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On the orbits of multiplicative pairs

Oleksiy Klurman and Alexander P. Mangerel

Dedicated to Imre Kátai on the occasion of his 80th birthday

We characterize all pairs of completely multiplicative functions $fg : \mathbb{N} \to \mathbb{T}$, where $\mathbb{T}$ denotes the unit circle, such that

$$\overline{\{(f(n), g(n+1))\}_{n\geq 1}} \neq \mathbb{T} \times \mathbb{T}.$$

In so doing, we settle an old conjecture of Zoltán Daróczy and Imre Kátai.

1. Introduction

In this paper, we will be concerned with demonstrating yet another instance of the expected general phenomenon that the multiplicative structure of positive integers should in general be "independent" of their additive structure. Of principal focus here will be the behavior of multiplicative functions at consecutive integers. Problems of this kind are widely open in general, though spectacular progress has recently been made as a consequence of the breakthrough of Matomäki and Radziwiłł [2016], and subsequent work of Matomäki, Radziwiłł and Tao [Matomäki et al. 2015], Tao [2016] and, more recently, Tao and Teräväinen [2019]. In particular, using the work in [Matomäki and Radziwiłł 2016], Tao [2016] established a weighted version of the binary Chowla conjecture in the form

$$\sum_{n\leq x} \frac{\lambda(n)\lambda(n+h)}{n} = o(\log x)$$

for all $h \geq 1$. For a comprehensive account of the recent developments in this direction, see [Matomäki and Radziwiłł 2019]. Let $\mathbb{U}$ denote the unit disc in $\mathbb{C}$ and let $\mathbb{T}$ denote the unit circle. Let $f, g : \mathbb{N} \to \mathbb{T}$ be completely multiplicative functions. We expect that as n varies through the set of positive integers, the values $f(n)$ and $g(n+1)$ should, roughly speaking, be independently distributed unless f and g satisfy some rigid relations. To be more precise, we shall investigate the following problem: if $\{f(n)\}_n$ and $\{g(n)\}_n$ are both dense in $\mathbb{T}$, but the sequence of pairs $\{(f(n), g(n+1))\}_n$ is *not* dense in $\mathbb{T}^2$, must there be a rigid relation between f and g? This multidimensional problem continues work on rigidity problems for additive and multiplicative functions initiated by the authors in [Klurman and Mangerel 2018]. This problem has a natural dynamical flavor, which explains the title of this paper. Let $T : \mathbb{N} \to \mathbb{N}$ denote the rightward shift map $T(n) := n + 1$. In this case, the above problem can be recast in terms of orbits of the

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pair (f, g_T), where $f, g : \mathbb{N} \to \mathbb{T}$ are semigroup homomorphisms that fix $n = 1$, and $g_T := g \circ T$. We seek a result of the kind that, unless f and g are specially chosen maps, the orbit closure of the point 1, i.e., the closure $\overline{\{(f(n), g(n+1))\}_n}$, is expected to be the same as the product of the closures of the marginal orbits $\{f \circ T^n\}_n$ and $\{g \circ T^n\}_n$. In this connection, we quote Conjecture 3 in the survey paper by Kátai [1989] (earlier formulated in [Daróczy and Kátai 1989]).

Conjecture 1.1. *Let $f, g : \mathbb{N} \to \mathbb{T}$ be completely multiplicative. Suppose $\{(f(n), g(n+1))\}_n$ is not dense in $\mathbb{T}^2$, yet $\{f(n)\}_n$ and $\{g(n)\}_n$ are both dense in $\mathbb{T}$. Then there are integers k and l such that $f(n)^k = g(n)^l$, with $f(n) = n^{it}$ for some t.*

As stated this conjecture is easily seen to be false, as we can construct the following two types of counterexamples:

(i) Let $h_1, h_2 : \mathbb{N} \to \mathbb{T}$ be completely multiplicative functions such that there are minimal positive integers $k, l \geq 2$ for which $h_1^k = h_2^l = 1$. Fixing an arbitrary $t \in \mathbb{R}\backslash\{0\}$ and setting $f(n) := h_1(n)n^{it}$ and $g(n) := h_2(n)n^{it}$ yields a pair of completely multiplicative functions such that $\{f(n)\}_n$ and $\{g(n)\}_n$ are dense, yet $\{(f(n), g(n+1))\}_n$ cannot be dense. On the other hand, it is true in this example that $f(n)^k = n^{it'}$ and $g(n)^l = n^{it''}$ for all n, where $t' = kt$ and $t'' = lt$.

(ii) Fix a prime p and distinct irrational numbers $\alpha, \beta \in \mathbb{R}$. Let $f, g : \mathbb{N} \to \mathbb{T}$ be the completely multiplicative functions defined on primes by

$$f(q) := \begin{cases} e(\alpha) & q = p, \\ 1 & q \neq p, \end{cases} \quad \text{and} \quad g(q) := \begin{cases} e(\beta) & q = p, \\ 1 & q \neq p. \end{cases}$$

It is easy to see that the sequence $\{(f(n), g(n+1))\}_n$ belongs to the union of the sets $\mathbb{T} \times \{1\}$, $\{1\} \times \mathbb{T}$ and $\{1\} \times \{1\}$ and thus cannot be dense. On the other hand we clearly have $\overline{\{f(p^m)\}_{m\geq 1}} = \overline{\{g(p^m)\}_{m\geq 1}} = \mathbb{T}$.

Given a completely multiplicative function h, let

$$T_h := \{p \text{ prime} : h(p) \neq 1\}.$$

The collection (i) of counterexamples suggests that we should relax the conclusion of Conjecture 1.1 by allowing $f(n)^k = n^{it}$ for some $k \geq 1$. The collection (ii) of counterexamples indicates that we should add a hypothesis to exclude those functions f and g such that both $|T_{f^k}||T_{g^k}| = 1$ and $T_{f^k} = T_{g^k}$ hold for all sufficiently large k.[1] We thus prove the following amendment of Conjecture 1.1, in which the above types of counterexamples are excluded.

Theorem 1.2 (amended Daróczy–Kátai conjecture). *Let $f, g : \mathbb{N} \to \mathbb{T}$ be completely multiplicative. Suppose $\overline{\{(f(n), g(n+1))\}_n} \neq \mathbb{T}^2$, yet $\overline{\{f(n)\}_n} = \overline{\{g(n)\}_n} = \mathbb{T}$. Suppose additionally that for infinitely many m we have $|T_{f^m} \cup T_{g^m}| > 1$. Then there are integers k and l such that $f(n)^k = g(n)^l$, with $f(n)^k = n^{it}$ for some nonzero real number t.*

[1] We note that the condition that $\overline{\{f(n)\}_n} = \mathbb{T}$ implies that $|T_{f^k}| > 0$ for all $k \in \mathbb{N}$.

In the case that $f = g$, we have the following immediate corollary.

Corollary 1.3. *Let $f : \mathbb{N} \to \mathbb{T}$ be a completely multiplicative function such that $\overline{\{f(n)\}_n} = \mathbb{T}$. Then $\overline{\{(f(n), f(n+1))\}_n} \neq \mathbb{T}^2$ if, and only if, one of the following conditions holds*:

(a) *There is a positive integer l such that $|T_{f^l}| = 1$, and for $p \in T_{f^l}$, $f(p) = e(\alpha)$ with $\alpha \notin \mathbb{Q}$.*

(b) *There is a positive integer k and a nonzero real number t for which $f(n)^k = n^{it}$.*

Proof. It is easy to see that if $f : \mathbb{N} \to \mathbb{T}$ is a completely multiplicative function satisfying either of conditions (a) or (b) then $\{f(n)\}_n$ is dense in $\mathbb{T}$ but $\{(f(n), f(n+1))\}_n$ is not dense in $\mathbb{T}^2$. We now assume that $\overline{\{f(n)\}_n} = \mathbb{T}$ but $\overline{\{(f(n), f(n+1))\}_n} \neq \mathbb{T}^2$. If $|T_{f^m}| > 1$ for infinitely many m then by Theorem 1.2 there is a $k \in \mathbb{Z}$ and a $t \neq 0 \in \mathbb{R}$ for which $f(n)^k = n^{it}$ for all $n \in \mathbb{N}$, which fulfills condition (b). Conversely, if $|T_{f^m}| = 1$ for all but finitely many m then there is some $M \in \mathbb{N}$ such that for all $m \geq M$, there is exactly one prime p with $f(p)^m \neq 1$. Thus, for all $p' \neq p$, $f(p') = e(a/b)$ for some $1 \leq a, b \leq M$. Putting $l := M!$, it follows that $f(p')^l = 1$, except when $p' = p$. Moreover, $f(p)^l$, and thus also $f(p)$, must have irrational argument, otherwise we could find a larger M' for which $f(p')^{M!M'} = 1$ for all p', contradicting the denseness of $\{f(n)\}_n$ in $\mathbb{T}$; such a function satisfies condition (a). □

In order to facilitate our discussion, we distinguish completely multiplicative functions according to their values on primes as follows.

Definition 1.4. A completely multiplicative function $f : \mathbb{N} \to \mathbb{T}$ is said to be *eventually rational* if there exist positive integers k and $N_0 = N_0(k)$ such that for all $p \geq N_0$ we have $f(p)^k = 1$. We will say that f is *irrational* otherwise.

An irrational function necessarily produces a sequence $\{f(n)\}_n$ that is dense. We stress, though, that the arguments of an irrational function at primes *need not be* irrational. For example, by our definition, the completely multiplicative function defined by $f(p) = e(1/p)$ for all primes p is irrational. To prove Theorem 1.2, we will treat two cases, depending on whether or not f or g is irrational (in the sense of Definition 1.4). In Section 4 we prove the following.

Proposition 1.5. *Suppose $f, g : \mathbb{N} \to \mathbb{T}$ are eventually rational, $\overline{\{f(n)\}_n} = \overline{\{g(n)\}_n} = \mathbb{T}$ and $|T_{f^k} \cup T_{g^k}| > 1$ for infinitely many k. Then $\overline{\{(f(n), g(n+1))\}_n} = \mathbb{T}^2$.*

Proposition 1.5 asserts that the only cases satisfying the assumptions of Theorem 1.2 are those for which either f or g is irrational. In this direction we prove the following in Section 6.

Theorem 1.6. *Suppose that $f, g : \mathbb{N} \to \mathbb{T}$ are completely multiplicative functions such that $\overline{\{f(n)\}_n} = \overline{\{g(n)\}_n} = \mathbb{T}$. Suppose furthermore that at least one of f and g is irrational. If $\overline{\{(f(n), g(n+1))\}_n} \neq \mathbb{T}^2$ then there are positive integers k, l and real numbers t, t' such that $f(n)^k = n^{it}$ and $g(n)^l = n^{it'}$.*

Having shown Theorem 1.6, in order to prove Theorem 1.2 it remains to prove that the real numbers t and t' satisfy $t = \lambda t'$, for $\lambda \in \mathbb{Q}$. This is the conclusion of Proposition 1.7, which we prove in Section 3.

Proposition 1.7. *Suppose there are $k, l \in \mathbb{N}$ and $t, t' \in \mathbb{R}$ with $tt' \neq 0$ such that $f(n)^k = n^{it}$ and $g(n)^l = n^{it'}$ for all $n \in \mathbb{N}$. Then $\overline{\{(f(n), g(n+1))\}} \neq \mathbb{T}^2$ if, and only if, $t = \lambda t'$ with $\lambda \in \mathbb{Q}$.*

The proof of Theorem 1.6, which represents the major substance of our analysis, uses arguments that extend those found in the proof of Theorems 1.1 and 1.4 in [Klurman and Mangerel 2018]. Let us recall the rough outline of this argument here. In Theorem 1.4 of [loc. cit.] (the proof of which establishes Theorem 1.1 there as well), it was shown that the sequence $\{f(n)\overline{f(n+1)}\}_n$ is dense in $\mathbb{T}$, except in predictable cases.[2] The objective was to show that for every $\varepsilon > 0$ and every $z \in \mathbb{T}$, the lower bound

$$|f(n)\overline{f(n+1)} - z| \geq \varepsilon \text{ for all } n \text{ sufficiently large} \tag{1}$$

cannot hold for "generic" completely multiplicative functions $f : \mathbb{N} \to \mathbb{T}$. To establish this, we noted that (1) implies that the sequence $\{f(n)\overline{f(n+1)}\}_n$ cannot be equidistributed, which led us (via the Erdős–Turán inequality and Tao's theorem [2016] on logarithmic averages of binary correlations; see Theorem 2.1 below) to a first conclusion that f is *pseudopretentious*, i.e., such that for some (minimal) $k \in \mathbb{N}$, some Dirichlet character χ and some $t \in \mathbb{R}$ (depending at most on ε), we have

$$\mathbb{D}(f^k, \chi n^{it}; x)^2 := \sum_{p \leq x} \frac{1 - \mathrm{Re}(f(p)^k \overline{\chi}(p) p^{-it})}{p} \ll_\varepsilon 1;$$

in particular, $\mathbb{D}(f, gn^{it}; x) \ll_\varepsilon 1$ for some completely multiplicative function g such that $g(n)^k = \chi(n)$ whenever $\chi(n) \neq 0$. Here the fact that $(n+1)^{it} \approx n^{it}$ for large n is used crucially. Roughly speaking, we then conditioned on a suitable subset of n (or positive logarithmic density) such that $g(n)\overline{g(n+1)}$ is constant, and thereby reduced our work to treating the 1-pretentious function $F = f\bar{g}n^{-it}$, showing that for small enough ε,

$$|F(n)\overline{F(n+1)} - z'| = |f(n)\overline{f(n+1)} - z| + O(\varepsilon^2) \gg \varepsilon$$

for some z' (where z and z' need not be the same) is untenable for all n sufficiently large (depending at most on ε). This reduction was a key part that made that argument work. In the context of Theorem 1.2, we must face several key differences in the argument. For example:

(i) The fact that[3] $\|(f(n), g(n+1)) - (z, w)\|_{\ell^1} \geq \varepsilon$ for all large n may mean that $|f(n) - z| \geq \varepsilon/2$ on a very dense set, or a very sparse set, and we cannot exclude either of these possibilities.

(ii) If f and g are both pseudopretentious in the above sense, say f is pretentious to $h_1 n^{it}$ and g is pretentious to $h_2 n^{it'}$, then it may be that $t \neq t'$, and we must then deal with the distribution in argument of the twist $n^{i(t-t')}$, unlike in the outline of the proof of Theorem 1.4 of [Klurman and Mangerel 2018].

With some additional ideas, we are able to address these issues. See especially Section 6 for further details.

[2] That is, except when $f(n) = g(n)n^{it}$, where g is a function taking values in bounded order roots of unity.

[3] For a pair $z = (z_1, z_2) \in \mathbb{C}^2$, we write $\|z\|_{\ell^1} := |z_1| + |z_2|$.

2. Auxiliary results towards Theorem 1.2

In this section we collect the definitions and lemmata that we shall use in the proof of Theorem 1.2. A crucial result on which our method relies is the following recent breakthrough result of Tao [2016].

Theorem 2.1 (Tao). *Let $f_1, f_2 : \mathbb{N} \to \mathbb{U}$ be multiplicative functions, such that for some $j \in \{1, 2\}$, we have*

$$\inf_{|t|\le x} \mathbb{D}(f_j, \chi n^{it}; x)^2 = \inf_{|t|\le x} \sum_{p\le x} \frac{1 - \mathrm{Re}(f_j(p)\overline{\chi}(p)p^{-it})}{p} \to \infty$$

for each fixed Dirichlet character χ. Then for any integers $a_1, a_2 \ge 1$ and $b_1, b_2 \ge 0$ with $a_1b_2 - a_2b_1 \ne 0$, we have

$$\frac{1}{\log x} \sum_{n\le x} \frac{f_1(a_1n + b_1) f_2(a_2n + b_2)}{n} = o(1).$$

A useful consequence of Theorem 2.1 is the following.

Proposition 2.2. *Let $f, g : \mathbb{N} \to \mathbb{U}$ be multiplicative functions. Suppose $k, l \in \mathbb{N}$ are minimal, such that*

$$\left| \frac{1}{\log x} \sum_{n\le x} \frac{f(n)^k g(n+1)^l}{n} \right| \gg 1, \tag{2}$$

as $x \to \infty$. Then there are Dirichlet characters χ_1, χ_2 with respective conductors $q_1, q_2 = O(1)$, and real numbers $t_1, t_2 = O(1)$ such that $\mathbb{D}(f, h_1n^{it_1}, \infty) < \infty$ and $\mathbb{D}(f, h_2n^{it_2}, \infty) < \infty$,[4] with $h_1(n)^k = \chi_1(n/(n, q_1^\infty))$ and $h_2(n)^l = \chi_2(n/(n, q_2^\infty))$.

Proof. From (2) and Theorem 2.1, it follows that for each x sufficiently large there is a pair (ξ_x, t_x), where ξ_x is a primitive Dirichlet character with $\mathrm{cond}(\xi_x) \ll 1$ and $t_x \ll x$, each estimate being uniform in x, such that $\mathbb{D}(f^k, \xi_x n^{it_x}; x) \ll 1$. By Lemma 2.5 of [Klurman and Mangerel 2018], it follows that there is a character χ_1 and a $t_1 \in \mathbb{R}$ such that $\mathbb{D}(f^k, \chi_1 n^{ikt_1}; x) \ll 1$. Combining this with Lemma 2.8 of [loc. cit.], it follows that there is a completely multiplicative function h_1 such that for all primes $p \nmid \mathrm{cond}(\chi_1)$ we have $h_1(p)^k = \chi_1(p)$, while if $p \mid \mathrm{cond}(\chi_1)$, $h_1(p) = 1$ and $\mathbb{D}(f, h_1n^{it_1}, \infty) < \infty$. This implies the claim about f. The claim about g follows in the same way. □

Presieving on level sets with Archimedean twists. Recall that a 1-bounded multiplicative function f is called *pseudopretentious* if there exists a (minimal) positive integer k, a primitive Dirichlet character χ modulo q, a real number t and a completely multiplicative function $h : \mathbb{N} \to \mathbb{T}$ such that $h(n)^k = \tilde{\chi}(n)$, and $\mathbb{D}(f, hn^{it}; x) \ll 1$. Here, $\tilde{\chi}$ is the unimodular completely multiplicative function defined on primes via $\tilde{\chi}(p) = \chi(p)$ if $p \nmid q$, and $\tilde{\chi}(p) = 1$ otherwise. Henceforth, we will refer to the function h here as a *pseudocharacter*, and refer to the modulus q of χ as the *conductor* of h. We emphasize that in this definition the minimality of k implies that $h(n)^j$ is nonpretentious for all $1 \le j \le k - 1$. This will be crucial in several of the arguments below. In order to control the behavior of h_1, h_2 coming from Proposition 2.2 on the corresponding progressions, we would like to eliminate the effect of the small

[4] Given $a, b \in \mathbb{N}$, we write $(a, b^\infty) := \prod_{p\mid b} p^{\nu_p(a)}$, where $a = \prod_{p\mid a} p^{\nu_p(a)}$.

primes by using presieving following the approach from [Klurman and Mangerel 2018]. To this end, given $\alpha_j \in h_j(\mathbb{N})$ for $j = 1, 2$ and $N, B \geq 1$, write

$$\mathcal{A}_{N,B}(h_1, h_2; \alpha_1, \alpha_2) := \{n \in \mathbb{N} : P^-(n(Bn+1)) > N, h_1(n) = \alpha_1, h_2(Bn+1) = \alpha_2\},$$

where $P^-(n)$ denotes the smallest prime factor dividing $n \in \mathbb{N}$. Furthermore, if $I, J \subseteq \mathbb{T}$ are arcs[5] and $u, v \in \mathbb{R}$ then

$$\mathcal{A}_{N,B,I}(h_1, h_2; \alpha_1, \alpha_2; u) := \{n \in \mathcal{A}_{N,B}(h_1, h_2; \alpha_1, \alpha_2) : n^{iu} \in I\},$$
$$\mathcal{A}_{N,B,I,J}(h_1, h_2; \alpha_1, \alpha_2; u, v) := \{n \in \mathcal{A}_{N,B}(h_1, h_2; \alpha_1, \alpha_2) : n^{iu} \in I, n^{iv} \in J\}.$$

Moreover, if $x, q \geq 1$ and a is a coprime residue class modulo q, put

$$\Phi_{N,B}(x; q, a) := |\{n \leq x : P^-(n(Bn+1)) > N \text{ and } n \equiv a(q)\}|.$$

When $q = 1$, we write $\Phi_{N,B}(x; q, a) = \Phi_{N,B}(x)$, and when $B = 1$ in addition, we write $\Phi_N(x)$. Finally, given a map $f : \mathbb{N} \to \mathbb{C}$ and a positive real $X \geq 2$, we shall write

$$\mathbb{E}^{\log}_{n \leq X} f(n) := \frac{1}{\log X} \sum_{n \leq X} \frac{f(n)}{n}.$$

Our first result, to be used throughout the paper, shows that the sets $\mathcal{A}_{N,B,I}$ and $\mathcal{A}_{N,B,I,J}$ given above, on which the values of two discrete functions h_1, h_2 are restricted as well as archimedean characters n^{iu} and n^{iv}, can be large in the sense that they have positive upper density in general.[6] Let μ_v denotes the set of v-th roots of unity.

Proposition 2.3. *Let χ_1, χ_2 be primitive Dirichlet characters of respective conductors q_1 and q_2 and orders k and l. Let $h_1, h_2 : \mathbb{N} \to \mathbb{T}$ be pseudocharacters modulo q_1 and q_2, respectively, such that $h_1(n)^r = \tilde{\chi}_1(n)$ and $h_2(n)^s = \tilde{\chi}_2(n)$, for some $r, s \in \mathbb{N}$:*

(a) *Let $u \in \mathbb{R}$, and let $(\alpha, \beta) \in \mu_{kr} \times \mu_{ls}$. Let $\delta > 0$, $z \in \mathbb{T}$ and let $I \subset \mathbb{T}$ be an arc with length δ about 1. Then, for any $B \geq 1$ satisfying $2q_1q_2 \mid B$,*

$$\mathbb{E}^{\log}_{n \leq x} 1_{\mathcal{A}_{N,B,I}(h_1,h_2;\alpha,\beta;u)}(n) \gg \frac{\delta}{krls} \frac{\Phi_{N,B}(x)}{x}$$

as $x \to \infty$. Moreover, if $u \neq 0$ then we may replace I above by any arc of length δ.

(b) *If $u, v \in \mathbb{R}$ are fixed and such that $u/v \notin \mathbb{Q}$, and J_1 and J_2 are arcs in $\mathbb{T}$ of respective lengths δ_1 and δ_2, then*

$$\mathbb{E}^{\log}_{n \leq x} 1_{\mathcal{A}_{N,B,J_1,J_2}(h_1,h_2;\alpha,\beta;u,v)} \gg \frac{\delta_1\delta_2}{krls} \frac{\Phi_{N,B}(x)}{x}.$$

[5] By an *arc* in $\mathbb{T}$, we mean the image of an interval $[a, b] \subseteq \mathbb{R}$ under the exponentiation map $t \mapsto e(t)$. Thus, a symmetric arc about 1, for example, refers to the image under exponentiation of any interval $[m - \eta, m + \eta]$ with $m \in \mathbb{Z}$.

[6] To be precise, Proposition 2.3 implies directly that these sets have positive upper *logarithmic density*, that is $\limsup_{x\to\infty} \mathbb{E}^{\log}_{n\leq x} 1_{\mathcal{A}}(n) > 0$; however, this also implies that the upper density $\limsup_{x\to\infty} \frac{1}{x} \sum_{n\leq x} 1_{\mathcal{A}}(n) > 0$ as well.

To prove Proposition 2.3, we need the following variant of Lemma 2.11 from [Klurman and Mangerel 2018], which is easily proven by the fundamental lemma of the sieve.

Lemma 2.4. *Let $q, B \geq 1$, $N \geq 2$ and let a be a residue class modulo q such that $(a(Ba+1), q) = 1$. Then as $x \to \infty$,*

$$\Phi_{N,B}(x; q, a) = \frac{x}{q} \prod_{\substack{p \leq N \\ p \mid B/(B,q^\infty)}} \left(1 - \frac{1}{p}\right) \prod_{\substack{3 \leq p \leq N \\ p \nmid qB}} \left(1 - \frac{2}{p}\right) + O(4^{\pi(N)}).$$

In the sequel, we write

$$\delta_{N,B,q} := \frac{1}{q} \prod_{\substack{p \leq N \\ p \mid B/(B,q^\infty)}} \left(1 - \frac{1}{p}\right) \prod_{\substack{3 \leq p \leq N \\ p \nmid qB}} \left(1 - \frac{2}{p}\right),$$

and set $\delta_{N,B} = \delta_{N,B,1}$.

Lemma 2.5. *Assume the hypotheses of Proposition 2.3. Furthermore, suppose $N > \max\{q_1, q_2\}$, and that h_1^j and h_2^m are both nonpretentious for all $1 \leq j \leq r-1$ and $1 \leq m \leq s-1$. Then as $x \to \infty$,*

$$\sum_{n \leq x} \frac{1_{A_{N,B}(h_1,h_2;\alpha,\beta)}(n)}{n^{1+iu}} = \begin{cases} \delta_{N,B}/(iukrls) + o(\log x) + O_{q_1}(|u| 4^{\pi(N)}) & \text{if } u \neq 0, \\ 1/(krls)(\delta_{N,B} + o(1)) \log x + O_{q_1}(4^{\pi(N)}) & \text{otherwise.} \end{cases}$$

Proof. Since $\alpha \in \mu_{kr}$ and $\beta \in \mu_{ls}$, we have the identities

$$1_{h_1(n)=\alpha} = \frac{1}{kr} \sum_{0 \leq j \leq kr-1} (h_1(n)\bar{\alpha})^j \quad \text{and} \quad 1_{h_2(Bn+1)=\beta} = \frac{1}{ls} \sum_{0 \leq m \leq ls-1} (h_2(Bn+1)\bar{\beta})^j.$$

It follows that

$$\begin{aligned} \sum_{n \leq x} \frac{1_{\mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}(n)}{n^{1+iu}} &= \sum_{\substack{n \leq x \\ P^-(n(Bn+1)) > N}} \frac{1_{h_1(n)=\alpha} 1_{h_2(Bn+1)=\beta}}{n^{1+iu}} \\ &= \frac{1}{klrs} \sum_{0 \leq j \leq kr-1} \sum_{0 \leq m \leq ls-1} \alpha^{-j} \beta^{-m} \sum_{\substack{n \leq x \\ P^-(n(Bn+1)) > N}} \frac{h_1(n)^j h_2(Bn+1)^m n^{-iu}}{n}. \quad (3) \end{aligned}$$

As h_1^j and h_2^m are both nonpretentious for all $1 \leq j \leq r-1$ and $1 \leq m \leq s-1$, it follows that the multiplicative functions

$$\begin{aligned} \phi_{j+ar}(n) &:= h_1(n)^{j+ar} 1_{P^-(n)>N} n^{-iu} = h_1(n)^j \tilde{\chi}_1(n)^a 1_{P^-(n)>N} n^{-iu} \\ \psi_{m+bs}(n) &:= h_2(n)^{m+bs} 1_{P^-(n)>N} = h_2(n)^m \tilde{\chi}_2(n)^b 1_{P^-(n)>N} \end{aligned}$$

are both nonpretentious for such j, m, and any $0 \le a \le k-1$, $0 \le b \le s-1$. By Theorem 2.1, it follows that

$$\Big| \sum_{\substack{0\le j\le kr-1\\ r\nmid j \text{ or } s\nmid m}} \sum_{0\le m\le ls-1} \alpha^{-j}\beta^{-m} \sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}} \frac{h_1(n)^j h_2(Bn+1)^m n^{-iu}}{n} \Big|$$
$$\le \sum_{\substack{0\le j\le kr-1\\ r\nmid j \text{ or } s\nmid m}} \sum_{0\le m\le ls-1} \left| \sum_{n\le x} \frac{\phi_j(n)\psi_m(Bn+1)}{n} \right| = o(\log x), \quad (4)$$

where the estimate depends on k, r, l and s. When $j = ra$ and $m = bs$, we instead have

$$\sum_{n\le x} \frac{\phi_{ar}(n)\psi_{bs}(Bn+1)}{n} = \sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}} \frac{\tilde{\chi}_1(n)^a \tilde{\chi}_2(Bn+1)^b}{n^{1+iu}} = \sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}} \frac{\tilde{\chi}_1(n)^a}{n^{1+iu}} \quad (5)$$

for each b, as $q_2 \mid B$. Now, as $N > q_1$, for each n with $P^-(n(Bn+1)) > B$ we must have $(n, q_1) = 1$. Thus, splitting the rightmost sum in (5) into coprime residue classes modulo q_1, the RHS of (5) becomes

$$\sum_{c(q_1)} \chi_1(c)^a \sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}} \frac{1_{n\equiv c(q_1)}}{n^{1+iu}}.$$

We first consider the case $u \neq 0$. Applying the previous lemma and partial summation, we get that for each coprime residue c modulo q_1,

$$\sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}} \frac{1_{n\equiv c(q_1)}}{n^{1+iu}} = \int_1^x \frac{1}{t^{1+iu}}\, d\{\Phi_{N,B}(t; q_1, c)\}$$
$$= \delta_{N,B,q_1} \int_1^x \frac{dt}{t^{1+iu}} + O_{q_1}\left(1 + |u| 4^{\pi(N)} \int_1^x \frac{dt}{t^2}\right) \quad (6)$$
$$= \frac{\delta_{N,B,q_1}}{iu}(1 - x^{-iu}) + O_{q_1}(1 + |u| 4^{\pi(N)}). \quad (7)$$

The main term being independent of c modulo q_1, we can invert the orders of summation and use orthogonality to get that whenever $a \neq 0$, we have

$$\sum_{c(q_1)} \chi_1(c)^a \sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}} \frac{1_{n\equiv c(q_1)}}{n^{1+iu}} = \frac{\delta_{N,B,q_1}}{iu}(1 - x^{-iu}) \sum_{c(q_1)} \chi_1(c)^a + O(q_1^2(1 + |u| 4^{\pi(N)}))$$
$$\ll q_1^2((1 + |u|)4^{\pi(N)}).$$

As such, it follows that whenever $a \neq 0$, we have

$$\sum_{n\le x} \frac{\phi_{ar}(n)\psi_{bs}(Bn+1)}{n} \ll q_1^2((1 + |u|)4^{\pi(N)}). \quad (8)$$

In the remaining case $a = 0$, we have

$$\sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}} \frac{1}{n^{1+iu}} = \frac{\delta_{N,B}}{iu}(1-x^{-iu}) + O((1+|u|)4^{\pi(N)}), \tag{9}$$

as this is the LHS of (6), but with q_1 replaced by 1. Combining (4), (8) and (9), and inserting these into (3), we find that

$$\sum_{n\le x} \frac{1_{\mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}(n)}{n^{1+iu}} = \frac{1}{krls}\left(\frac{\delta_{N,B}}{iu}(1-x^{-iu}) + O(q_1(1+|u|)4^{\pi(N)}) + o(\log x)\right)$$

when $u \neq 0$, as claimed. The case $u = 0$ follows the same lines, but is simpler. The proof is now complete. □

Before proceeding to the proof of Proposition 2.3, we pause to recall the following objects of relevance to it. Given $K \in \mathbb{N}$, let $B_K : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ denote the degree K *Beurling polynomial* (see Section 1.2 of [Montgomery 1994] for a definition and some of its properties). Recall that it satisfies the properties that:

(i) If $\psi(t) := \{t\} - \frac{1}{2}$ is the sawtooth function then $-B_K(-t) \le \psi(t) \le B_K(t)$. As such, given an interval $I \subseteq \mathbb{R}/\mathbb{Z}$ with endpoints $0 \le a < b < 1$, we have functions

$$f_I(t) := |I| - B_K(b-t) - B_K(t-a) \quad \text{and} \quad g_I(t) := |I| + B_K(t-b) + B_K(a-t),$$

for which

$$f_I(t) \le 1_I(t) \le g_I(t) \text{ for all } t \in \mathbb{R}/\mathbb{Z}. \tag{10}$$

(ii) the Fourier coefficients[7] $\hat{B}_K(m) := \int_0^1 B_K(t)e(-mt)\,dt$ satisfy

$$\hat{B}_K(m) = \begin{cases} 0 & \text{for } |m| > K,\\ 1/(2(K+1)) & \text{if } m = 0,\\ O(1/m) & \text{for all } m \neq 0, \end{cases}$$

the implicit constant in the last estimate being absolute.

Proof of Proposition 2.3. (a) When $u = 0$ it is clear that

$$\mathcal{A}_{N,B,I}(h_1,h_2;\alpha,\beta;u) = \{n \in \mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta) : 1 \in I\} = \mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta),$$

since $1 \in I$. By Lemma 2.4, we get

$$\delta_{N,B} = \frac{\Phi_{N,B}(x)}{x} + O(4^{\pi(N)}x^{-1}). \tag{11}$$

Proposition 2.3 in the case $u = 0$ thus follows from Lemma 2.5, since for large enough x, we have

$$\sum_{n\le x} \frac{1_{\mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}(n)}{n} = \left(\frac{\delta_{N,B,1}}{krls} + o(1)\right)\log x + O(4^{\pi(N)}) \gg \frac{\delta}{krls}\frac{\Phi_{N,B}(x)}{x}\log x.$$

[7] As usual, for $t \in \mathbb{R}$ we write $e(t) := e^{2\pi i t}$.

We now assume that $u \neq 0$. Here, we no longer assume that I is an arc around 1, but rather any arc of length δ (the proof being the same regardless of the center point of the arc). Let K be a large integer, and suppose x is sufficiently large in terms of K. Write $I' \subseteq \mathbb{R}/\mathbb{Z}$ to be the interval that maps onto I under exponentiation; thus, there is a constant $c > 0$ for which $|I'| \geq c\delta$. From (10), it follows that

$$\begin{aligned}
&\sum_{n\leq x} \frac{1_{\mathcal{A}_{N,B,I}(h_1,h_2;\alpha,\beta;u)}(n)}{n} \\
&\quad = \sum_{\substack{n\leq x \\ n\in\mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}} \frac{1_{I'}(\{u\log n/(2\pi)\})}{n} \\
&\quad \geq |I'| \sum_{n\leq x} \frac{1_{\mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}(n)}{n} \\
&\qquad - \sum_{|m|\leq K} \hat{B}_K(m)\left(e(mb) \sum_{n\leq x} \frac{1_{\mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}(n)}{n^{1+imu}} + e(-ma) \sum_{n\leq x} \frac{1_{\mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}(n)}{n^{1-imu}} \right) \\
&\quad \geq \frac{c\delta\delta_{N,B}}{krls} \log x - 2 \sum_{|m|\leq K} |\hat{B}_K(m)| \left| \sum_{n\leq x} \frac{1_{\mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}(n)}{n^{1+imu}} \right|. \qquad (12)
\end{aligned}$$

We consider the second term on the RHS of (12). Applying Lemma 2.5 for each m and summing, we get

$$\begin{aligned}
\sum_{1\leq|m|\leq K} |\hat{B}_K(m)| \left| \sum_{n\leq x} \frac{1_{\mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}(n)}{n^{1+imu}} \right| &\ll \sum_{1\leq m\leq K} \frac{1}{m} \left(\frac{\delta_{N,B}}{m|u|} + o(\log x) + (1+m|u|)4^{\pi(N)} \right) \\
&\ll |u|^{-1} + o((\log K)(\log x)) + K^2|u|4^{\pi(N)}.
\end{aligned}$$

The term with $k = 0$ gives

$$|\hat{B}_K(0)| \sum_{\substack{n\leq x \\ n\in\mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}} \frac{1}{n} \leq \frac{\log x}{2K}.$$

As such, we get

$$\sum_{\substack{n\leq x \\ n\in\mathcal{A}_{N,B,I}(h_1,h_2;u)}} \frac{1}{n} \geq \left(\frac{c\delta\delta_{N,B}}{krls} - \frac{1}{2K} - o(\log K) \right) \log x - O(|u|^{-1} + K^2|u|4^{\pi(N)}).$$

Choosing $K = K(x)$ tending to infinity with x, but sufficiently slow-growing so that $o((\log K)(\log x)) = o(\log x)$ and $K^2 4^{\pi(N)} = o(\log x)$, proves the claim.

(b) The case when $uv = 0$ is similar to (a) and we assume $uv \neq 0$. We are interested in

$$\sum_{\substack{n\leq x \\ n\in\mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}} \frac{1_{J_1}(n^{iu})1_{J_2}(n^{iv})}{n}. \qquad (13)$$

We minorize each indicator function by a Beurling polynomial of degree K in a similar way as to what was done just above; we shall therefore merely sketch the argument, highlighting the differences in the

argument and the relevance of the condition $u/v \notin \mathbb{Q}$. As above, let us denote the minorants of 1_{J_1} and 1_{J_2} as f_{J_1}, f_{J_2}, and the corresponding majorants as g_{J_1}, g_{J_2}. Now put[8]

$$F := f_{J_1} f_{J_2} - (g_{J_1} - f_{J_1})(g_{J_2} - f_{J_2}) = f_{J_1} g_{J_2} + f_{J_2} g_{J_1} - g_{J_1} g_{J_2}.$$

That $F \le 1_{J_1} 1_{J_2}$ is a minorant can be seen from the inequality

$$\begin{aligned} 1_{J_1}(y)1_{J_2}(y) - F(y) &= g_{J_2}(y)(g_{J_1}(y) - f_{J_1}(y)) + 1_{J_1}(y)1_{J_2}(y) - f_{J_2}(y)g_{J_1}(y) \\ &\ge g_{J_2}(y)(g_{J_1}(y) - f_{J_1}(y)) - f_{J_2}(y)(g_{J_1}(y) - 1_{J_1}(y)) \\ &\ge \begin{cases} (g_{J_2}(y) - f_{J_2}(y))(g_{J_1}(y) - f_{J_1}(y)) & \text{if } f_{J_2}(y) \ge 0, \\ g_{J_2}(y)(g_{J_1}(y) - f_{J_1}(y)) & \text{if } f_{J_2}(y) < 0, \end{cases} \end{aligned}$$

which implies that $1_{J_1}(y)1_{J_2}(y) - F(y) \ge 0$ for all y since $g_{J_2} \ge 0$ and $g_{J_r} \ge f_{J_r}$ for $r = 1, 2$. We note that for $j = 1, 2$, $\widehat{(g_{J_j} - f_{J_j})}(k) = 0$ for $|k| > K$ or $k = 0$, and otherwise $\|\widehat{g_{J_j} - f_{J_j}}\|_\infty \le 2/(K+1)$. Noting that $k_1 u \ne -k_2 v$ for all $1 \le |k_1|, |k_2| \le K$ since $u/v \notin \mathbb{Q}$, Lemma 2.5 gives

$$\begin{aligned} &\Bigg| \sum_{\substack{n \le x \\ n \in \mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}} \frac{1}{n}(g_{J_1} - f_{J_1})\left(\left\{\frac{u \log n}{2\pi}\right\}\right)(g_{J_2} - f_{J_2})\left(\left\{\frac{v \log n}{2\pi}\right\}\right)\Bigg| \\ &\qquad \le \frac{4}{(K+1)^2} \sum_{1 \le |k_1|, |k_2| \le K} \left| \sum_{n \le x} \frac{1_{\mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}(n)}{n^{1+i(k_1 u + k_2 v)}} \right| \ll \max_{1 \le |k_1|, |k_2| \le K} \left| \sum_{n \le x} \frac{1_{\mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}(n)}{n^{1+i(k_1 u + k_2 v)}} \right| \\ &\qquad = o(\log x), \end{aligned}$$

provided that x is sufficiently large (in terms of $\max_{1 \le |k_1|, |k_2| \le K} |k_1 u + k_2 v|^{-1}$). Thus, by the triangle inequality, (13) is bounded from below by

$$\begin{aligned} &|J_1||J_2| \sum_{\substack{n \le x \\ n \in \mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}} \frac{1}{n} - o(\log x) \\ &\quad - \Bigg(\sum_{1 \le |m_1| \le K} \frac{1}{|m_1|} \Bigg| \sum_{\substack{n \le x \\ n \in \mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}} \frac{1}{n^{1+im_1 u}} \Bigg| + \sum_{1 \le |m_2| \le K} \frac{1}{|m_2|} \Bigg| \sum_{\substack{n \le x \\ n \in \mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}} \frac{1}{n^{1+im_2 v}} \Bigg| \Bigg) - o(\log x) \\ &\qquad - \sum_{1 \le |m_1|, |m_2| \le K} \frac{1}{|m_1 m_2|} \Bigg| \sum_{\substack{n \le x \\ n \in \mathcal{A}_{N,B}(h_1,h_2;\alpha,\beta)}} \frac{1}{n^{1+i(m_1 u + m_2 v)}} \Bigg| - O\left(\frac{\log x}{K}\right) - o(\log x). \end{aligned}$$

Similarly, each of the terms here are $o(\log x)$ besides the first term. The claim of part (b) thus follows just as that of part (a) did. □

Reduction to pseudopretentious functions. Let $f, g : \mathbb{N} \to \mathbb{T}$ be completely multiplicative functions. Assume, as per the hypotheses of Theorem 1.2, that $\overline{\{(f(n), g(n+1))\}_n} \ne \mathbb{T}^2$. Using Theorem 2.1, we shall show in this subsection that f and g must both be pseudopretentious.

[8]This is inspired by the construction of vector sieve weights as found in [Brüdern and Fouvry 1996]. We thank the anonymous referee for this suggestion.

Proposition 2.6. *Let $f, g : \mathbb{N} \to \mathbb{T}$ be completely multiplicative functions. Suppose $\{(f(n), g(n+1))\}_n$ is such that there is an $\varepsilon > 0$ and a pair $(z, w) \in \mathbb{T}^2$ for which*

$$\|(f(n), g(n+1)) - (z, w)\|_{\ell_1} \geq \varepsilon \text{ for all } n \text{ sufficiently large}.$$

Then there exist primitive Dirichlet characters χ_1, χ_2 to respective moduli $q_1, q_2 = O_\varepsilon(1)$, minimal positive integers $k, l = O_\varepsilon(1)$, real numbers $t_1, t_2 = O_\varepsilon(1)$ and completely multiplicative functions $h_1, h_2 : \mathbb{N} \to \mathbb{T}$ such that:

(i) $\mathbb{D}(f, h_1 n^{it_1}; x), \mathbb{D}(g, h_2 n^{it_2}; x) \ll_\varepsilon 1$.

(ii) $h_1^k = \tilde{\chi}_1$ *and* $h_2^l = \tilde{\chi}_2$.

This is a two-dimensional analogue of the reduction argument used at the beginning of Section 2 of [Klurman and Mangerel 2018]. To appropriately formalize the arguments leading to Proposition 2.6, we recall the following (essentially standard) definitions.

Definition 2.7. Let $d, N \geq 1$ and let $\{a_n\}_n \subset \mathbb{T}^d$. The *logarithmic discrepancy* (*of height N*) of $\{a_n\}_n$ is the quantity

$$D_N(\{a_n\}_n) := \sup_{\substack{I_1,\dots,I_d \subseteq \mathbb{T} \\ I_j \text{ arcs}}} \left| \frac{1}{\log N} \sum_{\substack{n \leq N \\ a_n \in B(\boldsymbol{I})}} \frac{1}{n} - \prod_{1 \leq j \leq d} |I_j| \right|,$$

where $B(\boldsymbol{I}) := \prod_{1 \leq j \leq d} I_j$. Similarly, we let $D_N^*(\{a_n\}_n)$ denote[9] the same quantity but where the sup is over all d-tuples of arcs I_j all of whose left endpoints are 1.

Definition 2.8. A sequence $\{a_n\}_n \subset \mathbb{T}^d$ is *logarithmically equidistributed* if $D_N(\{a_n\}_n) = o(1)$, as $N \to \infty$.

It is easy to see that logarithmically equidistributed sequences in $\mathbb{T}^d$ are also dense in $\mathbb{T}^d$. Before launching into the proof Proposition 2.6, we must exclude the following degenerate case from consideration.

Proposition 2.9. *Let $f, g : \mathbb{N} \to \mathbb{T}$ be completely multiplicative. Suppose exactly one of f and g is pseudopretentious. Then $\overline{\{(f(n), g(n+1))\}_n} = \mathbb{T}^2$.*

To prove Proposition 2.9, we shall need a concentration estimate, showing that if a completely multiplicative function $F : \mathbb{N} \to \mathbb{T}$ is 1-pretentious then $F(n)$ is roughly constant for *most* $n \leq x$. This will be used in the proof of Lemma 2.11 below, which will allow us to approximate pseudopretentious functions pointwise on a positive upper density subset of integers n with $P^-(n(Bn+1)) > N$ (with B and N fixed). Given a completely multiplicative function $h : \mathbb{N} \to \mathbb{T}$ and $N \geq 1$, set

$$\mathfrak{I}_h(x; N) := \sum_{N < p \leq x} \frac{\mathrm{Im}(h(p))}{p}.$$

[9] By the triangle inequality, it is easy to check that if $\{a_n\}_n \subset \mathbb{T}^d$ then

$$D_N^*(\{a_n\}_n) \leq D_N(\{a_n\}_n) \leq 2^d D_N^*(\{a_n\}_n). \tag{14}$$

We will use this relationship between D_N and D_N^* below.

For $x \geq N \geq 1$, we define $\mathbb{D}(f, 1; N, x) := (\mathbb{D}(f, 1; x)^2 - \mathbb{D}(f, 1; N)^2)^{1/2}$.

Lemma 2.10. *Let $N, B \geq 1$ with $2 \mid B$. Then as $x \to \infty$,*

$$\sum_{\substack{n \leq x \\ P^-(n(Bn+1))>N}} |f(n) - e^{i\mathfrak{I}_f(x;N)}|^2 \ll \Phi_{N,B}(x)\left(\mathbb{D}(f, 1; N, x)^2 + \frac{(\log N)^2}{N}\right) + 4^{\pi(N)} x \frac{\log_2 x}{\log x}.$$

The same estimate holds when $f(n)$ is replaced by $f(Bn+1)$.

Proof. This is a simple extension of Proposition 2.3 in [Klurman 2017], but we give the details for the reader's convenience. The claim is trivial if $\mathbb{D}(f, 1; N, x) \geq 1$. Thus, in what follows we shall assume that $\mathbb{D}(f, 1; N, x) < 1$. Define an additive function $h : \mathbb{N} \to \mathbb{C}$ on prime powers via $h(p^k) := f(p^k) - 1$. By repeatedly applying triangle inequality we have that for all $|z_i|, |w_i| \leq 1$

$$\left| \prod_{1 \leq i \leq n} z_i - \prod_{1 \leq i \leq n} w_i \right| \leq \sum_{1 \leq i \leq n} |z_i - w_i|. \tag{15}$$

Note $e^{z-1} = z + O(|z-1|^2)$ for $|z| \leq 1$. Thus, for each $n \in \mathbb{N}$, using (15) we derive

$$\begin{aligned} f(n) &= \prod_{p^k \parallel n} f(p^k) \\ &= \prod_{p^k \parallel n} (1 + h(p^k)) \\ &= e^{h(n)} + O\bigg(\sum_{p^k \parallel n} |e^{h(p^k)} - (1 + h(p^k))| \bigg) \\ &= e^{h(n)} + O\bigg(\sum_{p^k \parallel n} |1 - f(p^k)|^2 \bigg) \\ &= e^{h(n)} + O\bigg(\sum_{p^k \parallel n} (1 - \mathrm{Re}(f(p^k))) \bigg). \end{aligned}$$

Summing over all $n \leq x$ with $P^-(n(Bn+1)) > N$, we get

$$\begin{aligned} &\sum_{\substack{n \leq x \\ P^-(n(Bn+1))>N}} |f(n) - e^{i\mathfrak{I}_f(x;N)}|^2 \\ &\qquad \ll \sum_{\substack{n \leq x \\ P^-(n(Bn+1))>N}} |e^{h(n)} - e^{i\mathfrak{I}_f(x;N)}|^2 + \sum_{p^k \leq x} (1 - \mathrm{Re}(f(p^k))) \sum_{\substack{n \leq x \\ P^-(n(Bn+1))>N}} 1_{p^k \parallel n} \\ &\qquad =: T_1 + T_2. \end{aligned}$$

Using Lemma 2.4, we have

$$\begin{aligned}T_2 &= \sum_{\substack{p\le x\\ p>N}}(1-\mathrm{Re}(f(p))) \sum_{\substack{m\le x/p\\ P^-(m(Bpm+1))>N}} 1+O\Bigg(\sum_{\substack{p^k\le x\\ p>N,k\ge 2}}\frac{1}{p^k}\Bigg)\\ &= x\prod_{\substack{p'\le N\\ p'\mid B}}\Big(1-\frac{1}{p'}\Big)\prod_{\substack{3\le p''\le N\\ p''\nmid B}}\Big(1-\frac{2}{p''}\Big)\sum_{N<p\le x}\frac{1-\mathrm{Re}(f(p))}{p}+O\Big(4^{\pi(N)}\pi(x)+\frac{x}{N}\Big)\\ &\ll \Phi_{N,B}(x)\Big(\mathbb{D}(f,1;N,x)^2+\frac{(\log N)^2}{N}\Big)+4^{\pi(N)}\frac{x}{\log x}.\end{aligned}$$

We next treat T_1. Define

$$\mu_h(x) := \sum_{\substack{p^k\le x\\ p>N}}\frac{h(p^k)}{p^k}\Big(1-\frac{1}{p}\Big),$$

which arises as the mean value of $h(n)$ in the estimate

$$\sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}} h(n) = \mu_h(x)\Phi_{N,B}(x)+O\Big(4^{\pi(N)}\pi(x)+\frac{x}{N}\Big),$$

using a similar argument as that which was used to bound T_2 (but keeping track of powers p^k with $k\ge 2$ as well). Writing $h(n)=\sum_{p^k\parallel n}h(p^k)$ and noticing that $\mathrm{Re}(h(p^k))\le 0$ for all primes p and all $k\ge 1$, it follows that $\mathrm{Re}(\mu_h(x))\le 0$, and we thus have

$$\sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}}|e^{h(n)}-e^{\mu_h(x)}|^2 \le \sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}}|h(n)-\mu_h(x)|^2. \tag{16}$$

Following the usual proof of the Turán–Kubilius inequality (e.g., as in Section III.2 of [Tenenbaum 1995]), we have

$$\begin{aligned}&\sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}}|h(n)-\mu_h(x)|^2\\ &\quad= \sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}}|h(n)|^2-\Phi_{N,B}(x)|\mu_h(x)|^2+O\Big(|\mu_h(x)|\Big(4^{\pi(N)}\pi(x)+\frac{1}{N}\Big)\Big)\\ &\quad= \sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}}\sum_{p^k,q^l\parallel n}h(p^k)\overline{h(q^l)}-\Phi_{N,B}(x)|\mu_h(x)|^2+O\Big(|\mu_h(x)|\Big(4^{\pi(N)}\pi(x)+\frac{1}{N}\Big)\Big).\end{aligned}$$

Denote by Σ the double sum in this last expression. Splitting the terms $p=q$ from $p\neq q$, Σ can be estimated with Lemma 2.4 to give

$$\begin{aligned}\Sigma &= \sum_{\substack{p^kq^l\le x\\ p\neq q,\,p,q>N}} h(p^k)\overline{h(q^l)} \sum_{\substack{m\le x/(p^kq^l)\\ P^-(m(Bp^kq^lm+1))>N}} 1_{(m,pq)=1} + \sum_{\substack{p^k\le x\\ p>N}} |h(p^k)|^2 \sum_{\substack{m\le x/p^k\\ P^-(m(Bp^km+1))>N}} 1_{(m,p)=1}\\ &= \Phi_{N,B}(x) \sum_{\substack{p^kq^l\le x\\ p\neq q,\,p,q>N}} \frac{h(p^k)\overline{h(q^l)}}{p^kq^l}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{q}\Big) + \Phi_{N,B}(x)\sum_{\substack{p^k\le x\\ p>N}} \frac{|h(p^k)|^2}{p^k}\Big(1-\frac{1}{p}\Big) + O(4^{\pi(N)}\pi_2(x)),\end{aligned}$$

where $\pi_2(x)$ is the number of integers $\le x$ that are product of at most two primes. Furthermore,

$$\begin{aligned}&\sum_{\substack{p^kq^l\le x\\ p\neq q,\,p,q>N}} \frac{h(p^k)\overline{h(q^l)}}{p^kq^l}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{q}\Big)\\ &\qquad = |\mu_h(x)|^2 + O\Big(\sum_{\substack{p^k,p^l\le x\\ p>N}} \frac{|h(p^k)||h(p^l)|}{p^{k+l}} + \sum_{\substack{p^k,q^l\le x\\ p^kq^l>x,\,p,q>N}} \frac{|h(p^k)||h(q^l)|}{p^kq^l}\Big)\\ &\qquad = |\mu_h(x)|^2 + O\Big(\sigma_h(x)^2 + \frac{1}{N} + x^{-1/2}\sigma_h(x)^2\Big),\end{aligned}$$

where we defined

$$\sigma_h(x)^2 := \sum_{\substack{p^k\le x\\ p>N}} \frac{|h(p^k)|^2}{p^k}\Big(1-\frac{1}{p}\Big),$$

and used Cauchy–Schwarz in the last line. It follows that

$$\begin{aligned}\Sigma - \Phi_{N,B}(x)|\mu_h(x)|^2 &\ll \Phi_{N,B}(x)\Big(\sigma_h(x)^2+\frac{1}{N}\Big) + 4^{\pi(N)}\pi_2(x)\\ &\ll \Phi_{N,B}(x)\Big(\sigma_h(x)^2+\frac{1}{N}\Big) + 4^{\pi(N)}x\frac{\log_2 x}{\log x}.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}} |h(n)-\mu_h(x)|^2 &\ll \Phi_{N,B}(x)\Big(\sigma_h(x)^2+\frac{1}{N}\Big) + 4^{\pi(N)}\Big(x\frac{\log_2 x}{\log x} + |\mu_h(x)|\pi(x)\Big)\\ &\ll \Big(\mathbb{D}(f,1;N,x)^2+\frac{1}{N}\Big)\Phi_{N,B}(x) + 4^{\pi(N)}x\frac{\log_2 x}{\log x}, \qquad (17)\end{aligned}$$

using $|\mu_h(x)|\le 2\sum_{p\le x} 1/p \le 2\log_2 x + O(1)$ and the estimate

$$\sigma_h(x)^2 = \sum_{\substack{p^k\le x\\ p>N}} \frac{|1-f(p^k)|^2}{p^k}\Big(1-\frac{1}{p}\Big) \ll \sum_{N<p\le x} \frac{1-\mathrm{Re}(f(p))}{p} + O\Big(\frac{1}{N}\Big) = \mathbb{D}(f,1;N,x)^2 + O\Big(\frac{1}{N}\Big).$$

Inserting (17) into (16), and using this in the definition of T_1, we get

$$\begin{aligned}\sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}} & |e^{h(n)}-e^{i\mathfrak{I}_f(x;N)}|^2\\ &\ll \sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}} |e^{\mu_h(x)}-e^{i\mathfrak{I}_f(x;N)}|^2+\sum_{\substack{n\le x\\ P^-(n(Bn+1))>N}} |e^{h(n)}-e^{\mu_h(x)}|^2\\ &\ll \Phi_{N,B}(x)|e^{\mu_h(x)}-e^{i\mathfrak{I}_f(x;N)}|^2+\Phi_{N,B}(x)\left(\mathbb{D}(f,1;N,x)^2+\frac{1}{N}\right)+4^{\pi(N)}x\frac{\log_2 x}{\log x}. \quad (18)\end{aligned}$$

Moreover, as

$$\begin{aligned}\mu_h(x)&=-\sum_{N<p\le x}\frac{1-\mathrm{Re}(f(p))}{p}+i\sum_{N<p\le x}\frac{\mathrm{Im}(f(p))}{p}+O(1/N)\\ &=-\mathbb{D}(f,1;N,x)^2+i\cdot\mathfrak{I}_f(x;N)+O\left(\frac{1}{N}\right),\end{aligned}$$

it follows that

$$|e^{i\mathfrak{I}_f(x;N)}-e^{\mu_h(x)}|^2=|1-e^{-\mathbb{D}(f,1;N,x)^2+O(1/N)}|^2\ll\mathbb{D}(f,1;N,x)^2+\frac{1}{N}.$$

Coupled with (18), this completes the proof of the first assertion of the lemma. The assertion with $f(Bn+1)$ in place of $f(n)$ is proved similarly, and we leave the details for the reader. □

Lemma 2.11. *Let $\eta>0$ and $C>1$. Let $B\ge 1$ be an even integer. Suppose furthermore that $f:\mathbb{N}\to\mathbb{T}$ is a pseudopretentious multiplicative function, with f pretending to be $h(n)n^{it}$ with h a pseudocharacter and $t\in\mathbb{R}$. Then there is an infinite sequence $\{x_j\}_j$ of positive real numbers and a large parameter N (depending at most on η) such that if $j=j(N)$ is chosen sufficiently large then*

$$f(n)=h(n)n^{it}+O(\eta)$$

for all but $O_C(\eta^C\Phi_{N,B}(x_j))$ choices of $n\le x_j$ with $P^-(n(Bn+1))>N$. Similarly, $f(Bn+1)=h(Bn+1)(Bn+1)^{it}+O(\eta)$ for all but $O_C(\eta^C\Phi_{N,B}(x_j))$ choices of $n\le x_j$ with $P^-(n(Bn+1))>N$.

Proof. The result is trivial for any $\eta\gg_C 1$, so in what follows we shall assume that η is smaller than any fixed bound depending on C. Let $F(n):=f(n)\overline{h(n)}n^{-it}$ for each n. We consider several cases, according to the behavior of the series

$$\mathfrak{I}_F(\infty):=\lim_{x\to\infty}\mathfrak{I}_F(x;1)=\lim_{x\to\infty}\sum_{p\le x}\frac{\mathrm{Im}(F(p))}{p}.$$

First, suppose $\mathfrak{I}_F(\infty)$ converges absolutely. Then, choosing N large enough in terms of η, it follows that $\mathfrak{I}_F(x;N)<\eta/2$ for all $x>N$. In this case, we shall choose $\{x_j\}_j$ to be the set of all $x>N$. Next, suppose that $\mathfrak{I}_F(\infty)$ is unbounded. In this case, since $\mathfrak{I}_F(n+1;N)-\mathfrak{I}_F(n;N)\to 0$ as $n\to\infty$, it follows that the sequence of fractional parts of $\mathfrak{I}_f(x;N)/2\pi$ must be dense in $[0,1]$, for any N. In this case, we may choose any large N and $\{x_j\}_j$ to be a sequence for which $\|\mathfrak{I}_f(x_j;N)/2\pi\|\to 0$, as $j\to\infty$. Lastly,

suppose $x \mapsto \mathfrak{I}_F(x;1)$ is bounded but not convergent. Let α be a limit point of $\mathfrak{I}_F(x;1)$, and choose $\{x_j\}_j$ to be a sequence for which $\mathfrak{I}_F(x_j;N) \to \alpha$. Picking j_0 sufficiently large in terms of η, then setting $N := x_{j_0}$ it follows that $\mathfrak{I}_F(x_j;N) < \eta/2$ for large enough $j > j_0$. As F is 1-pretentious, we can assume N is chosen large enough in terms of η (as in the analysis above) so that $\mathbb{D}(F,1;N,\infty)^2 + 1/N \ll \eta^{C+2}$. Furthermore, as discussed above we have selected a sequence $\{x_j\}_j$ such that $\|\mathfrak{I}_F(x_j;N)/2\pi\| < \eta/2$. Taking j sufficiently large in terms of N, Lemma 2.10 and Chebyshev's inequality give

$$\begin{aligned}
&\left|\left\{n \le x_j : P^-(n(Bn+1)) > N, |F(n) - e^{i\mathfrak{I}_F(x_j;N)}| > \frac{\eta}{2}\right\}\right| \\
&\qquad \le 4\eta^{-2} \sum_{\substack{n\le x \\ P^-(n(Bn+1))>N}} |F(n) - e^{i\mathfrak{I}_F(x_j;N)}|^2 \\
&\qquad \ll \eta^{-2}\Phi_{N,B}(x_j)\left(\mathbb{D}(F,1;N,x_j)^2 + \frac{1}{N}\right) + \eta^{-2}4^{\pi(N)}x_j\frac{\log_2 x_j}{\log x_j} \\
&\qquad \ll \Phi_{N,B}(x_j)\eta^C \\
&\qquad < \Phi_{N,B}(x_j)\frac{\eta}{2}, \qquad (19)
\end{aligned}$$

provided that η is sufficiently small (since $C > 1$). For all other $n \le x_j$ with $P^-(n(Bn+1)) > N$ we choose j larger if necessary so that $\|\mathfrak{I}_f(x_j;N)/2\pi\| < \eta/2$. It follows that

$$|f(n) - h(n)n^{it}| = |F(n) - 1| \le |F(n) - e^{i\mathfrak{I}_F(x_j;N)}| + |e^{i\mathfrak{I}_F(x_j;N)} - 1| < \frac{\eta}{2} + \frac{\eta}{2} = \eta,$$

which implies the first claim. The claim with $f(Bn+1)$ follows in the same way (with the same subsequence $\{x_j\}_j$) from Lemma 2.10 (which holds with $f(Bn+1)$ as well). □

Remark 2.12. In the sequel, we will apply Lemma 2.11 twice in a context in which a common subsequence will be sought for a pair of functions f and g (see the proofs of Lemma 3.2 and Proposition 1.5 below), and it is not immediately clear why such a subsequence is available in general. Fortunately, in both of these contexts, the functions f and g will be finite-valued, say taking values in μ_K for some $K \in \mathbb{N}$. A particular feature of this case is that if h_1 and h_2 are pseudocharacters associated with f and g respectively, and $F := f\overline{h_1}$ and $G := g\overline{h_2}$ then the fact that $\mathbb{D}(F,1;x), \mathbb{D}(G,1;x) \ll 1$ immediately implies that $F(p) = 1 = G(p)$ except on a set S of primes for which $\sum_{p\in S} 1/p < \infty$. This means, in particular, that $\mathfrak{I}_F(x), \mathfrak{I}_G(x)$ both converge absolutely as $x \to \infty$. Thus, according to the above proof, we can take our subsequence $\{x_j\}_j$ for f to consist of all sufficiently large x, and the same subsequence is admissible for g. Thus, a choice of common infinite subsequence for both f and g necessarily exists.

Proof of Proposition 2.9. Suppose there is an $\varepsilon > 0$ and a pair $(z,w) \in \mathbb{T}^2$ such that $(f(n), g(n+1)) \notin B_\varepsilon((z,w))$ for all large n.[10] Let h be a pseudocharacter such that f is hn^{it}-pretentious, and assume it has order kr (i.e., $h : \mathbb{N} \to \mu_{kr}$). Select $\{n_j\}_j$ on which $f(2n_j) \to z$, and hence for j chosen large enough

[10] Here and elsewhere, we write $B_\varepsilon((z,w))$ to denote the ε-ball about (z,w) in $\mathbb{C}^2$ with respect to the ℓ^1-norm.

we have $f(2n_j) = z + O(\varepsilon^2)$. By Lemma 2.11, we can choose N sufficiently large in terms of ε and a sequence $\{x_\ell\}_\ell$ such that,

$$f(2n_j m) = zh(m)m^{it} + O(\varepsilon^2),$$

for all but $O(\varepsilon^3 \Phi_{N,2n_j}(x_\ell))$ integers $m \le x_\ell$ with $P^-(m(2n_j m+1)) > N$. Thus, if ε is sufficiently small then for all but a small number of exceptions, we have

$$(h(m)m^{it}, g(2n_j m+1)) \notin B_{2\varepsilon/3}((1,w)).$$

Let I be a symmetric arc of length $\varepsilon/4$ about 1, and let J be a symmetric arc of the same length about w. To yield a contradiction, we shall presently show that there exists a sufficiently dense subset of integers $m \le x_\ell$ that satisfies the following properties:

(i) $h(m) = 1$.

(ii) $P^-(m(2n_j m+1)) > N$.

(iii) $g(2mn_j+1) \in J$.

(iv) $m^{it} \in I$.

The proof of this argument is similar to that of Proposition 2.3. For convenience, put $X := x_\ell$, for some sufficiently large index ℓ (depending at most on ε). Then we wish to bound

$$\mathfrak{G} = \sum_{\substack{m \le X \\ P^-(m(2n_j m+1))>N}} \frac{1_I(m^{it}) 1_{h(m)=1} 1_J(g(2mn_j+1))}{m}.$$

We write the arcs I and J as the images of intervals $[a,b]$ and $[c,d]$ (say) in $\mathbb{R}$ under the map $t \mapsto e(t)$, such that $[a,b]$ contains an integer (so that I contains 1). We consider two cases, depending on t. First, if $t = 0$ then $1_I(m^{it}) = 1$ for all m. Using the properties of Beurling polynomials, we have the minorization

$$\begin{aligned} 1_J(g(2n_j m+1)) &\ge \left(|J| - \sum_{0<|l|\le K} \hat{B}_K(l)(e(dl)g(2n_j m+1)^{-l} + e(-cl)g(2n_j m+1)^l)\right) \\ &\ge \left(|J| - \sum_{1\le|l|\le K} \hat{B}_K(l)(e(dl)g(2n_j m+1)^{-l} + e(-cl)g(2n_j m+1)^l)\right) - \frac{1}{K+1}, \quad (20) \end{aligned}$$

for any parameter $K = K(X)$ to be chosen. Inserting this into the definition of $\mathfrak{G}$ and summing over $m \le X$ with the given conditions, we get

$$\mathfrak{G} \ge |J| \sum_{\substack{m \le X \\ P^-(m(2n_j m+1))>N}} \frac{1_{h(m)=1}}{m} - \sum_{1\le|l|\le K} \hat{B}_K(l)(e(dl)M_{-l}(X) + e(-cl)M_l(X)) - O\left(\frac{\log X}{K}\right),$$

where for any $l \in \mathbb{Z}$ we have put

$$M_l(X) := \sum_{\substack{m \le X \\ P^-(m(2n_j m+1))>N}} \frac{1_{h(m)=1} g(2n_j m+1)^{-l}}{m}.$$

Expressing $1_{h(m)=1} = \frac{1}{kr}\sum_{\nu(kr)} h(m)^\nu$, we have

$$M_l(X) = \frac{1}{kr}\sum_{\nu(kr)} \sum_{\substack{m \le X \\ P^-(m(2n_j m+1))>N}} \frac{h(m)^\nu g(2n_j m+1)^{-l}}{m} =: \frac{1}{kr}\sum_{0\le\nu\le kr-1} M_l(X;\nu),$$

for any $l \in \mathbb{Z}$. Inserting this into our lower bound for $\mathfrak{G}$, we find

$$\begin{aligned}\mathfrak{G} \ge \frac{1}{kr}\sum_{0\le\nu\le kr-1} |J| \sum_{\substack{m \le X \\ P^-(m(2n_j m+1))>N}} \frac{h(m)^\nu}{m} - O((\log X)/K) \\ - \frac{1}{kr}\sum_{0\le\nu\le kr-1}\sum_{1\le|l|\le K} \hat{B}_K(l)(e(dl)M_{-l}(X;\nu) + e(-cl)M_l(X;\nu)) \\ =: \frac{1}{kr}\sum_{0\le\nu\le kr-1} \mathcal{S}_\nu - O((\log X)/K). \qquad (21)\end{aligned}$$

Fix $0 \le \nu \le kr-1$. Now, by hypothesis, g is not pseudopretentious, so that $g^l 1_{P^->N}$ is nonpretentious for all $l \in \mathbb{Z}\backslash\{0\}$. Hence, Theorem 2.1 implies that

$$M_l(X;\nu) = \sum_{m\le X} \frac{(h1_{P^->N})(m)^\nu (g1_{P^->N})(2n_j m+1)^l}{m} = o(\log X),$$

for all $1 \le |l| \le K$. Multiplying by $1/|l|$ and summing over $1 \le l \le K$ in the above estimate, we get

$$\mathcal{S}_\nu = |J| \sum_{\substack{m \le X \\ P^-(m(2n_j m+1))>N}} \frac{h(m)^\nu}{m} + o((\log X)(\log K)),$$

for all $0 \le \nu \le kr-1$. Recall that h^j is nonpretentious for all $1 \le j \le r-1$, and thus h^{tr+j} is nonpretentious for all $0 \le t \le q-1$. In particular, by Theorem 2.1 we have

$$\sum_{m\le X} \frac{h^{tr+v}1_{P^->N}(m)1_{P^->N}(2n_j m+1)}{m} = o(\log X).$$

Now suppose $\nu = vr$, for $v \ne 0$. Then $h^\nu = \chi^v$, and thus splitting m according to residue classes modulo q (noting that $N > q$), we get

$$\sum_{m\le X} \frac{\chi^v(m)1_{P^->N}(m)1_{P^->N}(2n_j m+1)}{m} = \sideset{}{^*}\sum_{a \pmod q} \chi(a)^v \sum_{\substack{m\le X \\ m\equiv a \pmod q}} \frac{1_{P^->N}(m)1_{P^->N}(2n_j m+1)}{m}.$$

If $(2n_ja+1,q)=1$ then Lemma 2.4 and partial summation implies that

$$\sum_{\substack{m\le X\\ m\equiv a \pmod q}} \frac{1_{P^->N}(m)1_{P^->N}(2n_jm+1)}{m} = \delta_{N,2n_j,q}\log X + O(4^{\pi(N)}).$$

On the other hand, if $(2n_ja+1,q)>1$ then the sum is 0. It follows that

$$\begin{aligned}\sum_{1\le v\le k-1}\sum_{m\le X}\frac{\chi^v(m)1_{P^->N}(m)1_{P^->N}(2n_jm+1)}{m}&\\ = \delta_{N,2n_j,q}(\log X)\sum_{1\le v\le k-1}\sum_{a \pmod q}\chi(a)^v\chi_0(2n_ja+1)+O_q(4^{\pi(N)})&\\ = o(\log X),&\end{aligned}$$

for x suitably large relative to N, since the Jacobi sum is 0 for all $1\le j\le k-1$. It follows that when $v\neq 0$, we have

$$S_v = o((\log X)(\log K)) = o(\log X),$$

if $K=K(X)$ is chosen to be tending to infinity with X sufficiently slowly, and therefore

$$\mathfrak{G} \ge \frac{|J|}{kr}\sum_{\substack{m\le X\\ P^-(m(2n_jm+1))>N}}\frac{1}{m} - o(\log X).$$

Next, suppose $t\neq 0$. Arguing as before, this time invoking a minorization like (20) with $1_I(n^{it})$ as well and using the triangle inequality, we have (see (21) and the lines preceding it)

$$\begin{aligned}\mathfrak{G} &\ge \frac{1}{kr}\sum_{0\le v\le kr-1}|I||J|\sum_{\substack{m\le X\\ P^-(m(2n_jm+1))>N}}\frac{h(m)^v}{m} - \frac{1}{kr}\sum_{0\le v\le kr-1}\sum_{\substack{0\le|l_1|,|l_2|\le m\\ \max\{|l_1|,|l_2|\}\ge 1}}\frac{1}{R(l_1,l_2)}\sum_{u,v\in\{-1,+1\}}|M_{ul_1,vl_2}(X;v)|\\ &=: \frac{1}{kr}\sum_{0\le v\le kr-1}T_v,\end{aligned}$$

where $R(l_1,l_2):=\max\{1,|l_1|\}\max\{1,|l_2|\}$ whenever $l_1l_2\neq 0$, and we have set

$$M_{l_1,l_2}(X;v) := \sum_{\substack{m\le X\\ P^-(m(2n_jm+1))>N}}\frac{h(m)^v g(2n_jm+1)^{l_1}m^{il_2t}}{m}.$$

Since $n\mapsto g(n)^{l_2}1_{P^-(n)>N}n^{il_1t}$ is nonpretentious for all $l_2\neq 0$, and $n\mapsto h(n)^v1_{P^-(n)>N}n^{-il_1t}$ is nonpretentious whenever $\max\{|l_1|,|l_2|\}\ge 1$, Theorem 2.1 implies that $M_{l_1,l_2}(X;v)=o(\log X)$ for all such l_1,l_2

and any ν. In particular,

$$T_\nu = |I||J| \sum_{\substack{m\le X\\ P^-(m(2n_jm+1))>N}} \frac{h(m)^\nu}{m} + o\Big((\log X)\sum_{1\le l_1,l_2\le K}\frac{1}{l_1l_2}\Big)$$
$$= |I||J| \sum_{\substack{m\le X\\ P^-(m(2n_jm+1))>N}} \frac{h(m)^\nu}{m} + o((\log X)(\log K)^2).$$

As above, if $\nu \neq 0$ then $T_\nu = o(\log X)$. Hence, we get

$$\mathfrak{G} \ge |I||J| \sum_{\substack{m\le X\\ P^-(m(2n_jm+1))>N}} \frac{1}{m} - o(\log X),$$

for $K = K(X) \to \infty$ sufficiently slowly. Invoking Lemma 2.4 (with $q = 1$) and partial summation, we get that when X is sufficiently large relative to N,

$$\sum_{\substack{m\le X\\ P^-(m(2n_jm+1))>N}} \frac{1}{m} = \delta_{N,B}\log X + o(\log X).$$

Since $|I| \ge |I||J| \gg \varepsilon^2$, this shows that

$$\mathfrak{G} \gg \frac{\varepsilon^2}{kr}\frac{\Phi_{N,B}(X)}{X}\log X.$$

Hence, assuming (as we may) that $\varepsilon \ll (kr)^{-2}$, the set of $n = 2mn_j$ with $h(m) = 1$, $|m^{it} - 1| < \varepsilon/4$, $|g(mn_j + 1) - w| < \varepsilon/4$ and $P^-(m(2n_jm + 1)) > N$ (i.e., satisfying properties (i)–(iv) above) intersects in a set of positive upper density with the set of $n \le 2n_jX$ where $f(m) = h(m)m^{it} + O(\varepsilon^2)$. It thus follows that for a set of integers n with positive upper density, we have

$$|f(n) - z| = |f(m) - 1| + O(\varepsilon^2) = |h(m)m^{it} - 1| + O(\varepsilon^2) = |m^{it} - 1| + O(\varepsilon^2) < \frac{\varepsilon}{3}$$

and $|g(n + 1) - w| = |g(2mn_j + 1) - w| < \varepsilon/4$. Consequently, $(f(n), g(n + 1)) \in B_{2\varepsilon/3}((z, w))$ which contradicts our initial assumption. This contradiction completes the proof. □

Having established the above proposition, we now know that if $\overline{\{(f(n), g(n+1))\}_n} \neq \mathbb{T}^2$ then either both f and g are pseudopretentious, or neither is. To complete the proof of Proposition 2.6, therefore, we shall show that the latter case is not possible.

Proof of Proposition 2.6. Let us assume for the sake of contradiction that one of f and g are not pseudopretentious. Proposition 2.9 implies that, then, both f *and* g are not pseudopretentious. Let $a, b : \mathbb{N} \to \mathbb{R}/\mathbb{Z}$ be completely additive functions for which $f(n) = e(a(n))$ and $g(n) = e(b(n))$. For each $M \in \mathbb{N}$ let

$$F_M(x_1, x_2) := \frac{1}{\log M} \sum_{\substack{n\le M\\ (a(n),b(n+1))\in[0,x_1]\times[0,x_2]}} \frac{1}{n},$$

and $G_M(x_1, x_2) = x_1x_2$ identically for all M. Applying Theorem 2 of [Niederreiter and Philipp 1973] gives that for any $K \geq 1$ we have

$$D_M^*(\{(f(n), g(n+1))\}_n) = \sup_{x_1,x_2\in[0,1)} |F_M(x_1, x_2) - G_M(x_1, x_2)| \tag{22}$$

$$\ll \frac{1}{K+1} + \frac{1}{\log M} \sum_{\substack{0\leq|m_1|,|m_2|\leq K\\(m_1,m_2)\neq(0,0)}} \frac{1}{R(m_1, m_2)} \left| \sum_{n\leq M} \frac{f(n)^{m_1} g(n+1)^{m_2}}{n} \right|, \tag{23}$$

where $R(m_1, m_2) = \max\{1, |m_1|\} \max\{1, |m_2|\}$ whenever $m_1m_2 \neq 0$. Now, by hypothesis, there is an $\varepsilon > 0$ and a point $(z, w) \in \mathbb{T}^2$ such that $(f(n), g(n+1)) \notin B_\varepsilon((z, w))$ for all large n. In particular, this means that if I_z and J_w are, respectively, the symmetric arcs of length 2ε about z and about w, then

$$\begin{aligned} D_M(\{f(n), g(n+1)\}_n) &= \sup_{\substack{I,J\subset\mathbb{T}\\ \text{arcs}}} \left| \frac{1}{\log M} \sum_{\substack{n\leq M\\(f(n),g(n+1))\in I\times J}} \frac{1}{n} - |I||J| \right| \\ &\geq \left| \frac{1}{\log M} \sum_{\substack{n\leq M\\(f(n),g(n+1))\in I_z\times J_w}} \frac{1}{n} - B_\varepsilon((z, w)) \right| \\ &= |I_z||J_w| \\ &\gg \varepsilon^2. \end{aligned}$$

From (14), we get $D_M^*(\{(f(n), g(n+1))\}_n) \gg \varepsilon^2$. Combining this with (22) and choosing $K = \lfloor C/\varepsilon^2 \rfloor$ with $C > 0$ a sufficiently large constant, the pigeonhole principle implies that for some pair $(m_1, m_2) \neq (0, 0)$ with $|m_1|, |m_2| \leq K$

$$\left| \sum_{n\leq M} \frac{f(n)^{m_1} g(n+1)^{m_2}}{n} \right| \gg_\varepsilon \log M.$$

Note that since neither f nor g is assumed to be pseudopretentious, we must have $m_1m_2 \neq 0$ as otherwise Halász' theorem would yield

$$\left| \sum_{n\leq M} \frac{f(n)^{m_1}}{n} \right|, \left| \sum_{n\leq M} \frac{g(n)^{m_2}}{n} \right| = o(\log M).$$

Thus, we may apply Proposition 2.2, which gives that f and g are indeed *both* pseudopretentious and the result follows. □

3. Proof of Proposition 1.7

Before addressing Propositions 1.5 and Theorem 1.6, we pause to prove Proposition 1.7, as its proof will be related to several subcases of Theorem 1.6 especially. Write $f(n) = f_0(n)n^{it}$ and $g(n) = g_0(n)n^{it'}$, where f_0 and g_0 are completely multiplicative functions taking values in μ_k and μ_l, respectively. We show in this section that if $\overline{\{f(n), g(n+1)\}_n} \neq \mathbb{T}^2$ then $t/t' \in \mathbb{Q}$. To that end, we will need the following lemma, which is a standard extension of a "repulsion" estimate for the pretentious distance due to Granville and

Soundararajan [2007] (see, for instance, Lemma C.1 in [Matomäki et al. 2015] for the case $k = 2$). We recall that for arithmetic functions $F, G : \mathbb{N} \to \mathbb{U}$, where $\mathbb{U}$ denotes the closed unit disc, and $x \geq 2$, we set

$$\mathbb{D}(F, G; x) := \left(\sum_{p \leq x} \frac{1 - \mathrm{Re}(F(p)\overline{G(p)})}{p} \right)^{1/2}.$$

Lemma 3.1. *Let $k \geq 1$ and let $f : \mathbb{N} \to \mu_k$ be a multiplicative function. Then*

$$\inf_{|\tau| \leq x} \mathbb{D}(f, n^{i\tau}; x) \geq \frac{1}{2k} \min\{\sqrt{\log_2 x}, \mathbb{D}(f, 1; x)\} - O_k(1).$$

Proof. We first assume that $\tau > 1$. The triangle inequality [Granville and Soundararajan 2007] gives

$$k\mathbb{D}(f, n^{i\tau}; x) \geq \mathbb{D}(f^k, n^{i\tau k}; x) = \mathbb{D}(1, n^{i\tau k}; x). \tag{24}$$

Now, the Vinogradov–Korobov zero-free region (see, for instance, Section 9.5 of [Montgomery 1994]) gives that

$$|\zeta(1 + 1/\log x + i\tau)| \ll (\log(2 + |\tau|))^{0.67} \text{ for all } 1 \leq |t| \leq x^2.$$

It follows that

$$\begin{aligned} \mathbb{D}(1, n^{i\tau k}; x)^2 &= \sum_{p \leq x} \frac{1 - \mathrm{Re}(p^{-ik\tau})}{p} \geq \log_2 x - \log \left| \zeta\left(1 + \frac{1}{\log x} + i\tau k\right) \right| - O(1) \\ &\geq \log_2 x - 0.67 \log_2 |k\tau| - O(1) \geq 0.33 \log_2 x - O_k(1). \end{aligned}$$

Inserting this estimate into (24), we get that

$$\mathbb{D}(f, n^{i\tau}; x) \geq \frac{1}{2k} \sqrt{\log_2 x} - O_k(1) \tag{25}$$

in this case. Suppose now that $|\tau| \leq 1$. We may assume that

$$\mathbb{D}(1, n^{i\tau k}; x) < \tfrac{1}{2}\mathbb{D}(f, 1; x) \tag{26}$$

since in the opposite case the lemma follows immediately from (24). By the prime number theorem, for each $|u| \leq k$ the asymptotic

$$\mathbb{D}(1, n^{iu}; x)^2 = \log(1 + |u| \log x) + O_k(1)$$

holds. In particular,

$$\mathbb{D}(1, n^{i\tau k}; x)^2 = \log(1 + k|\tau| \log x) + O_k(1) = \log(1 + |\tau| \log x) + O_k(1) = \mathbb{D}(1, n^{i\tau}; x)^2 + O_k(1),$$

for $|\tau| \leq 1$. Applying the triangle inequality once again, then invoking (24) and (26), we find that

$$\begin{aligned} \mathbb{D}(f, 1; x) &\leq \mathbb{D}(1, n^{i\tau}; x) + \mathbb{D}(f, n^{i\tau}; x) \\ &= \mathbb{D}(1, n^{i\tau k}; x) + \mathbb{D}(f, n^{i\tau}; x) + O_k(1) \\ &< \tfrac{1}{2}\mathbb{D}(f, 1; x) + k\mathbb{D}(f, n^{i\tau}; x) + O_k(1). \end{aligned}$$

Combined with (25), this yields the claim. □

The next result shows that for most n with $P^-(n(Bn+1))$ large, we can reduce to the case that $f_0 = h_1$ and $g_0 = h_2$, where h_1, h_2 defined as in Proposition 2.6. This will allow us to use the results of previous sections in proving Proposition 1.7.

Lemma 3.2. *Let $f(n) = f_0(n)n^{it}$ and $g(n) = g_0(n)n^{it'}$, where f_0 and g_0 are completely multiplicative functions taking values in μ_k and μ_l, respectively. Let $\eta > 0$ be sufficiently small (in terms of k and l). Then there exist pseudocharacters h_1, h_2 with the following properties. Let B be an even integer. Then there is a subsequence $\{x_j\}_j$ and a parameter N depending at most on η such that if j is sufficiently large (in terms of η) then the following holds: for all but $O(\eta\Phi_{N,B}(x_j))$ integers $n \le x_j$ with $P^-(n(Bn+1)) > N$, we have $f_0(n) = h_1(n)$ and $g_0(Bn+1) = h_2(Bn+1)$.*

Proof. Since $f(n)^k = n^{it}$, we have

$$\sum_{n\le x}\frac{f(n)^k\overline{f(n+1)^k}}{n} = \sum_{n\le x}\frac{1}{n} + O(k|t|) \gg \log x.$$

A similar estimate holds with g^l in place of f^k. Thus, by Proposition 2.2 we have that f and g are both pseudopretentious, and that there are real numbers $t_1, t_2 \in \mathbb{R}$ with $t_1, t_2 = O(1)$, primitive characters χ_1 and χ_2 with conductors q_1 and q_2 and $q_1, q_2 = O(1)$ and completely multiplicative functions h_1, h_2 such that $h_1(n)^k = \chi_1(n/(n, q_1^\infty))$ and $h_2(n)^l = \chi_2(n/(n, q_2^\infty))$, and such that

$$\mathbb{D}(f_0, h_1n^{i(t_1-t)}; x) = \mathbb{D}(f, h_1(n)n^{it_1}; x) \ll 1$$
$$\mathbb{D}(g_0, h_2n^{i(t_2-t')}; x) = \mathbb{D}(f, h_1(n)n^{it_1}; x) \ll 1.$$

Suppose that when χ_1 and χ_2 do not vanish, they take values in μ_r and μ_s, respectively. We begin by showing that $t_1 = t$ and $t_2 = t'$. Applying Lemma 3.1 with $F = f_0\overline{h_1}$,

$$\begin{aligned} 1 \gg \mathbb{D}(F, n^{i(t_1-t)}; x) &\ge \inf_{|u|\le x}\mathbb{D}(F, n^{iu}; x) \\ &\ge \frac{1}{2kr}\min\{\sqrt{\log_2 x}, \mathbb{D}(F, 1; x)\} - O_k(1) \\ &= \frac{1}{2kr}\mathbb{D}(F, 1; x) - O_k(1), \end{aligned}$$

It follows that $\mathbb{D}(f_0, h_1; x) \ll_{k,r} 1$, and hence the triangle inequality yields

$$\mathbb{D}(1, n^{i(t_1-t)}; x) \le \mathbb{D}(f_0\overline{h_1}, 1; x) + \mathbb{D}(f_0\overline{h_1}, n^{i(t_1-t)}; x) \ll_{k,r} 1.$$

Arguing as in the proof of Lemma 3.1 it is clear that for x sufficiently large,

$$\mathbb{D}(1, n^{i(t_1-t)}; x) = \log_2 x - \log\left|\zeta\left(1+\frac{1}{\log x}+i(t_1-t)\right)\right| + O(1) \ge \log_2 x - O\left(1+\log\left(\frac{\log x}{1+|t_1-t|\log x}\right)\right)$$

can only be bounded if $t_1 = t$. Similarly, we must take $t_2 = t$. Now, set $F_0(n) := f_0(n)\overline{h_1(n)}$ and $G_0(n) := g_0(n)\overline{h_2(n)}$. Lemma 2.11 implies that for a suitable choice of N (depending only on η, f and

g) and a common subsequence $\{x_j\}_j$ (see Remark 2.12 for an explanation of why this exists), we get that for large enough j,

$$|\{n \le x_j : P^-(n(Bn+1)) > N, |F_0(n) - 1| > \eta\}| \ll \eta \Phi_{N,B}(x_j)$$
$$|\{n \le x_j : P^-(n(Bn+1)) > N, |G_0(Bn+1) - 1| > \eta\}| \ll \eta \Phi_{N,B}(x_j).$$

It follows that for all but $O(\eta \Phi_{N,B}(x_j))$ integers $n \le x_j$ with $P^-(n(Bn+1)) > N$, we have that $F_0(n) = 1 + O(\eta)$, and $G_0(Bn+1) = 1 + O(\eta)$, for N sufficiently large. But since F_0 and G_0 take discrete values, for these n, $F_0(n) = G_0(Bn+1) = 1$ when η is sufficiently small (in terms of k and l). In light of the definition of F_0 and G_0, this implies the claim. □

Proof of Proposition 1.7. We suppose that $f(n)^k = n^{it}$ and $g(n) = n^{it'}$, for all $n \in \mathbb{N}$. Let $f(n) = f_0(n)n^{it_1}$ and $g(n) = g_0(n)n^{it_2}$, where $f_0^k = 1 = g_0^l$, $t_1 := t/k$ and $t_2 := t'/l$. As $(n+1)^{it_2} = n^{it_2} + O(|t_2|/n)$, it is equivalent to consider when the sequence of pairs $(f_0(n)n^{it_1}, g_0(n+1)n^{it_2})$ is, or is not dense for n sufficiently large. We first suppose that $t/t' = r_1/s_1 \in \mathbb{Q}\backslash\{0\}$. In this case, since

$$\overline{(f(n)^{ks_1}, g(n+1)^{lr_1})} = \overline{(n^{is_1t_1}, (n+1)^{ir_1t_2})} = \{(z, z) \in \mathbb{T}^2\} \ne \mathbb{T}^2,$$

we have $\overline{(f(n), g(n+1)} \ne \mathbb{T}^2$. Suppose now that $t/t' \notin \mathbb{Q}$, and that $\{(f_0(n)n^{it_1}, g(n+1)n^{it_2})\}_n$ is not dense in $\mathbb{T}^2$. By Proposition 2.6, we know that f and g must be pseudopretentious. Thus, let h_1 and h_2 be pseudocharacters with conductors q_1 and q_2 respectively, and $u, v \in \mathbb{R}$ such that f is $h_1 n^{iu}$-pretentious and g is $h_2 n^{iv}$-pretentious. From the proof of Lemma 3.2, it follows that $u = t_1$ and $v = t_2$. Now, replacing n by Bn, where $B = 2q_1q_2$, it follows that

$$(f_0(n)n^{it_1}, g_0(Bn+1)n^{it_2}) \notin B_\varepsilon(z\overline{f(B)}B^{-it_1}, wB^{-it_2}).$$

Set $(z', w') = (z\overline{f(B)}B^{-it_1}, wB^{-it_2})$. Now, according to Lemma 3.2 (applied with $\eta = \varepsilon^3$), we can choose x sufficiently large (belonging to a prescribed infinite subsequence) and N suitably large in terms of ε, such that $f_0(n) = h_1(n)$ and $g_0(Bn+1) = h_2(Bn+1)$, for all but $O(\varepsilon^3 \Phi_{N,B}(x))$ elements $n \le x$ with $P^-(n(Bn+1)) > N$. Hence, we know that

$$(h_1(n)n^{it_1}, h_2(Bn+1)n^{it_2}) \notin B_\varepsilon((z', w'))$$

for all but $O(\varepsilon^3 \Phi_{N,B}(x))$ of those n. On the other hand by Proposition 2.3 there are $\gg \varepsilon^2/(kl)\Phi_{N,B}(x)$ numbers $n \le x$ on which $\max\{|n^{it_1} - z'|, |n^{it_2} - w'|\} \le \varepsilon/4$ and $h_1(n) = h_2(Bn+1) = 1$, whence

$$(h_1(n)n^{it_1}, h_2(Bn+1)n^{it_2}) = (n^{it}, n^{it'}) \in B_{\varepsilon/2}((z', w')).$$

This must intersect with the set of n where $(f_0(n), g_0(Bn+1)) = (h_1(n), h_2(Bn+1))$, provided that ε is sufficiently small (in terms of k and l alone). This contradicts the earlier claim, and proves the proposition. □

4. The eventually rational case

In this section, we prove Proposition 1.5. That is, we assume that f and g are both eventually rational functions, i.e., for some N sufficiently large there is an $m \in \mathbb{N}$ such that for all primes $p > N$ we have $f(p)^m = 1$, and that $\{f(n)\}_n$ and $\{g(n)\}_n$ are both dense. Appealing to Proposition 2.6 once again, we may assume that f is $h_1 n^{it_1}$-pretentious and g is $h_2 n^{it_2}$-pretentious, k and l are minimal positive integers such that $h_1^k = \tilde{\chi}_1$ and $h_2^l = \tilde{\chi}_2$. As above, we let r and s denote the respective orders of $\tilde{\chi}_1$ and $\tilde{\chi}_2$. To prove Proposition 1.5, we need the following technical result, which extends Lemma 2.12 of [Klurman and Mangerel 2018] (which is the special case $M' = 1$ and $M'' = P_N := \prod_{p \leq N} p$). In what follows, for a positive integer n we write $\mathrm{rad}(n) := \prod_{p \mid n} p$ to denote the squarefree kernel of n.

Lemma 4.1. *Let $N > 2q_1q_2$ be a positive integer, and let m, m', m'' be coprime squarefree positive integers such that $P_N = mm'm''$, with $(m', 2q_1q_2) = 1$. Let M' and M'' be integers with $\mathrm{rad}(M') = m'$ and $\mathrm{rad}(M'') = m''$ and M'' odd, and suppose M is a multiple of $2q_1q_2$ with $\mathrm{rad}(M/2q_1q_2) = m$. Then*

$$\sum_{\substack{n \leq x \\ M \mid n}} \frac{1_{n \equiv -\overline{M}(M')} 1_{(n(n+1), M'') = 1} 1_{h_1(n)=1} 1_{h_2(n+1)=1}}{n} = \left(\frac{1}{MM'klrs} \prod_{p \mid M''} \left(1 - \frac{2}{p}\right) + o(1) \right) \log x.$$

Proof. The proof of Lemma 4.1 is similar to that of Lemma 2.12 of [Klurman and Mangerel 2018], and we merely sketch the proof here. If $\mathcal{S}$ denotes the LHS above then, as M, M' and M'' are coprime, using orthogonality modulo M' gives

$$\mathcal{S} = \frac{1}{krls} \sum_{\substack{n' \leq x/M \\ n'M \equiv -1 (M')}} \frac{1_{(n'(Mn'+1), M'') = 1} 1_{h_1(n'M)=1} 1_{h_2(n'M+1)=1}}{n'M} \left(\sum_{0 \leq a \leq kr-1} \sum_{0 \leq b \leq ls-1} h_1(n'M)^a h_2(n'M+1)^b \right)$$

$$= \frac{1}{krls} \sum_{\chi (M')} \frac{\chi(M)}{\phi(M')} \sum_{0 \leq a \leq kr-1} h_1(M)^a \sum_{0 \leq b \leq ls-1} \sum_{n' \leq x/M} \frac{\chi(n') h_1(n')^a h_2(Mn'+1)^b 1_{(n', M'')=1} 1_{(Mn'+1, M'')=1}}{n'M}.$$

One proceeds as in the proof of Lemma 2.12 in [Klurman and Mangerel 2018] (using Theorem 2.1) to show that whenever $ab \neq 0$, the contributions to the full sum are $o(\log x)$, irrespective of χ. In the remaining case $a = b = 0$, and χ is *nonprincipal* modulo M', the contribution to the full sum is $o(\log x)$ as well. It thus follows that

$$\mathcal{S} = \frac{1}{M\phi(M')krls} \sum_{n' \leq x/M} \frac{1_{(n', M''M')=1} 1_{(Mn'+1, M'')=1}}{n'} + o(\log x)$$

$$= \frac{1}{MM'krls} \prod_{p \mid M''} \left(1 - \frac{2}{p}\right) \log x + o(\log x),$$

the main term arising from the coprimality of M and M'' provided that x is large enough in terms of M (using, e.g., the fundamental lemma of the sieve). This proves the claim. □

Proof of Proposition 1.5. Let N be such that for all $p > N$, we have $f(p)^k = g(p)^l = 1$. Assume for contradiction that $(f(n), g(n+1)) \notin B_\varepsilon(z, w)$, for some $\varepsilon > 0$ and some pair $(z, w) \in \mathbb{T}^2$. By Proposition 2.6,

we know that $\mathbb{D}(f, h_1 n^{it}; x), \mathbb{D}(g, h_2 n^{it'}; x) \ll_\varepsilon 1$, for some h_1, h_2 completely multiplicative taking values in roots of unity. By multiplying the characters χ_1 and χ_2 corresponding to h_1 and h_2 by the principal character modulo P_N, we may assume that h_1 and h_2 are equal to 1 at all of the primes $p \le N$. Furthermore, as $f\overline{h_1}$ and $g\overline{h_2}$ also take values in roots of unity for all but finitely many primes, the proof of Lemma 3.1 implies that $t = t' = 0$. Now since $\{f(n)\}_n$ and $\{g(n)\}_n$ are dense, and $f(p)$, $g(p)$ take values in a set of bounded order roots of unity for all but finitely many primes, we know that there is (at least) a prime p and a prime p' at which the argument of $f(p) = e(\alpha)$ and $g(p') = e(\beta)$, with $\alpha, \beta \notin \mathbb{Q}$. The additional hypothesis of the proposition implies that, in fact, $p \ne p'$. With $\varepsilon > 0$ and $z, w \in \mathbb{T}$ given above, we may apply Kronecker's theorem (with α and with β, separately) to get $a, b \in \mathbb{N}$ such that

$$\|(f(p)^a, g(p')^b) - (z\overline{f(2q_1q_2)}, w)\|_{\ell_1} < \varepsilon.$$

Now set $B = 2q_1q_2p^a$. Thus, in the remainder of the proof, in order to achieve a contradiction it will suffice to check that we can find n such that $f(Bn) = f(2q_1q_2)f(p)^a$ and $g(nB+1) = g(p')^b$. Let $\eta > 0$ be a parameter to be chosen depending at most on ε, k, r, l and s. By Lemma 2.11 (see also Remark 2.12), there is a suitable infinite sequence of x (and a possibly larger choice of N) for which we have $f(n) = h_1(n)$ and $g(Bn+1) = h_2(Bn+1)$ for all but $O(\eta\Phi_{N,B}(x/B))$ integers $n \le x$ with $P^-(n(Bn+1)) > N$. Thus, it suffices to show that there are $\gg \varepsilon^2\Phi_{N,B}(x/B)$ integers $n \le x$ with $P^-(n(Bn+1)) > N$ that satisfy:

(i) $f(nB/(nB, P_N^\infty)) = 1 = g((nB+1)/(nB+1, P_N^\infty))$.

(ii) $(p')^b \parallel (nB+1)$ but $(nB(nB+1), P_N/pp') = 2q_1q_2$.

To do this, we shall apply Lemma 4.1. Let $m = p$, $m' = p'$ and $m'' = P_N/pp'$, and set $M = B$, $M' = (p')^b$ and $M'' = m''$. As $f(nB/(nB, P_N^\infty)) = h_1(nB)$ and $g((Bn+1)/(Bn+1, P_N^\infty)) = h_2(Bn+1)$ for all but $O(\varepsilon^3\Phi_{N,B}(x/B))$ choices of $n \le x/B$, Lemma 4.1 implies that the number of $n \le x$ with $B \mid n$ and satisfying the other required properties is

$$\gg \frac{1}{B(p')^{b-1}(p'-1)krls} \prod_{\substack{p'' \le N \\ p'' \nmid pp'B}} \left(1 - \frac{2}{p}\right) \gg \frac{1}{(p')^b krls}\Phi_{N,B}(x/B).$$

Choosing $\eta = C((p')^b krls)^{-1}$ with a sufficiently small constant $C = C(\varepsilon) > 0$ (noting that this quantity depends only on ε) implies the claim. □

5. Some preliminary cases of Theorem 1.6

In this section, we shall dispose of several special cases of Theorem 1.6. In this way, we shall be able to reduce our work in proving Theorem 1.6 in Section 6 to focusing on functions with certain prescribed behavior. In this section, we shall assume that either $f(p)^k = p^{it}$ or $g(p)^l = p^{it'}$ for all *sufficiently large* primes p. Since the argument in the first case (i.e., with f) is symmetric to that with g, we shall focus only on the case with f. We consider three subcases of this case, according to the behavior of g:

subcase (i): We assume that $f(p)^k = p^{it}$ for all large primes p, and g is an eventually rational function (see Lemma 5.1).

subcase (ii): We assume that $f(p)^k = p^{it}$ holds *only* for large primes p and $g(p)^l = p^{it'}$ holds for large primes p (see Lemma 5.2 for a more precise formulation).

subcase (iii): We assume that $f(p)^k = p^{it}$ for *all* p, and $g(p)^l = p^{it'}$ *only* for large p (see the discussion following the proof of Lemma 5.2). In what follows, we recall that for each $n \in \mathbb{N}$, $f_0(n) := f(n)n^{-it/k}$ an $g_0(n) := g(n)n^{-it'/l}$.

Lemma 5.1. *Suppose there are positive integers N and k and a real t such that for all $p > N$, $f(p)^k = p^{it}$. Suppose moreover that g is eventually rational. Then $\{(f(n), g(n+1))\}_n$ is dense in $\mathbb{T}^2$.*

The same result holds with the roles of f and g reversed (and replacing $n \mapsto n+1$ in the analysis below by $n \mapsto n-1$).

Proof. By choosing N larger if necessary, and replacing k by some multiple of k, we may assume that $g(p)^k = 1$ for all $p > N$. Now suppose $(f(n-1), g(n)) \notin B_\varepsilon(z, w)$ for n sufficiently large. By Proposition 2.6, f and g are both pseudopretentious, and following the argument in the previous section, one can show that f is $h_1 n^{it/k}$-pretentious, with h_1 a pseudocharacter with conductor q_1, and similarly g is h_2-pretentious where h_2 is a pseudocharacter with conductor q_2.[11] We may assume that $t \neq 0$ here, otherwise f and g are both eventually rational, which is a case dealt with by Proposition 1.5, proven in the previous section. Since $\{g(n)\}_n$ is dense, but with rational argument for all but finitely many primes, there exists a prime p be such that $g(p) = e(\alpha)$ with $\alpha \notin \mathbb{Q}$. We now fix an even integer $B = 2q_1q_2p^r$, where the exponent r is chosen (via Kronecker's theorem) such that $g(p)^r = \overline{g(2q_1q_2)}w + O(\varepsilon^2)$. Now, if m is chosen so that $P^-(m(Bm-1)) > N$, it follows that for m sufficiently large,

$$f(Bm-1) = f_0(Bm-1)(Bm-1)^{it/k} = f_0(Bm-1)B^{it/k}m^{it/k} + O(\varepsilon^2).$$

Thus, it follows that

$$(f_0(Bm-1)m^{it/k}, g_0(m)) \notin B_{\varepsilon/2}(zB^{-it/k}, 1) \tag{27}$$

for all m sufficiently large with $P^-(m(Bm-1)) > N$. Now, by Lemma 3.2,[12] for all but $O(\varepsilon^3\Phi_{N,B}(x))$ integers $m \leq x$ (with x chosen from appropriate infinite increasing sequence and N suitably large) with $P^-(m(Bm-1)) > N$ we have

$$(f_0(Bm-1), g_0(m)) = (h_1(Bm-1), h_2(m)) + O(\varepsilon^2).$$

[11] An application of Lemma 3.1, as in the previous section, implies that $g\bar{h}_2$, which takes finite roots of unity as values, is 1-pretentious.

[12] Strictly speaking, Lemmata 3.2 and 2.3 deal only with the pair of linear forms $(n \mapsto n, n \mapsto Bn+1)$; however, the same results can be derived with $(m \mapsto Bm-1, m \mapsto m)$ with minimal change to the proofs.

Moreover, by Proposition 2.3 there are $\gg \varepsilon^2 \Phi_{N,B}(x)$ such integers for which $(h_1(Bm-1), h_2(m)) = (1, 1)$, and such that $m^{it/k} = zB^{-it/k} + O(\varepsilon^2)$. Hence, it follows that

$$(f_0(Bm-1)m^{it/k}, g_0(m)) = (h_1(Bm-1)m^{it/k}, h_2(m)) = (zB^{-it/k}, 1) + O(\varepsilon^2),$$

for a positive upper density set of m. But this contradicts (27). The contradiction implies

$$\overline{\{f(n-1), g(n)\}_n} = \overline{\{f(n), g(n+1)\}_n} = \mathbb{T}^2,$$

as claimed. □

Lemma 5.2. *Suppose there are positive integers N, k, l and real numbers t, t' such that for all $p > N$, $f(p)^k = p^{it}$ and $g(p)^l = p^{it'}$, but for any $m \in \mathbb{N}$ and any $u \in \mathbb{R}$ there is an integer n such that $f(n)^m \neq n^{iu}$. Then $\{(f(n), g(n+1))\}_n$ is dense.*

Proof. If $t/t' \notin \mathbb{Q}$ then the proof is the same as the corresponding case in the proof of Proposition 1.7 (which only used data about integers n such that $n(Bn+1) > N$ for suitable B). Conversely, suppose $t = at'/b$ for some $a, b \in \mathbb{Z}$, with $b \neq 0$, and suppose there is an $\varepsilon > 0$ and a pair $(z, w) \in \mathbb{T}^2$ such that $(f(n), g(n+1)) \notin B_\varepsilon(z, w)$ for all n large. We may assume that $tt' \neq 0$, since otherwise f or g is eventually rational, and this was covered in the previous lemma. Furthermore, since we may replace k by kb and l by la, we may assume that $t = t'$. By hypothesis, we can choose a prime $p_0 \leq N$ for which $f(p_0)^k / p_0^{it}$ is not a root of unity of any order. Indeed, if every $p \leq N$ satisfied $f(p)^k / p^{it} \in \mu_m$ for some $m \geq 1$, we could replace k with km and t with tm so that $f(p)^{km} = p^{itm}$. Making these replacements for k and for t iteratively at every prime $p \leq N$ would imply that $f(n)^{k'} = n^{iu}$ for *all* n, where $k' \in \mathbb{N}$ and $u \in \mathbb{R}$. This contradicts our initial hypothesis regarding f. Thus, as per the above paragraph we can choose $p_0 \leq N$ such that $f(2p_0)(2p_0)^{-it/k} = e(\alpha)$, where $\alpha \notin \mathbb{Q}$. We may thus choose ℓ such that

$$(f(2p_0)(2p_0)^{-it})^\ell = e(\alpha\ell) = z\overline{w} + O(\varepsilon^2).$$

Now,[13] we pick m with $P^-(m((2p_0)^\ell m + 1)) > N$ and such that $m^{it} = w + O(\varepsilon^2)$. By assumption, it follows that

$$g((2p_0)^\ell m + 1) = ((2p_0)^\ell m + 1)^{it} = (2p_0)^{i\ell t} m^{it} + O(\varepsilon^2),$$

for sufficiently large m. Then, setting $n = (2p_0)^\ell m$ and assuming that m is sufficiently large, we have

$$\begin{aligned}(f(n), g(n+1)) &= (f(2p_0)^\ell m^{it}, (n+1)^{it}) \\ &= n^{it}((f(2p_0)(2p_0)^{-it})^\ell, 1) + O(\varepsilon^2) \\ &= w \cdot (z\overline{w}, 1) + O(\varepsilon^2) \\ &= (z, w) + O(\varepsilon^2),\end{aligned}$$

in contradiction to the claim. □

[13] This can be done, for example, by applying Proposition 2.3 with every pair of values of $h_1(n)$ and $h_2(Bn+1)$, then combining all of these contributions.

Turning to subcase (iii), it remains to consider the case that $f(n)^k = n^{it}$ for some $k \in \mathbb{N}$ and $t \in \mathbb{R}$, and $g(p)^l = p^{it'}$ for all $p > N$ but such that for each $m \in \mathbb{N}$ and $u \in \mathbb{R}$ there is $n \in \mathbb{N}$ with $g(n)^m \neq n^{iu}$. The argument is symmetric to that given in subcase (ii) (perhaps up to considering the sequence $\{f(Bn-1), g(n)\}_n$, for a suitable choice of B). We leave the details to the interested reader.

6. Proof of Theorem 1.6: The remaining cases

In this section, we prove Theorem 1.6, except in the exceptional cases that were dealt with in the previous section. By symmetry, we may assume that for one of the functions, say f, we have that for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$ there are infinitely many primes p such that $f(p)^k \neq p^{it}$. First, suppose for contradiction that there is an $\varepsilon > 0$ and a point $(z, w) \in \mathbb{T}^2$ such that

$$(f(n), g(n+1)) \notin B_\varepsilon((z,w)) \text{ for all sufficiently large} n.$$

Applying Proposition 2.6, we know that f and g are, respectively, $h_1 n^{it}$- and $h_2 n^{it'}$-pseudopretentious, where the conductors of the pseudocharacters h_1 and h_2 are q_1 and q_2, respectively. We need two technical results in order to proceed in the proof of Theorem 1.6. The first will allow us to simultaneously control the angular distribution of the values of the irrational function f at prime powers p^m, as well as the angular distribution of p^{imt}, for $t \in \mathbb{R}$.

Lemma 6.1. *Let $f : \mathbb{N} \to \mathbb{T}$ be completely multiplicative function such that for all $t \in \mathbb{R}$ and $l \in \mathbb{N}$ there are infinitely many primes p such that $f(p)^l \neq p^{it}$. Let $z \in \mathbb{T}$, $u \in \mathbb{R}$ and $\delta, \eta \in (0,1)$. Let $I \subset \mathbb{T}$ be the symmetric arc about 1 with length 2δ. Then for any $k, N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ with $P^-(n) > N$ and $m \in \mathbb{N}$ such that the following hold*:

(i) $|f(n)^m - z| < \eta$.

(ii) $n^{ium} \in I$.

(iii) $k \mid m$.

Moreover if $u \neq 0$ then for any $w \in \mathbb{T}$ we can choose I to be a symmetric arc about w of length 2δ.

Proof. We consider two cases. First, suppose there is a prime $p > N$ for which $f(p)$ and p^{iu} are such that if $f(p) = e(\alpha)$ and $p^{iu} = e(\beta)$ then $\{1, \alpha, \beta\}$ are $\mathbb{Q}$-linearly independent (thus, $u \neq 0$ necessarily). Equivalently, $\{1, k\alpha, k\beta\}$ is $\mathbb{Q}$-linearly independent. In this case, we can apply Kronecker's theorem to find m_0 such that

$$\|(f(p)^{km_0}, p^{iukm_0}) - (z,w)\|_{\ell_1} \leq \min\{\eta, \delta\}^2,$$

and the claim follows with $m = km_0$ and $n = p$. Now, suppose that for all primes $p > N$ we have that $f(p)$ and p^{iu} have $\mathbb{Q}$-linearly dependent arguments. Equivalently, we can find a root of unity ξ_p such that $f(p) = \xi_p p^{iu}$ for each prime $p > N$. We consider two subcases of this case. First, if $\{\xi_p\}_{p>N}$ is a set of roots of unity of *bounded* order, say M, then it follows that $f(p)^{M!} = p^{iuM!}$ for all $p > N$. This contradicts our initial assumption that $f(p)^k \neq p^{it}$ for infinitely many p (with $k = M!$ and $t = uM!$).

Next, suppose that $\{\xi_p\}_{p>N}$ is such that for any $M \geq 1$ we can find a prime $p > N$ such that $\xi_p = e(a/b)$ with $b > Mk$ and $(a, b) = 1$. Pick $M > 2\eta^{-2}$, and choose $p = p(M)$ in this way. For $g, r \in \mathbb{N}$ parameters to be chosen, note that

$$f(p)^{k(gb+r)} = e(ark/b)p^{ik(gb+r)u}.$$

Writing $z = e(\gamma)$, we may select $0 \leq \ell \leq b-1$ such that $\gamma \in [k\ell/b, k(\ell+1)/b]$, whence

$$|\gamma - k\ell/b| \leq k/b \leq 1/M < \tfrac{1}{2}\eta^2.$$

We will thus pick r such that $ar \equiv \ell(b)$, so that $|f(p)^{k(gb+r)} - zp^{ik(gb+r)u}| < \eta^2/2$. Now, if $u = 0$ and $w = 1$ then we are done, as we can take $n = p$ and $m = kr$ (i.e., with $g = 0$). Otherwise, if $u \neq 0$ (and w is not necessarily 1), the sequence $\{p^{ikbug}\}_g$ is dense in $\mathbb{T}$. Thus, having already chosen r, it follows that we can pick g such that $|p^{ikgbu} - wp^{-ikru}| \leq \frac{1}{2}\min\{\eta^2, \delta^2\}$. Hence, we get that

$$\begin{aligned} |p^{ik(gb+r)u} - w| &= |p^{ikgbu} - wp^{-ikru}| \leq \delta^2, \\ |f(p)^{k(gb+r)} - z| &\leq |f(p)^{k(gb+r)} - zp^{-ik(gb+r)u}| + |z||p^{-ik(gb+r)u} - 1| \leq \eta^2. \end{aligned}$$

Thus, selecting $m = k(gb+r)$ and $n = p$ suffices to prove the claim in this case. □

Lemma 6.2. *Let $f, g : \mathbb{N} \to \mathbb{T}$ be completely multiplicative functions such that $\overline{\{f(n)\}}_n = \overline{\{g(n)\}}_n = \mathbb{T}$, but that $\overline{\{(f(n), g(n+1))\}}_n \neq \mathbb{T}^2$, and furthermore that f satisfies the hypotheses of Lemma 6.1. Suppose that f and g are, respectively, $h_1 n^{it}$- and $h_2 n^{it'}$-pretentious, where h_1 and h_2 are pseudocharacters of conductor q_1 and q_2, respectively, and $h_1^{kr} = h_2^{ls} = 1$. Finally, let $N, B \geq 1$, with $2q_1q_2 \mid B$, and let $u \in \mathbb{R}$ and $z \in \mathbb{T}$. Then for any $\eta > 0$ sufficiently small (in terms of u) there is a positive upper density subset $T_z = T_z(\eta)$ of $\mathcal{A}_{N,B,I}(h_1, h_2; 1, 1; u)$ on which $|f(n) - z| < \eta$, where I is the symmetric arc of length 2η about 1.*

In words, the lemma states that we can find a "large" set of integers n satisfying $P^-(n(Bn+1)) > N$ such that $f(n)$ is close to z, $h_1(n) = h_2(n) = 1$, and $n^{iu} \in I$.

Proof. Let $w \in \mathbb{T}$ be a parameter to be chosen (based on t and u). By Lemma 6.1, choose n_0 with $P^-(n_0) > N$ and m a multiple of kr such that $f(n_0)^m = z + O(\eta^2)$ and $(n_0)^{imu} = w + O(\eta^2)$, and set $B' := Bn_0^m$. Since f is pretentious to $h_1(n)n^{it}$, by Lemma 2.11 we can choose x from a suitable infinite sequence (and N slightly larger if necessary) such that for all but $O(\eta^3/(klrsn_0^m)\Phi_{N,B'}(x))$ integers $n \leq x$ with $P^-(n(B'n+1)) > N$ we have $f(n) = h_1(n)n^{it} + O(\eta^2)$. Now, furthermore, Proposition 2.3 gives

$$\left|\mathcal{A}_{N,B',I,I}(h_1, h_2; 1, 1; t, u) \cap \left[1, \frac{x}{n_0^m}\right]\right| \gg \frac{\eta^2}{klrs}\Phi_{N,B'}(x/n_0^m) \asymp \frac{\eta^2}{klrsn_0^m}\Phi_{N,B'}(x),$$

provided that $t/u \notin \mathbb{Q}$. Choosing $w = 1$ in this case, it follows that for all n in this set we have $f(nn_0^m) = zh(n)n^{it} + O(\eta^2) = z + O(\eta^2)$, and moreover $(nn_0^m)^{iu} = 1 + O(\eta^2)$. We define, in the case $t/u \notin \mathbb{Q}$, the set

$$T_z := \{nn_0^m : n \in \mathcal{A}_{N,B',I,I}(h_1, h_2; 1, 1; t, u)\}.$$

If, instead, $t/u = a/b$ and $a < b$ and $\eta^{-1/2} > b/a$ (without loss of generality), we can conclude that if $n^{it} = 1 + O(\eta^2 b/a)$ then $n^{iu} = \zeta_a^c + O(\eta^{3/2})$, where ζ_a is a fixed primitive a-th root of unity, and $0 \leq c \leq a-1$. By Proposition 2.3 once again, we have

$$\left|\mathcal{A}_{N,B,I'}(h_1, h_2; 1, 1; t) \cap \left[1, \frac{x}{n_0^m}\right]\right| \gg \frac{\eta^2}{klrsn_0^m}\Phi_{N,B'}(x),$$

where $I' \subseteq I$ is a symmetric subinterval of I about 1 of length η. By the pigeonhole principle, there is a choice $0 \leq c_0 \leq a-1$ such that $n^{iu} = \zeta_a^{c_0} + O(\eta^{3/2})$ for at least a proportion $\geq 1/a$ of these n. Choosing $w = \zeta_a^{-c_0}$ in this case, it follows that for all $n \in \mathcal{A}_{N,B,I'}(h_1, h_2; 1, 1; t)$ we have $f(nn_0^m) = f(n_0)^m h(n) n^{it} = 1 + O(\eta^2)$ and $(nn_0^m)^{iu} = 1 + O(\eta^{3/2})$. In this case, we define

$$T_z := \{nn_0^m : n \in \mathcal{A}_{N,B,I'}(h_1, h_2; 1, 1; t)\}.$$

We now observe that for any $n' \in T_z$ (regardless of the nature of t/u), $h_1(n') = h_1(n_0)^m h_1(n) = 1$, since $kr \mid m$ and $h^{kr} = 1$. Furthermore, we have $h_2(Bn'+1) = h_2(B'n+1) = 1$ by choice of B', and as we saw above, $f(n') = z + O(\eta^2)$ and $(n')^{iu} = 1 + O(\eta^{3/2})$ simultaneously. It follows that T_z has positive upper density, that $T_z \subseteq \mathcal{A}_{N,B,I}(h_1, h_2; 1, 1; u)$ (provided η is small enough) and that on T_z we have $f(n') = z + O(\eta^2)$. This implies the claim if η is small enough. □

Proof of Theorem 1.6, Part 2. We assume here that f is both irrational, and such that for each $k \in \mathbb{N}$ and $t \in \mathbb{R}$, there are infinitely many primes p for which $f(p)^k \neq p^{it}$; we can do this since the other cases where f is irrational were treated in the previous section. Now, suppose that $\{(f(n), g(n+1)\}_n$ stays outside of an arc of radius ε (in ℓ^1) about $(a, b) \in \mathbb{T}^2$. This means that $|f(n) - a| + |g(n+1) - b| \geq \varepsilon$ for all large n. By Proposition 2.6, there are minimal positive integers k, l, r and s, pseudocharacters h_1, h_2 with conductors $q_1, q_2 = O_\varepsilon(1)$ taking values in μ_{kr} and μ_{ls} respectively, and $t_1, t_2 \in \mathbb{R}$ such that $\mathbb{D}(f, h_1 n^{it_1}; x), \mathbb{D}(g, h_2 n^{it_2}; x) \ll_\varepsilon 1$. Let I be a symmetric interval of length ε^2 about 1, and let B be an integer of our choosing, chosen subject only to the constraint that $2q_1q_2 \mid B$. Consider those $n \in \mathcal{A}_{N,B,I}(h_1, h_2; 1, 1; t_1 - t_2) =: T$. By Proposition 2.3, T has positive upper density. We have three possible scenarios:

(i) There is a positive upper density subset of T on which $|f(Bn) - a| \geq 3\varepsilon/4$ and $|g(Bn+1) - b| < \varepsilon/4$.

(ii) There is a positive upper density subset of T on which $|g(Bn+1) - b| \geq 3\varepsilon/4$ and $|f(Bn) - a| < \varepsilon/4$.

(iii) Except on an upper density zero subset of T we have $|f(Bn) - a| \geq \varepsilon/4$ and $|g(Bn+1) - b| \geq \varepsilon/4$.

Consider alternative (iii). It is clearly true that $|f(n) - a\overline{f(B)}| \geq \varepsilon/4$ for all but a zero upper density subset of $\mathbb{N}$. By Lemma 6.2 (taking $u = t_1 - t_2$ and $z = a\overline{f(B)}$ in the notation there), we can find a positive upper density subset $T_{a\overline{f(B)}} \subset T$ on which $|f(n) - a\overline{f(B)}| < \varepsilon/4$, contradicting the conditions of case (iii). Thus, (iii) can clearly not occur. Suppose next that we are in case (i). Then, writing

$$f(Bn)\overline{g(Bn+1)} - a\bar{b} = \overline{g(Bn+1)}(f(Bn) - a) + a\overline{(g(Bn+1) - b)}$$

and using the unimodularity of $g(Bn+1)$ and a together with the triangle inequality, we get that

$$|f(Bn)\overline{g(Bn+1)} - a\bar{b}| \geq |f(Bn) - a| - |g(Bn+1) - b| \geq \frac{\varepsilon}{2}$$

on a subset of positive upper density in T. The same condition occurs in case (ii) as well (by essentially the same argument). Now, put $F(n) := f(n)\overline{h_1(n)}n^{-it_1}$ and $G(n) := g(n)\overline{h_2(n)}n^{-it_2}$. We recall that for $n \in T$, $h_1(n) = h_2(n) = 1$ and $n^{i(t_2-t_1)} = 1 + O(\varepsilon^2)$. For each $n \in T$ sufficiently large we have

$$\begin{aligned} F(n)\overline{G(Bn+1)} &= f(n)\overline{g(Bn+1)}n^{-it_1}(Bn+1)^{it_2} \\ &= f(Bn)\overline{g(Bn+1)} \cdot n^{i(t_2-t_1)}\overline{f(B)}B^{it_2} + O(\varepsilon^2) \\ &= f(Bn)\overline{g(Bn+1)} \cdot \overline{f(B)}B^{it_2} + O(\varepsilon^2). \end{aligned}$$

It follows that on some positive upper density subset of T, we have

$$|F(n)\overline{G(Bn+1)} - \bar{b}a\overline{f(B)}B^{it_2}| \geq \frac{\varepsilon}{3},$$

for ε sufficiently small. At this point, we will select B using the following remarks. First, we are supposing that for any $k \in \mathbb{N}$ and $t \in \mathbb{R}$ there is a prime p such that $f(p)^k \neq p^{it}$. As such, for any $M \geq 1$ we can find P sufficiently large in terms of M such that $f(P)P^{-it_2} = e(\alpha)$ where either $\alpha \notin \mathbb{Q}$ or else $\alpha \in \mathbb{Q}$ with denominator at least M. Taking $M \asymp \varepsilon^{-2}$, we can choose r such that αr lies in the same $O(\varepsilon^2)$-length arc of $\mathbb{T}$ as the argument of $(2q_1q_2)^{it_2}b\overline{af(2q_1q_2)}$. Consequently, with this choice of r we find that

$$f(P)^r P^{-irt_2} = b(2q_1q_2)^{it_2}\overline{af(2q_1q_2)} + O(\varepsilon^2).$$

Finally, we choose $B = 2q_1q_2P^r$, and with this choice

$$f(B)B^{-it_2} = b\bar{a} + O(\varepsilon^2).$$

It then follows that

$$|F(n)\overline{G(Bn+1)} - 1| \geq \frac{\varepsilon}{4}$$

for each n belonging to a positive upper density subset of $T' \subseteq T$. Now, with N large we select by Szemerédi's theorem (see [Gowers 2001]) a long arithmetic progression $S \subset T'$; in particular, $|S| \to \infty$ as $x \to \infty$. As in the proof of Theorem 2.1 in [Klurman and Mangerel 2018], we have that on one hand

$$\begin{aligned} \tfrac{1}{16}\varepsilon^2|S \cap [1,x]| &\leq \sum_{\substack{n \leq x \\ n \in S}} |F(n)\overline{G}(Bn+1) - 1|^2 \\ &= 2|S \cap [1,x]|\left(1 - \mathrm{Re}\left(\frac{1}{|S \cap [1,x]|}\sum_{\substack{n \leq x \\ n \in S}} F(n)\overline{G(Bn+1)}\right)\right). \end{aligned}$$

Now write $S = \{a + jd : 0 \leq j \leq |S| - 1\}$, for some $d \in \mathbb{N}$ and $a \geq 0$. Let L_1 and L_2 denote the integer-valued linear forms $L_1(j) := a + jd$ and $L_2(j) := Ba + 1 + Bjd$, so that if $n \in S$ then there is a

$n' \in \mathbb{N}$ for which $n = L_1(n')$ and $Bn + 1 = L_2(n')$. For every prime p define

$$M_p(F(L_1), \overline{G}(L_2)) := \lim_{X\to\infty} \frac{1}{X} \sum_{\nu_1,\nu_2\geq 0} F(p)^{\nu_1}\overline{G}(p)^{\nu_2} \sum_{\substack{n\leq X\\ p^{\nu_1}\parallel L_1(n), p^{\nu_2}\parallel L_2(n)}} 1.$$

Theorem 1.3 of [Klurman 2017] (see also Remark 2.12 in [Klurman and Mangerel 2018]) shows that as $x \to \infty$ along a suitable infinite subsequence,

$$\sum_{\substack{n\leq x\\ n\in S}} F(n)\overline{G(Bn+1)}$$
$$= |S\cap[1,x]|\bigg(\prod_{p\leq x} M_p(F(L_1), \overline{G(L_2)}) + O\bigg(\mathbb{D}(F,1;N,Bx) + \mathbb{D}(G,1;N,Bx) + \frac{1}{N} + \frac{1}{\log\log x}\bigg)\bigg).$$

A simple calculation shows that $M_p(F(L_1), G(L_2)) = 0$ if $p \leq N$ and for $p > N$ we have

$$M_p(F(L_1), \overline{G(L_2)}) = \begin{cases} 1 + (F(p)-1)/p + (\overline{G(p)}-1)/p + O(1/p^2) & \text{if } p\nmid Bd,\\ 1 + (F(p)-1)/p + O(1/p^2) & \text{if } p \mid B,\ p\nmid d,\\ 1_{p\nmid a(Ba+1)} + F(p)^{\min\{\nu_p(a),\nu_p(d)\}} + \overline{G(p)}^{\min\{\nu_p(Ba+1),\nu_p(d)\}} & \text{if } p \mid d. \end{cases}$$

whence it follows that

$$\prod_{p\leq x} M_p(F(L_1), \overline{G(L_2)}) = 1 + O(\mathbb{D}(F,1;N,x)^2 + \mathbb{D}(G,1;N,x)^2 + 1/N).$$

Since F and G are both 1-pretentious then, provided x and N are large enough in terms of ε this yields the inequality

$$\tfrac{1}{16}\varepsilon^2|S\cap[1,x]| \ll \varepsilon^3|S\cap[1,x]|,$$

which is patently false when ε is sufficiently small. Note that we can argue in precisely the same way with the roles of f and g reversed to conclude that case (ii) cannot occur provided that $g(n)^l \neq n^{it'}$ for some t' (looking at the forms $(g(n), f(n-1))$ instead). This contradiction completes the proof of Theorem 1.6 in those cases not covered in the previous section. □

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