

# *Algebra & Number Theory*

Volume 14

2020

No. 2

**Positivity results for  
spaces of rational curves**

Roya Beheshti and Eric Riedl



# Positivity results for spaces of rational curves

Roya Beheshti and Eric Riedl

Let  $X$  be a very general hypersurface of degree  $d$  in  $\mathbf{P}^n$ . We investigate positivity properties of the spaces  $R_e(X)$  of degree  $e$  rational curves in  $X$ . We show that for small  $e$ ,  $R_e(X)$  has no rational curves meeting the locus of smooth embedded curves. We show that for  $n \leq d$ , there are no rational curves other than lines in the locus  $Y \subset X$  swept out by lines. We exhibit differential forms on a smooth compactification of  $R_e(X)$  for every  $e$  and  $n - 2 \geq d \geq \frac{1}{2}(n + 1)$ .

## 1. Introduction

We work over  $\mathbf{C}$ , the field of complex numbers. Let  $X$  be a smooth hypersurface of degree  $d$  in  $\mathbf{P}^n$ , and for  $e \geq 1$ , denote by  $R_e(X)$  the space of smooth rational curves of degree  $e$  on  $X$ . In this paper, we study some geometric properties of  $R_e(X)$ .

**Question 1.1.** *For which  $d, n$  and  $e$  with  $n \geq d$  does the very general hypersurface  $X \subset \mathbf{P}^n$  of degree  $d$  have a rational curve in  $R_e(X)$ ?*

One major motivation for considering [Question 1.1](#) is to study rational surfaces in Fano hypersurfaces. A rational curve in  $R_e(X)$  gives a rational surface in  $X$ , and conversely a rational surface in  $X$  gives a nonconstant map from  $\mathbf{P}^1$  to a compactification of  $R_e(X)$  for some  $e \geq 1$ . It is known that if  $d \ll \sqrt{n}$  every smooth hypersurface of degree  $d$  contains rational surfaces, but it is not known if the same holds for higher degree Fano hypersurfaces. It is conjectured that when  $d = n \geq 5$ ,  $X$  is not covered by rational surfaces:

**Conjecture 1.2.** *A very general hypersurface of degree  $n$  in  $\mathbf{P}^n$  is not covered by rational surfaces if  $n \geq 5$ .*

[Conjecture 1.2](#) has important implications in understanding birational properties of varieties. Recall that a variety is unirational if it is rationally dominated by projective space, and it is rationally connected if there is a rational curve through two general points. It is uniruled if there is a rational curve through a general point. Every unirational variety is rationally connected, and indeed, is swept out by rational surfaces. It is expected that there exist rationally connected varieties which are not unirational, but to date, no examples of this have been proven. Proving [Conjecture 1.2](#) would prove that a very general

MSC2010: 14E08.

Keywords: hypersurface, rational curve, rational surface, birational geometry.

hypersurface of degree  $n$  in  $\mathbf{P}^n$  is not unirational. Since every hypersurface of degree  $d \leq n$  is rationally connected, this would give an example of a variety which is rationally connected but not unirational.

Riedl and Yang [2016] answered Question 1.1 for  $e = 1$  by showing that if  $n \leq \frac{1}{6}(d^2 + 3d + 6)$  there is no rational curve in  $R_1(X)$ , the space of lines on  $X$ . Beheshti [2012] proved that  $R_e(X)$  is not uniruled for  $\frac{1}{2}(n+1) \leq d \leq n-3$ . Beheshti and Starr [2008] considered the special case  $d = n$  and proved that  $X$  is not swept out by del Pezzo surfaces or rational surfaces ruled by curves of degree up to  $n$ . However, Question 1.1 remains open and is presumably quite difficult in general.

In this paper, we prove a few results about the positivity of  $R_e(X)$ . In Section 2, we consider the Kontsevich moduli space of stable maps  $\overline{\mathcal{M}}_e(X)$ , a compactification of  $R_e(X)$ , and show the following:

**Theorem 1.3.** *Fix an integer  $n \geq 20$  and  $e$ , and suppose  $d \leq n$  satisfies*

$$\begin{aligned} d^2 + (2e-1)d &\geq e(e+1)n - 3e(e-1) + 2 && \text{if } e \geq 3, \\ d^2 + (2e+1)d &\geq (e+1)(e+2)n - 3e(e+1) + 2 && \text{if } e = 1, 2. \end{aligned}$$

*Then for a very general hypersurface  $X$  of degree  $d$  in  $\mathbf{P}^n$ , there is no nonconstant morphism  $\mathbf{P}^1 \rightarrow \overline{\mathcal{M}}_e(X)$  whose image intersects the locus of embedded rational curves in  $X$ .*

This is a substantial generalization of results in [Riedl and Yang 2016] since it applies to curves with  $e > 1$ . Unlike results in [Beheshti and Starr 2008], it applies to hypersurfaces with  $n > d$ , and proves that there are no rational curves meeting the locus of smooth curves in  $R_e(X)$ , instead of merely proving that the locus of  $R_e(X)$  covered by rational curves does not sweep out  $X$ . It also generalizes results of [Beheshti 2012].

Starr [2003, Corollary 10.10] computed the  $\mathcal{Q}$ -divisor class of the first chern class of the dualizing sheaf of  $\overline{\mathcal{M}}_e(X)$  for a general hypersurface  $X$ , and showed that the canonical divisor of  $\overline{\mathcal{M}}_e(X)$  is big for  $n$  less than about  $d^2$ . If  $\overline{\mathcal{M}}_e(X)$  were irreducible, of the expected dimension, and had canonical singularities, then  $\overline{\mathcal{M}}_e(X)$  would be of general type for  $d$  large,  $n \geq d+6$ , and  $2n+2 \leq d^2+d$ . Results of Harris, Roth, and Starr [Harris et al. 2004], Beheshti and Kumar [2013], and Riedl and Yang [2019] prove that  $\overline{\mathcal{M}}_e(X)$  is irreducible and of the expected dimension for  $d \leq n-2$ . However,  $\overline{\mathcal{M}}_e(X)$  is not always known to have canonical singularities. The best current results are for  $e+d \leq n$ , due to [Starr 2003].

Theorem 1.3, along with results of [Starr 2003; Beheshti 2012], suggests that there should be nonzero pluri-canonical forms on  $\overline{\mathcal{M}}_e(X)$  if  $d \geq \frac{1}{2}(n+1)$  or if  $d < \frac{1}{2}(n+1)$  and the inequality of Theorem 1.3 is satisfied. De Jong and Starr [2004] gave a general construction of differential forms on any desingularization of  $\overline{\mathcal{M}}_e(X)$ , the coarse moduli scheme of  $\overline{\mathcal{M}}_e(X)$ , and used this to construct nonzero pluri-canonical forms on these schemes when  $X$  is a general cubic fourfold and  $e \geq 5$  is an odd number. In Section 3, we use their construction and show:

**Proposition 1.4.** *If  $\frac{1}{2}(n+1) \leq d \leq n-2$  and  $X$  is any smooth hypersurface of degree  $d$  in  $\mathbf{P}^n$ , then there are nonzero differential forms on any desingularization of  $\overline{\mathcal{M}}_e(X)$  for every  $e \geq 1$ .*

It is interesting to investigate whether these forms can be used to construct nonzero pluri-canonical forms when  $d \geq \frac{1}{2}(n+1)$ .

Finally, in [Section 4](#) we generalize results of [\[Riedl and Yang 2016\]](#) in a different direction, and show:

**Theorem 1.5.** *If  $n \leq \frac{1}{6}(d(d+3)) + 1 - \frac{1}{3}k$ , then a very general degree  $d$  hypersurface  $X$  in  $\mathbf{P}^n$  contains no  $k$ -gonal curves in  $F_1(X)$ .*

If  $n \leq d$ , the lines on  $X$  sweep out a proper subvariety  $Y \subset X$ . Results of Clemens and Ran [\[2004\]](#) seem to suggest that  $Y$  is the “most negative” subvariety of  $X$ . They prove that for  $d \geq \frac{1}{2}(3n+1)$ , any subvarieties of  $X$  without effective canonical bundle (such as rational curves) must lie in  $Y$ . The following corollary of [Theorem 1.5](#) proves that in contrast to this,  $Y$  never contains rational curves for  $n \leq d$ .

**Corollary 1.6.** *Let  $X \subset \mathbf{P}^n$  be a very general degree  $d$  hypersurface with  $d \geq n$ . Let  $Y \subset X$  be the locus swept out by lines. Then  $Y$  contains no  $k$ -gonal curves other than lines if  $k \leq (d^2 + 3d + 6 - 6n)/(2(n-1))$ . In particular,  $Y$  contains no rational curves other than lines.*

## 2. Rational curves in $R_e(X)$

For a hypersurface  $X \subset \mathbf{P}^n$ , we denote by  $\overline{\mathcal{M}}_e(X)$  the Kontsevich moduli space of stable maps of degree  $e$  from curves of genus 0 to  $X$ . The goal of this section is to prove the following.

**Theorem 2.1.** *Fix an integer  $e$  and  $n$  ( $n \geq 20$ ), and suppose  $d \leq n$  satisfies*

$$\begin{aligned} d^2 + (2e-1)d &\geq e(e+1)n - 3e(e-1) + 2 && \text{if } e \geq 3, \\ d^2 + (2e+1)d &\geq (e+1)(e+2)n - 3e(e+1) + 2 && \text{if } e = 1, 2. \end{aligned}$$

*Then for a very general hypersurface  $X$  of degree  $d$  in  $\mathbf{P}^n$ , there is no nonconstant morphism  $\mathbf{P}^1 \rightarrow \overline{\mathcal{M}}_e(X)$  whose image intersects the locus of embedded smooth rational curves in  $X$ .*

Before proving [Theorem 2.1](#), we set some notation. Let  $S$  be a smooth rational surface admitting a map to  $\mathbf{P}^1$  with general fiber  $C$  isomorphic to  $\mathbf{P}^1$ . Let  $f : S \rightarrow \mathbf{P}^n$  be a generically finite morphism whose image is contained in a smooth hypersurface  $X$  of degree  $d$  in  $\mathbf{P}^n$ . We denote by  $N_{f, \mathbf{P}^n}$  the normal sheaf of  $f$ , i.e., the cokernel of the map  $T_S \rightarrow f^*T_{\mathbf{P}^n}$ . Similarly, we denote by  $N_{f, X}$  the normal sheaf of  $f$  considered as a morphism from  $S$  to  $X$ . There is a short exact sequence

$$0 \rightarrow N_{f, X} \rightarrow N_{f, \mathbf{P}^n} \rightarrow f^*\mathcal{O}_{\mathbf{P}^n}(d) \rightarrow 0 \tag{1}$$

The strategy of the proof will be to compute, for various values of  $t$ , the Euler characteristic of  $N_{f, X}(t) \otimes I_{C/S}$ . The main technique will be to argue that for  $X$  a general hypersurface of the appropriate degree and for  $t$  carefully chosen, the Euler characteristic is positive. We contrast this with the following direct computation of the Euler characteristic.

**Proposition 2.2.** *Assume  $B$  is a smooth projective curve and  $S$  a smooth surface admitting a fibration  $\pi : S \rightarrow B$  with rational fibers. Let  $f : S \rightarrow X$  be a map from  $S$  to a smooth degree  $d$  hypersurface  $X \subset \mathbf{P}^n$ , and let  $C$  be a general fiber of  $\pi$ . Suppose  $f|_C$  is an embedding. Let  $H = f^*\mathcal{O}(1)$  and  $K$  be the canonical class of  $S$ . Then we have*

$$\chi(N_{f, X}(t) \otimes I_{C/S}) = \frac{1}{2}b_t H^2 + \frac{1}{2}c_t H \cdot K - 2K^2 + d_t,$$

where

$$\begin{aligned} b_t &= (n - 3)t^2 + 2(n + 1 - d)t + n + 1 - d^2, \\ c_t &= -(n - 5)t + d - n - 1, \\ d_t &= (-(n - 3)t - n - 1 + d) H \cdot C + (n + 9) \chi(\mathcal{O}_S) - n + 5. \end{aligned}$$

*Proof.* This is a calculation involving additivity of the Euler characteristic. Using the exact sequence (1), we see

$$\chi(N_{f,X} \otimes I_{C/S}(t)) = \chi(N_{f,P^n} \otimes I_{C/S}(t)) - \chi(f^* \mathcal{O}_{P^n}(d+t) \otimes I_{C/S}).$$

Using the sequence defining  $N_{f,P^n}$  and the Euler sequence for  $P^n$ , we see that

$$\begin{aligned} \chi(N_{f,X} \otimes I_{C/S}(t)) &= \chi(T_{P^n} \otimes I_{C/S}(t)) - \chi(T_S \otimes I_{C/S}(t)) - \chi(f^* \mathcal{O}_{P^n}(d+t) \otimes I_{C/S}) \\ &= (n+1)\chi(f^* I_{C/S}(t+1)) - \chi(I_{C/S}(t)) - \chi(f^* \mathcal{O}_{P^n}(d+t) \otimes I_{C/S}) - \chi(T_S \otimes I_{C/S}(t)). \end{aligned}$$

Using Hirzebruch–Riemann–Roch, this becomes

$$\begin{aligned} (n+1) \frac{((t+1)H-C) \cdot ((t+1)H-C-K)}{2} - \frac{(tH-C) \cdot (tH-C-K)}{2} \\ - \frac{((t+d)H-C) \cdot ((t+d)H-C-K)}{2} + (n-1)\chi(\mathcal{O}_S) - \chi(T_S \otimes I_{C/S}(t)). \end{aligned} \tag{2}$$

We have

$$\begin{aligned} c_1(T_S \otimes I_{C/S}(t)) &= c_1(T_S) + 2(-C + tH) = -K + 2(-C + tH), \\ c_2(T_S \otimes I_{C/S}(t)) &= c_2(T_S) - K \cdot (-C + tH) + (-C + tH)^2. \end{aligned}$$

Via the splitting principle, we can introduce variables  $\alpha$  and  $\beta$  with  $\alpha + \beta = c_1(T_S \otimes I_{C/S}(t))$  and  $\alpha\beta = c_2(T_S \otimes I_{C/S}(t))$ . By Hirzebruch–Riemann–Roch, we get

$$\begin{aligned} \chi(T_S \otimes I_{C/S}(t)) &= 2\chi(\mathcal{O}_S) + \frac{1}{2}\alpha(\alpha - K) + \frac{1}{2}\beta(\beta - K) \\ &= 2\chi(\mathcal{O}_S) - \alpha\beta + \frac{1}{2}((\alpha + \beta)^2 - (\alpha + \beta) \cdot K). \end{aligned}$$

Plugging in the chern classes of  $T_S \otimes I_{C/S}(t)$  for  $\alpha + \beta$  and  $\alpha\beta$ , we obtain

$$\chi(T_S \otimes I_{C/S}(t)) = K^2 - c_2(T_S) - 2tH \cdot K + 2K \cdot C + t^2 H^2 - 2tH \cdot C + 2\chi(\mathcal{O}_S).$$

We have  $K \cdot C = -2$ . So putting the above equation in (2), collecting like terms, and using Noether’s formula  $c_2(T_S) = 12\chi(\mathcal{O}_S) - K^2$ , we obtain the result.  $\square$

**Corollary 2.3.** Assume  $t \geq 1, n \geq 20$  and  $d \leq n$ . If  $S$  contains no  $(-1)$ -curves contracted by both  $f$  and  $\pi$ , and

$$d^2 + d(2t + 1) \geq n(t + 2)(t + 1) - 3t^2 - 3t + 2,$$

then  $\chi(N_{f,X} \otimes I_{C/S}(t)) < 0$ .

*Proof.* This is an intersection theory calculation on  $S$ . First observe that  $2H + 2C + K$  is basepoint free, and hence, nef. If it were not, then by Reider's theorem [1988], there would be an effective divisor  $E$  on  $S$  with either  $E \cdot (2H + 2C) = 1$  and  $E^2 = 0$  or  $E \cdot (2H + 2C) = 0$  and  $E^2 = -1$ . The first case is impossible since  $E \cdot (2H + 2C)$  must be even. In the second case, we would have  $H \cdot E = 0 = C \cdot E$ , which implies that  $E$  is a  $(-1)$ -curve contracted by both  $\pi$  and  $f$ , and contradicts our assumption. It follows that  $2H + 2C + K$  is nef. So

$$0 \leq (2H + 2C + K)^2 = 4H^2 + K^2 + 4H \cdot K + 8H \cdot C - 8 \tag{3}$$

since  $K \cdot C = -2$  and  $C^2 = 0$ . Note that  $H^1(f^*\mathcal{O}(-1)) = 0$  by Kodaira vanishing, so by Hirzebruch–Riemann–Roch and the fact that  $\chi(\mathcal{O}_S) \leq 1$  we see that

$$H \cdot (H + K) = 2\chi(f^*\mathcal{O}_X(-1)) - 2\chi(\mathcal{O}_S) \geq -2\chi(\mathcal{O}_S) \geq -2. \tag{4}$$

By Proposition 2.2 and the relation (3) we see that

$$\begin{aligned} \chi(N_{f,X}(t) \otimes I_{C/S}) &= \frac{1}{2}b_t H^2 + \frac{1}{2}c_t H \cdot K - 2K^2 + d_t \\ &\leq \frac{1}{2}b_t H^2 + \frac{1}{2}c_t H \cdot K + 8H \cdot (H + K) + 16H \cdot C - 16 + d_t \\ &= \frac{1}{2}(b_t - c_t)H^2 + \frac{1}{2}(c_t + 16)(H \cdot (H + K) + 2) + d_t - c_t + 16H \cdot C - 32. \end{aligned}$$

Note that since by our assumption  $d \leq n$  and  $n \geq 20$ , we have  $c_t + 16 \leq 0$ , so the above inequality and (4) give

$$\chi(N_{f,X}(t) \otimes I_{C/S}) \leq \frac{1}{2}(b_t - c_t)H^2 + d_t - c_t + 16H \cdot C - 32.$$

We see that  $b_t - c_t \leq 0$  precisely when  $(n - 3)t^2 + (3n - 3 - 2d)t + 2n + 2 - d^2 - d \leq 0$ , or

$$d^2 + d(2t + 1) \geq n(t + 2)(t + 1) - 3t^2 - 3t + 2.$$

It remains to show that  $d_t - c_t + 16H \cdot C - 32 < 0$ . By Proposition 2.2 we have

$$\begin{aligned} d_t - c_t + 16H \cdot C - 32 &= -(n - d + t(n - 3) - 15)(H \cdot C - 1) - 2t + (n + 9)\chi(\mathcal{O}_S) - (n - 5) - 16 \\ &\leq -(n - d + t(n - 3) - 15)(H \cdot C - 1) - 2t - 2. \end{aligned}$$

We see that this is negative if  $n - d + t(n - 3) - 15 \geq 0$ , which happens for  $n \geq 20$  since  $t \geq 1$  and  $d \leq n$ .  $\square$

**Proposition 2.4.** Fix  $e \geq 1$ . Let  $m = e - 1$  if  $e \geq 3$  and  $m = e$  if  $e = 1, 2$ . Assume that for a very general hypersurface  $X$  of degree  $d$ , there is a map  $\phi : \mathbf{P}^1 \rightarrow \overline{\mathcal{M}}_e(X)$  whose image intersects the locus of embedded smooth rational curves. Then there is a smooth surface  $S$  with two morphisms  $f : S \rightarrow X$  and  $\pi : S \rightarrow \mathbf{P}^1$  such that  $f$  maps a general fiber  $C$  of  $\pi$  isomorphically to a smooth rational curve of degree  $e$  in  $X$  and

- (a) the image of the pull-back map  $H^0(X, \mathcal{O}_X(d)) \rightarrow H^0(S, f^*\mathcal{O}_{\mathbf{P}^n}(d))$  is contained in the image of the map

$$H^0(S, N_{f,\mathbf{P}^n}) \rightarrow H^0(S, f^*\mathcal{O}_{\mathbf{P}^n}(d)),$$

- (b) the restriction map  $H^0(S, N_{f,X}(m)) \rightarrow H^0(C, N_{f,X}(m)|_C)$  is surjective, and
- (c) the Euler characteristic  $\chi(N_{f,X}(m) \otimes I_{C/S}) \geq 0$ .

*Proof.* Our assumptions imply that there is an irreducible quasiprojective variety  $Z$  and a morphism  $\phi : Z \times \mathbf{P}^1 \rightarrow \overline{\mathcal{M}}_e(\mathbf{P}^n)$  such that the following hold:

- For each  $z_0 \in Z$ , the morphism

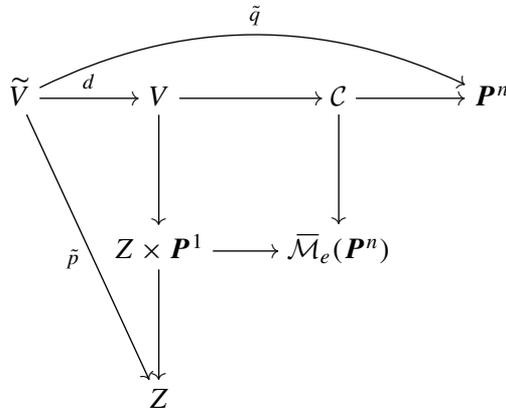
$$\phi_{z_0} := \phi(z_0, b) : \mathbf{P}^1 \rightarrow \overline{\mathcal{M}}_e(\mathbf{P}^n)$$

is a nonconstant morphism whose image intersects the locus of embedded smooth rational curves.

- For a very general  $X$ , there is  $z \in Z$  such that the image of  $\phi_z$  is contained in  $\overline{\mathcal{M}}_e(X)$ .

Replacing  $Z$  with an open subset we may assume  $Z$  is nonsingular.

Let  $\mathbf{P}^N$  be the projective space parametrizing hypersurfaces of degree  $d$  in  $\mathbf{P}^n$  and  $U \subset \mathbf{P}^N$  the open subset parametrizing smooth hypersurfaces. Denote by  $I \subset Z \times U$  the incidence correspondence parametrizing pairs  $(z, [X])$  such that the image of  $\phi_z : \mathbf{P}^1 \rightarrow \overline{\mathcal{M}}_e(\mathbf{P}^n)$  is contained in  $\overline{\mathcal{M}}_e(X)$ . Denote by  $\pi_1$  and  $\pi_2$  the projection maps from  $I$  to  $Z$  and  $U$  respectively. By our assumption  $\pi_2$  is dominant. Replacing  $I$  with an irreducible component which maps dominantly to  $U$  under  $\pi_2$ , we may assume  $I$  is irreducible. Let  $V$  be the pullback of the universal curve  $\mathcal{C} \rightarrow \overline{\mathcal{M}}_e(\mathbf{P}^n)$  to  $Z \times \mathbf{P}^1$ . Denote by  $q : V \rightarrow \mathbf{P}^n$  the pullback of the universal map  $\mathcal{C} \rightarrow \mathbf{P}^n$  to  $V$  and by  $p : V \rightarrow Z$  the projection to the first factor. Let  $d : \tilde{V} \rightarrow V$  be a desingularization, and set  $\tilde{p} = p \circ d$  and  $\tilde{q} = q \circ d$ .



Let  $(z, [X]) \in I$  be a general point. Denote the fiber of  $\tilde{p}$  over  $z$  by  $S$ . Since  $z$  is general in  $Z$ , by generic smoothness,  $S$  is a smooth surface and if  $f$  is the restriction of  $\tilde{q}$  to  $S$ , then  $f : S \rightarrow X$  maps a general fiber of  $S \rightarrow \mathbf{P}^1$  isomorphically onto a curve in  $X$ .

Denote by  $N_{(\tilde{p}, \tilde{q})}$  the normal sheaf of the map  $(\tilde{p}, \tilde{q}) : \tilde{V} \rightarrow Z \times \mathbf{P}^n$ . We get a sequence of maps

$$\rho : T_{Z,z} \rightarrow H^0(S, \tilde{p}^* T_Z|_S) \rightarrow H^0(S, (\tilde{p}, \tilde{q})^* T_{Z \times \mathbf{P}^n}|_S) \rightarrow H^0(S, N_{(\tilde{p}, \tilde{q})}|_S).$$

Note  $(\tilde{p}, \tilde{q})$  is generically finite and  $z$  is general, therefore  $T_{\tilde{v}}|_S \rightarrow (\tilde{p}, \tilde{q})^*T_{Z \times \mathbf{P}^n}|_S$  is injective. So  $N_{(\tilde{p}, \tilde{q})}|_S$  is isomorphic to  $N_{f, \mathbf{P}^n}$ :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & T_S & \longrightarrow & f^*T_{\mathbf{P}^n} & \longrightarrow & N_{f, \mathbf{P}^n} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & T_{\tilde{v}}|_S & \longrightarrow & (\tilde{p}, \tilde{q})^*T_{Z \times \mathbf{P}^n}|_S & \longrightarrow & N_{(\tilde{p}, \tilde{q})}|_S \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & N_{S/\tilde{v}} & \xrightarrow{=} & \tilde{p}^*T_Z|_S & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

There is a commutative diagram

$$\begin{array}{ccc}
 & T_{I, (z, [X])} & \\
 & \swarrow d\pi_1 & \searrow d\pi_2 \\
 T_{Z, z} & & T_{\mathbf{P}^n, [X]} = H^0(X, \mathcal{O}_X(d)) \\
 \downarrow \rho & & \downarrow \\
 H^0(S, N_{f, \mathbf{P}^n}) & \longrightarrow & H^0(S, f^*\mathcal{O}_X(d))
 \end{array}$$

Since  $\pi_2$  is dominant,  $d\pi_2$  is surjective, and part (a) follows.

To prove part (b), we tensor sequence (1) with  $\mathcal{O}_X(m)$  to get the short exact sequence

$$0 \rightarrow N_{f, X}(m) \rightarrow N_{f, \mathbf{P}^n}(m) \rightarrow f^*\mathcal{O}_X(d+m) \rightarrow 0.$$

Let  $C$  be a general fiber of the map  $\pi : S \rightarrow \mathbf{P}^1$ . Restricting the above short exact sequence to  $C$ , we get a short exact sequence of  $\mathcal{O}_C$ -modules

$$0 \rightarrow N_{f, X}(m)|_C \rightarrow N_{f, \mathbf{P}^n}(m)|_C \rightarrow f^*\mathcal{O}_X(d+m)|_C \rightarrow 0.$$

We have the following commutative diagram:

$$\begin{array}{ccccc}
 H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m+1)^{n+1}) & \longrightarrow & H^0(\mathbf{P}^n, T_{\mathbf{P}^n}(m)) & \longrightarrow & H^0(S, N_{f, \mathbf{P}^n}(m)) \\
 \downarrow & & \downarrow & & \downarrow \\
 H^0(C, \mathcal{O}_{\mathbf{P}^n}(m+1)^{n+1}|_C) & \longrightarrow & H^0(C, T_{\mathbf{P}^n}(m)|_C) & \longrightarrow & H^0(C, N_{f, \mathbf{P}^n}(m)|_C)
 \end{array}$$

Since  $f(C)$  is  $(m+1)$ -normal, the left vertical map is surjective. Using the Euler sequence and the fact that  $H^1(C, \mathcal{O}_C(m)) = 0$  and  $H^1(C, T_S(m)|_C) = 0$ , we see that the two lower horizontal maps are also surjective. So we have a surjective map

$$H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m+1)^{n+1}) \rightarrow H^0(C, N_{f, \mathbf{P}^n}(m)|_C).$$

Pick  $\alpha \in H^0(C, N_{f, X}(m)|_C)$ , and denote the image of  $\alpha$  in  $H^0(C, N_{f, \mathbf{P}^n}(m)|_C)$  by  $\beta$ . Let  $\tilde{\beta}$  be a lift of  $\beta$  to  $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m+1)^{n+1})$  under the above map and  $\tilde{\beta}$  the image of  $\tilde{\beta}$  in  $H^0(S, N_{f, \mathbf{P}^n}(m))$ . Let  $\gamma \in H^0(X, \mathcal{O}_X(m+d))$  be the image of  $\tilde{\beta}$  under the composition

$$\begin{aligned}
 H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m+1)^{n+1}) &\rightarrow H^0(\mathbf{P}^n, T_{\mathbf{P}^n}(m)) \\
 &\rightarrow H^0(X, T_{\mathbf{P}^n}(m)|_X) \rightarrow H^0(X, \mathcal{O}_X(m+d)),
 \end{aligned}$$

where the last map is induced by the sequence

$$0 \rightarrow T_X \rightarrow T_{\mathbf{P}^n}|_X \rightarrow \mathcal{O}_X(d) \rightarrow 0.$$

Let  $\tilde{\gamma}$  be a preimage of  $\gamma$  in  $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d+m))$ . Then  $\tilde{\beta}|_C = \beta$  and  $f^*\tilde{\gamma}|_C = 0$ . Since  $\tilde{\gamma}|_{f(C)} = 0$ , we can view  $\tilde{\gamma}$  as an element of  $H^0(\mathbf{P}^n, I_{f(C)/\mathbf{P}^n}(d+m))$ . Since  $I_{f(C)/\mathbf{P}^n}$  is an  $m$ -regular sheaf, the multiplication map

$$H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) \otimes H^0(\mathbf{P}^n, I_{f(C)/\mathbf{P}^n}(m)) \rightarrow H^0(\mathbf{P}^n, I_{f(C)/\mathbf{P}^n}(d+m))$$

is surjective, so  $\tilde{\gamma}$  can be written as

$$\tilde{\gamma} = \tilde{\gamma}_1 \tilde{\eta}_1 + \cdots + \tilde{\gamma}_k \tilde{\eta}_k,$$

where  $\tilde{\gamma}_i \in H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d))$  and  $\tilde{\eta}_i \in H^0(\mathbf{P}^n, I_{f(C)/\mathbf{P}^n}(m))$  for each  $i$ . So if we view  $\tilde{\eta}_i$  as an element of  $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m))$  we see that  $f^*\tilde{\eta}_i|_C = 0$  for each  $i$ . By part (a) the image of the map

$$f^* : H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) \rightarrow H^0(S, f^*\mathcal{O}_{\mathbf{P}^n}(d))$$

is contained in the image of the map

$$h : H^0(S, N_{f, \mathbf{P}^n}) \rightarrow H^0(S, f^*\mathcal{O}_{\mathbf{P}^n}(d)).$$

So  $f^*\tilde{\gamma}_i = h(\bar{\mu}_i)$  for some  $\bar{\mu}_i \in H^0(S, N_{f, \mathbf{P}^n})$ . Let

$$\bar{\mu} = \bar{\mu}_1 f^*\tilde{\eta}_1 + \cdots + \bar{\mu}_k f^*\tilde{\eta}_k \in H^0(S, N_{f, \mathbf{P}^n}(m)).$$

Then  $\bar{\mu}|_C = 0$  and since  $h(\bar{\mu} - \bar{\beta}) = 0$ , we have that  $\bar{\mu} - \bar{\beta}$  is the image of a section of  $N_{f,X}(m)$  whose restriction to  $C$  is  $\alpha$ .

To prove (c), note that by the Leray spectral sequence, to show the Euler characteristic is nonnegative, it is enough to show that for a general fiber  $C$  of  $\pi$ , (1)  $H^1(\mathbf{P}^1, \pi_*(N_{f,X}(m) \otimes I_{C/S})) = 0$  and (2)  $R^1\pi_*(N_{f,X}(m) \otimes I_{C/S}) = 0$ .

By part (b), the map  $H^0(S, N_{f,X}(m)) \rightarrow H^0(C, N_{f,X}(m)|_C)$  is surjective, and  $H^1(C, N_{f,X}(m)|_C) = 0$ , so

$$H^1(S, N_{f,X}(m) \otimes I_{C/S}) = H^1(S, N_{f,X}(m)).$$

Thus

$$H^1(\mathbf{P}^1, \pi_*N_{f,X}(m) \otimes \mathcal{O}_{\mathbf{P}^1}(-1)) = H^1(\mathbf{P}^1, \pi_*N_{f,X}(m)),$$

so  $H^1(\mathbf{P}^1, \pi_*N_{f,X}(m) \otimes \mathcal{O}_{\mathbf{P}^1}(-1)) = 0$ . This shows (1).

To show (2), we note that since  $X$  is very general, for any morphism  $g : \mathbf{P}^1 \rightarrow X$ ,  $g^*T_X(1)$  is globally generated by the main result of [Clemens 1986]; see also [Ein 1988; Voisin 1996]. So the restriction of  $f^*T_X(m)$  to every irreducible component of every fiber of  $\pi$  is globally generated. So  $R^1\pi_*T_X(m) = 0$  and  $R^1\pi_*N_{f,X}(m) = 0$ . □

*Proof of Theorem 2.1.* Assume to the contrary that for a very general  $X$ , there is a nonconstant map  $\mathbf{P}^1 \rightarrow \bar{\mathcal{M}}_e(X)$  whose image intersects the locus of embedded smooth rational curves in  $X$ . Let  $m = e - 1$  if  $e \geq 3$  and  $m = e$  if  $e = 1$  or  $e = 2$ . By Proposition 2.4, there is a surface  $S$  and map  $f : S \rightarrow X$  and  $\pi : S \rightarrow \mathbf{P}^1$  such that  $\chi(N_{f,X}(m) \otimes I_{C/S}) \geq 0$ . Blowing down, we may assume that there is no  $(-1)$ -curve in any fiber of  $\pi$  which is contracted by  $f$ . This contradicts Corollary 2.3. □

A modification of the proof of Proposition 2.4 shows that the statement of Theorem 2.1 remains true for morphisms  $\mathbf{P}^1 \rightarrow \bar{\mathcal{M}}_e(X)$  whose image intersects the locus of embedded reducible nodal rational curves in  $X$ . We sketch the proof here. Fix  $e \geq 1$ , let  $m = e - 1$  if  $e \geq 3$  and  $m = e$  if  $e = 1$  or  $2$ , and assume that for a very general hypersurface  $X$  of degree  $d$ , there is a nonconstant map  $\phi : \mathbf{P}^1 \rightarrow \bar{\mathcal{M}}_e(X)$  whose image intersects the locus of embedded nodal rational curves in  $X$ . Then there are positive integers  $r$  and  $s$  such that  $rs \leq e$  and for a very general hypersurface  $X$  of degree  $d$ , there is a smooth curve  $B$ , a degree  $r$  morphism  $B \rightarrow \mathbf{P}^1$ , and a morphism  $B \rightarrow \bar{\mathcal{M}}_s(X)$  such that the image intersects the locus of embedded smooth rational curves of degree  $s$  in  $X$ .

This implies that there are smooth, irreducible, and quasiprojective varieties  $P$  and  $Z$  ( $P$  is just a point in the case of Proposition 2.4), a proper morphism  $p_1 : W \rightarrow P$  whose fibers are smooth projective curves which are degree  $r$  covers of  $\mathbf{P}^1$  ( $r = 1$  and  $W$  is the projective line in the case of Proposition 2.4), and morphisms  $p_2 : Z \rightarrow P$  and  $\phi : Z \times_P W \rightarrow \bar{\mathcal{M}}_s(\mathbf{P}^n)$  with the following property: for every  $z \in Z$ ,

$$\phi_z : p_1^{-1}(p_2(z)) \rightarrow \bar{\mathcal{M}}_s(\mathbf{P}^n)$$

is a morphism which intersects the locus of embedded smooth rational curves, and for a very general  $X$ , there is  $z$  such that  $\phi_z$  parametrizes stable maps which are mapped to  $X$ . We proceed now as in the proof of

**Proposition 2.4** and let  $I \subset Z \times U$  be a dominating irreducible component of the incidence correspondence where  $U$  is the locus of smooth hypersurfaces of degree  $d$  in  $\mathbf{P}^n$ . We also let  $V, S$ , and  $z$  be as before.

We conclude that if  $B = p_1^{-1}(p_2(z))$ , then there is a morphism of degree  $r$   $g : B \rightarrow \mathbf{P}^1$ , and there are morphisms  $f : S \rightarrow X$  and  $\pi : S \rightarrow B$  such that  $f$  maps a general fiber  $C$  of  $\pi$  isomorphically onto a smooth rational curve of degree  $s$  on  $X$ ,  $rs \leq e$ , and

- (a) the image of the map  $H^0(X, \mathcal{O}_{\mathbf{P}^n}(d)) \rightarrow H^0(S, f^*\mathcal{O}_{\mathbf{P}^n}(d))$  is contained in the image of the map

$$H^0(S, N_{f, \mathbf{P}^n}(d)) \rightarrow H^0(S, f^*\mathcal{O}_{\mathbf{P}^n}(d)),$$

- (b) the map  $H^0(S, N_{f, X}(m)) \rightarrow H^0(D, N_{f, X}(m)|_D)$  is surjective where  $D$  is a general fiber of  $g \circ \pi : S \rightarrow \mathbf{P}^1$ , and
- (c) the Euler characteristic  $\chi(N_{f, X}(m) \otimes I_{D/S}) \geq 0$ .

Part (b) follows from the proof of **Proposition 2.4** since the image of  $D$  is  $e$ -regular in  $X$ . Part (c) follows from a similar argument as in **Proposition 2.4** and the fact that if  $F$  is a sheaf on  $B$  with  $H^1(B, F \otimes I_{g^{-1}(p)}) = H^1(B, F)$  for a general  $p \in \mathbf{P}^1$ , then  $H^1(B, F) = 0$ . Applying **Corollary 2.3** to  $S$  gives the desired result.

### 3. Differential forms on Kontsevich moduli space

Let  $X$  be a smooth hypersurface of degree  $d$  in  $\mathbf{P}^n$ . Let  $\overline{\mathcal{M}}_e(X)$  be the Kontsevich moduli space of stable maps of degree  $e$  from curves of genus zero to  $X$ , and let  $\overline{M}_e(X)$  be the corresponding coarse moduli scheme. There is a universal curve  $\pi : \mathcal{C} \rightarrow \overline{\mathcal{M}}_e(X)$  and an evaluation map  $ev : \mathcal{C} \rightarrow X$ .

In this section, we use the construction of de Jong and Starr [2004] to show that there are nonzero differential forms on any desingularization of  $\overline{M}_e(X)$  when  $\frac{1}{2}(n + 1) \leq d \leq n - 3$ . By [de Jong and Starr 2004, Corollary 4.3], for every  $i, j \geq 1$ , there is a  $\mathbf{C}$ -linear map

$$\alpha_{i,j} : H^i(X, \Omega_X^j) \rightarrow H^{i-1}(\overline{\mathcal{M}}_e(X), \Omega_{\overline{\mathcal{M}}_e(X)}^{j-1}).$$

We consider the map  $\alpha_{1,n-2}$ , so the above map gives  $(n-3)$ -forms on the Kontsevich moduli stack. Let  $\overline{N}_e(X)$  be a desingularization of  $\overline{M}_e(X)$ . By [de Jong and Starr 2004, Proposition 3.6], for every  $j \geq 0$ , there is a linear map

$$H^0(\overline{\mathcal{M}}_e(X), \Omega_{\overline{\mathcal{M}}_e(X)}^j) \rightarrow H^0(\overline{N}_e(X), \Omega_{\overline{N}_e(X)}^j).$$

Composing this with  $\alpha_{1,n-2}$ , we get a map from  $H^1(X, \Omega_X^{n-2})$  to the space of  $(n-3)$ -forms on  $\overline{N}_e(X)$ .

**Proposition 3.1.** *Assume  $\frac{1}{2}(n + 1) \leq d \leq n - 3$  and  $X$  is a smooth hypersurface of degree  $d$  in  $\mathbf{P}^n$ . If  $\overline{N}_e(X)$  is a desingularization of  $\overline{M}_e(X)$ , then the map  $H^1(X, \wedge^{n-2}\Omega_X) \rightarrow H^0(\overline{N}_e(X), \wedge^{n-3}\Omega_{\overline{N}_e(X)})$  is nonzero for every  $e$ .*

*Proof.* Fix  $e$  and  $X$ , and set  $\mathcal{M} = \overline{\mathcal{M}}_e(X)$  and  $N = \overline{N}_e(X)$  for simplicity. By [de Jong and Starr 2004, Corollaries 4.2 and 4.3] the map  $\alpha_{1,n-2}$  factors through the maps

$$H^1(X, \Omega_X^{n-2}) \rightarrow H^1(\mathcal{C}, \Omega_{\mathcal{C}}^{n-2}) \rightarrow H^1(\mathcal{C}, \pi^*\Omega_{\mathcal{M}}^{n-3} \otimes \omega_\pi) \rightarrow H^0(\mathcal{M}, \Omega_{\mathcal{M}}^{n-3}), \tag{5}$$

where

- the first map comes from the map  $ev^*\Omega_X \rightarrow \Omega_C$ ,
- the second map comes from a map of  $\mathcal{O}_C$ -modules

$$\Omega_C^{n-2} \rightarrow \pi^*\Omega_{\mathcal{M}}^{n-3} \otimes \omega_\pi$$

which fits into the following short exact sequence over the locus  $U$  of embedded smooth curves

$$\pi^*\Omega_{\mathcal{M}}^{n-2}|_U \rightarrow \Omega_C^{n-2}|_U \rightarrow \pi^*\Omega_{\mathcal{M}}^{n-3} \otimes \omega_\pi|_U \rightarrow 0,$$

- and the last map comes from the Leray spectral sequence and the fact that  $R^1\pi_*\omega_\pi = \mathcal{O}_{\mathcal{M}}$ .

Since  $d < n$ , there is an irreducible component of  $\mathcal{M}$  whose general point parametrizes an embedded smooth free rational curve of degree  $e$  on  $X$ . Let  $C$  be a such a curve. We denote the stable map corresponding to the isomorphism from  $\mathbf{P}^1$  onto  $C$  by  $[C] \in \mathcal{M}$ , and identify the fiber of  $\pi : \mathcal{C} \rightarrow \mathcal{M}$  over  $[C]$  with  $C$ . Since  $C$  is free,  $\mathcal{M}$  is smooth at  $[C]$  and  $T_{\mathcal{M}}|_{[C]} = H^0(C, N_{C/X})$ . Restricting sequence (5) to  $C$ , we get the diagram

$$\begin{array}{ccc} H^1(X, \Omega_X^{n-2}) & \xrightarrow{\alpha_{1,n-2}} & H^0(\mathcal{M}, \Omega_{\mathcal{M}}^{n-3}) \\ \downarrow & & \downarrow \\ H^1(C, \Omega_X^{n-2}|_C) \rightarrow H^1(C, \Omega_C^{n-2}|_C) \rightarrow H^1(C, \pi^*\Omega_{\mathcal{M}}^{n-3} \otimes \omega_\pi|_C) & \xrightarrow{\cong} & \Omega_{\mathcal{M}}^{n-3}|_{[C]} \end{array}$$

and we have  $\pi^*\Omega_{\mathcal{M}}^{n-3} \otimes \omega_\pi|_C = \wedge^{n-3} I_{C/X} \otimes \Omega_C$ . In order to show the statement, it suffices to prove that in the above diagram the composition of the maps

$$\begin{aligned} H^1(X, \Omega_X^{n-2}) &\rightarrow H^1(C, \Omega_X^{n-2}|_C) \rightarrow H^1(C, \Omega_C^{n-2}|_C) \\ &\rightarrow H^1(C, \pi^*\Omega_{\mathcal{M}}^{n-3} \otimes \omega_\pi|_C) = H^1(C, \wedge^{n-3} I_{C/X} \otimes \Omega_C) \end{aligned}$$

is nonzero. Since  $\Omega_{\mathcal{M}}^{n-3}|_{[C]} = \Omega_N^{n-3}|_{[C]}$ , this would show the assertion of the theorem. From the short exact sequence

$$0 \rightarrow I_{C/X} \otimes \mathcal{O}_C \rightarrow \Omega_X|_C \rightarrow \Omega_C \rightarrow 0,$$

we get the short exact sequence

$$0 \rightarrow \wedge^{n-2} I_{C/X} \otimes \mathcal{O}_C \rightarrow \Omega_X^{n-2}|_C \rightarrow \wedge^{n-3} I_{C/X} \otimes \Omega_C \rightarrow 0.$$

Similarly, there is an exact sequence

$$0 \rightarrow \wedge^{n-2} I_{C/X} \otimes \mathcal{O}_C \rightarrow \Omega_C^{n-2}|_C \rightarrow \wedge^{n-3} I_{C/X} \otimes \Omega_C \rightarrow 0,$$

and a commutative diagram

$$\begin{array}{ccc}
 H^1(C, \Omega_X^{n-2}|_C) & \longrightarrow & H^1(C, \wedge^{n-3} I_{C/X} \otimes \Omega_C) \\
 \downarrow & & \downarrow \\
 H^1(C, \Omega_C^{n-2}|_C) & \longrightarrow & H^1(C, \wedge^{n-3} I_{C/C} \otimes \Omega_C)
 \end{array}$$

So to show the assertion, we show that the composition of the maps

$$\begin{aligned}
 H^1(X, \Omega_X^{n-2}) &\rightarrow H^1(C, \Omega_X^{n-2}|_C) \\
 &\rightarrow H^1(C, \wedge^{n-3} I_{C/X} \otimes \Omega_C) \rightarrow H^1(C, \wedge^{n-3} I_{C/C} \otimes \Omega_C)
 \end{aligned} \tag{6}$$

is nonzero. Note that

$$\wedge^{n-3} I_{C/C} \otimes \mathcal{O}_C = \Omega_{\mathcal{M}}^{n-3}|_{[C]} \otimes \mathcal{O}_C = \wedge^{n-3} T_{\mathcal{M}}^\vee|_{[C]} \otimes \mathcal{O}_C = \wedge^{n-3} H^0(C, N_{C/X})^\vee \otimes \mathcal{O}_C.$$

So by Serre duality, the last map in sequence (6) is the dual of the map

$$\wedge^{n-3} H^0(C, N_{C/X}) \rightarrow H^0(C, \wedge^{n-3} N_{C/X}).$$

Since  $C$  is free, the above map is surjective, so the last map in sequence (6) is injective. Hence it is enough to show that under our assumptions, the composition of the maps

$$H^1(X, \Omega_X^{n-2}) \rightarrow H^1(C, \Omega_X^{n-2}|_C) \rightarrow H^1(C, \wedge^{n-3} I_{C/X} \otimes \Omega_C)$$

is nonzero.

To prove this we consider the short exact sequence

$$0 \rightarrow \mathcal{O}_X(-d) \rightarrow \Omega_{\mathbb{P}^n}|_X \rightarrow \Omega_X \rightarrow 0$$

which gives the following short exact sequence

$$0 \rightarrow \Omega_X^{n-2} \rightarrow \Omega_{\mathbb{P}^n}^{n-1}|_X \otimes \mathcal{O}_X(d) \rightarrow \mathcal{O}_X(2d - n - 1) \rightarrow 0. \tag{7}$$

There is also a short exact sequence on  $C$ ,

$$0 \rightarrow \wedge^{n-3} I_{C/X} \otimes \Omega_C \rightarrow \wedge^{n-2} I_{C/\mathbb{P}^n} \otimes \Omega_C \otimes \mathcal{O}_C(d) \rightarrow \mathcal{O}_C(2d - n - 1) \rightarrow 0, \tag{8}$$

and sequence (7) maps to sequence (8). Taking the long exact sequence of cohomology we get a commutative diagram

$$\begin{array}{ccc}
 H^0(X, \mathcal{O}_X(2d - n - 1)) & \longrightarrow & H^1(X, \Omega_X^{n-2}) \\
 \downarrow & & \downarrow \\
 H^0(C, \mathcal{O}_C(2d - n - 1)) & \longrightarrow & H^1(C, \wedge^{n-3} I_{C/X} \otimes \Omega_C)
 \end{array}$$

and since  $2d - n - 1 \geq 0$ , the left vertical map is nonzero. To show the desired result, we show that the bottom map is injective. This follows if we show  $H^0(C, \wedge^{n-2} I_{C/\mathbf{P}^n} \otimes \Omega_C \otimes \mathcal{O}_C(d)) = 0$ . Let

$$N_{C/\mathbf{P}^n} = \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_{n-1}).$$

We have  $\wedge^{n-2} I_{C/\mathbf{P}^n} \otimes \Omega_C \otimes \mathcal{O}_C(d) = N_{C/\mathbf{P}^n} \otimes \mathcal{O}_{\mathbf{P}^1}(e(d - n - 1))$ , so we need to show  $a_i < e(n + 1 - d)$  for each  $i$ . Since  $\sum_i a_i = e(n + 1) - 2$  and each  $a_i$  is at least  $e$ , we see that  $a_i \leq 3e - 2$  for all  $i$ . Thus,  $a_i < e(n + 1 - d)$  provided  $d \leq n - 2$ . This proves the result.  $\square$

#### 4. Gonality and the space swept out by lines

Our goal is to prove the following:

**Theorem 4.1.** *Let  $X \subset \mathbf{P}^n$  be a very general degree  $d \geq n$  hypersurface. Let  $Y \subset X$  be the locus swept out by lines. Then  $Y$  contains no  $k$ -gonal curves other than lines if  $k \leq (d^2 + 3d + 6 - 6n)/(2(n - 1))$ . In particular,  $Y$  contains no rational curves other than lines.*

We begin with a description of the Fano scheme  $F_1(X)$ .

**Proposition 4.2.** *Let  $X \subset \mathbf{P}^n$  be a general hypersurface. If  $2n - d - 3 \geq 0$ , then the Fano scheme  $F_1(X)$  of lines on  $X$  is smooth of dimension  $2n - d - 3$ . If  $2n - d - 3 < 0$ , then the canonical bundle of  $F_1(X)$  is  $(\frac{1}{2}d(d + 1) - n - 1)\sigma_1$ , where  $\sigma_1$  is the restriction of the divisor on  $\mathbb{G}(1, n)$  of lines meeting a fixed codimension 2 space.*

Now we recall the following definitions and results of Bastianelli, De Poi, Ein, Lazarsfeld and Ullery [Bastianelli et al. 2017]. A divisor  $D$  is *birationally very ample* to order  $k$  ( $BVA_k$ ) if  $D = E + kA$ , where  $E$  is effective and  $A$  is very ample.

**Theorem 4.3** [Bastianelli et al. 2017, Theorem 1.10]. *If a smooth variety  $Z$  which satisfies  $K_Z$  is  $BVA_k$ , then  $Z$  is not swept out by  $(k+1)$ -gonal curves.*

**Corollary 4.4.** *If  $X$  in  $\mathbf{P}^n$  is a general degree  $d$  hypersurface with  $n \leq \frac{1}{2}d(d + 1) - k$ , then  $F_1(X)$  is not swept out by  $k$ -gonal curves.*

We need a few basic results about the space of degree  $d$  hypersurfaces containing a fixed variety.

**Lemma 4.5.** *Any set of  $k$  distinct points in  $\mathbf{P}^n$  with  $k \leq d + 1$  imposes  $k$  conditions on the space of hypersurfaces of degree  $d$ .*

*Proof.* By a degeneration argument, it suffices to prove the result for  $k$  points that lie along a line  $\ell$ . Then, we can consider the map  $\alpha : H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) \rightarrow H^0(\mathcal{O}_{\mathbf{P}^1}(d))$ . Since  $\ell$  is normal, we see that  $\alpha$  is a surjective linear map. Hence, if we fix  $k$  points on  $\ell$  with  $k \leq d + 1$ , then the space of degree  $d$  hypersurfaces in  $\mathbf{P}^n$  containing those  $k$  points is the preimage under  $\alpha$  of all sections of  $\mathcal{O}_{\ell}(d)$  that vanish on those  $k$  points. The result follows.  $\square$

We also need the following standard lemma, described in [Riedl and Yang 2019].

**Lemma 4.6.** *Let  $Z$  be a variety of dimension  $k$ . The space of hypersurfaces containing  $Z$  is codimension at least  $\binom{n+d}{d}$  in the space of all hypersurfaces.*

We need the following proposition from [Riedl and Yang 2016].

**Proposition 4.7.** *Let  $C \subsetneq \mathbb{G}(k-1, n)$  be a nonempty variety of  $(k-1)$ -planes and let  $B \subset \mathbb{G}(k, n)$  be the set of  $k$ -planes containing the planes in  $C$ . Then if the codimension of  $C$  in  $\mathbb{G}(k-1, n)$  is  $\epsilon$ , the codimension of  $B$  in  $\mathbb{G}(k, n)$  is at most  $\epsilon - 1$ .*

We use the following corollary which follows immediately from Proposition 4.7.

**Corollary 4.8.** *Let  $\ell \subset \mathbf{P}^n$  be a line and let  $S_k$  be the variety of  $k$ -planes containing  $\ell$ . If  $C \subset S_{k-1}$  is a nonempty variety of  $(k-1)$ -planes of codimension  $\epsilon > 0$ , and  $B \subset S_k$  is the set of  $k$ -planes that contain a plane of  $C$ , then the codimension of  $B$  in  $S_k$  is at most  $\epsilon - 1$ .*

We also discuss the notion of a parametrized  $k$ -plane. A parametrized  $k$ -plane is a map  $\Lambda : \mathbf{P}^k \rightarrow \mathbf{P}^n$  defined by linear equations. The set of all parametrized  $k$ -planes is naturally a  $PGL_k$  bundle over the Grassmannian of  $k$ -planes in  $\mathbf{P}^n$ . Given a parametrized  $k$ -plane  $\Lambda$ , we have a natural map  $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) \rightarrow H^0(\mathbf{P}^k, \mathcal{O}_{\mathbf{P}^k}(d))$  given by pulling polynomials back to  $\mathbf{P}^k$  along  $\Lambda$ . We say that  $X' \subset \mathbf{P}^k$  is a parametrized  $k$ -plane section of  $X \subset \mathbf{P}^n$  if  $X'$  is cut out by the restriction of  $f$  to  $\mathbf{P}^k$ , where  $X = V(f)$ .

We now show that  $F_1(X)$  contains no  $k$ -gonal curves for certain ranges of  $n, d$  and  $k$ .

**Theorem 4.9.** *If  $n \leq \frac{1}{6}d(d+3) + 1 - \frac{1}{3}k$ , then a very general degree  $d$  hypersurface  $X$  in  $\mathbf{P}^n$  contains no  $k$ -gonal curves in  $F_1(X)$ .*

*Proof.* Let  $\mathcal{UL}_{n,d}$  be the set of pairs  $(\ell, X)$  of lines  $\ell$  lying in a hypersurface  $X$  of degree  $d$  in  $\mathbf{P}^n$ . Let  $R_{n,d,k}$  be the space of  $(\ell, X)$  such that  $F_1(X)$  has a  $k$ -gonal curve passing through  $[\ell]$ . We see that  $R_{n,d,k}$  will be a countable union of varieties. If for some tuple  $(n, d, k)$ , the codimension of (each component of)  $R_{n,d,k}$  in  $\mathcal{UL}_{n,d}$  is at least  $2n - d - 3$ , then a general hypersurface  $X$  of degree  $d$  in  $\mathbf{P}^n$  contains no  $k$ -gonal curves in  $F_1(X)$ .

Let  $d \geq 3$  be an integer, let  $m = \frac{1}{2}d(d+1) - k$ , and let  $X \subset \mathbf{P}^m$  be a general hypersurface of degree  $d$  in  $\mathbf{P}^m$ . By Corollary 4.4, we see that  $R_{m,d,k} \subset \mathcal{UL}_{m,d}$  has codimension at least 1, so we can find some pair  $(\ell_0, X_0)$  where  $F_1(X_0)$  has no  $k$ -gonal curves through  $[\ell_0]$ .

Let  $(\ell_1, X_1)$  be a general point of a component of  $R_{n,d,k}$ . We find a subvariety  $S \subset \mathcal{UL}_{n,d}$  containing  $(\ell_1, X_1)$  such that  $S \cap R_{n,d,k}$  is of codimension at least  $2n - d - 3$  in  $S$ . Since  $(\ell_1, X_1)$  could have been on any component, it follows that  $R_{n,d,k}$  has codimension at least  $2n - d - 3$  in  $\mathcal{UL}_{n,d}$ . We now construct  $S$ .

We claim that we can find a hypersurface  $Y \subset \mathbf{P}^M$  containing a line  $\ell_2$  such that  $(\ell_0, X_0)$  and  $(\ell_1, X_1)$  are both parametrized linear sections of  $(Y, \ell_2)$ . To see this, choose coordinates  $x_0, x_1, \dots, x_n$  on  $\mathbf{P}^n$  so that  $\ell_1$  is given by the vanishing of  $x_2, \dots, x_n$ , and choose coordinates  $x_0, x_1, y_2, \dots, y_m$  on  $\mathbf{P}^m$  so that  $\ell_0$  is given by the vanishing of  $y_2, \dots, y_m$ . Write  $X_0 = V(f_0)$  and  $X_1 = V(f_1)$ . Then we may take  $Y = V(f_0 + f_1)$  in  $\mathbf{P}^{m+n-1}$ , which has the desired properties.

Let  $S_n$  be the closure of the set of parametrized  $n$ -plane sections of  $(Y, \ell_2)$ . We see by the fact that  $(\ell_0, X_0)$  has no  $k$ -gonal curves in  $F_1(X_0)$  passing through  $[\ell_0]$  that  $S_m \cap R_{m,d,k}$  is codimension at least one in  $S_m$ . By [Corollary 4.8](#), we see that  $S_{m-1} \cap R_{m-1,d,k}$  has codimension at least 2 in  $S_{m-1}$ . Thus,  $S_n \cap R_{n,d,k}$  has codimension at least  $m - n + 1$  in  $S_n$ . Therefore, if  $m - n + 1 \geq 2n - d - 3$ , we see that there is no  $k$ -gonal curves in  $F_1(X)$  for  $X$  a general hypersurface of degree  $d$  in  $\mathbf{P}^n$ . This holds if

$$n \leq \frac{1}{3}(m + d + 4) = \frac{1}{6}d(d + 3) + \frac{4}{3} - \frac{1}{3}k. \quad \square$$

**Proposition 4.10.** *If  $X \subset \mathbf{P}^n$  is a general degree  $d \geq n$  hypersurface, then any  $p \in X$  has at most  $n - 1$  lines in  $X$  passing through  $p$ .*

*Proof.* It suffices to prove the result for  $d = n$ . Let  $\mathcal{U}$  be the incidence correspondence of pairs  $(p, X)$  where  $X \subset \mathbf{P}^n$  is a degree  $n$  hypersurface and  $p \in X$  is a point. Let  $\mathcal{L}_m \subset \mathcal{U}$  be the set of pairs  $(p, X)$  such that  $X$  contains  $m$  lines passing through  $p$ . We wish to show that  $\mathcal{L}_m$  has codimension at least  $n$  in  $\mathcal{U}$  for  $m \geq n$ , from which the result will follow. Consider a point  $p \in \mathbf{P}^n$ . Choose coordinates on  $\mathbf{P}^n$  so that  $p$  is the point  $[1, 0, \dots, 0]$ . Then given an  $X = V(f)$  containing  $p$ , we can expand the equation of  $f$  around  $p$ , writing  $f = f_1x_0^{d-1} + f_2x_0^{d-2} + \dots + f_d$ . Then the space of lines in  $X$  passing through  $p$  is  $\Sigma_p = V(f_1, \dots, f_d) \subset \mathbf{P}^{n-1}$ , where  $\Sigma_p$  is naturally contained in the  $\mathbf{P}^{n-1}$  of lines in  $\mathbf{P}^n$  passing through  $p$ . We consider the codimension of the locus of  $f$  for which  $V(f_1, \dots, f_i)$  has larger than expected dimension. Many of the ideas here are adapted from [\[Harris et al. 2004; Riedl and Yang 2019\]](#).

If  $i \leq n - 1$ , the locus of  $f$  where  $V(f_i)$  contains a component of  $V(f_1, \dots, f_{i-1})$  has codimension at least

$$\binom{n-i+i}{i} = \binom{n}{i} \geq n,$$

so the locus  $S \subset \mathcal{U}$  of the set of pairs  $(p, X)$  where there is a positive dimensional family of lines through  $p$  in  $X$  has codimension at least  $n$ . Thus, we may assume  $V(f_1, \dots, f_{n-1})$  is a finite set of points. For  $f_n$  to contain  $k$  of those points is  $k$  conditions by [Lemma 4.5](#), so the locus  $\mathcal{L}_m \subset \mathcal{U}$  is codimension at least  $m$ . Thus, a general hypersurface  $X$  of degree  $d$  in  $\mathbf{P}^n$  cannot have  $n$  lines passing through a single point of  $X$ . The result follows.  $\square$

*Proof of Corollary 1.6.* We simply put together the pieces. Let  $\mathcal{U} \rightarrow F_1(X)$  be the universal line on  $X$ , mapping to  $F_1(X)$  via a map  $\pi_1$ . Then  $\mathcal{U}$  maps surjectively to  $Y$  via a map  $\pi_2$ . Note that  $\pi_2$  is finite by [Proposition 4.10](#). Suppose  $C \subset Y$  is a  $k$ -gonal curve, and let  $D$  be an irreducible component of  $\pi_2^{-1}(C)$ . Then by [Proposition 4.10](#), the degree of  $\pi_2|_D$  is at most  $n - 1$ , and so  $D$  has gonality at most  $k(n - 1)$ . If  $D$  is contracted by  $\pi_1$ , then  $C$  must have been a line in  $X$ . If  $D$  is not contracted by  $\pi_1$ , then its image is a curve in  $F_1(X)$  of gonality at most  $k(n - 1)$ . By [Theorem 4.9](#), this means  $k(n - 1) > \frac{1}{2}(d^2 + 3d + 6) - 3n = \frac{1}{2}(d^2 + 3d + 6 - 6n)$ . Thus,  $k > (d^2 + 3d + 6 - 6n)/(2(n - 1))$ , so  $Y$  contains no  $k$ -gonal curves other than lines for  $k \leq (d^2 + 3d + 6 - 6n)/(2(n - 1))$ . In particular, this bound for  $k$  is at least one for every  $d \geq n \geq 3$ .  $\square$

## Acknowledgements

We are grateful to the anonymous referee for numerous helpful suggestions and corrections. Riedl was partially supported by the NSF RTG grant number DMS-1246844 while doing much of this research.

## References

- [Bastianelli et al. 2017] F. Bastianelli, P. De Poi, L. Ein, R. Lazarsfeld, and B. Ullery, “Measures of irrationality for hypersurfaces of large degree”, *Compos. Math.* **153**:11 (2017), 2368–2393. [MR](#) [Zbl](#)
- [Beheshti 2012] R. Beheshti, “Nonuniruledness results for spaces of rational curves in hypersurfaces”, *Algebra Number Theory* **6**:4 (2012), 669–687. [MR](#) [Zbl](#)
- [Beheshti and Kumar 2013] R. Beheshti and N. M. Kumar, “Spaces of rational curves on complete intersections”, *Compos. Math.* **149**:6 (2013), 1041–1060. [MR](#) [Zbl](#)
- [Beheshti and Starr 2008] R. Beheshti and J. M. Starr, “Rational surfaces in index-one Fano hypersurfaces”, *J. Algebraic Geom.* **17**:2 (2008), 255–274. [MR](#) [Zbl](#)
- [Clemens 1986] H. Clemens, “Curves on generic hypersurfaces”, *Ann. Sci. École Norm. Sup. (4)* **19**:4 (1986), 629–636. [MR](#) [Zbl](#)
- [Clemens and Ran 2004] H. Clemens and Z. Ran, “Twisted genus bounds for subvarieties of generic hypersurfaces”, *Amer. J. Math.* **126**:1 (2004), 89–120. [MR](#) [Zbl](#)
- [Ein 1988] L. Ein, “Subvarieties of generic complete intersections”, *Invent. Math.* **94**:1 (1988), 163–169. [MR](#) [Zbl](#)
- [Harris et al. 2004] J. Harris, M. Roth, and J. Starr, “Rational curves on hypersurfaces of low degree”, *J. Reine Angew. Math.* **571** (2004), 73–106. [MR](#) [Zbl](#)
- [de Jong and Starr 2004] A. J. de Jong and J. Starr, “Cubic fourfolds and spaces of rational curves”, *Illinois J. Math.* **48**:2 (2004), 415–450. [MR](#) [Zbl](#)
- [Reider 1988] I. Reider, “Vector bundles of rank 2 and linear systems on algebraic surfaces”, *Ann. of Math. (2)* **127**:2 (1988), 309–316. [MR](#) [Zbl](#)
- [Riedl and Yang 2016] E. Riedl and D. Yang, “Rational curves on general type hypersurfaces”, 2016. To appear in *J. Differential Geom.* [arXiv](#)
- [Riedl and Yang 2019] E. Riedl and D. Yang, “Kontsevich spaces of rational curves on Fano hypersurfaces”, *J. Reine Angew. Math.* **748** (2019), 207–225. [MR](#) [Zbl](#)
- [Starr 2003] J. Starr, “The Kodaira dimension of spaces of rational curves on low degree hypersurfaces”, preprint, 2003. [arXiv](#)
- [Voisin 1996] C. Voisin, “On a conjecture of Clemens on rational curves on hypersurfaces”, *J. Differential Geom.* **44**:1 (1996), 200–213. [MR](#) [Zbl](#)

Communicated by Ravi Vakil

Received 2019-04-30    Revised 2019-08-14    Accepted 2019-09-16

[beheshti@wustl.edu](mailto:beheshti@wustl.edu)

*Mathematics and Statistics, Washington University in St. Louis, Saint Louis, MO, United States*

[eriedl@nd.edu](mailto:eriedl@nd.edu)

*Department of Mathematics, University of Notre Dame, Notre Dame, IN, United States*

# Algebra & Number Theory

[msp.org/ant](http://msp.org/ant)

## EDITORS

### MANAGING EDITOR

Bjorn Poonen  
Massachusetts Institute of Technology  
Cambridge, USA

### EDITORIAL BOARD CHAIR

David Eisenbud  
University of California  
Berkeley, USA

### BOARD OF EDITORS

Bhargav Bhatt	University of Michigan, USA	Martin Olsson	University of California, Berkeley, USA
Richard E. Borcherds	University of California, Berkeley, USA	Raman Parimala	Emory University, USA
Antoine Chambert-Loir	Université Paris-Diderot, France	Jonathan Pila	University of Oxford, UK
J-L. Colliot-Thélène	CNRS, Université Paris-Sud, France	Irena Peeva	Cornell University, USA
Brian D. Conrad	Stanford University, USA	Anand Pillay	University of Notre Dame, USA
Samit Dasgupta	Duke University, USA	Michael Rapoport	Universität Bonn, Germany
Hélène Esnault	Freie Universität Berlin, Germany	Victor Reiner	University of Minnesota, USA
Gavril Farkas	Humboldt Universität zu Berlin, Germany	Peter Sarnak	Princeton University, USA
Hubert Flenner	Ruhr-Universität, Germany	Joseph H. Silverman	Brown University, USA
Sergey Fomin	University of Michigan, USA	Michael Singer	North Carolina State University, USA
Edward Frenkel	University of California, Berkeley, USA	Christopher Skinner	Princeton University, USA
Wee Teck Gan	National University of Singapore	Vasudevan Srinivas	Tata Inst. of Fund. Research, India
Andrew Granville	Université de Montréal, Canada	J. Toby Stafford	University of Michigan, USA
Ben J. Green	University of Oxford, UK	Shunsuke Takagi	University of Tokyo, Japan
Joseph Gubeladze	San Francisco State University, USA	Pham Huu Tiep	University of Arizona, USA
Christopher Hacon	University of Utah, USA	Ravi Vakil	Stanford University, USA
Roger Heath-Brown	Oxford University, UK	Michel van den Bergh	Hasselt University, Belgium
János Kollár	Princeton University, USA	Akshay Venkatesh	Institute for Advanced Study, USA
Philippe Michel	École Polytechnique Fédérale de Lausanne	Marie-France Vignéras	Université Paris VII, France
Susan Montgomery	University of Southern California, USA	Melanie Matchett Wood	University of California, Berkeley, USA
Shigefumi Mori	RIMS, Kyoto University, Japan	Shou-Wu Zhang	Princeton University, USA

## PRODUCTION

[production@msp.org](mailto:production@msp.org)

Silvio Levy, Scientific Editor

---

See inside back cover or [msp.org/ant](http://msp.org/ant) for submission instructions.

The subscription price for 2020 is US \$415/year for the electronic version, and \$620/year (+\$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

---

ANT peer review and production are managed by EditFLOW<sup>®</sup> from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2020 Mathematical Sciences Publishers

# Algebra & Number Theory

Volume 14    No. 2    2020

---

<a href="#">On the definition of quantum Heisenberg category</a>	275
JONATHAN BRUNDAN, ALISTAIR SAVAGE and BEN WEBSTER	
<a href="#">Characteristic cycles and Gevrey series solutions of <math>A</math>-hypergeometric systems</a>	323
CHRISTINE BERKESCH and MARÍA-CRUZ FERNÁNDEZ-FERNÁNDEZ	
<a href="#">Singularity categories of deformations of Kleinian singularities</a>	349
SIMON CRAWFORD	
<a href="#">Iwasawa main conjecture for Rankin–Selberg <math>p</math>-adic <math>L</math>-functions</a>	383
XIN WAN	
<a href="#">Positivity results for spaces of rational curves</a>	485
ROYA BEHESHTI and ERIC RIEDL	
<a href="#">Generalized Schur algebras</a>	501
ALEXANDER KLESHCHEV and ROBERT MUTH	