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Samuel Le Fourn

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Tubular approaches to Baker's method for curves and varieties

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Baker's method, relying on estimates on linear forms in logarithms of algebraic numbers, allows one to prove in several situations the effective finiteness of integral points on varieties. In this article, we generalize results of Levin regarding Baker's method for varieties, and explain how, quite surprisingly, it mixes (under additional hypotheses) with Runge's method to improve some known estimates in the case of curves by bypassing (or more generally reducing) the need for linear forms in p-adic logarithms. We then use these ideas to improve known estimates on solutions of S-unit equations. Finally, we explain how a finer analysis and formalism can improve upon the conditions given, and give some applications to the Siegel modular variety $A_2(2)$.

1. Introduction

One of the main concerns of number theory is solving polynomial equations in integers, which amounts to determining the integral points on the variety defined by those equations. For a smooth projective curve over a number field, Siegel's theorem says that there are generally only finitely many integral points on this curve, but this result is in general deeply ineffective in that it does not provide us with any way to actually determine this set of integral points.

We focus here on Baker's method (and to a lesser extent Runge's method), which are both *effective*: when applicable, they give a bound on the height of the integral points considered. Our work is based on Bilu's conceptual approach [1995] for curves, and its generalization to higher-dimensional varieties by Levin [2014]. It is also heavily inspired (sometimes implicitly) by a previous article [Le Fourn 2019], dealing with Runge's method. Before stating the main results, let us give some notations and motivations.

K is a fixed number field, L is a finite extension of K with set of places M_L divided into its archimedean places M_L^∞ and its finite places M_L^f, and S a finite set of places of L containing M_L^∞ (the pair (L, S) will be allowed to change). The ring of S-integers of L is denoted by $\mathcal{O}_{L,S}$ and the regulator of $\mathcal{O}_{L,S}^*$ by R_S. We also denote by P_S the largest norm of an ideal coming from a finite place of S (equal to 1 if $S = M_L^\infty$).

The notion of integral point on a projective variety X will be precisely defined (model-theoretically) in Section 2A, but the result is compatible with all reasonable definitions, e.g., the one in [Vojta 1987, Section I.4] and the intuition we give below. For Z a closed algebraic subvariety of X, the set $(X\setminus Z)(\mathcal{O}_{L,S})$ will thus be the set of S-integral points of $X \setminus Z$. It can be interpreted in the following way. Every place

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v of M_L defines a v-adic topology on $X(K_v)$ where K_v is the completion of K at v, and in this topology we can say if a point $P \in X(K_v)$ is "v-close" to the subvariety Z or not. The point P is then S-integral with respect to Z if for every place v of M_L except maybe the ones of S, P is v-far from Z.

Furthermore, as $X(K_v)$ is compact, given two algebraic (hence closed) subvarieties Z, Z' of X, a point P is v-close to both Z and Z' if and only if it is v-close to $Z \cap Z'$, in particular if this intersection is empty, P can be v-close to only one of them. This simple fact is very useful in our arguments.

The basic context (C) (which will be made more complex later) of our arguments will be the following:

- X is a smooth projective variety over a number field K.
- $D_1, \dots, D_n$ are effective divisors on X and $D = \bigcup_{i=1}^n D_i$.
- The point P belongs to $(X \setminus D)(\mathcal{O}_{L,S})$.

One of the first ideas to prove finiteness of integral points is the following: if a point P is v-far from D_i for *every* place v of M_L, the global height $h_{D_i}(P)$ relative to D_i can be bounded, so under some geometric property called *ampleness* of D_i, P must belong to a finite set that can be in principle effectively determined. The goal of many finiteness methods is thus to find conditions which ensure automatically that such a situation as above happens for P and one of the D_i, which would then entail finiteness. Notice that if $P \in (X \setminus D)(\mathcal{O}_{L,S})$, this is automatic for places not in S, so we only need to prove it for the (finitely many) places of S.

On the other hand, Baker's method proceeds quite differently: it relies on the fact (based on linear forms in logarithms) that a unit $x \in \mathcal{O}_{L,S}^* \setminus \{1\}$ cannot be too v-close to 1 (this depending on the global height of x itself), so if we force it somehow to be v-close, we obtain in return a bound on the global height of x, which in turn gives effective finiteness. For $P \in (X \setminus D)(\mathcal{O}_{L,S})$ as before, the game is then to ensure that there is some place $v \in S$ and some point P_0 in an explicit finite set (independent of P) such that P is very v-close to P_0, and there is a rational function ϕ sending P_0 to 1 and P in $\mathcal{O}_{L,S}^*$. This approach looks quite orthogonal to the previous paragraph but it turns out they can complement each other in diverse situations, and this is the topic of the present paper.

Let us start from (C), and fix $m_Y \geq 1$ an integer and Y the union of intersections of any $m_Y + 1$ divisors amongst $D_1, \dots, D_n$ (the notation might seem backwards, but it is because one can also first fix a closed subvariety Y and define m_Y from it, which is a natural way to proceed in practical cases). We also define $m_{\mathcal{B}} \geq 1$ such that any intersection of $m_{\mathcal{B}}$ distinct divisors amongst $D_1, \dots, D_n$ is a finite set. The main point of Theorem 1.1 is proving that, for any finite sets of places $M_L^\infty \subset S' \subset S$ the set of points of $(X \setminus D)(\mathcal{O}_{L,S}) \cap (X \setminus Y)(\mathcal{O}_{L,S'})$ is effectively finite (up to an explicit proper closed subset) when

$$(m_{\mathcal{B}} - 1)|S'| + m_Y |S \setminus S'| < n.$$

The basic idea (to which the reduction is nevertheless quite technical, and we forget the exceptional cases for simplicity here) is the following: given a point $P \in (X \setminus D)(\mathcal{O}_{L,S}) \cap (X \setminus Y)(\mathcal{O}_{L,S'})$, for $v \in S \setminus S'$, if P is v-close to more than m_Y divisors D_i, it is v-close to Y, which is ruled out by S'-integrality with respect to Y. On another hand, if $v \in S'$ and P is v-close to $m_{\mathcal{B}}$ divisors, it is close to some point in their intersection which is finite, and we can then apply Baker's method. Now, the inequality above guarantees that if none of these situations happens, then by the pigeonhole principle there is some D_i such that P is v-far away from D_i for every place $v \in M_L$, which proves effective finiteness by the arguments above.

We now state the complete version of this result.

Theorem 1.1. *Let X be a smooth projective variety over K and $D_1, \dots, D_n$ be ample effective divisors on X, $D = \bigcup_{i=1}^n D_i$, and h_D a choice of absolute logarithmic height relative to D.*

The number $m_{\mathcal{B}}$ (assumed to exist) is the smallest integer such that for any set $I \subset \{1, \dots, n\}$ with $|I| = m_{\mathcal{B}}$, the intersection $T_I = \bigcap_{i \in I} \mathrm{Supp}(D_i)(\overline{K})$ is finite. For any point P in the finite set

$$T := \bigcup_{|I|=m_{\mathcal{B}}} T_I,$$

assume $(H)_P$:

> *There exists $\phi_P \in K(X)$ nonconstant whose support is included in $(\mathrm{Supp}\, D) \setminus P$ (and $\phi_P(P) = 1$ for simplicity).*

Such a function will be fixed in the following.

Let $m_Y \geq 1$ be an integer and

$$Y = \bigcup_{|I|>m_Y} \bigcap_{i \in I} \mathrm{Supp}(D_i)(\overline{K}).$$

Then, there exists an effectively computable constant $C > 0$ and an explicit function $C_1(d, s)$ such that for any triple (L, S, S') with $M_L^\infty \subset S' \subset S$ finite such that $[L : \mathbb{Q}] = d$, $|S| = s$ and

$$(m_{\mathcal{B}} - 1)|S'| + m_Y |S \setminus S'| < n, \tag{1-1}$$

for every $Q \in (X \setminus D)(\mathcal{O}_{L,S}) \cap (X \setminus Y)(\mathcal{O}_{L,S'})$,

$$h_D(Q) \leq C \cdot C_1(d, s) h_L R_S P_{S'} \log^*(h_L R_S), \tag{1-2}$$

where $\log^*(x) = \max(\log x, 1)$, *unless*

$$Q \in Z := \bigcup_{P \in T} Z_{\phi_P}, \quad Z_{\phi_P} := \overline{\{Q \in (X \setminus \mathrm{Supp}\, \phi_P)(\overline{K}),\ \phi_P(Q) = 1\}},$$

this set Z being an effective strict closed subset of X independent of (L, S, S') and the bar denoting the Zariski closure.

A remark developed below is that the dependence on S' in (1-2) is the factor $P_{S'}$, so an ideal choice for S' is for it to be as large as possible to satisfy (1-1) but with its finite places as small as possible. This leeway in the choice of S' allows for improved bounds even on the well-trodden ground of curves, for which one obtains the following corollary.

Corollary 1.2. *Let C be a smooth projective curve over K and $\phi \in K(C)$ nonconstant.*

Assume that every pole P of ϕ satisfies $(H)_P$ (and to simplify, is defined over K). Then, there are a constant $C > 0$ and an explicit function $C_1(d, s)$ such that for any triple (L, S, S') satisfying

$$|S \setminus S'| < n, \tag{1-3}$$

any point $Q \in C(L)$ such that $\phi(Q) \in \mathcal{O}_{L,S}$ satisfies

$$h(\phi(Q)) \leq C \cdot C_1(d, s) h_L R_S \log^*(h_L R_S) P_{S'}.$$

Remarks 1.3. Let us make some comments about these results:

- For $S' = M_L^\infty$, under the assumption (1-1) that $|S \setminus M_L^\infty|$ is small (exactly translated by (1-3) for curves), one obtains a bound on the height which only grows in $R_S^{1+\varepsilon}$. In particular, there is no linear dependence on P_S (which would come from estimates of linear forms in p-adic logarithms in a straightforward application of Baker's method), but rather in $\log P_S$ (implicitly contained in R_S), which might prove useful for some applications.

- The set Z of Theorem 1.1 can actually be made smaller: as done in [Levin 2014], we can replace each Z_{ϕ_P} by the intersection of *all* Z_{ϕ_P} where ϕ_P runs through all functions satisfying the hypotheses of $(H)_P$.

- Theorem 1.1 applied for $m_Y = 0$ and $S = S'$ retrieves Levin's result [2014, Theorem 1], and for smaller S' (hence more hypotheses) improves upon the quantitative estimates it implicitly gave.

- For general m_Y, Theorem 1.1 improves qualitatively (when the hypotheses $(H)_P$ hold) upon a previous result based on Runge's method [Le Fourn 2019, Theorem 5.1 and Remark 5.2(b)], as condition (1-1) is generally weaker than the tubular Runge condition defined there (the choice of set $(X \setminus Y)(\mathcal{O}_{L,S'})$ is inspired by the notion of tubular neighborhood defined in that article, see section 3 there). Indeed, the $m = m_\varnothing$ in the statement of [Le Fourn 2019, Remark 5.2] is not $m_{\mathcal{B}}$, and in general one only knows that $m_{\mathcal{B}} \leq m + 1$. If $m_{\mathcal{B}} = m + 1$, the original Runge's method can be applied as in [Levin 2008] and gives better (and uniform) estimates than [Levin 2014] (and the same holds for the tubular variants we propose), but if $m_{\mathcal{B}} \leq m$ which is the most likely situation, the condition (1-1) is indeed more easily satisfied than for Runge's method.

- The words "effectively computable" for the constant C deserve to be made more precise. One requires to know an embedding of X in a projective space $\mathbb{P}_K^N$, explicit equations and formulas for X, the D_i, D, h_D, the points of T and expressions of ϕ_P relative to this embedding. With this data, the effectivity boils down to an effective Nullstellensatz such as e.g., [Masser and Wüstholz 1983, Theorem IV]. Now, the functions $C_1(d, s)$ (as well as $R_S \log^*(R_S)$) are coming from the theory of linear forms in logarithms, and are as such completely explicit. The addition of h_L is a technicality due to the necessity of slightly increasing the ring of units $\mathcal{O}_{L,S}^*$ upon which to apply Baker's estimates, and can often be removed in special cases.

• Unless we are in the case of curves, $m_{\mathcal{B}} > 1$ and then (1-1) bounds d and s in terms of n, which allows us to replace $C_1(d, s)$ by an explicit function $C_1(n)$.

• As will be discussed in Section 5, in some situations one can apply the same methods as in the proof of Theorem 1.1 without having (1-1), and one can also devise some more uniform variants of this result in intermediary cases.

As an illustration of the effectivity of the method, we prove the following result on the S-unit equation: fix L to be a number field of degree d, $S \supset M_L^\infty$ a set of places of L of cardinality s and $\alpha, \beta \in L^*$. We consider the S-unit equation

$$\alpha x + \beta y = 1, \quad x, y \in \mathcal{O}_{L,S}^*. \tag{1-4}$$

Theorem 1.4. *Let L, S, α, β be as above*:

- *If S contains at most two finite places, all solutions of* (1-4) *satisfy*

$$\max(h(x), h(y)) \leq 2c(d, s) R_S \log^*(R_S) H,$$

where $H = \max(h(\alpha), h(\beta), 1, \pi/d)$ and $c(d, s)$ is the constant defined as $c_{26}(s, d)$ in formula (30) *of [Győry and Yu 2006].*

- *For any set of places S, all solutions of* (1-4) *satisfy*

$$\max(h(x), h(y)) \leq 2c'(d, s) P'_S R_S (1 + \log^*(R_S)/\log^* P'_S) H,$$

where $c'(d, s) = c_1(s, d)$ from Theorem 1 *of [Győry and Yu 2006], and P'_S the third largest value of the norms of ideals coming from finite places of S.*

This result provides an improvement on known bounds for solutions of the S-unit equations. More precisely, its dependence on P'_S (instead of P_S, the largest norm of an ideal coming from a place of S) becomes particularly interesting when there are at most two places of S of large relative norm, and by construction it improves Theorem 1 of [Győry and Yu 2006]. One can also remark that such an estimate is likely to be close to optimal in terms of dependence on the primes in S, as replacing $R_S \log^*(R_S)$ by $o(R_S)$ in the first bound would imply that there are only finitely many Mersenne primes. Notice that there is an additional factor 2 in the inequalities of Theorem 1.4 when compared to the reference [Győry and Yu 2006], which is due to a special case in the proof.

On another hand, Theorem 4.1.7 of [Evertse and Győry 2015], based on slightly different Baker-type estimates, has a better dependence on s and d. It is possible to combine the strategy of proof of the latter theorem with our own to obtain an improvement of both results, essentially replacing again P_S by P'_S. This is achieved in a recent preprint of Győry [2019] (which also takes into account and deals with the factor 2 discussed above).

After proving Theorem 1.1 and Corollary 1.2 in Section 3 (Section 2 gathering the necessary reminders and tools for the proof), we prove Theorem 1.4 in Section 4. This application is heavily based on computations undertaken in [Győry and Yu 2006], hence we have chosen to refer to it whenever possible, and focus on pointing out where the improvements come from our approach.

In the last part of this paper (Section 5), inspired by comments from the referees, we discuss a rewording of the elementary ideas behind Runge and Baker's method in terms of a graph defined by the divisors D_i, which leads in some situations to hypotheses of application weaker than e.g., (1-1). In the spirit of [Le Fourn 2019] and as an example of the potential for improvement it reveals, we apply these ideas to the Siegel modular variety $A_2(2)^S$ and obtain finiteness of abelian surfaces over quadratic fields with full 2-torsion and satisfying conditions on their places of bad reduction (Propositions 5.5 and 5.7).

2. Reminders on Baker's theory and local heights

For any place w of L, the norm $|\cdot|_w$ associated to w is normalized to extend the norm on $\mathbb{Q}$ defined by v_0 below w, where $|\cdot|_\infty$ is the usual norm on $\mathbb{Q}$ and for every prime p and nonzero fraction a/b,

$$\left|\frac{a}{b}\right|_p = p^{\operatorname{ord}_p b - \operatorname{ord}_p a}.$$

We also define $n_w = [L_w : \mathbb{Q}_{v_0}]$ the local degree of L at w.

In all discussions below, X is a fixed projective smooth algebraic variety over the number field K and closed subset of X will mean a closed algebraic K-subvariety of X.

Regarding the integrality, we choose a model-theoretic definition as follows. Assume $\mathcal{X}$ is a proper model of X over $\mathcal{O}_K$, fixed until the end of this article. For every closed subset Y of X, denote by $\mathcal{Y}$ the Zariski closure of Y in $\mathcal{X}$. The set of integral points $(X \setminus Y)(\mathcal{O}_{L,S})$ will then implicitly denote the set of points $P \in X(L)$ whose reduction in $\mathcal{X}_v(\kappa(w))$ for a place w of $M_L \setminus S$ above $v \in M_K$ (well-defined by the valuative criterion of properness) never belongs to $\mathcal{Y}$.

2A. ***M_K-constants and M_K-bounded functions.*** The arguments below will be much simpler to present with the formalism of M_K-constants and M_K-functions briefly recalled here.

Definition 2.1. • An M_K*-constant* is a family $(c_v)_{v \in M_K}$ of nonnegative real numbers, all but finitely many of them being zero.

• An M_K*-function* f (on X) is a function defined on a subset E of $X(\overline{K}) \times M_{\overline{K}}$ with real values (typically, a local height function as below). Equivalently, it is defined as a function on a subset of $\bigsqcup_{K \subset L \subset \overline{K}} X(L) \times M_L$, consistently in the sense that if f is defined at (P, w) with $P \in X(L)$ and $w \in M_L$, then it is defined at (P, w') with $w' \mid w \in M_{L'}$ for any extension L' of L, and $f(P, w) = f(P, w')$.

• An M_K-function $f : E \to \mathbb{R}$ is M_K*-bounded* if there exists an M_K-constant $(c_v)_{v \in M_K}$ for which for all $(P, w) \in E$,

$$|f(P, w)| \leq c_v \quad (w \mid v).$$

The notation $O_{M_K}(1)$ will be used for an M_K-bounded function depending on the context (in particular, its domain E will often be implicit but obvious).

• Two M_K-functions $f, g : E \to \mathbb{R}$ are *M_K-proportional* when there is an absolute constant $C > 0$ and a M_K-constant $(c_v)_{v \in M_K}$ for which for all $(P, w) \in E$,

$$\tfrac{1}{C}|f(P,w)| - c_v \leq |g(P,w)| \leq C|f(P,w)| + c_v \quad (w \mid v).$$

• Two functions f, g defined on an open subset O of $X(\overline{K})$ (typically, global height functions) are *proportional* if there are absolute constants $C_1, C_2 > 0$ such that for every $P \in O$

$$\tfrac{1}{C_1} f(P) - C_2 \leq g(P) \leq C_1 f(P) + C_2.$$

2B. ***Local heights associated to closed subsets.*** We will now define explicitly local height functions relative to closed subsets of a projective variety X:

• For any point $P \in \mathbb{P}^N(L)$, one denotes by $x_P = (x_{P,0}, \dots, x_{P,n}) \in L^{n+1}$ a choice of coordinates representing P and $\|x_P\|_w = \max_i |x_{P,i}|_w$.

• For a polynomial $g \in L[X_0, \dots, X_N]$ and $w \in M_L$, the norm $\|g\|_w$ is the maximum norm of its coefficients for $|\cdot|_w$.

• Given a closed subset Y of $\mathbb{P}^N_K$ and homogeneous polynomials $g_1, \dots, g_m \in K[X_0, \dots, X_N]$ generating the ideal of definition of Y, for any $w \in M_L$ and any $P \in (\mathbb{P}^N \setminus Y)(L)$, one defines explicitly a choice of local height of P at Y for w by

$$h_{Y,w}(P) := -\min_i \log \frac{|g_i(x_P)|_w}{\|g_i\|_w \|x_P\|_w^{\deg g_i}}, \tag{2-1}$$

and the global height by

$$h_Y(P) := \frac{1}{[L:\mathbb{P}]} \sum_{w \in M_L} n_w \cdot h_{Y,w}(P).$$

With this normalization, for any $w \in M_L^f$ and $P \in (\mathbb{P}^N \setminus Y)(L)$, $h_{Y,w}(P) \geq 0$ and it is positive if and only if P reduces in Y modulo w.

Let us now sum up the main properties of those functions that we will need.

Proposition 2.2 (local heights). *Let X be a smooth projective variety over K, with an implicit embedding in a $\mathbb{P}^n_K$ and fixed choices of local heights as in (2-1) for all closed subsets considered below*:

(a) *For any closed subsets Y, Y' of X the functions $h_{Y \cap Y', w}$ and* $\min(h_{Y,w}, h_{Y,w'})$ *are M_K-proportional on $(X \setminus (Y \cup Y'))(\overline{K}) \times M_{\overline{K}}$.*

(b) *For a disjoint union $Y \sqcup Y'$ of closed subsets of X, one has*

$$h_{Y,w}(P) + h_{Y',w}(P) = h_{Y \sqcup Y', w}(P) + O_{M_K}(1)$$

on $(X \setminus (Y \cup Y'))(\overline{K}) \times M_{\overline{K}}$.

(c) *For $Y \subset Y'$ closed subsets, one has*

$$h_{Y,w}(P) \leq h_{Y',w}(P) + O_{M_K}(1)$$

on $(X \setminus Y')(\overline{K}) \times M_{\overline{K}}$.

(d) *If* $\phi : X' \to X$ *is a morphism of projective varieties, the functions* $(P, w) \mapsto h_{Y,w}(\phi(P))$ *(resp.* $h_{\phi^{-1}(Y),w}(P)$*) are* M_K*-proportional on* $(X' \setminus \phi^{-1}((Y))(\overline{K}) \times M_{\overline{K}}$.

(e) *For any closed subset* Y *of* X, *the function* $(P, w) \mapsto h_{Y,w}(P)$ *is* M_K*-bounded on the set of pairs satisfying* $P \in (X \setminus Y)(\mathcal{O}_{L,w})$ *(independently of the number field* L*).*

(f) *For any effective divisor* D *on* X *and any function* $\phi \in K(X)$ *with support of poles included in* $\operatorname{Supp} D$, *the function* $(P, w) \mapsto |\phi(P)|_w$ *is* M_K*-bounded on the set of pairs* $(P, w) \in X(L) \times M_L^f$ *satisfying* $P \in (X \setminus D)(\mathcal{O}_{L,w})$ *(independently of the number field* L*).*

(g) *If* D *and* D' *are two ample divisors on* X, *for any two choices of global heights* h_D *and* $h_{D'}$, *they are proportional on* $(X \setminus \operatorname{Supp}(D \cup D'))(\overline{K})$.

Furthermore, all the implied M_K*-constants and constants are effective.*

Proof. This proposition is mostly a reformulation of results of [Silverman 1987] already quoted in [Levin 2014]. First, (2-1) indeed defines local heights associated to closed subsets by [Silverman 1987, Proposition 2.4] so most of the proposition is contained in [loc. cit., Theorem 2.1]. Let us point out the slight differences and explain how it is effective. In that article, local height functions are more precisely defined by their ideal sheaves, whereas we consider closed subsets hence reduced closed subschemes. Now, if two ideal sheaves $\mathcal{I}$ and $\mathcal{I}'$ have the same support, their local height functions are M_K-proportional. More concretely, let us fix $Y \subset Y'$ closed subsets of $X \subset \mathbb{P}^N_K$ and two systems of homogeneous generators $g_1, \dots, g_m \in K[X_0, \dots, X_N]$ and $h_1, \dots, h_p$ of ideal sheaves with respective supports Y and Y' in $\mathbb{P}^N_K$. After multiplying by a suitable $n \geq 1$, one can assume all those polynomials' coefficients belong to $\mathcal{O}_K$, and such an n can be made effective in terms of the $\|g_i\|_v$ and $\|h_j\|_v$ for $v \in M_K^f$. Now, an effective Nullstellensatz (e.g., [Masser and Wüstholz 1983] applied to multiples of those generators), translated in the projective case, will give relations

$$a g_i^k = \sum_{j=1}^{p} f_{i,j} h_j$$

with $a \in \mathcal{O}_K$ nonzero, all the $f_{i,j}$ with coefficients in $\mathcal{O}_K$ and bounded $\|f_{i,j}\|_v$ and $|a|_v$ in terms of the norms of the polynomials for all $v \in M_K^\infty$. Furthermore, the power k is effectively bounded in terms of $[K : \mathbb{Q}]$,N and the degrees of the polynomials. This will clearly give an effective inequality

$$h_{Y,w}(P) \leq k \cdot h_{Y',w}(P) + O_{M_K}(1).$$

for all $(P, w) \in (X \setminus Y')(\overline{K}) \times M_{\overline{K}}$, and this argument works for parts (a) to (d) of the Proposition (the inequality in (c) without a factor k coming from the fact that we can extend generators of an ideal sheaf for Y' to an ideal sheaf for Y).

The only parts remaining to be proven are now (e), (f) and (g). Part (e), essentially saying that local height functions detect integral points up to some M_K-bounded error, is classical (see [Vojta 1987, Proposition 1.4.7]) and in fact automatic for the exact definition given in (2-1). Part (f) then comes from

(e) and Lemma 11 of [Levin 2014]. Finally, part (g) is a classical result on heights (e.g., [Lang 1983, Chapter IV, Proposition 5.4], and there also, the constants implied can be made effective. □

2C. *Baker's theory of linear forms in logarithms.* Let us now give our second main tool: estimates from Baker's theory in a special form sufficient for our purposes.

For any place w of M_L, $N(w)$ is defined to be 2 if w is archimedean and the norm of the associated prime ideal otherwise.

Proposition 2.3. *Define* $\log^*(x) = \max(\log(x), 1)$ *for* $x > 0$.

Let $d = [L : \mathbb{Q}]$ *and* $s = |S|$. *There is an effectively computable function* $C(d, s)$ *such that for any pair* (L, S), *any* $\alpha \in \mathcal{O}^*_{L,S} \setminus \{1\}$ *and any* $w \in M_L$,

$$\log|\alpha - 1|_w \geq -C(d, s)\frac{N(w)}{\log N(w)} R_S \log^*(N(w)h(\alpha)). \tag{2-2}$$

In terms of local heights, one can choose the local height $h_{1,w}(\alpha)$ *to be* $\max(-\log|\alpha - 1|_w, 0)$, *which gives us*

$$h_{1,w}(\alpha) \leq C(d, s)\frac{N(w)}{\log N(w)} R_S \log^*(N(w)h(\alpha)).$$

Proof. This result, although natural when one knows estimates for linear forms in logarithms, is not often presented in this form, so the following proof will explain how one can get to such an expression with known results. First, let us assume $s \geq 2$ (for $s = 1$, the result is trivial). By Lemma 1 of [Bugeaud and Győry 1996] ($\log h$ there is our logarithmic height here), one can choose a family of fundamental units $\varepsilon_1, \ldots, \varepsilon_{s-1}$ of $\mathcal{O}^*_{L,S}$ such that

$$\prod_{i=1}^{s-1} h(\varepsilon_i) \leq c_1(s) R_S,$$

where $c_1(s) = ((s-1)!)^2/(2^{s-1}d^{s-2})$.

Now, by Theorem 4.2.1 of [Evertse and Győry 2015] applied to $\Gamma = \mathcal{O}^*_{L,S}$ gives us the bound (taking into account our normalization of $|\cdot|_w$) with $C(d, s) = c_1(s)c_8$ where c_8 is defined as in the reference. □

3. Proof of the main theorem

We now have all the tools to prove the theorem. We keep the notations from its statement, and assume that we have an embedding $X \subset \mathbb{P}^N_K$ from which all local heights considered below are defined. Recall that $s = |S|$, $d = [L : \mathbb{Q}]$ and n_w is the local degree of L at w. The constants c_i below are absolute and can be made effective.

First, let us notice that for every point $P \in T$, as the support of ϕ_P is in D, by Proposition 2.2(f) applied to ϕ_P and ϕ_P^{-1}, there is an absolute positive integer m (independent on the choice of (L, S) and $P \in T$) such that for every $Q \in (X \setminus D)(\mathcal{O}_{L,S})$, one has $m\phi_P(Q) \in \mathcal{O}_{L,S}$ and $m\phi_P(Q)^{-1} \in \mathcal{O}_{L,S}$. Defining S_m the set of primes of L dividing m, one thus has $\phi_P(Q) \in \mathcal{O}^*_{L,S \cup S_m}$ for all $Q \in (X \setminus D)(\mathcal{O}_{L,S})$.

By Proposition 2.2(e), the map $(Q, w) \mapsto h_{D_i,w}(Q)$ is M_K-bounded on pairs (Q, w) with $w \in M_L \setminus S$ and $Q \in (X \setminus D)(\mathcal{O}_{L,S})$, and $(Q, w) \mapsto h_{Y,w}(Q)$ is M_K-bounded on pairs (Q, w) with $w \in M_L \setminus S'$ and $Q \in (X \setminus Y)(\mathcal{O}_{L,S'})$.

Let us assume now that $Q \in (X \setminus D)(\mathcal{O}_{L,S}) \cap (X \setminus Y)(\mathcal{O}_{L,S'})$. The previous paragraphs imply that for every $i \in \{1, \dots, n\}$

$$h_{D_i}(Q) = \frac{1}{[L:\mathbb{Q}]} \sum_{w \in S} n_w h_{D_i,w}(Q) + O(1)$$

where $O(1)$ is absolutely (and effectively bounded) on the set of such points Q (even if (L, S) is allowed to change).

Thus, for all $i \in \{1, \dots, n\}$, there is $w \in S$ such that

$$h_{D_i,w}(Q) \geq \frac{n_w}{[L:\mathbb{Q}]} h_{D_i,w}(Q) \geq \frac{1}{s} h_{D_i}(Q) + O(1).$$

After choosing for every $i \in \{1, \dots, n\}$ such a $w \in S$, we obtain a function $\{1, \dots, n\} \to S$. Now, if the fiber above a place $w \in S \setminus S'$ was a set J with $|J| > m_Y$, by Proposition 2.2(a), one would obtain an absolute effective (computable) upper bound on the minimum of such $h_{D_j,w}(Q)$, therefore on $h_{D_i}(Q)/s$ and $h_D(Q)/s$ by Proposition 2.2(g).

We can thus assume from now on that this is not the case. Therefore, the fibers of this function are of cardinality at most m_Y above $S \setminus S'$. Consequently, by hypothesis (1-1), one of the fibers above S', defined as I, has to be of cardinality at least $m_{\mathcal{B}}$ (if it's more, we extract a subset of cardinality $m_{\mathcal{B}}$), which gives $w \in S'$ such that

$$\min_{i \in I} h_{D_i,w}(Q) \geq \frac{1}{s} \min_{i \in I} h_{D_i}(Q) \geq \frac{c_1}{s} h_D(Q) + O(1).$$

Now, by Proposition 2.2(a) again and construction of T (if $T = \varnothing$, we directly obtain an absolute M_K-constant bound), there exists $P \in T$ such that

$$h_{P,w}(Q) \geq \frac{c_2}{s} h_D(Q) + O_{M_K}(1) \geq \frac{c_3}{s} h(\phi_P(Q)) + O_{M_K}(1), \tag{3-1}$$

using Proposition 2.2(g), with absolute effective constants $c_1, c_2, c_3 > 0$. Moreover, by Proposition 2.2(d), as $\phi_P(P) = 1$, if ϕ_P can be evaluated at Q and $\phi_P(Q) \neq 1$,

$$h_{1,w}(\phi_P(Q)) \geq c_4 \cdot h_{P,w}(Q) + O_{M_K}(1) \geq \frac{c_3 c_4}{s} h(\phi_P(Q)) + O_{M_K}(1), \tag{3-2}$$

for $c_4 > 0$ absolute effective and $O_{M_K}(1)$ computable in terms of the initial data of embeddings and equations (but bounding it crudely by an absolute constant would suffice in the following argument).

On another hand, applying Proposition 2.3 to $\mathcal{O}^*_{L,S \cup S_m}$, we get

$$h_{1,w}(\phi_P(Q)) \leq C(d, s + |S_m|) \cdot R_{S \cup S_m} \frac{N(w)}{\log N(w)} \cdot \log^*(N(w) h(\phi_P(Q))). \tag{3-3}$$

By formula (1.8.3) of [Evertse and Győry 2015], one has

$$R_{S\cup S_m} \le h_L R_S \prod_{\mathfrak{P}\in S_m\setminus S} \log N(\mathfrak{P})) \le h_L R_S \prod_{p\,|\,m} e^{d/e \log p} \le h_L R_S m^{d/e} \tag{3-4}$$

after optimizing the products of logarithms.

Combining (3-2), (3-3) and (3-4) and with some care about the logarithmic terms, we obtain an affine bound of the shape (1-2) for $h(\phi_P(Q))$, hence on $h_{P,v}(Q)$ by Proposition 2.2(d) applied the other way, which finally gives a bound on $h_D(Q)$ by (3-1) (there is a constant term which we can absorb in the linear one as it is effectively boundable).

4. Applications to the S-unit equation in the case of curves

In this section, we realize our method in the practical case of the S-unit equation (1-4), to prove Theorem 1.4.

This problem is related to finding the integral points of $(\mathbb{P}^1 \setminus \{0, 1, \infty\})(\mathcal{O}_{L,S})$ (up to taking into account the factors α, β), and this is the interpretation we will follow below to illustrate the main theorem. We follow closely the definitions and lemmas of [Győry and Yu 2006] (except their normalizations of norms), as our improvements intervene only at the beginning of the proof. As in that article, define

$$d = [L:\mathbb{Q}], \quad H = \max(h(\alpha), h(\beta), 1, \pi/d), \quad s = |S|.$$

For any $t \in L$, define $h_w(t) := h_{0,w}(t) = \log^+(1/|t|_w)$.

For the sake of symmetry of the exposition, we will do most computations with αx and βy, before coming back to H. This means we deal with $h_w(P)$ for

$$P \in E = \left\{\alpha x, \beta y, \frac{1}{\alpha x}\right\}.$$

Lemma 4.1. *For any $x, y \in L$ with $\alpha x + \beta y = 1$:*

- *For any place $w \in M_L$, at most one value of $h_w(P)$ for $P \in E$ can exceed $\delta_w \log 2$, where $\delta_w = 1$ if w is infinite, 0 otherwise.*
- *The maximum modulus of the difference of logarithmic heights of any two of them amongst $h(x)$, $h(y)$, $h(\alpha x)$, $h(\beta y)$ is at most $3H$, and even $2H$ except for $|h(x) - h(y)|$.*
- *If $x, y \in \mathcal{O}_{L,S}^*$ and $h = \max(h(x), h(y))$, we always have, for $P \in E$,*

$$\sum_{w\in S} \frac{n_w}{[L:\mathbb{Q}]} h_w(P) \ge h(P) - H \ge h - 3H.$$

Proof. The first item is the translation of the fact that if $z + z' = 1$ one of z, z' has to have norm at least 1 if the norm is ultrametric, and at least $\frac{1}{2}$ if it is archimedean.

The second item uses that for any nonzero algebraic numbers z, z', $h(zz') \leq h(z)+h(z')$ and $h(z+z') \leq h(z)+h(z')+\log 2$. For example, we obtain $|h(x)-h(\alpha x)| \leq H$ and $|h(\alpha x)-h(\beta y)| \leq \log 2$, and by symmetric role this leads to all other bounds on difference of heights, as $\log 2 \leq H$.

For the third item, in each of the three cases,

$$\sum_{w \in S} \frac{n_w}{[L:\mathbb{Q}]} h_w(P) = h(P) - \sum_{w \notin S} \frac{n_w}{[L:\mathbb{Q}]} \log^+(1/|P|_w)$$

but for $w \notin S$ and each of our three P's, the contribution of x or y to $1/|P|_w$ is by a factor 1 so this sum is bounded by H. The second inequality follows directly from the second item. □

Proof of Theorem 1.4. First, notice that if $s \leq 2$, Lemma 4.1 alone gives immediately that there is $P \in E$ such that $h_w(P) \leq \delta_w \log 2$ for all $w \in S$, and elsewhere we have $h_w(P) = h_w(\alpha)$, $h_w(\beta)$ or $h(\beta/\alpha)$ depending on the value of P, because x and y are S-units. Consequently, $h(P) \leq 2H + \log 2$, hence $h \leq 4H + \log 2$ in this case. We can now assume that $s \geq 3$.

For the first part of Theorem 1.4, the assumption amounts to saying that (1-1) holds in this case for $S' = M_L^\infty$. By Lemma 4.1, for any choice of $P \in E$, there is $w \in S$ such that

$$\frac{n_w}{[L:\mathbb{Q}]} h_w(P) \geq \frac{1}{|S|}(\max(h(x), h(y)) - 3H). \tag{4-1}$$

We want to fall back on a case where for one of the three choices of P, one can impose that $w \in M_L^\infty$. If that is not possible, by the pigeonhole principle and our hypothesis on S, there is a finite place w and two points $P, Q \in \{E\}$ distinct with

$$\frac{n_w}{[L:\mathbb{Q}]} \min(h_w(P), h_w(Q)) \geq \frac{1}{|S|}(\max(h(x), h(y)) - 3H).$$

By the same lemma, we get $\max(h(x), h(y)) \leq 3H$. This bound will be readily checked to be smaller than the other case.

One can thus assume from now on that for some $w \in M_L^\infty$ and $P \in E$, (4-1) holds.

The only thing to do is then to get back to the situation of [Győry and Yu 2006, page 24] in all three cases, after which we will obtain the exact same bounds. We fix a fundamental system $\varepsilon_1, \ldots, \varepsilon_{s-1}$ of units of $\mathcal{O}_{L,S}^*$ with the properties of [Győry and Yu 2006, Lemma 2].

- Assume first $P = \alpha x$, and write

$$y = \zeta \varepsilon_1^{b_1} \cdots \varepsilon_{s-1}^{b_{s-1}}, \tag{4-2}$$

with ζ a root of unity in L and $b_i \in \mathbb{Z}$ for all i. By the arguments of [Bugeaud and Győry 1996, page 76], we obtain that

$$B = \max(|b_1|, \ldots, |b_{s-1}|) \leq c_1(d,s) h(y)$$

with

$$c_1(d,s) = \begin{cases} ((s-1)!)^2/(2^{s-3}\log 2) & \text{if } d = 1, \\ ((s-1)!)^2/2^{s-2}) \log(3d)^3 & \text{if } d \geq 2. \end{cases}$$

We set $\alpha_s = \zeta\beta$ and $b_s = 1$ so that

$$|\alpha x|_w = |1 - \varepsilon_1^{b_1} \cdots \varepsilon_{s-1}^{b_{s-1}} \alpha_s^{b_s}|_w.$$

We set the A_i and A_s as in [Győry and Yu 2006, (31)] and can make the same assumption (otherwise, we obtain a smaller bound). By [loc. cit., Proposition 4 and Lemma 5], we thus obtain

$$h_w(P) = -\log|\alpha x|_w < c_2(d,s)c_3(d,s)R_S H \log\left(\frac{c_1(d,s)h(y)}{\sqrt{2}H}\right) \tag{4-3}$$

(as we always have $s \geq 3$ here), with

$$c_2(d,s) = d^3 \log(ed) \min(1.451(30\sqrt{2})^{s+4}(s+1)^{5.5}, \pi 2^{6.5s+27}),$$
$$c_3(d,s) = e\sqrt{s-2}(((s-1)!)^2/(2^{s-2}))\pi^{s-2} \cdot \begin{cases} 8.5 & \text{if } d = 1, \\ 29d\log d & \text{if } d \geq 2. \end{cases}$$

Let us now define $h = \max(h(x), h(y))$. We use inequality (4-1), and replace $h(y)$ by h in the right-hand side of (4-3) to obtain

$$\frac{h}{s} - H \leq \frac{h - 3H}{s} \leq \frac{n_w}{[L:\mathbb{Q}]} c_2(d,s)c_3(d,s)R_S H \log\left(\frac{c_1(d,s)h}{\sqrt{2}H}\right) \tag{4-4}$$

and these are equivalent to the two inequalities used on page 24 of [Győry and Yu 2006] to obtain the result.

- Assume $P = \beta y$. We apply the same argument by symmetry, replacing α by β and x by y everywhere, to finally obtain the same bound.
- Assume $P = \frac{1}{\alpha x}$. We thus write

$$h_w(P) = -\log\left|\frac{1}{\alpha x}\right|_w = -\log\left|1 - \frac{\beta y}{\alpha x}\right|_w.$$

Let us fix then

$$\frac{y}{x} = \zeta \varepsilon_1^{b_1} \cdots \varepsilon_{s-1}^{b_{s-1}}, \quad \alpha_s = \frac{\beta}{\alpha},$$

and proceed in the same fashion as before, with a loss of precision because $h(\alpha_s) \leq 2H$ and not H. This is the reason for the factor 2 in the final result of Theorem 1.4, and has been taken into account in [Győry 2019, end of proof of Theorem 1].

For the second part of Theorem 1.4, we can play the same game, by defining S' the set of places S deprived of its two prime ideals with largest norm. The same elimination work as before will then give $w \in S'$ and $P \in E$ satisfying (4-1), and from there we can apply for $P = \alpha x$ the exact method of [Győry and Yu 2006, page 25] with a prime ideal $\mathfrak{p}$ from S'. This finally leads to the same estimates with P'_S instead of P_S, using again Lemma 4.1. ☐

5. A general framework for integrality methods

This section is motivated by comments from the referees asking for comparisons with other results, which inspired the author to present an attempt at conceptualizing more closely the current approaches.

The formalism presented here does not bring anything completely new in this regard but allows to understand many versions of Runge's or Baker's method simultaneously, and sheds some light on how they can possibly be combined. There is also some degree of equivalence with preexisting statements in the literature, which we will try to emphasize.

5A. ***Graph-theoretic definitions.*** The context (C) is still the same as before:

- X is a (smooth, to simplify) projective variety over the number field K.
- $D_1, \dots, D_n$ are reduced effective pairwise distinct divisors on X.

We will interpret everything in terms of a directed graph, which motivates the following definitions.

Definition 5.1.
- A *descending directed graph* is a directed graph $\mathcal{G}$ with finitely many vertices and edges such that:
  - Every vertex $\mathfrak{v} \in V$ is given a *depth* $d_{\mathfrak{v}} \in \mathbb{N}$.
  - If there is an edge $\mathfrak{v}' \mapsto \mathfrak{v}$, $d_{\mathfrak{v}'} < d_{\mathfrak{v}}$ (and we then say that $\mathfrak{v}$ is a child of $\mathfrak{v}'$). If $\mathfrak{v}'$ has no children, it is called *extremal*.
- For a given vertex $\mathfrak{v}$, the *cone of ancestors* of (we will also say *originating in*) $\mathfrak{v}$ is defined as

$$\mathcal{C}_{\mathfrak{v}} = \{\mathfrak{v}\} \cup \{\mathfrak{v}', \exists \text{ path } \mathfrak{v}' \mapsto \cdots \mapsto \mathfrak{v}\},$$

  and its *depth* is also defined as $d_{\mathfrak{v}}$.
- A family of cones of ancestors $(\mathcal{C}_{\mathfrak{v}})_{\mathfrak{v}}$ *spans the graph in depth 1* when the union of those cones contains all vertices of depth 1. If it does not, its *remainder* $\mathcal{R}$ is the set of unspanned vertices of depth 1.

Such a graph is particularly easy to draw and arrange by rows with fixed depth, hence its name.

Definition 5.2 (intersection graph). The *intersection graph* $\mathcal{G}$ of $X, D_1, \dots, D_n$ is the descending directed graph defined as such:

- For every subset $I \subset \{1, \dots, n\}$, we define

$$Z_I := \bigcap_{i \in I} \operatorname{Supp}(D_i)$$

and say that I is *optimal* if $Z_I \neq \varnothing$ and there is no $I' \supsetneq I$ such that $Z_{I'} = Z_I$, in other words no $i \notin I$ such that $Z_I \subset \operatorname{Supp}(D_i)$. In this way, every nonempty set-theoretic intersection of the divisors corresponds to a unique optimal set of indices

• The vertices $\mathfrak{v}$ of $\mathcal{G}$ are indexed by the optimal sets of indices. We thus can associate to each $\mathfrak{v}$ its optimal set $I_{\mathfrak{v}}$ and $Z_{\mathfrak{v}} := Z_{I_{\mathfrak{v}}}$. The depth of $\mathfrak{v}$ is defined as $|I_{\mathfrak{v}}|$, in other words the maximal number of divisors whose intersection defines $Z_{\mathfrak{v}}$. In particular, the unique vertex of depth 0 corresponds to X and the vertices of depth 1 correspond bijectively to the divisors $D_1, \ldots, D_n$ (unless one divisor contains another which we can assume to never hold).

• There is a directed arrow from $\mathfrak{v}'$ to $\mathfrak{v}$ if and only if $Z_{\mathfrak{v}'} \supsetneq Z_{\mathfrak{v}}$ with no intermediary set $Z_{\mathfrak{v}''}$, which by our construction is equivalent to saying that $I_{\mathfrak{v}'} \subsetneq I_{\mathfrak{v}}$ with no intermediary optimal set of indices (this is mainly to reduce the number of arrows in the graph). Considering a vertex $\mathfrak{v}$, the ancestors of depth 1 of $\mathfrak{v}$ correspond to the divisors D_i containing $Z_{\mathfrak{v}}$.

• Extremal vertices correspond to minimal nonempty intersections of the divisors (and optimal sets of indices which are maximal for the inclusion).

Remark 5.3. The number m in Runge's method [Levin 2008, Theorem 4] is exactly the maximal depth of a vertex in $\mathcal{G}$. When the divisors D_i are ample and in general position and X is of dimension d, the graph is particularly simple: for every $i \leq d$, it has $\binom{n}{i}$ edges of depth i, and the maximal depth is d. Notice also that the finite sets are exactly the ones of depth d in this case. This ideal situation would not need the formalism of the graph above, but our purpose is precisely to deal with the situations where m is larger than it should be.

5B. ***Runge and Baker methods for integrality and their variations.*** To proceed with integrality, the intuition is that for a point $P \in (X \setminus D)(\mathcal{O}_{L,S})$, for every place $v \in S$, one considers the set of closed subsets Z_I which are v-close to S. If the notion of closeness is defined rigorously and consistently (which we postpone until later), the following property $(\mathcal{P})$ holds:

> *For every place v and every point $P \in X(K_v)$, the set of vertices $\mathfrak{v}$ such that P is v-close to $Z_{\mathfrak{v}}$ is the cone of ancestors of a vertex $\mathfrak{v}_o$.*

Notice that we can also do the same with a point P in $X(K_v)$ and look at the $\mathfrak{v}$ for which $Z_{\mathfrak{v}}$ contains P (instead of merely being close to it), and we again get cones of ancestors. More precisely, approximating P v-adically by points $P_v \in X(K_v)$, we can obtain by compactness (when P_v is close enough) this same cone as the set of vertices $\mathfrak{v}$ such that $Z_{\mathfrak{v}}$ is v-close to P_v.

Let us now, as before, fix (L, S) and $P \in (X \setminus D)(\mathcal{O}_{L,S})$. To each $v \in S$ we thus associate a cone of ancestors $\mathcal{C}(P, v)$ corresponding to P seen in $X(K_v)$.

It now turns out that Runge and Baker's method can each be applied under conditions on those cones of ancestors (and a geometric condition on their remainder) in a very straightforward way, which we call terminal conditions (because if they apply, we can finish the analysis via computations quite independent from the graph then).

For each case, we provide an hypothesis which would make termination conditions automatically realized.

We start with the classical versions of Runge and Baker's method:

(1) (a) Termination for Runge. The cones of ancestors $\mathcal{C}(P, v)$, $v \in S$ do not span the intersection graph in depth 1 and the remainder $\mathcal{R}$ satisfies the geometric condition $(G)_R$:

The sum $D_{\mathcal{R}}$ of the divisors $D_{\mathfrak{v}}$, $\mathfrak{v} \in \mathcal{R}$ is ample (resp. big, of positive Kodaira–Iitaka dimension).

By construction, every divisor $D_{\mathfrak{v}}$ with $\mathfrak{v} \in \mathcal{R}$ is v-far from P for every place $v \in S$, and the global height $h_{D_{\mathcal{R}}}(P)$ is thus sufficiently controlled to ensure finiteness (resp. finiteness outside of an explicit proper algebraic subset independent of (L, S), non-Zariski density) by $(G)_R$.

(b) Automatic guarantee for Runge. $|S| = s$, and a family of s cones of ancestors (of extremal vertices) cannot span the graph in depth 1, and every divisor D_i is ample (resp. big, of positive dimension)

It is sufficient here (and more convenient for computations) to prove this for cones of extremal vertices because otherwise one might take a strictly larger cone, which would thus span at least the same depth 1 vertices.

Remark 5.4. Let us first compare it with the original higher-dimensional version of Runge's method, found in Theorem 4 of [Levin 2008]. In this context, recall that the number m is the maximal depth of the graph, and by definition of the intersection graph, a vertex $\mathfrak{v}$ of the intersection graph has exactly $d_{\mathfrak{v}}$ ancestors of depth 1. If $ms < n$, then any given s cones of ancestors cannot span the graph in depth 1, which means that $ms < n$ implies the automatic guarantee for Runge (in the three cases ample, big, positive Kodaira–Iitaka dimension of that theorem).

Now, this statement (for the case of positive Kodaira–Iitaka dimension) is in fact equivalent to Proposition 4.2 of [Corvaja et al. 2015], which formulates it in terms of the maximal number s of points $P_1, \dots, P_s$ such that the sum of divisors D_i avoiding all of them make up a positive-dimensional divisor. This is exactly what $D_{\mathcal{R}}$ is meant to be above, hence the equivalence. One can also see with this graph-theoretic interpretation that the worst-case scenario is when each P_i belongs to a $Z_{\mathfrak{v}}$ where $\mathfrak{v}$ is an extremal vertex. One difficulty regarding the divisors dealt with in [Corvaja et al. 2015] is that they are not one by one of positive Kodaira dimension, and there is no obvious way to choose positive-dimensional sums of those divisors for which a classical Runge condition $ms < n$ can be applied, which is another reason why the improvement brought there by Proposition 4.2 is needed for the end result of that paper.

We give below a concrete example of application of this weaker Runge condition for a Siegel modular variety, following [Le Fourn 2019].

Proposition 5.5. *There is an absolute effectively computable constant $C > 0$ such that the following holds for any pair (K, S) with K quadratic and $S \supset M_K^\infty$ with $|S| = 2$.*

Let A be a principally polarized abelian surface over K with $A[2] \subset A(K)$ and such that the semistable reduction of A is always the jacobian of a smooth curve of genus 2 except at the finite places of S. Then, the stable Faltings height of A satisfies $h_{\mathcal{F}}(A) \leq C$. In particular, the set of all such abelian surfaces A up to isomorphism is effectively finite.

Proof. This amounts to a problem of $\mathcal{O}_{K,S}$-integral points on $A_2(2)^{\mathrm{Sa}} \setminus D$ when $|S| \leq 2$, where $A_2(2)^{\mathrm{Sa}}$ is the Satake compactification of the Siegel modular variety $A_2(2)$ of degree 2 and level 2 and D is the union of the ten divisors of moduli of products of elliptic curves. For all details, we refer to the author's previous work in [Le Fourn 2019, Section 8] with only some reminders here (every claimed fact with no reference can be found there). On this variety, the ten even theta constants define (as divisors of zeroes) ten divisors $D_1, \dots, D_{10}$ whose union D is exactly the boundary $\partial A_2(2) := A_2(2)^{\mathrm{Sa}} \setminus A_2(2)$ of the compactification together with the locus of products of elliptic curves. Furthermore, the fourth powers of these theta constants define an embedding

$$\psi : A_2(2)^{\mathrm{Sa}} \to \mathbb{P}^9,$$

for which the equations of the image are, with canonical ordering of the coordinates in $\mathbb{P}^9$:

$$x_1 - x_2 - x_6 - x_9 = 0 \tag{5-1}$$

$$x_1 - x_4 - x_5 - x_8 = 0 \tag{5-2}$$

$$x_2 - x_3 + x_5 - x_7 = 0 \tag{5-3}$$

$$x_3 - x_4 - x_6 + x_{10} = 0 \tag{5-4}$$

$$x_7 - x_8 - x_9 + x_{10} = 0 \tag{5-5}$$

$$\left(\sum_{i=1}^{10} x_i^2\right)^2 - 4\sum_{i=1}^{10} x_i^4 = 0, \tag{5-6}$$

which can also be seen as a quartic in $\mathbb{P}^4$ [Igusa 1964a, page 397]. The inverse image of the coordinate hyperplanes in $A_2(2)$ is thus the locus of moduli of products of elliptic curves. This description also holds for the semistable reduction modulo a prime $\mathfrak{P}$ of $\mathcal{O}_K$ of a point $P \in A_2(2)(K)$, in the sense that the reduction of the image by ψ of the modulus of a point given by A lands inside a coordinate hyperplane if and only if the semistable reduction of A is multiplicative, or a product of elliptic curves. This is true only up to some $M_{\mathbb{Q}}$-constant error, in particular the case of primes above 2 adds a lot of technical subtleties (see [Le Fourn 2019, paragraphs 8B to 8D]) so we do not try to obtain the constant C explicitly here as new computations would be needed (depending on the type of reduction of a curve in Théorème 1 of [Liu 1993]). By moduli arguments and the given equations for $\operatorname{Im}\psi$, one sees that a point Q in $\operatorname{Im}\psi$ has at most one zero coordinate if $\psi^{-1}(Q) \in A_2(2)$, and at most 6 if $\psi^{-1}(Q) \in \partial A_2(2)$. This gave the strong Runge condition $6|S| < 10$, hence in turn S had to be reduced to the unique archimedean place.

The termination condition for Runge written above will allow us to do a bit better; indeed, every D_i is ample and thus it is enough to ensure that the remainder is nonempty. To do this, it is possible to describe completely the intersection graph of the divisors D_i by first looking for the optimal sets of indices and then study the graph itself. Every time we "compute with equations" below, it means we proceed manually with the equations for $\operatorname{Im}\psi$ given above (and it will always be straightforward).

We recall from [Igusa 1964b, page 227] or [Streng 2010] that the canonical action of $\mathrm{Sp}_4(\mathbb{Z})$ on $A_2(2)^{\mathrm{Sa}}$ is transitive on their divisors D_i because the action on the ten fourth powers of theta constants is up to multiplication factors. Even better, this action is actually 2-transitive, which allows us to say that all

45 sets of indices $I \subset \{1, \dots, 10\}$ with $|I| = 2$ are optimal (it is enough to check it for one of them). Notice that as for all arguments below, it amounts to say that assuming that if two of the x_i are zero, the equations above do not imply that another one is. Now, the 120 sets of indices with $|I| = 3$ are of two different types: they can be *syzygous* or *azygous* [Igusa 1964a, page 403], each case happening 60 times, and the action of $\mathrm{Sp}_4(\mathbb{Z})$ is transitive on syzygous (resp. azygous) unordered triples so again it is enough to see what happens for one of each type. The syzygous triples are optimal by computing the equations (and define a nonirreducible curve), whereas for any azygous triple I, there is a unique $j \notin I$ such that $Z_I \subset D_j$ and then $\tilde{I} = I \cup \{j\}$ completes I and is optimal, and $Z_{\tilde{I}}$ is irreducible.

Now, for each syzygous triple I, there is a unique $j \notin I$ completing it into a "Göpel quadruple" (quadruple such that every triple in it is syzygous), and it defines an empty intersection (again using the equations and transitivity). For any other j, the set of indices is not optimal and one needs to add two more indices. In other words, the vertex associated to a syzygous triple $\mathfrak{v}'$ is the starting point of two arrows corresponding to a unique partition in triples of the remaining 6 indices, and for each child $\mathfrak{v}$, the set $Z_{\mathfrak{v}}$ is reduced to a point.

For each completion of an azygous triple, the 6 remaining indices split into three pairs who each define an extremal vertex, and a point again.

To sum up the situation for the intersection graph, there are:

- 10 vertices of depth 1, defining divisors, each with 9 children of depth 2.
- 45 vertices of depth 2, defining nonirreducible curves in the boundary of $A_2(2)$, each with 4 children of depth 3 and 2 children of depth 4.
- 60 vertices of depth 3, corresponding to the syzygous triples, defining curves with two components in the boundary, and each having 2 children of depth 6.
- 15 vertices of depth 4 corresponding to azygous quadruples [van der Geer 1982, page 337], defining irreducible curves in the boundary, and each having 3 children of depth 6.
- 15 vertices of depth 6, the extremal ones, corresponding to complements of Göpel quadruples, each defining a unique point in the boundary.

Now, given the explicit list of the Göpel quadruples, it is easy to see that two of them always have nonempty intersection. This means that the cones of ancestors of any two extremal vertices cannot span the graph in depth 1. Therefore, if we fix $|S| = 2$, there is a remaining divisor D_i which is v-far away from our integral point P at every place by our hypotheses, hence giving an absolute bound on the height. □

(2) (a) Termination for Baker. Either the condition of termination for Runge holds, or one of the cones of ancestors $\mathcal{C}(P, v)$ comes from a $Z_{\mathfrak{v}}$ which is finite *and* its remainder $\mathcal{R}_{\mathfrak{v}}$ (with respect to only this cone) satisfies the following condition $(G)_B$:

> *There is a nonconstant rational function ϕ on $\overline{K}(X)$ with support in $\mathcal{R}_{\mathfrak{v}}$ (which is thus disjoint from $Z_{\mathfrak{v}}$).*

We then obtain a bound on the height of points of $(X \setminus D)(\mathcal{O}_{L,S})$ depending on (L, S) and outside a fixed effectively computable proper subvariety of X.

(b) Automatic guarantee for Baker. With the same notations, s cones of ancestors (none of them giving a finite $Z_{\mathfrak{v}}$) cannot span the graph in depth 1, the D_i are all ample *and* $(B)_{\text{rank}}$ the rank of the group of principal divisors with support in $\bigcup_{i=1}^n D_i$ is larger than the depth of any vertex $\mathfrak{v}$ with $Z_{\mathfrak{v}}$ finite.

Remark 5.6. Again, the condition $(m_{\mathcal{B}} - 1)s < n$ in [Levin 2014, Theorem 1] together with $(G)_{\mathcal{B}}$ imply the automatic guarantee; indeed, for every edge $\mathfrak{v}$ of depth at least $m_{\mathcal{B}}$, the set $Z_{\mathfrak{v}}$ is finite by definition, so if we restrict to the other cones of ancestors, each vertex is below at most $m_{\mathcal{B}} - 1$ divisors and a pigeonhole principle applies again.

Another remark that might deserve to be pointed out is that $(G)_{\mathcal{B}}$ is made to be able to send any point of $Z_{\mathfrak{v}}$ to 1 via one of finitely many rational functions ϕ, and thus go back to the 1-dimension situation of Baker's method. It turns out, following the proof of Levin (especially [Levin 2014, Lemma 10]) that one can do the same process with nonfinite $Z_{\mathfrak{v}}$ assuming that *for each of the irreducible components of $Z_{\mathfrak{v}}$, there is nonconstant ϕ with support in $\mathcal{R}_v$ sending it to a single point*. This hypothesis is of course very strong (even more so than $(G)_{\mathcal{B}}$ itself), but it can sometimes be satisfied. Indeed, if X is embedded in $\mathbb{P}^n$ in such a way that the D_i become the coordinate hyperplanes and $\mathcal{I}_X$ is the homogeneous ideal of definition of X in $\mathbb{P}^n$, this hypothesis is true for a given $I_{\mathfrak{v}}$ if there are disjoint sets of indices I, J (disjoint with $I_{\mathfrak{v}}$) such that there is a nonzero homogenous polynomial

$$P_I - P_J \in (\mathcal{I}_X, x_i \ (i \in I_{\mathfrak{v}}))$$

where P_I (resp. P_J) is a monomial in the x_i, $i \in I$ (resp. $i \in J$). If we go back to the case of $A_2(2)$ as in Proposition 5.5, using again the equations, we prove easily that for any optimal set $I_{\mathfrak{v}}$ with $|I_{\mathfrak{v}}| \geq 2$, one can find such $j, k \notin I_{\mathfrak{v}}$ such that $x_j - x_k$ or $x_j + x_k$ belong to $(\mathcal{I}_X, x_i \ (i \in I_{\mathfrak{v}}))$: for example, taking $I_{\mathfrak{v}} = \{1, 2\}$, the first equation gives $x_6 - x_9$. For the syzygous triple $\{1, 2, 3\}$, the difference $x_6 - x_8$ works, and for the azygous quadruple $\{1, 2, 6, 9\}$, the sum $x_4 + x_8$ works. All this implies that one can in fact apply the termination condition for Baker's method as soon as one cone of ancestors is of depth at least 2! This allows us to modify this condition which in practice amounts to considering that $m_{\mathcal{B}} = 2$, to obtain the following statement:

Proposition 5.7. *For any pair (L, S), with $M_L^\infty \subset S$ and $|S| < 10$, there is an effectively computable bound $C(L, S)$ such that the following holds*:

For a principally polarized abelian surface A defined over L with $A[2] \subset A(L)$, if the semistable reduction of A is the jacobian of a smooth curve at every prime except maybe the ones in S, then $h_{\mathcal{F}}(A) \leq C(L, S)$ unless some two theta coordinates of a point of $A_2(2)$ (seen in $\mathbb{P}^9$) representing A are equal or opposite.

The equality up to sign of those theta coordinates comes from the need to take out an exceptional subset (where the auxiliary rational functions, which are exactly the functions $(x_j/x_i) \circ \psi$ here, can be equal to 1) already appearing in [Levin 2014] for the same reasons.

We will now consider variants of those methods, the tubular ones devised by the author and then the "reductions" devised by Levin [2018], and compare them.

Let us add to the context (C) an integer $m_Y \geq 1$ and the corresponding closed subset Y as in Theorem 1.1: notice that it amounts to drawing an horizontal line between depths m_Y and $m_Y + 1$. We denote by $\mathcal{G}_Y$ the part of the graph below this line, and consider triples (L, S, S') with $M_L^\infty \subset S' \subset S$ finite, $s = |S|$, $s' = |S'|$, and points $P \in (X \setminus D)(\mathcal{O}_{L,S}) \cap (X \setminus Y)(\mathcal{O}_{L,S'})$. Of course, when $Y = \varnothing$, this intersection is the set $(X \setminus D)(\mathcal{O}_{L,S})$ again.

(3) (a) Termination for tubular Runge [Le Fourn 2019, Theorem 5.1]. Either one of the $s - s'$ cones of ancestors $\mathcal{C}(P, v)$, $v \in S \setminus S'$ originates in $\mathcal{G}_Y$ or the s cones do not span $\mathcal{G}$ in depth 1 and the termination condition for Runge holds.

(b) Automatic guarantee for tubular Runge. When we consider s cones of ancestors with at most s' of them of depth $> m_Y$, they do not span the graph in depth 1 and each divisor is ample (resp. big, of positive dimension).

Remark 5.8. Following the now usual ideas, this guarantee holds when $m \cdot s' + m_Y(s - s') < n$, which is exactly equation (4) of [Le Fourn 2019]. Notice also that in this context, as a byproduct of Runge's strategy and the tools used here, the bound on the height is still uniform.

Notice that one cannot improve upon Proposition 5.5 in this tubular context, because when $s = s' = 2$, there can be only one divisor in the remainder (see the proof) and so no room for increasing s.

(4) (a) Termination for tubular Baker. Either the termination condition for tubular Runge with m_Y and $\mathcal{G}_Y$ holds, or one of the s' cones of ancestors coming from S' originates in a finite $Z_{\mathfrak{v}}$ and the remainder of this cone satisfies $(G)_B$.

(b) Automatic guarantee for tubular Baker. One cannot span the graph in depth 1 with s cones of ancestors when $s - s'$ of them originate outside $\mathcal{G}_Y$ and the s' other ones do not originate at a finite $Z_{\mathfrak{v}}$, and the condition $(B)_{\mathrm{rank}}$ holds.

Remark 5.9. Again, this guarantee is ensured whenever $(m_{\mathcal{B}} - 1)s' + m_Y(s - s') < n$ under the hypothesis $(H)_P$ of Theorem 1.1, where we recall that $m_{\mathcal{B}}$ is the depth starting from which every $Z_{\mathfrak{v}}$ is finite. As a byproduct of the proof made here, each case of the termination condition is uniform in the choice of (L, S, S') except the one where one cone from S' originates from a finite $Z_{\mathfrak{v}}$.

In the case $Y = \varnothing$, which amounts to $m_Y = m$ as in Runge's method, Baker's method applies when s, s' are such that $s - s'$ cones of ancestors (of any possible depth) together with s' ones not originating from finite $Z_{\mathfrak{v}}$'s cannot span the graph in depth 1. If we define for $s - s'$, $n_{s-s'}$ the minimal cardinality of the remainder of $s - s'$ cones of ancestors, this automatically holds then when $(m_{\mathcal{B}} - 1)s' < n_{s-s'}$.

For results from [Levin 2018, especially Theorem 5.15], we are yet in another context (we modified slightly the notations there for consistency here): S_0 is a fixed set of places of K, $S_{L,0}$ the set of places above S_0 in L of cardinality s' and we consider (L, S) and s such that $s = |S|$ this time.

(5) (a) Levin's reduction. The $s-s'$ cones of ancestors coming from $S \setminus S_{L,0}$ do not span the graph in depth 1, and the remainder $\mathcal{R}$ is such that a further method could be applied to its intersection graph, i.e., to a set of $S_{L,0}$-integral points on $(X \setminus D_{\mathcal{R}})$.

(b) Automatic guarantee for Levin's reduction. Any union of $s-s'$ cones of ancestors leave a remainder large enough to apply another method to its graph.

Remark 5.10. The big gain in this reduction is uniformity in the choice of $S \setminus S_{L,0}$. Of course, when S_0 is empty, one retrieves exactly Runge's method in higher dimension.

Notice the link with the case $Y = \varnothing$ of tubular Baker above: on the graph-theoretic side, the condition of reduction of Levin (and then the ability to apply Baker's condition) are very close, in particular

$$(m_{\mathcal{B}} - 1)s' + m(s - s')$$

implies both. The statements are not completely equivalent though: when Levin's reduction applies, it provides, as stated, uniformity in the choice of $S \setminus S_{L,0}$, which in practical estimates will give a bound only depending on the primes of S_0, $h_L R_L$ and $[L : \mathbb{Q}]$. By comparison, one can see in (1-2) that our tubular Baker estimates do depend on the primes of S and of the regulator of S in particular.

On the other hand, there are cases where Levin's reduction cannot be applied for effective results, starting with the S-unit equation: one cannot eliminate even one point and still satisfy the hypothesis $(G)_B$.

It thus seems that the best way to proceed (in the case $Y = \varnothing$, i.e., when one does not want to add another hypothesis of integrality) is to figure out the maximal number of cones of ancestors one can choose so that their remainder contains the support of a nonconstant rational function (which is, as we repeat, a *uniform* process of reduction) and then apply our tubular Baker method.

Notice that in the case of curves, Levin [2018, Theorems 7.19 and 7.20] uses reduction to respectively Siegel's theorem for $g \geq 1$ (thus ineffective) and Thue–Mahler equations for $\mathbb{P}^1$. As for Theorem 7.22 of [loc. cit.], the number s is exactly the number $s - s'$ mentioned above for which one can take this many cones of ancestors and still ensure there is a rational function with support in $D_{\mathcal{R}}$.

Finally, Levin's reduction and our tubular Baker approach can naturally be combined, making full use of the notion of tubular neighborhoods in [Le Fourn 2019]. In other words, start with $P \in (X \setminus D)(\mathcal{O}_{L,S})$ which is v-far away from Y at every $v \notin S'$. We then reproduce Levin's reduction for the places $v \in S \setminus S'$, which generate cone of ancestors of depth at most m_Y by hypothesis on P, and we are allowed to consider $|S \setminus S'|$ of them as long as for the remainder of their union, the associated graph satisfies $(G)_B$ with s' cones of ancestors. The end result will then be, as Levin's reduction, only depending on s, S' and L.

An example of application, not undertaken here, would be our pet example $A_2(2)^{\mathrm{Sa}}$ with Y being the boundary. We would obtain more uniformity than in Proposition 5.7 but at the cost of an hypothesis of potentially good reduction of the abelian surface at every prime except a bounded number of them.

5C. *Formalizing the closeness condition.* To prove rigorously all previous statements, one needs, as explained in the previous subsection, a rigorous definition of v-closeness that satisfies $\mathcal{P}$. Let us start with the divisors (i.e., depth 1 vertices). A first look gives us two rather different possibilities:

(1) For Runge's method, one wants to end up with one of the divisors D_i such that $h_{D_i,v}(P) \geq c_v$ for all $v \in M_K$, for an explicit M_K-constant $(c_v)_v$.

It is thus natural to define the v-closeness of P to $Z_{\mathfrak{v}}$ as $h_{Z_{\mathfrak{v}},v}(P) \geq c_{\mathfrak{v},v}$ for an M_K-constant deduced from the one above, such that the v-closeness is stable by intersection. Such M_K-constants exists by Proposition 2.2(a), and all arguments hold in this case.

(2) For Baker's method, given that we want a bound of the type $h \ll \log^* h$ for some global height at the end (see Section 3), we want to define v-closeness rather as something of the shape

$$h_{D_i,v}(P) > \lambda_i \cdot h_{D_i}(P).$$

(in fact any function of the height against which the logarithm would be dominated would suffice, but we always choose it linear here).

Again, we can make this v-closeness property inheritable thanks to Proposition 2.2(a), and v-closeness is then given by

$$h_{Z_{\mathfrak{v}},v}(P) > \lambda_{\mathfrak{v}} \cdot \min_{i \in \mathcal{C}(\mathfrak{v})} h_{D_i}(P),$$

where i goes through all ancestors of $\mathfrak{v}$ of depth 1 and $\lambda_{\mathfrak{v}} > 0$ is a well-chosen value in terms of the λ_i. Notice also that to apply Baker's method (and not fall back to the Runge case), one needs to ensure that every v does create a cone of ancestors, hopefully deep enough. The best way to guarantee that is fixing $\lambda_i < 1/|S|$ because the sum of $h_{D_i,v}(P)$ for $v \in S$ is almost $h_{D_i}(P)$ for our integral points.

To combine the principles behind the proofs above, one needs a consistent definition of closeness that fits both definitions. The most natural way to do this is

$$h_{D_i,v}(P) > \lambda_i \cdot h_{D_i}(P) + c_v$$

and deduced estimates for the $h_{Z_{\mathfrak{v}},v}(P)$. In passing, one can remark that the classical Runge condition of closeness is in fact too strong: a linear condition as above would also be amenable to the (classical) Runge method as long as $|S|\lambda_i = 1 - \varepsilon$, $\varepsilon > 0$, and the obtained bounds would then depend on $1/\varepsilon$.

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samuel.le-fourn@univ-grenoble-alpes.fr *Institut Fourier, Université Grenoble Alpes, CNRS, Grenoble, France*

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