Algebra & Number Theory

Volume 14
2020
No. 10

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ALGEBRA AND NUMBER THEORY 14:10 (2020)
https://doi.org/10.2140/ant.2020.14.2773

# Relative crystalline representations and $p$-divisible groups in the small ramification case

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Let $k$ be a perfect field of characteristic $p > 2$, and let $K$ be a finite totally ramified extension over $W(k)\left[\frac{1}{p}\right]$ of ramification degree $e$. Let $R_0$ be a relative base ring over $W(k)\langle t_1^{\pm 1}, \dots, t_m^{\pm 1}\rangle$ satisfying some mild conditions, and let $R = R_0 \otimes_{W(k)} \mathcal{O}_K$. We show that if $e < p-1$, then every crystalline representation of $\pi_1^{\text{ét}}\left(\operatorname{Spec} R\left[\frac{1}{p}\right]\right)$ with Hodge–Tate weights in [0, 1] arises from a $p$-divisible group over $R$.

## 1. Introduction

Let $k$ be a perfect field of characteristic $p > 2$, and let $W(k)$ be its ring of Witt vectors. Let $K$ be a finite totally ramified extension over $W(k)\left[\frac{1}{p}\right]$ with ramification degree $e$, and denote by $\mathcal{O}_K$ its ring of integers. If $G$ is a $p$-divisible group over $\mathcal{O}_K$, then it is well-known that its Tate module $T_p(G)$ is a crystalline $\operatorname{Gal}(\overline{K}/K)$-representation with Hodge–Tate weights in [0, 1]. Conversely, Kisin [2006] showed the following result.

**Theorem 1.1** [Kisin 2006, Corollary 2.2.6]. *Let $T$ be a crystalline* $\operatorname{Gal}(\overline{K}/K)$*-representation finite free over $\mathbb{Z}_p$ whose Hodge–Tate weights lie in* [0, 1]*. Then there exists a $p$-divisible group $G$ over $\mathcal{O}_K$ such that $T_p(G) \cong T$ as* $\operatorname{Gal}(\overline{K}/K)$*-representations.*

The result in Theorem 1.1 for the case $e \leq p-1$ was first proved in [Laffaille 1980], in which the low ramification assumption is used to directly associate certain modules equipped with filtration and Frobenius endomorphism to $p$-divisible groups. This was one of the starting points of $p$-adic Hodge theory, to classify crystalline representations by weakly admissible filtered $\varphi$-modules and establish their connections to algebraic geometric objects.

The goal of this paper is to study the statement analogous to Theorem 1.1 in the relative case. When we work over a relative base ring, the situation becomes much more complicated, and it is unknown how to characterize crystalline representations by linear algebraic data. For example, [Hartl 2013] shows that a naive generalization of weakly admissible modules is not sufficient. In this paper, we obtain a partial result towards this direction for crystalline representations of Hodge–Tate weights in [0, 1].

Let $R_0$ be a base ring over $W(k)\langle t_1^{\pm 1}, \dots, t_m^{\pm 1}\rangle$ given as in Section 2A, and let $R = R_0 \otimes_{W(k)} \mathcal{O}_K$. Let $\mathcal{G}_R$ be the étale fundamental group of $\operatorname{Spec}\left(R\left[\frac{1}{p}\right]\right)$. For representations of $\mathcal{G}_R$, the condition of being *crystalline* is well-defined by [Brinon 2008; Kim 2015]. If $G_R$ is a $p$-divisible group over $R$, its

*MSC2010:* primary 11F80; secondary 11S20, 14L05.
*Keywords:* crystalline representation, $p$-divisible group, relative $p$-adic Hodge theory.

Tate module $T_p(G_R)$ is a crystalline $\mathcal{G}_R$-representation with Hodge–Tate weights in [0, 1] (see [Kim 2015]). Conversely, when the ramification index $e$ is small, we prove that crystalline representations of Hodge–Tate weights in [0, 1] can be associated with the linear algebraic data called Kisin modules of height 1, and show the following:

**Theorem 1.2.** *Suppose $e < p-1$. Let $T$ be a crystalline $\mathcal{G}_R$-representation finite free over $\mathbb{Z}_p$ whose Hodge–Tate weights lie in* [0, 1]. *Then there exists a $p$-divisible group $G_R$ over $R$ such that $T_p(G_R) \cong T$ as $\mathcal{G}_R$-representations.*

As an immediate corollary using the results in [Moon 2020], we obtain the following result on the geometry of the locus of crystalline $\mathcal{G}_R$-representations with Hodge–Tate weights in [0, 1]. For a fixed absolutely irreducible $\mathbb{F}_p$-representation $V_0$ of $\mathcal{G}_R$, there exists a universal deformation ring which parametrizes the deformations of $V_0$ [de Smit and Lenstra 1997]. By [Moon 2020, Theorem 5.7], we deduce:

**Corollary 1.3.** *Suppose $R$ has Krull dimension* 2 *and $e < p-1$. Then the locus of crystalline representations of $\mathcal{G}_R$ with Hodge–Tate weights in* [0, 1] *cuts out a closed subscheme of the universal deformation space.*

We give a more precise statement of Corollary 1.3 in Section 6. The assumption that $R$ has Krull dimension 2 appears in [Moon 2020, Theorem 5.7], since the construction of Barsotti–Tate deformation ring in [Moon 2020, Section 5] uses the result in [de Smit and Lenstra 1997] and relies on the assumption.

We now explain the major ingredients for the proof of Theorem 1.2. Firstly, Kim [2015] generalized the Breuil–Kisin classification in the relative setting, and showed that the category of $p$-divisible groups over $R$ is anti-equivalent to the category of Kisin modules of height 1 over $R_0[\![u]\!]$. Using the classification, we reduce our problem to constructing desired Kisin modules. Secondly, Brinon and Trihan [2008] proved the generalization of Theorem 1.1 for the case when the base is a complete discrete valuation ring whose residue field has a finite $p$-basis. To construct appropriate Kisin modules, we use their result together with the fact that the $p$-adic completion of $R_{0,(p)}$ is an example of such a ring. We remark that our construction of Kisin modules relies on the assumption that the ramification index is small.

**1A.** ***Notations.*** We will reserve $\varphi$ for various Frobenius. To be more precise, let $A$ be an $W(k)$-algebra on which the arithmetic Frobenius $\varphi$ on $W(k)$ extends, and $M$ an $A$-module. We denote $\varphi_A : A \to A$ for such an extension. Let $\varphi_M : M \to M$ be a $\varphi_A$-semilinear map. This is equivalent to having an $A$-linear map $1 \otimes \varphi_M : \varphi_A^* M \to M$, where $\varphi_A^* M$ denotes $A \otimes_{\varphi_A, A} M$. We always drop the subscripts $A$ and $M$ from $\varphi$ if no confusion arises. Let $f : A \to B$ be a ring map compatible with Frobenius, that is, $f \circ \varphi_A = \varphi_B \circ f$. Then $\varphi_M$ naturally extends to $\varphi_{M_B} : M_B \to M_B$ for $M_B := B \otimes_A M$. It is easy to check that $\varphi_B^* M_B = B \otimes_A \varphi_A^* M$ and $1 \otimes \varphi_{M_B} : \varphi_B^* M_B \to M_B$ is equal to $B \otimes_A (1 \otimes \varphi_M)$.

## 2. Relative $p$-adic Hodge theory and étale $\varphi$-modules

**2A.** ***Base ring and crystalline period ring in the relative case.*** We follow the same notations as in the Introduction. We recall the assumptions on the base rings and the construction of crystalline period ring in relative $p$-adic Hodge theory in [Kim 2015] (see also [Brinon 2008]), together with an additional assumption which will be needed later. Let $R_0$ be a ring obtained from $W(k)\langle t_1^{\pm 1}, \dots, t_m^{\pm 1}\rangle$ by a finite number of iterations of the following operations:

- $p$-adic completion of an étale extension;
- $p$-adic completion of a localization;
- completion with respect to an ideal containing $p$.

We suppose that $R_0$ is an integral domain separated and complete with respect to some ideal $J \subset R_0$ containing $p$, such that $R_0/J$ is finitely generated over some field $k'$ (see [Kim 2015, Section 2.2.2]). We further assume that $R_0/pR_0$ is a unique factorization domain.

$R_0/pR_0$ has a finite $p$-basis given by $\{t_1, \dots, t_m\}$ in the sense of [de Jong 1995, Definition 1.1.1]. The Witt vector Frobenius on $W(k)$ extends (not necessarily uniquely) to $R_0$, and we fix such a Frobenius endomorphism $\varphi : R_0 \to R_0$. Let $\widehat{\Omega}_{R_0} := \varprojlim_n \Omega_{(R_0/p^n)/W(k)}$ be the module of $p$-adically continuous Kähler differentials. By [Brinon 2008, Proposition 2.0.2], $\widehat{\Omega}_{R_0} \cong \bigoplus_{i=1}^m R_0 \cdot dt_i$. We work over the base ring $R$ given by $R := R_0 \otimes_{W(k)} \mathcal{O}_K$.

Let $\bar{R}$ denote the union of finite $R$-subalgebras $R'$ of a fixed separable closure of $\mathrm{Frac}(R)$ such that $R'[\frac{1}{p}]$ is étale over $R[\frac{1}{p}]$. Then $\mathrm{Spec}\, \bar{R}[\frac{1}{p}]$ is a pro-universal covering of $\mathrm{Spec}\, R[\frac{1}{p}]$, and $\bar{R}$ is the integral closure of $R$ in $\bar{R}[\frac{1}{p}]$. Let $\mathcal{G}_R := \mathrm{Gal}(\bar{R}[\frac{1}{p}]/R[\frac{1}{p}]) = \pi_1^{\text{ét}}(\mathrm{Spec}\, R[\frac{1}{p}])$. By a representation of $\mathcal{G}_R$, we always mean a finite continuous representation.

The crystalline period ring $B_{\mathrm{cris}}(R)$ is constructed as follows. Let $\bar{R}^\flat = \varprojlim_\varphi \bar{R}/p\bar{R}$. There exists a natural $W(k)$-linear surjective map $\theta : W(\bar{R}^\flat) \to \widehat{\bar{R}}$ which lifts the projection onto the first factor. Here, $\widehat{\bar{R}}$ denotes the $p$-adic completion of $\bar{R}$. Let $\theta_{R_0} : R_0 \otimes_{W(k)} W(\bar{R}^\flat) \to \widehat{\bar{R}}$ be the $R_0$-linear extension of $\theta$. Define the integral crystalline period ring $A_{\mathrm{cris}}(R)$ to be the $p$-adic completion of the divided power envelope of $R_0 \otimes_{W(k)} W(\bar{R}^\flat)$ with respect to $\ker(\theta_{R_0})$. Choose compatibly $\epsilon_n \in \bar{R}$ such that $\epsilon_0 = 1$, $\epsilon_n = \epsilon_{n+1}^p$ with $\epsilon_1 \neq 1$, and let $\tilde{\epsilon} = (\epsilon_n)_{n \geq 0} \in \bar{R}^\flat$. Then $\tau := \log[\tilde{\epsilon}] \in A_{\mathrm{cris}}(R)$. Define $B_{\mathrm{cris}}(R) = A_{\mathrm{cris}}(R)[\frac{1}{\tau}]$. $B_{\mathrm{cris}}(R)$ is equipped naturally with $\mathcal{G}_R$-action and Frobenius endomorphism, and $B_{\mathrm{cris}}(R) \otimes_{R_0[\frac{1}{p}]} R[\frac{1}{p}]$ is equipped with a natural filtration by $R[\frac{1}{p}]$-submodules. Furthermore, we have a natural integrable connection $\nabla : B_{\mathrm{cris}}(R) \to B_{\mathrm{cris}}(R) \otimes_{R_0} \widehat{\Omega}_{R_0}$ such that Frobenius is horizontal and Griffiths transversality is satisfied.

For a $\mathcal{G}_R$-representation $V$ over $\mathbb{Q}_p$, let $D_{\mathrm{cris}}(V) := \mathrm{Hom}_{\mathcal{G}_R}(V, B_{\mathrm{cris}}(R))$. The natural morphism

$$\alpha_{\mathrm{cris}} : D_{\mathrm{cris}}(V) \otimes_{R_0[\frac{1}{p}]} B_{\mathrm{cris}}(R) \to V^\vee \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(R)$$

is injective. We say $V$ is *crystalline* if $\alpha_{\mathrm{cris}}$ is an isomorphism. When $V$ is crystalline, then $D_{\mathrm{cris}}(V)$ is a finite projective $R_0[\frac{1}{p}]$-module, and $D_{\mathrm{cris}}(V) \otimes_{R_0[\frac{1}{p}]} R[\frac{1}{p}]$ has the filtration induced by that on $B_{\mathrm{cris}}(R) \otimes_{R_0[\frac{1}{p}]} R[\frac{1}{p}]$. We define the Hodge–Tate weights similarly as in the classical $p$-adic Hodge theory. Frobenius and connection on $B_{\mathrm{cris}}(R)$ induce those structures on $D_{\mathrm{cris}}(V)$; for the Frobenius endomorphism on $D_{\mathrm{cris}}(V)$, $1 \otimes \varphi : \varphi^* D_{\mathrm{cris}}(V) \to D_{\mathrm{cris}}(V)$ is an isomorphism, and the connection $\nabla : D_{\mathrm{cris}}(V) \to D_{\mathrm{cris}}(V) \otimes_{R_0} \widehat{\Omega}_{R_0}$ is integrable and topologically quasinilpotent. Furthermore, Griffiths transversality is satisfied and $\varphi$ is horizontal. For a $\mathcal{G}_R$-representation $T$ which is free over $\mathbb{Z}_p$, we say it is crystalline if $T[\frac{1}{p}]$ is crystalline.

Suppose $S_0$ is another relative base ring over $W(k)\langle t_1^{\pm 1}, \dots, t_m^{\pm 1}\rangle$ satisfying the above conditions and equipped with a choice of Frobenius, and let $b : R_0 \to S_0$ be a $\varphi$-equivariant $W(k)\langle t_1^{\pm 1}, \dots, t_m^{\pm 1}\rangle$-algebra map. We also denote $b : R = R_0 \otimes_{W(k)} \mathcal{O}_K \to S := S_0 \otimes_{W(k)} \mathcal{O}_K$ the map induced $\mathcal{O}_K$-linearly. By choosing a common geometric point, this induces a map of Galois groups $\mathcal{G}_S \to \mathcal{G}_R$, and also a map of crystalline period rings $B_{\mathrm{cris}}(R) \to B_{\mathrm{cris}}(S)$ compatible with all structures. If $V$ is a crystalline representation of $\mathcal{G}_R$ with certain Hodge–Tate weights, then via these maps $V$ is also a crystalline representation of $\mathcal{G}_S$ with the same Hodge–Tate weights, and the construction of $D_{\mathrm{cris}}(V)$ is compatible with the base change.

We will consider the following base change maps in later sections. Let $\mathcal{O}_{L_0}$ be the $p$-adic completion of $R_{0,(p)}$, and let $b_L : R_0 \to \mathcal{O}_{L_0}$ be the natural $\varphi$-equivariant map. This induces $b_L : R \to \mathcal{O}_L := \mathcal{O}_{L_0} \otimes_{W(k)} \mathcal{O}_K$. Note that $L = \mathcal{O}_L\big[\frac{1}{p}\big]$ is an example of a complete discrete valuation field with a residue field having a finite $p$-basis, studied in [Brinon and Trihan 2008]. On the other hand, for each maximal ideal $\mathfrak{q} \in \mathrm{mSpec}\, R_0$, let $\widehat{R_{0,\mathfrak{q}}}$ be the $\mathfrak{q}$-adic completion of $R_{0,\mathfrak{q}}$. By the structure theorem of complete regular local rings, we have $\widehat{R_{0,\mathfrak{q}}} \cong \mathcal{O}_{\mathfrak{q}}[\![s_1, \dots, s_l]\!]$ where $\mathcal{O}_{\mathfrak{q}}$ is a Cohen ring with the maximal ideal $(p)$ and $l \geq 0$ is an integer ($\widehat{R_{0,\mathfrak{q}}}$ is understood to be $\mathcal{O}_{\mathfrak{q}}$ when $l = 0$). We consider the natural $\varphi$-equivariant morphism $b_{\mathfrak{q}} : R_0 \to \widehat{R_{0,\mathfrak{q}}}$, which induces $b_{\mathfrak{q}} : R \to R_{\mathfrak{q}} := \widehat{R_{0,\mathfrak{q}}} \otimes_{W(k)} \mathcal{O}_K$.

**2B. *Étale $\varphi$-modules.*** We study étale $\varphi$-modules and associated Galois representations. Most of the material in this section is a review of [Kim 2015, Section 7], and the underlying geometry is based on perfectoid spaces as in [Scholze 2012].

Let $R_0$ be a relative base ring over $W(k)\langle t_1^{\pm 1}, \dots, t_m^{\pm 1}\rangle$ and let $R = R_0 \otimes_{W(k)} \mathcal{O}_K$ as above. Choose a uniformizer $\varpi \in \mathcal{O}_K$. For integers $n \geq 0$, we choose compatibly $\varpi_n \in \overline{K}$ such that $\varpi_0 = \varpi$ and $\varpi_{n+1}^p = \varpi_n$, and let $K_\infty$ be the $p$-adic completion of $\bigcup_{n \geq 0} K(\varpi_n)$. Then $K_\infty$ is a perfectoid field and $\big(\widehat{\overline{R}}\big[\frac{1}{p}\big], \widehat{\overline{R}}\big)$ is a perfectoid affinoid $K_\infty$-algebra. Let $K_\infty^\flat$ denote the tilt of $K_\infty$ as defined in [Scholze 2012], and let $\underline{\varpi} := (\varpi_n) \in K_\infty^\flat$.

Let $\mathfrak{S} := R_0[\![u]\!]$ equipped with the Frobenius extending that on $R_0$ by $\varphi(u) = u^p$. Let $E_{R_\infty}^+ = \mathfrak{S}/p\mathfrak{S}$, and let $\widetilde{E}_{R_\infty}^+$ be the $u$-adic completion of $\varinjlim_\varphi E_{R_\infty}^+$. Let $E_{R_\infty} = E_{R_\infty}^+\big[\frac{1}{u}\big]$ and $\widetilde{E}_{R_\infty} = \widetilde{E}_{R_\infty}^+\big[\frac{1}{u}\big]$. By [Scholze 2012, Proposition 5.9], $(\widetilde{E}_{R_\infty}, \widetilde{E}_{R_\infty}^+)$ is a perfectoid affinoid $K_\infty^\flat$-algebra, and we have the natural injective map $(\widetilde{E}_{R_\infty}, \widetilde{E}_{R_\infty}^+) \hookrightarrow \big(\overline{R}^\flat\big[\frac{1}{\underline{\varpi}}\big], \overline{R}^\flat\big)$ given by $u \mapsto \underline{\varpi}$.

Let

$$\widetilde{R}_\infty := W(\widetilde{E}_{R_\infty}^+) \otimes_{W(K_\infty^{\flat\circ}), \theta} \mathcal{O}_{K_\infty}. \tag{2-1}$$

By [Scholze 2012, Remark 5.19], $\big(\widetilde{R}_\infty\big[\frac{1}{p}\big], \widetilde{R}_\infty\big)$ is a perfectoid affinoid $K_\infty$-algebra whose tilt is $(\widetilde{E}_{R_\infty}, \widetilde{E}_{R_\infty}^+)$. Furthermore, it is shown in [Kim 2015] that we have a natural injective map

$$\Big(\widetilde{R}_\infty\Big[\frac{1}{p}\Big], \widetilde{R}_\infty\Big) \hookrightarrow \Big(\widehat{\overline{R}}\Big[\frac{1}{p}\Big], \widehat{\overline{R}}\Big)$$

whose tilt is $(\widetilde{E}_{R_\infty}, \widetilde{E}_{R_\infty}^+) \hookrightarrow \big(\overline{R}^\flat\big[\frac{1}{\underline{\varpi}}\big], \overline{R}^\flat\big)$. For $\mathcal{G}_{\widetilde{R}_\infty} := \pi_1^{\text{ét}}\big(\mathrm{Spec}\, \widetilde{R}_\infty\big[\frac{1}{p}\big]\big)$, we then have a continuous map of Galois groups $\mathcal{G}_{\widetilde{R}_\infty} \to \mathcal{G}_R$, which is a closed embedding by [Gabber and Ramero 2003, Proposition 5.4.54]. By the almost purity theorem in [Scholze 2012], $\overline{R}^\flat\big[\frac{1}{\underline{\varpi}}\big]$ can be canonically identified with the

$\underline{\varpi}$-adic completion of the affine ring of a pro-universal covering of Spec $\widetilde{E}_{R_\infty}$, and letting $\mathcal{G}_{\widetilde{E}_{R_\infty}}$ be the Galois group corresponding to the pro-universal covering, there exists a canonical isomorphism $\mathcal{G}_{\widetilde{E}_{R_\infty}} \cong \mathcal{G}_{\widetilde{R}_\infty}$.

**Lemma 2.1.** *Consider the map of Galois groups $\mathcal{G}_{\mathcal{O}_L} \to \mathcal{G}_R$ induced by choosing a common geometric point for the base change map $b_L : R \to \mathcal{O}_L$ in Section 2A. Then the images of $\mathcal{G}_{\mathcal{O}_L}$ and $\mathcal{G}_{\widetilde{R}_\infty}$ inside $\mathcal{G}_R$ generate the group $\mathcal{G}_R$.*

*Proof.* $E^+_{R_\infty}$ has a finite $p$-basis given by $\{t_1, \dots, t_m, u\}$. Note that for any element of $g \in \mathcal{G}_R$, there exists an element $h \in \mathcal{G}_{\mathcal{O}_L}$ whose image in $\mathcal{G}_R$ induces the same actions on $t_1^{1/p^\infty}, \dots, t_m^{1/p^\infty}, \varpi^{1/p^\infty}$. Since $\widetilde{R}_\infty = W(\widetilde{E}^+_{R_\infty}) \otimes_{W(K^\flat_\infty),\theta} \mathcal{O}_{K_\infty}$, the actions of $g$ and $h$ are the same on the elements of $\widetilde{R}_\infty$. Hence, the assertion follows. □

Now, let $\mathcal{O}_\mathcal{E}$ be the $p$-adic completion of $\mathfrak{S}\big[\frac{1}{u}\big]$. Note that $\varphi$ on $\mathfrak{S}$ extends naturally to $\mathcal{O}_\mathcal{E}$.

**Definition 2.2.** An *étale $(\varphi, \mathcal{O}_\mathcal{E})$-module* is a pair $(\mathcal{M}, \varphi_\mathcal{M})$ where $\mathcal{M}$ is a finitely generated $\mathcal{O}_\mathcal{E}$-module and $\varphi_\mathcal{M} : \mathcal{M} \to \mathcal{M}$ is a $\varphi$-semilinear endomorphism such that $1 \otimes \varphi_\mathcal{M} : \varphi^*\mathcal{M} \to \mathcal{M}$ is an isomorphism. We say that an étale $(\varphi, \mathcal{O}_\mathcal{E})$-module is *projective* (resp. *torsion*) if the underlying $\mathcal{O}_\mathcal{E}$-module $\mathcal{M}$ is projective (resp. $p$-power torsion).

Let $\mathrm{Mod}_{\mathcal{O}_\mathcal{E}}$ denote the category of étale $(\varphi, \mathcal{O}_\mathcal{E})$-modules whose morphisms are $\mathcal{O}_\mathcal{E}$-module maps compatible with Frobenius. Let $\mathrm{Mod}^{\mathrm{pr}}_{\mathcal{O}_\mathcal{E}}$ and $\mathrm{Mod}^{\mathrm{tor}}_{\mathcal{O}_\mathcal{E}}$ respectively denote the full subcategories of projective and torsion objects. Note that we have a natural notion of a subquotient, direct sum, and tensor product for étale $(\varphi, \mathcal{O}_\mathcal{E})$-modules, and duality is defined for projective and torsion objects.

**Lemma 2.3.** *Let $\mathcal{M} \in \mathrm{Mod}^{\mathrm{tor}}_{\mathcal{O}_\mathcal{E}}$ be a torsion étale $\varphi$-module annihilated by $p$. Then $\mathcal{M}$ is a projective $\mathcal{O}_\mathcal{E}/p\mathcal{O}_\mathcal{E}$-module.*

*Proof.* This follows from essentially the same proof as in [Andreatta 2006, Lemma 7.10]. □

We consider $W\big(\bar{R}^\flat\big[\frac{1}{\underline{\varpi}}\big]\big)$ as an $\mathcal{O}_\mathcal{E}$-algebra via mapping $u$ to the Teichmüller lift $[\underline{\varpi}]$ of $\underline{\varpi}$, and let $\mathcal{O}^{\mathrm{ur}}_\mathcal{E}$ be the integral closure of $\mathcal{O}_\mathcal{E}$ in $W\big(\bar{R}^\flat\big[\frac{1}{\underline{\varpi}}\big]\big)$. Let $\widehat{\mathcal{O}}^{\mathrm{ur}}_\mathcal{E}$ be its $p$-adic completion. Since $\mathcal{O}_\mathcal{E}$ is normal, we have $\mathrm{Aut}_{\mathcal{O}_\mathcal{E}}(\mathcal{O}^{\mathrm{ur}}_\mathcal{E}) \cong \mathcal{G}_{E_{R_\infty}} := \pi_1^{\text{ét}}(\mathrm{Spec}\, E_{R_\infty})$, and by [Gabber and Ramero 2003, Proposition 5.4.54] and the almost purity theorem, we have $\mathcal{G}_{E_{R_\infty}} \cong \mathcal{G}_{\widetilde{E}_{R_\infty}} \cong \mathcal{G}_{\widetilde{R}_\infty}$. This induces $\mathcal{G}_{\widetilde{R}_\infty}$-action on $\widehat{\mathcal{O}}^{\mathrm{ur}}_\mathcal{E}$. The following is proved in [Kim 2015].

**Lemma 2.4** [Kim 2015, Lemmas 7.5 and 7.6]. *We have $(\widehat{\mathcal{O}}^{\mathrm{ur}}_\mathcal{E})^{\mathcal{G}_{\widetilde{R}_\infty}} = \mathcal{O}_\mathcal{E}$ and the same holds modulo $p^n$. Furthermore, there exists a unique $\mathcal{G}_{\widetilde{R}_\infty}$-equivariant ring endomorphism $\varphi$ on $\widehat{\mathcal{O}}^{\mathrm{ur}}_\mathcal{E}$ lifting the $p$-th power map on $\widehat{\mathcal{O}}^{\mathrm{ur}}_\mathcal{E}/(p)$ and extending $\varphi$ on $\mathcal{O}_\mathcal{E}$. The inclusion $\widehat{\mathcal{O}}^{\mathrm{ur}}_\mathcal{E} \hookrightarrow W\big(\bar{R}^\flat\big[\frac{1}{\underline{\varpi}}\big]\big)$ is $\varphi$-equivariant where the latter ring is given the Witt vector Frobenius.*

Let $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{\widetilde{R}_\infty})$ be the category of $\mathbb{Z}_p$-representations of $\mathcal{G}_{\widetilde{R}_\infty}$, and let $\mathrm{Rep}^{\mathrm{pr}}_{\mathbb{Z}_p}(\mathcal{G}_{\widetilde{R}_\infty})$ and $\mathrm{Rep}^{\mathrm{tor}}_{\mathbb{Z}_p}(\mathcal{G}_{\widetilde{R}_\infty})$ respectively denote the full subcategories of free and torsion objects. For $\mathcal{M} \in \mathrm{Mod}_{\mathcal{O}_\mathcal{E}}$ and $T \in \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{\widetilde{R}_\infty})$, we define $T(\mathcal{M}) := (\mathcal{M} \otimes_{\mathcal{O}_\mathcal{E}} \widehat{\mathcal{O}}^{\mathrm{ur}}_\mathcal{E})^{\varphi=1}$ and $\mathcal{M}(T) := (T \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}^{\mathrm{ur}}_\mathcal{E})^{\mathcal{G}_{\widetilde{R}_\infty}}$. For a torsion étale $\varphi$-module $\mathcal{M} \in \mathrm{Mod}^{\mathrm{tor}}_{\mathcal{O}_\mathcal{E}}$, we define its *length* to be the length of $\mathcal{M} \otimes_{\mathcal{O}_\mathcal{E}} (\mathcal{O}_\mathcal{E})_{(p)}$ as an $(\mathcal{O}_\mathcal{E})_{(p)}$-module.

**Proposition 2.5** [Kim 2015, Proposition 7.7]. *The assignments $T(\cdot)$ and $\mathcal{M}(\cdot)$ are exact equivalences (inverse of each other) of $\otimes$-categories between* $\mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}$ *and* $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{\widetilde{R}_\infty})$. *Moreover, $T(\cdot)$ and $\mathcal{M}(\cdot)$ restrict to rank-preserving equivalence of categories between* $\mathrm{Mod}^{\mathrm{pr}}_{\mathcal{O}_{\mathcal{E}}}$ *and* $\mathrm{Rep}^{\mathrm{pr}}_{\mathbb{Z}_p}(\mathcal{G}_{\widetilde{R}_\infty})$ *and length-preserving equivalence of categories between* $\mathrm{Mod}^{\mathrm{tor}}_{\mathcal{O}_{\mathcal{E}}}$ *and* $\mathrm{Rep}^{\mathrm{tor}}_{\mathbb{Z}_p}(\mathcal{G}_{\widetilde{R}_\infty})$. *In both cases, $T(\cdot)$ and $\mathcal{M}(\cdot)$ commute with taking duals.*

*Proof.* This is [Kim 2015, Proposition 7.7]. We remark here for some additional details. Note that $E_{R_\infty}$ is a normal domain and $\pi_1^{\text{ét}}(\mathrm{Spec}\,\mathbb{E}_{R_\infty}) \cong \mathcal{G}_{\widetilde{R}_\infty}$. Given Lemma 2.3, the assertion therefore follows from the usual dévissage and [Katz 1973, Lemma 4.1.1]. Note that both functors $T(\cdot)$ and $\mathcal{M}(\cdot)$ are a priori left exact by definition, and exactness can be proved by the same argument as in the proof of [Andreatta 2006, Theorem 7.11]. □

Suppose $S_0$ is another relative base ring over $W(k)\langle t_1^{\pm 1}, \dots, t_m^{\pm 1}\rangle$ as in Section 2A equipped with a choice of Frobenius, and suppose $b : R_0 \hookrightarrow S_0$ be a $\varphi$-equivariant $W(k)\langle t_1^{\pm 1}, \dots, t_m^{\pm 1}\rangle$-algebra map which is injective. Let $b : R = R_0 \otimes_{W(k)} \mathcal{O}_K \hookrightarrow S := S_0 \otimes_{W(k)} \mathcal{O}_K$ be the induced injective map. By choosing a common geometric point we have an injective map $\overline{R} \hookrightarrow \overline{S}$, and this induces an embedding $\widetilde{R}_\infty \hookrightarrow \widetilde{S}_\infty$ by the constructions given in (2-1). Hence, the corresponding map of Galois groups $\mathcal{G}_S \to \mathcal{G}_R$ restricts to $\mathcal{G}_{\widetilde{S}_\infty} \to \mathcal{G}_{\widetilde{R}_\infty}$. Let $\mathfrak{S}_S = S_0[\![u]\!]$ and let $\mathcal{O}_{\mathcal{E},S}$ be the $p$-adic completion of $\mathfrak{S}_S\big[\frac{1}{u}\big]$. Let $\mathcal{M}_S(\cdot)$ be the functor for the base ring $S$ constructed similarly as above. Let $T \in \mathrm{Rep}^{\mathrm{pr}}_{\mathbb{Z}_p}(\mathcal{G}_{\widetilde{R}_\infty})$. Then $T$ is also a $\mathcal{G}_{\widetilde{S}_\infty}$-representation via the map $\mathcal{G}_{\widetilde{S}_\infty} \to \mathcal{G}_{\widetilde{R}_\infty}$, and we have the natural isomorphism $\mathcal{M}(T) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E},S} \cong \mathcal{M}_S(T)$ as étale $(\varphi, \mathcal{O}_{\mathcal{E},S})$-modules by the definition of the functors $\mathcal{M}(\cdot)$ and $T(\cdot)$ and by Proposition 2.5.

## 3. Relative Breuil–Kisin classification

We now explain the classification of $p$-divisible groups over Spec $R$ via Kisin modules, which is proved in [Kisin 2006] when $R = \mathcal{O}_K$ and generalized in [Kim 2015] for the relative case. Denote by $E(u)$ the Eisenstein polynomial for the extension $K$ over $W(k)\big[\frac{1}{p}\big]$, and let $\mathfrak{S} = R_0[\![u]\!]$ as above.

**Definition 3.1.** Denote by $\mathrm{Kis}^1(\mathfrak{S})$ the category of pairs $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ where

- $\mathfrak{M}$ is a finitely generated projective $\mathfrak{S}$-module;
- $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ is a $\varphi$-semilinear map such that $\mathrm{coker}(1 \otimes \varphi_{\mathfrak{M}})$ is annihilated by $E(u)$.

The morphisms are $\mathfrak{S}$-module maps compatible with Frobenius.

Note that for $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in \mathrm{Kis}^1(\mathfrak{S})$, $1 \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is injective since $\mathfrak{M}$ is finite projective over $\mathfrak{S}$ and $\mathrm{coker}(1 \otimes \varphi_{\mathfrak{M}})$ is killed by $E(u)$. Consider the composite $\mathfrak{S} \twoheadrightarrow \mathfrak{S}/u\mathfrak{S} = R_0 \xrightarrow{\varphi} R_0$.

**Definition 3.2.** A *Kisin module* of height 1 is a tuple $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla_{\mathfrak{M}})$ such that:

- $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in \mathrm{Kis}^1(\mathfrak{S})$.
- Let $\mathcal{N} := \mathfrak{M} \otimes_{\mathfrak{S},\varphi} R_0$ equipped with the Frobenius $\varphi_{\mathfrak{M}} \otimes \varphi_{R_0}$. Then $\nabla_{\mathfrak{M}} : \mathcal{N} \to \mathcal{N} \otimes_{R_0} \widehat{\Omega}_{R_0}$ is a topologically quasinilpotent integrable connection commuting with Frobenius.

Here, $\nabla_{\mathfrak{M}}$ being topologically quasinilpotent means that the induced connection on $\mathcal{N}/p\mathcal{N}$ is nilpotent. Denote by $\mathrm{Kis}^1(\mathfrak{S}, \nabla)$ the category of Kisin modules of height 1 whose morphisms are $\mathfrak{S}$-module maps compatible with Frobenius and connection.

The following theorem classifying the $p$-divisible groups is proved in [Kim 2015].

**Theorem 3.3** [Kim 2015, Corollary 6.7 and Remark 6.9]. *There exists an exact anti-equivalence of categories*

$$\mathfrak{M}^* : \{p\text{-divisible groups over } \operatorname{Spec} R\} \to \mathrm{Kis}^1(\mathfrak{S}, \nabla).$$

*Let $S_0$ be another base ring satisfying the condition as in Section 2A and equipped with a Frobenius, and let $b : R_0 \to S_0$ be a $\varphi$-equivariant map. Then the formation of $\mathfrak{M}^*$ commutes with the base change $R \to S := S_0 \otimes_{W(k)} \mathcal{O}_K$ induced $\mathcal{O}_K$-linearly from $b$.*

Note that if $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in \mathrm{Kis}^1(\mathfrak{S})$, then $(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}, \varphi_{\mathfrak{M}} \otimes \varphi_{\mathcal{O}_{\mathcal{E}}})$ is a projective étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-module since $1 \otimes \varphi_{\mathfrak{M}}$ is injective and its cokernel is killed by $E(u)$ which is a unit in $\mathcal{O}_{\mathcal{E}}$. If $G_R$ is a $p$-divisible group over $R$, its Tate module is given by $T_p(G_R) := \mathrm{Hom}_{\bar{R}}(\mathbb{Q}_p/\mathbb{Z}_p, G_R \times_R \bar{R})$, which is a finite free $\mathbb{Z}_p$-representation of $\mathcal{G}_R$. By [Kim 2015, Corollary 8.2], we have a natural $\mathcal{G}_{\tilde{R}_\infty}$-equivariant isomorphism $T^\vee(\mathfrak{M}^*(G_R) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) \cong T_p(G_R)$, where $T^\vee(\mathfrak{M}^*(G_R) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}})$ denotes the dual of $T(\mathfrak{M}^*(G_R) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}})$.

## 4. Construction of Kisin modules

In this section, we will assume $e < p - 1$ from Proposition 4.3. We denote $\mathfrak{S}_n := \mathfrak{S}/p^n\mathfrak{S}$ for positive integers $n \geq 1$. Let $T$ be a crystalline $\mathcal{G}_R$-representation which is free over $\mathbb{Z}_p$ of rank $d$ with Hodge–Tate weights in $[0, 1]$. Let $\mathcal{M} := \mathcal{M}^\vee(T)$ be the associated étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-module, where $\mathcal{M}^\vee(T)$ denotes the dual of $\mathcal{M}(T)$. For each integer $n \geq 1$, denote $\mathcal{M}_n = \mathcal{M}/p^n\mathcal{M}$. Note that $\mathcal{M}_n \cong \mathcal{M}^\vee(T/p^nT)$. On the other hand, consider the map $b_L : R \to \mathcal{O}_L$ as in Section 2A. $T$ is also a crystalline $\mathcal{G}_{\mathcal{O}_L}$-representation with Hodge–Tate weights in $[0, 1]$, so by [Brinon and Trihan 2008, Theorem 6.10], there exists a $p$-divisible group $G_{\mathcal{O}_L}$ over $\mathcal{O}_L$ such that $T_p(G_{\mathcal{O}_L}) \cong T$ as $\mathcal{G}_{\mathcal{O}_L}$-representations. Let $(\mathfrak{M}_{\mathcal{O}_L}, \nabla_{\mathfrak{M}_{\mathcal{O}_L}}) := \mathfrak{M}^*(G_{\mathcal{O}_L}) \in \mathrm{Kis}^1(\mathfrak{S}_{\mathcal{O}_L}, \nabla)$ be the associated Kisin module over $\mathfrak{S}_{\mathcal{O}_L}$. Denote $\mathfrak{M}_{\mathcal{O}_L,n} = \mathfrak{M}_{\mathcal{O}_L}/p^n\mathfrak{M}_{\mathcal{O}_L}$. The map between the Galois groups $\mathcal{G}_{\mathcal{O}_L} \to \mathcal{G}_R$ restricts to $\mathcal{G}_{\tilde{\mathcal{O}}_{L,\infty}} \to \mathcal{G}_{\tilde{R}_\infty}$. Hence, we have the natural isomorphism $\mathcal{M} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E},\mathcal{O}_L} \cong \mathfrak{M}_{\mathcal{O}_L} \otimes_{\mathfrak{S}_{\mathcal{O}_L}} \mathcal{O}_{\mathcal{E},\mathcal{O}_L}$ of étale $(\varphi, \mathcal{O}_{\mathcal{E},\mathcal{O}_L})$-modules. Let $\mathcal{M}_{\mathcal{O}_L} := \mathcal{M} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E},\mathcal{O}_L}$ and $\mathcal{M}_{\mathcal{O}_L,n} := \mathcal{M}_{\mathcal{O}_L}/p^n\mathcal{M}_{\mathcal{O}_L}$.

For each $n \geq 1$, we define

$$\mathfrak{M}_n := \mathcal{M}_n \cap \mathfrak{M}_{\mathcal{O}_L,n},$$

where the intersection is taken as $\mathfrak{S}$-submodules of $\mathcal{M}_{\mathcal{O}_L,n}$. The Frobenius endomorphisms on $\mathcal{M}_n$ and $\mathfrak{M}_{\mathcal{O}_L,n}$ induce a Frobenius endomorphism on $\mathfrak{M}_n$. Since the Frobenius on $\mathcal{M}_{\mathcal{O}_L,n}$ is injective, we have the injective $\mathfrak{S}$-module morphism

$$1 \otimes \varphi : \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_n \to \mathfrak{M}_n$$

for each $n$.

**Lemma 4.1.** *$\mathfrak{M}_n$ is a finitely generated $\mathfrak{S}_n$-module. Furthermore, we have $\varphi$-equivariant isomorphisms*

$$\mathfrak{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \cong \mathcal{M}_n \quad \textit{and} \quad \mathfrak{M}_n \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathcal{O}_L} \cong \mathfrak{M}_{\mathcal{O}_L,n}.$$

*Proof.* We first prove that $\mathfrak{M}_n$ is finite over $\mathfrak{S}_n$. Note that $\mathfrak{M}_{\mathcal{O}_L,n}$ is free over $\mathfrak{S}_{\mathcal{O}_L,n}$ of rank $d$, and choose a basis $\{e_1, \dots, e_d\}$ of $\mathfrak{M}_{\mathcal{O}_L,n}$. On the other hand, since $\mathcal{M}_n$ is projective over $\mathfrak{S}_n\big[\frac{1}{u}\big]$ of rank $d$, there exists a nonzero divisor $g \in \mathfrak{S}_n$ such that $\mathcal{M}_n\big[\frac{1}{g}\big]$ is free of rank $d$ over $\mathfrak{S}_n\big[\frac{1}{u}\big]\big[\frac{1}{g}\big]$. Since $\mathcal{M}_n$ is finite over $\mathfrak{S}_n\big[\frac{1}{u}\big]$, we can choose a basis $\{f_1, \dots, f_d\}$ of $\mathcal{M}_n\big[\frac{1}{g}\big]$ over $\mathfrak{S}_n\big[\frac{1}{u}\big]\big[\frac{1}{g}\big]$ such that letting $\mathfrak{N}$ to be the $\mathfrak{S}_n$-submodule of $\mathcal{M}_n\big[\frac{1}{g}\big]$ generated by $f_1, \dots, f_d$, we have $\mathcal{M}_n \subset \mathfrak{N}\big[\frac{1}{u}\big]$ as $\mathfrak{S}_n\big[\frac{1}{u}\big]$-modules. It suffices to show that $\mathfrak{M}_n \subset \frac{1}{u^h} \cdot \mathfrak{N}$ as $\mathfrak{S}_n$-modules for some integer $h \geq 1$. We have

$$(f_1, \dots, f_d)^t = A \cdot (e_1, \dots, e_d)^t,$$

where $A$ is an invertible $d \times d$ matrix with entries in $\mathfrak{S}_{\mathcal{O}_L,n}\big[\frac{1}{u}\big]\big[\frac{1}{g}\big]$. Consider the intersection $\mathfrak{N}\big[\frac{1}{u}\big] \cap \mathfrak{M}_{\mathcal{O}_L,n}$ as submodules of $\mathfrak{M}_{\mathcal{O}_L,n}\big[\frac{1}{u}\big]\big[\frac{1}{g}\big]$. For an element $x = b_1 f_1 + \cdots + b_d f_d \in \mathfrak{N}\big[\frac{1}{u}\big]$ with $b_1, \dots, b_d \in \mathfrak{S}_n\big[\frac{1}{u}\big]$, we have $x \in \mathfrak{M}_{\mathcal{O}_L,n}$ if and only if

$$(b_1, \dots, b_d) \cdot A = (c_1, \dots, c_d)$$

for some $c_1, \dots, c_d \in \mathfrak{S}_{\mathcal{O}_L,n}$. Then $(b_1, \dots, b_d) = (c_1, \dots, c_d) A^{-1}$, which implies that $\mathfrak{N}\big[\frac{1}{u}\big] \cap \mathfrak{M}_{\mathcal{O}_L,n} \subset \frac{1}{u^h} \cdot \mathfrak{N}$ as $\mathfrak{S}_n$-modules for some integer $h \geq 1$. Since $\mathfrak{M}_n \subset \mathfrak{N}\big[\frac{1}{u}\big] \cap \mathfrak{M}_{\mathcal{O}_L,n}$, this shows the first statement.

We have

$$\mathfrak{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \cong \mathfrak{M}_n\Big[\frac{1}{u}\Big] \cong \mathcal{M}_n \cap \mathcal{M}_{\mathcal{O}_L,n} = \mathcal{M}_n$$

and hence the first isomorphism. On the other hand, since $\mathfrak{S} \to \mathfrak{S}_{\mathcal{O}_L}$ is flat and $\mathfrak{M}_{\mathcal{O}_L,n}$ is finite free over $\mathfrak{S}_{\mathcal{O}_L,n}$, we have

$$\begin{aligned}\mathfrak{M}_n \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathcal{O}_L} &\cong (\mathcal{M}_n \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathcal{O}_L}) \cap (\mathfrak{M}_{\mathcal{O}_L,n} \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathcal{O}_L}) = \mathcal{M}_{\mathcal{O}_L,n} \cap (\mathfrak{M}_{\mathcal{O}_L,n} \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathcal{O}_L}) \\ &\cong \Big(\mathfrak{M}_{\mathcal{O}_L,n} \otimes_{\mathfrak{S}_n} \mathfrak{S}_n\Big[\frac{1}{u}\Big]\Big) \cap (\mathfrak{M}_{\mathcal{O}_L,n} \otimes_{\mathfrak{S}_n} \mathfrak{S}_{\mathcal{O}_L,n}) \cong \mathfrak{M}_{\mathcal{O}_L,n}\end{aligned}$$

by $\mathfrak{S}_n\big[\frac{1}{u}\big] \cap \mathfrak{S}_{\mathcal{O}_L,n} = \mathfrak{S}_n$. □

**Lemma 4.2.** *The cokernel of the $\mathfrak{S}$-module map $1 \otimes \varphi : \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_n \to \mathfrak{M}_n$ is killed by $E(u)$.*

*Proof.* Let $x \in \mathfrak{M}_n$. There exists a unique $y_1 \in \mathcal{O}_{\mathcal{E}} \otimes_{\varphi,\mathcal{O}_{\mathcal{E}}} \mathcal{M}_n \cong \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathcal{M}_n$ such that $(1 \otimes \varphi)(y_1) = E(u)x$. On the other hand, there exists a unique $y_2 \in \mathfrak{S}_{\mathcal{O}_L} \otimes_{\varphi,\mathfrak{S}_{\mathcal{O}_L}} \mathfrak{M}_{\mathcal{O}_L,n}$ such that $(1 \otimes \varphi)(y_2) = E(u)x$. Then we have $y_1 = y_2 \in (\mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathcal{M}_n) \cap (\mathfrak{S}_{\mathcal{O}_L} \otimes_{\varphi,\mathfrak{S}_{\mathcal{O}_L}} \mathfrak{M}_{\mathcal{O}_L,n})$.

Since $\mathcal{O}_{L_0}/p\mathcal{O}_{L_0}$ has a finite $p$-basis given by $t_1, \dots, t_m \in R_0/pR_0$ which also gives a $p$-basis of $R_0/pR_0$, the natural map $\mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_{\mathcal{O}_L,n} \to \mathfrak{S}_{\mathcal{O}_L} \otimes_{\varphi,\mathfrak{S}_{\mathcal{O}_L}} \mathfrak{M}_{\mathcal{O}_L,n}$ is an isomorphism. Hence

$$y_1 \in (\mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathcal{M}_n) \cap (\mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_{\mathcal{O}_L,n}) \cong \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} (\mathcal{M}_n \cap \mathfrak{M}_{\mathcal{O}_L,n}) = \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_n$$

since $\varphi : \mathfrak{S} \to \mathfrak{S}$ is flat by [Brinon 2008, Lemma 7.1.8]. This proves the assertion. □

For any finite $\mathfrak{S}$-module $\mathfrak{N}$ equipped with a $\varphi$-semilinear endomorphism $\varphi : \mathfrak{N} \to \mathfrak{N}$, say $\mathfrak{N}$ has *$E(u)$-height* $\leq 1$ if there exists an $\mathfrak{S}$-module map $\psi : \mathfrak{N} \to \varphi^*\mathfrak{N} = \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}$ such that the composite

$$\varphi^*\mathfrak{N} \xrightarrow{1\otimes\varphi} \mathfrak{N} \xrightarrow{\psi} \varphi^*\mathfrak{N}$$

is $E(u) \cdot \mathrm{Id}_{\varphi^*\mathfrak{N}}$. By Lemma 4.2, $\mathfrak{M}_n$ has $E(u)$-height $\leq 1$.

For each maximal ideal $\mathfrak{q} \in \mathrm{mSpec} R_0$, consider $b_{\mathfrak{q}} : R \to R_{\mathfrak{q}}$ as in Section 2A. By choosing a common geometric point, we have the induced map of Galois groups $\mathcal{G}_{R_{\mathfrak{q}}} \to \mathcal{G}_R$ which restricts to $\mathcal{G}_{\widetilde{R}_{\mathfrak{q},\infty}} \to \mathcal{G}_{\widetilde{R},\infty}$, and $T$ is a crystalline $\mathcal{G}_{R_{\mathfrak{q}}}$-representation with Hodge–Tate weights in $[0, 1]$. Denote $\mathfrak{S}_{\mathfrak{q}} := \widehat{R_{0,\mathfrak{q}}}[\![u]\!]$.

**Proposition 4.3.** *Assume $e < p - 1$. For each integer $n \geq 1$, $\mathfrak{M}_n$ is projective over $\mathfrak{S}_n$ of rank $d$.*

*Proof.* Let $\mathfrak{q}$ be a maximal ideal of $R_0$, and let $\mathfrak{N}_n := \mathfrak{M}_n \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathfrak{q}}$ equipped with the induced Frobenius endomorphism. Then we have the induced $\mathfrak{S}_{\mathfrak{q}}$-linear map $\psi : \mathfrak{N}_n \to \mathfrak{S}_{\mathfrak{q}} \otimes_{\varphi,\mathfrak{S}_{\mathfrak{q}}} \mathfrak{N}_n$ such that the composite

$$\mathfrak{S}_{\mathfrak{q}} \otimes_{\varphi,\mathfrak{S}_{\mathfrak{q}}} \mathfrak{N}_n \xrightarrow{1\otimes\varphi} \mathfrak{N}_n \xrightarrow{\psi} \mathfrak{S}_{\mathfrak{q}} \otimes_{\varphi,\mathfrak{S}_{\mathfrak{q}}} \mathfrak{N}_n$$

is $E(u) \cdot \mathrm{Id}$. For the isomorphism $\widehat{R_{0,\mathfrak{q}}} \cong \mathcal{O}_{\mathfrak{q}}[\![s_1, \dots, s_l]\!]$ as above, let us consider the projection $\mathfrak{S}_{\mathfrak{q}} \to \mathfrak{S}_{\mathfrak{q}}/(p, s_1, \dots, s_l) \cong k_{\mathfrak{q}}[\![u]\!]$, where $k_{\mathfrak{q}} := \mathcal{O}_{\mathfrak{q}}/(p)$. Denote $\overline{\mathfrak{N}}_n = \mathfrak{N}_n \otimes_{\mathfrak{S}_{\mathfrak{q}}} k_{\mathfrak{q}}[\![u]\!]$ equipped with the induced Frobenius. Then we have the induced $k_{\mathfrak{q}}[\![u]\!]$-linear map $\psi : \overline{\mathfrak{N}}_n \to k_{\mathfrak{q}}[\![u]\!] \otimes_{\varphi,k_{\mathfrak{q}}[\![u]\!]} \overline{\mathfrak{N}}_n$ such that the composite

$$k_{\mathfrak{q}}[\![u]\!] \otimes_{\varphi,k_{\mathfrak{q}}[\![u]\!]} \overline{\mathfrak{N}}_n \xrightarrow{1\otimes\varphi} \overline{\mathfrak{N}}_n \xrightarrow{\psi} k_{\mathfrak{q}}[\![u]\!] \otimes_{\varphi,k_{\mathfrak{q}}[\![u]\!]} \overline{\mathfrak{N}}_n$$

is $u^e \cdot \mathrm{Id}$. Since $k_{\mathfrak{q}}[\![u]\!]$ is a principal ideal domain, $\overline{\mathfrak{N}}_n$ is a direct sum of its free part and $u$-torsion part $\overline{\mathfrak{N}}_n \cong \overline{\mathfrak{N}}_{n,\mathrm{free}} \oplus \overline{\mathfrak{N}}_{n,\mathrm{tor}}$ as $k_{\mathfrak{q}}[\![u]\!]$-modules. Furthermore, $\varphi$ maps $\overline{\mathfrak{N}}_{n,\mathrm{tor}}$ into $\overline{\mathfrak{N}}_{n,\mathrm{tor}}$, and hence the above maps induce

$$k_{\mathfrak{q}}[\![u]\!] \otimes_{\varphi,k_{\mathfrak{q}}[\![u]\!]} \overline{\mathfrak{N}}_{n,\mathrm{tor}} \xrightarrow{1\otimes\varphi} \overline{\mathfrak{N}}_{n,\mathrm{tor}} \xrightarrow{\psi} k_{\mathfrak{q}}[\![u]\!] \otimes_{\varphi,k_{\mathfrak{q}}[\![u]\!]} \overline{\mathfrak{N}}_{n,\mathrm{tor}}$$

whose composite is $u^e \cdot \mathrm{Id}$.

We claim that $\overline{\mathfrak{N}}_{n,\mathrm{tor}} = 0$. Suppose otherwise. Then $\overline{\mathfrak{N}}_{n,\mathrm{tor}} \cong \bigoplus_{i=1}^{b} k_{\mathfrak{q}}[\![u]\!]/(u^{a_i})$ for some integers $a_i \geq 1$, and $k_{\mathfrak{q}}[\![u]\!] \otimes_{\varphi,k_{\mathfrak{q}}[\![u]\!]} \overline{\mathfrak{N}}_{n,\mathrm{tor}} \cong \bigoplus_{i=1}^{b} k_{\mathfrak{q}}[\![u]\!]/(u^{pa_i})$. By taking the appropriate wedge product and letting $a = a_1 + \cdots + a_b$, the above maps induce the map of $k_{\mathfrak{q}}[\![u]\!]$-modules

$$k_{\mathfrak{q}}[\![u]\!]/(u^{pa}) \xrightarrow{1\otimes\varphi} k_{\mathfrak{q}}[\![u]\!]/(u^{a}) \xrightarrow{\psi} k_{\mathfrak{q}}[\![u]\!]/(u^{pa})$$

whose composite is equal to $u^{eb} \cdot \mathrm{Id}$. Let $(1\otimes\varphi)(1) = f(u) \in k_{\mathfrak{q}}[\![u]\!]/(u^a)$, and $\psi(1) = h(u) \in k_{\mathfrak{q}}[\![u]\!]/(u^{pa})$. Then $u^{pa} \mid u^a h(u)$, so $u^{(p-1)a} \mid h(u)$. On the other hand, $f(u)h(u) = u^{eb}$ in $k_{\mathfrak{q}}[\![u]\!]/(u^{pa})$. This implies $u^{(p-1)a} \mid u^{eb}$. But $e < p - 1$ and $a \geq b$, so we get a contradiction. Hence, $\overline{\mathfrak{N}}_{n,\mathrm{tor}} = 0$ and $\overline{\mathfrak{N}}_n$ is free over $k_{\mathfrak{q}}[\![u]\!]$ of rank $d$, since by Lemma 4.1 $\overline{\mathfrak{N}}_n\left[\frac{1}{u}\right] \cong (\mathcal{M}_n \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathfrak{q}}) \otimes_{\mathfrak{S}_{\mathfrak{q}}} k_{\mathfrak{q}}[\![u]\!]$ which is projective over $k_{\mathfrak{q}}(\!(u)\!)$

of rank $d$. Let $b_1, \dots, b_d \in \mathfrak{N}_n$ be a lift of a basis elements of $\overline{\mathfrak{N}}_n$. By Nakayama's lemma, we have a surjection of $\mathfrak{S}_{\mathfrak{q},\mathfrak{n}}$-modules

$$f : \bigoplus_{i=1}^{d} \mathfrak{S}_{\mathfrak{q},n} \cdot e_i \twoheadrightarrow \mathfrak{N}_n$$

given by $e_i \mapsto b_i$. Since $\mathfrak{N}_n\bigl[\frac{1}{u}\bigr] \cong \mathcal{M}_n \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathfrak{q}}$ is projective over $\mathfrak{S}_{\mathfrak{q},n}\bigl[\frac{1}{u}\bigr]$ of rank $d$, $f$ is also injective. Thus, $\mathfrak{N}_n = \mathfrak{M}_n \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathfrak{q}}$ is projective over $\mathfrak{S}_{\mathfrak{q},n}$ of rank $d$. Since this holds for every $\mathfrak{q} \in \mathrm{mSpec}\, R_0$, it proves the assertion. □

**Lemma 4.4.** *Assume $e < p-1$. Let $\mathfrak{N}$ and $\mathfrak{N}'$ be finite $u$-torsion free $\mathfrak{S}$-modules equipped with Frobenius endomorphisms such that $\mathfrak{N}\bigl[\frac{1}{u}\bigr]$ and $\mathfrak{N}'\bigl[\frac{1}{u}\bigr]$ are torsion étale $\varphi$-modules. Suppose that $\mathfrak{N}$ and $\mathfrak{N}'$ have $E(u)$-height $\le 1$ and $\mathfrak{N}\bigl[\frac{1}{u}\bigr] = \mathfrak{N}'\bigl[\frac{1}{u}\bigr]$ as étale $\varphi$-modules. Then $\mathfrak{N} = \mathfrak{N}'$.*

*Proof.* Consider $\mathfrak{N}$ and $\mathfrak{N}'$ as $\mathfrak{S}$-submodules of $\mathfrak{N}\bigl[\frac{1}{u}\bigr]$. Let $\mathfrak{L}$ be the cokernel of the embedding $\mathfrak{N} \hookrightarrow \mathfrak{N}+\mathfrak{N}'$ of $\mathfrak{S}$-modules. Note that $\mathfrak{S} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{N}+\mathfrak{N}') \cong \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N} + \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}'$ since $\varphi : \mathfrak{S} \to \mathfrak{S}$ is flat. Thus, $\mathfrak{N}+\mathfrak{N}'$ has $E(u)$-height $\le 1$, and $\mathfrak{L}$ has $E(u)$-height $\le 1$. Since $\mathfrak{L}\bigl[\frac{1}{u}\bigr] = 0$, we deduce similarly as in the proof of Proposition 4.3 that $\mathfrak{L} = 0$. So $\mathfrak{N} = \mathfrak{N}+\mathfrak{N}'$. Similarly, $\mathfrak{N}' = \mathfrak{N}+\mathfrak{N}'$. □

It is clear that both $p\mathfrak{M}_{n+1}$ and $\mathfrak{M}_n$ are $u$-torsion free, have $E(u)$-height $\le 1$ and

$$p\mathfrak{M}_{n+1}\Bigl[\frac{1}{u}\Bigr] = p\mathcal{M}_{n+1} \cong \mathcal{M}_n = \mathfrak{M}_n\Bigl[\frac{1}{u}\Bigr].$$

We conclude the following:

**Proposition 4.5.** *Assume $e < p-1$. For each $n \ge 1$, we have a $\varphi$-equivariant isomorphism*

$$p\mathfrak{M}_{n+1} \cong \mathfrak{M}_n.$$

By Lemma 4.2, Proposition 4.3 and 4.5, if we suppose $e < p-1$ and define the $\mathfrak{S}$-module

$$\mathfrak{M} := \varprojlim_n \mathfrak{M}_\mathrm{n},$$

then $\mathfrak{M} \in \mathrm{Kis}^1(\mathfrak{S})$. Note that we have a $\varphi$-equivariant isomorphism $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathcal{O}_L} \cong \mathfrak{M}_{\mathcal{O}_L}$ by Lemma 4.1.

**Remark 4.6.** Analogous statements hold when $T$ is a crystalline $\mathcal{G}_R$-representation with Hodge–Tate weights in $[0, r]$ for the case $er < p-1$, since [Brinon and Trihan 2008] constructs more generally a functor from crystalline representations with Hodge–Tate weights in $[0, r]$ to Kisin modules of height $r$ when the base is a complete discrete valuation field whose residue field has a finite $p$-basis.

To study connections for $\mathfrak{M}$, we first consider the following general situation. Let $A_0$ be a $k$-algebra which is an integral domain. Consider $n$-variables $x_1, \dots, x_n$, and denote $\underline{x} = (x_1, \dots, x_n)^t$ and $\underline{x}^{[p]} := (x_1^p, \dots, x_n^p)^t$. An Artin–Schreier system of equations in $n$ variables over $A_0$ is given by

$$\underline{x} = B\underline{x}^{[p]} + C, \tag{4-1}$$

where $B = (b_{ij})_{1\le i,j\le n} \in M_{n\times n}(A_0)$ is an $n \times n$ matrix with entries in $A_0$ and $C = (c_i)_{1\le i\le n} \in M_{n\times 1}(A_0)$.

Let

$$A_1 := A_0[x_1, \dots, x_n]/\Big(x_1 - c_1 - \sum_{i=1}^{n} b_{1i}x_i^p, \dots, x_n - c_n - \sum_{i=1}^{n} b_{ni}x_i^p\Big),$$

which is the $A_0$-algebra parametrizing the solutions of (4-1). $A_0 \to A_1$ is étale by [Vasiu 2013, Theorem 2.4.1(a)].

**Lemma 4.7.** *There exists a nonzero element $f \in A_0$ which depends only on $B$ (and not on $C$) such that $A_1\big[\frac{1}{f}\big]$ is finite étale over $A_0\big[\frac{1}{f}\big]$.*

*Proof.* We induct on $n$. Suppose $n = 1$. If $\det B \neq 0$, then (4-1) is equivalent to

$$x_1^p = B^{-1}x_1 - B^{-1}C,$$

so the assertion holds with $f = \det B$. If $\det B = 0$, then $B = 0$ and $A_1 \cong A_0$, so the assertion holds trivially.

For $n \geq 2$, if $\det B \neq 0$, then (4-1) is equivalent to

$$\underline{x}^{[p]} = B^{-1}\underline{x} - B^{-1}C.$$

Hence, with $f = \det B$, $A_1\big[\frac{1}{f}\big]$ is finite étale over $A_0\big[\frac{1}{f}\big]$. Suppose $\det B = 0$. Denote by $B^{(i)}$ the $i$-th row of $B$. Then up to renumbering the index for $x_i$'s, we have

$$\sum_{i=1}^{n} e_i B^{(i)} = 0$$

for some nonzero $f_1 \in A_0$ depending only on $B$ and some $e_i \in A_0\big[\frac{1}{f_1}\big]$ with $e_n = 1$. From (4-1), we get

$$x_n = -\sum_{i=1}^{n-1} e_i x_i + c_n + \sum_{i=1}^{n-1} c_i e_i.$$

Hence, denoting $\underline{x}' = (x_1, \dots, x_{n-1})^t$, (4-1) is equivalent to an Artin–Schreier system of equations in $n-1$ variables over $A_0\big[\frac{1}{f_1}\big]$

$$\underline{x}' = B'\underline{x}'^{[p]} + C'$$

where $B' \in M_{(n-1)\times(n-1)}\big(A_0\big[\frac{1}{f_1}\big]\big)$ and $C' \in M_{(n-1)\times 1}\big(A_0\big[\frac{1}{f_1}\big]\big)$. Note that $B'$ depends only on $B$ and not on $C$. Hence, the assertion follows by induction. □

Let $\mathcal{N} := \mathfrak{M} \otimes_{\mathfrak{S},\varphi} R_0$ equipped with the Frobenius $\varphi_{\mathfrak{M}} \otimes \varphi_{R_0}$. From [Kim 2015, Equations (6.1), (6.2) and Remark 3.13], we have the $R_0$-submodule $\mathrm{Fil}^1\mathcal{N} \subset \mathcal{N}$ associated with $\mathfrak{M} \in \mathrm{Kis}^1(\mathfrak{S})$ such that $p\mathcal{N} \subset \mathrm{Fil}^1\mathcal{N}$, $\mathcal{N}/\mathrm{Fil}^1\mathcal{N}$ is projective over $R_0/(p)$, and $(1 \otimes \varphi)(\varphi^*\mathrm{Fil}^1\mathcal{N}) = p\mathcal{N}$ as $R_0$-modules (see [Kim 2015, Definitions 3.4 and 3.6] for the frame $(R_0, pR_0, R_0/(p), \varphi_{R_0}, \varphi_{R_0}/p)$). Fix an $R_0$-direct factor $\mathcal{N}^1 \subset \mathcal{N}$ which lifts $\mathrm{Fil}^1\mathcal{N}/p\mathcal{N} \subset \mathcal{N}/p\mathcal{N}$, and let

$$\widetilde{\mathcal{N}} := R_0 \otimes_{\varphi, R_0} \Big(\mathcal{N} + \frac{1}{p}\mathcal{N}^1\Big) \subset R_0\Big[\frac{1}{p}\Big] \otimes_{\varphi, R_0} \mathcal{N}.$$

Let $\mathrm{Spf}(A, p) \to \mathrm{Spf}(R_0, p)$ be an étale morphism. Note that $A$ is equipped with a unique Frobenius lifting that on $R_0$, and $\widehat{\Omega}_A \cong A\widehat{\otimes}_{R_0}\widehat{\Omega}_{R_0} \cong \bigoplus_{i=1}^m A \cdot dt_i$. For a connection

$$\nabla_{A,n} : A/(p^n) \otimes_{R_0} \mathcal{N} \to (A/(p^n) \otimes_{R_0} \mathcal{N}) \otimes_A \widehat{\Omega}_A$$

on $A/(p^n) \otimes_{R_0} \mathcal{N}$, we say that the Frobenius is *horizontal* if the following diagram commutes:

$$\begin{array}{ccc} A/(p^n) \otimes_A \widetilde{\mathcal{N}} & \xrightarrow{\varphi^*(\nabla_{A,n})} & A/(p^n) \otimes_A \widetilde{\mathcal{N}} \otimes_A \widehat{\Omega}_A \\ \downarrow{\scriptstyle 1\otimes\varphi} & & \downarrow{\scriptstyle (1\otimes\varphi)\otimes \mathrm{id}_{\widehat{\Omega}_A}} \\ A/(p^n) \otimes_A \mathcal{N} & \xrightarrow{\nabla_{A,n}} & A/(p^n) \otimes_A \mathcal{N} \otimes_A \widehat{\Omega}_A \end{array}$$

Here, $\varphi^*(\nabla_{A,n})$ is given by choosing an arbitrary lift of $\nabla_{A,n}$ on $A/(p^{n+1}) \otimes_A \mathcal{N}$, and $\varphi^*(\nabla_{A,n})$ does not depend on the choice of such a lift (see [Vasiu 2013, Section 3.1.1, Equation (9)]).

**Proposition 4.8.** *There exists $\tilde{f} \in R_0$ with $\tilde{f} \notin pR_0$ such that the following holds*:

*Let $S_0$ be the $p$-adic completion of $R_0\big[\frac{1}{\tilde{f}}\big]$ equipped with the induced Frobenius, and let $\mathfrak{S}_S = S_0[\![u]\!]$. Let $\mathfrak{M}_S = \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_S$ equipped with the induced Frobenius, so $\mathfrak{M}_S \in \mathrm{Kis}^1(\mathfrak{S}_S)$. Then there exists a topologically quasinilpotent integrable connection*

$$\nabla_{\mathfrak{M}_S} : (S_0 \otimes_{\varphi,\mathfrak{S}_S} \mathfrak{M}_S) \to (S_0 \otimes_{\varphi,\mathfrak{S}_S} \mathfrak{M}_S) \otimes_{S_0} \widehat{\Omega}_{S_0}$$

*such that $\varphi$ is horizontal, and thus $(\mathfrak{M}_S, \nabla_{\mathfrak{M}_S}) \in \mathrm{Kis}^1(\mathfrak{S}_S, \nabla)$. Furthermore, we can choose $\nabla_{\mathfrak{M}_S}$ so that $\mathfrak{M}_S \otimes_{\mathfrak{S}_S} \mathfrak{S}_{\mathcal{O}_L}$ equipped with the induced Frobenius and connection is isomorphic to $(\mathfrak{M}_{\mathcal{O}_L}, \nabla_{\mathfrak{M}_{\mathcal{O}_L}})$ as Kisin modules over $\mathfrak{S}_{\mathcal{O}_L}$.*

*Proof.* Without loss of generality, we may pass to a Zariski open set of $\mathrm{Spf}(R_0, p)$ if necessary so that $\mathcal{N}^1$ and $\mathcal{N}/\mathcal{N}^1$ are free over $R_0$. Fix an $R_0$-basis of $\mathcal{N}$ adapted to the direct factor $\mathcal{N}^1$. Let $\mathrm{Spf}(A, p) \to \mathrm{Spf}(R_0, p)$ be an étale morphism. Consider a connection

$$\nabla_{A,1} : A/(p) \otimes_{R_0} \mathcal{N} \to (A/(p) \otimes_{R_0} \mathcal{N}) \otimes_A \widehat{\Omega}_A$$

such that the Frobenius is horizontal. By [Vasiu 2013, Section 3.2, Basic Theorem] and its proof, the set of such connections $\nabla_{A,1}$ corresponds to the set of solutions over $A/(p)$ of an Artin–Schreier system of equations

$$\underline{x} = B\underline{x}^{[p]} + C_1$$

for $\underline{x} = (x_1, \ldots, x_{dm})^t$, where $B \in M_{dm\times dm}(R_0/(p))$ and $C_1 \in M_{dm\times 1}(R_0/(p))$. When $A = \mathcal{O}_{L_0}$, it has a solution given by $\nabla_{\mathfrak{M}_{L_0}}$. Since $\mathcal{O}_{L_0}/(p) \cong \mathrm{Frac}(R_0/(p))$ and $R_0/(p)$ is a unique factorization domain, the solution lies in $(R_0/(p))\big[\frac{1}{f}\big]$ for some nonzero $f \in R_0/(p)$ depending only on $B$ by Lemma 4.7 and its proof. Let $\tilde{f} \in R_0$ be a lift of $f$, and let $S_0$ be the $p$-adic completion of $R_0\big[\frac{1}{\tilde{f}}\big]$.

For $n \geq 1$, suppose we are given a connection

$$\nabla_{S_0,n} : S_0/(p^n) \otimes_{R_0} \mathcal{N} \to (S_0/(p^n) \otimes_{R_0} \mathcal{N}) \otimes_{S_0} \widehat{\Omega}_{S_0}$$

such that the Frobenius is horizontal and inducing $\nabla_{\mathfrak{M}_{L_0}} (\mathrm{mod}\ p^n)$ via the natural map $S_0 \to \mathcal{O}_{L_0}$. By [Vasiu 2013, Section 3.2, Basic Theorem] and its proof, for the choice of a basis of $\mathcal{N}$ as above, the set of connections

$$\nabla_{S_0,n+1} : S_0/(p^{n+1}) \otimes_{R_0} \mathcal{N} \to (S_0/(p^{n+1}) \otimes_{R_0} \mathcal{N}) \otimes_{S_0} \widehat{\Omega}_{S_0}$$

such that the Frobenius is horizontal and lifting $\nabla_{S_0,n}$ corresponds to the set of solutions over $S_0/(p)$ of an Artin–Schreier system of equations

$$\underline{x} = B\underline{x}^{[p]} + C_{n+1},$$

where $B$ is the same matrix as above and $C_{n+1} \in M_{dm\times 1}(S_0/(p))$. The solution over $\mathcal{O}_{L_0}/(p)$ given by $\nabla_{\mathfrak{M}_{L_0}}$ lies in $S_0/(p)$ by Lemma 4.7 and its proof. This proves the assertion. ☐

**Proposition 4.9.** *Let $S_0$ be a ring as given in Proposition 4.8, and let $S = S_0 \otimes_{W(k)} \mathcal{O}_K$. Then there exists a $p$-divisible group $G_S$ over $S$ such that $T_p(G_S) \cong T$ as $\mathcal{G}_S$-representations.*

*Proof.* Let $G_S$ be the $p$-divisible group over $S$ given by $(\mathfrak{M}_S, \nabla_{\mathfrak{M}_S})$ in Proposition 4.8. Since $\mathfrak{M}_S \otimes_{\mathfrak{S}_S} \mathfrak{S}_{\mathcal{O}_L} \cong \mathfrak{M}_{\mathcal{O}_L}$ as Kisin modules, we have $T_p(G_S) \cong T$ as $\mathcal{G}_{\mathcal{O}_L}$-representations. On the other hand, $\mathfrak{M}_S \otimes_{\mathfrak{S}_S} \mathcal{O}_{\mathcal{E},S} \cong \mathcal{M} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E},S}$ as étale $\varphi$-modules. Hence, $T_p(G_S) \cong T$ as $\mathcal{G}_{\widetilde{S},\infty}$-representations. Since $\mathcal{G}_{\widetilde{S},\infty}$ and $\mathcal{G}_{\mathcal{O}_L}$ generate the Galois group $\mathcal{G}_S$ by Lemma 2.1, we have $T_p(G_S) \cong T$ as $\mathcal{G}_S$-representations. ☐

## 5. Proof of the main theorem

In this section, we finish the proof of Theorem 1.2. We begin by recalling the following well-known lemma about $p$-divisible groups.

**Lemma 5.1.** *Let $R_1$ be an integral domain over $W(k)$ such that* $\mathrm{Frac}(R_1)$ *has characteristic* 0. *Then via the Tate module functor $T_p(\cdot)$, the category of $p$-divisible groups over $R_1\left[\frac{1}{p}\right]$ is equivalent to the category of finite free $\mathbb{Z}_p$-representations of $\mathcal{G}_{R_1} = \pi_1^{\text{ét}}\left(\mathrm{Spec}\ R_1\left[\frac{1}{p}\right]\right)$. Furthermore, such an equivalence is functorial in the following sense*:

*Let $R_1 \to R_2$ be a map of integral domains over $W(k)$ such that* $\mathrm{Frac}(R_1)$ *and* $\mathrm{Frac}(R_2)$ *have characteristic* 0. *Let $G_{R_1}$ be a $p$-divisible group over $R_1$. Then $T_p(G_{R_1}) \cong T_p(G_{R_1} \times_{R_1} R_2)$ as $\mathcal{G}_{R_2}$-representations.*

We first consider the case when $R$ is a formal power series ring of dimension 2.

**Proposition 5.2.** *Suppose $R_0 = \mathcal{O}[\![s]\!]$ for a Cohen ring $\mathcal{O}$, and $e \leq p-1$. Let $T$ be a crystalline $\mathcal{G}_R$-representation which is finite free over $\mathbb{Z}_p$ and has Hodge–Tate weights in* $[0, 1]$. *Then there exists a $p$-divisible group $G_R$ over $R$ such that $T_p(G_R) \cong T$ as $\mathcal{G}_R$-representations.*

*Proof.* Let $G$ be a $p$-divisible group over $R\left[\frac{1}{p}\right]$ given by Lemma 5.1 such that $T_p(G) \cong T$ as $\mathcal{G}_R$-representations. It suffices to show that $G$ extends to a $p$-divisible group $G_R$ over $R$.

By [Brinon and Trihan 2008, Theorem 6.10], there exists a $p$-divisible group $G_{\mathcal{O}_L}$ over $\mathcal{O}_L$ extending $G \times_{R[\frac{1}{p}]} L$. For each integer $n \geq 1$, let $A_n$ be the Hopf algebra over $R\big[\frac{1}{s}\big]\big[\frac{1}{p}\big]$ for the finite flat group scheme $\big(G \times_{R[\frac{1}{p}]} R\big[\frac{1}{s}\big]\big[\frac{1}{p}\big]\big)[p^n]$, and let $B_n$ be the Hopf algebra over $\mathcal{O}_L$ for the finite flat group scheme $G_{\mathcal{O}_L}[p^n]$. Identify $A_n \otimes_{R[\frac{1}{s}][\frac{1}{p}]} L = B_n \otimes_{\mathcal{O}_L} L$ as Hopf algebras over $L$. Note that the $p$-adic completion of $R\big[\frac{1}{s}\big]$ is isomorphic to $\mathcal{O}_L$. By [Beauville and Laszlo 1995, Main Theorem] and its proof, the $R\big[\frac{1}{s}\big]$-subalgebra $C_n := A_n \cap B_n \subset B_n \otimes_{\mathcal{O}_L} L$ is finite flat over $R\big[\frac{1}{s}\big]$. Moreover, $C_n$ is equipped with the Hopf algebra structure induced from $(A_n, B_n)$ such that $C_n \otimes_{R[\frac{1}{s}]} R\big[\frac{1}{s}\big]\big[\frac{1}{p}\big] \cong A_n$ and $C_n \otimes_{R[\frac{1}{s}]} \mathcal{O}_L \cong B_n$. Hence, the datum of finite flat group schemes

$$\left(\left(G \times_{R[\frac{1}{p}]} R\left[\frac{1}{s}\right]\left[\frac{1}{p}\right]\right)[p^n], G_{\mathcal{O}_L}[p^n]\right)$$

descends to a finite flat group scheme over $R\big[\frac{1}{s}\big]$ (up to a unique isomorphism by [Beauville and Laszlo 1995, Main Theorem]).

Thus, we obtain a system of finite flat group schemes $(G_{U,n})_{n\geq 1}$ over $U := \operatorname{Spec} R \setminus \mathrm{pt}$ extending $(G[p^n])_{n\geq 1}$. Here, pt denotes the closed point given by the maximal ideal of $R$. The natural induced sequence of finite flat group schemes

$$0 \to G_{U,1} \to G_{U,n+1} \xrightarrow{\times p} G_{U,n} \to 0$$

is exact by fpqc descent. So $(G_{U,n})_{n\geq 1}$ is a $p$-divisible group over $U$ extending $G$. Since $e \leq p-1$, $G_U$ extends to a $p$-divisible group $G_R$ over $R$ by [Vasiu and Zink 2010, Theorem 3]. □

**Remark 5.3.** As illustrated in the above proof, this special case can be shown without using Kisin modules. However, the purity result for $p$-divisible groups [Vasiu and Zink 2010, Theorem 3] is proved only when $R$ is regular local of Krull dimension 2 with low ramification (see [Vasiu and Zink 2010, Section 5.1]). So we use the construction of Kisin modules to show Theorem 1.2 for more general $R$ with arbitrary dimensions.

Now, let $R_0$ be a general ring satisfying the assumptions in Section 2A, and let $R = R_0 \otimes_{W(k)} \mathcal{O}_K$ with $e < p-1$. Let $T$ be a crystalline $\mathcal{G}_R$-representation free over $\mathbb{Z}_p$ with Hodge–Tate weights in [0, 1]. Denote by $\mathfrak{M}_{\mathfrak{S}}(T)$ the $\mathfrak{S}$-module in $\mathrm{Kis}^1(\mathfrak{S})$ constructed from $T$ as in Section 4. Let $\tilde{f} \in R_0$ be an element as in Proposition 4.8, and let $S_0$ be the $p$-adic completion of $R_0\big[\frac{1}{\tilde{f}}\big]$ as in Proposition 4.9. Let $f \in R_0/pR_0$ be the image of $\tilde{f}$ in the projection $R_0 \to R_0/(p)$. Note that if $f$ is a unit in $R_0/pR_0$, then $\tilde{f}$ is a unit in $R_0$ since $R_0$ is $p$-adically complete. So for such a case, $S_0 = R_0$ and Theorem 1.2 follows from Proposition 4.9. Now consider the case when $f$ is not a unit in $R_0/(p)$. Since $R_0/(p)$ is a UFD, there exist prime elements $\bar{s}_1, \dots, \bar{s}_l$ of $R_0/(p)$ dividing $f$. Let $s_1, \dots, s_l \in R_0$ be any preimages of $\bar{s}_1, \dots, \bar{s}_l$ respectively.

For each $i = 1, \dots, l$, consider the prime ideal $\mathfrak{p}_i = (p, s_i) \subset R_0$ and let $R_0^{(i)} := \widehat{R_{0,\mathfrak{p}_i}}$ be the $\mathfrak{p}_i$-adic completion of $R_{0,\mathfrak{p}_i}$. Note that $R_0^{(i)}$ is a formal power series ring over a Cohen ring with Krull dimension 2. We consider the natural $\varphi$-equivariant map $b_i : R_0 \to R_0^{(i)}$, which induces $b_i : R \to R^{(i)} := R_0^{(i)} \otimes_{W(k)} \mathcal{O}_K$. On the other hand, let $k_c$ be a field extension of $\mathrm{Frac}(R_0/pR_0)$ which is a composite of the fields

$\mathrm{Frac}(R_0^{(i)}/(p))$ for $i=1,\dots,l$, and let $k_c^{\mathrm{perf}}=\varinjlim_\varphi k_c$ be its direct perfection. By the universal property of $p$-adic Witt vectors, there exists a unique $\varphi$-equivariant map $b_c: R_0\to W(k_c^{\mathrm{perf}})$. Moreover, for each $i=1,\dots,l$, we have a unique $\varphi$-equivariant embedding $R_0^{(i)}\to W(k_c^{\mathrm{perf}})$ whose composite with $b_i$ is equal to $b_c$. We claim that $S_0/(p)\cap\bigcap_{i=1}^l(R_0^{(i)}/(p))=R_0/(p)$ inside $k_c^{\mathrm{perf}}$. To see the claim, let $x$ be a nonzero element of $S_0/(p)=(R_0/(p))\bigl[\frac{1}{f}\bigr]$ such that it also lies in $\bigcap_{i=1}^l(R_0^{(i)}/(p))$. We can write $x=\bigl(\prod_{i=1}^l\bar{s}_i^{n_i}\bigr)\cdot a$ for some integers $n_i$ and some $a\in R_0/(p)$ which is not in the ideal $(\bar{s}_1,\dots,\bar{s}_l)$ of $R_0/(p)$. Suppose $n_1<0$. Then $(1/\bar{s}_1^{n_1})x$ lies in the maximal ideal of $R_0^{(1)}/(p)$. But $(1/\bar{s}_1^{n_1})x=\bigl(\prod_{i=2}^l\bar{s}_i^{n_i}\bigr)\cdot a$ is a unit in $R_0^{(1)}/(p)$, which is a contradiction. So $n_1\ge 0$, and similarly $n_i\ge 0$ for each $i=1,\dots,l$. So $x\in R_0/(p)$, which proves the claim. This implies that the natural embedding $R_0\to S_0\cap\bigcap_{i=1}^l R_0^{(i)}$ as subrings of $W(k_c^{\mathrm{perf}})$ is bijective.

By Proposition 5.2, there exists a $p$-divisible group $G_i$ over $R^{(i)}$ such that $T_p(G_i)\cong T$ as $\mathcal{G}_{R^{(i)}}$-representations. We have

$$(\mathfrak{M}_{\mathfrak{S}}(T)\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}})\otimes_{\mathcal{O}_{\mathcal{E}}}\mathcal{O}_{\mathcal{E},R^{(i)}}\cong\mathfrak{M}^*(G_i)\otimes_{\mathfrak{S}_{R^{(i)}}}\mathcal{O}_{\mathcal{E},R^{(i)}}$$

as étale $(\varphi,\mathcal{O}_{\mathcal{E},R^{(i)}})$-modules. Applying Lemma 4.4, we can deduce that $\mathfrak{M}_{\mathfrak{S}}(T)\otimes_{\mathfrak{S}}\mathfrak{S}_{R^{(i)}}\cong\mathfrak{M}^*(G_i)$ compatibly with Frobenius.

Let $D=D_{\mathrm{cris}}\bigl(T\bigl[\frac{1}{p}\bigr]\bigr)$, and denote $\mathfrak{M}=\mathfrak{M}_{\mathfrak{S}}(T)$ and $\mathcal{N}=\mathfrak{M}\otimes_{\mathfrak{S},\varphi}R_0$. Let $\nabla: D\to D\otimes_{R_0}\widehat{\Omega}_{R_0}$ be the connection given by the functor $D_{\mathrm{cris}}(\cdot)$.

**Proposition 5.4.** *There exists a natural $\varphi$-equivariant embedding*

$$h:\mathcal{N}\hookrightarrow D$$

*of $R_0$-modules. Furthermore, if we consider $\mathcal{N}$ as an $R_0$-submodule of $D$ via $h$, then $\nabla$ maps $\mathcal{N}$ into $\mathcal{N}\otimes_{R_0}\widehat{\Omega}_{R_0}$. Hence, $\mathfrak{M}$ is a Kisin module of height* 1.

*Proof.* By [Kim 2015, Corollaries 5.3 and 6.7], there exists a natural $\varphi$-equivariant embedding

$$h_i:\mathcal{N}\to D\otimes_{R_0,b_i}R_0^{(i)}$$

for each $i=1,\dots,l$ such that the connections given by $\mathfrak{M}^*(G_i)$ and $D$ are compatible, and there exists a natural $\varphi$-equivariant embedding $h_c:\mathcal{N}\to D\otimes_{R_0,b_c}W(k_c^{\mathrm{perf}})$. Moreover, by Proposition 4.9, there exists a natural $\varphi$-equivariant embedding $h_S:\mathcal{N}\to D\otimes_{R_0}S_0$ such that the connections given by $\mathfrak{M}^*(G_S)$ and $D$ are compatible. Since the construction of those natural maps is compatible with $\varphi$-equivariant base changes (see [Kim 2015, Section 5.5]), we deduce that the maps $h_1,\dots,h_l$ and $h_S$ are compatible with one another, in the sense that their composites with the embedding into $D\otimes_{R_0,b_c}W(k_c^{\mathrm{perf}})$ are all equal to $h_c$. Hence, we obtain a $\varphi$-equivariant embedding

$$h:\mathcal{N}\hookrightarrow\Bigl(D\otimes_{R_0[\frac{1}{p}]}S_0\Bigl[\frac{1}{p}\Bigr]\Bigr)\cap\Bigl(\bigcap_{i=1}^{l}D\otimes_{R_0[\frac{1}{p}],b_i}R_0^{(i)}\Bigl[\frac{1}{p}\Bigr]\Bigr)\cong D\otimes_{R_0[\frac{1}{p}]}\Bigl(S_0\Bigl[\frac{1}{p}\Bigr]\cap\bigcap_{i=1}^{l}R_0^{(i)}\Bigl[\frac{1}{p}\Bigr]\Bigr)=D,$$

since $D$ is flat over $R_0\bigl[\frac{1}{p}\bigr]$ and $S_0\bigl[\frac{1}{p}\bigr]\cap\bigcap_{i=1}^l R_0^{(i)}\bigl[\frac{1}{p}\bigr]=R_0\bigl[\frac{1}{p}\bigr]$.

Now, identify $\widehat{\Omega}_{R_0} = \bigoplus_{j=1}^m R_0 \cdot dt_j$. Then $\nabla$ maps $\mathcal{N}$ to $\mathcal{N} \otimes_{R_0} \big(\bigoplus_{j=1}^m R_0\big[\frac{1}{p}\big] \cdot dt_j\big)$. On the other hand, by Propositions 4.8, 5.2, and the compatibility of $D_{\mathrm{cris}}(\cdot)$ with respect to $\varphi$-compatible base changes, we have that $\nabla$ maps $\mathcal{N}$ into $\mathcal{N} \otimes_{R_0} \big(\bigoplus_{j=1}^m S_0 \cdot dt_j\big)$ and also into $\mathcal{N} \otimes_{R_0} \big(\bigoplus_{j=1}^m R_0^{(i)} \cdot dt_j\big)$ for each $i = 1, \dots, l$. Since $\mathcal{N}$ is flat over $R_0$ and $S_0 \cap \bigcap_{i=1}^l R_0^{(i)} = R_0$, $\nabla$ maps $\mathcal{N}$ into $\mathcal{N} \otimes_{R_0} \big(\bigoplus_{j=1}^m R_0 \cdot dt_j\big)$. □

**Theorem 5.5.** *There exists a $p$-divisible group $G_R$ over $R$ such that $T_p(G_R) \cong T$ as $\mathcal{G}_R$-representations.*

*Proof.* By Proposition 5.4, we have $\mathfrak{M} \in \mathrm{Kis}^1(\varphi, \nabla)$. Furthermore, $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathcal{O}_L} \cong \mathfrak{M}_{\mathcal{O}_L}$ as Kisin modules over $\mathfrak{S}_{\mathcal{O}_L}$, since the Frobenius and connection structure on $\mathfrak{M}$ agree with those on $D$. Thus, if $G_R$ is the $p$-divisible group corresponding to $\mathfrak{M}$, then $T_p(G_R) \cong T$ as $\mathcal{G}_{\mathcal{O}_L}$-representations as well as $\mathcal{G}_{\widetilde{R}_\infty}$-representations. The assertion follows from Lemma 2.1. □

## 6. Barsotti–Tate deformation ring

As an application of Theorem 5.5, we study the geometry of the locus of crystalline representations with Hodge–Tate weights in [0, 1] by using the results in [Moon 2020]. Note that in [Moon 2020, Section 2], $R_0$ is assumed to satisfy the additional conditions that $W(k)\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle \to R_0$ has geometrically regular fibers or $R_0$ has Krull dimension less than 2, and that $k \to R_0/pR_0$ is geometrically integral. These assumptions are only used to have the crystalline period ring as in [Brinon 2008]. However, the additional conditions are not necessary by [Kim 2015, Section 4], and the results in [Moon 2020] hold in our setting.

Denote by $\mathcal{C}$ the category of topological local $\mathbb{Z}_p$-algebras $A$ satisfying the following conditions:

- The natural map $\mathbb{Z}_p \to A/\mathfrak{m}_A$ is surjective, where $\mathfrak{m}_A$ denotes the maximal ideal of $A$.
- The map from $A$ to the projective limit of its discrete artinian quotients is a topological isomorphism.

By the first condition, the residue field of $A$ is $\mathbb{F}_p$. The second condition is equivalent to that $A$ is complete and its topology is given by a collection of open ideals $\mathfrak{a} \subset A$ for which $A/\mathfrak{a}$ is artinian. Morphisms in $\mathcal{C}$ are continuous $\mathbb{Z}_p$-algebra morphisms.

For $A \in \mathcal{C}$, we mean by an *$A$-representation of $\mathcal{G}_R$* a finite free $A$-module equipped with a continuous $A$-linear $\mathcal{G}_R$-action. Fix an $\mathbb{F}_p$-representation $V_0$ of $\mathcal{G}_R$ which is absolutely irreducible. For $A \in \mathcal{C}$, a *deformation* of $V_0$ in $A$ is defined to be an isomorphism class of $A$-representations of $V$ of $\mathcal{G}_R$ satisfying $V \otimes_A \mathbb{F}_p \cong V_0$ as $\mathbb{F}_p[\mathcal{G}_R]$-modules. Denote by $\mathrm{Def}(V_0, A)$ the set of such deformations. A morphism $f : A \to A'$ in $\mathcal{C}$ induces a map $f_* : \mathrm{Def}(V_0, A) \to \mathrm{Def}(V_0, A')$ sending the class of an $A$-representation $V$ to the class of $V \otimes_{A,f} A'$. The following theorem on universal deformation ring is proved in [de Smit and Lenstra 1997].

**Theorem 6.1** [de Smit and Lenstra 1997, Theorem 2.3]. *There exists a universal deformation ring $A_{\mathrm{univ}} \in \mathcal{C}$ and a deformation $V_{\mathrm{univ}} \in \mathrm{Def}(V_0, A_{\mathrm{univ}})$ such that for all $A \in \mathcal{C}$, we have a bijection*

$$\mathrm{Hom}_{\mathcal{C}}(A_{\mathrm{univ}}, A) \xrightarrow{\cong} \mathrm{Def}(V_0, A) \tag{6-1}$$

*given by $f \mapsto f_*(V_{\mathrm{univ}})$.*

We deduce that when $R$ has dimension 2 and $e$ is small, the locus of crystalline representations with Hodge–Tate weights in [0, 1] cuts out a closed subscheme of Spec $A_{\mathrm{univ}}$ in the following sense.

**Theorem 6.2.** *Suppose that $e < p-1$ and that the Krull dimension of $R$ is 2. Then there exists a closed ideal $\mathfrak{a}_{\mathrm{BT}} \subset A_{\mathrm{univ}}$ such that the following holds*:

*For any finite flat $\mathbb{Z}_p$-algebra $A$ equipped with the $p$-adic topology and any continuous $\mathbb{Z}_p$-algebra map $f : A_{\mathrm{univ}} \to A$, the induced representation $V_{\mathrm{univ}} \otimes_{A_{\mathrm{univ}}, f} A\left[\frac{1}{p}\right]$ of $\mathcal{G}_R$ is crystalline with Hodge–Tate weights in* $[0, 1]$ *if and only if $f$ factors through the quotient $A_{\mathrm{univ}}/\mathfrak{a}_{\mathrm{BT}}$.*

*Proof.* This follows directly from Theorem 5.5 and [Moon 2020, Theorem 5.7]. Note that [Moon 2020, Theorem 5.7] assumes the Krull dimension of $R$ is 2. The assumption was necessary in the argument of [Moon 2020, Section 5] to construct Barsotti–Tate deformation ring using the result in [de Smit and Lenstra 1997]. □

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Communicated by Kiran S. Kedlaya
Received 2019-10-02 Revised 2020-05-14 Accepted 2020-06-24



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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

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