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Let K be a number field or a p-adic field and F the function field of a curve over K. Let ℓ be a prime. Suppose that K contains a primitive ℓ-th root of unity. If $\ell = 2$ and K is a number field, then assume that K is totally imaginary. In this article we show that every element in $H^3(F, \mu_\ell^{\otimes 3})$ is a symbol. This leads to the finite generation of the Chow group of zero-cycles on a quadric fibration of a curve over a totally imaginary number field.

1. Introduction

Let F be a field and ℓ a prime not equal to the characteristic of F. For $n \geq 1$, let $H^n(F, \mu_\ell^{\otimes n})$ be the n-th Galois cohomology group with coefficients in $\mu_\ell^{\otimes n}$. We have $F^*/F^{*\ell} \simeq H^1(F, \mu_\ell)$. For $a \in F^*$, let $(a) \in H^1(F, \mu_\ell)$ denote the image of the class of a in $F^*/F^{*\ell}$. Let $a_1, \ldots, a_n \in F^*$. The cup product $(a_1) \cdots (a_n) \in H^n(F, \mu_\ell^{\otimes n})$ is called a *symbol*. A theorem of Voevodsky [2003] asserts that every element in $H^n(F, \mu_\ell^{\otimes n})$ is a sum of symbols. Let $\alpha \in H^n(F, \mu_\ell^{\otimes n})$. The *symbol length* of α is defined as the smallest m such that α is a sum of m symbols in $H^n(F, \mu_\ell^{\otimes n})$.

Let K be a p-adic field. Then it is well-known that every element in $H^2(K, \mu_\ell^{\otimes 2})$ is a symbol and $H^n(K, \mu_\ell^{\otimes n}) = 0$ for all $n \geq 3$. Let F be the function of a curve over K. Suppose that K contains a primitive ℓ-th root of unity. If $\ell \neq p$, then it was proved in [Suresh 2010] (see [Brussel and Tengan 2014]) that the symbol length of every element in $H^2(F, \mu_\ell^{\otimes 2})$ is at most 2. If $p \neq \ell$, then it was proved in [Parimala and Suresh 2010] (see [Parimala and Suresh 2016]) that every element in $H^3(F, \mu_\ell^{\otimes 3})$ is a symbol. If $\ell = p$, then it was proved in [Parimala and Suresh 2014] that for every central simple algebra A over F, the index of A divides the square of the period of A. In particular if $p = 2$, then the symbol length of every element in $H^2(F, \mu_2^{\otimes 2})$ is at most 2. Since $u(F) = 8$ [Heath-Brown 2010; Leep 2013] (see [Parimala and Suresh 2014]), it follows that every element in $H^3(F, \mu_2^{\otimes 3})$ is a symbol.

If F is the function field of a curve over a global field of positive characteristic p, $\ell \neq p$ and F contains a primitive ℓ-th root of unity, then it was proved in [Parimala and Suresh 2016] that every element in $H^3(F, \mu_\ell^{\otimes 3})$ is a symbol.

Let K be a number field. A consequence of class field theory is that every element in $H^n(K, \mu_\ell^{\otimes n})$ is a symbol. A classical lemma of Tate states that given finitely many elements $\alpha_1, \ldots, \alpha_r \in H^2(K, \mu_\ell^{\otimes 2})$, there

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exist $a, b_i \in K^*$ such that $\alpha_i = (a)\cdot(b_i)$. Let F be the function field of a curve over K. Suresh [2004] proved a higher dimensional version of this lemma over F: given finitely any elements $\alpha_1, \ldots, \alpha_r \in H^3(F, \mu_2^{\otimes 3})$, there exists $f \in F^*$ such that $\alpha_i = (f)\cdot\beta_i$ for some $\beta_i \in H^2(F, \mu_2^{\otimes 2})$. In particular if there exists an integer N such that the symbol length of every element in $H^2(F, \mu_2^{\otimes 2})$ is bounded by N, then the symbol length of every element in $H^3(F, \mu_2^{\otimes 3})$ is bounded by N. In [Lieblich et al. 2014], it was proved that such an integer N exists under the hypothesis that a conjecture of Colliot-Thélène on the Hasse principle for the existence of 0-cycles of degree 1 holds. However, unconditionally the existence of such N is still open.

In this paper we prove the following (see Corollary 7.8):

Theorem 1.1. *Let K be a global field or a local field and F the function field of a curve over K. Let ℓ be a prime not equal to* char(K). *Suppose that K contains a primitive ℓ-th root of unity and one of the following holds*:

(i) $\ell \neq 2$.

(ii) *K is a local field.*

(iii) *K is a totally imaginary number field.*

Then every element in $H^3(F, \mu_\ell^{\otimes 3})$ is a symbol.

The above theorem for K a p-adic field and $\ell \neq p$ is proved in [Parimala and Suresh 2010] (see [Parimala and Suresh 2016]). Our method in this paper is uniform, it covers both global and local fields at the same time and we do not exclude the case $\ell = p$.

We have the following (see Corollary 8.3):

Corollary 1.2. *Let K be a totally imaginary number field and F the function field of a curve over K. Let q be a quadratic form over F and $\lambda \in F^*$. If the dimension of q is at least* 5, *then $q \otimes \langle 1, -\lambda\rangle$ is isotropic.*

Let L be a field of characteristic not equal to 2 and $u(L)$ be the u-invariant of L. By a theorem of Pfister if $u(L) \leq 2^n$ for some n, then every element in $H^n(L, \mu_2^{\otimes n})$ is a symbol. Let K be a totally imaginary number field. Then it is well-known that $u(K)$ is 4. Let F be a function field over K of transcendence degree n. It is a wide open question whether $u(F) = 2^{n+2}$. The finiteness of $u(F)$ is not known even for $n = 1$. In the perspective of Pfister's theorem, the conclusion from (iii) of Theorem 1.1 strengthens the expectations that $u(F)$ is 8 for function fields of curves over totally imaginary number fields

In a related direction Colliot-Thélène raised the question whether every element of $H^{n+2}(F, \mu_\ell^{\otimes(n+2)})$ is a symbol if F is a function field of transcendence degree n over a totally imaginary number field. Our main theorem gives an affirmative answer to this question for function fields of curves.

For a smooth integral variety X over a field k, let $\mathrm{CH}_0(X)$ be the Chow group of 0-cycles modulo rational equivalence. If k is a number field and X a smooth projective geometrically integral curve, the Mordell–Weil theorem implies that $\mathrm{CH}_0(X)$ is finitely generated.

Let C be a smooth projective geometrically integral curve over a field k. Let $X \to C$ be an (admissible) quadric fibration (see [Colliot-Thélène and Skorobogatov 1993]). Let $\mathrm{CH}_0(X/C)$ be the kernel of the natural homomorphism $\mathrm{CH}_0(X) \to \mathrm{CH}_0(C)$. If char$(k) \neq 2$, Colliot-Thélène and Skorobogatov identified

$CH_0(X/C)$ with a certain subquotient of $k(C)^*$ [Colliot-Thélène and Skorobogatov 1993]. From this identification it follows that $CH_0(X/C)$ is a 2-torsion group. Thus $CH_0(X/C)$ is finitely generated if and only if it is finite. Suppose that k is a number field. If $\dim(X) \leq 2$, then the finiteness of $CH_0(X/C)$ is a result of Gros [1987]. If $\dim(X) = 3$, then it was proved in [Colliot-Thélène and Skorobogatov 1993; Parimala and Suresh 1995] that $CH_0(X/C)$ is finite. Thus for $\dim(X) \leq 3$, $CH_0(X)$ is finitely generated. As a consequence of Corollary 1.2, we prove the following conjecture of Colliot-Thélène and Skorobogatov (see Theorem 8.4).

Theorem 1.3. *Let K be a totally imaginary number field, C a smooth projective geometrically integral curve over K. Let $X \to C$ be an admissible quadric fibration. If* $\dim(X) \geq 4$, *then* $CH_0(X/C) = 0$. *In particular* $CH_0(X)$ *is finitely generated.*

Let K be a global field of positive characteristic p or a local field with the characteristic of the residue field p. Let F be the function field of a curve over K and ℓ a prime not equal to p. Let us recall that the main ingredient in the proof of the fact that every element in $H^3(F, \mu_\ell^{\otimes 3})$ is a symbol [Parimala and Suresh 2010], is a certain local-global principle for divisibility of an element of $H^3(F, \mu_\ell^{\otimes 3})$ by a symbol in $H^2(F, \mu_\ell^{\otimes 2})$ [Parimala and Suresh 2010; 2016]. In fact it was proved that for a given $\zeta \in H^3(F, \mu_\ell^{\otimes 3})$ and a symbol $\alpha \in H^2(F, \mu_\ell^{\otimes 2})$ if for every discrete valuation ν of F there exists $f_\nu \in F^*$ such that $\zeta - \alpha \cdot (f_\nu)$ is unramified at ν, then there exists $f \in F^*$ such that $\zeta = \alpha \cdot (f)$. In the proof of this local-global principle, the existence of residue homomorphisms on $H^2(F, \mu_\ell^{\otimes 2})$ and $H^3(F, \mu_\ell^{\otimes 3})$ is used. However note that if K is a global field or a p-adic field with $\ell = p$, then there is no "residue homomorphism" on $H^2(F, \mu_\ell^{\otimes 2})$ which can be used to describe the unramified Brauer group.

We now briefly explain the main ingredients of our result. Let K be a global field or a local field and F the function field of a curve over K. Let ℓ be a prime not equal to characteristic of K. Suppose that K contains a primitive ℓ-th root of unity. Let ν be a discrete valuation on F and $\kappa(\nu)$ the residue field at ν. Then Kato [1986, Section 1] defined a residue homomorphism $H^3(F, \mu_\ell^{\otimes 3}) \to {}_\ell\mathrm{Br}(\kappa(\nu))$. Let $\zeta \in H^3(F, \mu_\ell^{\otimes 3})$ and $\alpha = [a, b) \in H^2(F, \mu_\ell^{\otimes 2})$. First we show that if there is a regular proper model $\mathscr{X}$ of F such that the triple $(\zeta, \alpha, \mathscr{X})$ satisfies certain assumptions, then there is a local global principle for the divisibility of ζ by α (see Theorem 6.5). One of the key assumptions is that $a \in F^*$ has some "nice" properties at closed points of $\mathscr{X}$ which are on the support of the prime ℓ and in the ramification of ζ or α (see Assumptions 5.1 and 6.3). These assumptions on a enable us to work in spite of the absence of a residue homomorphisms on $H^2(F, \mu_\ell^{\otimes 2})$ for discrete valuations with residue fields of characteristic ℓ and also enable us to blow up the given model so that there are no chilly loops (as defined by Saltman).

Let $\zeta \in H^3(F, \mu_\ell^{\otimes 3})$. First we choose a regular proper model $\mathscr{X}$ of F where the ramification of ζ and the support of ℓ is a union of regular curves with normal crossings on $\mathscr{X}$. For each irreducible curve C on $\mathscr{X}$ which is in the union of the ramification of ζ and support of ℓ, let β_C be the residue of ζ at C. Since the residue field $\kappa(C)$ at C is either a global field or a local field, β_C is a cyclic algebra. Using the class field theory and weak approximation, we write $\beta_C = [a_C, b_C)$ with some conditions on a_C and b_C at finitely many closed points of the model. Then we lift these a_C and b_C to $a, b \in F^*$ which satisfy some "nice" conditions and let $\alpha = [a, b)$. By the choice of a and b, α is unramified at all irreducible curves in the support of ℓ and also unramified at some predetermined finitely many closed points of the model. Suppose that $\ell \neq 2$ or K is a local field or K is a global field without real places. Then we show that there exists a sequence of blow-ups $\mathscr{Y}$ of $\mathscr{X}$ such that $\alpha = [a, b) \in H^2(F, \mu_\ell^{\otimes 2})$ and $\mathscr{Y}$ satisfies the assumption of Section 6. Thus, by the local global principle for the divisibility, there exists $f \in F^*$ such that $\zeta - \alpha \cdot (f)$ is unramified on $\mathscr{X}$. Then, using a result of Kato [1986], we arrive at the proof of Theorem 7.7.

2. Preliminaries

Lemma 2.1 [Colliot-Thélène 1999, Proposition 4.1.2(i)]. *Let K be a field with a discrete valuation ν and κ the residue field at ν. Let m be the maximal ideal of the valuation ring R at ν. Suppose that* $\mathrm{char}(K) = 0$ *and* $\mathrm{char}(\kappa) = \ell > 0$. *Suppose that K contains a primitive ℓ-th root of unity ρ. Then $\ell = x(\rho - 1)^{\ell-1}$ for some unit x at ν with $x \equiv -1$ modulo m. In particular $\nu(\rho - 1) = \nu(\ell)/(\ell - 1)$.*

Proof. The congruence $x \equiv -1$ modulo m holds according to the proof of [Colliot-Thélène 1999, Proposition 4.1.2(i)]. □

Lemma 2.2. *Suppose R is a discrete valuation ring with field of fractions K and residue field κ. Suppose that* $\mathrm{char}(K) = 0$, $\mathrm{char}(\kappa) = \ell > 0$ *and K contains a primitive ℓ-th root of unity ρ. Let $u \in R$ and $\bar{u} \in \kappa$ the image of u. If $1 - u(\rho - 1)^\ell \in R^\ell$, then $X^\ell - X + \bar{u}$ has a root in κ. The converse is true if R is complete.*

Proof. Let m be the maximal ideal of R. Suppose that $u \in m$. Then $\bar{u} = 0$ and $X^\ell - X$ has a root in κ.

Suppose that $u \in R$ is a unit. Suppose $1 - u(\rho - 1)^\ell \in R^\ell$. Let $z \in R$ with $z^\ell = 1 - u(\rho - 1)^\ell \in R$. Since $\rho - 1 \in m$, $1 - u(\rho - 1)^\ell$ is a unit in R and hence z is a unit in R with $z^\ell \equiv 1$ modulo m. Since $\mathrm{char}(\kappa) = \ell$, $z \equiv 1$ modulo m. Thus $z = 1 + d$ for some $d \in m$. Since $z^\ell = (1 + d)^\ell = 1 + \ell d + \cdots + d^\ell$, all the nontrivial binomial coefficients are divisible by ℓ and $d \in m$, we have $z^\ell = 1 + \ell d y + d^\ell$ for some unit $y \in R$ with $y \equiv 1$ modulo m. Since $z^\ell = 1 - u(\rho - 1)^\ell$, we have $\ell d y + d^\ell = -u(\rho - 1)^\ell$.

We claim that $\nu(d) = \nu(\rho - 1)$. Suppose that $\nu(\ell d) = \nu(d^\ell)$. Then $\nu(\ell) + \nu(d) = \ell\nu(d)$ and hence $\nu(d) = \nu(\ell)/(\ell - 1) = \nu(\rho - 1)$ (Lemma 2.1). Suppose that $\nu(\ell d) < \nu(d^\ell)$. Then $\nu(\ell d y + d^\ell) = \nu(\ell d) = \nu(\ell) + \nu(d)$. Since $\ell d y + d^\ell = -u(\rho - 1)^\ell$, $\nu(\ell) + \nu(d) = \ell\nu(\rho - 1)$ and hence $\nu(d) = \ell\nu(\rho - 1) - \nu(\ell) = \ell\nu(\ell)/(\ell - 1) - \nu(\ell) = \nu(\ell)/(\ell - 1) = \nu(\rho - 1)$. Suppose that $\nu(\ell d y) > \nu(d^\ell)$. Then $\ell\nu(\rho - 1) = \nu(d^\ell) = \ell\nu(d)$ and hence $\nu(d) = \nu(\rho - 1)$.

Since $\nu(d) = \nu(\rho - 1)$, we have $d = w(\rho - 1)$ for some unit $w \in R$. By Lemma 2.1, we have $\ell = x(\rho - 1)^{\ell-1}$ with $x \equiv -1$ modulo m. Thus

$$-u(\rho - 1)^\ell = \ell d y + d^\ell = x y w(\rho - 1)^\ell + w^\ell(\rho - 1)^\ell$$

and hence

$$-u = w^\ell + x y w.$$

Since $x \equiv -1$ modulo m and $y \equiv 1$ modulo m, we have $\overline{w}^\ell - \overline{w} + \bar{u} = 0$. In particular $X^\ell - X + \bar{u}$ has a root in κ.

Suppose R is complete and $X^\ell - X + \bar{u}$ has a root in κ. Since $\text{char}(\kappa) = \ell$, $X^\ell - X + \bar{u}$ has ℓ distinct roots in κ. Since R is complete, $X^\ell - X + u$ has a root w in R. Let $d = w(\rho - 1) \in R$. Then, as above, we have $(1+d)^\ell = 1 + \ell d y + d^\ell$ for some $y \in R$ with $y \equiv 1$ modulo m_R. By Lemma 2.1, we have $\ell = x(\rho - 1)^{\ell-1}$ for some $x \in R$ with $x \equiv -1$ modulo m_R. Since $w^\ell = w - u$ and $d = w(\rho - 1)$, we have

$$\begin{aligned}(1+d)^\ell = 1 + \ell d y + d^\ell &= 1 + \ell w(\rho-1)y + w^\ell(\rho-1)^\ell \\ &= 1 + \ell w(\rho-1)y + w(\rho-1)^\ell - u(\rho-1)^\ell \\ &= 1 + xyw(\rho-1)^\ell + w(\rho-1)^\ell - u(\rho-1)^\ell \\ &= 1 + w(\rho-1)^\ell(xy+1) - u(\rho-1)^\ell.\end{aligned}$$

Since $xy+1 \equiv 0$ modulo m, we have $(1+d)^\ell = 1-u(\rho-1)^\ell$ modulo $(\rho-1)^\ell m$ and hence $1-u(\rho-1)^\ell \in R^{*\ell}$ (see [Epp 1973, Section 0.3]). □

Let R be a regular domain with field of fractions K and let L/K be a finite separable extension. Let S be the integral closure of R in L. We say that L/K is *unramified* at a prime ideal P of R, if S_P/PS_P is a separable algebra over the field R_P/PR_P, where $S_P = S \otimes_R R_P$ is the same as the integral closure of the local ring R_P in L. We say that L/K is *unramified* on R if it is unramified at every prime ideal of R. If L/K is unramified at a prime ideal P of R, the separable R_P/PR_P-algebra S_P/PS_P is called the *residue field* of L at P. Note that S_P/PS_P is a product of separable field extensions of R_P/PR_P. If R is a regular local ring, then L/K is unramified at R if and only if the discriminant of L/K is a unit in R (see [Milne 1980, Exercise 3.9, page 24]). Thus in particular, L/K is unramified on R if and only is L/K is unramified at all height one prime ideals of R. If L is a product of fields L_i with $K \subset L_i$, then we say that L/K is *unramified* on R if each L_i/K is unramified on R.

We have the following (see [Epp 1973, Proposition 1.4]):

Proposition 2.3. *Suppose R is a discrete valuation ring with field of fractions K and residue field κ. Suppose that* $\text{char}(K) = 0$, $\text{char}(\kappa) = \ell > 0$ *and K contains a primitive ℓ-th root of unity ρ. Let $u \in R$ and $L = K[X]/(X^\ell - (1 - u(\rho-1)^\ell))$. Let S be the integral closure of R in L. Then L/K is unramified on R and*:

- *If $X^\ell - X + \bar{u}$ is irreducible in $\kappa[X]$, then S has a unique maximal ideal, it is generated by the maximal ideal m_R of R, and $S/m_R S \simeq \kappa[X]/(X^\ell - X + \bar{u})$, where $\bar{u}$ is the image of u in κ.*
- *If $X^\ell - X + \bar{u}$ is reducible in $\kappa[X]$, then $m_R S$ is the product of ℓ distinct maximal ideals of S and again $S/m_R S \simeq \kappa[X]/(X^\ell - X + \bar{u})$.*

Proof. Without loss of generality we assume that R is complete. If L is not a field, which happens if and only if $X^\ell - X - \bar{u}$ is reducible in $\kappa[X]$ by Lemma 2.2, then the result is clearly true. So we further assume that L is a field and $X^\ell - X - \bar{u}$ is irreducible in $\kappa[X]$. Then S is a complete discrete valuation ring. Let m_R be the maximal ideal of R and m_S the maximal ideal of S. Since $1 - u(\rho-1)^\ell \in S^\ell$, by

Lemma 2.2, $X^\ell - X - \bar{u}$ has a root in S/m_S. Since $[S/m_S : \kappa] \leq \ell$, $S/m_S \simeq \kappa[X]/(X^\ell - x + \bar{u})$ and hence the ramification index of S over R is 1 and $m_S = m_R S$. It follows that L/K unramified on R. □

Corollary 2.4. *Suppose that A is a regular local ring of dimension two with field of fractions F, maximal ideal m and residue field κ. Suppose that* $\mathrm{char}(F) = 0$, $\mathrm{char}(\kappa) = \ell > 0$ *and F contains a primitive ℓ-th root of unity ρ. Let $u \in A$ and $L = F[X]/(X^\ell - (1 - u(\rho - 1)^\ell))$. Suppose that L is a field. Let S be the integral closure of A in L. Then L/F is unramified on A and $S/mS \simeq \kappa[X]/(X^\ell - X + \bar{u})$, where $\bar{u}$ is the image of u in κ.*

Proof. Since $\mathrm{char}(\kappa) = \ell$ and $\rho^\ell = 1$, $1 - \rho$ is in the maximal ideal of A and hence $1 - u(\rho - 1)^\ell$ is a unit in A. Let P be a prime ideal of A of height one. Suppose $\mathrm{char}(A/P) \neq \ell$. Since $1 - u(\rho - 1)^\ell$ is a unit in A, L/F is unramified at P. If $\mathrm{char}(A/P) = \ell$, then by Proposition 2.3, L/F is unramified at P. Thus L/F is unramified on A.

Let $m = (\pi, \delta)$ be the maximal ideal of A. Since L/F is unramified on A, $S/\pi S$ is a regular semilocal ring (see [Milne 1980, Proposition 3.17, page 27]). Suppose that $\mathrm{char}(A/(\pi)) \neq \ell$. Since $1 - u(\rho - 1)^\ell$ is a unit at π, L/F is unramified at π and $S \otimes_A A_{(\pi)}/(\pi) \simeq (A_{(\pi)}/(\pi))[X]/(X^\ell - (1 - \bar{u}(\bar{\rho} - 1)^\ell))$, where $\bar{\ }$ denotes the image modulo (π). Hence by Proposition 2.3, $S/(\pi, \delta)S = \kappa[X]/(X^\ell - X + \bar{u})$. Suppose that $\mathrm{char}(A/(\pi)) = \ell$. Then, by Proposition 2.3, the field of fractions of $S/\pi S$ is the field of fractions of $(A/(\pi))[X]/(X^\ell - X + \bar{u})$. Since u is a unit in $A/(\pi)$, $A/(\pi)[X]/(X^\ell - X + \bar{u})$ is a regular local ring and hence $S/\pi S \simeq A/(\pi)[X]/(X^\ell - X + \bar{u})$. Hence $S/(\pi, \delta)S = \kappa[X]/(X^\ell - X + \bar{u})$. □

Let K be a field and ℓ a prime. Then every nontrivial element in $H^1(K, \mathbb{Z}/\ell)$ is represented by a pair (L, σ), where L/K is a cyclic field extension of degree ℓ and σ a generator of $\mathrm{Gal}(L/K)$.

Suppose $\ell \neq \mathrm{char}(K)$ and K contains a primitive ℓ-th root of unity. Fix a primitive ℓ-th root of unity $\rho \in K$. Let L/K be a cyclic extension of degree ℓ. Then, by Kummer theory, we have $L = K(\sqrt[\ell]{a})$ for some $a \in K^*$ and $\sigma \in \mathrm{Gal}(L/K)$ given by $\sigma(\sqrt[\ell]{a}) = \rho\sqrt[\ell]{a}$ is a generator of $\mathrm{Gal}(L/K)$. Thus we have an isomorphism $K^*/K^{*\ell} \to H^1(K, \mathbb{Z}/\ell\mathbb{Z})$ given by sending the class of a in $K^*/K^{*\ell}$ to the pair (L, σ), where $L = K[X]/(X^\ell - a)$ and $\sigma(\sqrt[\ell]{a}) = \rho\sqrt[\ell]{a}$. Let $a \in K^*$. If the image of the class of a in $H^1(F, \mathbb{Z}/\ell\mathbb{Z})$ is (L, σ) and i is coprime to ℓ, then the image of a^i is (L, σ^i). In particular $(L, \sigma)^i = (L, \sigma^i)$ for all i coprime to ℓ.

Suppose $\mathrm{char}(K) = \ell$ and L/K is a cyclic extension of degree ℓ. Then, by Artin–Schreier theory, $L = K[X]/(X^\ell - X + a)$ for some $a \in K$. The element $\sigma \in \mathrm{Gal}(L/K)$ given by $\sigma(x) = x + 1$, where $x \in L$ is the image of X in L, is a generator of $\mathrm{Gal}(L/K)$. Let $\wp : K \to K$ be the Artin–Schreier map $\wp(b) = b^\ell - b$. We have an isomorphism $K/\wp(K) \to H^1(K, \mathbb{Z}/\ell\mathbb{Z})$ given by sending the class of a to the pair (L, σ), where $L = K[X]/(X^\ell - X + a)$ and $\sigma(x) = x + 1$. We note that if the image the class of a is (L, σ), then the image of the class of ia is (L, σ^i) for all $1 \leq i \leq \ell - 1$.

In either case ($\mathrm{char}(K) \neq \ell$ or $\mathrm{char}(K) = \ell$), for $a \in K^*$ (or K), the pair (L, σ) is denoted by $[a)$. Sometimes, by abuse of notation, we also denote the cyclic extension L by $[a)$.

Let R be a regular ring of dimension at most 2 with field of fractions K and ℓ a prime. If ℓ is not equal to $\mathrm{char}(K)$, then assume that K contains a primitive ℓ-th root of unity ρ. Suppose $L = [a)$ is a cyclic extension of K of degree ℓ. Let P be a prime ideal of R, $\kappa(P) = R_P/PR_P$ and S_P the integral closure of R_P in L. Suppose $\mathrm{char}(\kappa(P)) \neq \ell$. Then $L = K[X]/(X^\ell - a)$ and hence $S_P/PS_P \simeq \kappa(P)[X]/(X^\ell - \bar{a})$ where $\bar{a}$ is the image of a in $\kappa(P)$. Suppose $\mathrm{char}(\kappa(P)) = \ell$, $\mathrm{char}(K) \neq \ell$ and $a = 1 - u(\rho - 1)^\ell$ for some $u \in R_P$. Then, by (Proposition 2.3 and Corollary 2.4), $S_P/PS_P \simeq \kappa(P)[X]/(X^\ell - X + \bar{u})$. Suppose $\mathrm{char}(\kappa(P)) = \mathrm{char}(K) = \ell$ and $a \in R_P$. Then $L = K[X]/(X^\ell - X + a)$ and hence $S_P/PS_P \simeq \kappa(P)[X]/(X^\ell - X + \bar{a})$. Thus, in either case, S_P/PS_P is either a cyclic field extension of degree ℓ over $\kappa(P)$ or the split extension of degree ℓ over $\kappa(P)$ and we denote these S_P/PS_P by $[a(P))$. If $P = (\pi)$ for some $\pi \in R$, then we also denote $[a(P))$ by $[a(\pi))$. If P induces a discrete valuation ν on K, then we also denote $[a(P))$ by $[a(\nu))$. For an element $b \in R$, we also denote the image of b in R/P by $b(P)$. If $b \in R$ and $c \in R/P$, we write $b = c \in R/P$ for $b \equiv c$ modulo P.

Lemma 2.5. *Let A be a semilocal regular ring of dimension at most two with field of fractions F. Let ℓ be a prime not equal to the characteristic of F. Suppose that F contains a primitive ℓ-th root of unity. For each maximal ideal m of A, let $[u_m)$ be a cyclic extension of A/m of degree ℓ. Then there exists $a \in A$ such that*:

- *$[a)$ is unramified on A with residue field $[u_m)$ at each maximal ideal m of A.*
- *If $\ell = 2$ and A/m is finite for all maximal ideals m of A, then a can be chosen to be a sum of two squares in A.*

Proof. Let $\rho \in F$ be a primitive ℓ-th root of unity. Let m be a maximal ideal of A. If $\mathrm{char}(A/m) \neq \ell$, then let $b_m = (1 - u_m/(\rho - 1)^\ell) \in A/m$. If $\mathrm{char}(A/m) = \ell$, then let $b_m = u_m \in A/m$. Choose $b \in A$ with $b = b_m \in A/m$ for all maximal ideals m of A and $a = 1 - b(\rho - 1)^\ell$. Let m be a maximal ideal of A. Suppose that $\mathrm{char}(A/m) \neq \ell$. Then, by the choice of a and b, we have $a = 1 - b_m(\rho - 1)^\ell = u_m \in A/m$. Thus $[a)$ is unramified on A_m with the residue field $[u_m)$ at m. Suppose that $\mathrm{char}(A/m) = \ell$. Then, by (Proposition 2.3 and Corollary 2.4), $[a)$ is unramified on A_m with the residue field $[\bar{b})$. Since $b = b_m = u_m \in A/m$, the residue field of $[a)$ at m is $[u_m)$.

Suppose $\ell = 2$ and A/m is a finite field for all maximal ideals m of A. Let m be a maximal ideal of A. Suppose that $\mathrm{char}(A/m) \neq 2$. Since every element of A/m is a sum of two squares in A/m [Scharlau 1985, page 39, 3.7], there exist $x_m, y_m \in A/m$ such that $x_m^2 + y_m^2 = 1 - 4u_m$. Suppose that $\mathrm{char}(A/m) = 2$. Since A/m is a finite field, every element in A/m is a square. Let $y_m \in A/m$ be such that $y_m^2 = u_m$. Let $x, y \in A$ be such that for every maximal ideal m of A:

- If $\mathrm{char}(A/m) \neq 2$, then $x = \frac{1}{4}(x_m - 1) \in A/m$ and $y = \frac{1}{2} y_m \in A/m$.
- If $\mathrm{char}(A/m) = 2$, then $x = 0 \in A/m$ and $y = y_m \in A/m$.

Let $a = (1 + 4x)^2 + (2y)^2 \in A$. Let m be a maximal ideal of A. Suppose $\mathrm{char}(A/m) \neq 2$. Then $a = x_m^2 + y_m^2 = u_m \in A/m$ and hence $[a)$ is unramified on A_m with residue field at m equal to $[u_m)$. Suppose that $\mathrm{char}(A/m) = 2$. Then $\frac{1}{4}(1 - a) = u_m \in A/m$ and hence $[a)$ is unramified on A_m with residue field $[u_m)$ (Proposition 2.3 and Corollary 2.4). □

Lemma 2.6. *Let R be a semilocal regular domain of dimension 1 and K its field of fractions. Let ℓ be a prime not equal to* $\operatorname{char}(K)$. *Suppose that K contains a primitive ℓ-th root of unity ρ. Let $L = K(\sqrt[\ell]{u})$ for some $u \in R$. Let $m_1, \dots, m_r, m_{r+1}, \dots, m_n$ be the maximal ideals of R. Suppose that* $\operatorname{char}(\kappa(m_j)) = \ell$ *and L/K is unramified at m_j for all $r+1 \le j \le n$. Then there exists $v \in R$ such that $L = K(\sqrt[\ell]{v})$, $v \equiv u$ modulo m_i for all $1 \le i \le r$ and $(1-v)/(\rho-1)^\ell \in R_{m_j}$ for all $r+1 \le j \le n$.*

Proof. For a maximal ideal m of R, let K_m denote the field of factions of the completion of R at m.

Let $r+1 \le j \le n$. Since $\operatorname{char}(\kappa(m_j)) = \ell$ and L/K unramified at m_j, the residue field of L at m_j is $\kappa(m_j)[X]/(X^\ell - X + \overline{w}_j)$ for some $w_j \in R_{m_j}$. Since the residue field of $K[X]/(X^\ell - (1 - w_j(\rho-1)^\ell))$ is isomorphic to $\kappa(m_j)[X]/(X^\ell - X + \overline{w}_j)$ (Proposition 2.3 and Corollary 2.4),

$$L \otimes K_{m_j} \simeq K_{m_j}[X]/(X^\ell - (1 - w_j(\rho-1)^\ell)).$$

Since $\operatorname{char}(K) \ne \ell$ and $L = K(\sqrt[\ell]{u})$, there exists $\theta_j \in K_{m_j}$ such that $u\theta_j^\ell = 1 - w_j(\rho-1)^\ell$. Let N be an integer larger than the sum of the valuations of u and $(\rho-1)^\ell$ at all m_i. By the weak approximation, there exists $\theta \in K$ such that $\theta \equiv 1$ modulo m_i for $1 \le i \le r$ and $\theta\theta_j^{-1} \equiv 1$ modulo m_j^{N+1} for $r+1 \le j \le n$.

Let $v = u\theta^\ell$. Let $1 \le i \le r$. Since $\theta \equiv 1$ modulo m_i, $v \equiv u$ modulo m_i. Let $r+1 \le j \le n$. Let $\pi_j \in R$ be a generator of the ideal m_j. Then $\theta^\ell\theta_j^{-\ell} = 1 + a_j\pi_j^{N+1}$ for some $a_j \in \hat{R}_{m_j}$. Since $u\theta_j^\ell = 1 - w_j(\rho-1)^\ell \in R_{m_j}$ is a unit and N is bigger than the sum of the valuations of u and $(\rho-1)^\ell$, we have $\theta_j^\ell a_j \pi_j^{N+1} = b_j(\rho-1)^\ell$ for some $b_j \in \hat{R}_{m_j}$. Hence

$$v = u\theta^\ell = u\theta_j^\ell + ub_j(\rho-1)^\ell = 1 - w_j(\rho-1)^\ell + ub_j(\rho-1)^\ell = 1 - c_j(\rho-1)^\ell$$

for some $c_j \in \hat{R}_{m_j}$ Since $c_j = (1-v)/(\rho-1)^\ell \in K \cap \hat{R}_{m_j} = R_{m_j}$, v has the required properties. □

The following is a generalization of a result of Saltman [2008, Proposition 0.3].

Lemma 2.7. *Let A be a UFD. For $1 \le i \le n$, let $I_i = (a_i) \subset A$ with $gcd(a_i, a_j) = 1$ for all $i \ne j$. For each $i < j$, let $I_{ij} = I_i + I_j$. Suppose that the ideals I_{ij} are comaximal. Then*

$$A \to \bigoplus_i A/I_i \to \bigoplus_{i<j} A/I_{ij}$$

is exact, where for $i < j$, the map from $A/I_i \oplus A/I_j \to A/I_{ij}$ is given by $(x, y) \mapsto x - y$.

Proof. Proof by induction on n. The case $n = 2$ is in [Saltman 2008, Lemma 0.2]. Assume that $n \ge 3$. Suppose $(x_i) \in A/I_i$ maps to zero in $\oplus A/I_{ij}$. By induction, there exists $b \in A$ such that $b = x_i \in A/I_i$ for $1 \le i \le n-1$. We claim that $I_1 \cap \cdots \cap I_{n-1} + I_n = (I_1 + I_n) \cap \cdots \cap (I_{n-1} + I_n)$. Since both sides contain I_n, it is enough to prove the equality modulo I_n. Since $\gcd(a_i, a_j) = 1$ for all $i \ne j$, we have $I_1 \cap \cdots \cap I_{n-1} = Aa_1 \cdots a_{n-1}$ and hence $I_1 \cap \cdots \cap I_{n-1} + I_n/I_n = (A/I_n)\bar{a}_1 \cdots \bar{a}_{n-1}$. Since I_{ij} are comaximal, $I_{in}/I_n = (A/I_n)\bar{a}_i$ are comaximal for $1 \le i \le n-1$ and hence $(A/I_n)\bar{a}_1 \cdots \bar{a}_{n-1} = (A/I_n)\bar{a}_1 \cap \cdots \cap (A/I_n)\bar{a}_{n-1}$. Let $b_1 \in A/(I_1 \cap \cdots \cap I_{n-1})$ be the image of b. Then, by the case $n = 2$, there exists $a \in A$ such that $a = b_1 \in A/I_1 \cap \cdots \cap I_{n-1}$ and $a = x_n \in A/I_n$.Thus a has the required properties. □

3. Central simple algebras

Let K be a field, L/K a cyclic extension of degree n with $\sigma \in \mathrm{Gal}(L/K)$ a generator and $b \in K^*$. Let (L, σ, b) denote the cyclic algebra $L \oplus Lx \oplus \cdots \oplus Lx^{n-1}$ with relations $x^n = b$, $x\lambda = \sigma(\lambda)x$ for all $\lambda \in L$. Then (L, σ, b) is a central simple algebra over K and represents an element in the n-torsion subgroup ${}_n\mathrm{Br}(K)$ of the Brauer group $\mathrm{Br}(K)$ [Albert 1939, Theorem 18, page 98]. Suppose that n is coprime to $\mathrm{char}(K)$ and K contains a primitive n-th root of unity. Then $L = K(\sqrt[n]{a})$ for some $a \in K^*$. Fix a primitive n-th root of unity ρ in K. Let σ be the generator of $\mathrm{Gal}(L/K)$ given by $\sigma(\sqrt[n]{a}) = \rho\sqrt[n]{a}$. Then, the cyclic algebra (L, σ, b) is denoted by $[a, b)$. Suppose that n is prime and equal to $\mathrm{char}(K)$. Then, $L = K[X]/(X^n - X + a)$ for some $a \in K$. If σ is the generator of $\mathrm{Gal}(L/K)$ given by $\sigma(x) = x + 1$, then the cyclic algebra (L, σ, b) is also denoted by $[a, b)$.

For any Galois module M over K, let $H^n(K, M)$ denote the Galois cohomology of K with coefficients in M. Let ℓ be a prime. Let $\mathbb{Z}/\ell(i)$ be the Galois modules over K as in [Kato 1986, Section 0]. We have canonical isomorphisms $H^1(K, \mathbb{Z}/\ell) \simeq \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(K^{\mathrm{ab}}/K), \mathbb{Z}/\ell)$ and ${}_\ell\mathrm{Br}(K) \simeq H^2(K, \mathbb{Z}/\ell(1))$, where K^{ab} is the maximal abelian extension of K [Kato 1986, Section 0].

Suppose A is a regular domain with field of fractions F. We say that an element $\alpha \in H^2(F, \mathbb{Z}/\ell(1))$ is *unramified* on A if α is represented by a central simple algebra over F which comes from an Azumaya algebra over A. If it is not unramified, then we say that α is *ramified* on A. Suppose P is a prime ideal of A and $\alpha \in H^2(F, \mathbb{Z}/\ell(1))$. We say that α is *unramified* at P if α is unramified on A_P. If α is not unramified at P, then we say that α is *ramified* at P. Suppose that α is unramified at P. Let $\mathscr{A}$ be an Azumaya algebra over A_P with the class of $\mathscr{A} \otimes_{A_P} F$ equal to α. The algebra $\bar{\alpha} = \mathscr{A} \otimes_{A_P} (A_P/PA_P)$ is called the *specialization* of α at P. Since A_P is a regular local ring, the class of $\bar{\alpha}$ is independent of the choice of $\mathscr{A}$. Let $a, b \in F$ and $\alpha = [a, b) \in H^2(F, \mathbb{Z}/\ell(1))$. If the cyclic extension $[a)$ is unramified at P and b is a unit at P, then α is unramified at P and the specialization of α at P is $[a(P), b(P))$, where $[a(P))$ is the residue field of $[a)$ at P and $b(P)$ is the image of b in A_P/PA_P.

Suppose that R is a discrete valuation ring with field of fractions K and residue field κ. Let ℓ be a prime not equal to $\mathrm{char}(K)$. Suppose that $\mathrm{char}(\kappa) \neq \ell$ or $\mathrm{char}(\kappa) = \ell$ with $\kappa = \kappa^\ell$. Then there is a *residue homomorphism* $\partial : H^2(K, \mathbb{Z}/\ell(1)) \to H^1(\kappa, \mathbb{Z}/\ell)$ [Kato 1986, Section 1]. Further a class $\alpha \in H^2(K, \mathbb{Z}/\ell(1))$ is unramified at R if and only if $\partial(\alpha) = 0$. Let $a, b \in K^*$. If $[a)$ is unramified at R, then $\partial([a, b)) = [a(\nu))^{\nu(b)}$, where ν is the discrete valuation on K. In particular if $[a)$ is unramified on R and ℓ divides $\nu(b)$, then $[a, b)$ is unramified on R.

Lemma 3.1 ([Auslander and Goldman 1960, Proposition 7.4], see [Lieblich et al. 2014, Lemma 3.1]). *Let A be a regular ring of dimension* 2 *and F its field of fractions. Let ℓ be a prime not equal to* $\mathrm{char}(F)$ *and $\alpha \in H^2(F, \mathbb{Z}/\ell(1))$. If α is unramified at all height one prime ideals of A, then α is unramified on A.*

Lemma 3.2. *Let R be a complete discrete valuation ring with field of fractions K and residue field κ. Let ℓ be a prime not equal to* $\mathrm{char}(\kappa)$. *Let D be a central simple algebra of index ℓ over K. Suppose that D is ramified at R. If L/K is the unramified extension of K with residue field equal to the residue of D at R, then $D \otimes L$ is a split algebra.*

Proof. We have $D = D_0 \otimes (L, \sigma, \pi)$ for some generator of $\mathrm{Gal}(L/K)$, π a parameter in R and D_0 unramified at R (see [Parimala et al. 2018, Lemma 4.1]). Further $\ell = \mathrm{ind}(D) = \mathrm{ind}(D_0 \otimes L)[L : K]$ (see [loc. cit., Lemma 4.2]). Since D is ramified at R, $[L : K] = \ell$ and hence $D_0 \otimes L = 0$. Hence $D_0 = (L, \sigma, u)$ for some $u \in K$ and $D = (L, \sigma, u\pi)$. Thus $D \otimes L$ is a split algebra. □

Lemma 3.3. *Let A be a complete regular local ring of dimension 2 with field of fractions F and residue field κ. Suppose that κ is a finite field. Let $m = (\pi, \delta)$ be the maximal ideal of A. Let ℓ be a prime not equal to* $\mathrm{char}(F)$ *and $\alpha = [a, b) \in H^2(F, \mathbb{Z}/\ell(1))$ for some $a, b \in F^*$. Suppose that*:

- *If* $\mathrm{char}(\kappa) = \ell$, *then the cyclic extension $[a)$ is unramified on A.*
- *α is unramified on A except possibly at δ.*
- *The specialization of α at π is unramified on $A/(\pi)$.*

Then $\alpha = 0$.

Proof. Suppose that $\mathrm{char}(\kappa) \neq \ell$. Then, it follows from [Reddy and Suresh 2013, Proposition 3.4] that $\alpha = 0$ (see [Parimala et al. 2018, Corollary 5.5]).

Suppose that $\mathrm{char}(\kappa) = \ell$. Since F is the field of fractions of A, without loss of generality, we assume that $b \in A$ and not divisible by θ^ℓ for any prime $\theta \in A$. Write $b = v\delta^n\theta_1^{n_1} \cdots \theta_r^{n_r}$ for some distinct primes $\theta_i \in A$ with $(\delta) \neq (\theta_i)$ for all i, $1 \leq n_i \leq \ell - 1$, $0 \leq n \leq \ell - 1$ and $v \in A$ a unit. Since κ is a finite field, A is complete and $[a)$ is unramified on A, we have $[a, v) = 0$ and hence $\alpha = [a, b) = [a, \delta^n\theta_1^{n_1} \cdots \theta_r^{n_r})$.

Since $[a)$ is unramified on A, for any prime $\theta \in A$, $[a, \theta)$ is unramified on A except possibly at θ. Let $1 \leq j \leq r$. Since $\alpha = [a, b) = [a, \delta^n) \prod [a, \theta_i^{n_i})$, $[a, \delta^n)$ and $[a, \theta_i^{n_i})$ are unramified at θ_j for all $i \neq j$, $[a, \theta_j^{n_j})$ is unramified at θ_j and hence $[a, \theta_j^{n_j})$ is unramified on A (see Lemma 3.1). Since κ is a finite field and A is complete, $[a, \theta_j^{n_j}) = 0$. Thus, we have $\alpha = [a, \delta^n)$.

If $n = 0$, then $\alpha = 0$. Suppose $1 \leq n \leq \ell - 1$. Let $\bar{\alpha}$ be the specialization of α at π. Since $\alpha = [a, \delta^n)$ and $[a)$ is unramified at π, we have $\bar{\alpha} = [a(\pi), \bar{\delta}^n)$, where $[a(\pi))$ is the residue field of $[a)$ at π and $\bar{\delta}$ is the image of δ in $A_P/(\pi)$. Since $\bar{\alpha}$ is unramified on $A/(\pi)$, A is complete and κ is a finite field, $\bar{\alpha} = [a(\pi), \bar{\delta}^n) = 0$. Since $\partial(\bar{\alpha}) = [a(m))^n = 1$ and n is coprime to ℓ, $[a(m)) = 0$. Since A is complete, $[a)$ is trivial and hence $\alpha = 0$. □

We now recall the chilly, cool, hot and cold points and the chilly loops associated to a central simple algebra, due to Saltman [2007; 2008]. Let $\mathscr{X}$ be a regular integral excellent scheme of dimension 2 and F its field of fractions. Let ℓ be a prime which is not equal to $\mathrm{char}(F)$. Suppose that F contains a primitive ℓ-th root of unity. Let $\alpha \in H^2(F, \mathbb{Z}/\ell(1))$. Suppose that $\mathrm{ram}_{\mathscr{X}}(\alpha) = \{D_1, \ldots, D_n\}$ for some regular irreducible curves D_i on $\mathscr{X}$ with normal crossings. Suppose $P \in D_i \cap D_j$ is a closed point. Let A_P be the local ring at P. Let $\pi_i, \pi_j \in A_P$ be primes defining D_i and D_j at P respectively. Suppose that $\mathrm{char}(\kappa(P)) \neq \ell$. Suppose that $\alpha = \alpha_0 + (u, \pi_i) + (v, \pi_j)$ for some α_0 unramified at P, u, v units at P. We say that P is a *chilly point* of α if $u(P)$ and $v(P)$ generate the same nontrivial subgroup of $\kappa(P)^*/\kappa(P)^{*\ell}$, a *cool point* of α if $u(P), v(P) \in \kappa(P)^{*\ell}$, a *hot point* of α if $u(P)$ and $v(P)$ generate different subgroup of $\kappa(P)^*/\kappa(P)^{*\ell}$. We say that P is a *cold point* of α If $\alpha = \alpha_0 + (u\pi_i, v\pi_j^s)$ for some α_0 unramified at P, u, v units at P and s coprime to ℓ.

Let Γ be a graph with vertices D_i's and edges as chilly points, i.e., two distinct vertices D_i and D_j have an edge between them if there is a chilly point in $D_i \cap D_j$. A loop in this graph is called a *chilly loop* on $\mathscr{X}$. Let $\mathscr{X}\left[\frac{1}{\ell}\right]$ be the open subscheme of $\mathscr{X}$ obtained by inverting ℓ. Since, by the definition of chilly point, char$(\kappa(P)) \neq \ell$ for any chilly point P, we have the following

Proposition 3.4 [Saltman 2007, Corollary 2.9]. *There exists a sequence of blow-ups $\mathscr{X}' \to \mathscr{X}$ centered at closed points $P \in \mathscr{X}\left[\frac{1}{\ell}\right]$ such that α has no chilly loops on $\mathscr{X}'$.*

Let K be a global field and ℓ a prime. Let $\beta \in {}_\ell\mathrm{Br}(K)$. Let ν be a discrete vacation of K, K_ν the completion of K at ν and $\kappa(\nu)$ the residue field at ν. Since K_ν is a local field, the invariant map gives an isomorphism $\partial_\nu : {}_\ell\mathrm{Br}(K_\nu) = H^2(K_\nu, \mathbb{Z}/\ell(1)) \to H^1(\kappa(\nu), \mathbb{Z}/\ell)$.

Proposition 3.5. *Let K be a global field and ℓ a prime. If ℓ is not equal to* char(K), *then assume that K contains a primitive ℓ-th root of unity ρ. Let $\beta \in {}_\ell\mathrm{Br}(K)$. Let S be a finite set of discrete valuations of K containing all the discrete valuations ν of K with $\partial_\nu(\beta) \neq 0$. Let S' be a finite set of discrete valuations of K with $S \cap S' = \varnothing$. Let $a \in K^*$ and for each $\nu \in S'$, let $n_\nu \geq 2$ be an integer. Suppose that for every $\nu \in S$, $[a)$ is unramified at ν with $\partial_\nu(\beta) = [a(\nu))$. Further assume that if $\ell = 2$, then $\beta \otimes K_\nu(\sqrt{a}) = 0$ for all real places ν of K. Then there exists $b \in K^*$ such that*:

- *$\beta = [a, b)$.*
- *If $\nu \in S$, then $\nu(b) = 1$.*
- *If $\nu \in S'$, then $\nu(b-1) \geq n_\nu$.*

Proof. Let $L = [a)$. Let $\nu \in S$. If $\partial_\nu(\beta) = 0$, then $\beta \otimes K_\nu = 0$ [Cassels and Fröhlich 1967, page 131]. Suppose that $\partial_\nu(\beta) \neq 0$. Then $[a(\nu))$ is a field extension of $\kappa(\nu)$ of degree ℓ and hence $L \otimes_K K_\nu$ is a degree ℓ field extension of K_ν. Thus $\beta \otimes_K (L \otimes K_\nu) = 0$ [loc. cit., page 131]. Suppose ν is a real place of K. Then, by the assumption on a, $\beta \otimes_K (L \otimes_K K_\nu) = 0$. Thus $\beta \otimes L = 0$ [loc. cit., page 187] and hence there exists $c \in K^*$ such that $\beta = [a, c)$ [Albert 1939, page 94].

Let R be the semilocal ring at the discrete valuations in $S \cup S'$. Replacing c by $c\theta^\ell$ for some $\theta \in K^*$, we assume that $c \in R$. For $\nu \in S \cup S'$, let $\pi_\nu \in R$ be a parameter at ν. Let $\nu \in S$. Since $[a)$ is unramified at ν, $\partial_\nu(\beta) = \partial_\nu([a, c)) = [a(\nu))^{\nu(c)}$. Suppose $[a(\nu))$ is nontrivial. Since, by the hypothesis, $\partial_\nu(\beta) = [a(\nu))$, $\nu(c) - 1$ is divisible by ℓ. Since $[L : K] = \ell$, $\pi_\nu^{\nu(c)-1}$ is a norm from $L \otimes_K K_\nu/K_\nu$. Suppose that $[a(\nu))$ is trivial. Then $L \otimes_K K_\nu$ is the split extension and hence every element of K_ν is a norm from $L \otimes_K K_\nu/K_\nu$. Thus for each $\nu \in S$, there exists $x_\nu \in L \otimes_K K_\nu$ with norm $\pi_\nu^{\nu(c)-1}$. Let $\nu \in S'$. Then $\partial_\nu(\beta) = 0$ and we have $\beta \otimes K_\nu = [a, c) \otimes K_\nu = 0$ [Cassels and Fröhlich 1967, page 131]. Hence c is a norm from $L \otimes_K K_\nu$. For each $\nu \in S'$, $x_\nu \in L \otimes_K K_\nu$ with norm c. Let $z \in L$ be sufficiently close to x_ν such that $\nu(N_{L\otimes_K K_\nu}(z) - \pi_\nu^{\nu(c)-1}) \geq \nu(c)$ for all $\nu \in S$ and $\nu(N_{L\otimes_K K_\nu}(z) - c) \geq \nu(c) + n_\nu$ for all $\nu \in S'$.

Let d be the norm of z and $b = cd^{-1}$. Then $\beta = [a, cd^{-1}) = [a, b)$. Let $\nu \in S$. Since $\nu(d - \pi_\nu^{\nu(c)-1}) \geq \nu(c)$, we have $\nu(d) = \nu(c) - 1$ and hence $\nu(b) = \nu(cd^{-1}) = 1$. Let $\nu \in S'$. Since $\nu(d - c) \geq \nu(c) + n_\nu \geq 2$, $\nu(d) = \nu(c)$ and $\nu(b - 1) = \nu(cd^{-1} - 1) \geq n_\nu$. □

4. A complex of Kato

Let K be a complete discrete valued field with residue field κ. Let ℓ be a prime not equal to characteristic of K. If $\ell = \text{char}(\kappa)$, then assume that $[\kappa : \kappa^\ell] \leq \ell$. Then, there is a residue homomorphism $\partial : H^3(K, \mathbb{Z}/\ell(2)) \to H^2(\kappa, \mathbb{Z}/\ell(1))$ [Kato 1986, Section 1]. We say that an element $\zeta \in H^3(K, \mathbb{Z}/\ell(2))$ is *unramified* at the discrete valuation of F if $\partial(\zeta) = 0$.

Let $\mathscr{X}$ be a two-dimensional regular integral excellent Noetherian scheme quasiprojective over some affine scheme and F the function field of $\mathscr{X}$. For $x \in \mathscr{X}$, let F_x be the field of fractions of the completion $\hat{A}_x$ of the local ring A_x at x on $\mathscr{X}$ and $\kappa(x)$ the residue field at x. Let $x \in \mathscr{X}$ and C be the closure of $\{x\}$ in $\mathscr{X}$. Then, we also denote F_x by F_C. If the dimension of C is one, then C defines a discrete valuation ν_C (or ν_x) on F. Let $\mathscr{X}_{(i)}$ be the set of points of $\mathscr{X}$ with the dimension of the closure of $\{x\}$ equal to i. Let ℓ be a prime not equal to char(F). Suppose that F contains a primitive ℓ-th root of unity. If $P \in \mathscr{X}_{(0)}$ is a closed point of $\mathscr{X}$ with char$(\kappa(P)) = \ell$, then we assume $\kappa(P) = \kappa(P)^\ell$. Let $x \in \mathscr{X}_{(1)}$. We have a *residue homomorphism*

$$\partial_x : H^3(F, \mathbb{Z}/\ell(2)) \to H^2(\kappa(x), \mathbb{Z}/\ell(1))$$

[Kato 1986, Section 1]. We say that an element $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$ is *unramified* at x (or C) if ζ is unramified at ν_x. Further if $P \in \mathscr{X}_{(0)}$ is in the closure of $\{x\}$, then we have a *residue homomorphism*

$$\partial_P : H^2(\kappa(x), \mathbb{Z}/\ell(1)) \to H^1(\kappa(P), \mathbb{Z}/\ell)$$

[Kato 1986, Section 1]. For $x \in \mathscr{X}_{(1)}$, if C is the closure of $\{x\}$, we also denote ∂_x by ∂_C. An element $\alpha \in H^2(\kappa(x), \mathbb{Z}/\ell(1)) \simeq {}_\ell\text{Br}(\kappa(x))$ is unramified at P if and only if $\partial_P(\alpha) = 0$. We use the additive notation for the group operations on $H^2(F, \mathbb{Z}/\ell(1))$ and $H^3(F, \mathbb{Z}/\ell(2))$ and multiplicative notation for the group operation on $H^1(F, \mathbb{Z}/\ell)$.

Proposition 4.1 [Kato 1986, Proposition 1.7]. *Then*

$$H^3(F, \mathbb{Z}/\ell(2)) \xrightarrow{\partial} \oplus_{x \in \mathscr{X}_{(1)}} H^2(\kappa(x), \mathbb{Z}/\ell(1)) \xrightarrow{\partial} \oplus_{P \in \mathscr{X}_{(0)}} H^1(\kappa(P), \mathbb{Z}/\ell).$$

is a complex, where the maps are given by the residue homomorphism.

Lemma 4.2 [Kato 1980, Section 3.2, Lemma 3; 1986, Lemma 1.4(3). *Let $x \in \mathscr{X}_{(1)}$ and ν_x be the discrete valuation on F at x. Then $\partial_x : H^3(F_x, \mathbb{Z}/\ell(2)) \to H^2(\kappa(x), \mathbb{Z}/\ell(1))$ is an isomorphism. Further if $\alpha \in H^2(F, \mathbb{Z}/\ell(1))$ is unramified at x and $f \in F^*$, then $\partial_x(\alpha \cdot (f)) = \bar{\alpha}^{\nu_x(f)}$.*

The following is a consequence of Proposition 4.1.

Corollary 4.3. *Let C_1 and C_2 be two irreducible regular curves in $\mathscr{X}$ intersecting at a closed point P. Let $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$. Suppose that ζ is unramified at all codimension one points of $\mathscr{X}$ passing through P except possibly at C_1 and C_2. Then*

$$\partial_P(\partial_{C_1}(\zeta)) = \partial_P(\partial_{C_2}(\zeta))^{-1}.$$

Corollary 4.4. *Let C be an irreducible curve on $\mathscr{X}$ and $P \in C$ with C regular at P. Let $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$. Suppose that ζ is unramified at all codimension one points of $\mathscr{X}$ passing through P except possibly at C. If $\kappa(P)$ is finite, then $\zeta \otimes F_P = 0$. In particular if $\kappa(P)$ is finite, then ζ is unramified at every discrete valuation of F centered at P.*

Proof. Since C is regular at P, there exists an irreducible curve C' passing through P and intersecting C transversely at P. Then, by Corollary 4.3, we have $\partial_P(\partial_C(\zeta)) = \partial_P(\partial_{C'}(\zeta))^{\ell-1}$. Since, by assumption, $\partial_{C'}(\zeta) = 0$, we have $\partial_P(\partial_C(\zeta)) = 1$.

Let $\pi \in A_P$ be a prime defining C at P. Since C is regular at P, $A_P/(\pi)$ is a discrete valued ring with residue field $\kappa(P)$ and $\kappa(C)$ is the field of fractions of $A_P/(\pi)$. Further π remains a regular prime in $\hat{A}_P$ and $\hat{A}_P/(\pi)$ is the completion of $A_P/(\pi)$. In particular the field of fractions of $\hat{A}_P/(\pi)$ is the completion $\kappa(C)_P$ of the field $\kappa(C)$ at the discrete valuation given by the discrete valuation ring $A_P/(\pi)$. Let $\tilde{\nu}$ be the discrete valuation on F_P given by the height one prime ideal (π) of $\hat{A}$ and ν the discrete valuation of F given by the height one prime ideal (π) of A. Then the restriction of $\tilde{\nu}$ to F is ν and the residue field $\kappa(\tilde{\nu})$ at $\tilde{\nu}$ is $\kappa(C)_P$.

Since $\partial_P(\partial_C(\zeta)) = 1$, we have $\partial_C(\zeta) \otimes \kappa(C)_P = 0$ [Kato 1986, Lemma 1.4(3)]. Hence

$$\partial_{\tilde{\nu}}(\zeta \otimes F_P) = \partial_C(\zeta) \otimes \kappa(C)_P = 0.$$

Let $F_{P,\tilde{\nu}}$ be the completion of F_P at $\tilde{\nu}$. Since $\partial_{\tilde{\nu}} : H^3(F_{P,\tilde{\nu}}, \mathbb{Z}/\ell(2)) \to H^2(\kappa(C)_P, \mathbb{Z}/\ell(2))$ is an isomorphism [loc. cit., Lemma 1.4(3)], $\zeta \otimes F_{P,\tilde{\nu}} = 0$.

Let ν' be a discrete valuation of F_P given by a height one prime ideal of $\hat{A}$ not equal to (π). Then, by the assumption on ζ, $\partial_{\nu'}(\zeta \otimes F_P) = 0$ and hence $\zeta \otimes F_{P,\nu'} = 0$ [loc. cit., Lemma 1.4(3)], where $F_{P,\nu'}$ is the completion of F_P at ν'. Hence, by [Saito 1987, Theorem 5.3], $\zeta \otimes F_P = 0$. □

5. A local global principle

Let $\mathscr{X}$, F and ℓ be as in Section 4. Let $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$. Let $\alpha = [a, b) \in H^2(F, \mathbb{Z}/\ell(1))$. In this section we show that under some additional assumptions on $\mathscr{X}$, ζ and α, there exists $f \in F^*$ such that $\partial_x(\zeta - \alpha \cdot (f))$ is unramified at all the discrete valuations of $\kappa(x)$ centered at closed points of $\overline{\{x\}}$ for all $x \in \mathscr{X}_{(1)}$ (see Theorem 5.7).

For the rest of this section, we assume the following.

Assumptions 5.1. Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies the following conditions:

(A1) $\mathrm{ram}_{\mathscr{X}}(\zeta) = \{C_1, \ldots, C_r\}$, the C_i are regular irreducible curves with normal crossings.

(A2) $\mathrm{ram}_{\mathscr{X}}(\alpha) = \{D_1, \ldots, D_n\}$, the D_j are regular curves with normal crossings and $C_i \neq D_j$ for all i, j.

By reindexing, we have $\mathrm{ram}_{\mathscr{X}}(\alpha) = \{D_1, \ldots, D_m, \ldots, D_n\}$, with $\mathrm{char}(\kappa(D_i)) = \ell$ for $1 \le i \le m$ and $\mathrm{char}(\kappa(D_j)) \neq \ell$ for $m+1 \le j \le n$:

(A3) $D_i \cap D_j = \varnothing$ for all $1 \le i \le m$ and $m+1 \le j \le n$.

(A4) If $P \in D_i \cap D_j$ for some $m+1 \le i < j \le n$, then $\mathrm{char}(\kappa(P)) \neq \ell$.

(A5) There are no chilly loops (see Section 3) for α on $\mathscr{X}$.

(A6) $\partial_{C_i}(\zeta)$ is the specialization of α at C_i for all i.

(A7) $C_i \cap D_j = \varnothing$ for all i and $1 \le j \le m$.

(A8) If $P \in C_i \cap D_s$ for some i and s, then $P \in C_i \cap C_j$ for some $i \neq j$.

(A9) For every $i \neq j$, through any point of $C_i \cap C_j$ there is at most one D_t.

(A10) In the representation $\alpha = [a, b)$ the element a can be chosen such that if $P \in \mathscr{X}_{(0)}$ with $\mathrm{char}(\kappa(P)) = \ell$ and $P \in D_i$ for some i, then $(1-a)/(\rho-1)^{\ell} \in A_P$.

(A11) If $P \in C_i \cap C_j \cap D_t$ for some $i < j$ and for some t, then D_t is given by a regular prime $u\pi_i^{\ell-1} + v\pi_j$ at P, for some prime π_i (resp. π_j) defining C_i (resp. C_j) at P and units u, v at P.

Let $\mathscr{P}$ be a finite set of closed points of $\mathscr{X}$ containing $C_i \cap C_j$, $D_i \cap D_j$ for all $i \neq j$, $C_i \cap D_j$ for all i, j and at least one point from each C_i and D_j. Let A be the regular semilocal ring at $\mathscr{P}$ on $\mathscr{X}$. For every $P \in \mathscr{P}$, let M_P be the maximal ideal of A at P. For $1 \le i \le r$ and $1 \le j \le n$, let $\pi_i \in A$ be a prime defining C_i on A and $\delta_j \in A$ a prime defining D_j on A.

Lemma 5.2. *For $1 \le j \le n$, let $n_j = \ell \nu_{D_j}(\ell) + 1$. Then there exists a unit $u \in A$ such that $u \prod \pi_i$ is an ℓ-th power modulo $\delta_j^{n_j}$ for all $1 \le j \le n$. In particular $u \prod \pi_i \in F_{D_j}^{\ell}$ for all j.*

Proof. Let $\pi = \prod_1^r \pi_i$ and $\delta = \prod_1^m \delta_j^{n_j}$. Since, by the assumption (A7), $C_i \cap D_j = \varnothing$ for all i and $1 \le j \le m$, the ideals $A\pi$ and $A\delta$ are comaximal in A. In particular the image of π in $A/(\delta)$ is a unit. Let $P \in \mathscr{P} \setminus \left(\left(\bigcup_1^r C_i\right) \cup \left(\bigcup_1^m D_j\right)\right)$. Then π is a unit at P and the ideals (π), (δ), m_P are comaximal. By the Chinese remainder theorem, there exists $u_1 \in A$ be such that $u_1 = \pi \in A/(\delta)$, $u_1 = 1 \in A/(\pi)$ and $u_1 = \pi \in A/M_P$ for all $P \in \mathscr{P} \setminus \left(\left(\bigcup_1^r C_i\right) \cup \left(\bigcup_1^m D_j\right)\right)$. Since the image of π in $A/(\delta)$ is a unit, u_1 is a unit in A. Let $\pi' = u_1^{-1}\pi$.

Let $m+1 \le s \le n$ and a_s be the image of π' in $A/(\delta_s)$. We claim that $a_s = w_s b_s^{\ell}$ for some $w_s, b_s \in A/(\delta_s)$ with w_s a unit in $A/(\delta_s)$ and $w_s(P) = 1$ for all $P \in D_s \cap D_{s'}$, $s \neq s'$. Let M be a maximal ideal of $A/(\delta_s)$. Then $M = M_P/(\delta_s)$ for some $P \in D_s \cap \mathscr{P}$. Suppose $P \notin C_i$ for all i. Then π' is a unit at P and hence a_s is a unit at M. Suppose $P \in C_i$ for some i. Then $P \in C_i \cap D_s$. Thus, by the assumption (A8), there exists $j \neq i$ such that $P \in C_i \cap C_j$. Suppose $i < j$. Then, by the assumption (A11), $\delta_s = v_i \pi_i^{\ell-1} + v_j \pi_j$ for some units v_i and v_j at P. Hence

$$a_s \equiv u_1^{-1}\Bigl(\prod_{t\neq i,j} \pi_t\Bigr)\pi_i\pi_j = u_1^{-1}\Bigl(\prod_{t\neq i,j} \pi_t\Bigr)\pi_i\Bigl(-\frac{v_i}{v_j}\pi_i^{\ell-1}\Bigr) = u_1^{-1}\Bigl(\prod_{t\neq i,j} \pi_t\Bigr)\Bigl(-\frac{v_i}{v_j}\Bigr)\pi_i^{\ell} \quad \text{modulo } \delta_s.$$

Since π_t, $t \neq i, j$, is a unit at P (assumption (A1)), $a_s \equiv w_P \pi_j^\ell$ modulo δ_s, for some $w_P \in A/(\delta_s)$ a unit at P. Suppose $i > j$. Then $\delta_s = v_j \pi_j + v_i \pi_i^{\ell-1}$ for some units v_i and v_j at P. Hence, as above, $a_s \equiv w_P \pi_i^\ell$ modulo δ_s, for some $w_P \in A/(\delta_s)$ a unit at P. Hence at every maximal ideal of $A/(\delta_s)$, a_s is a product of a unit and an ℓ-th power. Since D_s is a regular curve on $\mathscr{X}$, $A/(\delta_s)$ is a semilocal regular ring and hence $A/(\delta_s)$ is an UFD. In particular $a_s = w_s b_s^\ell$ for some $w_s, b_s \in A/(\delta_s)$ with w_s a unit.

Let $P \in D_s \cap D_{s'}$ for some $s' \neq s$. Since $m+1 \leq s \leq n$, by the assumption (A3), $P \notin D_i$ for all $1 \leq i \leq m$. By the assumptions (A8) and (A9), $P \notin C_i$ for all i. Thus, by the choice of u_1, $\pi'(P) = 1$. In particular $a_s(P) = 1$ and hence $w_s(P) = b_s(P)^{-\ell}$. Let $\tilde{w}_s \in A/(\delta_s)$ be a unit such that $\tilde{w}_s(P) = b_s(P)$ for all $P \in D_s \cap D_{s'}$, $s \neq s'$. Since $a_s = w_s \tilde{w}_s^\ell (\tilde{w}_s^{-1} b_s)^\ell$ and $w_s \tilde{w}_s^\ell(P) = 1$, replacing w_s by $w_s \tilde{w}_s^\ell$ and b_s by $\tilde{w}_s^{-1} b_s$, we assume that $a_s = w_s b_s^\ell$ with $w_s(P) = 1$ for all $P \in D_s \cap D_{s'}$, $s \neq s'$. Since $m+1 \leq s \leq n$, by the assumption (A3), $(\delta_s, \delta) = A$. Hence, by Lemma 2.7, there exists $w \in A$ such that $w = 1 \in \kappa(P)$ for all $P \in \mathscr{P} \setminus \left(\bigcup_1^n D_i\right)$, $w = 1 \in A/(\delta)$ and $w = w_s \in A/(\delta_s)$. Since $w_s \in A/(\delta_s)$ is a unit, w is a unit in A.

Let $u = w^{-1} u_1^{-1}$. Since u_1 and w are units in A, $u \in A$ is a unit. We have $u \prod \pi_i = w^{-1} \pi' \equiv w_s^{-1} a_s = b_s^\ell$ modulo δ_s for $m+1 \leq s \leq n$ and $u \prod \pi_i = w^{-1} \pi' = w_\delta^{-\ell} \in A/(\delta)$. Since $\nu_{D_j}(\ell) = 0$ for $m+1 \leq j \leq n$ (assumption (A2)), $u \prod \pi_i$ is an ℓ-th power in $A/(\delta_j^{n_j})$ for $1 \leq j \leq n$. Since $n_j = \ell \nu_{D_j}(\delta_j) + 1$, $u \prod \pi_i \in F_{D_j}^\ell$ for all j (see [Epp 1973, Section 0.3]). □

Let $u \in A$ be a unit as in Lemma 5.2 and $\pi = u \prod_1^r \pi_i \in A$. Then $\operatorname{div}_{\mathscr{X}}(\pi) = \sum C_i + \sum_1^d t_s E_s$ for some irreducible curves E_s with $E_s \cap \mathscr{P} = \varnothing$. In particular $C_i \neq E_s$, $D_j \neq E_s$ for all i, j and s. Let $\mathscr{P}'$ be a finite set of points of $\mathscr{X}$ containing $\mathscr{P}$, $C_i \cap E_s$, $D_j \cap E_s$ for all i, j and s and at least one point from each E_s. Let A' be the semilocal ring at $\mathscr{P}'$. For $1 \leq i \leq n$, let $\delta_i' \in A'$ be a prime defining D_i on A'. Note that $\delta_i A \cap A' = \delta_i' A'$ for all i.

Lemma 5.3. *There exists $v \in A'$ such that*:

- *v is a unit and $F(\sqrt[\ell]{v})/F$ is unramified at all the points $P \in \mathscr{P}'$ except possible at the points P in $D_i \cap D_j$ for all $i \neq j$ with $\operatorname{char}(\kappa(P)) \neq \ell$.*
- *If $\operatorname{char}(\kappa(D_j)) \neq \ell$, then the extension $F(\sqrt[\ell]{v})/F$ is unramified at D_j with the residue field of $F(\sqrt[\ell]{v})$ at D_j equal to $\partial_{D_j}(\alpha)$.*
- *If $\operatorname{char}(\kappa(D_j)) = \ell$, then $F_{D_j}(\sqrt[\ell]{v}) \simeq F_{D_j}(\sqrt[\ell]{a})$. In particular $\alpha \otimes F_{D_j}(\sqrt[\ell]{v})$ is trivial.*

Proof. For $1 \leq i \leq n$, we show that there exists $u_i \in A'/(\delta_i') \subset \kappa(D_i)$ which patch to get an element in A' having the required properties.

Let $1 \leq i \leq m$. Then $\operatorname{char}(\kappa(D_i)) = \ell$. By the assumption (A10), $(a-1)/(\rho-1)^\ell \in A_P$ for all $P \in D_i$. In particular $(a-1)/(\rho-1)^\ell$ is regular at D_i and the image of $(a-1)/(\rho-1)^\ell$ in $\kappa(D_i)$ is in $A'/(\delta_i')$. Let u_i be the image of $(1-a)/(\rho-1)^\ell$ in $A'/(\delta_i')$.

Let $m+1 \leq i \leq n$. Then $\operatorname{char}(\kappa(D_i)) \neq \ell$. If $\operatorname{char}(\kappa(P)) = \ell$ for all $P \in D_i$, then let $w_i \in \kappa(D_i)$ be such that $\partial_{D_i}(\alpha) = [w_i]$.

Suppose there exists $P \in D_i$ with $\mathrm{char}(\kappa(P)) \neq \ell$. By [Saltman 2008, Proposition 7.10], there exists $w_i \in \kappa(D_i)^*$ such that:

- $\partial_{D_i}(\alpha) = \kappa(D_i)(\sqrt[\ell]{w_i})$.
- w_i is defined at all $P \in \mathscr{P}' \cap D_i$ with $\mathrm{char}(\kappa(P)) \neq \ell$.
- w_i is a unit at all $P \in (\mathscr{P}' \cap D_i) \setminus \big(\bigcup_{j \neq i} D_j\big)$ with $\mathrm{char}(\kappa(P)) \neq \ell$.
- $w_i(P) = w_j(P)$ for all $P \in D_i \cap D_j$, $i \neq j$ with P a chilly point or a cold point.

Let $P \in D_i \cap D_j$ for some $i \neq j$. Then, by assumptions (A3) and (A4), $\mathrm{char}(\kappa(P)) \neq \ell$. Suppose P is neither a chilly point nor a cold point. Since α is a symbol, there are no hot points [Saltman 2007, Theorem 2.5]. Hence P is a cool point. Since $\partial_{D_i}(\alpha) = \kappa(D_i)(\sqrt[\ell]{w_i})$, by the definition of a cool point, it follows that $w_i \in \kappa(D_i)_P^{*\ell}$. Write $w_i = w'^{\ell}_{iP}$ for some $w'_{iP} \in \kappa(D_i)^*_P$. Let $w'_i \in \kappa(D_i)^*$ be such that w'_i is close to w'_{iP} for all cool points $P \in D_i$ and w'_i is close to 1 for all other $P \in D_i \cap \mathscr{P}'$. Then, replacing w_i by $w_i w_i'^{-\ell}$, we assume that $w_i(P) = w_j(P)$ at all $P \in D_i \cap D_j$ with $\mathrm{char}(\kappa(P) \neq \ell$.

Let $P \in \mathscr{P}' \cap D_i$. Suppose $\mathrm{char}(\kappa(P)) = \ell$. Then, by the assumptions (A10), $[a)$ is unramified at P (see Proposition 2.3). Since $\alpha = [a, b)$, $\partial_{D_i}(\alpha) = [a(D_i))^{\nu_{D_i}(b)}$. In particular $\partial_{D_i}(\alpha) = \kappa(D_i)(\sqrt[\ell]{w_i})$ is unramified at P. Thus, by Lemma 2.6, we assume that $(1 - w_i)/(\rho - 1)^{\ell}$ is regular at all $P \in \mathscr{P}' \cap D_i \setminus \big(\bigcap_{j \neq i} D_j\big)$ with $\mathrm{char}(\kappa(P)) = \ell$. Since $\mathrm{char}(\kappa(D_i)) \neq \ell$, by assumptions (A3) and (A4), if $P \in D_i \cap D_j$ for some $j \neq i$, then $\mathrm{char}(\kappa(P)) \neq \ell$. Thus $(1 - w_i)/(\rho - 1)^{\ell} \in A'/(\delta_i)$. Let $u_i = (1 - w_i)/(\rho - 1)^{\ell} \in A'/(\delta'_i)$.

Let $P \in D_i \cap D_j$ for some $i \neq j$. Suppose $\mathrm{char}(\kappa(P)) = \ell$. Then, by the assumption (A3) and (A4), $1 \leq i, j \leq m$ and hence by the choice of u_i, we have $u_i(P) = u_j(P) \in \kappa(P)$. Suppose $\mathrm{char}(\kappa(P)) \neq \ell$. Then, $m + 1 \leq i, j \leq n$ and hence by the choice of w_i, we have $u_i(P) = u_j(P)$. Thus, by Lemma 2.7, there exists $u' \in A'$ such that $u' = u_i$ modulo (δ'_i) for all i. By the Chinese remainder theorem, we get $v' \in A'$ such that $v' = u' \in A'/\big(\prod \delta'_i\big)$ and $v' = 0 \in \kappa(P)$ for all $P \in \mathscr{P}'$ with $P \notin D_i$ for all i.

We now show that $v = 1 - (\rho - 1)^{\ell} v'$ has all the required properties.

Let $P \in \mathscr{P}'$. Suppose $\mathrm{char}(\kappa(P)) = \ell$. Then $\rho - 1 \in M_P$. Since $v' \in A'$, v is a unit at P and $F(\sqrt[\ell]{v})$ is unramified at P (Corollary 2.4). Suppose $\mathrm{char}(\kappa(P)) \neq \ell$. Suppose that $P \notin D_i$ for all i. Then, by the choice of v', $v' \in M_P$ and hence v is a unit at P and $F(\sqrt[\ell]{v})/F$ is unramified at P. Suppose that $P \in D_i$ for some i. Since $\mathrm{char}(\kappa(P)) \neq \ell$, $\mathrm{char}(\kappa(D_i)) \neq \ell$. Thus, by the choice of v', we have $v' = u' = u_i = (1 - w_i)/(\rho - 1)^{\ell} \in A'/(\delta'_i)$. Hence $v = w_i \in A'/(\delta'_i)$. Suppose $P \notin D_j$ for all $j \neq i$. Then, by the choice w_i is a unit at P and hence v is a unit at P. In particular $F(\sqrt[\ell]{v})/F$ is unramified at P. Thus v is a unit and $F(\sqrt[\ell]{v})/F$ is unramified at all $P \in \mathscr{P}'$ except possibly at $P \in D_i \cap D_j$ with $\mathrm{char}(\kappa(P)) \neq \ell$.

Suppose $\mathrm{char}(\kappa(D_i)) \neq \ell$. Then, by the choice of v, we have $v = 1 - (\rho - 1)^{\ell} v' = 1 - (\rho - 1)^{\ell} u_i = w_i \in A'/(\delta'_i) \subset \kappa(D_i)$. Since $w_i \neq 0$, v is a unit at δ_i and $F(\sqrt[\ell]{v})$ is unramified at D_i with residue field $\kappa(D_i)(\sqrt[\ell]{w_i}) = \partial_{D_i}(\alpha)$.

Suppose that $\mathrm{char}(\kappa(D_i)) = \ell$. Since $v = 1 - (\rho - 1)^{\ell} v'$ and $v' = u_i = w_i \in A'/(\delta'_i)$, $F(\sqrt[\ell]{v})$ is unramified at D_i with residue field equal to $\kappa(D_i)[X]/(X^{\ell} - X + w_i)$ (Proposition 2.3). Since w_i is the image of $(1-a)/(\rho-1)^\ell$ in $A'/(\delta_i')$, the residue field of $F(\sqrt[\ell]{a})$ at δ_i' is $\kappa(D_i)[X]/(X^\ell - X + w_i)$ (Proposition 2.3). Hence $F_{D_i}(\sqrt[\ell]{v}) \simeq F_{D_i}(\sqrt[\ell]{a})$. Since $\alpha = [a, b)$, $\alpha \otimes F_{\delta_i'}(\sqrt{v})$ is trivial. □

Remark 5.4. If ℓ is a unit in A', then the extension $F(\sqrt[\ell]{v})/F$ given in the above lemma is the lift of the residues of α which is in the sense of [Saltman 2008, Proposition 7.11].

Let $v \in A'$ be as in Lemma 5.3. Let $V_1, \ldots, V_q$ be the irreducible curves in $\mathscr{X}$ where $F(\sqrt[\ell]{v\pi})$ is ramified. Since $\pi \in F_{D_j}^\ell$ Lemma 5.2 and $F(\sqrt[\ell]{v})$ is unramified at D_j Lemma 5.3 for all j, $V_i \neq D_j$ for all i and j. Let $\mathscr{P}'' = \mathscr{P} \cup (\cup(D_i \cap E_s)) \cup (\cup(D_i \cap V_j))$. After reindexing E_s, we assume that there exists $d_1 \leq d$ such that $E_s \cap \mathscr{P}'' \neq \varnothing$ for $1 \leq s \leq d_1$ and $E_s \cap \mathscr{P}'' = \varnothing$ for $d_1 + 1 \leq s \leq d$.

Lemma 5.5. *There exists $h \in F^*$ which is a norm from the extension $F(\sqrt[\ell]{v\pi})$ such that*

$$\operatorname{div}_{\mathscr{X}}(h) = -\sum_{1}^{d_1} t_i E_i + \sum r_i E_i',$$

where $E_j' \cap \mathscr{P}'' = \varnothing$ for all j.

Proof. Let A'' be the regular semilocal ring at $\mathscr{P}''$. Let $L = F(\sqrt[\ell]{v\pi})$ and T be the integral closure of A'' in L.

Let $1 \leq s \leq d_1$ and $P \in \mathscr{P}'' \cap E_s$. Since $E_s \cap \mathscr{P} = \varnothing$, $P \in D_i \cap E_s$ for some i. Since v is a unit at all $P \in (\mathscr{P}' \setminus \mathscr{P})$ Lemma 5.3 and $D_i \cap E_s \subset \mathscr{P}'$, v is a unit at P and hence v is a unit at E_s.

Let e_s and f_s be the ramification index and the residue degree of L/F at E_s respectively. Suppose that $e_s = \ell$. Then there is a unique curve $\tilde{E}_s$ in T lying over E_s and let $t_s' = t_s$. Suppose that $e_s = 1$. Since $\operatorname{div}_{\mathscr{X}}(\pi) = \sum C_i + \sum_1^d t_s E_s$ and v is a unit at E_s, ℓ divides t_s. Suppose that $f_s = 1$. Let $t_s' = t_s/\ell$ and $\tilde{E}_s = t_s' \sum E_{s,i}$, where $E_{s,i}$ are the irreducible divisors in T which lie over E_s. Suppose that $f_s = \ell$. Then there is a unique curve $\tilde{E}_s$ in T lying over E_s and let $t_s' = t_s$.

Let $\tilde{E} = -\sum t_s' \tilde{E}_s$. Then the pushforward of $\tilde{E}$ from T to A'' is $-\sum_1^d t_s E_s$. We claim that $\tilde{E}$ is a principal divisor on T. Since T is normal it is enough to check this at every maximal ideal of T. Let M be a maximal ideal of T. Then $M \cap A'' = M_P$ for some $P \in \mathscr{P}''$. Suppose $P \notin E_s$ for all $1 \leq s \leq d_1$. Then $\tilde{E}$ is trivial at M. Suppose that $P \in E_s$ for some s with $1 \leq s \leq d_1$. Then, as we have seen above, $P \in D_i \cap E_s$ for some i. Since $D_i \cap C_j \in \mathscr{P}$ for all i and j and $\mathscr{P} \cap E_s = \varnothing$, $P \notin C_i$ for all i. Hence $\operatorname{div}_{A_P}(\pi) = \sum_{P \in E_i} t_i E_i$. Since v is a unit at P Lemma 5.3, $\operatorname{div}_{A_P}(v\pi) = \operatorname{div}_{A_P}(\pi)$ and hence $\tilde{E} = -\operatorname{div}(\sqrt[\ell]{v\pi})$ at M. In particular $\tilde{E}$ is principal at M. Hence $\tilde{E} = div_T(g)$ for some $g \in L$. Let $h = N_{L/F}(g)$. Since the pushforward of $\tilde{E}$ from T to A'' is $-\sum_1^d t_s E_s$, $\operatorname{div}_{A''}(h) = -\sum_1^{d_1} t_i E_i$ and hence h has the required properties. □

Lemma 5.6. *Let $h \in F^*$ be as in Lemma 5.5 with $\operatorname{div}_{\mathscr{X}}(h) = -\sum_1^{d_1} t_i E_i + \sum r_j E_j'$. Then α is unramified at E_j'. Further, if r_j is coprime to ℓ for some j, then the specialization of α at E_j' is unramified at every discrete valuation of $\kappa(E_j')$ which is centered on E_j'.*

Proof. Since $E_j' \cap \mathscr{P}'' = \varnothing$ and $D_i \cap \mathscr{P}'' \neq \varnothing$ for all i, $E_j' \neq D_i$ for all i. Hence, by the assumption (A2), α is unramified at E_j'.

Let P be a closed point of E'_j for some j with r_j coprime to ℓ. Let $L = F(\sqrt[\ell]{v\pi})$ and B_P be the integral closure of A_P in L. We first show that there exists an Azumaya algebra $\mathscr{A}_P$ over B_P such that $\alpha \otimes_F L$ is the class of $\mathscr{A}_P \otimes_{B_P} L$.

Suppose $P \notin D_i$ for all i. Then α is unramified at P (assumption (A2)). Hence there exists an Azumaya algebra $\mathscr{A}'_P$ over A_P such that α is the class of $\mathscr{A}'_P \otimes_{A_P} F$ (see Lemma 3.1). Let $\mathscr{A}_P = \mathscr{A}'_P \otimes_{A_P} B_Q$. Then $\alpha \otimes_F L$ is the class of $\mathscr{A}_P \otimes_{B_P} L$.

Suppose $P \in D_i$ for some i. Since $E'_j \cap \mathscr{P}'' = \varnothing$ Lemma 5.5, $P \notin \mathscr{P}''$. Since $\cup(V_{i'} \cap D_i) \subset \mathscr{P}''$, $P \notin \cup V_{i'}$ for all i' and hence L is unramified at P. Hence B_P is a regular semilocal domain. Let $Q \subset B_P$ be a height one prime ideal and $Q_0 = Q \cap A_P$. Then Q is a height one prime ideal of A_P. If α is unramified at Q_0, then $\alpha \otimes_F L$ is unramified at Q. Suppose that α is ramified at Q_0. Since $P \notin D_j$ for $j \neq i$, Q_0 is the prime ideal corresponding to D_i. Since $\pi \in F^{\ell}_{D_i}$ (Lemma 5.2), $F_{D_i}(\sqrt[\ell]{v\pi}) = F_{D_i}(\sqrt[\ell]{v})$. Suppose that $\mathrm{char}(\kappa(D_i)) \neq \ell$. Since L/F is unramified at D_i with residue field equal to $\partial_{D_i}(\alpha)$ (Lemma 5.3), $\alpha \otimes_F L$ is unramified at Q (see [Parimala et al. 2018, Lemma 4.1]). Suppose that $\mathrm{char}(\kappa(D_i)) = \ell$. Since $\alpha \otimes F_{D_i}(\sqrt[\ell]{v})$ is trivial (Lemma 5.3), $\alpha \otimes_F L$ is unramified at Q. Since B_P is a regular semilocal ring of dimension two, $\alpha \otimes F(\sqrt[\ell]{v\pi})$ is unramified at B_P (see Lemma 3.1). Hence there exists an Azumaya algebra $\mathscr{A}_P$ over B_P such that $\alpha \otimes_F L$ is the class of $\mathscr{A}_P \otimes_{B_P} L$.

Let $\beta \in H^2(\kappa(E'_j), \mathbb{Z}/\ell(1))$ be the specialization of α at E'_j. Suppose that r_j is coprime to ℓ. Let ν be a discrete valuation of $\kappa(E'_j)$ centered on a closed point P of E'_j. Let $Q_0 \subset A_P$ be the prime ideal defining E_j at P. Let $Q \subset B_P$ be a height one prime ideal of B_P lying over Q_0. Since E'_j is in the support of h, r_j is coprime to ℓ and h is a norm from L, the valuation on F given by Q_0 is either ramified or splits in L. Hence $A_P/Q_0 \subseteq B_P/Q \subset \kappa(E'_j)$. Thus β is the class of $\mathscr{A}_P \otimes_{B_P/Q} \kappa(E'_j)$. Since B_P/Q is integral over A_P/Q_0, the ring of integers at ν contains B_P/Q. In particular β is unramified at ν. □

Theorem 5.7. *Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies Assumptions 5.1. Then there exists $f \in K^*$ such that for every $x \in \mathscr{X}_{(1)}$, $\partial_x(\zeta - \alpha \cdot (f))$ is unramified at every discrete valuation of $\kappa(x)$ centered on the closure of $\{x\}$.*

Proof. We use the same notation as above and let $h \in F^*$ be as in Lemma 5.5. We claim that $f = h\pi$ has the required properties, i.e., $\partial_x(\zeta - \alpha \cdot (f))$ is unramified at every discrete valuation of $\kappa(x)$ for all $x \in \mathscr{X}_{(1)}$.

Let $x \in \mathscr{X}_{(1)}$ and D be the closure of $\{x\}$. Suppose $D = C_i$ for some i. Then h is a unit at C_i (Lemma 5.5), α is unramified at C_i (assumption (A2)) and π is a parameter at C_i, we have $\partial_{C_i}(\alpha \cdot (f))$ is the specialization of α at C_i (Lemma 4.2). Hence, by the assumption (A6), $\partial_{C_i}(\zeta - \alpha \cdot (f)) = 0$.

Suppose that $D = D_j$ for some j. By the assumption (A2), $\partial_{D_j}(\zeta) = 0$ and α is ramified at D_j. If $\mathrm{char}(\kappa(D_j)) = \ell$, then by the choice $\alpha \otimes F_{D_j}(\sqrt[\ell]{v}) = 0$ (Lemma 5.3). Suppose that $\mathrm{char}(\kappa(D_j)) \neq \ell$. Since $F_{D_j}(\sqrt[\ell]{v})$ is unramified with residue field equal to $\partial_{D_j}(\alpha)$ (Lemma 5.3), we have $\alpha \otimes F_{D_j}(\sqrt[\ell]{v}) = 0$ (Lemma 3.2). In particular, in either case, $\alpha \cdot (g) = 0 \in H^3(F_{D_j}(\sqrt{v}), \mathbb{Z}/\ell(2))$. Since $\pi \in F^{\ell}_{D_i}$ (Lemma 5.2), $L \otimes F_{D_j} = F_{D_j}(\sqrt[\ell]{v})$ and $\alpha \cdot (\pi) = 0 \in H^3(F_{D_j}, \mathbb{Z}/\ell(2))$. Thus $\alpha \cdot (h) = \mathrm{cor}_{L/F}(\alpha \cdot (g)) = 0 \in H^3(F_{D_j}, \mathbb{Z}/\ell(2))$ and $\partial_{D_j}(\alpha \cdot (h)) = 0$. Hence $\partial_{D_j}(\zeta - \alpha \cdot (f)) = 0$.

Suppose $D \neq C_i$ and D_j for all i and j. Then $\partial_D(\zeta) = 0$ and α is unramified at D. If $\nu_D(f)$ is a multiple of ℓ, then $\partial_D(\alpha \cdot (f)) = 0$. Suppose that $\nu_D(f)$ is coprime to ℓ. Since $\mathrm{div}_{\mathscr{X}}(\pi) =$

$\sum C_i + \sum_1^d t_i E_i$ (Lemma 5.2), $\mathrm{div}_{\mathscr{X}}(h) = -\sum_1^{d_1} t_s E_s + \sum r_i E_i'$ (Lemma 5.5) and $f = h\pi$, we have $\mathrm{div}_{\mathscr{X}}(f) = \sum C_i + \sum_{d_1+1}^d t_s E_s + \sum r_i E_i'$. Since $\nu_D(f)$ is coprime to ℓ and $D \neq C_i$ for all i, $D = E_s$ for some $d_1 + 1 \leq s \leq d$ or $D = E_i'$ for some i.

If $D = E_i'$, then by Lemma 5.6, the specialization $\bar{\alpha}$ of α at D is unramified at every discrete valuation of $\kappa(D)$ centered on D. Suppose $D = E_s$ for some $d_1+1 \leq s \leq d$. Then by the choice of d_1, $E_s \cap \mathscr{P}'' = \varnothing$ and hence $E_s \cap D_j = \varnothing$ for all j. Let $P \in E_s$. Then α is unramified at P (assumption (A2)) and hence $\bar{\alpha}$ is unramified at P. In particular $\bar{\alpha}$ is unramified at every discrete valuation of $\kappa(E_s)$ centered at P. Since α is unramified at E_s, $\partial_{E_s}(\alpha \cdot (f)) = \bar{\alpha}^{\nu_{E_s}(f)}$ (Lemma 4.2). Since $\bar{\alpha}$ is unramified at every discrete valuation of $\kappa(E_s)$ centered on E_s, $\partial_{E_s}(\alpha \cdot (f))$ is unramified at every discrete valuation of $\kappa(E_s)$ centered on E_s. Hence f has the required property. □

6. Divisibility of elements in H^3 by symbols in H^2

Let K be a global field or a local field and F the function field of a curve over K. If K is a number field or a local field, let R be the ring of integers in K. If K is a global field of positive characteristic, let R be the field of constants of K. Let $\mathscr{X}$ be a regular proper model of F over $\mathrm{Spec}(R)$. Let ℓ be a prime not equal to $\mathrm{char}(K)$. Suppose that K contains a primitive ℓ-th root of unity ρ. Then for any $P \in \mathscr{X}_{(0)}$, $\kappa(P)$ is a finite field. Hence if $\mathrm{char}(\kappa(P)) = \ell$, then $\kappa(P) = \kappa(P)^\ell$.

Thus we have a complex (see Proposition 4.1)

$$0 \to H^3(F, \mathbb{Z}/\ell(2)) \xrightarrow{\partial} \oplus_{x \in \mathscr{X}_{(1)}} H^2(\kappa(x), \mathbb{Z}/\ell(1)) \xrightarrow{\partial} \oplus_{P \in \mathscr{X}_{(0)}} H^1(\kappa(P), \mathbb{Z}/\ell).$$

Let $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$ and $\alpha = [a, b) \in H^2(F, \mathbb{Z}/\ell(1))$. In this section we prove (see Theorem 6.5) a certain local global principle for divisibility of ζ by α if $(\mathscr{X}, \zeta, \alpha)$ satisfies certain assumptions (see Assumptions 6.3).

For a sequence of blow-ups $\eta : \mathscr{Y} \to \mathscr{X}$ and for an irreducible curve C in $\mathscr{X}$, we denote the strict transform of C in $\mathscr{Y}$ by C itself.

We begin with the following:

Lemma 6.1. *Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumption* (A1) *of Assumptions 5.1. Let $\mathscr{Y} \to \mathscr{X}$ be a sequence of blow-ups centered on closed points of $\mathscr{X}$ which are not in $C_i \cap C_j$ for all $i \neq j$. Let $1 \leq I \leq 11$ with $I \neq 3, 5, 7$. If $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumption* (AI) *of Assumptions 5.1, then $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the assumption* (AI).

Proof. Let Q be a closed point of $\mathscr{X}$ which is not in $C_i \cap C_j$ for $i \neq j$ and $\eta : \mathscr{Y} \to \mathscr{X}$ a simple blow-up at Q. It is enough to prove the lemma for $(\mathscr{Y}, \zeta, \alpha)$.

Let E be the exceptional curve in $\mathscr{Y}$. Since $Q \notin C_i \cap C_j$ for $i \neq j$ and $(\mathscr{X}, \zeta, \alpha)$ satisfies (A1) of Assumptions 5.1, by Corollary 4.4, ζ is unramified at E.

Let $1 \leq I \leq 11$ with $I \neq 3, 5, 7$. Suppose further $I \neq 4, 10$. Since the exceptional curve E is not in $\mathrm{ram}_{\mathscr{Y}}(\zeta)$, if $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumption (AI) of Assumptions 5.1, then $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the same assumption.

Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumption (A4) of Assumptions 5.1. Suppose $\mathrm{char}(\kappa(Q)) = \ell$. Then $\mathrm{char}(\kappa(E)) = \ell$ and hence $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the assumption (A4) of Assumptions 5.1. Suppose $\mathrm{char}(\kappa(Q)) \neq \ell$. Then $\mathrm{char}(\kappa(P)) \neq \ell$ for all $P \in E$ and hence $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the assumption (A4) of Assumptions 5.1.

Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumption (A10) of Assumptions 5.1. If $\mathrm{char}(\kappa(Q)) \neq \ell$, then $\mathrm{char}(\kappa(P)) \neq \ell$ for all $P \in E$ and hence $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the assumption (A10) of Assumptions 5.1. Suppose that $\mathrm{char}(\kappa(Q)) = \ell$. If $Q \notin D_i$ for any i, then α is unramified at Q and hence α is unramified at E. In particular $E \notin \mathrm{ram}_{\mathscr{Y}}(\alpha)$ and hence $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the assumption (A10) of Assumptions 5.1. Suppose $Q \in D_i$ for some i. Since $(\mathscr{X}, \zeta, \alpha)$ satisfies (A10) of Assumptions 5.1, $(1-a)/(\rho-1)^\ell \in A_Q$. Let $P \in E$. Since $A_Q \subset A_P$, $(1-a)/(\rho-1)^\ell \in A_P$. Hence $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the assumption (A10) of Assumptions 5.1. □

Lemma 6.2. *Let $\mathscr{Y} \to \mathscr{X}$ be a sequence of blow-ups centered on closed points Q of $\mathscr{X}$ with* $\mathrm{char}(\kappa(Q)) \neq \ell$. *Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfy the assumptions* (A1) *and* (A2). *If $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumption* (A3) *or* (A7) *of Assumptions 5.1, then $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the same assumption.*

Proof. Let Q be a closed point of $\mathscr{X}$ with $\mathrm{char}(\kappa(Q)) \neq \ell$ and E the exceptional curve in $\mathscr{Y}$. Since $\mathrm{char}(\kappa(E)) \neq \ell$ and for any closed point P of E $\mathrm{char}(\kappa(P)) \neq \ell$, the lemma follows. □

Assumptions 6.3. Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies the following:

(B1) $\mathrm{ram}_{\mathscr{X}}(\zeta) = \{C_1, \ldots, C_r\}$, the C_i are irreducible regular curves with normal crossings.

(B2) $\mathrm{ram}_{\mathscr{X}}(\alpha) = \{D_1, \ldots, D_n\}$ with the D_j irreducible curves such that $C_i \neq D_j$ for all i and j.

(B3) If $D_s \cap C_i \cap C_j \neq \varnothing$ for some s, $i \neq j$, then $\mathrm{char}(\kappa(D_s)) \neq \ell$.

(B4) If $P \in D_j$ for some $1 \le j \le n$ with $\mathrm{char}(\kappa(P)) = \ell$, then $(1-a)/(\rho-1)^\ell \in A_P$.

(B5) $\partial_{C_i}(\zeta)$ is the specialization of α at C_i for all i.

(B6) If $\ell = 2$, then $\zeta \otimes F \otimes K_\nu$ is trivial for all real places ν of K.

(B7) If $\ell = 2$, then a is a sum of two squares in F.

(B8) For $1 \le i < j \le r$, through any point of $C_i \cap C_j$ there passes at most one D_s and if $P \in D_s \cap C_i \cap C_j$, then D_s is defined by $u\pi_i^{\ell-1} + v\pi_j$ at P for some units u and v at P and π_i, π_j primes defining C_i and C_j at P.

Lemma 6.4. *Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies Assumptions 6.3. Let $\mathscr{Y} \to \mathscr{X}$ be a sequence of blow-ups centered on closed points of $\mathscr{X}$ which are not in $C_i \cap C_j$ for $i \neq j$. Then $(\mathscr{Y}, \zeta, \alpha)$ also satisfies Assumptions 6.3.*

Proof. Let Q be a closed point of $\mathscr{X}$ which is not in $C_i \cap C_j$ for $i \neq j$ and $\eta : \mathscr{Y} \to \mathscr{X}$ a simple blow-up at Q. It is enough to show that $(\mathscr{Y}, \zeta, \alpha)$ satisfies Assumptions 6.3.

Since (B1), (B4), (B5) and (B8) are restatements of (A1), (A10), (A6) and (A9), (A11), by Lemma 6.1, $(\mathscr{Y}, \zeta, \alpha)$ satisfies (B1), (B4), (B5) and (B8). Let E be the exceptional curve in $\mathscr{Y}$. Since $Q \notin C_i \cap C_j$

for $i \neq j$, by Corollary 4.4, ζ is unramified at E. Hence $\mathrm{ram}_{\mathscr{Y}}(\zeta) = \{C_1, \ldots, C_r\}$. Since $\mathrm{ram}_{\mathscr{Y}}(\alpha) \subset \{D_1, \ldots, D_n, E\}$, $(\mathscr{Y}, \zeta, \alpha)$ satisfies (B2). Since $E \cap C_i \cap C_j = \varnothing$ for all $i \neq j$, $(\mathscr{Y}, \zeta, \alpha)$ satisfies (B3).

Since (B6) and (B7) do not depend on the model, $(\mathscr{Y}, \zeta, \alpha)$ satisfies all Assumptions 6.3. □

Theorem 6.5. *Let K, F and $\mathscr{X}$ be as above. Let $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$ and $\alpha = [a, b) \in H^2(F, \mathbb{Z}/\ell(1))$. Suppose that F contains a primitive ℓ-th root of unity. If $(\mathscr{X}, \zeta, \alpha)$ satisfies Assumptions 6.3, then there exists $f \in F^*$ such that $\zeta = \alpha \cdot (f)$.*

Proof. Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies Assumptions 6.3. First we show that there exists a sequence of blow-ups $\eta : \mathscr{Y} \to \mathscr{X}$ such that $(\mathscr{Y}, \zeta, \alpha)$ satisfies Assumptions 5.1.

Let $P \in \mathscr{X}_{(0)}$. Suppose $P \in D_s$ for some s and D_s is not regular at P or $P \in D_s \cap D_t$ for some $s \neq t$. Then, by the assumption (B8), $P \notin C_i \cap C_j$ for all $i \neq j$. Thus, there exists a sequence of blow-ups $\mathscr{X}' \to \mathscr{X}$ at closed points which are not in $C_i \cap C_j$ for all $i \neq j$ such that $\mathrm{ram}_{\mathscr{X}'}(\alpha)$ is a union of regular with normal crossings. By Lemma 6.4, $\mathscr{X}'$ also satisfies Assumptions 6.3. Thus, replacing $\mathscr{X}$ by $\mathscr{X}'$ we assume that $(\mathscr{X}, \zeta, \alpha)$ satisfies Assumptions 6.3, D_i's are regular with normal crossings and D_s, C_i have normal crossings at all $P \notin C_j$ for all $j \neq i$. In particular $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumptions (A1) and (A2) of Assumptions 5.1.

Suppose there exists $i \neq j$ and $P \in D_i \cap D_j$ such that $\mathrm{char}(\kappa(D_i)) \neq \ell$, $\mathrm{char}(\kappa(D)_j) \neq \ell$ and $\mathrm{char}(\kappa(P)) = \ell$. Let $\mathscr{X}' \to \mathscr{X}$ be the blow-up at P and E the exceptional curve in $\mathscr{X}'$. Then $\mathrm{char}(\kappa(E)) = \mathrm{char}(\kappa(P)) = \ell$ and $D_i \cap D_j \cap E = \varnothing$ in $\mathscr{X}'$. By the assumption (B8), $P \notin C_{i'} \cap C_{j'}$ for all $i' \neq j'$ and hence $\mathscr{X}'$ satisfies Assumptions 6.3 (see Lemma 6.4) and assumptions (A1) and (A2) of Assumptions 5.1 (see Lemma 6.1). Thus replacing $\mathscr{X}$ by a sequence of blow-ups at closed points in $D_i \cap D_j$ for $i \neq j$, we assume that $\mathscr{X}$ satisfies Assumptions 6.3 and assumptions (A1), (A2) and (A4) of Assumptions 5.1.

Since $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumptions (B4), (B5) and (B8) of Assumptions 6.3, $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumptions (A6), (A9), (A10) and (A11) of Assumptions 5.1.

Suppose $P \in C_i \cap D_s$ for some i, s and $P \notin C_j$ for all $j \neq i$. Since ζ is unramified at P except at C_i, $\partial_{C_i}(\zeta)$ is zero over $\kappa(C_i)_P$ (Corollary 4.4). By the assumption (B5), we have $\partial_{C_i}(\zeta) = \bar{\alpha}$. Since $P \notin C_j$ for all $j \neq i$, C_i and D_s have normal crossings at P and $P \notin D_{s'}$ for all $s' \neq s$. Thus, by Lemma 3.3, $\alpha \otimes F_P = 0$. Let $\mathscr{X}' \to \mathscr{X}$ be the blow-up at P and E the exceptional curve in $\mathscr{X}'$. Since $\alpha \otimes F_P = 0$ and $F_P \subset F_E$, α is unramified at E and hence $\mathrm{ram}_{\mathscr{X}'}(\alpha) = \{D_1, \ldots, D_n\}$. Since $\zeta \otimes F_P = 0$, $\mathrm{ram}_{\mathscr{X}'}(\zeta) = \{C_1, \ldots, C_r\}$. Note that $C_i \cap D_s = \varnothing$ in $\mathscr{X}'$. Hence $(\mathscr{X}', \zeta, \alpha)$ satisfies assumption (A8) of Assumptions 5.1. Since $P \notin C_j$ for all $j \neq i$, $(\mathscr{X}', \zeta, \alpha)$ satisfies Assumptions 6.3, 6.4, 5.1, except possibly (A3), (A5) and (A7), and 6.1. Thus, replacing $\mathscr{X}$ by $\mathscr{X}'$ we assume that $(\mathscr{X}, \zeta, \alpha)$ satisfies Assumptions 6.3 and 5.1 except possibly (A3), (A5) and (A7).

Let $\mathrm{ram}_{\mathscr{X}}(\alpha) = \{D_1, \ldots, D_m, D_{m+1}, \ldots, D_n\}$ with $\mathrm{char}(\kappa(D_s)) = \ell$ for $1 \leq s \leq m$ and $\mathrm{char}(\kappa(D_t)) \neq \ell$ for $m+1 \leq t \leq n$. Suppose $D_s \cap D_t \neq \varnothing$ for some $1 \leq s \leq m$ and $m+1 \leq t \leq n$. Let $P \in D_s \cap D_t$. Then $\mathrm{char}(\kappa(P)) = \ell$ and hence $(a-1)/(\rho-1)^{\ell} \in A_P$ (assumption (B4)). In particular $[a)$ is unramified at P (see Proposition 2.3). Since α is ramified at D_t, $\nu_{D_t}(b)$ is coprime to ℓ and hence there exists i such that $\nu_{D_s}(b) + i\nu_{D_t}(b)$ is divisible by ℓ. Let $\mathscr{X}_1 \to \mathscr{X}$ be the blow-up at P and E_1 the exceptional curve in $\mathscr{X}_1$.

We have $\nu_{E_1}(b)=\nu_{D_s}(b)+\nu_{D_t}(b)$. Let Q_1 be the point in $E_1\cap D_t$ and $\mathscr{X}_2\to\mathscr{X}_1$ be the blow-up at Q_1. Let E_2 be the exceptional curve in $\mathscr{X}_2$. We have $\nu_{E_2}(b)=\nu_{E_1}(b)+\nu_{D_t}(b)=\nu_{D_s}(b)+2\nu_{D_t}(b)$. Continue this process i times and get $\mathscr{X}_i\to\mathscr{X}_{i-1}$ and E_i the exceptional curve in $\mathscr{X}_i$. Then $\nu_{E_i}(b)=\nu_{D_t}(b)+i\nu_D(b)$ is divisible by ℓ. Since $[a)$ is unramified at P, α is unramified at E_i. Since $\mathrm{char}(\kappa(E_j))=\ell$ for all j, $E_{i-1}\cap D_t=\varnothing$ in $\mathscr{X}_i$ and E_i is in not in $\mathrm{ram}_{\mathscr{X}_i}(\alpha)$. Since $P\notin C_i\cap C_j$ for all $i\neq j$ (assumption (B4)), $\mathscr{X}_i$ satisfies Assumptions 6.3 (see Lemma 6.4). Thus, replacing $\mathscr{X}$ by $\mathscr{X}_i$, we assume that $D_s\cap D_t=\varnothing$ for all $1\le s\le m$ and $m+1\le t\le n$ and $\mathscr{X}$ satisfies Assumptions 6.3. Thus $\mathscr{X}$ satisfies all the assumptions of Assumptions 5.1 expect possibly (A5) and (A7) (see Lemma 6.1).

Suppose $C_i\cap D_t\neq\varnothing$ for some i and t. Since $(\mathscr{X},\zeta,\alpha)$ satisfies (A8) and (A9) of Assumptions 5.1, there exists $j\neq i$ such that $C_i\cap C_j\cap D_t\neq\varnothing$. Since $(\mathscr{X},\zeta,\alpha)$ satisfies the assumption (B3) of Assumptions 6.3, $\mathrm{char}(\kappa(D_t))\neq\ell$. Hence $C_i\cap D_t=\varnothing$ for all i and $1\le t\le m$. In particular $(\mathscr{X},\zeta,\alpha)$ satisfies (A7) of Assumptions 5.1 and hence $(\mathscr{X},\zeta,\alpha)$ satisfies all the assumptions of Assumptions 5.1 except possibly (A5).

Let $P\in\mathscr{X}_{(0)}$. Suppose that P is a chilly point for α. Then $P\in D_s\cap D_t$ for some $D_s, D_t\in\mathrm{ram}_{\mathscr{X}}(\alpha)$ with $D_s\neq D_t$ with $\mathrm{char}(\kappa(P))\neq\ell$. In particular $P\notin C_i\cap C_j$ for all $i\neq j$ (assumption (B8)). Since there is a sequence of blow-ups $\mathscr{Y}\to\mathscr{X}$ centered on chilly points of α on $\mathscr{X}$ with no chilly loops on $\mathscr{Y}$ (Proposition 3.4), by Lemmas 6.1 and 6.2, replacing $\mathscr{X}$ by $\mathscr{Y}$ we assume that $(\mathscr{X},\zeta,\alpha)$ satisfies Assumptions 6.3 and 5.1.

Thus, by Theorem 5.7, there exists $f\in F^*$ such that for every $x\in\mathscr{X}_{(1)}$, $\partial_x(\zeta-\alpha\cdot(f))$ is unramified at every discrete valuation of $\kappa(x)$ centered at a closed point of the closure $\overline{\{x\}}$ of $\{x\}$. Since $\kappa(x)$ is a global field or a local field, every discrete valuation of $\kappa(x)$ is centered on a closed point of $\overline{\{x\}}$. Hence $\partial_x(\zeta-\alpha\cdot(f))$ is unramified at every discrete valuation of $\kappa(x)$.

For place ν of K, let K_ν be the completion of K at ν and $F_\nu=F\otimes_K K_\nu$.

Let ν be a real place of K. Since a is a sum of two squares in F, a is a norm from the extension $F_\nu(\sqrt{-1})$. Let $\tilde{a}\in F_\nu(\sqrt{-1})$ with norm equal to a. Since $H^2(F_\nu(\sqrt{-1}),\mathbb{Z}/2(1))=0$ [Serre 1997, page 80] and $\mathrm{cor}_{F_\nu(\sqrt{-1})/F_\nu}[\tilde{a},b)=[a,b)\otimes F_\nu$, $\alpha=[a,b)=0\in H^2(F_\nu,\mathbb{Z}/2(1))$. Since, by assumption $\zeta\otimes F_\nu=0$,

$$\zeta-\alpha\cdot(f)=0\in H^3(F_\nu,\mathbb{Z}/2(2)).$$

Let $x\in\mathscr{X}_{(1)}$. Since $\zeta-\alpha\cdot(f)=0\in H^3(F_\nu,\mathbb{Z}/2(2))$ for all real places ν of K, it follows that $\partial_x(\zeta-\alpha\cdot(f))=0\in H^2(\kappa(x)_{\nu'},\mathbb{Z}/2(1))$ for all real places ν' of $\kappa(x)$. Since $\partial_x(\zeta-\alpha\cdot(f))$ is unramified at every discrete valuation of $\kappa(x)$, $\partial_x(\zeta-\alpha\cdot(f))=0$ [Cassels and Fröhlich 1967, page 130]. Hence $\zeta-\alpha\cdot(f)$ is unramified on $\mathscr{X}$.

Let ν be a finite place of K. Since $\zeta-\alpha\cdot(f)$ is unramified on $\mathscr{X}$,

$$(\zeta-\alpha\cdot(f))\otimes_F F_\nu)=0\in H^3(F_\nu,\mathbb{Z}/\ell(2))$$

[Kato 1986, Corollary page 145]. Hence $\zeta=\alpha\cdot(f)$ [loc. cit., Theorem 0.8(2)]. □

7. Main theorem

In this section we prove our main result Theorem 7.7. Let K be a global field or a local field and F the function field of a curve over K. Let ℓ be a prime not equal to char(K). Suppose that F contains a primitive ℓ-th root of unity ρ. If K is a number field or a local field, let R be the ring of integers in K. If K is a global field of positive characteristic, let R be the field of constants of K.

To prove our main result Theorem 7.7, we first show Proposition 7.6 that given $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$ with $\zeta \otimes_F (F \otimes_K K_\nu) = 0$ for all real places ν of K, there exist $\alpha = [a, b) \in H^2(F, \mathbb{Z}/\ell(1))$ and a regular proper model $\mathscr{X}$ of F over R such that the triple $(\mathscr{X}, \zeta, \alpha)$ satisfies Assumptions 6.3.

Let $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$ be such that $\zeta \otimes_F (F \otimes_K K_\nu) = 0$ for all real places ν of K. Choose a regular proper model $\mathscr{X}$ of F over R [Saltman 1997, page 38] such that:

- $\mathrm{ram}_{\mathscr{X}}(\zeta) \cup \mathrm{supp}_{\mathscr{X}}(\ell) \subset \{C_1, \dots, C_{r_1}, \dots, C_r\}$, where the C_i are irreducible regular curves with normal crossings.
- For $i \neq j$, C_i and C_j intersect at most at one closed point.
- $C_i \cap C_j = \varnothing$ if $i, j \leq r_1$ or $i, j > r_1$.

For $x \in \mathscr{X}_{(1)}$, let $\beta_x = \partial_x(\zeta)$. Let $\mathscr{P}_0 \subset \cup C_i$ be a finite set of closed points of $\mathscr{X}$ containing $C_i \cap C_j$ for $1 \leq i < j \leq r$, and at least one closed point from each C_i. Let A be the regular semilocal ring at the points of $\mathscr{P}_0$. Let $Q \in C_i$ be a closed point. Since C_i is regular on $\mathscr{X}$, Q gives a discrete valuation ν_Q^i on $\kappa(C_i)$.

Lemma 7.1. *There exists $a \in A$ such that*:

- *$(a-1)/(\rho-1)^\ell \in A$ and $[a)$ is unramified on A.*
- *For $1 \leq i \leq r_1$ and $P \in C_i \cap \mathscr{P}_0$, $\partial_P(\beta_{x_i}) = [a(P))$.*
- *For $r_1 + 1 \leq i \leq r$ and $P \in C_i \cap \mathscr{P}_0$, $\partial_P(\beta_{x_i}) = [a(P))^{-1}$.*
- *If $P \in \mathscr{P}_0$ and $P \notin C_i \cap C_j$ for all $i \neq j$, then $[a(P))$ is the trivial extension.*
- *If $\ell = 2$, then a is a sum of two squares in A.*

Proof. Let $P \in \mathscr{P}_0$. Suppose $P \in C_i \cap C_j$ for some $i < j$. Then, by the choice of $\mathscr{X}$, the pair (i, j) is uniquely determined by P. Let $u_P \in \kappa(P)$ be such that $\partial_P(\partial_{x_i}(\zeta)) = [u_P)$. If $P \notin C_i \cap C_j$ for all $i \neq j$, let $u_P \in \kappa(P)$ with $[u_P)$ the trivial extension.

Then, by Lemma 2.5, there exists $a \in A$ such that for every $P \in \mathscr{P}_0$, the cyclic extension $[a)$ over F is unramified on A with the residue field $[a(P))$ of $[a)$ at P is $[u_P)$. Further if $\ell = 2$, choose a to be a sum of two squares in A (Lemma 2.5). From the proof of Lemma 2.5, we have $(a-1)/(\rho-1)^\ell \in A$.

Let $P \in \mathscr{P}_0$. Suppose that $P \in C_i$ for some i and $P \notin C_j$ for all $i \neq j$. Then $\partial_P(\partial_{x_i}(\zeta)) = 1$ (Corollary 4.3) and by the choice of a and u_P, we have $[a(P)) = [u_P) = 1$. Suppose that $P \in C_i \cap C_j$ for some $i \neq j$. Suppose $i < j$. Then by the choice of a and u_P we have $\partial_P(\partial_{x_i}(\zeta)) = [u_P) = [a(P))$. Suppose $i > j$. Then by the choice of a and u_P we have $\partial_P(\partial_{x_j}(\zeta)) = [u_P) = [a(P))$. Since $\partial_P(\partial_{x_i}(\zeta)) = \partial_P(\partial_{x_j}(\zeta))^{-1}$ (Corollary 4.3), we have $\partial_P(\partial_{x_i}(\zeta)) = [a(P))^{-1}$. Thus a has the required properties. □

Let $a \in A$ be as in Lemma 7.1. Let $L_1, \dots, L_d$ be the irreducible curves in $\mathscr{X}$ which are in the ramification of $[a)$ or $\nu_{L_i}((a-1)/(\rho-1)^\ell) < 0$.

Lemma 7.2. *Then $L_i \cap \mathscr{P}_0 = \varnothing$ for all i. In particular $L_i \neq C_j$ for all i, j and* $\operatorname{char}(\kappa(L_i)) \neq \ell$.

Proof. By the choice of a, $[a)$ is unramified on A and $(a-1)/(\rho-1)^\ell \in A$ (Lemma 7.1). Hence $\mathscr{P}_0 \cap L_i = \varnothing$ for all i. Since $\mathscr{P}_0$ contains at least one point from each C_j, $L_i \neq C_j$ for all i and j. Since $\operatorname{supp}_{\mathscr{X}}(\ell) \subset \{C_1, \dots, C_r\}$, $\operatorname{char}(\kappa(L_i)) \neq \ell$ for all i. □

Let $\mathscr{P}_1 \subset \bigcup_j L_j$ be a finite set of closed points of $\mathscr{X}$ consisting of $L_i \cap L_j$ for $i \neq j$, $L_i \cap C_j$, one point from each L_i. Since $L_i \cap \mathscr{P}_0 = \varnothing$ for all i (Lemma 7.2), $\mathscr{P}_0 \cap \mathscr{P}_1 = \varnothing$.

Let $\mathscr{P} = \mathscr{P}_0 \cup \mathscr{P}_1$ and B be the semilocal ring at $\mathscr{P}$ on $\mathscr{X}$. For each i and j, let $\pi_i \in B$ be a prime defining C_i and $\delta_j \in B$ a prime defining L_j.

Lemma 7.3. *For each $P \in C_i \cap \mathscr{P}_1$, let n_i^P be a positive integer. Then for each i, $1 \le i \le r$, there exists $b_i \in B/(\pi_i) \subset \kappa(C_i)$ such that*:

- $\partial_{C_i}(\zeta) = [a(C_i), b_i)$.
- $\nu_P^i(b_i) = 1$ *for all $P \in C_i \cap \mathscr{P}_0$, $1 \le i \le r_1$.*
- $\nu_P^i(b_i) = \ell - 1$ *for all $P \in C_i \cap \mathscr{P}_0$, $r_1 + 1 \le i \le r$.*
- $\nu_P^i(b_i - 1) \ge n_i^P$ *for all $P \in \mathscr{P}_1 \cap C_i$ for all i.*

Proof. Let $1 \le i \le r$. Let $\beta_{x_i} = \partial_{x_i}(\zeta) \in H^2(\kappa(C_i), \mathbb{Z}/\ell(1))$ and $a_i = a(C_i)$.

Suppose $1 \le i \le r_1$. By Lemma 7.1, $\partial_P(\beta_{x_i}) = [a_i(P))$ for all $P \in C_i \cap \mathscr{P}_0$. If $P \notin \mathscr{P}_0$, then $\partial_P(\beta_{x_i}) = 0$ for all i (Corollary 4.3). By the assumption, $\beta_{x_i} \otimes \kappa(C_i)_\nu = 0$ for all real places ν of $\kappa(C_i)$. Thus, by Proposition 3.5, there exists $b_i \in \kappa(C_i)^*$ such that $\beta_{x_i} = [a_i, b_i)$, with $\nu_P^i(b_i) = 1$ for all $P \in C_i \cap \mathscr{P}_0$ and $\nu_P^i(b_i - 1) \ge n_i^P$ for all $P \in C_i \cap \mathscr{P}_1$. In particular b_i is regular at all $P \in C_i \cap \mathscr{P}$ and hence $b_i \in B/(\pi_i)$.

Suppose $r_1 + 1 \le i \le r$. Let $P \in C_i \cap \mathscr{P}_0$. Since $\partial_P(\beta_{x_i}) = [a(P))^{-1}$ for all $P \in C_i \cap \mathscr{P}_0$ (Lemma 7.1), $\partial_P(\beta_{x_i}^{-1}) = [a(P))$. Thus, as above, by Proposition 3.5, there exists $c_i \in B/(\pi_i)$ such that $\beta_{x_i}^{-1} = [a_i, c_i)$, with $\nu_P^i(c_i) = 1$ for all $P \in C_i \cap \mathscr{P}_0$ and $\nu_P^i(c_i - 1) \ge n_i^P$ for all $P \in C_i \cap \mathscr{P}_1$. Let $b_i = c_i^{\ell-1} \in B/(\pi_i)$. Then $\beta_{x_i} = [a_i, b_i)$. Let $P \in C_i \cap \mathscr{P}_1$. Since $c_i \in B/(\pi_i)$ and $\nu_P^i(c_i - 1) \ge n_i^P$, it follows tat $\nu_P^i(b_i - 1) \ge n_i^P$. Thus b_i has the required properties. □

Let $\delta = \prod \delta_j \in B$. For $1 \le i \le r$, let $\bar{\delta}(i) \in B/(\pi_i)$ be the image of δ. Let d be an integer greater than $\nu_P^i(\bar{\delta}(i)) + 1$ for all i and $P \in C_i \cap \mathscr{P}$.

Lemma 7.4. *Let $b_i \in B/(\pi_i)$ be as in Lemma 7.3 for $n_i^P = d$ for all $P \in C_i \cap \mathscr{P}$. Then there exists $b \in B$ such that*:

- $b = b_i$ *modulo π_i for all i.*
- $b = 1$ *modulo δ_j for all j.*
- b *is a unit at all $P \in \mathscr{P}_1$.*

Proof. For $1 \leq i \leq r$, let $I_i = (\pi_i) \subset B$ and $I_{r+1} = (\delta) \subset B$. Clearly the $\gcd(\pi_i, \pi_j) = 1$ and $\gcd(\pi_i, \delta) = 1$ for all $1 \leq i < j \leq r$. For $1 \leq i < j \leq r$, $I_{ij} = I_i + I_j$ is either maximal ideal or equal to B. For $1 \leq i \leq r$, we have $I_{i(r+1)} = (\pi_i, \delta)$. Since $L_s \cap \mathscr{P}_0 = \varnothing$ for all s, $(\delta_s, \pi_i, \pi_j) = A$ for all $1 \leq i < j \leq r$ and for all s. Thus the ideals I_{ij}, $1 \leq i < j \leq r+1$, are coprime. Let $b_{r+1} = 1 \in B/(I_{r+1})$.

Let $1 \leq i < j \leq r$. Suppose $(\pi_i, \pi_j) \neq B$. Then (π_i, π_j) is a maximal ideal of B corresponding to a point $P \in C_i \cap C_j$. Since $P \in \mathscr{P}_0$, by the choice of b_i and b_j (see Lemma 7.4), we have $\nu_P^i(b_i) = 1$, $\nu_P^i(b_j) = \ell - 1$ and hence $b_i = b_j = 0 \in B/(\pi_i, \pi_j) = B/I_{ij}$.

Suppose $I_{i(r+1)} \neq B$ for some $1 \leq i \leq r$. Then we claim that $b_i = 1 \in B/I_{i(r+1)}$. For each $P \in L_j \cap C_i$, let M_P be the maximal ideal of B at P. Since $\mathscr{X}$ is regular and C_i is regular on $\mathscr{X}$, we have $M_P = (\pi_i, \pi_{i,P})$ for some $\pi_{i,P} \in M_P$ and the image of $\pi_{i,P}$ in $B/(\pi_i)$ is a parameter at the discrete valuation ν_P^i. Since $d > \nu_P^i(\bar{\delta}(i))$, we have $\left(\pi_i, \prod \pi_{i,P}^d\right) \subset (\pi_i, \delta) = I_{i(r+1)}$. Since $B/\left(\pi_i, \prod \pi_{i,P}^d\right) \simeq \prod_P B/(\pi_i, \pi_{i,P}^d)$ and $\nu_P^i(b_i - 1) \geq d$, we have $b_i = 1 \in B/\left(\pi_i, \prod \pi_{i,P}^d\right)$. Since $B/I_i + I_{r+1}$ is a quotient of $B/I_i + \left(\prod_P \pi_{i,P}\right)^d$, it follows that $b_i = b_{r+1} = 1 \in B/I_i + I_{r+1} = B/I_{i(r+1)}$.

Thus, by Lemma 2.7, there exists $b \in B$ such that $b = b_i \in B/(\pi_i)$ for all i and $b = 1 \in B/I_{r+1}$. Since $I_{r+1} = (\delta) \subset (\delta_j)$ and $b = 1 \in B/(\delta)$, we have $b = 1 \in B/(\delta_j)$ for all j. Let $P \in \mathscr{P}_1$. Then $P \in L_j$ for some j. Since $b = 1 \in B/(\delta_j)$, b is a unit at P. Thus b has all the required properties. □

Lemma 7.5. *Let a be as in Lemma 7.1 and b as in Lemma 7.4 and $\alpha = [a, b)$. Then α is unramified at all C_i, L_j and at all $Q \in \mathscr{P}_1$. Further $\partial_{C_i}(\zeta)$ is the specialization of α at C_i for all $1 \leq i \leq r$.*

Proof. Since $[a)$ is unramified at C_i (Lemma 7.1) and b is a unit at C_i for all i (Lemma 7.4), α is unramified at C_i and the specialization of α at C_i is $[a(C_i), b_i) = \partial_{C_i}(\zeta)$ (Lemmas 7.3 and 7.4). Since $\mathrm{char}(\kappa(L_j)) \neq \ell$ (Lemma 7.2) and $b = 1$ modulo δ_j (Lemma 7.4), b is an ℓ-th power in F_{L_j} and hence $\alpha \otimes F_{L_j} = 0$. In particular α is unramified at L_j.

Let $Q \in \mathscr{P}_1$. Then b is a unit at Q (Lemma 7.4). Let x be a dimension one point of $\mathrm{Spec}(B_Q)$. Then b is a unit at x. If $[a)$ is unramified at x, then α is unramified at x. Suppose $[a)$ is ramified at x. Then, by the choice of the L_j, x is the generic point of L_j for some j and hence α is unramified at x. Thus α is unramified at Q (see Lemma 3.1). □

Proposition 7.6. *The triple $(\mathscr{X}, \zeta, [a, b))$ satisfies Assumptions 6.3.*

Proof. By the choice of $\mathscr{X}$, (B1) of Assumptions 6.3 is satisfied. Let $\mathrm{ram}_{\mathscr{X}}(\alpha) = \{D_1, \ldots, D_n\}$. Since α is unramified at all C_i (Lemma 7.5), (B2) of Assumptions 6.3 is satisfied. Since $\mathrm{supp}_{\mathscr{X}}(\ell) \subset \{C_1, \ldots, C_r\}$ and $D_i \neq C_j$ for all i and j, $\mathrm{char}(\kappa(D_i)) \neq \ell$ for all i and hence (B3) of Assumptions 6.3 is satisfied.

Let $P \in D_j$ some j with $\mathrm{char}(\kappa(P)) = \ell$. Since $\mathrm{supp}_{\mathscr{X}}(\ell) \subset \{C_1, \ldots, C_r\}$, $P \in C_i$ for some i. Since α is unramified at all $Q \in \mathscr{P}_1$ (Lemma 7.5), $P \notin \mathscr{P}_1$. Since $C_i \cap L_s \subset \mathscr{P}_1$ for all s, $P \notin L_s$ for all s and hence $(a-1)/(\rho-1)^\ell \in A_P$. Thus (B4) of Assumptions 6.3 is satisfied.

Since $\partial_{C_i}(\zeta)$ is the specialization of α at C_i (Lemma 7.5), (B5) of Assumptions 6.3 is satisfied.

By the assumption on ζ, (B6) of Assumptions 6.3 is satisfied. If $\ell = 2$, then, by the choice of a (Lemma 7.1), (B7) of Assumptions 6.3 is satisfied.

Let $P \in C_i \cap C_j$ for some $i < j$. Then, by the choice of b_i and b_j (Lemma 7.3), we have $b_i = \bar{u}_j \bar{\pi}_j$ for some unit u_j at P and $b_j = \bar{u}_i \bar{\pi}_i^{\ell-1}$ for some unit u_i at P. Since $b = b_i$ modulo π_i and $b = b_j$ modulo π_j, we have $b = v_i \pi_i^{\ell-1} + v_j \pi_j$ for some units v_i, v_j at P. In particular b is a regular prime at P. Since $[a)$ is unramified at P (Lemma 7.1) and b being a prime at P, α is unramified at P except possibly at b. Thus there is at most one D_s with $P \in D_s$ and such a D_s is defined by $b = v_i \pi_i^{\ell-1} + v_j \pi_j$ for some units v_i, v_j at P. In particular (B8) of Assumptions 6.3 is satisfied. □

Theorem 7.7. *Let K be a global field or a local field and F the function field of a curve over K. Let ℓ be a prime not equal to the characteristic of K. Suppose that K contains a primitive ℓ-th root of unity. Let $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$. Suppose that $\zeta \otimes_F (F \otimes_K K_\nu)$ is trivial for all real places ν of K. Then there exist $a, b, f \in F^*$ such that $\zeta = [a, b) \cdot (f)$.*

Proof. By Proposition 7.6, there exist $a, b \in F^*$ and regular proper model $\mathscr{X}$ of F such that the triple $(\mathscr{X}, \zeta, \alpha)$ satisfy the Assumptions 6.3. Thus, by Theorem 6.5, there exists $f \in F^*$ such that $\zeta = \alpha \cdot (f) = [a, b) \cdot (f)$. □

Corollary 7.8. *Let K be a global field or a local field and F the function field of a curve over K. Let ℓ be a prime not equal to the characteristic of K. Suppose that K contains a primitive ℓ-th root of unity. Suppose that either $\ell \neq 2$ or K has no real places. Then for every element $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$, there exist $a, b, c \in F^*$ such that $\zeta = [a, b) \cdot (c)$.*

8. Applications

In this section we given some applications of our main result to quadratic forms and Chow group of zero-cycles.

Let K be a field of characteristic not equal to 2. Let $W(K)$ denote the Witt group of quadratic forms over K and $I(K)$ the fundamental ideal of $W(K)$ consisting of classes of even dimensional forms [Scharlau 1985, Chapter 2]. For $n \geq 1$, let $I^n(K)$ denote the n-th power of $I(K)$. For $a_1, \dots, a_n \in F^*$, let $\langle\langle a_1, \dots, a_n \rangle\rangle$ denote the n-fold Pfister form $\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$ [loc. cit., Chapter 4].

Theorem 8.1. *Let k be a totally imaginary number field and F the function field of a curve over k. Then every element in $I^3(F)$ is represented by a 3-fold Pfister form. In particular if the class of a quadratic form q is in $I^3(F)$ and dimension of q is at least 9, then q is isotropic.*

Proof. Since every element in $H^3(F, \mathbb{Z}/2(3))$ is a symbol (Corollary 7.8) and $\mathrm{cd}_2(F) \leq 3$, it follows from [Arason et al. 1986, Theorem 2] that every element in $I^3(F)$ is represented by a 3-fold Pfister form (see the proof of [Parimala and Suresh 1998, Theorem 4.1]). □

Proposition 8.2. *Let F be a field of characteristic not equal to 2 with $cd_2(F) \leq 3$ Suppose that every element in $H^3(F, \mathbb{Z}/2(3))$ is a symbol. If q is a quadratic form over F of dimension at least 5 and $\lambda \in F^*$, then $q \otimes \langle 1, -\lambda \rangle$ is isotropic.*

Proof. Without loss of generality we assume that dimension of q is 5. By scaling we also assume that $q = \langle -a, -b, ab, c, d \rangle$ for some $a, b, c, d \in F^*$. Let $q' = \langle -a, -b, ab, c, d, -cd \rangle \otimes \langle 1, -\lambda \rangle$. Since

$\langle -a, -b, ab, c, d, -cd\rangle \in I^2(K)$ [Scharlau 1985, page 82], $q' \in I^3(F)$. Hence, by Theorem 8.1, q' is represented by 3-fold Pfister form. Since $q' \otimes F(\sqrt{\lambda}) = 0$, $q' = \langle 1, -\lambda\rangle \otimes \langle 1, \mu\rangle \otimes \langle 1, \mu'\rangle$ for some $\mu, \mu' \in F^*$ (see [Scharlau 1985, Theorem 5.2 on page 45, Corollary 1.5 on page 143 and Theorem 1.4 on page 144]). Since $H^4(F, \mathbb{Z}/2(4)) = 0$, $I^4(F) = 0$ [Arason et al. 1986, Corollary 2], we have $q' = -cd\langle 1, -\lambda\rangle \otimes \langle 1, \mu\rangle \otimes \langle 1, \mu'\rangle$.

Thus we have

$$\begin{aligned}\langle -a, -b, ab, c, d\rangle \otimes \langle 1, -\lambda\rangle &= -cd\langle 1, -\lambda\rangle \otimes \langle 1, \mu\rangle \otimes \langle 1, \mu'\rangle + cd\langle 1, -\lambda\rangle \\ &= -cd\langle 1 - \lambda\rangle \otimes \langle \mu, \mu', \mu\mu'\rangle.\end{aligned}$$

In particular $\langle -a, -b, ab, c, d\rangle \otimes \langle 1, -\lambda\rangle$ is isotropic [Scharlau 1985, page 34]. □

Corollary 8.3. *Let K be a totally imaginary number field and F the function field a curve over K. Let q be a quadratic forms over F of dimension at least 5. Let $\lambda \in F^*$. Then the quadratic form $q \otimes \langle 1, -\lambda\rangle$ is isotropic.*

Proof. Since K is a totally imaginary number field and F is a function field of a curve over k, we have $H^4(F, \mathbb{Z}/2(4)) = 0$. Since every element in $H^3(F, \mathbb{Z}/2(3))$ is a symbol (Corollary 7.8), $q \otimes \langle 1, -\lambda\rangle$ is isotropic (Proposition 8.2). □

The following was conjectured by Colliot-Thélène and Skorobogatov [1993].

Theorem 8.4. *Let k be a totally imaginary number field and C a smooth projective geometrically integral curve over K. Let $\eta : X \to C$ be an admissible quadric fibration. If* $\dim(X) \geq 4$, *then* $\mathrm{CH}_0(X)$ *is a finitely generated abelian group.*

Proof. Let q be a quadratic form over $k(C)$ defining the generic fiber of $\eta : X \to C$. Let $N_q(k(C))$ be the subgroup of $k(C)^*$ generated by fg with $f, g \in k(C)^*$ represented by q. Let $\lambda \in k(C)^*$. Since $\dim(X) \geq 4$, the dimension of q is at least 5. Thus, by Corollary 8.3, $q \otimes \langle 1, -\lambda\rangle$ is isotropic. Hence λ is a product of two values of q. In particular $\lambda \in N_q(k(C))$ and $k(C)^* = N_q(k(C))$.

Let $\mathrm{CH}_0(X/C)$ be the kernel of the induced homomorphism $\mathrm{CH}_0(X) \to \mathrm{CH}_0(C)$. Then, by [Colliot-Thélène and Skorobogatov 1993], $\mathrm{CH}_0(X/C)$ is a subquotient of the group $k(C)^*/N_q(k(C))$ and hence $\mathrm{CH}_0(X/C) = 0$. In particular $\mathrm{CH}_0(X)$ is isomorphic to a subgroup of $\mathrm{CH}_0(C)$. Since, by a theorem of Mordell–Weil, $\mathrm{CH}_0(C)$ is finitely generated, $\mathrm{CH}_0(X)$ is finitely generated. □

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