

Algebra & Number Theory

Volume 14

2020

No. 4

On the motivic class of an algebraic group

Federico Scavia



On the motivic class of an algebraic group

Federico Scavia

Let F be a field of characteristic zero admitting a biquadratic field extension. We give an example of a torus G over F whose classifying stack BG is stably rational and such that $\{BG\} \neq \{G\}^{-1}$ in the Grothendieck ring of algebraic stacks over F . We also give an example of a finite étale group scheme A over F such that BA is stably rational and $\{BA\} \neq 1$.

1. Introduction

Let F be a field. The Grothendieck ring of algebraic stacks $K_0(\text{Stacks}_F)$ was introduced by Ekedahl [2009b], following up on earlier works [Behrend and Dhillon 2007; Joyce 2007; Toën 2005]. It is a variant of the Grothendieck ring of varieties $K_0(\text{Var}_F)$. By definition, $K_0(\text{Stacks}_F)$ is generated as an abelian group by the equivalence classes $\{X\}$ of all algebraic stacks X of finite type over F with affine stabilizers. These classes are subject to the scissor relations $\{X\} = \{Y\} + \{X \setminus Y\}$ for every closed substack $Y \subseteq X$, and the relations $\{E\} = \{\mathbb{A}^n \times X\}$ for every vector bundle E of rank n over X . The product is defined by $\{X\} \cdot \{Y\} := \{X \times Y\}$, and extended by linearity.

Given a group scheme G over F , we may consider the class $\{BG\}$ of its classifying stack in $K_0(\text{Stacks}_F)$. The problem of computing $\{BG\}$ appears to be related to the problem of the stable rationality of BG , although no direct implications are known. Recall that BG is stably rational if for one (equivalently, every) generically free representation V of G , the rational quotient V/G is stably rational. An equivalent terminology is that the Noether problem for stable rationality has a positive solution for G ; see [Florence and Reichstein 2018, Section 3]. The case of a finite (constant) group G was considered in [Ekedahl 2009a]: it frequently happens that $\{BG\} = 1$ (notably for the symmetric groups, see [loc. cit., Theorem 4.3]), although there are examples of finite groups G for which $\{BG\} \neq 1$; see [loc. cit., Corollaries 5.2 and 5.8]. Further work on the triviality of $\{BG\}$ for finite groups G has been done in [Martino 2016; 2017]. So far, all the known examples of finite group schemes G for which $\{BG\} \neq 1$ are such that BG is not stably rational. This suggests the following question.

Question 1.1 (see [Ekedahl 2009a, Section 6]). Is it true that, for a finite group scheme G , the following two conditions are equivalent?

- BG is stably rational.
- $\{BG\} = 1$ in $K_0(\text{Stacks}_F)$.

MSC2010: primary 14L15; secondary 14D23.

Keywords: motivic class, Grothendieck ring of stacks, classifying stack, algebraic torus.

We will answer [Question 1.1](#) in the negative in [Theorem 1.6](#).

Now let G be a connected linear algebraic group. Recall that G is special if every G -torsor is Zariski-locally trivial. For example, GL_n , SL_n and Sp_{2n} are special; see [\[Serre 1958\]](#). It was shown by Ekedahl that if $P \rightarrow S$ is a torsor under the special group G , then $\{P\} = \{G\}\{S\}$. This is immediate if S is a scheme, but less obvious when S is a stack; see [\[Bergh 2016, Corollary 2.4\]](#). Applying this to the universal G -torsor $\mathrm{Spec} F \rightarrow BG$, one obtains $\{BG\}\{G\} = 1$.

The equality $\{BG\} = \{G\}^{-1}$ appears to be the analogue for connected groups of the relation $\{BG\} = 1$ for finite group schemes. In [\[Bergh 2016\]](#), these equalities are referred to as *expected class formulas*, and there is a sense in which they are “almost” true. Ekedahl [\[2009b, Section 2\]](#) defines a generalized Euler characteristic

$$\chi_c : K_0(\mathrm{Stacks}_F) \rightarrow K_0(\mathrm{Coh}_F)$$

taking values in a Grothendieck ring $K_0(\mathrm{Coh}_F)$ of Galois representations over F . If G is a finite group scheme, the equality $\chi_c(\{BG\}) = 1$ always holds [\[Ekedahl 2009a, Proposition 3.1\]](#). On the other hand, if G is connected, then $\chi_c(\{BG\}\{G\}) = 1$; see [\[Bergh 2016, Section 2.2\]](#). Since $\{BG\} \neq 1$ for some finite groups G , the following question naturally arises:

Question 1.2. Let F be a field. Is it true that

$$\{BG\} = \{G\}^{-1} \tag{1.3}$$

in $K_0(\mathrm{Stacks}_F)$ for every connected group G ?

In [Theorem 1.5](#), we show that the answer to [Question 1.2](#) is also negative. Computations for nonspecial G have been carried out for PGL_2 and PGL_3 in [\[Bergh 2016\]](#), for SO_n and n odd in [\[Dhillon and Young 2016\]](#), for SO_n and n even (and O_n for any n) in [\[Talpo and Vistoli 2017\]](#), and for Spin_7 , Spin_8 and G_2 in [\[Pirisi and Talpo 2019\]](#). In each of these cases, (1.3) was found to be true. The expectation was that, for a connected linear algebraic group G over a field F of characteristic 0, [Question 1.4](#) below should have an affirmative answer. If F is an algebraically closed field, then there are no examples of connected G where BG is known not to be stably rational. If F is not assumed to be algebraically closed, then such examples exist. The following variant of [Question 1.1](#) seems natural in this context.

Question 1.4 (see [\[Talpo and Vistoli 2017, Section 1\]](#) and [\[Pirisi and Talpo 2019, Remark 4.1\]](#)). Is it true that, for a connected linear algebraic group G , the following two conditions are equivalent?

- BG is stably rational.
- $\{BG\} = \{G\}^{-1}$ in $K_0(\mathrm{Stacks}_F)$.

Our first result gives a negative answer to [Questions 1.2](#) and [1.4](#).

Theorem 1.5. *Let F be a field of characteristic zero which admits a biquadratic field extension K , let E_1 and E_2 be two distinct quadratic subextensions of K/F , and set $G := R_{E_1 \times E_2/F}^{(1)}(\mathbb{G}_m)$. Then*

- (a) BG is stably rational, and
- (b) $\{BG\} \neq \{G\}^{-1}$ in $K_0(\text{Stacks}_F)$.

The torus G is an example of a norm-one torus; see [Section 2](#) for the definition. It follows from [Theorem 1.5](#) that counterexamples H to (1.3) exist in any dimension $\dim H \geq 3$: consider for example $H := G \times \mathbb{G}_m^r$ for $r \geq 0$.

The key ingredient in the proof of [Theorem 1.5](#) is the *refined Euler characteristic* of Ekedahl, introduced in [\[Ekedahl 2009b, Sections 6 and 3\]](#); see [Section 4](#).

Our second result gives a negative answer to [Question 1.1](#).

Theorem 1.6. *Let F be a field of characteristic zero which admits a biquadratic field extension K , and let E_1 and E_2 be two distinct quadratic subextensions of K/F . Define $G := R_{E_1 \times E_2/F}^{(1)}(\mathbb{G}_m)$, and let $A := G[2]$ be the 2-torsion subgroup of G . Then*

- (a) BA is stably rational, and
- (b) $\{BA\} \neq 1$ in $K_0(\text{Stacks}_F)$.

Questions [1.1](#), [1.2](#) and [1.4](#) remain open in the case, where the base field F is assumed to be algebraically closed. Our arguments do not shed any new light in this setting.

The remainder of this paper is structured as follows. In [Section 2](#) we review well known computations of motivic classes for nonsplit tori. In [Section 3](#) we obtain explicit formulas for the motivic classes of G and BG , and in [Section 4](#) we give the required background on the refined Euler characteristic. In [Section 5](#) we prove [Theorem 1.5](#), and in [Section 6](#) we prove [Theorem 1.6](#).

2. Preliminaries

Let F be a field. We will write \mathbb{L} for the class $\{\mathbb{A}^1\}$ in $K_0(\text{Var}_F)$ or $K_0(\text{Stacks}_F)$. If E is an étale algebra over F , we will denote by $\{E\}$ the class $\{\text{Spec } E\}$ in $K_0(\text{Var}_F)$ or $K_0(\text{Stacks}_F)$. If X is a quasiprojective scheme over E , we will denote by $R_{E/F}(X)$ the *Weil restriction* of X to F . By definition, for every F -scheme S one has $R_{E/F}(X)(S) = X(S_E)$. We refer the reader to [\[Voskresensky 1998, Section 3.12\]](#) for an account of the main properties of the Weil restriction.

Let G be a linear algebraic group over F , and $\alpha \in H^1(F, G)$ be represented by a G -torsor $P \rightarrow \text{Spec } F$. For every quasiprojective F -scheme Z , we denote by ${}^\alpha Z$ the *twist* of Z by P , that is,

$${}^\alpha Z := (Y \times P)/G,$$

where G acts diagonally. We refer the reader to [\[Florence 2008, Section 2\]](#) for the definition and the basic properties of the twisting operation.

We will write C_2 for the cyclic group of two elements and S_n for the symmetric group on n symbols. The following observations will be repeatedly used in the sequel.

Lemma 2.1. *Let X be a scheme over F , E an étale algebra of degree n over F , $\alpha \in H^1(F, S_n)$ the class corresponding to E/F .*

(a) *Let S_n act on the disjoint union $\coprod_{i=1}^n X$ by permuting the n copies of X . Then*

$$\alpha\left(\coprod_{i=1}^n X\right) \cong X_E.$$

(b) *Let S_n act on X^n by permuting the n factors. Then*

$$\alpha(X^n) \cong R_{E/F}(X).$$

Proof. (a) Let $Y := \coprod_{i=1}^n X$, and let S_n act on Y by permuting the copies of X . By definition,

$$\alpha Y = (Y \times \text{Spec } E)/S_n \cong (Y \times_X X_E)/S_n,$$

where S_n acts diagonally. This shows that αY is the twist of X_E by the trivial S_n -torsor $Y \rightarrow X$ in the category of X -schemes, which implies $\alpha Y \cong X_E$.

(b) See the bottom of page 5 in [Florence et al. 2017]. □

Lemma 2.2. *Let*

$$1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1$$

be an exact sequence of group schemes over F , and assume that G is special. Then

$$\{BN\} = \{H\}/\{G\}.$$

Proof. See [Bergh 2016, Proposition 2.9]. □

Let F_s be a separable closure of F . Recall that a group scheme T over F is called a *torus* if $T_{F_s} \cong \mathbb{G}_{m, F_s}^n$ for some $n \geq 0$. The *character lattice* of T is the finitely generated \mathbb{Z} -free $\text{Gal}(F)$ -module $\text{Hom}_{F_s}(T_{F_s}, \mathbb{G}_{m, F_s})$. The character lattice induces an antiequivalence between the category of F -tori and the category of $\text{Gal}(F)$ -lattices, i.e., finitely generated \mathbb{Z} -free continuous $\text{Gal}(F)$ -modules; see [Favi and Florence 2008, Section 2]. Similarly, for every separable finite extension L/F , we have an antiequivalence between $\text{Gal}(L/F)$ -lattices and F -tori T split by L , i.e., such that $T_L \cong \mathbb{G}_{m, L}^n$ for some $n \geq 0$. The *dual torus* of T is the torus T' whose character lattice is dual to that of T .

Let E be an étale algebra over F . If G is a group scheme over E , then $R_{E/F}(G)$ is a group scheme over F . The group $R_{E/F}(\mathbb{G}_m) := R_{E/F}(\mathbb{G}_{m, E})$ is an F -torus. Tori of this kind are called *quasisplit*. They are special groups, and they correspond to permutation $\text{Gal}(F)$ -lattices, that is, lattices admitting a \mathbb{Z} -basis that is permuted by $\text{Gal}(F)$; see [Voskresensky 1998, Section 3.12, Example 19].

Lemma 2.3. *Let T be an algebraic torus over F , and let T' be its dual. Assume that T is stably rational. Then:*

(a) *BT' is stably rational.*

(b) *$\{BT'\}\{T\} = 1$ in $K_0(\text{Stacks}_F)$.*

Proof. Since T is stably rational, by [Voskresensky 1998, Section 4.7, Theorem 2] there is a short exact sequence

$$1 \rightarrow T_1 \rightarrow T_2 \rightarrow T \rightarrow 1 \tag{2.4}$$

where T_1 and T_2 are quasisplit. Since quasisplit tori are isomorphic to their dual, the sequence dual to (2.4),

$$1 \rightarrow T' \rightarrow T_2 \rightarrow T_1 \rightarrow 1, \tag{2.5}$$

shows that T' embeds in T_2 . We may view T_2 as a maximal torus inside GL_n , where $n = \mathrm{rank} T_2$. This gives a faithful representation of T' with quotient birational to T_1 . Since quasisplit tori are rational, it follows that BT' is stably rational.

Quasisplit tori are special, so we may apply Lemma 2.2 to (2.4) and (2.5). We obtain $\{T\} = \{T_2\}/\{T_1\}$ and $\{BT'\} = \{T_1\}/\{T_2\}$, so $\{BT'\}\{T\} = 1$. □

Let E/F be an étale algebra, and let $R_{E/F}(\mathbb{G}_m)$ be the associated quasisplit torus. The kernel of the norm homomorphism $R_{E/F}(\mathbb{G}_m) \rightarrow \mathbb{G}_m$ is called a *norm-one torus*, and is denoted by $R_{E/F}^{(1)}(\mathbb{G}_m)$. Its dual torus is isomorphic to $R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m$.

Lemma 2.6. *Assume that $\mathrm{char} F \neq 2$. Let $E := F(\sqrt{m})$ be a separable quadratic field extension, and let α denote the class of E/F in $H^1(F, C_2)$. Then:*

- (a) $R_{E/F}^{(1)}(\mathbb{G}_m) \cong R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m$.
- (b) Let $\mathrm{Gal}(E/F)$ act on \mathbb{P}^1 via $z \mapsto z^{-1}$. Then ${}^\alpha\mathbb{P}^1 \cong \mathbb{P}^1$.
- (c) $R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m$ is rational and

$$\{R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m\} = \{B(R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m)\}^{-1} = \mathbb{L} - \{E\} + 1.$$

- (d) $\{R_{E/F}(\mathbb{G}_m)\} = \{BR_{E/F}(\mathbb{G}_m)\}^{-1} = (\mathbb{L} - 1)(\mathbb{L} - \{E\} + 1)$.
- (e) $\{R_{E/F}(\mathbb{P}^1)\} = \mathbb{L}^2 + \{E\}\mathbb{L} + 1$.

Proof. (a) Both tori correspond to the unique nontrivial $\mathrm{Gal}(E/F)$ -lattice of rank 1. Here $\mathrm{Gal}(E/F) \cong C_2$.

(b) The C_2 -action on \mathbb{P}^1 has a fixed point $z = 1$, hence ${}^\alpha\mathbb{P}^1$ has an F -point. By Châtelet’s theorem [Gille and Szamuely 2006, Theorem 5.1.3], a form of \mathbb{P}^n which admits an F -point is trivial (the case $n = 1$ is particularly simple, see [loc. cit., Remark 1.3.5]). We conclude that ${}^\alpha\mathbb{P}^1 \cong \mathbb{P}^1$.

(c) Let $T := R_{E/F}^{(1)}(\mathbb{G}_m) \cong R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m$. The open embedding $\mathbb{G}_m \hookrightarrow \mathbb{P}^1$, as the complement of $Z := \{0, \infty\}$, is equivariant under the C_2 -action on \mathbb{G}_m and \mathbb{P}^1 given by $z \mapsto z^{-1}$. Twisting by α , we obtain by (b) an open embedding of T in \mathbb{P}^1 as the complement of ${}^\alpha Z$. In particular, T is rational. By Lemma 2.1(a), ${}^\alpha Z \cong \mathrm{Spec} E$, so

$$\{T\} = \{\mathbb{P}^1\} - \{{}^\alpha Z\} = \mathbb{L} + 1 - \{E\}.$$

Now (c) follows from Lemma 2.3(b).

(d) The first equality holds because $R_{E/F}(\mathbb{G}_m)$ is special. Consider the short exact sequence

$$1 \rightarrow \mathbb{G}_m \rightarrow R_{E/F}(\mathbb{G}_m) \rightarrow T \rightarrow 1.$$

Since $R_{E/F}(\mathbb{G}_m)$ is special, [Lemma 2.2](#) yields

$$\{R_{E/F}(\mathbb{G}_m)\} = (\mathbb{L} - 1)\{BT\}^{-1},$$

thus (d) follows from (c).

(e) Write $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$, and consider the C_2 -equivariant decomposition

$$(\mathbb{P}^1)^2 = (\mathbb{A}^1)^2 \amalg (\mathbb{A}^1 \times \{\infty\} \cup \{\infty\} \times \mathbb{A}^1) \amalg \{(\infty, \infty)\}.$$

By Hilbert’s Theorem 90 and [Lemma 2.1\(a\)](#), twisting by α gives

$$R_{E/F}(\mathbb{P}^1) = \mathbb{A}^2 \amalg \mathbb{A}_E^1 \amalg \text{Spec } F,$$

thus $\{R_{E/F}(\mathbb{P}^1)\} = \mathbb{L}^2 + \{E\}\mathbb{L} + 1$. □

3. The classes of G and BG

Let F be a field of characteristic not 2, and assume that there exists a biquadratic extension

$$K := F(\sqrt{m_1}, \sqrt{m_2})$$

of F . Let

$$E_1 := F(\sqrt{m_1}), \quad E_2 := F(\sqrt{m_2}), \quad E_{12} := F(\sqrt{m_1 m_2}), \quad E := E_1 \times E_2,$$

and let $\Gamma := \text{Gal}(K/F) \cong C_2^2$ be the Galois group of K/F . We define the torus

$$G := R_{E/F}^{(1)}(\mathbb{G}_m)$$

and let

$$G' := R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m$$

be the dual torus of G . By definition, we have a short exact sequence

$$1 \rightarrow G \rightarrow R_{E/F}(\mathbb{G}_m) \xrightarrow{N} \mathbb{G}_m \rightarrow 1, \tag{3.1}$$

where N is the norm homomorphism.

The purpose of this section is the proof of [Proposition 3.7](#), which expresses $\{BG\}$ and $\{G\}$ as rational functions in \mathbb{L} , with coefficients classes of étale algebras.

Let σ_1 and σ_2 be generators for Γ such that $E_1 = K^{\sigma_1}$ and $E_2 = K^{\sigma_2}$. Consider the Γ -action on \mathbb{G}_m^2 , where $\sigma_1(u, v) = (v^{-1}, u^{-1})$ and $\sigma_2(u, v) = (v, u)$, and set

$$T := {}^\alpha(\mathbb{G}_m^2), \tag{3.2}$$

where $\alpha \in H^1(F, \Gamma)$ corresponds to the extension K/F .

Lemma 3.3. *We have*

$$\{T\} = \mathbb{L}^2 + (\{E_{12}\} - \{K\})\mathbb{L} + \{K\} - \{E_1\} - \{E_2\} + 1.$$

Proof. The embedding of \mathbb{G}_m in \mathbb{P}^1 as the complement of $Z := \{0, \infty\}$ gives an open embedding $\mathbb{G}_m^2 \hookrightarrow (\mathbb{P}^1)^2$ such that the Γ -action on \mathbb{G}_m^2 extends to $(\mathbb{P}^1)^2$. By definition

$$\alpha(\mathbb{P}^1)^2 = ((\mathbb{P}^1)^2 \times \text{Spec } K) / \Gamma,$$

where $\Gamma = \langle \sigma_1, \sigma_2 \rangle$ acts diagonally. We first take the quotient by the subgroup $\langle \sigma_1 \sigma_2 \rangle$. Since $\sigma_1 \sigma_2(u, v) = (u^{-1}, v^{-1})$ and $E_{12} = K^{\sigma_1 \sigma_2}$, by [Lemma 2.6\(b\)](#)

$$\alpha(\mathbb{P}^1)^2 = ((\mathbb{P}^1)^2 \times \text{Spec } E_{12}) / C_2,$$

where C_2 acts on $(\mathbb{P}^1)^2$ by switching the two factors. Here we are using the fact that every automorphism of $(\mathbb{P}^1)^2$ must respect the ruling (because it respects the intersection form), and so $\text{Aut}((\mathbb{P}^1)^2) = (\text{Aut}(\mathbb{P}^1))^2 \rtimes C_2$, where C_2 switches the two factors. By [Lemma 2.1\(b\)](#) we deduce that $\alpha(\mathbb{P}^1)^2 \cong R_{E_{12}/F}(\mathbb{P}^1)$, so by [Lemma 2.6\(e\)](#)

$$\{\alpha(\mathbb{P}^1)^2\} = \mathbb{L}^2 + \{E_{12}\}\mathbb{L} + 1. \tag{3.4}$$

We may partition $(\mathbb{P}^1)^2 \setminus \mathbb{G}_m^2$ into two strata

$$Z_1 := Z \times Z, \quad Z_2 := (Z \times \mathbb{G}_m) \amalg (\mathbb{G}_m \times Z).$$

The Γ -action on Z_1 has two orbits, and Γ acts on Z_2 by transitively permuting the components as the Klein subgroup of S_4 . By [Lemma 2.1\(a\)](#), $\alpha Z_1 = \text{Spec } E_1 \amalg \text{Spec } E_2$ and $\alpha Z_2 = \mathbb{G}_m \times \text{Spec } K$. By [\(3.4\)](#)

$$\begin{aligned} \{T\} &= \{\alpha(\mathbb{P}^1)^2\} - \{\alpha Z_1\} - \{\alpha Z_2\} \\ &= \mathbb{L}^2 + \{E_{12}\}\mathbb{L} + 1 - \{E_1\} - \{E_2\} - \{K\}(\mathbb{L} - 1) \\ &= \mathbb{L}^2 + (\{E_{12}\} - \{K\})\mathbb{L} + \{K\} - \{E_1\} - \{E_2\} + 1. \end{aligned} \quad \square$$

Proposition 3.5. *There is a short exact sequence of tori*

$$1 \rightarrow \mathbb{G}_m \rightarrow G \rightarrow T \rightarrow 1,$$

where T is the torus of [\(3.2\)](#).

Proof. Let P , M and \mathbb{Z} be the character lattices of $R_{E/F}(\mathbb{G}_m)$, G and \mathbb{G}_m , respectively. We may view P as the Γ -lattice with a basis e_1, e_2, e_3, e_4 , such that σ_1 acts by switching e_1 with e_2 and fixing e_3 and e_4 , and σ_2 switches e_3 with e_4 and fixes e_1 and e_2 . The sequence of Γ -lattices dual to [\(3.1\)](#) identifies M with the cokernel of the Γ -homomorphism $\mathbb{Z} \rightarrow P$ given by $1 \mapsto e_1 + e_2 + e_3 + e_4$; denote by $\bar{X}_i \in M$ the projection of e_i . Following Kunyavskii [[1987](#), Section 3, Proposition 1(b)], we consider an exact sequence of Γ -lattices

$$0 \rightarrow N \rightarrow M \xrightarrow{\pi} \mathbb{Z} \rightarrow 0. \tag{3.6}$$

The map π is defined by $\pi(\sum a_i \bar{X}_i) = a_1 + a_2 - a_3 - a_4$, and $N := \text{Ker } \pi$. A basis for N is given by $v_1 := \bar{X}_1 + \bar{X}_3$ and $v_2 := \bar{X}_1 + \bar{X}_4$. With respect to the basis (v_1, v_2) , the Γ -action on N is given by $\sigma_1(a, b) = (-b, -a)$ and $\sigma_2(a, b) = (b, a)$. It is now clear that N is the character lattice of the torus T of (3.2), hence the proof is complete. □

Proposition 3.7. (a) BG is stably rational.

(b) $\{BG\}\{G'\} = 1$ in $K_0(\text{Stacks}_F)$.

Proof. Consider the sequence

$$1 \rightarrow \mathbb{G}_m \rightarrow G' \rightarrow (R_{E_1/F}(\mathbb{G}_m)/\mathbb{G}_m) \times (R_{E_2/F}(\mathbb{G}_m)/\mathbb{G}_m) \rightarrow 1, \tag{3.8}$$

which exhibits G' as a \mathbb{G}_m -torsor over a rational variety, by Lemma 2.6(c). We deduce that G' is rational, and now (a) and (b) follow from Lemma 2.3. □

Proposition 3.9. We have

$$\{G\} = (\mathbb{L} - 1)(\mathbb{L}^2 + (\{E_{12}\} - \{K\})\mathbb{L} + \{K\} - \{E_1\} - \{E_2\} + 1) \tag{3.10}$$

and

$$\{BG\}^{-1} = (\mathbb{L} - 1)(\mathbb{L} - \{E_1\} + 1)(\mathbb{L} - \{E_2\} + 1) \tag{3.11}$$

in $K_0(\text{Stacks}_F)$.

Proof. By Proposition 3.5, G is a \mathbb{G}_m -torsor over T . Since \mathbb{G}_m is special, $\{G\} = (\mathbb{L} - 1)\{T\}$. The class of T was determined in Lemma 3.3.

By Proposition 3.7(b), $\{BG\}^{-1} = \{G'\}$. Since \mathbb{G}_m is special, by (3.8),

$$\{G'\} = (\mathbb{L} - 1)\{R_{E_1/F}^{(1)}(\mathbb{G}_m)\}\{R_{E_2/F}^{(1)}(\mathbb{G}_m)\}.$$

Now (3.11) follows from Lemma 2.6(c). □

4. The refined Euler characteristic

Let F be a field of characteristic zero. Using the computations of the previous section, we will reduce Theorem 1.5(b) to the assertion that a certain polynomial in \mathbb{L} with coefficients motivic classes of étale algebras is a nonzero element of $K_0(\text{Var}_F)$. To prove the assertion, we will use a simplified version of the refined Euler characteristic, introduced by Ekedahl [2009b].

Fix a prime number p , and let \mathcal{G} be a profinite group. The representation ring $a_p(\mathcal{G})$ of \mathcal{G} is the Grothendieck ring of continuous \mathcal{G} -representations $[M]$ of finite dimension over \mathbb{F}_p , subject to the relations $[M \oplus N] = [M] + [N]$. Note that no relations for nonsplit short exact sequences are imposed. The product structure on $a_p(\mathcal{G})$ is given by tensor product of representations. The next observation is well-known when \mathcal{G} is assumed to be finite; see [Benson 1991, Section 5.1].

Lemma 4.1. As an abelian group, $a_p(\mathcal{G})$ is freely generated by the set of isomorphism classes of indecomposable representations.

Proof. It is clear that $a_p(\mathcal{G})$ is generated by isomorphism classes of indecomposable representations. Assume that $\sum a_i[M_i] - \sum b_j[N_j] = 0$ in $a_p(\mathcal{G})$, for some positive integers a_i, b_j and some pairwise nonisomorphic indecomposable \mathcal{G} -representations M_i and N_j .

As a group, $a_p(\mathcal{G})$ is the quotient group F/I , where F is the free abelian group with one generator $\langle P \rangle$ for every isomorphism class of \mathcal{G} -representations P , and I is the subgroup generated by all elements of the form $\langle P \oplus Q \rangle - \langle P \rangle - \langle Q \rangle$. It follows that we may find a \mathcal{G} -representation X such that

$$(\oplus_i M_i^{\oplus a_i}) \oplus X \cong (\oplus_j N_j^{\oplus b_j}) \oplus X.$$

Let \mathcal{G}_0 be a finite quotient of \mathcal{G} such that \mathcal{G} acts on M_i, N_j and X through \mathcal{G}_0 . Then $M \oplus X \cong N \oplus X$ as \mathcal{G}_0 -representations, where $M := \oplus_i M_i^{\oplus a_i}$ and $N := \oplus_j N_j^{\oplus b_j}$. By the Krull–Schmidt Theorem applied to the group algebra $\mathbb{F}_p[\mathcal{G}_0]$, this implies $M \cong N$ as \mathcal{G}_0 -modules, hence as \mathcal{G} -modules. This is impossible, because the indecomposable representations M_i and N_j are pairwise nonisomorphic. \square

Proposition 4.2. *Let F be a field of characteristic zero, let $\text{Gal}(F)$ be the absolute Galois group of F , and let $R_p := a_p(\text{Gal}(F))$. There is a ring homomorphism*

$$\mu : K_0(\text{Var}_F) \rightarrow R_p[t]$$

such that for every smooth complete variety X we have $\mu(X) = \sum_i [H^i(\bar{X}_{\acute{e}t}, \mathbb{F}_p)]t^i$.

Proof. See the proof of [Ekedahl 2009b, Proposition 3.2(i)]. To show that μ is well-defined, one needs to assume that $\text{char } F = 0$ in order to invoke Bittner’s presentation of $K_0(\text{Var}_F)$; see [Bittner 2004, Theorem 3.1]. \square

5. Proof of Theorem 1.5

Theorem 1.5(a) was proved in Proposition 3.7(a), so we will focus on Theorem 1.5(b). We maintain the notation given at the beginning of Section 3.

Proof of Theorem 1.5(b). Assume by contradiction that $G = R_{E/F}^{(1)}(\mathbb{G}_m)$ satisfies (1.3). Then by Proposition 3.9 we have

$$(\mathbb{L} - 1)(\mathbb{L} - \{E_1\} + 1)(\mathbb{L} - \{E_2\} + 1) = (\mathbb{L} - 1)(\mathbb{L}^2 + (\{E_{12}\} - \{K\})\mathbb{L} + \{K\} - \{E_1\} - \{E_2\} + 1)$$

in $K_0(\text{Stacks}_F)$. Since $\mathbb{L} - 1$ is invertible in $K_0(\text{Stacks}_F)$, we may divide by $\mathbb{L} - 1$ on both sides. Subtracting \mathbb{L}^2 on the left and on the right, we arrive at

$$(2 - \{E_1\} - \{E_2\})\mathbb{L} + (1 - \{E_1\})(1 - \{E_2\}) = (\{E_{12}\} - \{K\})\mathbb{L} + \{K\} - \{E_1\} - \{E_2\} + 1,$$

that is

$$(\{K\} - \{E_1\} - \{E_2\} - \{E_{12}\} + 2)\mathbb{L} = 0$$

in $K_0(\text{Stacks}_F)$.

Recall that $K_0(\text{Stacks}_F)$ is the localization of $K_0(\text{Var}_F)$ at \mathbb{L} and the cyclotomic polynomials in \mathbb{L} ; see [Ekedahl 2009b, Theorem 1.2]. It follows that

$$(\{K\} - \{E_1\} - \{E_2\} - \{E_{12}\} + 2)f(\mathbb{L}) = 0 \tag{5.1}$$

in $K_0(\text{Var}_F)$, where $f(x) \in \mathbb{Z}[x]$ is a monic polynomial of some degree n .

In order to obtain a contradiction, we now want to apply the homomorphism μ of Proposition 4.2, with respect to the prime $p = 2$. If L/F is an étale algebra of degree n , $\mu(\{L\})$ consists of the permutation representation of $\text{Gal}(F)$ associated to L , concentrated in degree 0. Since we have chosen $p = 2$, $\mu(\{\mathbb{P}^1\})$ consists of one copy of the trivial representation in degree 0 and 2 (in the case $p > 2$ one would need a Tate twist in degree 2). Since $\mathbb{L} = \{\mathbb{P}^1\} - 1$, we deduce that $\mu(\mathbb{L}) = t^2$, and hence $\mu(f(\mathbb{L})) = f(t^2)$.

If X is a finite $\text{Gal}(F)$ -set, we denote by $\mathbb{F}_2[X]$ the permutation representation over \mathbb{F}_2 associated to X . Recall from Section 3 that we denote $\text{Gal}(K/F)$ by $\Gamma = \langle \sigma_1, \sigma_2 \rangle$. Applying μ to (5.1) and looking at degree $2n$, we obtain

$$[\mathbb{F}_2[\Gamma]] - [\mathbb{F}_2[\Gamma/\langle \sigma_1 \rangle]] - [\mathbb{F}_2[\Gamma/\langle \sigma_2 \rangle]] - [\mathbb{F}_2[\Gamma/\langle \sigma_{12} \rangle]] + 2[\mathbb{F}_2] = 0$$

in R_2 . This is a nontrivial relation of linear dependence in R_2 among classes of indecomposable representations. This is in contradiction with Lemma 4.1, hence $\{BG\} \neq \{G\}^{-1}$, as desired. \square

Remark 5.2. By [Voskresensky 1998, Section 4.9, Example 7] every torus of rank 2 is rational, so by Proposition 3.5 the torus G is rational. By Lemma 2.3, BG' is stably rational and $\{BG'\} = \{G\}^{-1}$. By Proposition 3.7(b) we have $\{BG\} = \{G'\}^{-1}$, so $\{BG'\}\{G'\} = \{BG\}^{-1}\{G\}^{-1}$. Since $\{BG\}\{G\} \neq 1$, the conclusions of Theorem 1.5(a) and (b) hold for G' as well.

6. Proof of Theorem 1.6

We maintain the notation of Section 3.

Proof of Theorem 1.6. Let $\Gamma := \text{Gal}(K/F)$, let M be the character lattice of G , so that $M/2M$ is the character module of A , and let P be the character lattice of $R_{E/F}(\mathbb{G}_m)$. As in the proof of Proposition 3.5, we view P as the lattice freely generated by e_1, e_2, e_3, e_4 , such that σ_1 acts by switching e_1 with e_2 , and σ_2 by switching e_3 with e_4 . Using (3.1), we may construct a commutative diagram of Γ -modules

$$\begin{CD} 0 @>>> \mathbb{Z} @>>> P @>>> M @>>> 0 \\ @. @VV\iota V @| @VV V @. \\ 0 @>>> N @>>> P @>\varphi>> M/2M @>>> 0. \end{CD} \tag{6.1}$$

with exact rows. Here \mathbb{Z} denotes the trivial one-dimensional Γ -lattice, $\iota(1) := e_1 + e_2 + e_3 + e_4$, and N is the kernel of φ , that is,

$$N = \left\{ \sum_{i=1}^4 a_i e_i : a_1 \equiv a_2 \equiv a_3 \equiv a_4 \pmod{2} \right\}.$$

Applying the snake lemma to (6.1), we obtain a short exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{\iota} N \rightarrow M \rightarrow 0.$$

Define $\pi : N \rightarrow \mathbb{Z}$ by sending $\sum a_i e_i$ to $(a_1 + a_2)/2$. Then π is a Γ -homomorphism and ι is a section of π . Therefore, we have an isomorphism $N \cong \mathbb{Z} \oplus M$.

Let S be an F -torus with character lattice N . Since $N \cong \mathbb{Z} \oplus M$, we have $S \cong \mathbb{G}_m \times G$. Thus, the bottom row of (6.1) corresponds to the short exact sequence of group schemes

$$1 \rightarrow A \rightarrow R_{E/F}(\mathbb{G}_m) \rightarrow \mathbb{G}_m \times G \rightarrow 1.$$

By Lemma 2.2, we have $\{BA\} = \{\mathbb{G}_m\}\{G\}/\{R_{E/F}(\mathbb{G}_m)\}$. Applying Lemma 2.2 to (3.1), we see that $\{BG\} = \{\mathbb{G}_m\}/\{R_{E/F}(\mathbb{G}_m)\}$. Therefore, $\{BA\} = \{BG\}\{G\}$. By Theorem 1.6 we have $\{BG\} \neq \{G\}^{-1}$, hence $\{BA\} \neq 1$, as desired. \square

Acknowledgments

I would like to thank my advisor Zinovy Reichstein for his guidance and for greatly improving the exposition, Mattia Talpo and Angelo Vistoli for helpful comments, and Boris Kunyavskii for sending me a copy of his paper [Kunyasvki 1987]. I am very grateful to the anonymous referee for finding a mistake in a previous version of the proof of Theorem 1.5, and for suggesting a fix.

References

- [Behrend and Dhillon 2007] K. Behrend and A. Dhillon, “On the motivic class of the stack of bundles”, *Adv. Math.* **212**:2 (2007), 617–644. [MR](#) [Zbl](#)
- [Benson 1991] D. J. Benson, *Representations and cohomology, II: Cohomology of groups and modules*, Cambridge Stud. Adv. Math. **31**, Cambridge Univ. Press, 1991. [MR](#) [Zbl](#)
- [Bergh 2016] D. Bergh, “Motivic classes of some classifying stacks”, *J. Lond. Math. Soc.* (2) **93**:1 (2016), 219–243. [MR](#) [Zbl](#)
- [Bittner 2004] F. Bittner, “The universal Euler characteristic for varieties of characteristic zero”, *Compos. Math.* **140**:4 (2004), 1011–1032. [MR](#) [Zbl](#)
- [Dhillon and Young 2016] A. Dhillon and M. B. Young, “The motive of the classifying stack of the orthogonal group”, *Michigan Math. J.* **65**:1 (2016), 189–197. [MR](#) [Zbl](#)
- [Ekedahl 2009a] T. Ekedahl, “A geometric invariant of a finite group”, preprint, 2009. [arXiv](#)
- [Ekedahl 2009b] T. Ekedahl, “The Grothendieck group of algebraic stacks”, preprint, 2009. [arXiv](#)
- [Favi and Florence 2008] G. Favi and M. Florence, “Tori and essential dimension”, *J. Algebra* **319**:9 (2008), 3885–3900. [MR](#) [Zbl](#)
- [Florence 2008] M. Florence, “On the essential dimension of cyclic p -groups”, *Invent. Math.* **171**:1 (2008), 175–189. [MR](#) [Zbl](#)
- [Florence and Reichstein 2018] M. Florence and Z. Reichstein, “The rationality problem for forms of $\overline{M}_{0,n}$ ”, *Bull. Lond. Math. Soc.* **50**:1 (2018), 148–158. [MR](#) [Zbl](#)
- [Florence et al. 2017] M. Florence, N. Hoffmann, and Z. Reichstein, “On the rationality problem for forms of moduli spaces of stable marked curves of positive genus”, preprint, 2017. [arXiv](#)
- [Gille and Szamuely 2006] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Stud. Adv. Math. **101**, Cambridge Univ. Press, 2006. [MR](#) [Zbl](#)
- [Joyce 2007] D. Joyce, “Motivic invariants of Artin stacks and ‘stack functions’”, *Q. J. Math.* **58**:3 (2007), 345–392. [MR](#) [Zbl](#)

- [Kunyavski 1987] B. È. Kunyavskii, “Three-dimensional algebraic tori”, pp. 90–111 in *Investigations in number theory*, edited by N. G. Chudakov, Saratov. Gos. Univ., 1987. In Russian; translated in *Selecta Math. Soviet.* **9**:1 (1990), 1–21. [MR](#) [Zbl](#)
- [Martino 2016] I. Martino, “The Ekedahl invariants for finite groups”, *J. Pure Appl. Algebra* **220**:4 (2016), 1294–1309. [MR](#) [Zbl](#)
- [Martino 2017] I. Martino, “Introduction to the Ekedahl invariants”, *Math. Scand.* **120**:2 (2017), 211–224. [MR](#) [Zbl](#)
- [Pirisi and Talpo 2019] R. Pirisi and M. Talpo, “On the motivic class of the classifying stack of G_2 and the spin groups”, *Int. Math. Res. Not.* **2019**:10 (2019), 3265–3298. [MR](#) [Zbl](#)
- [Serre 1958] J.-P. Serre, “Espaces fibrés algébriques”, exposé 1 in *Anneaux de Chow et applications*, Séminaire Claude Chevalley **3**, Secrétariat Math., Paris, 1958.
- [Talpo and Vistoli 2017] M. Talpo and A. Vistoli, “The motivic class of the classifying stack of the special orthogonal group”, *Bull. Lond. Math. Soc.* **49**:5 (2017), 818–823. [MR](#) [Zbl](#)
- [Toën 2005] B. Toën, “Grothendieck rings of Artin n -stacks”, preprint, 2005. [arXiv](#)
- [Voskresensky 1998] V. E. Voskresenskii, *Algebraic groups and their birational invariants*, Transl. Math. Monogr. **179**, Amer. Math. Soc., Providence, RI, 1998. [MR](#) [Zbl](#)

Communicated by Martin Olsson

Received 2018-08-07 Revised 2019-07-16 Accepted 2019-12-19

scavia@math.ubc.ca

University of British Columbia, Vancouver BC, Canada

Algebra & Number Theory

msp.org/ant

EDITORS

MANAGING EDITOR

Bjorn Poonen
Massachusetts Institute of Technology
Cambridge, USA

EDITORIAL BOARD CHAIR

David Eisenbud
University of California
Berkeley, USA

BOARD OF EDITORS

Bhargav Bhatt	University of Michigan, USA	Martin Olsson	University of California, Berkeley, USA
Richard E. Borcherds	University of California, Berkeley, USA	Raman Parimala	Emory University, USA
Antoine Chambert-Loir	Université Paris-Diderot, France	Jonathan Pila	University of Oxford, UK
J-L. Colliot-Thélène	CNRS, Université Paris-Sud, France	Irena Peeva	Cornell University, USA
Brian D. Conrad	Stanford University, USA	Anand Pillay	University of Notre Dame, USA
Samit Dasgupta	Duke University, USA	Michael Rapoport	Universität Bonn, Germany
Hélène Esnault	Freie Universität Berlin, Germany	Victor Reiner	University of Minnesota, USA
Gavril Farkas	Humboldt Universität zu Berlin, Germany	Peter Sarnak	Princeton University, USA
Hubert Flenner	Ruhr-Universität, Germany	Michael Singer	North Carolina State University, USA
Sergey Fomin	University of Michigan, USA	Christopher Skinner	Princeton University, USA
Edward Frenkel	University of California, Berkeley, USA	Vasudevan Srinivas	Tata Inst. of Fund. Research, India
Wee Teck Gan	National University of Singapore	J. Toby Stafford	University of Michigan, USA
Andrew Granville	Université de Montréal, Canada	Shunsuke Takagi	University of Tokyo, Japan
Ben J. Green	University of Oxford, UK	Pham Huu Tiep	University of Arizona, USA
Joseph Gubeladze	San Francisco State University, USA	Ravi Vakil	Stanford University, USA
Christopher Hacon	University of Utah, USA	Michel van den Bergh	Hasselt University, Belgium
Roger Heath-Brown	Oxford University, UK	Akshay Venkatesh	Institute for Advanced Study, USA
János Kollár	Princeton University, USA	Marie-France Vignéras	Université Paris VII, France
Philippe Michel	École Polytechnique Fédérale de Lausanne	Melanie Matchett Wood	University of California, Berkeley, USA
Susan Montgomery	University of Southern California, USA	Shou-Wu Zhang	Princeton University, USA
Shigefumi Mori	RIMS, Kyoto University, Japan		

PRODUCTION

production@msp.org

Silvio Levy, Scientific Editor

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2020 is US \$415/year for the electronic version, and \$620/year (+\$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2020 Mathematical Sciences Publishers

Algebra & Number Theory

Volume 14 No. 4 2020

The distribution of p -torsion in degree p cyclic fields	815
JACK KLYS	
On the motivic class of an algebraic group	855
FEDERICO SCAVIA	
A representation theory approach to integral moments of L -functions over function fields	867
WILL SAWIN	
Deformations of smooth complete toric varieties: obstructions and the cup product	907
NATHAN ILTEN and CHARLES TURO	
Mass equidistribution on the torus in the depth aspect	927
YUEKE HU	
The basepoint-freeness threshold and syzygies of abelian varieties	947
FEDERICO CAUCCI	
On the Ekedahl–Oort stratification of Shimura curves	961
BENJAMIN HOWARD	
A moving lemma for relative 0-cycles	991
AMALENDU KRISHNA and JINHYUN PARK	