifies the equality of (2) for geodesic 2-spheres. If equality holds in (2), then Σ is of multiplicity 1, i.e., $k = 1$ at every point of $x \in \Sigma$ by Corollary 1. Also, there is some point $y \in \Sigma$ such that $\boldsymbol{H}(y) \neq 0$. Most importantly, by (3) the mean curvature vector $\boldsymbol{H}$ at $y \in \Sigma$ can be evaluated in terms of

$$-\frac{1}{2}\boldsymbol{H}(y) = \frac{\sinh\rho}{w(\rho)}\tau^{\perp}, \tag{19}$$

where ρ and $\tau \in T_y\mathbb{H}^n$ are, respectively, the length and the velocity vector at y of the geodesic segment from any $x \in \Sigma$ to y. Let $\{e_1, e_2\}$ span $T_y\Sigma$; note that (19) says that τ is a linear combination of $\{e_1, e_2, \boldsymbol{H}(y)\}$, and hence every point $x \in \Sigma$ lies in an $\mathbb{H}^3$ subspace of $\mathbb{H}^n$. $\mathbb{H}^3$ is totally geodesic in $\mathbb{H}^n$; we can then consider $n = 3$ only. Now we identify every point and the tangent space as elements in $\mathbb{R}^{1,3}$, and from (17),

$$\tau = x\sinh\rho + z\cosh\rho.$$

Let ν be the outward pointing normal of Σ in $\mathbb{H}^3$ at y. This is possible since Σ is embedded in $\mathbb{H}^3$ and, hence, oriented. By (18),

$$\begin{aligned}
-\frac{1}{2}\boldsymbol{H}(y) &= \frac{\sinh\rho}{w(\rho)}\langle x\sinh\rho + z\cosh\rho, \nu\rangle\nu \\
&= \frac{\sinh^2\rho}{w(\rho)}\langle x, \nu\rangle\nu + \frac{\cosh\rho}{w(\rho)}\langle y + x\langle x, y\rangle, \nu\rangle\nu \\
&= \frac{\sinh^2\rho}{w(\rho)}\langle x, \nu\rangle\nu + \frac{\cosh\rho}{w(\rho)}\langle x, \nu\rangle\langle x, y\rangle\nu \\
&= \frac{\sinh^2\rho}{w(\rho)}\langle x, \nu\rangle\nu - \frac{\cosh^2\rho}{w(\rho)}\langle x, \nu\rangle\nu = -\frac{1}{w(\rho)}\langle x, \nu\rangle\nu.
\end{aligned}$$

Since $\boldsymbol{H} = -H\nu$, the mean curvature H at y of Σ immersed in $\mathbb{H}^3$ is

$$\frac{1}{2}H(y) = -\frac{\langle x, \nu\rangle}{w(\rho)} = \frac{\langle x, \nu\rangle}{1 - \cosh\rho} = \frac{\langle x, \nu\rangle}{1 + \langle x, y\rangle}. \tag{20}$$

We prove now that $H(y)$ cannot be less than 0. Assume on the contrary that $H(y) = -2\coth t < 0$ where $t > 0$; we fix the coordinates of $\mathbb{R}^{1,3}$ now by setting the point $\exp_y(-t\nu)$ to be $o = (1, 0, 0, 0) \in \mathbb{R}^{1,3}$ where $\exp_y$ is the exponential map of $\mathbb{H}^3$ at y. Note that o is the origin under polar coordinates

$$\begin{aligned}[0, \infty) \times \mathbb{S}^2 &\to \mathbb{R}^{1,3},\\ (s, \theta) &\mapsto (\cosh s, \theta \sinh s),\end{aligned}$$

where $\theta \in \mathbb{S}^2 \subset \mathbb{R}^3$. We assume that $y = (\cosh t, \theta \sinh t)$ since the distance from o to y is t, and $x = (\cosh \bar{t}, \bar{\theta} \sinh \bar{t})$, $\bar{t} > 0$. Then ν is $(\sinh t, \theta \cosh t)$. By inserting the values of y, ν and x to the identity (20), we get

$$-\coth t = \frac{-\cosh \bar{t} \sinh t + \sinh \bar{t} \cosh t\theta \cdot \bar{\theta}}{1 - \cosh t \cosh \bar{t} + \sinh t \sinh \bar{t}\theta \cdot \bar{\theta}}.$$

Here $\theta \cdot \bar{\theta}$ is the standard $\mathbb{R}^3$ inner product. This readily reduces to

$$\begin{aligned}0 &= 2 \sinh \bar{t} \sinh t \cosh t\theta \cdot \bar{\theta} - \cosh \bar{t} \sinh^2 t + \cosh t - \cosh^2 t \cosh \bar{t}\\ &= \sinh \bar{t} \sinh(2t)\theta \cdot \bar{\theta} - \cosh \bar{t} \cosh(2t)\end{aligned}$$

which is however not possible since $\theta \cdot \bar{\theta} \leqslant 1$.

So $H(y) > 0$. We can set instead $H(y) = 2\coth t > 0$ with $t > 0$. Using this t and doing the same thing as before, we get

$$\coth t = \frac{-\cosh \bar{t} \sinh t + \sinh \bar{t} \cosh t\theta \cdot \bar{\theta}}{1 - \cosh t \cosh \bar{t} + \sinh t \sinh \bar{t}\theta \cdot \bar{\theta}},$$

and finally $t = \bar{t}$. Then Σ has to be a geodesic sphere of radius t. □

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