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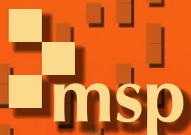
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We establish a short exact sequence about depth-graded motivic double zeta values of even weight relative to μ_2 . We find a basis for the depth-graded motivic double zeta values relative to μ_2 of even weight and a basis for the depth-graded motivic triple zeta values relative to μ_2 of odd weight. As an application of our main results, we prove Kaneko and Tasaka's conjectures about the sum odd double zeta values and the classical double zeta values. We also prove an analogue of Kaneko and Tasaka's conjecture in depth three. Finally, we formulate a conjecture which is related to sum odd multiple zeta values in higher depth.

1. Introduction

Multiple zeta values are defined by the convergent series

$$\zeta(n_1, \dots, n_r) = \sum_{0 < k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}}, \quad n_1, \dots, n_{r-1} > 0, \quad n_r > 1.$$

In particular, when $r = 1$ they are the classical Riemann zeta values. We call r the depth, and $N = n_1 + \dots + n_r$ the weight of the above multiple zeta value.

Denote by \mathcal{Z}_N the \mathbb{Q} -vector space generated by all the weight N multiple zeta values for $N > 0$, and $\mathcal{Z}_0 = \mathbb{Q}$. Then $\mathcal{Z}_{N_1} \cdot \mathcal{Z}_{N_2} \subseteq \mathcal{Z}_{N_1+N_2}$, and

$$\mathcal{Z} = \mathcal{Z}_0 + \mathcal{Z}_1 + \dots + \mathcal{Z}_N + \dots$$

is an algebra with the shuffle product. The weight structure is conjectured to be a grading. There is a depth filtration \mathcal{D} on \mathcal{Z}_N ,

$$\mathcal{D}_r \mathcal{Z}_N = \langle \zeta(n_1, \dots, n_k) \in \mathcal{Z}_N, k \leq r \rangle_{\mathbb{Q}},$$

where for $x_i \in \mathbb{R}, i \in A$, $\langle x_i, i \in A \rangle_{\mathbb{Q}}$ mean the \mathbb{Q} -linear subspace generated by $x_i, i \in A$ in \mathbb{R} .

The double zeta values generate the subspace $\mathcal{D}_2 \mathcal{Z}$ of \mathcal{Z} . Gangl, Kaneko, Zagier [Gangl et al. 2006] found an interesting connection between period polynomials of $SL_2(\mathbb{Z})$ and the double shuffle relations among $\mathcal{D}_2 \mathcal{Z}$.

Brown [2012] defined the motivic multiple zeta values algebra \mathcal{H} . Its elements can be written as \mathbb{Q} -linear combinations of motivic multiple zeta values $\zeta^m(n_1, \dots, n_r)$. There is a surjective algebra homomorphism:

$$\eta : \mathcal{H} \rightarrow \mathcal{Z}, \quad \zeta^m(n_1, \dots, n_r) \mapsto \zeta(n_1, \dots, n_r).$$

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Brown proved that the set $\{\zeta^m(n_1, \dots, n_r), n_i \in 2, 3\}$ is a basis for nonzero weight subspace of \mathcal{H} , thus proving that every multiple zeta value is a \mathbb{Q} -linear combination of $\zeta(n_1, \dots, n_r), n_i \in 2, 3$ (Conjecture C in [Hoffman 1997]). Because of the period homomorphism, we can study the multiple zeta values by studying these motivic multiple zeta values.

Motivic multiple zeta values also satisfy the double shuffle relations by [Soudères 2010]. By Gangl, Kaneko, Zagier’s results, there are also period polynomial relations among motivic double zeta values of even weight. This fact was reinterpreted as a short exact sequence which involves motivic double zeta values of even weight with a slight modification and period polynomials in the second author’s paper [Li 2019].

Furthermore, Li [2019] proposed two exact sequence conjectures which relate the depth-graded version of motivic multiple zeta values and period polynomials of $SL_2(\mathbb{Z})$. Li verified the two conjectures in low depth. Besides, Li and Liu [2020] established a short exact sequence about motivic double zeta values of odd weight.

After Brown, Glanois [2016] considered the motivic multiple zeta values relative to μ_N , which is a generalization of \mathcal{H} for the cyclotomic field, where μ_N is the set of all N -th roots of unity. She gave a basis of motivic multiple zeta values relative to μ_N for $N = 2, 3, 4, 6, 8$. We will give a brief introduction to Glanois’ work in the next section in the case of $N = 2$.

Ma [2015] studied motivic double zeta values relative to μ_N for $N = 2, 3$. He found various connections between some special matrices which come from motivic Galois action on motivic double zeta values relative to μ_N , Hecke operators and newforms of $\Gamma_0(N)$ for $N = 2, 3$.

In the rest of this paper, we only consider the motivic multiple zeta values relative to μ_2 , and denote by \mathcal{H} the \mathbb{Q} -algebra generated by them rather than the motivic multiple zeta algebra of Brown for convenience.

For positive integers $n_1 \geq 1, n_2 \geq 2$, define

$$\zeta^o(n_1, n_2) = \sum_{0 < k_1 < k_2, \text{ odd}} \frac{1}{k_1^{n_1} k_2^{n_2}}.$$

It is obvious to see

$$\zeta^o(n_1, n_2) = \frac{1}{4}(\zeta(n_1, n_2) - \zeta(\bar{n}_1, n_2) - \zeta(n_1, \bar{n}_2) + \zeta(\bar{n}_1, \bar{n}_2)),$$

where

$$\zeta(\bar{n}_1, n_2) = \sum_{0 < k_1 < k_2} \frac{(-1)^{k_1}}{k_1^{n_1} k_2^{n_2}}, \quad \zeta(n_1, \bar{n}_2) = \sum_{0 < k_1 < k_2} \frac{(-1)^{k_2}}{k_1^{n_1} k_2^{n_2}}, \quad \zeta(\bar{n}_1, \bar{n}_2) = \sum_{0 < k_1 < k_2} \frac{(-1)^{k_1+k_2}}{k_1^{n_1} k_2^{n_2}}.$$

Denote by \mathcal{Z}^2 the \mathbb{Q} -vector space generated by

$$\zeta \left(\begin{matrix} n_1, \dots, n_r \\ \epsilon_1, \dots, \epsilon_r \end{matrix} \right) = \sum_{0 < k_1 < \dots < k_r} \frac{\epsilon_1^{k_1} \dots \epsilon_r^{k_r}}{k_1^{n_1} \dots k_r^{n_r}}, \quad \epsilon_i \in \{1, -1\}, (n_r, \epsilon_r) \neq (1, 1),$$

and we call $N = n_1 + \dots + n_r$ and r its weight and depth respectively. Denote by $\mathcal{D}_r \mathcal{Z}^2$ the subspace of \mathcal{Z}^2 spanned by elements of depth $\leq r$. According to the previous equality, we have $\zeta^o(n_1, n_2) \in \mathcal{D}_2 \mathcal{Z}^2$.

Similarly we have

$$\zeta^o(n_1, n_2, \dots, n_r) = \sum_{0 < k_1 < k_2 < \dots < k_r, \text{ odd}} \frac{1}{k_1^{n_1} k_2^{n_2} \dots k_r^{n_r}} = \frac{1}{2^r} \sum_{\epsilon_i \in \{\pm 1\}, 1 \leq i \leq r} \epsilon_1 \epsilon_2 \dots \epsilon_r \zeta \binom{n_1, \dots, n_r}{\epsilon_1, \dots, \epsilon_r}.$$

We call $\zeta^o(n_1, n_2, \dots, n_r)$ a sum odd multiple zeta value.

In the case of even weight, Kaneko and Tasaka [2013] found the following result:

Theorem 1.1 (Kaneko–Tasaka). *For any even integer $N \geq 4$, denote by $S_N(2)$ the space of cusp forms for $\Gamma_0(2) = \{\gamma \in SL_2(\mathbb{Z}); \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, c \equiv 0 \pmod{2}\}$ of weight N , then we have*

$$\dim_{\mathbb{Q}} \langle \zeta^o(2r, N - 2r); 1 \leq r \leq N/2 - 1 \rangle_{\mathbb{Q}} \leq N/2 - 1 - \dim_{\mathbb{C}} S_N(2).$$

Besides, for N even, they also conjectured that elements

$$\zeta^o(2r - 1, N - 2r + 1), \quad 1 \leq r \leq N/2 - 1$$

are \mathbb{Q} -linear independent, and each element $\zeta^o(r, N - r), 1 \leq r \leq N - 2$ can be written as a \mathbb{Q} -linear combination of

$$\zeta^o(2r - 1, N - 2r + 1), \quad 1 \leq r \leq N/2 - 1, \zeta(N).$$

In this paper we will reinterpret Kaneko and Tasaka results on the motivic level.

There are weight grading and depth filtration structures on \mathcal{H} which are compatible with the usual weight and depth structures on classical multiple zeta values relative to μ_2 . Denote by $gr_r^{\mathcal{D}} \mathcal{H}_N$ the weight N depth r part of the depth-graded motivic multiple zeta values, and

$$gr \cdot \mathcal{H} = \mathbb{Q} \oplus \bigoplus_{r \geq 1} \mathcal{D}_r \mathcal{H} / \mathcal{D}_{r-1} \mathcal{H}.$$

Denote by $\zeta^{o,m}(n_1, n_2)$ the motivic sum odd double zeta value, which we will introduce later. Let $\mathcal{P}_{ev}^{o,m}$ be the space generated by the images of

$$\{\zeta^{o,m}(n_1, n_2), n_1, n_2 \geq 2, \text{ even}\}$$

in the quotient $gr_2^{\mathcal{D}} \mathcal{H}$.

Our first main result (in a rough version) is the following:

Theorem 1.2. *There is an exact sequence with respect to sum odd motivic double zeta values*

$$0 \rightarrow \mathcal{P}_{ev}^{o,m} \rightarrow (gr_1^{\mathcal{D}} \mathcal{H}^{\text{odd}} \otimes gr_1^{\mathcal{D}} \mathcal{H}^{\text{odd}})^b \rightarrow \mathbb{P}(\Gamma_0(2))^{\vee} \rightarrow 0,$$

where \mathcal{H} is the algebra of motivic multiple zeta values relative to μ_2 .

Details of the above notations will be introduced in Sections 3 and 4. Theorem 1.2 gives a description of the space of motivic sum odd multiple zeta values of the form

$$\zeta^{m,o}(n_1, n_2), \quad n_1, n_2 \geq 2, \text{ even},$$

and from it we recover Theorem 1.1 immediately.

We can also discuss the case of odd n_1, n_2 , and obtain the following theorem:

Theorem 1.3. (i) *For an even integer $N \geq 4$, the set of the images of*

$$\{\zeta^{o,m}(n_1, n_2), n_1 + n_2 = N, 1 \leq n_i \leq N - 1, \text{ odd}\}$$

in $gr_2^{\mathcal{D}}\mathcal{H}$ is a basis for $gr_2^{\mathcal{D}}\mathcal{H}_N$.

(ii) *For an odd integer $N \geq 5$, the set of the images of*

$$\{\zeta^{o,m}(n_1, n_2, n_3), n_1 + n_2 + n_3 = N, 1 \leq n_i \leq N - 2, \text{ odd}\}$$

in $gr_3^{\mathcal{D}}\mathcal{H}$ is a basis for $gr_3^{\mathcal{D}}\mathcal{H}_N$.

In the above theorem, $\zeta^{o,m}(n_1, n_2, n_3)$ means the motivic version of the sum odd multiple zeta value $\zeta^o(n_1, n_2, n_3)$. Its definition will be given in Section 2.

From the explicit calculations in the proof of Theorems 1.2 and 1.3, we also obtain the following theorem, which was conjectured in [Kaneko and Tasaka 2013, Section 3.2, Remark 2].

Theorem 1.4. (i) *For an even integer $N \geq 4$, the space*

$$\langle \zeta^o(r, N - r); 1 \leq r \leq N - 2 \rangle_{\mathbb{Q}}$$

is spanned by

$$\{\zeta(N), \zeta^o(r, N - r); 1 \leq r \leq N - 3, \text{ odd}\}.$$

(ii) *For an even integer $N \geq 6$, we have*

$$\begin{aligned} \langle \zeta(n_1, n_2); n_1 + n_2 = N, n_2 \geq 2 \rangle_{\mathbb{Q}} &\subseteq \langle \zeta^o(n_1, n_2); n_1 + n_2 = N, n_2 \geq 2 \rangle_{\mathbb{Q}}, \\ \langle \zeta^o(n_1, n_2); n_1 + n_2 = N, n_i \text{ even} \rangle_{\mathbb{Q}} &\subseteq \langle \zeta(n_1, n_2); n_1 + n_2 = N, n_2 \geq 2 \rangle_{\mathbb{Q}}. \end{aligned}$$

We can also give some information in higher depth cases, as in the case of depth 3:

Theorem 1.5. *For a given odd integer $N \geq 5$, and $n_1 + n_2 + n_3 = N, n_1, n_2 \geq 1, n_3 \geq 2$, the element*

$$\zeta^o(n_1, n_2, n_3)$$

can be written as a \mathbb{Q} -linear combination of

$$\zeta^o(m_1, m_2, m_3), \quad m_1 + m_2 + m_3 = N, \quad m_1, m_2 \geq 1, m_3 \geq 3, m_i \text{ odd}$$

and lower depth multiple zeta values relative to μ_2 .

It seems that Theorem 1.3 should also be true for higher depth. We calculate the depth-graded motivic Galois action for sum odd motivic multiple zeta values explicitly in higher depth. Assuming the invertibility of a specific matrix, we can prove the higher depth analogue of Theorem 1.3.

Our paper is divided as follows. In Section 2A, we introduce mixed Tate motives over $\mathbb{Z}[\frac{1}{2}]$. In Section 2B, we introduce motivic multiple zeta values relative to μ_2 , which was considered by Glanois [2016], following Deligne and Goncharov’s work [2005]. We consider the motivic Galois action and show the way to do the calculation in Section 2C. Then we give a brief introduction to period polynomials in Section 3. The proofs of our main results will be given in Sections 4 and 5.

2. Motivic multiple zeta values relative to μ_2

As said in the introduction, the motivic multiple zeta values relative to μ_N are the generalization of Brown’s motivic multiple zeta values. In this section, we only define them in the case of $N = 2$. The main references for this section are [Deligne 2010; Glanois 2016; Gil and Fresán 2018].

2A. Mixed Tate motives over $\mathbb{Z}[\frac{1}{2}]$. Consider the category of mixed Tate motives over $\mathbb{Z}[\frac{1}{2}]$; denote it by \mathcal{MT}_2 . It is a Tannakian category with the natural fiber functor

$$\omega : \mathcal{MT}_2 \rightarrow \text{Vec}_{\mathbb{Q}}; M \mapsto \oplus \omega_r(M),$$

where

$$\omega_r(M) = \text{Hom}_{\mathcal{MT}_2}(\mathbb{Q}(r), gr_{-2r}^{\omega}(M)).$$

Let $\mathcal{G}^{\mathcal{MT}_2}$ be the Tannakian fundamental group (the motivic Galois group) of \mathcal{MT}_2 with respect to this fiber functor ω , and $\mathcal{U}^{\mathcal{MT}_2}$ be the pro-unipotent radical of $\mathcal{G}^{\mathcal{MT}_2}$. We have

$$\mathcal{G}^{\mathcal{MT}_2} \cong \mathbb{G}_m \ltimes \mathcal{U}^{\mathcal{MT}_2}.$$

By Proposition 1.9 in [Deligne and Goncharov 2005], the extension group $\text{Ext}_{\mathcal{MT}_2}^1(\mathbb{Q}(0), \mathbb{Q}(n))$ is nontrivial only when $n \geq 1$, odd and

$$\begin{aligned} \text{Ext}_{\mathcal{MT}_2}^1(\mathbb{Q}(0), \mathbb{Q}(n)) &\cong \mathbb{Q}, & n \geq 1, \text{ odd,} \\ \text{Ext}_{\mathcal{MT}_2}^2(\mathbb{Q}(0), \mathbb{Q}(n)) &= 0, & \text{for all } n. \end{aligned}$$

By Appendix A in [Deligne and Goncharov 2005], there is a set of symbols $\{f_{2n+1}; n \geq 0\}$ such that (noncanonical isomorphism)

$$\mathcal{O}(\mathcal{U}^{\mathcal{MT}_2}) \cong \mathbb{Q}\langle f_1, f_3, \dots, f_{2n+1}, \dots \rangle,$$

where $\mathbb{Q}\langle f_1, f_3, \dots, f_{2n+1}, \dots \rangle$ denotes the noncommutative polynomial ring with variables $f_1, f_3, \dots, f_{2n+1}, \dots$ under the shuffle product.

Let \mathfrak{g} be the Lie algebra of $\mathcal{U}^{\mathcal{MT}_2}$, then $\mathfrak{g} = (m/m^2)^\vee$, where $m \subseteq \mathcal{O}(\mathcal{U}^{\mathcal{MT}_2})$ is the maximal ideal. It is a free Lie algebra with a set of generators $\{\sigma_{2n+1}; n \geq 0\}$.

Denote by ${}_0\Pi_1 = \pi_1^{dR}(\mathbb{P}^1 - \{0, 1, -1, \infty\}, \overrightarrow{1_0}, \overleftarrow{-1_1})$ the motivic torsor of paths from 0 to 1 on $\mathbb{P}^1 - \{0, \pm 1, \infty\}$, with tangential base point given by the tangent vectors 1 at 0 and -1 at 1. It is a functor. For any \mathbb{Q} -algebra R , denote by $R\langle\langle e_0, e_{-1}, e_1 \rangle\rangle$ the noncommutative formal power series in e_0, e_{-1}, e_1 with coefficients in R and

$$\Delta : R\langle\langle e_0, e_{-1}, e_1 \rangle\rangle \rightarrow R\langle\langle e_0, e_{-1}, e_1 \rangle\rangle \otimes_R R\langle\langle e_0, e_{-1}, e_1 \rangle\rangle$$

the co-product on $R\langle\langle e_0, e_{-1}, e_1 \rangle\rangle$ satisfying $\Delta e_i = e_i \otimes 1 + 1 \otimes e_i$ for $i \in \{0, \pm 1\}$. Letting $R\langle\langle e_0, e_{-1}, e_1 \rangle\rangle^\times$ be the set of nonzero elements of $R\langle\langle e_0, e_{-1}, e_1 \rangle\rangle$, we have

$${}_0\Pi_1(R) = \{S \in R\langle\langle e_0, e_{-1}, e_1 \rangle\rangle^\times; \Delta S = S \otimes S\},$$

i.e., ${}_0\Pi_1(R)$ is the set of group-like elements in $R\langle\langle e_0, e_{-1}, e_1 \rangle\rangle^\times$.

Denote by e^i the canonical dual of e_i for $i \in \{0, 1, -1\}$. The affine ring of regular functions of ${}_0\Pi_1$ is the graded algebra with the shuffle product

$$\mathcal{O}({}_0\Pi_1) \cong \mathbb{Q}\langle e^0, e^1, e^{-1} \rangle.$$

The symbol ${}_01_1$ is the point ${}_01_1 : \text{Spec } \mathbb{Q} \rightarrow {}_0\Pi_1$ whose function ring homomorphism maps every nonempty word in e^0, e^1, e^{-1} to 0.

More generally, for $x, y \in \{0, \pm 1\}$, denote by ${}_x\Pi_y$ the motivic fundamental groupoid from the tangential point $\vec{1}$ at x to the tangential point $\vec{1}$ at y .

Let V be the automorphism subgroup of the motivic fundamental groupoid (all basepoints are tangential points at $\{0, \pm 1\}$) of $\mathbb{P}^1 - \{0, \pm 1, \infty\}$ satisfying the following properties:

- (i) Elements of V are compatible with the composition law on the motivic groupoid of $\mathbb{P}^1 - \{0, \pm 1, \infty\}$.
- (ii) Elements of V fix $\exp(e_i) \in {}_i\Pi_i$ for $i \in \{0, \pm 1\}$.
- (iii) Elements of V are equivariant with the $\{\pm 1\}$ -action on the motivic groupoid.

By Proposition 5.11 in [Deligne 2010], the map

$$\xi : V \rightarrow {}_0\Pi_1, \quad a \mapsto a({}_01_1)$$

is an isomorphism of schemes and

$$\text{Lie } V = (\mathbb{L}(e_0, e_1, e_{-1}), \{ , \}),$$

where $\mathbb{L}(e_0, e_1, e_{-1})$ is the free Lie algebra generated by the three symbols e_0, e_1, e_{-1} , and $\{ , \}$ denotes the Ihara Lie bracket on $\mathbb{L}(e_0, e_1, e_{-1})$.

The action of $\mathcal{U}^{\mathcal{M}T_2}$ on ${}_x\Pi_y, x, y \in \{0, \pm 1\}$ factors through V . So there is a natural Lie algebra homomorphism

$$i : \mathfrak{g} \rightarrow \text{Lie } V = (\mathbb{L}(e_0, e_1, e_{-1}), \{ , \}).$$

By the main results of [Deligne 2010], the map i is injective.

For any element w in $\mathbb{L}(e_0, e_1, e_{-1})$, let $\text{depth}(w)$ be the smallest number of total occurrences of e_1 and e_{-1} in w . It induces a depth filtration \mathcal{D} on $\mathbb{L}(e_0, e_1, e_{-1})$ as follows:

$$\mathcal{D}^r \mathbb{L}(e_0, e_1, e_{-1}) = \{w \in \mathbb{L}(e_0, e_1, e_{-1}); \text{depth}(w) \geq r\}.$$

According to [Deligne 2010], the map i satisfies

$$i(\sigma_1) = e_{-1}, \quad i(\sigma_{2n+1}) = (1 - 2^{2n})\text{ad}(e_0)^{2n}e_{-1} + 2^{2n}\text{ad}(e_0)^{2n}e_1 + \text{HDT},$$

where HDT means the higher depth term.

The motivic Lie algebra \mathfrak{g} has an induced depth filtration $\mathcal{D}^r \mathfrak{g}$ from the injective map i . Since the Ihara bracket is compatible with the depth filtration, we know that the depth-graded space

$$\partial \mathfrak{g} = \bigoplus_{r \geq 1} \mathcal{D}^r \mathfrak{g} / \mathcal{D}^{r+1} \mathfrak{g}$$

is a Lie algebra with the induced Ihara bracket. Furthermore, from the main results of [Deligne 2010], \mathfrak{dg} is a free Lie algebra with generators

$$\overline{i(\sigma_1)} = e_{-1}, \quad \overline{i(\sigma_{2n+1})} = (1 - 2^{2n})\text{ad}(e_0)^{2n} e_{-1} + 2^{2n} \text{ad}(e_0)^{2n} e_1$$

in the depth one part.

We will use them in the style of Lie polynomial in $\mathbb{Q}\langle e_0, e_1, e_{-1} \rangle$ rather than Lie words in the rest of this paper for convenience:

$$i(\sigma_{2n+1}) = (1 - 2^{2n}) \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{2n-r} e_{-1} e_0^r + 2^{2n} \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{2n-r} e_1 e_0^r + \text{HDT}. \quad (1)$$

2B. Motivic multiple zeta values. Similar to Brown’s work, Glanois [2016] considered motivic iterated integral I^m and motivic multiple zeta values $\zeta^m \binom{x_1, x_2, \dots, x_p}{\epsilon_1, \epsilon_2, \dots, \epsilon_p}$, $\epsilon_i \in \mu_N$ relative to the set of N -th roots of unity μ_N from [Deligne and Goncharov 2005]. We denote by $\mathcal{H} = \mathcal{H}^2$ the \mathbb{Q} -vector space of motivic multiple zeta values relative to $\mu_2 = \{1, -1\}$. Here we only give the definition in the case of $N = 2$.

Let us consider the map

$$dch : \mathbb{Q}\langle e^0, e^1, e^{-1} \rangle \rightarrow \mathbb{R}.$$

For words $u_i \in \{e^0, e^1, e^{-1}\}$, $i = 1, \dots, k$ satisfying $u_1 \neq e^0, u_k \neq e^1$, define

$$dch(u_1 \cdots u_k) = \int_{0 < t_1 < \dots < t_k < 1} \omega_{u_1}(t_1) \cdots \omega_{u_k}(t_k),$$

where $\omega_{e^0}(t) = dt/t$, $\omega_{e^i}(t) = dt/(i - t)$, $i \in \{1, -1\}$. By Appendix A in [Le and Murakami 1996], we know that

$$\int_{\epsilon < t_1 < \dots < t_k < 1 - \eta} \omega_{u_1}(t_1) \cdots \omega_{u_k}(t_k) = P(\log(\epsilon), \log(\eta)) + O(\sup(\epsilon |\log(\epsilon)|^A + \eta |\log(\eta)|^B)),$$

where P is a polynomial. For a general word sequence $u_1 \cdots u_k$, define

$$dch(u_1, \dots, u_k) = P(0, 0).$$

By the shuffle product of iterated integral, dch is a \mathbb{Q} -algebra homomorphism. Since $\mathcal{O}(\mathfrak{O}\Pi_1)$ is an inductive object in the category of mixed Tate motives \mathcal{MT}_2 , there is a natural action of $\mathcal{G}^{\mathcal{MT}_2}$ on $\mathbb{Q}\langle e^0, e^1, e^{-1} \rangle$. Denote by \mathcal{I} the largest graded subideal of $\text{Ker } dch$ which is stable under the action of $\mathcal{G}^{\mathcal{MT}_2}$. The motivic multiple zeta algebra \mathcal{H} is $\mathcal{O}(\mathfrak{O}\Pi_1)/\mathcal{I}$.

Denote by \mathcal{I}^m the natural quotient map

$$\mathcal{I}^m : \mathcal{O}(\mathfrak{O}\Pi_1) = \mathbb{Q}\langle e^0, e^1, e^{-1} \rangle \rightarrow \mathcal{H}$$

and by per the map $per : \mathcal{H} \rightarrow \mathbb{R}$ satisfying $per \circ \mathcal{I}^m = dch$.

The motivic multiple zeta value $\zeta^m \binom{x_1, x_2, \dots, x_p}{\epsilon_1, \epsilon_2, \dots, \epsilon_p}$ is

$$\mathcal{I}^m(e^{(\epsilon_1 \cdots \epsilon_p)^{-1}} (e^0)^{x_1-1} e^{(\epsilon_2 \cdots \epsilon_p)^{-1}} (e^0)^{x_2-1} \dots e^{(\epsilon_p)^{-1}} (e^0)^{x_p-1}).$$

It's obvious to check that

$$\text{per}\left(\zeta^m\begin{pmatrix} x_1, \dots, x_p \\ \epsilon_1, \dots, \epsilon_p \end{pmatrix}\right) = \zeta\begin{pmatrix} x_1, \dots, x_p \\ \epsilon_1, \dots, \epsilon_p \end{pmatrix}.$$

Define $\zeta^{o,m}(n_1, \dots, n_r)$ as

$$\zeta^{o,m}(n_1, \dots, n_r) = \frac{1}{2^r} \sum_{\epsilon_i \in \{\pm 1\}, 1 \leq i \leq r} \epsilon_1 \cdots \epsilon_r \zeta^m\begin{pmatrix} n_1, \dots, n_r \\ \epsilon_1, \dots, \epsilon_r \end{pmatrix}.$$

It's clear that the image of $\zeta^{o,m}(n_1, \dots, n_r)$ under the period map per is the sum odd multiple zeta values $\zeta^o(n_1, \dots, n_r)$.

In $\mathcal{O}({}_0\Pi_1) = \mathbb{Q}\langle e^0, e^1, e^{-1} \rangle$, for any word $u_1 \cdots u_k, u_i \in \{e^0, e^1, e^{-1}\}$, k is called its weight and the total number of occurrences of e^1 and e^{-1} is called its depth. Denote by $\mathcal{D}_r\mathbb{Q}\langle e^0, e^1, e^{-1} \rangle$ the subspace spanned by elements of depth $\leq r$.

Since the depth filtration on $\mathcal{O}({}_0\Pi_1)$ is motivic [Deligne and Goncharov 2005], it induces a natural depth filtration on \mathcal{H} . The depth filtration on \mathcal{H} is compatible with the depth filtration on \mathbb{Z}^2 through the map per .

Denote by $gr_r^{\mathcal{D}}\mathcal{H} = \mathcal{D}_r\mathcal{H}/\mathcal{D}_{r-1}\mathcal{H}$. The following formula in the case of depth 1 follows from the main results in [Deligne and Goncharov 2005]. From this formula we can deduce a basis for $gr_1^{\mathcal{D}}\mathcal{H}$:

Lemma 2.1. (Deligne–Goncharov) *We have the distribution formula*

$$\zeta^m\begin{pmatrix} n \\ -1 \end{pmatrix} = (2^{-n+1} - 1)\zeta^m\begin{pmatrix} n \\ 1 \end{pmatrix}, \quad \text{for all } n \geq 2.$$

Lemma 2.2. (Deligne–Goncharov) *There is a basis of $gr_1^{\mathcal{D}}\mathcal{H}$: $\{\zeta^m\begin{pmatrix} r \\ -1 \end{pmatrix}, r \geq 1 \text{ odd}\}$.*

Remark 2.3. We will always write $\zeta^m\begin{pmatrix} n_1, n_2 \\ -1, 1 \end{pmatrix}$ as $\zeta^m(\bar{n}_1, n_2)$, similarly $\zeta^m(n_1, \bar{n}_2)$, $\zeta^m(\bar{n}_1, \bar{n}_2)$, $\zeta^m(\bar{k})$ for convenience.

2C. Motivic Galois action. In this subsection we explain how to calculate the depth-graded motivic Galois action of the motivic Lie algebra of $\mathcal{MT}(\mathbb{Z}[\frac{1}{2}])$ on the motivic multiple zeta values relative to μ_2 . Then we give the definition of the map ∂ and deduce its injectivity from the results of [Brown 2012; Deligne 2010; Glanois 2016].

Since the expression of $i(\sigma_{2n+1})$ in $(\mathbb{L}(e_0, e_1, e_{-1}), \{, \})$ has canonical depth one part, σ_{2n+1} in $\mathfrak{g} = \text{Lie } \mathcal{U}^{\mathcal{MT}_2}$ induces a well-defined derivation

$$\partial_{2n+1} : gr_r^{\mathcal{D}}\mathcal{H} \rightarrow gr_{r-1}^{\mathcal{D}}\mathcal{H}.$$

Denote by $gr_1^{\mathcal{D}}\mathcal{H}^{\text{odd}}$ the subspace of $gr_1^{\mathcal{D}}\mathcal{H}$ generated by the images of

$$\zeta^m(\bar{1}), \zeta^m(\bar{3}), \dots, \zeta^m(\overline{2n+1}), \dots$$

in the quotient space $gr_1^{\mathcal{D}}\mathcal{H}$. In this section we will show how to calculate the map ∂_{2n+1} explicitly.

Since $\mathcal{O}({}_0\Pi_1)$ is an ind-object in the category \mathcal{MT}_2 , there is an action of the motivic Lie algebra

$$\mathfrak{g} \times \mathcal{O}({}_0\Pi_1) \rightarrow \mathcal{O}({}_0\Pi_1).$$

Denote by $\mathfrak{h} = \text{Lie } V = (\mathbb{L}(e_0, e_1, e_{-1}), \{ , \})$. The action of \mathfrak{g} on $\mathcal{O}({}_0\Pi_1)$ factors through the action of \mathfrak{h} on $\mathcal{O}({}_0\Pi_1)$.

Denote by $\mathcal{U}\mathfrak{h}$ the universal enveloping algebra of \mathfrak{h} . Then

$$\mathcal{U}\mathfrak{h} = (\mathbb{Q}\langle e_0, e_1, e_{-1} \rangle, \circ),$$

where \circ denotes the new product on $\mathbb{Q}\langle e_0, e_1, e_{-1} \rangle$ transformed from the natural concatenation product on $\mathcal{U}\mathfrak{h}$.

The product \circ is difficult to calculate in general, but by the same reasoning as Proposition 2.2 in [Brown 2013], for any $a \in \mathfrak{h}$, any words w in e_0, e_1, e_{-1} , and any $n \geq 0$, we have

$$a \circ (e_0^n e_i w) = e_0^n (([i]a)e_i + e_i([i]a)^*)w + e_0^n e_i (a \circ w), \quad i \in \{1, -1\}, \tag{2}$$

where

$$a \circ e_0^n = e_0^n a, \quad [i](e_0^a e_i^b e_{-i}^c \cdots) = e_0^a e_{-i}^b e_1^c \cdots, \quad (a_1 \cdots a_n)^* = (-1)^n (a_n \cdots a_1), \quad a_i \in \{e_0, e_1, e_{-1}\}.$$

From the correspondence between unipotent algebraic group and nilpotent Lie algebra, we know that for any $a \in \mathfrak{h}$, the natural action of a on $\mathcal{O}({}_0\Pi_1)$,

$$\mathcal{O}({}_0\Pi_1) = \mathbb{Q}\langle e^0, e^1, e^{-1} \rangle \xrightarrow{a} \mathcal{O}({}_0\Pi_1) = \mathbb{Q}\langle e^0, e^1, e^{-1} \rangle, \quad x \mapsto a(x),$$

is dual to the following action of a on $\mathcal{U}\mathfrak{h}$:

$$\mathcal{U}\mathfrak{h} = \mathbb{Q}\langle e_0, e_1, e_{-1} \rangle \xrightarrow{a} \mathcal{U}\mathfrak{h} = \mathbb{Q}\langle e_0, e_1, e_{-1} \rangle, \quad y \mapsto a \circ y.$$

By the definition of \mathcal{H} and ∂_{2n+1} , we have the commutative diagram

$$\begin{array}{ccc} gr_r^{\mathcal{D}}\mathbb{Q}\langle e^0, e^1, e^{-1} \rangle & \xrightarrow{\overline{\partial_{2n+1}}} & gr_{r-1}^{\mathcal{D}}\mathbb{Q}\langle e^0, e^1, e^{-1} \rangle \\ \downarrow \Downarrow & & \downarrow \Downarrow \\ gr_r^{\mathcal{D}}\mathcal{H} & \xrightarrow{\partial_{2n+1}} & gr_{r-1}^{\mathcal{D}}\mathcal{H} \end{array}$$

where $\overline{\partial_{2n+1}}$ is the depth-graded version of the action of $i(\sigma_{2n+1})$ on $\mathbb{Q}\langle e^0, e^1, e^{-1} \rangle$. Thus in order to write out the maps $\overline{\partial_{2n+1}}$ and ∂_{2n+1} clearly, we need to compute the action $\circ : \mathfrak{h} \times \mathcal{U}\mathfrak{h} \rightarrow \mathcal{U}\mathfrak{h}$ first.

There is a well-defined map

$$\partial : gr_r^{\mathcal{D}}\mathcal{H} \rightarrow gr_1^{\mathcal{D}}\mathcal{H}^{\text{odd}} \otimes gr_{r-1}^{\mathcal{D}}\mathcal{H}; \quad \partial = \sum_{n \geq 0} \zeta^n \overline{(2n+1)} \otimes \partial_{2n+1}.$$

The following proposition is crucial to our analysis.

Proposition 2.4. *For $r \geq 2$, the map ∂ is injective.*

Proof. By exactly the same method as that in [Brown 2012, Section 2.3], it follows that $\mathcal{H} \cong \mathcal{O}(\mathcal{U}^{\mathcal{M}T_2})[t]$ (t is a weight 2, depth 1 element with trivial action of \mathfrak{g}) as a \mathfrak{g} -module. Moreover $t^n, n \geq 1$ are all depth 1 elements.

So we have

$$gr_r \mathcal{H} \cong gr_r \mathcal{O}(\mathcal{U}^{\mathcal{MT}_2}) \oplus \bigoplus_{n \geq 1} gr_{r-1} \mathcal{O}(\mathcal{U}^{\mathcal{MT}_2}) t^n.$$

It suffices to prove that $\partial|_{gr_r \mathcal{O}(\mathcal{U}^{\mathcal{MT}_2})}$ is injective. By the main results of [Deligne 2010], the depth-graded motivic Lie algebra \mathfrak{dg} is a free Lie algebra with generators in the depth one part. By the correspondence between nilpotent Lie algebra and unipotent algebraic group, $\partial|_{gr_r \mathcal{O}(\mathcal{U}^{\mathcal{MT}_2})}$ is injective. \square

Remark 2.5. Proposition 2.4 is not true for Brown’s original motivic multiple zeta values, since, in that case, the depth-graded motivic Lie algebra of $\mathcal{MT}(\mathbb{Z})$ is not a free Lie algebra and it has generators in higher depth part. See [Brown 2013; Enriquez and Lochak 2016; Li 2020] for some conjectural descriptions of the depth-graded motivic Lie algebra of $\mathcal{MT}(\mathbb{Z})$.

3. Period polynomials

In this section, we review the theory of period polynomials, and define $\mathbb{P}(\Gamma_0(2))^\vee$ in Theorem 1.2. The main reference is [Kaneko and Tasaka 2013].

As we know, $\Gamma_0(2) = \{\gamma \in \text{SL}_2(\mathbb{Z}); \gamma \equiv 0 \pmod{2}\}$ is generated by two elements

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$$

For a positive even integer N , denote by V_N the space of homogeneous polynomials with two variables X, Y of degree $N - 2$:

$$V_N = \left\{ P(X, Y) \in \mathbb{Q}[X, Y]; P(X, Y) = \sum_{i=0}^{N-2} a_i X^i Y^{N-2-i} \right\}.$$

The group $\Gamma = \Gamma_0(2)$ acts on V_N naturally: for any polynomial $P(X, Y) \in V_N$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$,

$$\gamma \circ P(X, Y) = P(aX + bY, cX + dY).$$

We write this action as $P(X, Y)|\gamma$ for convenience. Consider the subspace W_N of V_N as follows:

$$W_N = \{P(X, Y) \in V_N[X, Y]; P|(1 - T)(1 + M) = 0\}.$$

Denote by $\mathcal{S}_N(2)$ the space of cusp forms of weight N for $\Gamma_0(2)$. For $f \in \mathcal{S}_N(2)$, the period polynomial $r_f(X, Y)$ of f is given by

$$r_f(X, Y) = \int_0^\infty f(z)(X - zY)^{N-2} dz.$$

It can be shown that

$$r_f(X, Y) \in W_N \otimes \mathbb{C}.$$

Now we decompose W_N into two parts. Let $\varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. It is obvious to see that $P|(1 \pm \varepsilon) \in W_N^\pm$ for any $P \in W_N$, thus we have the direct sum decomposition

$$W_N = W_N^+ \oplus W_N^-,$$

where W_N^+ (resp. W_N^-) is the even (resp. odd) part of W_N ,

$$W_N^\pm = \{P \in V_N; P|\varepsilon = \pm P, P|(1 - T)(1 + M) = 0\}.$$

For $f \in S_N(2)$, denote by r_f^\pm the even and odd parts of the map r_f ,

$$r^\pm : S_N(2) \rightarrow W_N^\pm \otimes \mathbb{C}; f \mapsto r_f^\pm(X, Y).$$

We can decompose W_N^+ further as

$$W_N^+ = \mathbb{Q} \cdot X^{N-2} \oplus W_N^{+,0} \oplus \mathbb{Q} \cdot Y^{N-2},$$

where

$$W_N^{+,0} = \left\{ P(X, Y) = \sum_{i=2 \text{ even}}^{N-4} a_i X^i Y^{N-2-i} \in V_N; P|(1 - T)(1 + M) = 0 \right\}.$$

Kaneko and Tasaka [2013] proved the following two propositions which describe the structure of $W_N^{+,0}$:

Proposition 3.1. *For any even integer N , there are two isomorphisms of vector spaces*

$$r^+ : S_N(2) \rightarrow W_N^{+,0} \otimes \mathbb{C} \quad \text{and} \quad r^- : S_N(2) \rightarrow W_N^- \otimes \mathbb{C}.$$

Proposition 3.2. *For any even integer n, k , denote by $\binom{n}{k}$ the binomial coefficient. The space $W_N^{+,0}$ is of the form*

$$\left\{ \sum_{i=2 \text{ even}}^{N-4} a_i X^i Y^{N-2-i}; \sum_{i=2 \text{ even}}^{N-4} \left(\binom{i}{j} - \binom{i}{N-2-j} \right) a_{N-2-j} = 0, 1 \leq j \leq N-3, \text{ odd} \right\}.$$

Remark 3.3. We let $\binom{n}{k} = 0$ when $k > n$ or $k < 0$.

Denote by $\mathbb{P}(\Gamma_0(2)) = \bigoplus_{N \text{ even}} W_N^{+,0}$. Then $\mathbb{P}(\Gamma_0(2))^\vee$ in Theorem 1.2 is the compact dual of $\mathbb{P}(\Gamma_0(2))$.

4. The depth two case

In this section we calculate the map

$$\partial : gr_r^{\mathcal{D}} \mathcal{H} \rightarrow gr_1^{\mathcal{D}} \mathcal{H}^{\text{odd}} \otimes gr_{r-1}^{\mathcal{D}} \mathcal{H}$$

in the case of $r = 2$ explicitly. Then we establish a short exact sequence about sum odd motivic double zeta values and we find a basis for the depth-graded motivic double zeta values relative to μ_2 by the explicit expression of the map ∂ in the case of $r = 2$. As an application of our results, we prove Kaneko and Tasaka’s conjectures [2013, Remark 2].

4A. The calculation in depth two. The following formulas come from direct calculation. We write $i(\overline{\sigma_{2n+1}})$ as $\overline{\sigma_{2n+1}}$ for short. When $n = 0$, $\overline{\sigma_{2n+1}} = \overline{\sigma_1}$, $a_1 \geq 0$, we have

$$\overline{\sigma_1} \circ (e_0^{a_0} e_1 e_0^{a_1}) = e_0^{a_0} e_{-1} e_1 e_0^{a_1} - e_0^{a_0} e_1 e_{-1} e_0^{a_1} + e_0^{a_0} e_1 e_0^{a_1} e_{-1}, \tag{3}$$

$$\overline{\sigma_1} \circ (e_0^{a_0} e_{-1} e_0^{a_1}) = e_0^{a_0} e_1 e_{-1} e_0^{a_1} - e_0^{a_0} e_{-1} e_1 e_0^{a_1} + e_0^{a_0} e_{-1} e_0^{a_1} e_{-1}. \tag{4}$$

When $n > 0$, $a_1 \geq 0$, we have

$$\begin{aligned} &\overline{\sigma_{2n+1}} \circ (e_0^{a_0} e_1 e_0^{a_1}) \\ &= e_0^{a_0} (\overline{\sigma_{2n+1}} e_1 - e_1 \overline{\sigma_{2n+1}}) e_0^{a_1} + e_0^{a_0} e_1 e_0^{a_1} \overline{\sigma_{2n+1}} \\ &= (1 - 2^{2n}) \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{a_0} e_0^{2n-r} e_{-1} e_0^r e_1 e_0^{a_1} + 2^{2n} \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{a_0} e_0^{2n-r} e_1 e_0^r e_1 e_0^{a_1} \\ &\quad - (1 - 2^{2n}) \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{a_0} e_1 e_0^r e_{-1} e_0^{2n-r} e_0^{a_1} - 2^{2n} \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{a_0} e_1 e_0^r e_1 e_0^{2n-r} e_0^{a_1} \\ &\quad + (1 - 2^{2n}) \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{a_0} e_1 e_0^{a_1} e_0^{2n-r} e_{-1} e_0^r + 2^{2n} \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{a_0} e_1 e_0^{a_1} e_0^{2n-r} e_1 e_0^r, \tag{5} \end{aligned}$$

and

$$\begin{aligned} &\overline{\sigma_{2n+1}} \circ (e_0^{a_0} e_{-1} e_0^{a_1}) \\ &= e_0^{a_0} ([-1](\overline{\sigma_{2n+1}}) e_{-1} - e_{-1} [-1](\overline{\sigma_{2n+1}})) e_0^{a_1} + e_0^{a_0} e_{-1} e_0^{a_1} \overline{\sigma_{2n+1}} \\ &= (1 - 2^{2n}) \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{a_0} e_0^{2n-r} e_1 e_0^r e_{-1} e_0^{a_1} + 2^{2n} \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{a_0} e_0^{2n-r} e_{-1} e_0^r e_{-1} e_0^{a_1} \\ &\quad - (1 - 2^{2n}) \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{a_0} e_{-1} e_0^r e_1 e_0^{2n-r} e_0^{a_1} - 2^{2n} \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{a_0} e_{-1} e_0^r e_{-1} e_0^{2n-r} e_0^{a_1} \\ &\quad + (1 - 2^{2n}) \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{a_0} e_{-1} e_0^{a_1} e_0^{2n-r} e_{-1} e_0^r + 2^{2n} \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{a_0} e_{-1} e_0^{a_1} e_0^{2n-r} e_1 e_0^r. \tag{6} \end{aligned}$$

By taking dual of formula (1) and (2) we have the following result:

Lemma 4.1. For positive even integers n_1, n_2 and $\epsilon_1, \epsilon_2 \in \{1, -1\}$,

$$\overline{\partial_1}(e^{\epsilon_1}(e^0)^{n_1-1} e^{\epsilon_2}(e^0)^{n_2-1}) = 0.$$

Proof. We calculate the map by taking dual of the action $\overline{\sigma_{2n+1}}$, thus we only need to find the terms $e_{\epsilon_1}(e_0)^{n_1-1} e_{\epsilon_2}(e_0)^{n_2-1}$ on the right-hand sides of equations (3) and (4). However, there are no such terms because n_1, n_2 are both even and thus $n_1 - 1, n_2 - 1$ are odd, it means that there is at least one e_0 between e_{ϵ_1} and e_{ϵ_2} , and one e_0 after e_{ϵ_2} . It follows that $\overline{\partial_1}(e^{\epsilon_1}(e^0)^{n_1-1} e^{\epsilon_2}(e^0)^{n_2-1}) = 0$ for all n_1, n_2 even. \square

Lemma 4.2. For positive even integers n_1, n_2 and $n > 0$, writing the words

$$\begin{aligned} & \binom{2n}{n_1-1} e^1 (e^0)^{n_1+n_2-2n-2}, & \binom{2n}{n_1-1} e^{-1} (e^0)^{n_1+n_2-2n-2}, \\ & \binom{2n}{n_2-1} e^1 (e^0)^{n_1+n_2-2n-2}, & \binom{2n}{n_2-1} e^{-1} (e^0)^{n_1+n_2-2n-2}, \end{aligned}$$

as $\Theta_1^{n_1}, \Theta_{-1}^{n_1}, \Theta_1^{n_2}, \Theta_{-1}^{n_2}$ respectively for convenience, we have

$$\overline{\partial_{2n+1}}(e^1 (e^0)^{n_1-1} e^1 (e^0)^{n_2-1}) = 2^{2n} (\Theta_1^{n_1} - \Theta_{-1}^{n_2}), \tag{7}$$

$$\overline{\partial_{2n+1}}(e^1 (e^0)^{n_1-1} e^{-1} (e^0)^{n_2-1}) = (1 - 2^{2n})(\Theta_1^{n_1} - \Theta_{-1}^{n_2}), \tag{8}$$

$$\overline{\partial_{2n+1}}(e^{-1} (e^0)^{n_1-1} e^1 (e^0)^{n_2-1}) = (1 - 2^{2n})\Theta_{-1}^{n_1} - 2^{2n}\Theta_{-1}^{n_2}, \tag{9}$$

$$\overline{\partial_{2n+1}}(e^{-1} (e^0)^{n_1-1} e^{-1} (e^0)^{n_2-1}) = 2^{2n}\Theta_{-1}^{n_1} - (1 - 2^{2n})\Theta_{-1}^{n_2}. \tag{10}$$

Proof. To calculate the term

$$\overline{\partial_{2n+1}}(e^1 (e^0)^{n_1-1} e^1 (e^0)^{n_2-1}),$$

we only need to find the term

$$e_1 (e_0)^{n_1-1} e_1 (e_0)^{n_2-1}$$

on the right-hand sides of equations (3) and (4). The only two possibilities are $a_0 = 0, r = n_1 - 1$ or $a_0 = 0, r = n_2 - 1$. Thus we have

$$\overline{\partial_{2n+1}}(e^1 (e^0)^{n_1-1} e^1 (e^0)^{n_2-1}) = 2^{2n} \left[\binom{2n}{n_1-1} - \binom{2n}{n_2-1} \right] e^1 (e^0)^{n_1+n_2-2n-2}.$$

Formula (5) is proved. By the same method we have (6)–(8). □

It is also useful for us to determine the case that n_1, n_2 are both odd. We use the same argument here and the result is a little different.

Lemma 4.3. For positive odd integers $n_1 \geq 1, n_2 \geq 1$, we have

$$\overline{\partial_1}(e^{i_1} (e^0)^{n_1-1} e^{i_2} (e^0)^{n_2-1}) = \begin{cases} e^{-1}, & \text{if } (i_1, i_2) = (1, -1), n_1 = 1, n_2 = 1, \\ -e^1 (e^0)^{n_2-1} + e^{-1} (e^0)^{n_2-1}, & \text{if } (i_1, i_2) = (1, -1), n_1 = 1, n_2 \geq 3, \\ e^1 (e^0)^{n_2-1} - e^{-1} (e^0)^{n_2-1}, & \text{if } (i_1, i_2) = (-1, 1), n_1 = 1, n_2 \geq 1, \\ e^1 (e^0)^{n_1-1}, & \text{if } (i_1, i_2) = (1, -1), n_1 \geq 3, n_2 = 1, \\ e^{-1} (e^0)^{n_1-1}, & \text{if } (i_1, i_2) = (-1, -1), n_1 \geq 1, n_2 = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Define $\delta \binom{m}{n} = 1$ if $m = n$, $\delta \binom{m}{n} = 0$ if $m \neq n$.

Lemma 4.4. For positive odd integers n_1, n_2 , let $\Theta_1^{n_1}, \Theta_{-1}^{n_1}, \Theta_1^{n_2}, \Theta_{-1}^{n_2}$ be as above and $n \geq 1$, we have

$$\begin{aligned} \overline{\partial_{2n+1}}(e^1(e^0)^{n_1-1}e^1(e^0)^{n_2-1}) &= -2^{2n}\left((\Theta_1^{n_1} - \Theta_1^{n_2}) - \delta\binom{2n}{n_1-1}\Theta_1^{n_1}\right), \\ \overline{\partial_{2n+1}}(e^1(e^0)^{n_1-1}e^{-1}(e^0)^{n_2-1}) &= -(1-2^{2n})\left((\Theta_1^{n_1} - \Theta_1^{n_2}) - \delta\binom{2n}{n_1-1}\Theta_{-1}^{n_1}\right), \\ \overline{\partial_{2n+1}}(e^{-1}(e^0)^{n_1-1}e^1(e^0)^{n_2-1}) &= -(1-2^{2n})\left(\Theta_{-1}^{n_1} - \delta\binom{2n}{n_1-1}\Theta_1^{n_1}\right) + 2^{2n}\Theta_{-1}^{n_2}, \\ \overline{\partial_{2n+1}}(e^{-1}(e^0)^{n_1-1}e^{-1}(e^0)^{n_2-1}) &= -2^{2n}\left(\Theta_{-1}^{n_1} - \delta\binom{2n}{n_1-1}\Theta_{-1}^{n_1}\right) + (1-2^{2n})\Theta_{-1}^{n_2}. \end{aligned}$$

With the above lemmas, we can calculate the maps $\tilde{\partial}$ and ∂ in the case of $r = 2$.

4B. Proofs of the main results. Now we are ready to state our main results. We have already defined the map ∂_{2n+1} for $n \geq 0$ in Section 2C and the space $\mathcal{P}_{ev}^{o,m}$ in Section 1, which is the subspace of $gr_2^{\mathcal{D}}\mathcal{H}$ generated by the set of images of $\{\zeta^{o,m}(n_1, n_2), n_1, n_2 \geq 2, \text{ even}\}$. Define

$$\begin{aligned} D : gr_1^{\mathcal{D}}\mathcal{H}^{odd} \otimes gr_1^{\mathcal{D}}\mathcal{H}^{odd} &\rightarrow (\mathbb{P}(\Gamma_0(2)))^\vee, \\ \zeta^m(\overline{2n_1+1}) \otimes \zeta^m(\overline{2n_2+1}) &\mapsto \frac{2^{2n_2}-1}{2^{2n_2+1}-1} \cdot v(2n_1+1, 2n_2+1), \quad n_1, n_2 \geq 0, \end{aligned}$$

where $v(2n_1+1, 2n_2+1)$ is a linear functional on $\mathbb{P}(\Gamma_0(2))$ satisfying

$$v(2n_1+1, 2n_2+1)(p) = p_{2n_1, 2n_2}$$

for

$$p = \sum p_{2m_1, 2m_2} X^{2m_1} Y^{2m_2} \in \mathbb{P}(\Gamma_0(2)).$$

Theorem 4.5. Denote by $(gr_1^{\mathcal{D}}\mathcal{H}^{odd} \otimes gr_1^{\mathcal{D}}\mathcal{H}^{odd})^b$ the subspace of $gr_1^{\mathcal{D}}\mathcal{H}^{odd} \otimes gr_1^{\mathcal{D}}\mathcal{H}^{odd}$ which is generated by $\zeta^m(\bar{n}_1) \otimes \zeta^m(\bar{n}_2), n_1, n_2 \geq 3, \text{ odd}$. Then

$$\partial(\mathcal{P}_{ev}^{o,m}) \subseteq (gr_1^{\mathcal{D}}\mathcal{H}^{odd} \otimes gr_1^{\mathcal{D}}\mathcal{H}^{odd})^b$$

and there is an exact sequence

$$0 \rightarrow \mathcal{P}_{ev}^{o,m} \xrightarrow{\tilde{\partial}} (gr_1^{\mathcal{D}}\mathcal{H}^{odd} \otimes gr_1^{\mathcal{D}}\mathcal{H}^{odd})^b \xrightarrow{\tilde{D}} \mathbb{P}(\Gamma_0(2))^\vee \rightarrow 0,$$

where the second map $\tilde{\partial}$ is induced from $\partial|_{\mathcal{P}_{ev}^{o,m}}$ and the third map \tilde{D} is induced from D defined as above.

Proof. By Lemmas 4.1 and 4.2 it's obvious to check that

$$\partial(\mathcal{P}_{ev}^{o,m}) \subseteq (gr_1^{\mathcal{D}}\mathcal{H}^{odd} \otimes gr_1^{\mathcal{D}}\mathcal{H}^{odd})^b.$$

The map $\tilde{\partial}$ is injective by Proposition 2.4. The surjectivity of \tilde{D} is trivial. We only need to show that $\text{Im } \tilde{\partial} = \text{Ker } \tilde{D}$.

The following diagram is commutative:

$$\begin{CD} gr_2^{\mathcal{D}}\mathbb{Q}\langle e^0, e^1, e^{-1} \rangle @>\bar{\partial}>> (\mathfrak{d}\mathfrak{g}_1)^\vee \otimes gr_1^{\mathcal{D}}\mathbb{Q}\langle e^0, e^1, e^{-1} \rangle \\ @VVV @VVV \\ gr_2^{\mathcal{D}}\mathcal{H} @>\partial>> gr_1^{\mathcal{D}}\mathcal{H} \otimes gr_1^{\mathcal{D}}\mathcal{H} \end{CD}$$

where

$$\bar{\partial} = \sum_{n \geq 0} (\overline{\sigma_{2n+1}})^\vee \otimes \bar{\partial}_{2n+1},$$

and $(\overline{\sigma_{2n+1}})^\vee, n \geq 0$, is the dual basis of $\overline{\sigma_{2n+1}}, n \geq 0$, in $(\mathfrak{d}\mathfrak{g}_1)^\vee$. The second column map transforms $(\overline{\sigma_{2n+1}})^\vee \otimes \bar{\partial}_{2n+1}(x)$ to $\zeta^m(2n+1) \otimes \partial_{2n+1}(\mathcal{L}^m(x))$.

Thus we can calculate the image of $\mathcal{P}_{ev}^{o,m}$ under ∂ by calculating its lift on

$$gr_2\mathbb{Q}\langle e^0, e^1, e^{-1} \rangle.$$

For even integers n_1, n_2 , the motivic double zeta value $\zeta^{o,m}(n_1, n_2)$ regarded as an element of $gr_2\mathcal{H}$ is equal to

$$\frac{1}{4}\mathcal{L}^m(e^1(e^0)^{n_1-1}e^1(e^0)^{n_2-1} - e^{-1}(e^0)^{n_1-1}e^1(e^0)^{n_2-1} - e^{-1}(e^0)^{n_1-1}e^{-1}(e^0)^{n_2-1} + e^1(e^0)^{n_1-1}e^{-1}(e^0)^{n_2-1}).$$

Define $\Lambda(n_1, n_2)$ to be

$$\frac{1}{4}[e^1(e^0)^{n_1-1}e^1(e^0)^{n_2-1} - e^{-1}(e^0)^{n_1-1}e^1(e^0)^{n_2-1} - e^{-1}(e^0)^{n_1-1}e^{-1}(e^0)^{n_2-1} + e^1(e^0)^{n_1-1}e^{-1}(e^0)^{n_2-1}].$$

Let $\Theta_i^{n_k}$ be as above, $n > 0$ and $s = n_1 + n_2 - 2n - 2$. According to Lemmas 4.1 and 4.2 we have

$$\overline{\partial_{2n+1}}(\Lambda(n_1, n_2)) = \frac{1}{4}(\Theta_1^{n_1} - \Theta_1^{n_2} - \Theta_{-1}^{n_1} + \Theta_{-1}^{n_2}) = \frac{1}{4}\left(\binom{2n}{n_1-1} - \binom{2n}{n_2-1}\right)(e^1(e^0)^s - e^{-1}(e^0)^s).$$

By Lemmas 2.1 and 2.2, if $s > 0$, we have

$$\partial_{2n+1}(\zeta^{o,m}(n_1, n_2)) = \frac{1 - 2^{s+1}}{4(2^s - 1)}\left(\binom{2n}{n_1-1} - \binom{2n}{n_2-1}\right)\zeta^m(\overline{s+1}).$$

Combining with the definition of \tilde{D} and Proposition 3.2, it is obvious that $\text{Im}(\tilde{\partial}) = \text{Ker}(\tilde{D})$. □

Kaneko and Tasaka [2013] proved that there are at least $\dim \mathcal{S}_N(\Gamma_0(2))$ -linear independent relations among the numbers $\{\zeta^o(n_1, n_2), n_1 + n_2 = N, n_1, n_2 \geq 2, \text{ even}\}$. From Theorem 4.5 we obtain

$$\dim_{\mathbb{Q}}\langle \zeta^o(n_1, n_2); n_1 + n_2 = N, n_1, n_2 \geq 2, \text{ even} \rangle_{\mathbb{Q}} \leq \frac{N}{2} - 1 - \dim \mathcal{S}_N(\Gamma_0(2))$$

immediately. The above inequality is compatible with Kaneko and Tasaka's result.

The next theorem gives an affirmative answer for part of Kaneko and Tasaka's conjectures in the motivic setting.

Theorem 4.6. *For an even integer $N \geq 4$. The elements*

$$\{\zeta^{o,m}(k, N - k), 1 \leq k \leq N - 1 \text{ odd}\}$$

are \mathbb{Q} -linear independent. Moreover, the set of their images in $gr_2^{\mathcal{D}}\mathcal{H}$ is a basis of $gr_2^{\mathcal{D}}\mathcal{H}_N$.

Proof. We will make use of the above calculations again. The case $N = 4$ is easy to check. Given an even integer $N \geq 6$, according to Lemmas 4.3 and 4.4, for any odd n_1, n_2 such that $n_1 + n_2 = N$, we have for all $n_1, n_2 > 1$, $\partial_1(\zeta^m(\binom{n_1, n_2}{\epsilon_1, \epsilon_2})) = 0$ and

$$\partial_1(\zeta^{o,m}(1, n_2)) = \frac{1}{2}\mathcal{I}^m[-e^1(e^0)^{n_2-1} + e^{-1}(e^0)^{n_2-1}], \quad \partial_1(\zeta^{o,m}(n_1, 1)) = \frac{1}{4}\mathcal{I}^m[e^1(e^0)^{n_1-1} - e^{-1}(e^0)^{n_1-1}].$$

Thus by the distribution formula, we have

$$\partial_1(\zeta^{o,m}(1, n_2)) = \frac{1 - 2^{n_2}}{2 - 2^{n_2}}\zeta^m(\bar{n}_2), \quad \partial_1(\zeta^{o,m}(n_1, 1)) = -\frac{1 - 2^{n_1}}{4(1 - 2^{n_1-1})}\zeta^m(\bar{n}_1).$$

For the same reason, if $n \geq 1$ and $s = n_1 + n_2 - 2n - 2 > 0$, the following formula holds:

$$\partial_{2n+1}(\zeta^{o,m}(n_1, n_2)) = \frac{2^{s+1} - 1}{4(2^s - 1)} \left[\binom{2n}{n_1 - 1} - \binom{2n}{n_2 - 1} + \delta \binom{2n}{n_1 - 1} (1 - 2^{2n+1}) \right] \zeta^m(\overline{s+1}).$$

If $n \geq 1$ and $s = n_1 + n_2 - 2n - 2 = 0$, we have

$$\partial_{2n+1}(\zeta^{o,m}(n_1, n_2)) = -\frac{1}{4}\delta \binom{2n}{n_1 - 1} (2^{2n+1} - 1) \zeta^m(\bar{1}).$$

In conclusion, we can write the map ∂ in the following form in the case of $n_1 + n_2 = N$:

$$\partial \begin{pmatrix} \zeta^{o,m}(1, N - 1) \\ \zeta^{o,m}(3, N - 3) \\ \vdots \\ \zeta^{o,m}(N - 1, 1) \end{pmatrix} = \tilde{M}B \begin{pmatrix} \zeta^m(\bar{1}) \otimes \zeta^m(\overline{N - 1}) \\ \zeta^m(\bar{3}) \otimes \zeta^m(\overline{N - 3}) \\ \vdots \\ \zeta^m(\overline{N - 1}) \otimes \zeta^m(\bar{1}) \end{pmatrix}.$$

In the above formula,

$$B = \text{diag} \left(\frac{1 - 2^{N-1}}{2 - 2^{N-1}}, \frac{2^{N-3} - 1}{4(2^{N-4} - 1)}, \dots, \frac{2^3 - 1}{4(2^2 - 1)}, -\frac{(2^{N-1} - 1)}{4} \right)$$

is a $(\frac{N}{2})$ -th invertible diagonal matrix, \tilde{M} is a square matrix of order $\frac{N}{2}$ in the form

$$\tilde{M} = \begin{pmatrix} 1 & \cdots & 0 \\ 0 & & 0 \\ \vdots & M & \vdots \\ 0 & & 0 \\ c & \cdots & 1 \end{pmatrix},$$

where M is an $(\frac{N}{2} - 2)$ -th square matrix in the middle of \tilde{M} . The matrix M equals $(a_{i,j})_{1 \leq i, j \leq \frac{N}{2} - 2}$, where

$$a_{i,j} = \binom{2j}{2i} - \binom{2j}{N - 2 - 2i} + \delta \binom{2i}{2j} (1 - 2^{2j+1}).$$

The theorem holds if M is invertible by the fact that ∂ is injective. M can be written as the form $D + A$, where $t = \frac{N}{2} - 1$,

$$D = \text{diag}(d_1, \dots, d_{t-1}), \quad A = (b_{i,j})_{1 \leq i, j \leq t-1}, \quad \text{and} \quad d_i = 1 - 2^{2i+1}, \quad b_{i,j} = \binom{2j}{2i} - \binom{2j}{2t-2i}.$$

Given j , notice that $b_{i,j} + b_{t+1-i,j} = 0$ and $b_{i,j} = 0$ for $j < i < t - j$, it's obvious to check that

$$\sum_{i=1}^{t-1} |b_{i,j}| = 2 \sum_{i=1}^{\min\{\frac{t-1}{2}, j-1\}} |b_{i,j}| \leq 2 \sum_{i=1}^{\min\{\frac{t-1}{2}, j-1\}} \binom{2j}{2i} \leq 2 \sum_{i=1}^{j-1} \binom{2j}{2i} < 2^{2j+1} - 1.$$

So clearly for $j = 1, \dots, t - 1$, we have

$$\sum_{i=1, i \neq j}^{t-1} |b_{i,j}| = \sum_{i=1}^{t-1} |b_{i,j}| - |b_{j,j}| < |d_j + b_{j,j}|.$$

By the following lemma, the matrix M , and furthermore \tilde{M} are invertible. □

Lemma 4.7. *For a real matrix $A = (a_{i,j})_{1 \leq i, j \leq n}$, if $|a_{i,i}| > \sum_{i \neq j} |a_{i,j}|$ for $i = 1, 2, \dots, n$, then $|A| \neq 0$.*

Proof. Denote by α_i the i -th column vector of A , if $|A| = 0$, there exist $\{k_1, \dots, k_n\} \neq \{0\}$ such that $k_1\alpha_1 + \dots + k_n\alpha_n = 0$ is the zero column vector.

Let

$$|k_l| = \max\{|k_1|, \dots, |k_n|\}.$$

Now consider the l -th variable in the above zero column vector. Because we have that $|a_{l,l}| > \sum_{l \neq j} |a_{l,j}|$, $k_1a_{l,1} + \dots + k_n a_{l,n} \neq 0$, we get a contradiction. □

Remark 4.8. Kaneko and Tasaka [2013] conjectured that for given $N \geq 4$, elements

$$\{\zeta^o(n_1, n_2); n_1 \geq 1, n_2 > 1, \text{ odd}, n_1 + n_2 = N\}$$

are \mathbb{Q} -linear independent. Theorem 4.6 gives a proof of the motivic version of Kaneko and Tasaka's conjecture.

As we know, for odd $n > 1$, the double zeta value $\zeta^o(n, 1)$ is not well-defined. However, the motivic sum odd double zeta value $\zeta^{o,m}(n, 1)$ is well-defined. We will calculate the period of $\zeta^{o,m}(n, 1)$. Recall that

$$\zeta^{o,m}(n, 1) = \frac{1}{4} [\zeta^m(n, 1) - \zeta^m(\bar{n}, 1) - \zeta^m(n, \bar{1}) + \zeta^m(\bar{n}, \bar{1})]. \tag{11}$$

Lemma 4.9. *For $n > 1$, odd, the period of $\zeta^{o,m}(n, 1)$ is*

$$\begin{aligned} \text{per}(\zeta^{o,m}(n, 1)) &= \frac{1}{4} [-\zeta(1, n) - \zeta(n+1) - \zeta(\bar{1}, n) + \zeta(\bar{n}+1) - \zeta(n, \bar{1}) + \zeta(\bar{n}, \bar{1})] \\ &= \frac{1}{4} [-\zeta(1, n) + \zeta(1, \bar{n}) - \zeta(n, \bar{1}) + \zeta(\bar{n}, \bar{1}) + (2^{-n} - 2)\zeta(n+1)]. \end{aligned}$$

Proof. It is direct to get the periods of $\zeta^m(n, \bar{1})$ and $\zeta^m(\bar{n}, \bar{1})$, we only need to determine the other two terms in (11). Consider the following regularized integral:

$$\int_{0 < t_1 < \dots < t_{n+1} < 1-\eta} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2} \dots \frac{dt_n}{t_n} \frac{dt_{n+1}}{1-t_{n+1}} = \sum_{0 < s < r} \frac{(1-\eta)^r}{s^n r} = \sum_{s=1}^{\infty} \frac{1}{s^n} \left(-\log(\eta) - \sum_{r=1}^s \frac{(1-\eta)^r}{r} \right).$$

Let $\log(\eta) = 0$. The above integral is equal to $-\sum_{s=1}^{\infty} \sum_{r=1}^s (1-\eta)^r / (rs^n)$. Letting $\eta \rightarrow 0$, we have

$$-\sum_{s=1}^{\infty} \sum_{r=1}^s \frac{1}{rs^n} = -\sum_{0 < r < s} \frac{1}{rs^n} - \sum_{s=1}^{\infty} \frac{1}{s^{n+1}} = -\zeta(1, n) - \zeta(n+1).$$

By the definition of *per*, we have

$$\text{per}(\zeta^m(n, 1)) = -\zeta(1, n) - \zeta(n+1).$$

Similarly, we have

$$\text{per}(\zeta^m(\bar{n}, 1)) = -\zeta(1, \bar{n}) - \zeta(\overline{n+1}).$$

Combined with (11), this proves the lemma. □

The following remark follows from Theorem 4.6 and Lemma 4.9 immediately.

Remark 4.10. Every element $\zeta \binom{n_1, n_2}{\epsilon_1, \epsilon_2}$, $n_1 + n_2 = N$, N even, $(n_2, \epsilon_2) \neq (1, 1)$ can be written as a \mathbb{Q} -linear combination of $\zeta^o(\text{odd}, \text{odd})$, $\zeta(N)$ and $\text{per}(\zeta^{o,m}(N-1, 1))$ as above.

4C. Kaneko and Tasaka’s three conjectures. Kaneko and Tasaka [2013] additionally conjectured that $\langle \zeta^o(r, N-r); 1 \leq r \leq N-2 \rangle_{\mathbb{Q}}$ is spanned by $\zeta^o(\text{odd}, \text{odd})$ and $\zeta(N)$. We will prove this statement as an application of the motivic method.

Theorem 4.11. For a given even integer $N \geq 4$, the space $\langle \zeta^o(r, N-r); 1 \leq r \leq N-2 \rangle_{\mathbb{Q}}$ is spanned by

$$\{\zeta(N), \zeta^o(r, N-r); 1 \leq r \leq N-3, \text{odd}\}.$$

Proof. Denote by $gr_2^{\mathcal{D}}\mathcal{H}_N$ the weight N part of $gr_2^{\mathcal{D}}\mathcal{H}$. According to the property of the period map *per*, we only need to prove that

$$\langle \zeta^{o,m}(r, N-r); 1 \leq r \leq N-2 \rangle_{\mathbb{Q}} = \text{span}\{\zeta^{o,m}(r, N-r); 1 \leq r \leq N-3, \text{odd}\}$$

in $gr_2^{\mathcal{D}}\mathcal{H}_N$. (Be aware that $\zeta^{o,m}(r, N-r); 1 \leq r \leq N-3, \text{odd}$, are elements of \mathcal{H} , in the above formula we mean their natural images in $gr_2^{\mathcal{D}}\mathcal{H}_N$.)

We use the same notation as in the proof of Theorem 4.6, there is a matrix E such that

$$\partial \begin{pmatrix} \zeta^{o,m}(1, N-1) \\ \zeta^{o,m}(3, N-3) \\ \vdots \\ \zeta^{o,m}(N-1, 1) \end{pmatrix} = E \begin{pmatrix} \zeta^m(\bar{1}) \otimes \zeta^m(\overline{N-1}) \\ \zeta^m(\bar{3}) \otimes \zeta^m(\overline{N-3}) \\ \vdots \\ \zeta^m(\overline{N-1}) \otimes \zeta^m(\bar{1}) \end{pmatrix},$$

where $E = \tilde{M}B$ is invertible.

Thus we have

$$\partial E^{-1} \begin{pmatrix} \zeta^{o,m}(1, N-1) \\ \zeta^{o,m}(3, N-3) \\ \vdots \\ \zeta^{o,m}(N-1, 1) \end{pmatrix} = \begin{pmatrix} \zeta^m(\bar{1}) \otimes \zeta^m(\overline{N-1}) \\ \zeta^m(\bar{3}) \otimes \zeta^m(\overline{N-3}) \\ \vdots \\ \zeta^m(\overline{N-1}) \otimes \zeta^m(\bar{1}) \end{pmatrix}.$$

On the other hand, according to Lemmas 4.1 and 4.2, we have

$$\partial \begin{pmatrix} \zeta^{o,m}(2, N-2) \\ \zeta^{o,m}(4, N-4) \\ \vdots \\ \zeta^{o,m}(N-2, 2) \end{pmatrix} = F \begin{pmatrix} \zeta^m(\bar{1}) \otimes \zeta^m(\overline{N-1}) \\ \zeta^m(\bar{3}) \otimes \zeta^m(\overline{N-3}) \\ \vdots \\ \zeta^m(\overline{N-1}) \otimes \zeta^m(\bar{1}) \end{pmatrix},$$

where F is a matrix of order $(\frac{N}{2} - 1, \frac{N}{2})$, thus

$$\partial \begin{pmatrix} \zeta^{o,m}(2, N-2) \\ \zeta^{o,m}(4, N-4) \\ \vdots \\ \zeta^{o,m}(N-2, 2) \end{pmatrix} = \partial F E^{-1} \begin{pmatrix} \zeta^{o,m}(1, N-1) \\ \zeta^{o,m}(3, N-3) \\ \vdots \\ \zeta^{o,m}(N-1, 1) \end{pmatrix}.$$

By the injectivity of ∂ , we have

$$\begin{pmatrix} \zeta^{o,m}(2, N-2) \\ \zeta^{o,m}(4, N-4) \\ \vdots \\ \zeta^{o,m}(N-2, 2) \end{pmatrix} = F E^{-1} \begin{pmatrix} \zeta^{o,m}(1, N-1) \\ \zeta^{o,m}(3, N-3) \\ \vdots \\ \zeta^{o,m}(N-1, 1) \end{pmatrix}$$

in $gr_2^D \mathcal{H}_N$. From the explicit calculation in Theorems 4.5 and 4.6, it's obvious to check that the last column of the matrix $F E^{-1}$ is 0. By using the period map, the theorem is proved. \square

Kaneko and Tasaka [2013] gave some other conjectures and we can prove them by the same motivic method as above.

Proposition 4.12. *For even integer $N \geq 6$, we have*

$$\langle \zeta(n_1, n_2); n_1 + n_2 = N, n_2 \geq 2 \rangle_{\mathbb{Q}} \subseteq \langle \zeta^o(n_1, n_2); n_1 + n_2 = N, n_i \geq 2 \rangle_{\mathbb{Q}},$$

$$\langle \zeta^o(n_1, n_2); n_1 + n_2 = N, n_i \text{ even} \rangle_{\mathbb{Q}} \subseteq \langle \zeta(n_1, n_2); n_1 + n_2 = N, n_2 \geq 2 \rangle_{\mathbb{Q}}.$$

Proof. We only need to prove this proposition in the motivic version. According to our calculations above, for $n \geq 0$, letting $s = N - 2n - 2$, we have

$$\partial_1(\zeta^m(n_1, n_2))=0, \quad \partial_{2n+1}(\zeta^m(n_1, n_2))=2^{2n} \left((-1)^{n_1} \binom{2n}{n_1-1} - (-1)^{n_2} \binom{2n}{n_2-1} + \delta \binom{2n}{n_1-1} \right) \zeta^m(s+1).$$

By the distribution formula, when $s \neq 0$ we have

$$\partial_{2n+1}(\zeta^m(n_1, n_2)) = \frac{2^{N-2}}{1-2^s} \left((-1)^{n_1} \binom{2n}{n_1-1} - (-1)^{n_2} \binom{2n}{n_2-1} + \delta \binom{2n}{n_1-1} \right) \zeta^m(\overline{s+1})$$

and when $s = 0$ we have

$$\partial_{2n+1}(\zeta^m(n_1, n_2)) = 0.$$

We have shown there is an invertible matrix S such that

$$\partial S^{-1} \begin{pmatrix} \zeta^{o,m}(3, N-3) \\ \vdots \\ \zeta^{o,m}(N-3, 3) \end{pmatrix} = \begin{pmatrix} \zeta^m(\overline{3}) \otimes \zeta^m(\overline{N-3}) \\ \vdots \\ \zeta^m(\overline{N-3}) \otimes \zeta^m(\overline{3}) \end{pmatrix}.$$

By the injectivity of ∂ , we have

$$\langle \zeta^m(n_1, n_2); n_1 + n_2 = N, n_2 \geq 2 \rangle_{\mathbb{Q}} \subseteq \langle \zeta^{o,m}(n_1, n_2); n_1 + n_2 = N, 2 \leq n_i \leq k-2 \rangle_{\mathbb{Q}}.$$

For $n_1, n_2 \geq 2$, even, if $s = 0$ or $n = 0$,

$$\partial_{2n+1}(\zeta^{o,m}(n_1, n_2)) = 0,$$

and if $s, n > 0$,

$$\partial_{2n+1}(\zeta^{o,m}(n_1, n_2)) = \frac{1}{4} \frac{1-2^{s+1}}{2^s-1} \left[\binom{2n}{n_1-1} - \binom{2n}{n_2-1} \right] \zeta^m(\overline{s+1}).$$

Since the map ∂ is injective, to prove

$$\langle \zeta^o(n_1, n_2); n_1 + n_2 = N, n_i \text{ even} \rangle_{\mathbb{Q}} \subseteq \langle \zeta(n_1, n_2); n_1 + n_2 = N, n_2 \geq 2 \rangle_{\mathbb{Q}},$$

it suffices to prove that there are numbers $d \binom{m_1, m_2}{n_1, n_2}$ which satisfy

$$\frac{1}{2^{2n}} \left[\binom{2n}{n_1-1} - \binom{2n}{n_2-1} \right] = \sum_{\substack{m_1+m_2=N \\ m_i \geq 1}} d \binom{m_1, m_2}{n_1, n_2} \left[(-1)^{m_1} \binom{2n}{m_1-1} - (-1)^{m_2} \binom{2n}{m_2-1} + \delta \binom{2n}{m_1-1} \right]$$

for all $n_1 + n_2 = N$, $n_1, n_2 \geq 2$, even, $3 \leq 2n+1 \leq N-3$. The above statement follows from Lemma 4.13 and Remark 4.14 below. □

Denote by

$$V_{N,2} = \langle x_1^{n_1-1} x_2^{n_2-1}; n_1 + n_2 = N, n_1, n_2 \geq 3, \text{ odd} \rangle_{\mathbb{Q}},$$

$$\mathbb{P}_{N,2} = \langle x_1^{n_1-1} x_2^{n_2-1}; n_1 + n_2 = N, n_1, n_2 \geq 1 \rangle_{\mathbb{Q}},$$

$$\mathbb{P}_{N,2}^{od} = \langle x_1^{n_1-1} x_2^{n_2-1}; n_1 + n_2 = N, n_1, n_2 \geq 2, \text{ even} \rangle_{\mathbb{Q}}.$$

For $p(x_1, x_2) \in V_{N,2}$, define

$$L_{1,1}(p)(x_1, x_2) = p(x_1, x_2) + p(x_1 - x_2, x_1) - p(x_1 - x_2, x_2),$$

$$L_{\frac{1}{2},1}(p)(x_1, x_2) = p\left(\frac{x_1}{2}, x_2\right) + p\left(\frac{x_1-x_2}{2}, x_1\right) - p\left(\frac{x_1-x_2}{2}, x_2\right).$$

Lemma 4.13. Denote by $i^{od} : \mathbb{P}_{N,2} \rightarrow \mathbb{P}_{N,2}^{od}$ the natural map which satisfies for $p(x_1, x_2) \in \mathbb{P}_{N,2}$,

$$i^{od}(p)(x_1, x_2) = p(x_1, x_2) - p(-x_1, x_2).$$

There is a linear map $j : \mathbb{P}_{N,2} \rightarrow \mathbb{P}_{N,2}^{od}$ such that the following diagram is commutative:

$$\begin{array}{ccc} V_{N,2} & \xrightarrow{L_{1,1}} & \mathbb{P}_{N,2} \\ \downarrow L_{\frac{1}{2},1} & & \downarrow j \\ \mathbb{P}_{N,2} & \xrightarrow{i^{od}} & \mathbb{P}_{N,2}^{od} \end{array}$$

Proof. Define $j_1 : \mathbb{P}_{N,2} \rightarrow \mathbb{P}_{N,2}$ as the \mathbb{Q} -linear map which is induced by

$$x_1 \mapsto \frac{x_1+x_2}{2}, \quad x_2 \mapsto x_2.$$

Define $j_2 : \mathbb{P}_{N,2} \rightarrow \mathbb{P}_{N,2}$ as the \mathbb{Q} -linear map which is induced by

$$x_1 \mapsto \frac{x_1+x_2}{2}, \quad x_2 \mapsto x_1.$$

Define $j = \frac{1}{2}i^{od} \circ (j_1 - j_2)$.

For $p \in V_{N,2}$,

$$\begin{aligned} i^{od} \circ L_{\frac{1}{2},1}(p)(x_1, x_2) &= L_{\frac{1}{2},1}(p)(x_1, x_2) - L_{\frac{1}{2},1}(p)(-x_1, x_2) \\ &= p\left(\frac{x_1-x_2}{2}, x_1\right) - p\left(\frac{x_1-x_2}{2}, x_2\right) - p\left(\frac{x_1+x_2}{2}, x_1\right) + p\left(\frac{x_1+x_2}{2}, x_2\right), \end{aligned}$$

$j \circ L_{1,1}(p)(x_1, x_2)$

$$\begin{aligned} &= \frac{1}{2} \left[(j_1 - j_2) \circ L_{1,1}(p)(x_1, x_2) - (j_1 - j_2) \circ L_{1,1}(p)(-x_1, x_2) \right] \\ &= \frac{1}{2} \left(j_1 \circ L_{1,1}(p)(x_1, x_2) - j_2 \circ L_{1,1}(p)(x_1, x_2) - j_1 \circ L_{1,1}(p)(-x_1, x_2) + j_2 \circ L_{1,1}(p)(-x_1, x_2) \right) \\ &= \frac{1}{2} \left(L_{1,1}(p)\left(\frac{x_1+x_2}{2}, x_2\right) - L_{1,1}(p)\left(\frac{x_1+x_2}{2}, x_1\right) - L_{1,1}(p)\left(\frac{-x_1+x_2}{2}, x_2\right) + L_{1,1}(p)\left(\frac{-x_1+x_2}{2}, -x_1\right) \right) \\ &= \frac{1}{2} \left[p\left(\frac{x_1+x_2}{2}, x_2\right) + p\left(\frac{x_1-x_2}{2}, \frac{x_1+x_2}{2}\right) - p\left(\frac{x_1-x_2}{2}, x_2\right) \right] \\ &\quad - \frac{1}{2} \left[p\left(\frac{x_1+x_2}{2}, x_1\right) + p\left(\frac{x_1-x_2}{2}, \frac{x_1+x_2}{2}\right) - p\left(\frac{x_1-x_2}{2}, x_1\right) \right] \\ &\quad - \frac{1}{2} \left[p\left(\frac{-x_1+x_2}{2}, x_2\right) + p\left(\frac{x_1+x_2}{2}, \frac{x_1-x_2}{2}\right) - p\left(\frac{x_1+x_2}{2}, x_2\right) \right] \\ &\quad + \frac{1}{2} \left[p\left(\frac{-x_1+x_2}{2}, x_1\right) + p\left(\frac{x_1+x_2}{2}, \frac{x_1-x_2}{2}\right) - p\left(\frac{x_1+x_2}{2}, x_1\right) \right] \\ &= p\left(\frac{x_1-x_2}{2}, x_1\right) - p\left(\frac{x_1-x_2}{2}, x_2\right) - p\left(\frac{x_1+x_2}{2}, x_1\right) + p\left(\frac{x_1+x_2}{2}, x_2\right). \end{aligned}$$

As a result of the above calculations, the lemma is proved. \square

Remark 4.14. Define $d\binom{m_1, m_2}{n_1, n_2}$ to be the coefficient of $x_1^{n_1-1}x_2^{n_2-1}$ in $\frac{1}{2}j(x_1^{m_1-1}x_2^{m_2-1})$, i.e.,

$$\frac{1}{2}j(x_1^{m_1-1}x_2^{m_2-1}) = \sum_{\substack{n_1+n_2=N \\ n_i \geq 2, \text{even}}} d\binom{m_1, m_2}{n_1, n_2} x_1^{n_1-1} x_2^{n_2-1}.$$

For $3 \leq 2n + 1 \leq N - 3$, by running the commutative diagram in Lemma 4.13 on

$$p = x_1^{2n} x_2^{N-2-2n} \in V_{N,2},$$

we have

$$\frac{1}{2^{2n}} \left[\binom{2n}{n_1-1} - \binom{2n}{n_2-1} \right] = \sum_{\substack{m_1+m_2=N \\ m_i \geq 1}} d\binom{m_1, m_2}{n_1, n_2} \left[(-1)^{m_1} \binom{2n}{m_1-1} - (-1)^{m_2} \binom{2n}{m_2-1} + \delta\binom{2n}{m_1-1} \right].$$

Remark 4.15. Assuming Grothendieck period conjecture, neither of the two inclusions in Theorem 1.4(ii) is an equality. To see this, we can count the dimensions of two sides on motivic level by the injective map ∂ .

By the motivic method we can prove the following, which was proved by Kaneko and Tasaka [2013]:

Proposition 4.16. *For odd integer $N > 6$, we have*

$$\langle \zeta^o(n_1, n_2); n_1 + n_2 = N, n_i \geq 2 \rangle_{\mathbb{Q}} \subseteq \langle \zeta(n_1, n_2); n_1 + n_2 = N, n_2 \geq 2 \rangle_{\mathbb{Q}}.$$

5. The higher depth case

In this section we calculate the map

$$\partial : gr_r^{\mathcal{D}} \mathcal{H} \rightarrow gr_1^{\mathcal{D}} \mathcal{H}^{\text{odd}} \otimes gr_{r-1}^{\mathcal{D}} \mathcal{H}$$

in the case of $r \geq 3$ for sum odd motivic multiple zeta values explicitly. As a corollary we obtain a basis for the depth-graded motivic triple zeta values of odd weight. What’s more, all elements of this basis are the natural images of sum odd motivic multiple zeta values in the depth-graded motivic triple zeta values of odd weight. At last we conjecture that a matrix appearing in the explicit calculation of ∂ on the sum odd motivic multiple zeta values is invertible.

Denote by

$$T_{N,r} = \{(n_1, \dots, n_r) \in \mathbb{Z}^r; n_1 + \dots + n_r = N, n_i \geq 1, \text{odd}, 1 \leq i \leq r\}.$$

Define $\delta\binom{m_1, \dots, m_r}{n_1, \dots, n_r} = 1$ if $(m_1, \dots, m_r) = (n_1, \dots, n_r)$, $\delta\binom{m_1, \dots, m_r}{n_1, \dots, n_r} = 0$ if $(m_1, \dots, m_r) \neq (n_1, \dots, n_r)$.

Proposition 5.1. *Let $N \equiv r \pmod{2}$ and $N \geq r + 2$. For $(k_1, \dots, k_r) \in T_{N,r}$ we have*

$$\partial(\zeta^{o,m}(k_1, \dots, k_r)) = \sum_{(n_1, \dots, n_r) \in T_{N,r}} e\binom{k_1, k_2, \dots, k_r}{n_1, n_2, \dots, n_r} \zeta^m(\bar{n}_1) \otimes \zeta^{o,m}(n_2, \dots, n_r),$$

where for $n_1 \geq 3$, odd,

$$e\binom{k_1, k_2, \dots, k_r}{n_1, n_2, \dots, n_r} = \left(2^{n_1-1} - \frac{1}{2}\right) \delta\binom{k_1, k_2, \dots, k_r}{n_1, n_2, \dots, n_r} + \frac{1}{2} \sum_{i=1}^{r-1} \left(\binom{n_1-1}{k_{i+1}-1} - \binom{n_1-1}{k_i-1} \right) \delta\binom{k_1, \dots, k_{i-1}, k_{i+2}, \dots, k_r}{n_2, \dots, n_i, n_{i+2}, \dots, n_r},$$

$$e\binom{k_1, k_2, \dots, k_r}{1, n_2, \dots, n_r} = -\delta\binom{k_1, k_2, \dots, k_r}{1, n_2, \dots, n_r} + \frac{1}{2} \delta\binom{k_1, \dots, k_{r-1}, k_r}{n_2, \dots, n_r, 1}.$$

Proof. Notice the following calculation:

$$\begin{aligned} &\zeta^{o,m}(n_1, \dots, n_r) \\ &= \frac{1}{2^r} \sum_{\epsilon_i \in \{\pm 1\}, 1 \leq i \leq r} \epsilon_1 \cdots \epsilon_r \zeta^m \left(\begin{matrix} n_1, \dots, n_r \\ \epsilon_1, \dots, \epsilon_r \end{matrix} \right) \\ &= \frac{1}{2^r} \sum_{\epsilon_i \in \{\pm 1\}, 1 \leq i \leq r} \epsilon_1 \cdots \epsilon_r \mathcal{I}^m(e^{\epsilon_1 \cdots \epsilon_r} (e^0)^{n_1-1} e^{\epsilon_2 \cdots \epsilon_r} (e^0)^{n_2-1} \cdots e^{\epsilon_r} (e^0)^{n_r-1}) \\ &= \frac{1}{2^r} \sum_{\epsilon_i \in \{\pm 1\}, 2 \leq i \leq r} \mathcal{I}^m \left[(e^1 (e^0)^{n_1-1} e^{\epsilon_2 \cdots \epsilon_r} (e^0)^{n_2-1} \cdots e^{\epsilon_r} (e^0)^{n_r-1}) \right. \\ &\qquad \qquad \qquad \left. - e^{-1} (e^0)^{n_1-1} e^{\epsilon_2 \cdots \epsilon_r} (e^0)^{n_2-1} \cdots e^{\epsilon_r} (e^0)^{n_r-1} \right] \\ &= \frac{1}{2^r} \sum_{\epsilon_i \in \{\pm 1\}, 2 \leq i \leq r} \mathcal{I}^m \left[(e^1 (e^0)^{n_1-1} e^{\epsilon_2} (e^0)^{n_2-1} \cdots e^{\epsilon_r} (e^0)^{n_r-1}) - e^{-1} (e^0)^{n_1-1} e^{\epsilon_2} (e^0)^{n_2-1} \cdots e^{\epsilon_r} (e^0)^{n_r-1} \right]. \end{aligned}$$

We have

$$\overline{\sigma_{2n+1}} \circ (e_0^{a_0} e_{i_1} e_0^{a_1} \cdots e_{i_r} e_0^{a_r}) = \sum_{j=1}^r e_0^{a_0} \cdots (\overline{\sigma_{2n+1}} \circ e_{i_j}) e_0^{a_j} \cdots e_{i_r} e_0^{a_r} + e_0^{a_0} \cdots e_{i_r} e_0^{a_r} \overline{\sigma_{2n+1}}.$$

Since, when $n = 0$, we have

$$\bar{\sigma}_1 = e_{-1}, e_{-1} \circ e_1 = e_{-1} e_1 - e_1 e_{-1}, e_{-1} \circ e_{-1} = e_1 e_{-1} - e_{-1} e_1,$$

it follows that

$$\begin{aligned} \bar{\partial}_1(e^{i_1} (e^0)^{a_1} \cdots e^{i_s} (e^0)^{a_s}) &= \delta \binom{a_1}{0} \delta \binom{i_1 i_2}{-1} i_1 (e^{-1} - e^1) (e^0)^{a_2} \cdots e^{i_s} (e^0)^{a_s} + \cdots \\ &\quad + \delta \binom{a_{s-1}}{0} \delta \binom{i_{s-1} i_s}{-1} i_{s-1} e^{i_1} (e^0)^{a_1} \cdots e^{i_{s-2}} (e^0)^{a_{s-2}} (e^{-1} - e^1) (e^0)^{a_s} \\ &\quad + \delta \binom{a_s}{0} \delta \binom{i_s}{-1} e^{i_1} (e^0)^{a_1} \cdots e^{i_{s-1}} (e^0)^{a_{s-1}}. \end{aligned}$$

As a result, we have

$$\begin{aligned} &\partial_1(\zeta^{o,m}(n_1, n_2, \dots, n_r)) \\ &= \frac{1}{2^r} \sum_{\epsilon_i \in \{\pm 1\}, 1 \leq i \leq r} \mathcal{I}^m \left[\bar{\partial}_1(\epsilon_1 e^{\epsilon_1} (e^0)^{n_1-1} e^{\epsilon_2} (e^0)^{n_2-1} \cdots e^{\epsilon_r} (e^0)^{n_r-1}) \right] \\ &= \frac{1}{2^r} \delta \binom{n_1}{1} \sum_{\epsilon_i \in \{\pm 1\}, 1 \leq i \leq r} \mathcal{I}^m \left(\delta \binom{\epsilon_1 \epsilon_2}{-1} \epsilon_1^2 (e^{-1} - e^1) (e^0)^{n_2-1} \cdots e^{\epsilon_r} (e^0)^{n_r-1} \right) + \cdots \\ &\quad + \frac{1}{2^r} \delta \binom{n_{r-1}}{1} \sum_{\epsilon_i \in \{\pm 1\}, 1 \leq i \leq r} \mathcal{I}^m \left[\delta \binom{\epsilon_{r-1} \epsilon_r}{-1} \epsilon_{r-1} \epsilon_r e^{\epsilon_1} (e^0)^{n_1-1} \cdots e^{\epsilon_{r-2}} (e^0)^{n_{r-2}-1} (e^{-1} - e^1) (e^0)^{n_r-1} \right] \\ &\quad + \frac{1}{2^r} \delta \binom{n_r}{1} \sum_{\epsilon_i \in \{\pm 1\}, 1 \leq i \leq r} \mathcal{I}^m \left(\delta \binom{\epsilon_r}{-1} \epsilon_1 e^{\epsilon_1} (e^0)^{n_1-1} \cdots e^{\epsilon_{r-1}} (e^0)^{n_{r-1}-1} \right) \\ &= -\delta \binom{n_1}{1} \zeta^{o,m}(n_2, \dots, n_r) + \frac{1}{2} \delta \binom{n_r}{1} \zeta^{o,m}(n_1, \dots, n_{r-1}). \end{aligned}$$

In the above calculation, the last equality is due to the fact that

$$\sum_{\epsilon_1, \epsilon_2 \in \{\pm 1\}} \delta \left(\begin{matrix} \epsilon_1 \epsilon_2 \\ -1 \end{matrix} \right) \epsilon_1 = 0.$$

Similarly for $n \geq 1$, from

$$\begin{aligned} \overline{\sigma_{2n+1}} &= (1 - 2^{2n}) \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{2n-r} e_{-1} e_0^r + 2^{2n} \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} e_0^{2n-r} e_1 e_0^r, \\ \overline{\sigma_{2n+1}} \circ e_1 &= (1 - 2^{2n}) \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} (e_0^{2n-r} e_{-1} e_0^r e_1 - e_1 e_0^{2n-r} e_{-1} e_0^r) \\ &\quad + 2^{2n} \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} (e_0^{2n-r} e_1 e_0^r e_1 - e_1 e_0^{2n-r} e_1 e_0^r), \\ \overline{\sigma_{2n+1}} \circ e_{-1} &= (1 - 2^{2n}) \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} (e_0^{2n-r} e_1 e_0^r e_{-1} - e_{-1} e_0^{2n-r} e_1 e_0^r) \\ &\quad + 2^{2n} \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} (e_0^{2n-r} e_{-1} e_0^r e_{-1} - e_{-1} e_0^{2n-r} e_{-1} e_0^r), \end{aligned}$$

we have

$$\begin{aligned} &\bar{d}_{2n+1}(e^{i_1}(e^0)^{a_1} \dots e^{i_s}(e^0)^{a_s}) \\ &= (1 - 2^{2n}) \delta \left(\begin{matrix} a_1 \\ 2n \end{matrix} \right) \delta \left(\begin{matrix} i_1 i_2 \\ -1 \end{matrix} \right) e^{-i_1}(e^0)^{a_2} e^{i_3}(e^0)^{a_3} \dots e^{i_s}(e^0)^{a_s} \\ &\quad - (1 - 2^{2n}) (-1)^{a_1} \binom{2n}{a_1} \delta \left(\begin{matrix} i_1 i_2 \\ -1 \end{matrix} \right) e^{i_1}(e^0)^{a_1+a_2-2n} e^{i_3}(e^0)^{a_3} \dots e^{i_s}(e^0)^{a_s} \\ &\quad + 2^{2n} \delta \left(\begin{matrix} a_1 \\ 2n \end{matrix} \right) \delta \left(\begin{matrix} i_1 i_2 \\ 1 \end{matrix} \right) e^{i_1}(e^0)^{a_2} e^{i_3}(e^0)^{a_3} \dots e^{i_s}(e^0)^{a_s} \\ &\quad - 2^{2n} (-1)^{a_1} \binom{2n}{a_1} \delta \left(\begin{matrix} i_1 i_2 \\ 1 \end{matrix} \right) e^{i_1}(e^0)^{a_1+a_2-2n} e^{i_3}(e^0)^{a_3} \dots e^{i_s}(e^0)^{a_s} \\ &\quad + \dots + (1 - 2^{2n}) (-1)^{a_{s-1}} \binom{2n}{a_{s-1}} \delta \left(\begin{matrix} i_{s-1} i_s \\ -1 \end{matrix} \right) e^{i_1}(e^0)^{a_1} \dots e^{i_{s-2}}(e^0)^{a_{s-2}+a_{s-1}-2n} e^{-i_{s-1}}(e^0)^{a_s} \\ &\quad - (1 - 2^{2n}) (-1)^{a_{s-1}} \binom{2n}{a_{s-1}} \delta \left(\begin{matrix} i_{s-1} i_s \\ -1 \end{matrix} \right) e^{i_1}(e^0)^{a_1} \dots e^{i_{s-2}}(e^0)^{a_{s-2}} e^{i_{s-1}}(e^0)^{a_{s-1}+a_s-2n} \\ &\quad + 2^{2n} (-1)^{a_{s-1}} \binom{2n}{a_{s-1}} \delta \left(\begin{matrix} i_{s-1} i_s \\ 1 \end{matrix} \right) e^{i_1}(e^0)^{a_1} \dots e^{i_{s-2}}(e^0)^{a_{s-2}+a_{s-1}-2n} e^{i_{s-1}}(e^0)^{a_s} \\ &\quad + 2^{2n} (-1)^{a_{s-1}} \binom{2n}{a_{s-1}} \delta \left(\begin{matrix} i_{s-1} i_s \\ 1 \end{matrix} \right) e^{i_1}(e^0)^{a_1} \dots e^{i_{s-2}}(e^0)^{a_{s-2}} e^{i_{s-1}}(e^0)^{a_{s-1}+a_s-2n} \\ &\quad + (1 - 2^{2n}) (-1)^{a_s} \binom{2n}{a_s} \delta \left(\begin{matrix} i_s \\ -1 \end{matrix} \right) e^{i_1}(e^0)^{a_1} \dots (e^0)^{a_{s-2}} e^{i_{s-1}}(e^0)^{a_{s-1}+a_s-2n} \\ &\quad + 2^{2n} (-1)^{a_s} \binom{2n}{a_s} \delta \left(\begin{matrix} i_s \\ 1 \end{matrix} \right) e^{i_1}(e^0)^{a_1} \dots (e^0)^{a_{s-2}} e^{i_{s-1}}(e^0)^{a_{s-1}+a_s-2n}. \end{aligned}$$

Thus for $(n_1, n_2, \dots, n_r) \in T_{N,r}$, we have

$$\begin{aligned}
 & \partial_{2n+1}(\zeta^{o,m}(n_1, \dots, n_r)) \\
 &= \frac{1}{2^r} \sum_{\epsilon_i \in \{\pm 1\}, 1 \leq i \leq r} \epsilon_1 \mathcal{T}^m[\bar{\delta}_{2n+1}(e^{\epsilon_1}(e^0)^{n_1-1} \dots e^{\epsilon_r}(e^0)^{n_r-1})] \\
 &= \frac{1}{2}(2^{2n} - 1)\delta\left(\begin{matrix} n_1 \\ 2n+1 \end{matrix}\right)\zeta^{o,m}(n_2, \dots, n_r) \\
 &\quad + \frac{(-1)^{n_1-1}}{2}(2^{2n} - 1)\binom{2n}{n_1-1}\zeta^{o,m}(n_1 + n_2 - 2n - 1, n_3, \dots, n_r) \\
 &\quad + \frac{2^{2n}}{2}\delta\left(\begin{matrix} n_1 \\ 2n+1 \end{matrix}\right)\zeta^{o,m}(n_2, \dots, n_r) - \frac{2^{2n}}{2}(-1)^{n_1-1}\binom{2n}{n_1-1}\zeta^{o,m}(n_1 + n_2 - 2n - 1, n_3, \dots, n_r) \\
 &\quad + \dots + \frac{1-2^{2n}}{2}(-1)^{n_{r-1}-1}\binom{2n}{n_{r-1}-1}\zeta^{o,m}(n_1, \dots, n_{r-3}, n_{r-2} + n_{r-1} - 2n - 1, n_r) \\
 &\quad - \frac{1-2^{2n}}{2}(-1)^{n_{r-1}-1}\binom{2n}{n_{r-1}-1}\zeta^{o,m}(n_1, \dots, n_{r-2}, n_{r-1} + n_r - 2n - 1) \\
 &\quad + \frac{2^{2n}}{2}(-1)^{n_{r-1}-1}\binom{2n}{n_{r-1}-1}\zeta^{o,m}(n_1, \dots, n_{r-3}, n_{r-2} + n_{r-1} - 2n - 1, n_r) \\
 &\quad - \frac{2^{2n}}{2}(-1)^{n_{r-1}-1}\binom{2n}{n_{r-1}-1}\zeta^{o,m}(n_1, \dots, n_{r-2}, n_{r-1} + n_r - 2n - 1) \\
 &\quad + \frac{1-2^{2n}}{2}(-1)^{n_r-1}\binom{2n}{n_r-1}\zeta^{o,m}(n_1, \dots, n_{r-2}, n_{r-1} + n_r - 2n - 1) \\
 &\quad + \frac{2^{2n}}{2}(-1)^{n_r-1}\binom{2n}{n_r-1}\zeta^{o,m}(n_1, \dots, n_{r-2}, n_{r-1} + n_r - 2n - 1) \\
 &= \left(1 - \frac{1}{2}\right)\delta\left(\begin{matrix} n_1 \\ 2n+1 \end{matrix}\right)\zeta^{o,m}(n_2, \dots, n_r) - \frac{1}{2}\binom{2n}{n_1-1}\zeta^{o,m}(n_1 + n_2 - 2n - 1, n_3, \dots, n_r) \\
 &\quad + \dots + \frac{1}{2}\binom{2n}{n_{r-1}-1}\zeta^{o,m}(n_1, \dots, n_{r-3}, n_{r-2} + n_{r-1} - 2n - 1, n_r) \\
 &\quad - \frac{1}{2}\binom{2n}{n_{r-1}-1}\zeta^{o,m}(n_1, \dots, n_{r-2}, n_{r-1} + n_r - 2n - 1) \\
 &\quad + \frac{1}{2}\binom{2n}{n_r-1}\zeta^{o,m}(n_1, \dots, n_{r-2}, n_{r-1} + n_r - 2n - 1) \\
 &= (2^{n_1-1} - \frac{1}{2})\delta\left(\begin{matrix} n_1 \\ 2n+1 \end{matrix}\right)\zeta^{o,m}(n_2, \dots, n_r) \\
 &\quad + \frac{1}{2} \sum_{i=1}^{r-1} \left(\binom{2n}{n_{i+1}-1} - \binom{2n}{n_i-1} \right) \cdot \zeta^{o,m}(n_1, \dots, n_{i-1}, n_i + n_{i+1} - 2n - 1, n_{i+2}, \dots, n_r).
 \end{aligned}$$

Thus the proposition holds. □

With the help of the above proposition, we can generalize Theorem 4.6 to the case of depth 3.

Theorem 5.2. For $r = 3, N \geq 5$ odd.

(i) The set of the images of elements

$$\{\zeta^{o,m}(n_1, n_2, n_3); n_1 + n_2 + n_3 = N, n_i \text{ odd}\}$$

in $gr_3^{\mathcal{D}}\mathcal{H}_N$ is a basis of the total space $gr_3^{\mathcal{D}}\mathcal{H}_N$.

(ii) Every element in

$$\mathcal{P}_{N,3}^o = \langle \zeta^o(n_1, n_2, n_3); n_1 + n_2 + n_3 = N, n_3 > 1 \rangle_{\mathbb{Q}}$$

can be written as a \mathbb{Q} -linear combination of some sum odd multiple zeta values of weight N , depth 3 and multiple zeta values relative to μ_2 of weight N , depth less than 3.

Proof. We have known that the set of elements

$$\{\zeta^{o,m}(n_1, n_2); n_1 + n_2 = k, n_i \geq 1, \text{ odd}\}$$

is a basis of the space $gr_2^{\mathcal{D}}\mathcal{H}_N$. Similar to the proofs of Theorems 4.6 and 4.11 we will use the above proposition to prove the first part. Using Lemma 4.7, we only need to prove that for any given $(n_1, n_2, n_3) \in T_{N,3}$,

$$\sum_{(k_1, k_2, \dots, k_r) \neq (n_1, n_2, \dots, n_r)} \left| e \binom{k_1, k_2, \dots, k_r}{n_1, n_2, \dots, n_r} \right| < \left| e \binom{n_1, n_2, \dots, n_r}{n_1, n_2, \dots, n_r} \right|.$$

When $n_1 = 1$, the above inequality is trivial.

When $n_1 \geq 3$, we have

$$\begin{aligned} & \sum_{(k_1, k_2, k_3) \in S_{N,3}} \left| \delta \binom{k_3}{n_3} \left(\binom{n_1-1}{k_1-1} - \binom{n_1-1}{k_2-1} \right) + \delta \binom{k_1}{n_2} \left(\binom{n_1-1}{k_2-1} - \binom{n_1-1}{k_3-1} \right) \right| \\ & \leq \sum_{(k_1, k_2, k_3) \in S_{N,3}} \left| \delta \binom{k_3}{n_3} \left(\binom{n_1-1}{k_1-1} - \binom{n_1-1}{k_2-1} \right) \right| + \left| \delta \binom{k_1}{n_2} \left(\binom{n_1-1}{k_2-1} - \binom{n_1-1}{k_3-1} \right) \right| \\ & \leq \sum_{(k_1, k_2, k_3) \in S_{N,3}} \delta \binom{k_3}{n_3} \left(\left| \binom{n_1-1}{k_1-1} \right| + \left| \binom{n_1-1}{k_2-1} \right| \right) + \delta \binom{k_1}{n_2} \left(\left| \binom{n_1-1}{k_2-1} \right| + \left| \binom{n_1-1}{k_3-1} \right| \right) - 2 \\ & \leq 4 \sum_{i \geq 0} \binom{n_1-1}{2i} - 2 < 2^{n_1} - 1. \end{aligned}$$

Thus the first statement holds.

As for the second part of this theorem, denote by

$$\mathcal{C} = \{\zeta^{o,m}(n_1, n_2, n_3); n_1 + n_2 + n_3 = N\} \setminus \{\zeta^{o,m}(n_1, n_2, n_3); n_1 + n_2 + n_3 = N, n_i \text{ odd}\}.$$

Assume that there is a lexicographical order on $T_{N,r}$, it induces an order on \mathcal{C} and $\{\zeta^{o,m}(n_1, n_2, n_3); n_1 + n_2 + n_3 = N\}$. Let α (resp. β) be the column vector whose i -th element is the i -th element in \mathcal{C}

(resp. $\{\zeta^{o,m}(n_1, n_2, n_3); n_1 + n_2 + n_3 = N\}$). The argument above and Proposition 5.1 show that there is a matrix P and an invertible matrix Q such that

$$\partial(\alpha) = P\gamma, \quad \partial(\beta) = Q\gamma,$$

where $\gamma = (\zeta^m(\bar{1}) \otimes \zeta^{o,m}(1, N-2), \dots, \zeta^m(\overline{N-2}) \otimes \zeta^{o,m}(1, 1))^T$.

The last column of Q is $(0, \dots, 0, 2^{N-3} - \frac{1}{2})^T$ obviously, and the last column of P is 0 because of the following equation:

$$\partial_{N-2}(\zeta^{o,m}(n_1, n_2, n_3)) = 0, \quad \text{for all } \zeta^{o,m}(n_1, n_2, n_3) \in \mathcal{C}.$$

By the injectivity of ∂ we have

$$\alpha = P Q^{-1} \beta,$$

and that the last row of $P Q^{-1}$ is 0. Thus the theorem holds. □

Furthermore we can put forward the following conjecture:

Conjecture 5.3. *For any $r \geq 4$, $N \geq r + 2$, $N - r \equiv 0 \pmod{2}$, the order $|T_{N,r}|$ matrix*

$$E = \left(e \begin{pmatrix} k_1, k_2, \dots, k_r \\ n_1, n_2, \dots, n_r \end{pmatrix} \right)$$

as in Proposition 5.1 is invertible.

Remark 5.4. If this conjecture is true we can directly generalize Theorem 5.2 to cases of higher depth by induction. Unfortunately in depth ≥ 4 , the matrix E is usually not a strictly diagonal dominant matrix any more. Thus Lemma 4.7 is not helpful in cases of higher depth. By explicit calculation we have checked that Conjecture 5.3 is true for $r = 4$, $N = 6, 8, 10$.

Remark 5.5. The motivic approach in this paper can also be used to study cyclotomic multiple zeta values for other roots of unity. By explicit calculation, the analogue of Theorem 1.3 in other roots of unity (at least for μ_3, μ_4, μ_6 and μ_8) is true. Unfortunately we can't find any short exact sequence either for other roots of unity or for μ_2 in higher depth.

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