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Unobstructedness of Galois deformation rings associated to regular algebraic conjugate self-dual cuspidal automorphic representations

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Let F be a CM field and let $(\bar{r}_{\pi,\lambda})_\lambda$ be the compatible system of residual $\mathcal{G}_n$-valued representations of Gal_F attached to a regular algebraic conjugate self-dual cuspidal (RACSDC) automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$, as studied by Clozel, Harris and Taylor (2008) and others. Under mild assumptions, we prove that the fixed-determinant universal deformation rings attached to $\bar{r}_{\pi,\lambda}$ are unobstructed for all places λ in a subset of Dirichlet density 1, continuing the investigations of Mazur, Weston and Gamzon. During the proof, we develop a general framework for proving unobstructedness (with future applications in mind) and an $R = T$-theorem, relating the universal crystalline deformation ring of $\bar{r}_{\pi,\lambda}$ and a certain unitary fixed-type Hecke algebra.

1. Introduction

This article studies unobstructedness of certain Galois deformation rings. For this introduction, let F be a number field, let k be a finite field of characteristic ℓ and fix an absolutely irreducible representation

$$\bar{\rho}\colon \mathrm{Gal}_{F,S} \to \mathrm{GL}_n(k),$$

where $S \subset \mathrm{Pl}_F$ is a finite set of places. Then assigning to a complete Noetherian local algebra A over the ring W of Witt vectors of k the set of all $\mathrm{GL}_n(A)$-valued deformations of $\bar{\rho}$ defines a functor, which is representable by a universal deformation ring $R_S(\bar{\rho})$, studied first by Mazur [1989].

If the cohomology group $H^2(\mathrm{Gal}_{F,S}, \mathrm{ad}\,\bar{\rho})$ vanishes, then $R_S(\bar{\rho})$ is easily seen to be formally smooth, i.e., isomorphic to a power series ring over W. In this sense, the group $H^2(\mathrm{Gal}_{F,S}, \mathrm{ad}\,\bar{\rho})$ can be interpreted as the obstruction to the smoothness of $R_S(\bar{\rho})$, and we say that $R_S(\bar{\rho})$ is *unobstructed* if $H^2(\mathrm{Gal}_{F,S}, \mathrm{ad}\,\bar{\rho}) = 0$.

We point out the following connection with a conjecture of Jannsen: Assume that $\bar{\rho}$ is the reduction of the ℓ-adic representation $\rho_{f,\ell}$ attached to a cuspidal modular eigenform f (see [Deligne 1973; Shimura 1971; Deligne and Serre 1974]). Then the Frobenius eigenvalues of $\rho_{f,\ell}$ are Weil-numbers of some fixed weight w, i.e., $\rho_{f,\ell}$ is pure of weight w. A conjecture of Jannsen [1989, Conjecture 1] (see also [Bellaïche 2009, Conjecture 5.1]) predicts the vanishing of $H^2(\mathrm{Gal}_{F,S}, \mathrm{ad}\,\rho)$. This implies that $H^2(\mathrm{Gal}_{F,S}, \Lambda)$

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is finite and torsion, where $\Lambda \subset \operatorname{ad}\rho$ denotes an integral $\mathrm{Gal}_{F,S}$-stable lattice. Now our residual H^2-vanishing implies the vanishing of $H^2(\mathrm{Gal}_{F,S}, \Lambda)$ by Nakayama's lemma. This, in turn, implies the vanishing of $H^2(\mathrm{Gal}_{F,S}, \operatorname{ad}\rho)$, as predicted by Jannsen. Besides this application, numerous uses of Galois deformation-theoretic methods in number theory indicate that the structure of universal deformation rings is of independent interest.

Unobstructedness for Galois representations attached to automorphic objects rarely can be expected to hold for all choices of ℓ. The best we can hope for is that unobstructedness holds for almost all primes (or for all primes in a subset of Dirichlet density 1), and this question has been studied (under different technical assumptions) in the following cases:

(a) For $\bar{\rho}$ the reduction of the representation $\rho_{E,\ell}$ attached to an elliptic curve E over $F = \mathbb{Q}$; see [Mazur 1989].

(b) For $\bar{\rho}$ the reduction of the representation $\rho_{f,\ell}$ attached to a newform f of weight $k \geq 3$ over $F = \mathbb{Q}$; see [Weston 2004] (but see also [Yamagami 2004; Hatley 2015]).

(c) For $\bar{\rho}$ the reduction of the representation $\rho_{f,\ell}$ attached to a Hilbert eigenform f over a totally real field F; see [Gamzon 2016].

Note that $n = 2$ in all these cases.

For an example of $n = 3$, in [Chenevier 2011, Appendix] unobstructedness is shown under GRH for $\bar{\rho} = \mathrm{Sym}^2 A[\ell](-1)|_{G_{E,S}}$ where A is an elliptic curve over $\mathbb{Q}$ of one of the ten isogeny classes listed in [loc. cit., Proposition 6.15], $\ell = 5$, $E = \mathbb{Q}(i)$ and S is the set of places containing ∞, ℓ and the primes dividing $\operatorname{disc}(E).\operatorname{cond}(A)$; see [loc. cit., Appendix].

In this article, we develop a general framework for proving unobstructedness. To this end, we adjust the arguments in the existing literature to deal with framings, build on [Shotton 2018] to understand minimal lifts at $\ell \neq p$, and use results of [Allen 2016] together with ideas of Khare–Wintenberger to bootstrap the $R[1/p]^{\mathtt{red}} = T[1/p]$ theorems of [Barnet-Lamb et al. 2014] to obtain an $R^{\mathtt{min}} = T^{\mathtt{min}}$ theorem. We apply this framework to the reduction of the Galois representation attached to a regular algebraic conjugate self-dual cuspidal (RACSDC) automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ with ramification set S, where F is a CM field.[1] To give a more precise statement, we have to recall that π gives rise, in first instance, not to GL_n-valued representations, but to morphisms $r_{\pi,\lambda}\colon \mathrm{Gal}_{F^+} \to \mathcal{G}_n(\overline{\mathbb{Q}}_{\ell(\lambda)})$, where λ runs through the places of the coefficient field of π, where $\mathcal{G}_n$ denotes the group scheme from [Clozel et al. 2008, Section 2.1] and where $\ell(\lambda)$ denotes the rational prime below λ.

We make the following assumption:

Assumption 1.1. The set of the λ for which the GL_n-valued representation $\bar{r}_{\pi,\lambda} \mid \mathrm{Gal}_F$ is absolutely irreducible has Dirichlet density 1.

[1] We remark that, in light of the results of [Barnet-Lamb et al. 2014], it should be possible to weaken the conjugate self-duality assumption to an essentially self-duality assumption, thus treating RACESDC automorphic representations.

We remark that this assumption is fulfilled, e.g., if $n \leq 5$ or if π is extremely regular, or would follow from absolute irreducibility of the ℓ-adic system $(r_{\pi,\lambda} \mid \mathrm{Gal}_F)$; see Remark 8.2. For the following, we fix for each λ a lift χ of the character $\mathtt{m} \circ \bar{r}_{\pi,\lambda}$ of Gal_F, where $\mathtt{m}$ is the multiplier character of the group $\mathcal{G}_n$; see Section 6A. By $R^{\chi}_{S_\ell}(\bar{r}_{\pi,\lambda})$ we denote the universal ring parametrizing deformations r of $\bar{r}_{\pi,\lambda}$ that are unramified outside the places that are in S or divide $\infty.\ell(\lambda)$ and that fulfill $\mathtt{m} \circ r = \chi$. The correct unobstructedness requirement is then the vanishing of $H^2(\mathrm{Gal}_{F,S_\ell}, \mathfrak{g}_n^{\mathtt{der}})$, where $\mathfrak{g}_n^{\mathtt{der}}$ denotes the Lie algebra of the derived subgroup of $\mathcal{G}_n$. Our main result is:

Theorem 1.2. *Assume that all Hodge–Tate weights of $r_{\pi,\lambda}$ (which are independent of λ, as the $r_{\pi,\lambda}$ form a compatible system) are nonconsecutive: if $a, b \in \mathbb{Z}$ show up as Hodge–Tate weights, then $|a-b| \neq 1$. Then, for all λ in a set of places of Dirichlet density* 1 *the universal deformation ring $R^{\chi}_{S_\ell}(\bar{r}_{\pi,\lambda})$ is unobstructed.*

Remark that we do not require a particular splitting behavior at the places in S. We also want to stress that the developed framework is flexible and in principle applicable to Galois representations with values in other groups and can be used to establish unobstructedness of universal deformation rings with imposed deformation conditions, which are more sophisticated than the fixed-determinant condition $\mathtt{m} \circ r = \chi$. Therefore, we hope that the framework will be useful for other applications, as better modularity lifting results become available in the future. We also remark that presently the condition on the Hodge–Tate weights is necessary for using a local unobstructedness property at the places above $\ell(\lambda)$; a technical inconvenience we expect to weaken in future work.

We give a short outline of the article: After some remarks about notation, we start in Section 3 with a collection of the general deformation theoretic methods we will use. Moreover, we will define a suitably flexible notion of unobstructedness for conditioned deformation functors (Definition 3.28). In Section 4, we state and prove the core framework (Theorem 4.2), which uses a list of six assumptions as input and provides unobstructedness as output. This framework is presented for local deformation conditions `crys`, `min`, `sm`, which have a purely formal meaning throughout Section 4. The main input is the formal smoothness of the deformation ring with respect to the conditions `min` and `crys`, which is the natural output of a suitable $R = T$-theorem, and the desired unobstructedness is then deduced by commutative algebra arguments and comparing dimensions. Section 5 introduces and studies useful local conditions that will go into the framework theorem later. After a reminder on the association of Galois deformations to automorphic forms, the additional results are provided in Section 7: We consider the deformation ring $R^{\mathtt{min,crys}} := R^{\chi,\mathtt{min,crys}}_{S_\ell}(\bar{r}_{\pi,\lambda})$ parametrizing those lifts that are minimally ramified (in the sense of Section 5D) at all places in S and crystalline (in the Fontaine–Laffaille range) at all places dividing ℓ. Moreover, we consider a corresponding Hecke algebra $\mathbb{T}^{\mathtt{min}}$ that is defined as the localization of a certain endomorphism algebra of automorphic forms of the same weight and level as π, and with a certain fixed type-requirement at the places in S. Then, using the modularity lifting results of [Barnet-Lamb et al. 2014], we show:

Theorem 1.3. $R^{\mathtt{min,crys}} \cong \mathbb{T}^{\mathtt{min}}$ *and, for almost all λ,* $\mathbb{T}^{\mathtt{min}} \cong W$.

This result is crucial to prove in Section 8 that, for almost all λ, there exists a suitable finite solvable extension F' of F such that the deformation ring $R^{\chi,\mathtt{min}}_{S_\ell}(\bar{r}_{\pi,\lambda} \mid \mathrm{Gal}_{F',S})$, parametrizing deformations of the base change of $\bar{r}_{\pi,\lambda}$ to F' that are minimally ramified at all places above S, is unobstructed. We go on to show that the minimally ramified condition can be waived for almost all λ (Theorem 7.10). It is important to keep track of the different field extensions necessary when running through all λ, so that we are left with a set of Dirichlet density 1 to which we can apply a result on potential unobstructedness (Lemma 4.8) and finally deduce Theorem 1.2.

2. Notation

Before we start with the main body of this article, let us make some remarks on the notation used: If F denotes a number field, we denote by Pl_F the set of places of F and by $\mathrm{Pl}_F^{\mathrm{fin}}$ the set of finite places of F. Moreover, we set $\Omega_\infty^F = \mathrm{Pl}_F \setminus \mathrm{Pl}_F^{\mathrm{fin}}$ and, for a rational prime ℓ, we denote by Ω_ℓ^F the set of places of F dividing ℓ. If F is understood, we will simply write Ω_∞ and Ω_ℓ. For a place $\lambda \in \mathrm{Pl}_F^{\mathrm{fin}}$ we define $\ell(\lambda)$ (or ℓ, if λ is understood) as the rational prime below λ. If $S \subset \mathrm{Pl}_F^{\mathrm{fin}}$ and ℓ is some rational prime, we set $S_\ell := S \cup \Omega_\infty \cup \Omega_\ell$.

We denote by $\bar{F}$ the Galois closure of F. When dealing with a quadratic extension $F \mid F^+$, we will denote by c the nontrivial element of the Galois group $\mathrm{Gal}(F \mid F^+)$. Moreover, for a rational prime ℓ, we denote by $\epsilon_\ell \colon \mathrm{Gal}_F \to \overline{\mathbb{Z}}_\ell^\times$ the ℓ-adic cyclotomic character and by $\bar{\epsilon}_\ell$ its mod-ℓ reduction.

If $L \mid F$ is a finite extension and S is a fixed set of places of F, then we will denote as well by S the set $\{v' \in \mathrm{Pl}_L : v' \text{ divides some } v \in S\}$. In a completely analogous way, if S is a subset of Pl_L, then we will denote as well by S the set $\{v' \in \mathrm{Pl}_F : v' \text{ is divided by some } v \in S\}$. If ρ is a representation of Gal_F and v a place of F, we will use the symbol ρ_v for the restriction of ρ to a decomposition subgroup at v.

For a topological group Γ and a topological ring R, we denote by $\mathrm{Rep}_R(\Gamma)$ the category of finitely generated R-modules with a continuous Γ-action. If A is a Γ-module, we denote by A^* the Pontryagin dual and by $A^\vee$ the Tate dual of A.

We will often make statements concerning variations of deformation rings and we will shorten this using brackets; e.g., we will use the notation $R^{(\chi),[\mathtt{min}]}(\bar{\rho}) = 0$ as a shortcut for the four statements $R(\bar{\rho}) = 0$, $R^\chi(\bar{\rho}) = 0$, $R^{\mathtt{min}}(\bar{\rho}) = 0$ and $R^{\chi,\mathtt{min}}(\bar{\rho}) = 0$. For cohomology groups, we abbreviate $h^i(*,*)$ for $\dim H^i(*,*)$.

Let k be a finite field of characteristic ℓ. For the valuation ring Λ of a finite extension of $\mathbb{Q}_\ell$ with residue field $k_\Lambda = k$, we will consider the category $\mathcal{C}_\Lambda$ of complete Noetherian local Λ-algebras A fulfilling $k_A = k$.

3. Liftings and deformations

In this section, which contains nothing original, we recall the main results on deformation theory. For general background literature, we refer the reader to [Tilouine 1996; Mauger 2000; Levin 2013; Balaji 2012; Bleher and Chinburg 2003]. Let us first fix a finite field k and denote $\ell = \mathrm{char}(k)$. We will denote the ring of Witt vectors over k by $W(k)$, or, if k is understood, by W. Moreover, let us fix a profinite group Γ which fulfills the ℓ-finiteness condition (Φ_ℓ) of [Mazur 1989]: For any open subgroup $H \subset \Gamma$, the maximal pro-ℓ quotient of H is topologically finitely generated.

Let G be a smooth linear algebraic group over W and fix a continuous group homomorphism $\bar{\rho}\colon \Gamma \to G(k)$, where $G(k)$ carries the discrete topology.

Basic facts on coefficient rings. Let us first state some basic facts on the category $\mathcal{C}_\Lambda$, whose proofs we leave to the reader: The pushout in $\mathcal{C}_\Lambda$ is realized by the completed tensor product $\widehat{\otimes}$; see [Mazur 1997, Section 12]. Consequently, if $C \leftarrow A \to B$ is a diagram in $\mathcal{C}_\Lambda$, then $\mathrm{Hom}_{\mathcal{C}_\Lambda}(B\widehat{\otimes}_A C, \_)$ is the pullback of the diagram of functors $\mathrm{Hom}_{\mathcal{C}_\Lambda}(C,\_) \to \mathrm{Hom}_{\mathcal{C}_\Lambda}(A,\_) \leftarrow \mathrm{Hom}_{\mathcal{C}_\Lambda}(B,\_)$. Consider a pushout diagram in $\mathcal{C}_\Lambda$ where one arrow (say, f) is surjective. This implies that the parallel arrow (say, g) is surjective as well, so taking $I = \ker(f)$ and $J = \ker(g)$ we can extend the orthogonal arrow (say, π) to a map of short exact sequences of Λ-modules:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ C & \xrightarrow[g]{} & P \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & 0 \\ & & {\scriptstyle\pi\,|\,I}\downarrow & & {\scriptstyle\pi}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & C & \xrightarrow[g]{} & P & \longrightarrow & 0 \end{array}$$

If $\mathfrak{I}$ is an ideal of some $D \in \mathcal{C}_\Lambda$ we denote cardinality of a minimal set of generators of $\mathfrak{I}$ by $\mathrm{gen}_D(\mathfrak{I}) := \dim_k \mathfrak{I}/\mathfrak{m}_D\mathfrak{I}$. Then, we easily see that the following holds for the above diagram:

Proposition 3.1. *In the above diagram*, $\mathrm{gen}_C(J) \le \mathrm{gen}_A(I)$.

Proof. This follows from the above extended diagram, using that both the map $I \to C\hat{\otimes}_A I$ induced by base change from A to C and the surjective module homomorphism $C\hat{\otimes}_A I \twoheadrightarrow J$ send systems of generators to systems of generators. □

Recall the following elementary facts about regular systems of parameters:

Proposition 3.2 [Serre 2000, Proposition 22 and the subsequent corollary]. (a) *Let $x_1, \dots, x_l$ be l elements of the maximal ideal $\mathfrak{m}_A$ of a regular local ring A. Then the following are equivalent*:

(i) *$x_1, \dots, x_l$ is a subset of a regular system of parameters of A.*

(ii) *The images of $x_1, \dots, x_l$ in $\mathfrak{m}_A/\mathfrak{m}_A^2$ are linearly independent over k.*

(iii) *The local ring $A/(x_1, \dots, x_l)$ is regular and has dimension* $\dim A - l$. (*In particular*, $(x_1, \dots, x_l)$ *is a prime ideal.*)

(b) *If $\mathfrak{I}$ is an ideal of a regular local ring A, the following properties are equivalent*:

(i) *$A/\mathfrak{I}$ is a regular local ring.*

(ii) *$\mathfrak{I}$ is generated by a subset of a regular system of parameters of A.*

Moreover, we have the following results, which follow easily from standard facts about regular systems of parameters (see [Serre 2000, Proposition 22] and its use in Section 2 of [Guiraud 2016]):

Lemma 3.3. *Suppose $A = \Lambda[\![x_1, \dots, x_a]\!]$, $B = \Lambda[\![x_1, \dots, x_b]\!] \in \mathcal{C}_\Lambda$ and let $J \subset A$ be an ideal of the form $J = (f_1, \dots, f_u)$ with $f_i \in A$ and $u \le a$. Suppose moreover that there exists a surjective morphism $f : A/J \twoheadrightarrow B$ and denote its kernel by I. Then the following are equivalent*:

- $A/J \cong \Lambda[\![x_1, \dots, x_{a-u}]\!]$.
- $\operatorname{gen}_{A/J}(I) = a - u - b$.
- $\operatorname{gen}_{A/J}(I) \le a - u - b$.

Proof. It is clear that there cannot be a negative number of generators of I. By Proposition 3.2(b), the ideal I can be generated by a subset (of, say, cardinality r) of a regular system of parameters of A. By part (a) of said proposition, the quotient A/I has dimension $\dim A - r = a + 1 - r$. We get $r = a - b$, which is thus an upper bound on $\operatorname{gen}(I)$. In order to derive a lower bound, consider the canonical surjection

$$\pi : A/\mathfrak{m}_A.I \twoheadrightarrow A/\mathfrak{m}_A^2.$$

The image of $I/\mathfrak{m}_A.I$ under π is $(I + \mathfrak{m}_A^2)/\mathfrak{m}_A^2 \cong I/(I \cap \mathfrak{m}_A^2)$. This implies $\operatorname{gen}(I) = \dim_k I/\mathfrak{m}_A I \ge \dim_k I/I \cap \mathfrak{m}_A^2 = r$, where the last equality is taken from the proof of [Serre 2000, Proposition 22]. □

Proposition 3.4. *Let $m \in \mathbb{N}$. Then $A \in \mathcal{C}_\Lambda$ is regular if and only if $A[\![x_1, \dots, x_m]\!]$ is regular.*

Proof. It is clearly sufficient to consider the case $m = 1$. The "only if" part is [Matsumura 1970, Proposition 24D]. For the other direction, assume that $A[\![x]\!]$ is regular. It is clear that x is not contained in $\mathfrak{m}_{A[\![x]\!]}^2 = (\mathfrak{m}_A, x)^2$, so implication (ii) ⇒ (iii) of Proposition 3.2(a) yields regularity of $A[\![x]\!]/(x) \cong A$. □

Proposition 3.5. *Let $f : A \to B$ be a morphism in $\mathcal{C}_\Lambda$. Then f is formally smooth* (*see* [EGA IV_1 1964, Section 19]) *if and only if B is isomorphic to a formal power series ring over A.*

Proof. This is the equivalence (i)⇔(ii) of [Sernesi 2006, Proposition C.6]. □

Lemma 3.6. *Let $A \in \mathcal{C}_\Lambda$, $m \in \mathbb{N}$ such that $\Lambda[\![x_1, \dots, x_m]\!] \cong A \widehat{\otimes}_\Lambda \Lambda[\![x]\!]$. Then $A \cong \Lambda[\![x_1, \dots, x_{m-1}]\!]$.*

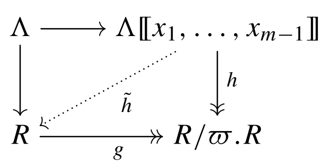
Proof. Let ϖ be a uniformizing element of Λ. Clearly, the unknown

$$x \in (R/\varpi.R)[\![x]\!] \cong k[\![x_1, \dots, x_m]\!]$$

is contained in a regular system of parameters, so

$$R/\varpi.R \cong k[\![x_1, \dots, x_{m-1}]\!]. \tag{1}$$

Now consider the diagram



where h and g are the projection maps modulo ϖ. As $\Lambda[[x_1, \dots, x_{m-1}]]$ is formally smooth over Λ, there exists a dotted map $\tilde{h}$. Because of the isomorphism (1), R modulo the maximal ideal of $\Lambda[[x_1, \dots, x_{m-1}]]$ is k and hence, by Nakayama's lemma, the map $\tilde{h}$ is surjective.

Now we see that $\tilde{h}$ must be an isomorphism: Assume, this is not the case. Then $\dim R < m$, which is in conflict with the isomorphism $\Lambda[[x_1, \dots, x_m]] \cong R\hat{\otimes}_\Lambda \Lambda[[x]] \cong R[[x]]$. □

Lemma 3.7. *Let $\Delta \in \mathcal{C}_\Lambda$ such that the structure morphism $\Lambda \to \Delta$ is flat and*

$$R = \Lambda[[x_1, \dots, x_d]]/(f_1, \dots, f_b) \tag{2}$$

for some $f_1, \dots, f_b$ in $\Lambda[[x_1, \dots, x_d]]$. Then R is formally smooth of relative dimension $d \in \mathbb{N}$ over Λ if and only if $\Delta\widehat{\otimes}_\Lambda R$ is formally smooth of relative dimension d over Δ.

Proof. Let $I := (f_1, \dots, f_b)$. Let $\mathfrak{m}$ denote the maximal ideal of R and $b = \dim I/\mathfrak{m}.I$, and let $\mathfrak{m}'$ denote the maximal ideal of $R' := \Delta\widehat{\otimes}_\Lambda R$ and $b' = \dim I/\mathfrak{m}'.I$.

To see that b equals $b' := \dim_{\Delta/\mathfrak{m}'} \Delta \otimes_\Lambda I/\mathfrak{m}'.\Delta \otimes_\Lambda I$, we use the isomorphism

$$\Delta \otimes_\Lambda I/\mathfrak{m}'.\Delta \otimes_\Lambda I \cong I/\mathfrak{m}.I \otimes_{\Lambda/\mathfrak{m}} \Delta/\mathfrak{m}'$$

and the fact that $\Lambda/\mathfrak{m} \to \Delta/\mathfrak{m}'$ is a monomorphism of fields:

$$b = \dim_{\Lambda/\mathfrak{m}} I/\mathfrak{m}.I = \dim_{\Delta/\mathfrak{m}'} \Delta \otimes_\Lambda I/\mathfrak{m}'.\Delta \otimes_\Lambda I = b'.$$ □

Liftings and deformations of G-valued representations.

Definition 3.8. (1) A lifting of $\bar{\rho}$ to an object $A \in \mathcal{C}_\Lambda$ is a continuous group homomorphism $\rho\colon \Gamma \to G(A)$ fulfilling $\mathrm{mod}_{\mathfrak{m}_A} \circ \rho = \bar{\rho}$, where $\mathrm{mod}_{\mathfrak{m}_A}\colon G(A) \to G(A/\mathfrak{m}_A) = G(k)$ is the canonical reduction.

(2) Denote by $D^\square_\Lambda(\bar{\rho})\colon \mathcal{C}_\Lambda \to \mathrm{Sets}$ the functor which assigns to an object $A \in \mathcal{C}_\Lambda$ the set of all liftings of $\bar{\rho}$ to A.

By [Balaji 2012, Theorem 1.2.2], $D^\square_\Lambda(\bar{\rho})$ is representable by an object $R^\square_\Lambda(\bar{\rho}) \in \mathcal{C}_\Lambda$. As an examination of its proof easily yields, we get (with respect to the ring of integers Λ' of some finite extension of $\mathrm{Quot}(\Lambda)$ with residue field $k_{\Lambda'} = k$) an isomorphism

$$R^\square_{\Lambda'}(\bar{\rho}) \cong \Lambda'\widehat{\otimes}_\Lambda R^\square_\Lambda(\bar{\rho}). \tag{3}$$

Definition 3.9. A lifting condition is a family $\mathcal{D} = (S(A))_{A\in\mathcal{C}_\Lambda}$ of subsets $S(A) \subset D^\square_\Lambda(\bar{\rho})(A)$ such that:

(1) $\bar{\rho} \in S(k)$.

(2) If $f\colon A \to B$ is a morphism in $\mathcal{C}_\Lambda$ and $\rho \in S(A)$, then $G(f) \circ \rho \in S(B)$.

(3) Let $f_1\colon A_1 \to A$, $f_2\colon A_2 \to A$ be morphisms in $\mathcal{C}_\Lambda$ and let ρ_3 be a lifting of $\bar{\rho}$ to $A_3 := A_1 \times_A A_2$. For $i = 1, 2$ denote by $\pi_i : A_3 \to A_i$ the canonical map and by ρ_i the lifting $G(\pi_i) \circ \rho_3$ of $\bar{\rho}$ to A_i. Then, $\rho_3 \in S(A_3)$ if and only if $\rho_1 \in S(A_1)$ and $\rho_2 \in S(A_2)$.

Condition (2) guarantees that $\mathcal{D}$ defines a subfunctor $D^{\square,\mathcal{D}}_\Lambda(\bar{\rho}) \subset D^\square_\Lambda(\bar{\rho})$. Condition (3) is a variation of the Mayer–Vietoris property, so a standard argument yields:

Proposition 3.10. *$D^{\square,\mathcal{D}}_{\Lambda}(\bar\rho)$ is a relatively representable subfunctor (in the sense of [Mazur 1997, Section 19]) of $D^{\square}_{\Lambda}(\bar\rho)$, i.e., representable by some $R^{\square,\mathcal{D}}_{\Lambda}(\bar\rho) \in \mathcal{C}_\Lambda$. On the other hand, any representable subfunctor $F \subset D^{\square}_{\Lambda}(\bar\rho)$ yields a lifting condition $\mathcal{D} = (S(A))_{A\in\mathcal{C}_\Lambda}$ via $S(A) := F(A)$.*

We have the following conditioned version of (3):

$$R^{\square,\mathcal{D}'}_{\Lambda'}(\bar\rho) \cong \Lambda'\widehat{\otimes}_\Lambda R^{\square,\mathcal{D}}_{\Lambda}(\bar\rho), \tag{4}$$

where the condition $\mathcal{D}'$ on the left is a truncated version of $\mathcal{D}$, i.e., denotes the family of those $S(A)$ as in the definition of $\mathcal{D}$ for which $A \in \mathcal{C}_{\Lambda'}$. We will often omit this distinction and write $\mathcal{D}$ in place of $\mathcal{D}'$.

Remark 3.11. Let Λ be as above and let ${}^*\mathcal{C}_\Lambda$ denote the category of complete Noetherian local Λ-algebras A such that $[k_A : k]$ is finite. Then one can extend $D^{\square}_{\Lambda}(\bar\rho)$ to a functor on ${}^*\mathcal{C}_\Lambda$ by considering A-valued liftings of $\bar\rho$ as continuous group homomorphisms $\rho : \Gamma \to G(A)$ which fulfill $\mathrm{mod}_{\mathfrak{m}_A} \circ \rho = \iota_{k\subset k_A} \circ \bar\rho$, where $\iota_{k\subset k_A} : G(k) \to G(k_A)$ is the map induced by the structure map $\Lambda \to A$. It is easy to check that this extended functor is representable by the same universal object $R^{\square}_{\Lambda}(\bar\rho)$ as the functor from Definition 3.8. Moreover, if Λ' is the ring of integers of some finite extension of $\mathrm{Quot}(\Lambda)$ such that $[k_{\Lambda'} : k] < \infty$, we have the following version of (3):

$$R^{\square}_{\Lambda'}(\iota_{k\subset k_A} \circ \bar\rho) \cong \Lambda'\widehat{\otimes}_\Lambda R^{\square}_{\Lambda}(\bar\rho).$$

Moreover, if $\mathcal{D}$ is an extended lifting condition, i.e., a family $(S(A))_{A\in{}^*\mathcal{C}_\Lambda}$ fulfilling the analogue conditions of Definition 3.9 (with $A, A_i, B \in {}^*\mathcal{C}_\Lambda$), we have the following conditioned version of (4):

$$R^{\square,\mathcal{D}}_{\Lambda'}(\iota_{k\subset k_A} \circ \bar\rho) \cong \Lambda'\widehat{\otimes}_\Lambda R^{\square,\mathcal{D}}_{\Lambda}(\bar\rho),$$

where $\mathcal{D}$ on the left hand side is to be understood as the Λ'-truncated version of the condition $\mathcal{D}$, i.e., a family indexed by ${}^*\mathcal{C}_{\Lambda'}$ instead of ${}^*\mathcal{C}_\Lambda$. Moreover, the statement of Lemma 3.7 holds if Λ' is in ${}^*\mathcal{C}_\Lambda$ instead of $\mathcal{C}_\Lambda$. (The content of this remark is strongly inspired by the treatment in [Conrad et al. 1999, Appendix A] and [Mazur 1997].)

Definition 3.12. (1) A deformation of $\bar\rho$ to $A \in \mathcal{C}_\Lambda$ is an equivalence class of liftings to A, where two lifts are taken to be equivalent if they are conjugate by some element of $\hat G(A) := \ker(\mathrm{mod}_{\mathfrak{m}_A})$.

(2) Denote by $D_\Lambda(\bar\rho) : \mathcal{C}_\Lambda \to \mathrm{Sets}$ the functor which assigns to an object $A \in \mathcal{C}_\Lambda$ the set of all deformations of $\bar\rho$ to A.

For the following, denote by Z_G the center of G and by $\mathfrak{g}$ (resp. by $\mathfrak{z}$) the Lie algebra of the special fiber of G (resp. of Z_G). We assume from now on that Z_G is formally smooth over Λ.

Theorem 3.13 [Tilouine 1996, Theorem 3.3]. *If $H^0(\Gamma, \mathfrak{g}) = \mathfrak{z}$ then $D_\Lambda(\bar\rho)$ is representable by an object $R_\Lambda(\bar\rho) \in \mathcal{C}_\Lambda$.*

Observe that in the case $G = \mathrm{GL}_n$, the condition of Theorem 3.13 becomes the usual centralizer condition $\mathrm{End}_{k[\Gamma]}(\bar\rho) = k$. In practice, this is often deduced from absolute irreducibility of $\bar\rho$ by Schur's lemma. This reasoning can be adopted to more general groups G as follows:

Definition 3.14 (Absolute irreducibility, see [Serre 1998]). We say that $\bar\rho$ is absolutely irreducible if there does not exist a proper parabolic subgroup $P \subsetneq G$ over $\bar k$ such that $\bar\rho(\Gamma) \subset P$.

Then the following can be deduced from [Bate et al. 2005, Proposition 2.13]:

Lemma 3.15 (Schur's lemma). *Assume that ℓ is very good for G (see [Bate et al. 2010, Section 2]) or that there exists an embedding $G \hookrightarrow \mathrm{GL}(V)$ such that $(\mathrm{GL}(V), G)$ is a reductive pair (in the sense of [Bate et al. 2005, Definition 3.32]). Then $H^0(\Gamma, \mathfrak{g}) = \mathfrak{z}$ if $\bar\rho$ is absolutely irreducible.*

We now give an appropriate version of Definition 3.9:

Definition 3.16. A deformation condition is a lifting condition in the sense of Definition 3.9 which fulfills additionally:

(4) If $\rho \in S(A)$ and $g \in \hat{G}(A)$, then $g\rho g^{-1} \in S(A)$.

This defines a relatively representable subfunctor $D_\Lambda^{\mathcal{D}}(\bar\rho)$ of $D_\Lambda(\bar\rho)$: If $D_\Lambda(\bar\rho)$ is representable, then so is $D_\Lambda^{\mathcal{D}}(\bar\rho)$ and the representing object $R_\Lambda^{\mathcal{D}}(\bar\rho)$ is a quotient of $R_\Lambda(\bar\rho)$. In addition to the conditions appearing in Section 5 below, we will be interested in the following conditions:

(1) If $\Delta \subset \Gamma$ is a profinite subgroup and $\bar\rho(\Delta) = \{1\}$, then the assignment $S(A) := \{\rho \mid \rho(\Delta) = \{1\}\}$ defines a deformation condition. In the case $\Gamma = \mathrm{Gal}_K$ for a local field K and $\Delta = I_K$, we call this the unramified lifting condition and write $D_\Lambda^{(\square),\mathtt{nr}}(\bar\rho)$ for the corresponding subfunctor.

(2) Fix a representation $\chi : \Gamma \to G^{\mathrm{ab}}(\Lambda)$ such that $d(k) \circ \bar\rho = \bar\chi$, where $d\colon G \to G^{\mathrm{ab}}$ is the canonical projection modulo the derived subgroup G^{der} and where $\bar\chi$ denotes the reduction of χ. In accordance with the case $G = \mathrm{GL}_n$, we call this the fixed determinant condition and write $D_\Lambda^{(\square),\chi}(\bar\rho)$ for the corresponding subfunctor.

(3) Let F be a number field and let $\Sigma \subset S \subset \mathrm{Pl}_F$ be a finite set of finite places. Let $\Gamma = \mathrm{Gal}_{F,S}$ be the Galois group of the maximal unramified outside S extension F_S of F, and fix for each $v \in \Sigma$ a local condition D_v of the functor $D_\Lambda^{(\square)}(\bar\rho_v)$, where $\bar\rho_v$ denotes the restriction of $\bar\rho$ to a decomposition group at v. Then the assignment $S(A) = \{\rho \mid \rho_v \in D_\Lambda^{(\square),D_v}(\bar\rho_v)\forall v \in \Sigma\}$ defines a global deformation condition, denoted by $\mathcal{D} = (D_v)_{v\in\Sigma}$. The afforded subfunctor of $D_\Lambda^{(\square)}(\bar\rho)$ is denoted by $D_\Lambda^{(\square),\mathcal{D}}(\bar\rho)$.

(4) If Γ, F, Σ are as above and if $\bar\rho$ is unramified outside Σ, then requiring that a lift ρ is unramified outside Σ defines a global deformation condition, and we denote the corresponding subfunctor by $D_{\Sigma,\Lambda}^{(\square)}(\bar\rho)$. It is easily seen that studying these lifts is equivalent to studying unconditioned lifts of $\bar\rho$, understood as a representation of the Galois group $\mathrm{Gal}_{F,\Sigma}$ of the maximal, unramified outside Σ, extension F_Σ of F.

It is easily seen that decreeing multiple conditions defines another condition, i.e., it makes sense to write for example $D_\Lambda^{\square,\chi,\mathtt{nr}}(\bar\rho)$.

Multiply framed deformations. Fix finite subsets $\Sigma \subset S \subset \mathrm{Pl}_F$ such that $\bar\rho$ is unramified outside S and continue to denote $\Gamma = \mathrm{Gal}_{F,S}$.

Definition 3.17. Following [Khare and Wintenberger 2009b, Section 4.1.1], we define the functor $D^{\square_\Sigma}_\Lambda(\bar\rho) : \mathcal{C}_\Lambda \to \mathrm{Sets}$ by mapping A to

$$\{(\rho, (\rho_v, \beta_v)_{v\in\Sigma}) \mid \rho \in D^{\square}_\Lambda(\bar\rho)(A),\ \rho_v \in D^{\square}_\Lambda(\bar\rho_v)(A),\ \beta_v \in \hat G(A) \text{ such that } \rho \mid \mathrm{Gal}(F_v) = \beta_v \rho_v \beta_v^{-1}\}/\sim$$

where $(\rho, (\rho_v, \beta_v)_{v\in\Sigma})$ and $(\rho', (\rho'_v, \beta'_v)_{v\in\Sigma})$ are taken to be equivalent if $\rho_v = \rho'_v$ for all v and if there is a $\gamma \in \hat G(A)$ such that $\rho' = \gamma\rho\gamma^{-1}$ and $\beta'_v = \gamma\beta_v$ for all v.

Note that specifying the ρ_v is not strictly necessary, as they can be obtained from ρ and β_v. We can impose a deformation condition $\mathcal{D} = (S(A))_{A\in\mathcal{C}_\Lambda}$ on multiply framed deformations in the same way we did for liftings and deformations, i.e., we allow only those triples $(\rho, (\rho_v, \beta_v)_{v\in\Sigma})$ for which $\rho \in S(A)$. The following assertions are immediate; see [Khare and Wintenberger 2009b, Proposition 4.1] or [Guiraud 2016, Proposition 2.62]:

Proposition 3.18. (1) *$D^{\square_\Sigma,(\chi),\mathcal{D}}_{[S],\Lambda}$ is representable and we denote the afforded deformation ring by $R^{\square_\Sigma,(\chi),\mathcal{D}}_{[S],\Lambda}$ (if $\Sigma = \varnothing$, we have to assume $H^0(\Gamma, \mathfrak{g}) = \mathfrak{z}$).*

(2) *If $\#\Sigma = 1$, then the functors $D^{\square_\Sigma,(\chi),\mathcal{D}}_{[S],\Lambda}$ and $D^{\square,(\chi),\mathcal{D}}_{[S],\Lambda}$ are naturally isomorphic.*

(3) *If $\Sigma \neq \varnothing$, then*

$$R^{\square_\Sigma,(\chi),\mathcal{D}}_{[S],\Lambda} \cong R^{\square,(\chi),\mathcal{D}}_{[S],\Lambda}[[x_1, \dots, x_t]]$$

and, if $H^0(\Gamma, \mathfrak{g}) = \mathfrak{z}$, then also

$$R^{\square,(\chi),\mathcal{D}}_{[S],\Lambda} \cong R^{(\chi),\mathcal{D}}_{[S],\Lambda}[[x_1, \dots, x_u]]$$

with $t = \dim(\mathfrak{g}).(\#\Sigma - 1)$, $u = \dim(\mathfrak{g}) - \dim(\mathfrak{z}) = \dim(\mathfrak{g}^{\mathrm{der}})$.

From now on, let us suppose

Assumption 3.19. $H^0(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathrm{der}}) = 0$.

With respect to a deformation condition $\mathcal{D} = (D_v)_{v\in\Sigma}$ as in Example (3) above, we set

$$R^{\mathrm{loc}_\Sigma,(\chi),\mathcal{D}}_\Lambda(\bar\rho) := \widehat{\bigotimes}_{v\in\Sigma} R^{\square,(\chi_v),D_v}_\Lambda(\bar\rho_v).$$

The following is essentially a special case of [Balaji 2012, Proposition 4.2.5] (which goes back to [Kisin 2007, Proposition 4.1.5]):

Proposition 3.20. *If Σ contains all infinite places, $H^0(\Gamma, \mathfrak{g}^{\mathrm{der},\wedge}) = 0$, and $D^{(\chi)}_\Lambda(\bar\rho)$ is representable, then*

$$R^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}(\bar\rho) \cong R^{\mathrm{loc}_\Sigma,(\chi),\mathcal{D}}_\Lambda(\bar\rho)[[x_1, \dots, x_{a+b}]]/(f_1, \dots, f_a)$$

for suitable $a \in \mathbb{N}$, $f_i \in R^{\mathrm{loc}_\Sigma,(\chi),\mathcal{D}}_\Lambda(\bar\rho)[[x_1, \dots, x_{a+b}]]$ and with $b = 0$ if the determinant is not fixed (resp. $b = (\#\Sigma - 1).\dim(\mathfrak{g}^{\mathrm{ab}})$ if the determinant is fixed).

Corollary 3.21. *Assume that each* $R^{\square,(\chi_v),D_v}(\bar\rho_v)$ *is a complete intersection ring of relative dimension* d_v *over* Λ. *Assume moreover that* $D^{\mathcal{D}}_{[S],\Lambda}(\bar\rho)$ *is representable and that* $d := \sum_{v\in\Sigma} d_v > \dim(\mathfrak{g}).\#\Sigma - \dim(\mathfrak{z}) - b$ (*with* b *as in Proposition 3.20*). *Then there exists a presentation*

$$R^{\mathcal{D}}_{[S],\Lambda}(\bar\rho) \cong \Lambda[\![x_1, \dots, x_m]\!]/(f_1, \dots, f_m)$$

for a suitable $m \in \mathbb{N}$.

Proof. Using Proposition 3.20 and the assumption on $\mathcal{D}$, we can write

$$R^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}(\bar\rho) \cong R^{\mathtt{loc}_\Sigma,(\chi),\mathcal{D}}_{\Lambda}(\bar\rho)[\![x_1, \dots, x_{a+b}]\!]/(f_1, \dots, f_a) \cong \Lambda[\![x_1, \dots, x_{a+b+c+d}]\!]/(f_1, \dots, f_{a+c})$$

for a, b as above and for a suitably chosen $c \in \mathbb{N}_0$. On the other hand, by Cohen's structure theorem we can write $R^{(\chi),\mathcal{D}}_{S,\Lambda}(\bar\rho) \cong \Lambda[\![x_1, \dots, x_u]\!]/(f_1, \dots, f_v)$ for suitable $u, v \in \mathbb{N}_0$ (and we assume that this is a minimal presentation, i.e., that the quantity $u - v$ is maximal among all ways to write $R^{(\chi),\mathcal{D}}_{S,\Lambda}(\bar\rho)$ as a quotient of a power series ring), so by the third part of Proposition 3.18 we have

$$R^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}(\bar\rho) \cong R^{(\chi),\mathcal{D}}_{S,\Lambda}(\bar\rho)[\![x_1, \dots, x_r]\!] \cong \Lambda[\![x_1, \dots, x_{r+u}]\!]/(f_1, \dots, f_v)$$

with $r = \dim(\mathfrak{g}).\#\Sigma - \dim(\mathfrak{z})$. Comparing these two presentations, we get

$$u - v + \dim(\mathfrak{g}).\#\Sigma - \dim(\mathfrak{z}) \ge b + d \Rightarrow u - v \ge b + d - \dim(\mathfrak{g}).\#\Sigma + \dim(\mathfrak{z}).$$

Thus, the claim follows immediately from our assumption on d. □

Tangent spaces and systems of local conditions. With respect to a deformation condition $\mathcal{D}$ will consider the tangent space $t_{D^{(\square)}} = D^{(\square),\mathcal{D}}_{\Lambda}(k[\epsilon])$, which we consider as a (finite-dimensional) k-vector space (see [Gouvêa 2001, Lecture 2]). There are canonical isomorphisms

$$t_{D^{(\chi)}} \cong Z^1(\Gamma, \mathfrak{g}^{(\mathrm{der})}),\ t_D \cong H^1(\Gamma, \mathfrak{g}) \quad \text{and} \quad t_{D^\chi} \cong H^1(\Gamma, \mathfrak{g}^{\mathrm{der}})' := \mathrm{im}(H^1(\Gamma, \mathfrak{g}^{\mathrm{der}}) \to H^1(\Gamma, \mathfrak{g})),$$

so via the embedding $D^{(\chi),\mathcal{D}}_{\Lambda}(k[\epsilon]) \hookrightarrow D^{(\chi)}_{\Lambda}(k[\epsilon])$ we are provided with an assignment $\mathcal{D} \mapsto L(\mathcal{D})^{(\chi)} := D^{(\chi),\mathcal{D}}_{\Lambda}(k[\epsilon])$ from deformation conditions to subspaces of $H^1(\Gamma, \mathfrak{g})$ (resp. $H^1(\Gamma, \mathfrak{g}^{\mathrm{der}})'$). In the case $\Gamma = \mathrm{Gal}_F$ for a number field F and if $\mathcal{D} = (D_v)_{v\in\Sigma}$, we call the afforded family $\mathcal{L}^{(\chi)} = (L(D_v))_{v\in \mathrm{Pl}_F}$ of subspaces of $H^1(\mathrm{Gal}_{F_v}, \mathfrak{g})$ (resp. of $H^1(\mathrm{Gal}_{F_v}, \mathfrak{g}^{\mathrm{der}})'$) a *system of local conditions*. Also note that there is an exact sequence

$$0 \to \mathfrak{g}/\mathfrak{g}^\Gamma \to t_{D^{(\chi)}} \to t_{D^{(\chi)}}$$

where, in case $\ell \gg 0$ (such that $\mathfrak{g} = \mathfrak{g}^{\mathrm{der}} \oplus \mathfrak{g}^{\mathrm{ab}}$), the object $\mathfrak{g}/\mathfrak{g}^\Gamma$ can be replaced by $\mathfrak{g}^{\mathrm{der}}/(\mathfrak{g}^{\mathrm{der}})^\Gamma$.

Liftings at infinity.

Proposition 3.22. *Assume* $\Gamma = \mathbb{Z}/2\mathbb{Z} = \{1, c\}$ *and* $\ell = \mathrm{char}(\mathbb{F}) \ne 2$. *Then*

$$R^{\square}_{\Lambda}(\bar\rho) \cong \Lambda[\![x_1, \dots, x_m]\!] \quad \textit{with } m = \dim(\mathfrak{g}^{c=-1}).$$

If ψ *is a lift of the determinant, then the same result holds for* $R^{\square,\psi}_{\Lambda}(\bar\rho)$ *after replacing* $\mathfrak{g}$ *by* $\mathfrak{g}^{\mathrm{der}}$.

Proof. We use the general formula $H^2(\mathbb{Z}/n\mathbb{Z}, M) = M^{\mathbb{Z}/n\mathbb{Z}}/\operatorname{im}(\varphi)$ with

$$\varphi : M \to M \quad m \mapsto \sum_{j=0}^{n-1} j.m.$$

Because $\ell > 2$, we have $H^2(\mathbb{Z}/2\mathbb{Z}, \mathfrak{g}) = H^2(\{1, c\}, \mathfrak{g}) = 0$ and the lifting ring is unobstructed. To get the number of variables we have to evaluate

$$Z^1(\{1, c\}, \mathfrak{g}) = \{f : \{1, c\} \to \mathfrak{g} \mid f(xy) = f(x) + {}^x f(y)\}.$$

Looking at $x = y = c$, we see that f is uniquely determined by a vector $v = f(c)$. Looking at $x = 1$, $y = c$, we see that $f(1) = v + {}^c v = 0$, i.e., that $v \in \mathfrak{g}^{c=-1}$. On the other hand, any such v defines an $f \in Z^1$ via $1 \mapsto 0$, $c \mapsto v$.

The modifications of this argument for the fixed-determinant case are straight-forward. □

A simple criterion for the vanishing of cohomology groups. Now assume that $\Gamma = \mathrm{Gal}_K$ for a local field K. Recall that, by local Tate duality, the Pontryagin dual of $H^2(\Gamma, \mathfrak{g})$ can be identified with $H^0(\Gamma, \mathfrak{g}^\vee) = (\mathfrak{g}^\vee)^\Gamma$. Together with the identification of $(\operatorname{ad}\bar\rho^{(0)})^\vee$ and $(\operatorname{ad}\bar\rho^{(0)})(1)$ via the trace pairing, this implies the following criterion for the vanishing of $H^2(\Gamma, \mathfrak{g}^{\mathrm{der}})$ in the case $G = \mathrm{GL}_n$:

Lemma 3.23 (Local case). *Let Γ be the absolute Galois group of a nonarchimedean local field, k be a finite field of characteristic ℓ and*

$$\bar\rho : \Gamma \to \mathrm{GL}_n(k)$$

a representation.

(1) *If $\mathrm{Hom}_\Gamma(\bar\rho, \bar\rho(1))$ vanishes, then $H^2(\Gamma, \operatorname{ad}\bar\rho)$ vanishes.*

(2) *Assume that $\ell \nmid n$. Then, if $\mathrm{Hom}_\Gamma(\bar\rho, \bar\rho(1))$ vanishes, also $H^2(\Gamma, \operatorname{ad}\bar\rho^0)$ vanishes.*

In the global case, there is no such duality and we record the following:

Lemma 3.24 (Global case). *Let $\Gamma = \mathrm{Gal}_{F,S}$ for a number field F and a (possibly) finite set S of places of F. Let k, $\bar\rho$ be as in Lemma 3.23 above.*

(1) *If $\mathrm{Hom}_\Gamma(\bar\rho, \bar\rho(1))$ vanishes, then $H^0(\Gamma, (\operatorname{ad}\bar\rho)^\vee)$ vanishes.*

(2) *Assume that $\ell \nmid n$. Then, if $\mathrm{Hom}_\Gamma(\bar\rho, \bar\rho(1))$ vanishes, also $H^0(\Gamma, (\operatorname{ad}\bar\rho^0)^\vee)$ vanishes.*

We easily deduce the following result, which also implies the vanishing of the error term δ in [Böckle 2013] (see Remark 5.2.3(d) of that work) for large ℓ:

Corollary 3.25. *There exists a constant C, depending only on n and F, such that Assumption 3.19 holds if $\mathrm{char}(k) > C$, $G = \mathrm{GL}_n$ and $\bar\rho$ is irreducible.*

Unobstructedness.

Definition 3.26. The functor $D_\Lambda^{(\square),[\chi]}(\bar\rho)$ is called unobstructed if $h^2(\Gamma, \mathfrak{g}^{[\mathrm{der}]}) = 0$.

Definition 3.27. A relatively representable subfunctor of $D_\Lambda^{(\square),[\chi]}(\bar\rho)$ is called smooth (of dimension m) if its representing object is isomorphic to $\Lambda[\![x_1, \dots, x_m]\!]$.

The most apparent application of the unobstructedness-property is that it implies the smoothness of the lifting/deformation ring; see [Böckle 2007]: Assume that $D_\Lambda^{\square,(\chi)}(\bar\rho)$ is smooth and (in the fixed-determinant case) that $\ell \gg 0$ and (in the unframed case) that $D_\Lambda^{(\chi)}(\bar\rho)$ is representable. Then

$$D_\Lambda^{\square,(\chi)}(\bar\rho) \cong \Lambda[\![x_1, \dots, x_{a(+c)}]\!] \quad \text{and} \quad D_\Lambda^{(\chi)}(\bar\rho) \cong \Lambda[\![x_1, \dots, x_{b(+c)}]\!]$$

with $b = h^1(\Gamma, \mathfrak{g})$, $c = h^1(\Gamma, \mathfrak{g}^{\mathrm{der}})' - b$, $a = b + \dim(\mathfrak{g}^{(\mathrm{der})}) - h^0(\Gamma, \mathfrak{g}^{(\mathrm{der})})$. The converse direction (i.e., that smoothness implies unobstructedness) is known not to hold (for general profinite groups Γ); see [Sprang 2013].

In order to relax this notion to functors corresponding to deformation conditions, we restrict to the case $\Gamma = \mathrm{Gal}_{F,S}$. Let $\mathcal{D}^{(\chi)} = (D_v^{(\chi)})_{v\in \mathrm{Pl}_F}$ be a system of deformation conditions and $\mathcal{L}^{(\chi)} = (L_v^{(\chi)})_{v\in \mathrm{Pl}_F}$ the corresponding system of local conditions.

Denote by $\mathfrak{g}^{(\mathrm{der}),\vee}$ the Tate dual of $\mathfrak{g}^{\mathrm{der}}$ and by $L_v^{(\chi),\perp}$ the annihilator of $L_v^{(\chi)}$ under the Tate pairing

$$H^i(F_v, \mathfrak{g}^{\mathrm{der},\vee}) \times H^{2-i}(F_v, \mathfrak{g}^{\mathrm{der}}) \to H^2(F_v, k(1)) \cong \mathbb{Q}/\mathbb{Z}$$

for $i = 1$; see [Neukirch et al. 2008, (7.2.6) Theorem]. Then we denote the corresponding dual Selmer group by

$$H^1_{\mathcal{L}^{(\chi),\perp}}(F, \mathfrak{g}^{(\mathrm{der}),\vee}) := \ker\Bigl(\bigoplus_{v\in \mathrm{Pl}} \mathrm{res}_v : H^1(F, \mathfrak{g}^{(\mathrm{der}),\vee}) \to \bigoplus_{v\in \mathrm{Pl}} H^1(F_v, \mathfrak{g}^{(\mathrm{der}),\vee})/L_v^{(\chi),\perp}\Bigr).$$

From now on, let us assume that $D_v^{(\chi)}$ for $v \notin S$ parametrizes unramified deformations.

Definition 3.28. We say that $D_{S,\Lambda}^{\mathcal{D}^{(\chi)}}(\bar\rho)$ (or $D_{S,\Lambda}^{\square,\mathcal{D}^{(\chi)}}(\bar\rho)$, or $D_{S,\Lambda}^{\square_\Sigma,\mathcal{D}^{(\chi)}}(\bar\rho)$ for some set of places Σ) has vanishing dual Selmer group if $H^1_{\mathcal{L}^{(\chi),\perp}}(F, \mathfrak{g}^{(\mathrm{der}),\vee}) = 0$.

Definition 3.29. Let $\boldsymbol{m} = (m_v)_{v\in S} \in \mathbb{N}_0^S$. We say that $D_{S,\Lambda}^{\mathcal{D}^{(\chi)}}(\bar\rho)$ (or $D_{S,\Lambda}^{\square,\mathcal{D}^{(\chi)}}(\bar\rho)$, or $D_{S,\Lambda}^{\square_\Sigma,\mathcal{D}^{(\chi)}}(\bar\rho)$) is globally unobstructed (of local dimensions $\boldsymbol{m}$) if its dual Selmer group vanishes and if each $D_\Lambda^{\square,\mathcal{D}^{(\chi_v)}}(\bar\rho_v)$ for $v \in S$ is smooth (of dimension m_v).

We remark that if $D_{S,\Lambda}^{\mathcal{D}^{(\chi)}}(\bar\rho)$ is globally unobstructed and representable, then by [Böckle 2007, Theorem 5.2] the representing object $R_{S,\Lambda}^{\mathcal{D}^{(\chi)}}(\bar\rho)$ is isomorphic to a power series ring in $h^1_{\mathcal{L}^{(\chi)}}(F, \mathfrak{g}^{(\mathrm{der})})^{(\prime)}$ variables.

Since $Ш^2_S(\mathfrak{g}^{(\mathtt{der})}) := H^1_{\mathcal{L}^\perp}(F, \mathfrak{g}^{(\mathtt{der}),\vee})^*$ vanishes,[2] the following short exact sequence results directly from that of [Böckle 2007, page 7]

$$0 \to Ш^2_S(\mathfrak{g}^{(\mathtt{der})}) \to H^2(\mathrm{Gal}_{F,S}, \mathfrak{g}^{(\mathtt{der})}) \to \bigoplus_{\nu\in S} H^2(F_\nu, \mathfrak{g}^{(\mathtt{der})}) \to H^0(F, \mathfrak{g}^{(\mathtt{der}),\vee})^* \to 0,$$

where $H^0(F, \mathfrak{g}^{(\mathtt{der}),\vee})^*$ vanishes for $\ell \gg 0$.

Proposition 3.30. *Assume that $D^{(\chi_\nu)}_\Lambda(\bar\rho_\nu)$ is unobstructed (for all $\nu \in S$) and that $D^{(\chi)}_{S,\Lambda}(\bar\rho)$ is globally unobstructed (without making an assumption on the dimension). Then $D^{(\chi)}_{S,\Lambda}(\bar\rho)$ is unobstructed in the sense of Definition 3.26. For $\ell \gg 0$, also the converse is true.*

4. A general framework for unobstructedness

For this section, we take the following static point of view: Let k be a finite field with ring of Witt vectors $W = W(k)$, let S be a finite set of finite places of F. We assume $\ell := \mathrm{char}(k) \notin S \cup \{2\}$. Then we fix a continuous representation

$$\bar\rho\colon\ \mathrm{Gal}_{F,S} \to G(k)$$

together with a lift $\chi\colon\ \mathrm{Gal}_{F,S} \to G^{\mathrm{ab}}(W)$ of the determinant. Let us moreover fix a Borel subgroup $B \subset G$ and denote by $\mathfrak{g}^{\mathtt{der}}$ (resp. $\mathfrak{b}^{\mathtt{der}}$) the Lie algebra of the derived subgroup $G^{\mathtt{der}}$ (resp. the Lie algebra of $B \cap G^{\mathtt{der}}$).

With respect to some choice of local deformation conditions[3]

- $\mathtt{min}$ of the restriction $\bar\rho_\nu$ of $\bar\rho$ to a decomposition group at $\nu \in S$,
- $\mathtt{sm}$ and $\mathtt{crys}$ of the restriction $\bar\rho_\nu$ of $\bar\rho$ to a decomposition group at $\nu \mid \ell$,

consider the following list of assumptions, where we leave out the W in the subscript of the occurring deformation functors and rings:

(sm/k) For each $\nu \mid \ell$, the subfunctor $D^{\square,\chi_\nu,\mathtt{sm}}(\bar\rho_\nu)$ of $D^{\square,\chi_\nu}(\bar\rho_\nu)$ is representable by a formally smooth (over W) object $R^{\square,\chi_\nu,\mathtt{sm}}_\nu$ (and we denote the relative dimension by $d^{\square,\mathtt{sm}}_\nu$).

(crys) For each $\nu \mid \ell$, the subfunctor $D^{\square,\chi_\nu,\mathtt{crys}}(\bar\rho_\nu)$ of $D^{\square,\chi_\nu}(\bar\rho_\nu)$ is representable by a formally smooth (over W) object $R^{\square,\chi_\nu,\mathtt{crys}}_\nu$ of relative dimension

$$d^{\square,\mathtt{crys}}_\nu = \dim(\mathfrak{g}^{\mathtt{der}}) + (\dim(\mathfrak{g}^{\mathtt{der}}) - \dim(\mathfrak{b}^{\mathtt{der}}))[F_\nu : \mathbb{Q}_\ell].$$

[2] We remark that the vanishing of the "Tate–Shafarevich group" $Ш^2_S(\mathfrak{g}^{(\mathtt{der})})$ implies that all obstructions for $D^{(\chi)}_\Lambda(\bar\rho_\nu)$ come from local obstructions, see [Böckle 2007, Theorem 3.1].

[3] During the following applications of the presented material, we will consider for $\mathtt{min}$ the condition of Section 5D, for $\mathtt{crys}$ the condition of Section 5C and for $\mathtt{sm}$ the unconditioned deformation condition. We stress, however, that for the purpose of this section we treat $\mathtt{min}$, $\mathtt{crys}$, $\mathtt{sm}$ purely formally as deformation conditions satisfying the listed assumptions of Definition 3.9.

(min) For each $v \in S$, the subfunctor $D^{\square,\chi_v,\mathtt{min}}(\bar\rho_v)$ of $D^{\square,\chi_v}(\bar\rho_v)$ is representable by a formally smooth (over W) object $R_v^{\square,\chi_v,\mathtt{min}}$ of relative dimension

$$d_v^{\square,\mathtt{min}} = \dim(\mathfrak{g}^{\mathtt{der}}).$$

(∞) For each $v \mid \infty$, the functor $D^{\square,\chi_v}(\bar\rho_v)$ is representable by an object (over W) of relative dimension $d_v^{\square} = \dim(\mathfrak{b}^{\mathtt{der}})$. (As $\ell > 2 = \#\operatorname{Gal}_{F_v}$, the strict ℓ-cohomological dimension $\operatorname{scd}_\ell(\operatorname{Gal}_{F_v})$ is zero, i.e., the representing object is automatically formally smooth over W.)

(Presentability) There exists a presentation

$$R_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min},\mathtt{sm}} \cong R_{S_\ell}^{\mathtt{loc},\mathtt{min},\mathtt{sm}}[\![x_1,\dots,x_a]\!]/(f_1,\dots,f_b)$$

for integers a, b fulfilling $a - b = (\#S_\ell - 1).\dim(\mathfrak{g}^{\mathrm{ab}})$. In this equation, we take

$$R_{S_\ell}^{\mathrm{loc},\mathtt{min},\mathtt{sm}} = \widehat{\bigotimes_{v\in S_\ell}} \tilde R_v \text{ with } \tilde R_v = \begin{cases} R_v^{\square,\chi_v,\mathtt{min}} & \text{if } v \in S; \\ R_v^{\square,\chi_v,\mathtt{sm}} & \text{if } v \mid \ell; \\ R_v^{\square,\chi_v} & \text{if } v \mid \infty. \end{cases} \tag{5}$$

$(\boldsymbol{R} = \boldsymbol{T})$ The ring $R_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min},\mathtt{crys}}$ is formally smooth of relative dimension

$$r_0 := \dim(\mathfrak{g}).\#S_\ell - \dim(\mathfrak{g}^{\mathrm{ab}}).$$

Remark 4.1 (Taylor–Wiles condition). Let $v \mid \infty$ so that $\operatorname{scd}_\ell(\operatorname{Gal}_{F_v}) = 0$, then it follows from condition (∞), $\operatorname{scd}_\ell(\operatorname{Gal}_{F_v}) = 0$ and the remark following Definition 3.26 that

$$\dim(\mathfrak{b}^{\mathtt{der}}) = \dim_W(R^\square) = h^1(\operatorname{Gal}_{F_v},\mathfrak{g}^{\mathtt{der}})' + \dim(\mathfrak{g}^{\mathtt{der}}) - h^0(\operatorname{Gal}_{F_v},\mathfrak{g}^{\mathtt{der}}) = \dim(\mathfrak{g}^{\mathtt{der}}) - h^0(\operatorname{Gal}_{F_v},\mathfrak{g}^{\mathtt{der}}).$$

This implies

$$\sum_{v\mid\infty} h^0(\operatorname{Gal}_{F_v},\mathfrak{g}^{\mathtt{der}}) = [F:\mathbb{Q}].(\dim\mathfrak{g}^{\mathtt{der}} - \dim(\mathfrak{b}^{\mathtt{der}})). \tag{6}$$

We can now state the main result of this section.

Theorem 4.2. *Suppose the six conditions are met and, for $v \mid \ell$, write $d_v^{\square,\mathtt{sm}} = \dim(\mathfrak{g}^{\mathtt{der}}).([F_v:\mathbb{Q}_\ell]+1)$.*

(1) *The ring $R_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min},\mathtt{sm}}$ is formally smooth of relative dimension*

$$\#S_\ell.\dim(\mathfrak{g}) - \dim(\mathfrak{g}^{\mathrm{ab}}) + [F:\mathbb{Q}].\dim(\mathfrak{b}^{\mathtt{der}}).$$

If the unframed deformation functor $D_{S_\ell}^{\chi,\mathtt{min},\mathtt{sm}}$ is representable, then $R_{S_\ell}^{\chi,\mathtt{min},\mathtt{sm}}$ is formally smooth of relative dimension $[F:\mathbb{Q}].\dim(\mathfrak{b}^{\mathtt{der}})$.

(2) *Let $\mathcal{L} := (L_v^\chi)_v$ be the system of local conditions corresponding to the deformation functor $D_{S_\ell}^{\chi,\mathtt{min},\mathtt{sm}}(\bar\rho)$. Assume:*

(a) $\mathfrak{g} = \mathfrak{g}^{\mathtt{der}} \oplus \mathfrak{g}^{\mathtt{ab}}$ *(e.g., because $\ell \gg 0$).*

(b) $H^0(\operatorname{Gal}_F, \mathfrak{g}^{\mathtt{der},\vee}) = 0$.

(c) *For $v \in S$, we have $\dim(L_v) = h^0(\operatorname{Gal}_{F_v},\mathfrak{g}^{\mathtt{der}})$.*

Then $H^1_{\mathcal{L}^\perp}(\operatorname{Gal}_{F,S},\mathfrak{g}^{\mathtt{der},\vee}) = H^0(\operatorname{Gal}_{F,S},\mathfrak{g}^{\mathtt{der}}) = 0$.

Remark 4.3. (1) As the deformation conditions $\mathtt{sm}$ and $\mathtt{crys}$ are relatively representable (see conditions (1) and (2)), $D_{S_\ell}^{\chi,\mathtt{min,sm}}$ is representable if $D_{S_\ell}^{\chi}$ is representable. For example, this is the case if $\bar{\rho}$ is absolutely irreducible (in the sense of Definition 3.14).

(2) For $v \notin S_\ell$, the equality $\dim(L_v) = h^0(\mathrm{Gal}_{F_v}, \mathfrak{g}^{\mathtt{der}})$ holds automatically if $\ell \gg 0$ (so that $\mathfrak{g} = \mathfrak{g}^{\mathtt{der}} \oplus \mathfrak{g}^{\mathtt{ab}}$).

Proof of Theorem 4.2. First remark that the second claim of part (1) follows by a straightforward lifting argument from Proposition 3.5, as $R_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min,sm}}$ is a power series ring over $R_{S_\ell}^{\chi,\mathtt{min,sm}}$, and from the formula $\dim \mathfrak{g} = \dim \mathfrak{g}^{\mathtt{der}} + \dim \mathfrak{g}^{\mathtt{ab}}$.

For the first sentence of (1), we use the shorthand notation $d_T^\star = \sum_{v\in T} d_v^\star$ for a subset T of Pl_F. Moreover, we write $d_\infty^\square$ for $d_{\Omega_\infty}^\square$ and $d_\ell^\star$ for $d_{\Omega_\ell}^\star$. Let us consider the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & R_{S_\ell}^{\mathtt{loc,min,sm}} & \xrightarrow{f} & R_{S_\ell}^{\mathtt{loc,min,crys}} & \longrightarrow & 0 \\ && \downarrow{\scriptstyle\pi} && \downarrow{\scriptstyle\pi} && \downarrow{\scriptstyle\pi'} && \\ 0 & \longrightarrow & J & \longrightarrow & R_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min,sm}} & \xrightarrow[g]{} & R_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min,crys}} & \longrightarrow & 0\end{array}$$

In this diagram, the right square is a pushout square, $R_{S_\ell}^{\mathtt{loc,min,crys}}$ is defined as in (5) (but with $\tilde{R}_v = R_v^{\square,\chi_v,\mathtt{crys}}$ for $v \mid \ell$) and f, g are the canonical projections. Moreover, $\pi = \otimes_{v\in S_\ell}\pi_v$ is induced from the natural transformations

$$D_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min,crys}} \to \tilde{D}_v,$$

where $\tilde{D}_v$ is the deformation functor corresponding to (i.e., represented by) the ring $\tilde{R}_v$ in (5) and, analogously, $\pi' = \otimes_{v\in S_\ell}\pi'_v$ is defined with $\mathtt{crys}$ in place of $\mathtt{sm}$.

Using the list of assumptions, we can rewrite the above diagram as:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & W[\![x_1,\dots,x_{d_\ell^{\square,\mathtt{sm}}+d_\infty^\square+d_S^{\square,\mathtt{min}}}]\!] & \xrightarrow{f} & W[\![x_1,\dots,x_{d_\ell^{\square,\mathtt{crys}}+d_\infty^\square+d_S^{\square,\mathtt{min}}}]\!] & \longrightarrow & 0 \\ && \downarrow{\scriptstyle\pi} && \downarrow{\scriptstyle\pi} && \downarrow{\scriptstyle\pi'} && \\ 0 & \longrightarrow & J & \longrightarrow & W[\![x_1,\dots,x_m]\!]/(f_1,\dots,f_{m-\gamma}) & \xrightarrow[g]{} & W[\![x_1,\dots,x_{r_0}]\!] & \longrightarrow & 0\end{array}$$

with $\gamma = (\#S_\ell - 1).\dim(\mathfrak{g}^{\mathtt{ab}}) + d_\ell^{\square,\mathtt{sm}} + d_\infty^\square + d_S^{\square,\mathtt{min}}$. By Lemma 3.3, $R_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min,sm}}$ is formally smooth if we can show $\mathrm{gen}(J) \le m - (m-\gamma) - r_0 = \gamma - r_0$. As f is a surjection of regular rings, it follows by the same token that $\mathrm{gen}(I) = d_\ell^{\square,\mathtt{sm}} - d_\ell^{\square,\mathtt{crys}}$. From the pushout property of the diagram, we can easily deduce that $\mathrm{gen}(J) \le \mathrm{gen}(I)$. Thus, we are left to show the inequality

$$\begin{aligned} d_\ell^{\square,\mathtt{sm}} - d_\ell^{\square,\mathtt{crys}} \le \gamma - r_0 &= (\#S_\ell - 1).\dim(\mathfrak{g}^{\mathtt{ab}}) + d_\ell^{\square,\mathtt{sm}} + d_\infty^\square + d_S^{\square,\mathtt{min}} - \dim(\mathfrak{g}).\#S_\ell + \dim(\mathfrak{g}^{\mathtt{ab}}) \\ &= \#S_\ell.(\dim(\mathfrak{g}^{\mathtt{ab}}) - \dim(\mathfrak{g})) + d_\ell^{\square,\mathtt{sm}} + d_\infty^\square + d_S^{\square,\mathtt{min}}. \end{aligned}$$

By assumptions **(min)** and (∞) and by the identity $\dim(\mathfrak{g}^{\mathrm{der}}) + \dim(\mathfrak{g}^{\mathrm{ab}}) = \dim(\mathfrak{g})$, this amounts to

$$d_\ell^{\square,\mathrm{crys}} \geq \dim(\mathfrak{g}^{\mathrm{der}}).(\#\Omega_\ell + [F:\mathbb{Q}]) - \dim(\mathfrak{b}^{\mathrm{der}})[F:\mathbb{Q}].$$

Assumption **(crys)** amounts precisely to the fact that this inequality is fulfilled (with equality), which implies the formal smoothness of $R_{S_\ell}^{\square_{S_\ell},\chi,\mathrm{min},\mathrm{sm}}$. Moreover, we easily check that the relative dimension of $R_{S_\ell}^{\square_{S_\ell},\chi,\mathrm{min},\mathrm{sm}}$ is

$$\begin{aligned}\gamma &= (\#S_\ell - 1).\dim(\mathfrak{g}^{\mathrm{ab}}) + d_\ell^{\square,\mathrm{sm}} + d_\infty^{\square} + d_S^{\square,\mathrm{min}} \\ &= \#S_\ell.\dim \mathfrak{g}^{\mathrm{ab}} - \dim \mathfrak{g}^{\mathrm{ab}} + \dim \mathfrak{g}^{\mathrm{der}}.([F:\mathbb{Q}] + \#\Omega_\ell) + [F:\mathbb{Q}].\dim(\mathfrak{b}^{\mathrm{der}}) + \#S.\dim(\mathfrak{g}^{\mathrm{der}}) \\ &= \#S_\ell.\dim(\mathfrak{g}) + [F:\mathbb{Q}].\dim(\mathfrak{b}^{\mathrm{der}}) - \dim(\mathfrak{g}^{\mathrm{ab}}).\end{aligned}$$

Concerning part (2), note that (using condition (a)) we have an exact sequence

$$0 \to \mathfrak{g}/\mathfrak{g}^{\mathrm{Gal}_{F_v}} = \mathfrak{g}^{\mathrm{der}}/(\mathfrak{g}^{\mathrm{der}})^{\mathrm{Gal}_{F_v}} \to t_{D_W^{\square,\chi_v,\mathrm{sm}}(\bar\rho_v)} \to t_{D_W^{\chi_v,\mathrm{sm}}(\bar\rho_v)} \to 0$$

for $v \mid \ell$. Therefore, using condition (2), we have for $v \mid \ell$ the following:

$$\dim(L_v) = \dim t_{D_W^{\chi_v,\mathrm{sm}}(\bar\rho_v)} = h^0(\mathrm{Gal}_{F_v}, \mathfrak{g}^{\mathrm{der}}) + [F_v:\mathbb{Q}_\ell].\dim(\mathfrak{g}^{\mathrm{der}}).$$

Recall the Greenberg–Wiles formula [Neukirch et al. 2008, Theorem 8.7.9]:

$$\begin{aligned}&\dim H^1_{\mathcal{L}}(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathrm{der}}) - \dim H^1_{\mathcal{L}^\perp}(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathrm{der},\vee}) \\ &\qquad = h^0(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathrm{der}}) - h^0(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathrm{der},\vee}) + \sum_{v\in S_\ell}(\dim(L_v) - h^0(\mathrm{Gal}_{F_v}, \mathfrak{g}^{\mathrm{der}}))\end{aligned}$$

By [Böckle 2007, Section 5], we know that $H^1_{\mathcal{L}}(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathrm{der}})$ can be identified with the tangent space of the functor $D_{S_\ell}^{\chi,\mathrm{min},\mathrm{sm}}$ and hence (by part (2)) equals $[F:\mathbb{Q}].\dim(\mathfrak{b}^{\mathrm{der}})$. For $v \mid \infty$, we have $L_v \subset H^1(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathrm{der}}) = 0$. Thus, using the Taylor–Wiles formula (6) and assumption (b), the sum evaluates to

$$\sum_{v\in S_\ell}(\dim(L_v) - h^0(\mathrm{Gal}_{F_v}, \mathfrak{g}^{\mathrm{der}})) = [F:\mathbb{Q}].\dim(\mathfrak{g}^{\mathrm{der}}) - [F:\mathbb{Q}].(\dim(\mathfrak{g}^{\mathrm{der}}) - \dim(\mathfrak{b}^{\mathrm{der}})).$$

Therefore we get

$$-\dim H^1_{\mathcal{L}^\perp}(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathrm{der},\vee}) = h^0(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathrm{der}}).$$

As neither quantity can be negative, they must both vanish and the result follows. □

From the exact sequence

$$H^1_{\mathcal{L}^\perp}(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathrm{der},\vee})^* \to Ш^2_{S_\ell}(\mathfrak{g}^{\mathrm{der}}) \to 0$$

(see, e.g., equation (9) on page 10 of [Böckle 2007]) we can deduce:

Corollary 4.4. *Under the assumptions of part* (2) *of Theorem 4.2*, $Ш^2_{S_\ell}(\mathfrak{g}^{\mathrm{der}})$ *vanishes. In particular, in view of Proposition 3.30 the unrestricted deformation functor* $D_{S_\ell}^{(\square_{S_\ell}),\chi}(\bar\rho)$ *is globally unobstructed precisely if the local deformation functors* $D^{(\square),\chi_v}(\bar\rho_v)$ *are relatively smooth for* $v \in S \cup \Omega_\ell$.

We remark that $D^{(\square),\chi_v}(\bar{\rho}_v)$ is relatively smooth for $v \in \Omega_\infty$ by Proposition 3.22, so Corollary 4.4 holds true with " …for $v \in S_\ell$" in place of " …for $v \in S \cup \Omega_\ell$".

Potential unobstructedness. We start with the following easy observation:

Proposition 4.5. *Let K be a local field and let K' be a finite extension of K such that ℓ does not divide the index $[K' : K]$. Let $\bar{\rho}$ be a G-valued residual representation of Gal_K and fix a lift χ of the determinant. Then unobstructedness of $D_\Lambda^{(\chi)}(\bar{\rho} \mid \mathrm{Gal}_{K'})$ implies unobstructedness of $D_\Lambda^{(\chi)}(\bar{\rho})$.*

Proof. This follows immediately from the injectivity of

$$\mathrm{res}_{K' \mid K} : H^2(K, \mathfrak{g}^{(\mathrm{der})}) \to H^2(K', \mathfrak{g}^{(\mathrm{der})});$$

see [Neukirch et al. 2008, Corollary (1.5.7)]. □

This proof is not directly applicable to the global situation, as we have to keep track of the set of places at which we allow ramification. Therefore, we first describe a more flexible method which can also handle conditioned deformation functors:

Definition 4.6 (Dual-pre condition). Let $F' \mid F$ be a finite extension of number fields:

(1) Let $v' \in \mathrm{Pl}_{F'}$, $v \in \mathrm{Pl}_F$ such that $v' \mid v$. Moreover, let $L_v \subset H^1(F_v, \mathfrak{g}^{(\mathrm{der})})$, $L'_{v'} \subset H^1(F'_{v'}, \mathfrak{g}^{(\mathrm{der})})$ be local conditions. We say that L_v is a dual-pre-$L'_{v'}$ condition if $\mathrm{res}^\vee_{v'}(L_v^\perp) \subset L'^\perp_{v'}$, where

$$\mathrm{res}^\vee_{v'} : H^1(F_v, \mathfrak{g}^{(\mathrm{der}),\vee}) \to H^1(F'_{v'}, \mathfrak{g}^{(\mathrm{der}),\vee})$$

denotes the usual restriction map.

(2) Let $\mathcal{L}' = (L'_{v'})_{v' \in \mathrm{Pl}_{F'}}$ be a system of local conditions for F'. We say that a system $\mathcal{L} = (L_v)_{v \in \mathrm{Pl}_F}$ of local conditions for F is dual-pre-$\mathcal{L}'$ if for each pair v, v' as above, L_v is a dual-pre-$L'_{v'}$ condition.

Example 4.7. Let F, F' be as in Definition 4.6 and fix a finite set $S \subset \mathrm{Pl}_F$ such that $\bar{\rho}$ is unramified outside S. Take for $\mathcal{L}$ the local system parametrizing all deformations which are unramified outside S, i.e., $L_v = H^1(F_v, \mathfrak{g}^{(\mathrm{der})})$ if $v \in S$ and $L_v = H^1(\mathrm{Gal}_{F_v}/I_{F_v}, \mathfrak{g}^{(\mathrm{der})})$ otherwise. Analogously, let $\mathcal{L}'$ the local system parametrizing all deformations which are unramified outside S. Then any lift of $\bar{\rho}$ which is unramified outside S is, after restriction to $\mathrm{Gal}_{F'}$, a lift of $\bar{\rho} \mid \mathrm{Gal}_{F'}$ which is unramified outside S. But this implies easily that the restriction map $\mathrm{res}_{v'} : H^1(F_v, \mathfrak{g}^{(\mathrm{der})}) \to H^1(F'_{v'}, \mathfrak{g}^{(\mathrm{der})})$ maps L_v into $L'_{v'}$ for any pair of places v, v' with $v' \mid v$. Using the fact that Tate duality is given by the cup product which sends unramified classes to zero, we see that $\mathcal{L}$ is dual-pre-$\mathcal{L}'$.

Lemma 4.8. *Let $\bar{\rho}$, F and F' be as above and assume $(\ell, [F' : F]) = 1$. Let $\mathcal{L} = (L_v)_{v \in \mathrm{Pl}_F}$, $\mathcal{L}' = (L'_{v'})_{v' \in \mathrm{Pl}_{F'}}$ be systems of local conditions (with associated deformation conditions $\mathcal{D}$ and $\mathcal{D}'$) such that $\mathcal{L}$ is dual-pre-$\mathcal{L}'$. Moreover, assume that $D^{[\square],(\chi),\mathcal{D}'}(\bar{\rho} \mid \mathrm{Gal}_{F'})$ has vanishing dual Selmer group. Then also $D^{[\square],(\chi),\mathcal{D}}(\bar{\rho})$ has vanishing dual Selmer group.*

Proof. As above, the invertibility of $[F' : F]$ implies that the restriction map

$$H^1(\mathrm{Gal}_F, \mathfrak{g}^{(\mathrm{der}),\vee}) \to H^1(\mathrm{Gal}_{F'}, \mathfrak{g}^{(\mathrm{der}),\vee})$$

is injective. Consider the diagram with exact rows:

$$\begin{array}{ccccc} H^1_{\mathcal{L}^\perp}(F, \mathfrak{g}^{(\mathrm{der}),\vee}) & \hookrightarrow & H^1(F, \mathfrak{g}^{(\mathrm{der}),\vee}) & \longrightarrow & \bigoplus_{v\in \mathrm{Pl}_F} H^1(F_v, \mathfrak{g}^{(\mathrm{der}),\vee})/L_v^\perp \\ \downarrow \varphi & & \downarrow & & \downarrow \\ H^1_{\mathcal{L}',\perp}(F', \mathfrak{g}^{(\mathrm{der}),\vee}) & \hookrightarrow & H^1(F', \mathfrak{g}^{(\mathrm{der}),\vee}) & \longrightarrow & \bigoplus_{v'\in \mathrm{Pl}_{F'}} H^1(F'_{v'}, \mathfrak{g}^{(\mathrm{der}),\vee})/L_{v'}^{',\perp} \end{array}$$

The vertical map on the right is defined because $\mathcal{L}$ is dual-pre-$\mathcal{L}'$, and this implies the well-definedness of φ. A simple diagram chase implies injectivity of φ, from which the claim follows. □

The following follows now directly from Example 4.7 and Lemma 4.8:

Corollary 4.9. *Let F be a number field and let F' be a finite extension of F such that ℓ does not divide the index $[F' : F]$. Let $\bar{\rho}$ be a G-valued residual representation of* Gal_F *which is unramified outside a finite set of places S and fix a lift χ of the determinant. Then unobstructedness of $D^{(\chi \mid \mathrm{Gal}_{F'})}_{\Lambda,S}(\bar{\rho} \mid \mathrm{Gal}_{F'})$ implies unobstructedness of $D^{(\chi)}_{\Lambda}(\bar{\rho})$.*

5. Local deformation conditions for $G = \mathrm{GL}_n$

Let K be a finite extension of $\mathbb{Q}_p$ and let k be a finite field of characteristic ℓ. In the following, we consider deformation conditions for a continuous representation $\bar{\rho} : \mathrm{Gal}_K \to \mathrm{GL}_n(k)$.

5A. ***Unrestricted deformations ($p \neq \ell$).*** In the case $p \neq \ell$, we have the following result:

Theorem 5.1. Spec $R^\square(\bar{\rho})$ *is a reduced complete intersection, flat and equidimensional of relative dimension n^2 over* Spec W.

Proof. This is Theorem 2.5 in [Shotton 2018]. □

5B. ***Unrestricted deformations ($p = \ell$).*** For the remainder of this subsection, we assume that $\bar{\rho}$ is the semisimplification of the reduction of a crystalline representation

$$\rho : \mathrm{Gal}_K \to \mathrm{GL}_n(L)$$

for a suitable finite extension L of $\mathbb{Q}_p$ with residue field k and for $p = \ell$. Denote the set of embeddings $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$ by $\mathbb{E}_K$ and for $\tau \in \mathbb{E}_K$ denote by $\mathrm{HT}_\tau(\rho)$ the multiset of Hodge–Tate weights of ρ with respect to τ.

Theorem 5.2. *Assume that $K \mid \mathbb{Q}_p$ is unramified and that for each $\tau \in \mathbb{E}_K$:*

(1) *There exists an $\alpha \in \mathbb{Z}$ such that all Hodge–Tate weights in* $\mathrm{HT}_\tau(\rho)$ *lie in the range* $[\alpha, \alpha + \ell - 3]$.

(2) *The Hodge–Tate weights of ρ are nonconsecutive, i.e., if two numbers $a, b \in \mathbb{Z}$ occur in* $\mathrm{HT}_\tau(\rho)$, *then* $|a - b| \neq 1$.

Then $R^\square(\bar{\rho}) \cong W[\![x_1, \ldots, x_m]\!]$ with $m = n^2.([K : \mathbb{Q}_\ell] + 1)$.

Before we come to the proof, recall the theory of Fontaine and Laffaille [1982], as normalized in [Clozel et al. 2008] (see also [Barnet-Lamb et al. 2014, Section 1.4]): We consider the category $\underline{\mathrm{FL}}_{\mathcal{O}_K,\mathcal{O}_L}$, consisting of $\mathcal{O}_K \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$-modules M, endowed with a decreasing filtration $(\mathrm{Fil}^i\, M)_{i\in\mathbb{Z}}$ with $\mathrm{Fil}^0\, M = M$ and $\mathrm{Fil}^{\ell-1}\, M = 0$ and a family of $\mathrm{Frob}\otimes 1$-linear maps $\mathrm{Fil}^i\, M \to M$ such that $\varphi^i \mid \mathrm{Fil}^{i+1} = \ell.\varphi^{i+1}$ and $\sum_i \varphi^i(\mathrm{Fil}^i\, M) = M$. Let $\underline{\mathrm{FL}}_{\mathcal{O}_K,k}$ denote the full subcategory of finite length objects which are annihilated by the maximal ideal $\varpi_L.\mathcal{O}_L$. We need the following well-known facts:

- There exists an exact, fully faithful, covariant and $\mathcal{O}_L$-linear functor
$$\mathtt{G}_K : \underline{\mathrm{FL}}_{\mathcal{O}_K,\mathcal{O}_L} \to \mathrm{Rep}_{\mathcal{O}_L}(\mathrm{Gal}_K).$$
The essential image is closed under taking subobjects and quotients. Moreover, $\mathtt{G}_K$ restricts to a functor
$$\underline{\mathrm{FL}}_{\mathcal{O}_K,k} \to \mathrm{Rep}_k(\mathrm{Gal}_K).$$
- For $\boldsymbol{M} \in \underline{\mathrm{FL}}_{\mathcal{O}_K,\mathcal{O}_L}$ projective over $\mathcal{O}_L$, we have
$$\mathrm{HT}_\tau(\mathtt{G}_K(\boldsymbol{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)) = \mathrm{FL}_\tau(\boldsymbol{M} \otimes_{\mathcal{O}_L} k),$$
where for $\boldsymbol{N} \in \underline{\mathrm{FL}}_{\mathcal{O}_K,k}$ we denote by $\mathrm{FL}_\tau(\boldsymbol{N})$ the multiset of integers i, such that
$$\mathrm{gr}^i(\boldsymbol{N}^\tau) = \mathrm{Fil}^i\, N \otimes_{\mathcal{O}_K\otimes_{\mathbb{Z}_p}\mathcal{O}_L,\tau\otimes 1} \mathcal{O}_L / \mathrm{Fil}^{i+1}\, N \otimes_{\mathcal{O}_K\otimes_{\mathbb{Z}_p}\mathcal{O}_L,\tau\otimes 1} \mathcal{O}_L$$
does not vanish, where i is counted with multiplicity $\dim_k \mathrm{gr}^i(\boldsymbol{N}^\tau)$.
- Assuming condition (1) of Theorem 5.2, any Gal_K-stable $\mathcal{O}_L$-lattice of ρ is in the image of $\mathtt{G}_K$, and so is its reduction $\Lambda/\varpi_L.\Lambda$.
- Morphisms in $\underline{\mathrm{FL}}_{\mathcal{O}_K,k}$ are strict with filtrations. If $f : \boldsymbol{M} \to \boldsymbol{N}$ is such a morphism, then $f(\mathrm{Fil}^i\, M) = f(M) \cap \mathrm{Fil}^i\, N$ for all $i \in \mathbb{Z}$. In particular, if $\boldsymbol{M}, \boldsymbol{N} \in \underline{\mathrm{FL}}_{\mathcal{O}_K,k}$ fulfill
$$\mathrm{FL}_\tau(\boldsymbol{M}) \cap \mathrm{FL}_\tau(\boldsymbol{N}) \tag{7}$$
for all $\tau \in \mathbb{E}_K$, then $\mathrm{Hom}_{\underline{\mathrm{FL}}_{\mathcal{O}_K,k}}(\boldsymbol{M}, \boldsymbol{N}) = 0$.

Proof of Theorem 5.2. As $h^2(K, \mathrm{ad}\,\bar\rho)$ is an upper bound on the number of generators of the kernel of a surjection $W[\![x_1, \dots, x_s]\!] \twoheadrightarrow R^\square(\bar\rho)$ with $s = \dim Z^1(K, \mathrm{ad}\,\bar\rho)$ (see [Allen 2016, Proposition 2.1.2]), we have to prove
$$H^2(K, \mathrm{ad}\,\bar\rho) = 0. \tag{8}$$
Moreover, using the exact sequence
$$0 \to \mathrm{ad}\,\bar\rho/(\mathrm{ad}\,\bar\rho)^{\mathrm{Gal}_K} \to t_{D_W^\square(\bar\rho)} \to t_{D_W(\bar\rho)} \to 0$$
and the local Euler–Poincaré formula, we can compute
$$s = h^1(K, \mathrm{ad}\,\bar\rho) + n^2 - h^0(K, \mathrm{ad}\,\bar\rho) = n^2 - \chi(K, \mathrm{ad}\,\bar\rho) = n^2([K : \mathbb{Q}_p] + 1).$$
Thus, (8) implies the claim.

As the trace pairing identifies $\operatorname{ad}\bar\rho^\vee$ and $\operatorname{ad}\bar\rho(1)$, we are finished if we can show that

$$H^2(K, \operatorname{ad}\bar\rho)^* \cong H^0(K, \operatorname{ad}\bar\rho^\vee) \cong H^0(K, \operatorname{ad}\bar\rho(1)) \cong \operatorname{Hom}_{\mathrm{Gal}_K}(\bar\rho, \bar\rho(1)) = 0.$$

Because $\operatorname{Hom}_{\mathrm{Gal}_K}(\bar\rho, \bar\rho(1)) \cong \operatorname{Hom}_{\mathrm{Gal}_K}(\bar\rho(1-\alpha), \bar\rho(2-\alpha))$, we can assume without loss of generality that $\alpha = 1$.

It is easy to see that we can choose a Gal_K-stable $\mathcal{O}_L$-lattice Λ of ρ such that its reduction is semisimple, i.e., $\Lambda/\varpi_L.\Lambda \cong \bar\rho$ (if necessary, after replacing ρ by a base change $\rho \otimes_L L'$ to a sufficiently ramified finite extension L' of L, which does not affect the validity of (8)). By our first assumption that all weights of ρ lie in the range $[1, \ell-2]$, it thus follows that $\bar\rho$ is of the form $\mathtt{G}_K(\boldsymbol{M})$ for a suitable $\boldsymbol{M} \in \underline{\mathrm{FL}}_{\mathcal{O}_K,k}$. By the same argument, $\bar\rho(1) = \mathtt{G}_K(\boldsymbol{N})$ for a suitable $\boldsymbol{N} \in \underline{\mathrm{FL}}_{\mathcal{O}_K,k}$. As the cyclotomic character shifts the weights by -1, the second condition translates precisely into the condition (7). Thus, using that $\mathtt{G}_K$ is fully faithful, we get

$$0 = \operatorname{Hom}_{\underline{\mathrm{FL}}_{\mathcal{O}_K,k}}(\boldsymbol{M}, \boldsymbol{N}) \cong \operatorname{Hom}_{\mathrm{Gal}_K}(\bar\rho, \bar\rho(1)). \qquad \square$$

5C. *Crystalline deformations ($\ell = p$).* Let K be unramified. Consider again a representation $\rho\colon \mathrm{Gal}_K \to \mathrm{GL}_n(L)$ which fulfills the conditions of Theorem 5.2. We will also make the additional assumption that all occurring Hodge–Tate weights of $\bar\rho$ have multiplicity one. We will consider the deformation problem $\mathtt{crys}$ of $\bar\rho$ consisting of those lifts $\tilde\rho : \mathrm{Gal}_K \to \mathrm{GL}_n(A)$ of $\bar\rho$ for which $\tilde\rho \otimes_A A'$ lies in the essential image of $\mathtt{G}_K$ for all Artinian quotients A' of A (see [Clozel et al. 2008, Section 2.4.1]). We refer to those lifts as FL-*crystalline lifts* of $\bar\rho$.

That $\mathtt{crys}$ defines a deformation condition in the sense of Definition 3.16 was already remarked in [Clozel et al. 2008] and follows easily from the Ramakrishna framework [1993]: We remarked already in Section 5B that the essential image of $\mathtt{G}_K$ is closed under subobjects and quotients. That the essential image is closed under direct sums follows immediately from the exactness of $\mathtt{G}_K$, since then $\mathtt{G}_K$ preserves direct sums (see [Freyd 1964, Theorem 3.12$^{(*)}$]). Thus we can record the following (where for part (2) we refer to the remark just below Proposition 3.10):

Lemma 5.3. *Let Λ be the ring of integers of a finite, totally ramified extension E of* $\mathrm{Quot}(W(k))$ *and let Λ' be the ring of integers of a finite, totally ramified extension of E (so that we have $k = k_\Lambda = k_{\Lambda'}$.) Then*:

(1) *The functor $D^{\square,\mathtt{crys}}_\Lambda(\bar\rho)$ is representable by a quotient $R^{\square,\mathtt{crys}}_\Lambda(\bar\rho)$ of $R^\square_\Lambda(\bar\rho)$.*

(2) *The functor $D^{\square,\mathtt{crys}}_{\Lambda'}(\bar\rho)$ is representable by*

$$R^{\square,\mathtt{crys}}_{\Lambda'}(\bar\rho) \cong \Lambda' \otimes_\Lambda R^{\square,\mathtt{crys}}_\Lambda(\bar\rho). \tag{9}$$

We remark that the condition $\mathtt{crys}$ fulfills the extended requirements as described in Remark 3.11, so that (9) holds even if $\infty > [k_{\Lambda'} : k_\Lambda] > 1$.

Lemma 5.4. *Under the above hypotheses*

$$R^{\square,\mathtt{crys}}_\Lambda(\bar\rho) \cong \Lambda[[x_1, \dots, x_m]]$$

with $m = n^2 + [K : \mathbb{Q}_\ell].n.(n-1)/2$.

Proof. This is a part of the statement of [Clozel et al. 2008, Corollary 2.4.3]. □

Let us also note the following useful compatibility with base change:

Lemma 5.5. *Let K' be a finite unramified extension of K with associated inclusion map $\iota_{K'|K} : \mathrm{Gal}_{K'} \to \mathrm{Gal}_K$. Set $\bar{\rho}' = \bar{\rho} \circ \iota_{K'|K}$. Let $\tilde{\rho}$ be a crystalline lift of $\bar{\rho}$. Then $\tilde{\rho}' = \tilde{\rho} \circ \iota_{K'|K}$ is a crystalline lift of $\bar{\rho}'$.*

In particular, the restriction map $\mathrm{res} : H^1(K, \mathrm{ad}\,\bar{\rho}) \to H^1(K', \mathrm{ad}\,\bar{\rho}')$ maps the tangent subspace associated to the crystalline deformation condition for $\bar{\rho}$ into the tangent subspace associated to the crystalline deformation condition for $\bar{\rho}'$.

Proof. This is a direct consequence of the following compatibility of the Fontaine–Laffaille functor with base change: Let $\boldsymbol{M} \in \underline{\mathrm{MF}}_{\mathcal{O}_K,\mathcal{O}_L}$, then $\mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \boldsymbol{M}$ defines an object of $\underline{\mathrm{MF}}_{\mathcal{O}_{K'},\mathcal{O}_L}$. It follows from the definition of the functors $\mathtt{G}_K, \mathtt{G}_{K'}$ and a calculation analogous to the one in Section 3.11 of [Fontaine and Laffaille 1982] that $\mathtt{G}_K(\boldsymbol{M})$ and $\mathtt{G}_{K'}(\mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \boldsymbol{M})$ are isomorphic as $\mathcal{O}_L$-modules and that this isomorphism commutes with the action of $\mathrm{Gal}_{K'}$. In other words,

$$r^K_{K'}(\mathtt{G}_K(\boldsymbol{M})) \cong \mathtt{G}_{K'}(\mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \boldsymbol{M})$$

where $r^K_{K'}$ denotes the restriction to $\mathrm{Gal}_{K'}$. □

5D. *Minimally ramified deformations ($p \neq \ell$).* For this subsection, recall from [Clozel et al. 2008, Section 2.4.4] the minimal ramification condition for a lift ρ of $\bar{\rho}$. Let P_K denote the kernel of one (hence, any) surjection $I_K \twoheadrightarrow \mathbb{Z}_\ell$. Moreover, let $\Delta_{\bar{\rho}}$ denote the set of equivalence classes of P_K-representations over k such that $\mathrm{Hom}_{P_K}(\tau, \bar{\rho}) \neq 0$. Then the following can easily be deduced from the material in [loc. cit., Section 2.4.4], in particular [loc. cit., Corollary 2.4.21]:

Proposition 5.6. *Assume that any $\tau \in \Delta_{\bar{\rho}}$ is absolutely irreducible. Then we have*:

(1) *The condition of being minimally ramified defines a lifting condition, denoted* min. *The representing universal object fulfills*

$$R^{\square,\mathtt{min}}_\Lambda(\bar{\rho}) \cong \Lambda[\![X_1, \ldots, X_{n^2}]\!].$$

(2) *If Λ is the ring of integers of some finite extension of* $\mathrm{Quot}(\Lambda)$ *with residue field $k_{\Lambda'} = k$, we have*

$$R^{\square,\mathtt{min}}_{\Lambda'}(\bar{\rho}') \cong \Lambda' \otimes_\Lambda R^{\square,\mathtt{min}}_\Lambda(\bar{\rho}).$$

We will be particularly interested in the case where $\bar{\rho}$ has *unipotent ramification* i.e., where $\bar{\rho}(P_K) = \{1\}$.[4] In the unipotent case, we have a strong connection between minimally ramified liftings and liftings of prescribed type as considered in [Shotton 2015]. In order to make this precise, let E denote the quotient field of Λ and $\bar{E}$ its algebraic closure.

[4]This notion is explained by the observation that $\bar{\rho}$ is unipotently ramified if and only if $\bar{\rho}(I_K)$ lies in a conjugate of the standard unipotent subgroup consisting of upper-triangular matrices in $\mathrm{GL}_n(k)$ with diagonal entries all equal to 1.

Definition 5.7 [Shotton 2015, Definition 2.10]. Let $\tau\colon I_K \to \mathrm{GL}_n(\bar{E})$ be a representation which extends to a continuous representation of the Weil group W_K of K (considered with the ℓ-adic topology). Then the isomorphism class of τ is called an *inertial type*. (*Warning*: This differs from the usual definition of an inertial type as e.g., in [Gee and Kisin 2014].)

Let ρ be a lift of $\bar{\rho}$ which has values in $\bar{E}$, then we say that ρ "is of type τ" if $\rho \mid I_K$ is isomorphic to τ.

For the following we consider a τ which is defined over E. Then we say that a morphism $x : \operatorname{Spec} \bar{E} \to \operatorname{Spec} R^{\square}_{\Lambda}(\bar{\rho})$ is of type τ if the associated $\bar{E}$-valued representation ρ_x is of type τ. This notion depends only on the image of x (because τ is defined over E).

Definition 5.8 (Fixed type deformation ring [Shotton 2015, Definiton 2.14]). Let $R^{\square,\tau}_{\Lambda}(\bar{\rho})$ be the reduced quotient of $R^{\square}_{\Lambda}(\bar{\rho})$ which is characterized by the requirement that $\operatorname{Spec} R^{\square,\tau}_{\Lambda}(\bar{\rho})$ is the Zariski closure of the $\bar{E}$-points of type τ in $\operatorname{Spec} R^{\square}_{\Lambda}(\bar{\rho})$.

A general classification of inertial types is given in Section 2.2.1 of [Shotton 2015]. Under the unipotent ramification assumption, this becomes particularly simple: The set $\mathcal{I}^{\mathtt{uni}}$ of the isomorphism classes of inertial types which are trivial on P_K is in bijection with the set $\mathcal{Y}_n$ of Young diagrams of size n. The partition $(l_1, \dots, l_k)$ (with $l_i \geq l_{i+1}$) corresponds (using the notation of [loc. cit.]) to the type given by the I_K-restriction of the W_K-representation

$$\bigoplus_{i=1}^{k} \mathrm{Sp}(\mathbf{1}, l_i),$$

where $\mathrm{Sp}(\bullet, \bullet)$ is defined as in [loc. cit., Section 3.1]. We can express this differently: Each member of $\mathcal{I}^{\mathtt{uni}}$ is uniquely characterized by (the conjugacy class of) its value on the generator $\zeta :=\zeta_{\mathrm{triv}}$ of I_K/P_K, and a bijection $\nabla : \mathcal{Y}_n \to \mathcal{I}^{\mathtt{uni}}$ is given by

$$(l_1, \dots, l_k) \overset{\nabla}{\longmapsto} \tau(\zeta) = \left[1 + \begin{pmatrix} \mathcal{B}_{l_1} & & & \\ & \mathcal{B}_{l_2} & & \\ & & \ddots & \\ & & & \mathcal{B}_{l_k} \end{pmatrix} \right] \quad \text{with} \quad \mathcal{B}_m = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \in \mathbb{M}_{m\times m}(E). \tag{10}$$

On the other hand, we can associate to a $\tau \in \mathcal{I}^{\mathtt{uni}}$ a partition of n by considering the kernel sequences:

$$\Theta\colon \mathcal{I}^{\mathtt{uni}} \to \mathcal{Y}_n, \qquad \tau \mapsto (s_1, \dots, s_r)$$

with

$$s_i := \dim\ker(\tau(\zeta) - \mathbf{1})^i - \dim\ker(\tau(\zeta) - \mathbf{1})^{i-1}$$

and

$$r := \min\{i \mid \dim\ker(\tau(\zeta) - \mathbf{1})^i = \dim\ker(\tau(\zeta) - \mathbf{1})^{i+1}\} = \min\{i \mid \ker(\tau(\zeta) - \mathbf{1})^i = V\}.$$

(Here, V is the vector space underlying τ and we use the convention that f^0 is the identity map for any linear map f.) It follows easily from the characterization of $\mathcal{I}^{\mathtt{uni}}$ in (10) that $s_i \geq s_{i+1}$, i.e., that Θ has values in $\mathcal{Y}_n$.

It is an easy combinatorial calculation to check that τ is uniquely characterized by its value under Θ and that each Young diagram occurs as a kernel sequence (i.e., that Θ is a bijection). More precisely, we have:

Lemma 5.9. *The map $\Theta \circ \nabla^{-1} : \mathcal{Y}_n \to \mathcal{Y}_n$ is given by the conjugation operation on Young diagrams* (*see* [Fulton and Harris 1991, Section 4.1] *or* [Harris et al. 2008, Section 2.8]). *In particular, for a given $\tau \in \mathcal{I}^{\mathtt{uni}}$, the block matrix structure of $\tau(\zeta)$* (*up to reordering blocks*) *as in* (10) *determines its kernel sequence and vice versa.*

Proof. Retaining the notation used in (10), we first remark that for $i \in \mathbb{N}_0$ we have

$$\dim \ker \mathcal{B}_m^i = \min(i, m).$$

Thus, setting $\mathcal{B} = \operatorname{diag}(\mathcal{B}_{l_1}, \dots, \mathcal{B}_{l_k})$, we get

$$\dim \ker \mathcal{B}^i = \sum_{j=1}^{k} \min(i, l_j).$$

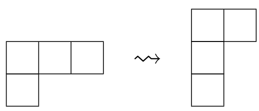
Consequently the kernel sequence $(s_1, \dots, s_r)$ associated to $(l_1, \dots, l_k)$ is given by

$$s_i = \sum_{j=1}^{k} \min(i, l_j) - \min(i-1, l_j) = \#\{j \mid l_j \geq i\} = \max\{j \mid l_j \geq i\}$$

and

$$r = \max\{l_j \mid j = 1, \dots, k\} = l_1.$$

Hence, the transition $(l_1, \dots, l_k) \rightsquigarrow (s_1, \dots, s_r)$ is precisely the conjugation operation of reflecting a Young diagram at the main diagonal (see [Harris et al. 2008, Section 2.8]), e.g.,



□

In order to state the desired comparison result, let us recall that we consider a residual representation $\bar{\rho} : \mathrm{Gal}_K \to \mathrm{GL}_n(k)$ with unipotent ramification. Let $\underline{\lambda} = (l_1, \dots, l_k) \in \mathcal{Y}_n$ such that $\bar{\rho}(\zeta) \sim \mathbf{1} + \operatorname{diag}(\mathcal{B}_{l_1}, \dots, \mathcal{B}_{l_k})$. Let $\tau = \nabla(\underline{\lambda}) \in \mathcal{I}^{\mathtt{uni}}$.

Theorem 5.10. *Assume $\bar{\rho}$ is unipotently ramified and τ as above. Then there is an isomorphism of the quotients*

$$R_{\Lambda}^{\square,\tau}(\bar{\rho}) \cong R_{\Lambda}^{\square,\mathtt{min}}(\bar{\rho}) \cong \Lambda[\![X_1, \dots, X_{n^2}]\!]$$

of $R_{\Lambda}^{\square}(\bar{\rho})$, i.e., a lifting of $\bar{\rho}$ is minimally ramified if and only if it is of type τ.

Proof. The diagram

$$\begin{array}{ccccc} & & R_\Lambda^{\square,\mathtt{min}}(\bar\rho) & & \\ & \nearrow & & \searrow & \\ R_\Lambda^{\square}(\bar\rho) & & & & \bar E \\ & \searrow & & \nearrow & \\ & & R_\Lambda^{\square,\tau}(\bar\rho) & & \end{array}$$

allows us to consider the $\bar E$-points of Spec $R_\Lambda^{\square,\mathtt{min}}(\bar\rho)$ and Spec $R_\Lambda^{\square,\tau}(\bar\rho)$ as subsets of the $\bar E$-points of Spec $R_\Lambda^{\square}(\bar\rho)$. We claim that they are equal: Translated into terms of $\bar E$-valued representations, we have to compare the sets

$$\Xi^{\mathtt{min}} = \left\{ \rho : \mathrm{Gal}_K \to \mathrm{GL}_n(\bar E) \;\middle|\; \begin{array}{l} \rho \text{ lifts } \bar\rho \text{ and has values in } \mathcal{O}_{\bar E}, \\ \dim\ker(\rho(\zeta)-\mathbf{1})^{i-1} - \dim\ker(\rho(\zeta)-\mathbf{1})^{i} = l_i \forall i \end{array} \right\}$$

and

$$\Xi^\tau = \{\rho : \mathrm{Gal}_K \to \mathrm{GL}_n(\bar E) \mid \rho \text{ lifts } \bar\rho \text{ and has values in } \mathcal{O}_{\bar E},\ \rho \mid I_K \cong \tau\}.$$

Lemma 5.9 implies that $\Xi^{\mathtt{min}} = \Xi^\tau$.

Now by definition of the ring $R_\Lambda^{\square,\tau}(\bar\rho)$ (as the schematic closure of the points in Ξ^τ) we have

$$\ker(R_\Lambda^{\square}(\bar\rho) \to R_\Lambda^{\square,\tau}(\bar\rho)) = \bigcap_{\rho\in\Xi^\tau} \ker(\rho).$$

Moreover, we clearly have

$$\ker(R_\Lambda^{\square}(\bar\rho) \to R_\Lambda^{\square,\mathtt{min}}(\bar\rho)) \subseteq \bigcap_{\rho\in\Xi^{\mathtt{min}}} \ker(\rho).$$

Hence, by $\Xi^\tau = \Xi^{\mathtt{min}}$ we get a factorization

$$R_\Lambda^{\square}(\bar\rho) \twoheadrightarrow R_\Lambda^{\square,\mathtt{min}}(\bar\rho) \overset{\varphi}{\twoheadrightarrow} R_\Lambda^{\square,\tau}(\bar\rho)$$

where the middle and the right ring have the same spectrum as topological spaces. Now we know by Proposition 5.6 that $R_\Lambda^{\square,\mathtt{min}}(\bar\rho)$ is formally smooth over Λ of relative dimension n^2 and that $\dim R_\Lambda^{\square,\tau}(\bar\rho) = n^2+1$ (combine Theorem 2.4 and Proposition 2.15 of [Shotton 2015]). Thus, φ is an isomorphism by Proposition 3.4 and the claim follows. □

5E. *Taylors deformation condition* $(1,\dots,1)$ $(\ell \neq p)$. We continue to consider a unipotently ramified residual representation $\bar\rho : \mathrm{Gal}_K \to \mathrm{GL}_n(k)$. If $A \in \mathcal{C}_{\mathcal{O}}$ is a coefficient ring, we say that an A-valued lift ρ of $\bar\rho$ fulfills the condition $(1,\dots,1)$ if $\mathrm{charPoly}(\rho(\xi)) = (T-1)^n$ for all $\zeta \in I_K$. By our assumption that $\bar\rho$ is unipotently ramified, it is sufficient to check the case where ξ is a topological generator of the tame inertia. This defines a deformation condition (and, in comparison to [Taylor 2008], we don't assume that $\bar{\rho}$ is trivial; see [Thorne 2012, Remark before Proposition 3.17]).

Proposition 5.11. *If a lift ρ is minimally ramified, it fulfills the Taylor condition. In particular, there is a canonical surjection*

$$R^{\square,(1,\dots,1)}(\bar{\rho}) \twoheadrightarrow R^{\square,\mathtt{min}}(\bar{\rho}),$$

and a morphism $R^{\square,(1,\dots,1)}(\bar{\rho}) \to A$ factors through this surjection if and only if the associated A-valued lift of $\bar{\rho}$ is minimally ramified.

Proof. By the unipotency assumption, we can assume that $\bar{\rho} \mid P_K$ is trivial and $\bar{\rho}(\zeta)$ is upper-triangular with each diagonal entry equal to 1 (where ζ is a topological generator of I_K/P_K). If a lift ρ is minimal, it follows that $\rho \mid P_K$ is trivial and that $\rho(\zeta)$ is unipotent; see [Clozel et al. 2008, Lemma 2.4.15, Assertion 3 $\Rightarrow$ 1]. It follows that $\rho(\sigma)$ is unipotent for any $\sigma \in I_K$. This proves the claim. □

Proposition 5.12. *Let L be a finite extension of K. Let*

$$\rho^{\square,(1,\dots,1)} : G_K \to \mathrm{GL}_n(R^{\square,(1,\dots,1)}(\bar{\rho}))$$

be the universal lifting of $\bar{\rho}$ with respect to the condition $(1,\dots,1)$ and let

$$\rho_L^{\square,(1,\dots,1)} : G_L \to \mathrm{GL}_n(R^{\square,(1,\dots,1)}(\bar{\rho} \mid G_L))$$

be the universal lifting of $\bar{\rho} \mid G_L$ with respect to the condition $(1,\dots,1)$. Then there exists a unique morphism of $\mathcal{C}_W$-algebras $\varphi : R^{\square,(1,\dots,1)}(\bar{\rho} \mid G_L)/(\ell) \to R^{\square,(1,\dots,1)}(\bar{\rho})/(\ell)$ such that

$$\overline{\rho^{\square,(1,\dots,1)}} \mid G_L = \varphi \circ \overline{\rho_L^{\square,(1,\dots,1)}}.$$

Proof. The lifting $\rho^{\square,(1,\dots,1)}$ fulfills the condition $(1,\dots,1)$, i.e., $\mathrm{charPoly}(\rho^{\square,(1,\dots,1)}(\sigma)) = (T-1)^n$ for all $\sigma \in I_K$. As $I_L \subset I_K$, $\rho^{\square,(1,\dots,1)} \mid G_L$ is a deformation of $\bar{\rho} \mid G_L$ which fulfills the condition $(1,\dots,1)$, i.e., factors via $\mathrm{GL}_n(R^{\square,(1,\dots,1)}(\bar{\rho} \mid G_L))$. This implies the existence of a map $R^{\square,(1,\dots,1)}(\bar{\rho} \mid G_L) \to R^{\square,(1,\dots,1)}(\bar{\rho})$ whose mod-ℓ reduction fulfills the required properties. □

Lemma 5.13. *Let $\tilde{\rho}$ be an A-valued lift, where we assume that A is reduced. Write $X = \tilde{\rho}(\zeta)$. Then $\chi_X := \mathrm{charPoly}(X)$ equals $(T-1)^n$ if $\ell \geq q^{n!}$.*

Proof. Assume first that A is an integral domain. By the condition $\varphi X \varphi^{-1} = X^q$ we see that raising to the q-th power permutes the eigenvalues of X (understood as a list of n elements). Thus, any eigenvalue of X must be a $(q^{\#S_n} - 1) = (q^{n!} - 1)$-th root of unity. Thus, if $Q(\mu)$ denotes the decomposition field of the polynomial $f(T) = T^{q^{n!}-1} - 1$ over the quotient field of A and $A(\mu)$ denotes the integral closure of A in $Q(\mu)$, then χ_X decomposes completely in $A(\mu)[T]$. On the other hand, each eigenvalue of X is sent to 1 by the canonical reduction map

$$\pi' : A(\mu) = A(\mu) \otimes_A A \to A(\mu) \otimes_A k.$$

As the kernel of π' is a pro-ℓ-subgroup and as $(\ell^m, q^{n!}-1) = 1$ for any $m \in \mathbb{N}$, it follows that any eigenvalue of X is 1, i.e., that $\chi_X = (T-1)^n$. The result for a general (reduced) A follows easily from using the embedding

$$A \hookrightarrow \prod_{\mathfrak{q}} A/\mathfrak{q},$$

where $\mathfrak{q}$ runs through the minimal primes of A. □

Corollary 5.14. *If $\ell \geq q^{n!}$, then $R^{\square,(1,\dots,1)}(\bar\rho) = R^{\square}(\bar\rho)$. In particular, $R^{\square,(1,\dots,1)}(\bar\rho)$ is reduced (see Theorem 5.1).*

Proof. By Lemma 5.13 (together with Theorem 5.1), we see that the identity map on $R^{\square}(\bar\rho)$ factors through $R^{\square,(1,\dots,1)}(\bar\rho)$. On the other hand, $R^{\square,(1,\dots,1)}(\bar\rho)$ is by definition a quotient of $R^{\square}(\bar\rho)$. Thus, we have found a surjective endomorphism of $R^{\square,(1,\dots,1)}(\bar\rho)$ (which must then be an isomorphism, as the rings in question are noetherian) which factors via $R(\bar\rho)^{\square}$. This proves the claim. □

6. On automorphic forms on unitary groups

6A. *The group $\mathcal{G}_n$.* For $n \in \mathbb{N}$ recall from [Clozel et al. 2008, Section 2.1] the definition of the group scheme $\mathcal{G}_n$ over $\mathbb{Z}$ and the multiplier character $\mathtt{m} : \mathcal{G}_n \to \mathrm{GL}_1$. We write $\mathcal{G}_n^0$ for the connected component of the identity and $\mathfrak{g}_n$ for the Lie algebra of $\mathcal{G}_n$ (where we differ in notation from [loc. cit.]). We have $\mathcal{G}_n^{\mathtt{der}} \cong \mathrm{GL}_n$ and $\mathcal{G}_n^{\mathtt{ab}} \cong \mathrm{GL}_1 \times \mathbb{Z}/2\mathbb{Z}$. If F is a CM-field with totally real subfield F^+, recall in particular the connection between GL_n-valued conjugate self-dual representations of Gal_F and $\mathcal{G}_n$-valued representations of Gal_{F^+}; see [loc. cit., Lemma 1.1.4] or [Gee 2011, Lemma 5.1.1].

We will be particularly interested in deformations of $\mathcal{G}_n$-valued residual representations. In the local split case, there is a substantial simplification possible: Let k be a finite field and let $\bar\rho$ be a GL_n-valued representation of Gal_F, let $\bar\chi$ a character such that $\bar\chi\bar\rho^\vee \cong \bar\rho^c$ and let $\bar r$ be the associated $\mathcal{G}_n(k)$-valued representation of Gal_{F^+}. Moreover, let Λ be the ring of integers of a finite extension of the quotient field of $W(k)$. The following proposition now follows easily from the definitions:

Proposition 6.1. *Let v be a place of F^+ which splits as $\tilde v\tilde v^c$ in F. Denote $\bar r_v := \bar r \mid \mathrm{Gal}_{F_v^+}$ and $\bar\rho_{\tilde v} := \bar\rho \mid \mathrm{Gal}_{F_{\tilde v}}$. Fix a lift $\chi_v : \mathrm{Gal}_{F_v^+} \to \Lambda^\times$ of $\mathtt{m} \circ \bar r_v$. Then*

$$R_\Lambda^{(\square),\chi_v}(\bar r_v) \cong R_\Lambda^{(\square)}(\bar\rho_{\tilde v}), \quad H^i(F_v^+, \mathfrak{g}_n^{\mathtt{der}}) \cong H^i(F_{\tilde v}, \mathfrak{gl}_n) \quad \textit{and} \quad Z^1(F_v^+, \mathfrak{g}_n^{\mathtt{der}}) \cong Z^1(F_{\tilde v}, \mathfrak{gl}_n).$$

This observation allows us to define local conditions for deformations of $\bar r$ at split places by GL_n-valued local conditions. In order to make this precise, let $\Sigma \subset \mathrm{Pl}_{F^+}^{\mathrm{fin}}$ be a finite set of places and assume that any place in Σ splits as $v = \tilde v\tilde v^c$ in the extension $F \mid F^+$ (so, in particular, we fix a place $\tilde v$ above v). Moreover, assume that $\bar r$ is unramified outside Σ, i.e., factors through $\mathrm{Gal}_{F^+,\Sigma}$. We set $\tilde\Sigma := \{\tilde v \mid v \in \Sigma\}$. Fix a character $\chi :\ \mathrm{Gal}_{F^+,\Sigma} \to \Lambda^\times$ lifting $\mathtt{m} \circ \bar r$. Moreover, for each $\tilde v \in \tilde\Sigma$ fix a deformation condition D_v of the GL_n-valued representation $\bar\rho_{\tilde v}$.

Definition 6.2 (Deformation problem, following [Clozel et al. 2008]). The collection

$$\mathscr{S} = (F \mid F^+, \Sigma, \tilde{\Sigma}, \Lambda, \bar{r}, \chi, \{D_\nu\}_{\nu\in\Sigma}),$$

parametrizing deformations r of $\bar{r}$ to $\mathcal{C}_\Lambda$ which fulfill $\mathtt{m} \circ r = \chi$, which are unramified outside Σ and fulfill D_ν (via Proposition 6.1) at $\nu \in \Sigma$, defines a global deformation condition.

We end this section by a remark on the conventions for multiple framings, in which we differ from [Clozel et al. 2008]. For this, let $T \subset \Sigma$ be a nonempty subset and recall our Definition 3.17 for the multiply framed deformation functor $D_\Lambda^{\square_T,\mathcal{S}}(\bar{r})$ and its representing object $R_\Lambda^{\square_T,\mathcal{S}}(\bar{r})$. Comparing this with the functor and representing object considered in [loc. cit., Definition 2.2.7], which we denote by $D_\Lambda^{\boxdot_T,\mathcal{S}}(\bar{r})$ and $R_\Lambda^{\boxdot_T,\mathcal{S}}(\bar{r})$, we easily get the following observation:

Proposition 6.3. *$D_\Lambda^{\square_T,\mathcal{S}}(\bar{r})$ is representable if and only if $D_\Lambda^{\boxdot_T,\mathcal{S}}(\bar{r})$ is representable, and in this case we have*

$$R_\Lambda^{\square_T,\mathcal{S}}(\bar{r}) \cong R_\Lambda^{\boxdot_T,\mathcal{S}}(\bar{r})[\![X_1, \dots, X_{\#T}]\!].$$

6B. ***Automorphic forms and Hecke algebras.*** For this subsection, let us assume that the extension $F \mid F^+$ is unramified at all finite places and, in case n is even, that $\frac{n}{2}[F^+ : \mathbb{Q}]$ is even. This allows us to fix a definite unitary group H over $\mathcal{O}_{F^+}$, as considered in [Guerberoff 2011, Section 2.11] or [Geraghty 2010, Section 1.1], whose key properties we recall here:

- The extension of scalars of H to F^+ is an outer form of GL_n / F^+, which becomes isomorphic to GL_n / F after extending scalars to F.
- H is quasisplit at every finite place of F^+.
- H is totally definite, i.e., $H(F^+_\infty)$ is compact and $H(F^+_\nu) \cong U_n(\mathbb{R})$ for all infinite places ν of F^+.
- For any finite place ν of F^+ which splits as $\tilde{\nu}\tilde{\nu}^c$ in F, we can choose an isomorphism $\iota_{\tilde{\nu}} : H(F^+_\nu) \to \mathrm{GL}_n(F_{\tilde{\nu}})$ whose restriction to $H(\mathcal{O}_{F^+_\nu})$ provides an isomorphism $H(\mathcal{O}_{F^+_\nu}) \cong \mathrm{GL}_n(\mathcal{O}_{F_{\tilde{\nu}}})$.

Level subgroups. Let us fix a finite subset $\mathcal{T} \subset \mathrm{Pl}^{\mathrm{fin}}_{F^+}$ such that each $\nu \in \mathcal{T}$ splits as $\tilde{\nu}\tilde{\nu}^c$ in F. For the remainder of this section, the letter U will denote an open compact subgroup of $H(\mathbb{A}^\infty_{F^+})$. For later applications, we will be particularly interested in the choice $U_{\mathcal{T}} := \prod_{\nu\in\mathrm{Pl}^{\mathrm{fin}}_{F^+}} U_\nu$ with:

- If ν is not split in $F \mid F^+$, then U_ν is a hyperspecial maximal compact subgroup of $H(F^+_\nu)$.
- If $\nu \notin \mathcal{T}$ splits, then $U_\nu = H(\mathcal{O}_{F^+_\nu})$.
- If $\nu \in \mathcal{T}$, then $U_\nu = \iota_{\tilde{\nu}}^{-1}(\mathrm{Iw})$, where $\mathrm{Iw} \subset \mathrm{GL}_n(\mathcal{O}_{F_{\tilde{\nu}}})$ denotes the Iwahori subgroup.

We remark that in many articles (e.g., [Clozel et al. 2008]) the set $\mathcal{T}$ is enlarged by a choice of auxiliary places at which a suitable level condition is imposed. Our arguments don't require such auxiliary places.

Weights. Recall the parametrization of complex and ℓ-adic representations of unitary and general linear groups, e.g., from [Guerberoff 2011]:

• To a tuple $\boldsymbol{\omega} = (\underline{\omega}_\tau) \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F^+,\mathbb{R})}$ we associate the representation

$$\xi_{\boldsymbol{\omega}} : H(F^+_\infty) = \prod_{\tau \in \mathrm{Hom}(F^+,\mathbb{R})} H(F^+_\tau) \cong \prod_{\tau \in \mathrm{Hom}(F^+,\mathbb{R})} U_n(\mathbb{R}) \xrightarrow{\varphi} \prod_{\tau \in \mathrm{Hom}(F^+,\mathbb{R})} \mathrm{GL}_n(W_{\underline{\omega}_\tau}) \subset \mathrm{GL}_n(W_{\boldsymbol{\omega}}),$$

where $W_{\boldsymbol{\omega}} = \otimes_\tau W_{\underline{\omega}_\tau}$ and where φ is the product of the highest weight representations $W_{\underline{\omega}_\tau}$ attached to the weight $\underline{\omega}_\tau$ (see e.g., [Bellaïche and Chenevier 2009; Guerberoff 2011; Geraghty 2010]).

• Let ℓ be a rational prime such that every place v of F^+ above ℓ splits in $F \mid F^+$ and fix for each such v a place $\tilde{v}$ of F above v. Let $\mathcal{K}$ be a finite extension of $\mathbb{Q}_\ell$ which is F-big enough and let $\boldsymbol{\omega} = (\underline{\omega}_\tau) \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F,\mathcal{K})}$. To each $\tau \in \mathrm{Hom}(F,\mathcal{K})$ we can associate a place v of F^+ above ℓ for which we have just fixed a place $\tilde{v}$. Denote this assignment $\mathrm{Hom}(F,\mathcal{K}) \to \Omega^F_\ell$ by $\tau \mapsto w_\tau$. Let

$$\begin{aligned}\xi^{\mathcal{K}}_{\boldsymbol{\omega}} : \prod_{v \in \Omega^F_\ell} H(F^+_v) &\cong \prod_{v \in \Omega^F_\ell} \mathrm{GL}_n(F_{\tilde{v}}) \\ &\xrightarrow{\prod d_v} \prod_{v \in \Omega^F_\ell} \prod_{\substack{\tau \in \mathrm{Hom}(F,\mathcal{K}) \\ w_\tau = \tilde{v}}} \mathrm{GL}_n(F_{\tilde{v}}) = \prod_{\tau \in \mathrm{Hom}(F,\mathcal{K})} \mathrm{GL}_n(F_{\tilde{v}}) \\ &\xrightarrow{\psi} \prod_{\tau \in \mathrm{Hom}(F,\mathcal{K})} \mathrm{GL}_n(W^{\mathcal{K}}_{\underline{\omega}_\tau}) \subset \mathrm{GL}_n(W^{\mathcal{K}}_{\boldsymbol{\omega}})\end{aligned}$$

be the representation where each d_v is the diagonal embedding, where $W^{\mathcal{K}}_{\boldsymbol{\omega}} = \otimes_\tau W^{\mathcal{K}}_{\underline{\omega}_\tau}$ and where ψ is the product of the highest weight representations $W^{\mathcal{K}}_{\underline{\omega}_\tau}$ attached to the weight $\underline{\omega}_\tau$. The representation $\xi^{\mathcal{K}}_{\boldsymbol{\omega}}$ admits an integral model over $\mathcal{O}_{\mathcal{K}}$, whose underlying finite free $\mathcal{O}_{\mathcal{K}}$-module we denote by $M^{\mathcal{O}_{\mathcal{K}}}_{\boldsymbol{\omega}}$.

Automorphic forms. We denote by

$$\mathcal{A}(H) = \bigoplus_\pi \pi^{m(\pi)}$$

the space of (complex) automorphic forms on H, which decomposes into isomorphism classes of irreducible representations of $H(\mathbb{A}_{F^+})$, each occurring with finite multiplicity $m(\pi)$ (see e.g., [Guerberoff 2011]).

Definition 6.4 (Vector-valued automorphic form). Let $\boldsymbol{\omega} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F^+,\mathbb{R})}$ be a weight, then we denote by $\mathcal{S}_{\boldsymbol{\omega}}$ the space of locally constant functions $f : H(\mathbb{A}^\infty_{F^+}) \to W^\vee_{\boldsymbol{\omega}}$ which fulfill

$$f(\gamma.h) = \gamma_\infty.f(h) \quad \forall h \in H(\mathbb{A}^\infty_{F^+}), \gamma \in H(F^+).$$

(We denote by γ_∞ the image of γ under the canonical embedding $H(F^+) \to H(F^+_\infty)$.) $H(\mathbb{A}^\infty_{F^+})$ acts on $\mathcal{S}_{\boldsymbol{\omega}}$ by right translation, and for a level subgroup U we denote by $\mathcal{S}_{\boldsymbol{\omega}}(U)$ the space of U-fixed vectors.

There exists an $H(\mathbb{A}_{F^+}) = H(\mathbb{A}_{F^+,\infty}) \times H(\mathbb{A}^\infty_{F^+})$-equivariant decomposition

$$\mathcal{A}(H) = \bigoplus_{\boldsymbol{\omega}} W_{\boldsymbol{\omega}} \otimes \mathcal{S}_{\boldsymbol{\omega}}.$$

Thus we can associate with $f \in \mathcal{S}_{\boldsymbol{\omega}}$ the automorphic representation $\langle f \rangle$ that is uniquely characterized by the condition that it contains all vectors of $W_{\boldsymbol{\omega}} \otimes f$. The main feature of the group H is the existence of avatars.

Theorem 6.5. *Let Π be a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\boldsymbol{\omega} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F,\mathbb{C})}$ in the sense of* [Clozel et al. 2008, Section 4]. *Then there exists an automorphic representation π_0 of $H(\mathbb{A}_{F^+})$ such that Π is a base change of π_0:*

- *For each archimedean place v of F^+ and each place $\tilde{v}$ of F above v, we have $\pi_{0,v} \cong \xi_{\underline{\omega}_{\tilde{v}}}$.*
- *For each finite place v of F^+ which splits as $\tilde{v}\tilde{v}^c$ in F, $\Pi_{\tilde{v}}$ is the local base change of $\pi_{0,v}$.*
- *If v is a finite place of F^+ which stays inert in F and for which Π_v is unramified, then π_v has a fixed vector for a maximal hyperspecial compact subgroup of $H(F_v^+)$.*

Proof. See [Guerberoff 2011, Theorem 2.2] and [Geraghty 2010, Lemma 2.2.7]. □

Hecke algebras. We continue to consider a fixed set of places $\mathcal{T}$ as above (with corresponding level subgroup $U = U_{\mathcal{T}}$) and a weight $\boldsymbol{\omega}$. For $j \in \{1, \dots, n\}$ and for w a place of F which is split over F^+ and does not divide an element of $\mathcal{T}$, we consider the following Hecke operator (acting on $\mathcal{S}_{\boldsymbol{\omega}}(U)$):

$$T_{F_w}^{(j)} = \left[U.\iota_w^{-1} \begin{pmatrix} \varpi_{F_w} \mathbf{1}_j & 0 \\ 0 & \mathbf{1}_{n-j} \end{pmatrix} .U \right]$$

For a finite set $\mathcal{T}' \subset \mathrm{Pl}_{F^+}^{\mathrm{fin}}$ containing $\mathcal{T}$ and a subring $\mathscr{R}$ of $\mathbb{C}$ we define the Hecke algebra

$${}^{\mathscr{R}}\boldsymbol{T}_{\boldsymbol{\omega}}^{\mathcal{T}'}(U) := \mathrm{im}\big(\mathscr{R}[T_{F_w}^{(j)} \mid j \in \{1, \dots, n\}, w \in \mathrm{Pl}_F^{\mathtt{split},\mathcal{T}'}] \to \mathrm{End}_{\mathbb{C}}(\mathcal{S}_{\boldsymbol{\omega}}(U))\big),$$

where $\mathrm{Pl}_F^{\mathtt{split},\mathcal{T}'}$ denotes the set of places of F which are split over F^+ and which do not divide an element of $\mathcal{T}'$. Besides the case $\mathscr{R} = \mathbb{Z}$ we will be interested in $\mathscr{R} = \mathcal{E}_f$ (the coefficient field of an eigenform f with respect to ${}^{\mathbb{Z}}\boldsymbol{T}_{\boldsymbol{\omega}}^{\mathcal{T}}(U)$)) and in $\mathscr{R} = \mathcal{E}(U) = \prod_f \mathcal{E}_f$, where the product (i.e., the field compositum operation) runs through all eigenforms of $\mathcal{S}_{\boldsymbol{\omega}}(U)$. We note the following well-known facts: There are only finitely many (one-dimensional) eigenspaces $\mathbb{C}.f_1, \dots, \mathbb{C}.f_r$ contained in $\mathcal{S}_{\boldsymbol{\omega}}(U)$, so $\mathcal{E}(U)$ is a number field. Moreover, $\mathcal{S}_{\boldsymbol{\omega}}(U)$ admits a basis of eigenforms, i.e., we can choose the f_i such that

$$\mathcal{S}_{\boldsymbol{\omega}}(U) \cong \mathbb{C}.f_1 \oplus \cdots \oplus \mathbb{C}.f_r$$

as a $\boldsymbol{T}_{\boldsymbol{\omega}}^{\mathcal{T}}(U)$-module (see decomposition (3.1.1) of [Guerberoff 2011]). By mapping a Hecke operator to its f-eigenvalue, any eigenform $f \in \mathcal{S}_{\boldsymbol{\omega}}(U)$ gives rise to a $\mathbb{Z}$-algebra-homomorphism

$$\varphi_f : {}^{\mathbb{Z}}\boldsymbol{T}_{\boldsymbol{\omega}}^{\mathcal{T}}(U) \to \mathcal{E}(U), \quad T_{F_w}^{(j)} \mapsto a_f(T_{F_w}^{(j)})$$

and it can be shown that $\mathrm{im}(\varphi_f) \subset \mathcal{O}_{\mathcal{E}(U)}$. The form f is uniquely characterized by φ_f (up to $\mathbb{C}$-multiples).

ℓ-adic models of automorphic forms. The following is based on Section 2.3 of [Guerberoff 2011]. For this paragraph, we fix a rational prime ℓ which does not lie below $\mathcal{T}$ and such that all places of F^+ above ℓ are split in the extension $F \mid F^+$ and consider the following setup: Let $\mathcal{K}$ be a finite extension of $\mathbb{Q}_\ell$ which is F-big enough and fix an isomorphism $\iota : \bar{\mathcal{K}} \cong \mathbb{C}$. Moreover, we fix an ℓ-adic weight $\boldsymbol{\omega}$, i.e., an element of

$$(\mathbb{Z}^{n,+})_c^{\mathrm{Hom}(F,\mathcal{K})} = \{\boldsymbol{\omega} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F,\mathcal{K})} \mid \underline{\omega}_{\tau^c,i} = -\underline{\omega}_{\tau,n-i+1} \forall \tau \in \mathrm{Hom}(F,\mathcal{K}), i \in \{1,\dots,n\}\}.$$

Definition 6.6. For $U \subset H(\mathbb{A}_{F^+}^\infty)$ a compact subgroup and an $\mathcal{O}_\mathcal{K}$-algebra A, suppose that either the projection of U to $H(F_\ell^+)$ is contained in $H(\mathcal{O}_{F_\ell^+})$ or that A is a $\mathcal{K}$-algebra. Then we define

$$S_{\boldsymbol{\omega}}(U, A) = \{f : H(F^+)\backslash H(\mathbb{A}_{F^+}^\infty) \to A \otimes_{\mathcal{O}_\mathcal{K}} M_{\boldsymbol{\omega}}^{\mathcal{O}_\mathcal{K}} \mid u_\ell . f(hu) = f(h) \forall u \in U, h \in H(\mathbb{A}_{F^+}^\infty)\},$$

where u_ℓ denotes the image of u under the projection map $H(\mathbb{A}_{F^+}^\infty) \to H(F_\ell^+)$.

We are primarily interested in the case that A is $\mathcal{O}_\mathcal{K}$-flat, so that we have $S_{\boldsymbol{\omega}}(U, A) \cong A \otimes_{\mathcal{O}_\mathcal{K}} S_{\boldsymbol{\omega}}(U, \mathcal{O}_\mathcal{K})$.

The main connection with complex automorphic forms is as follows (see also [Guerberoff 2011, Section 2.3]): The isomorphism ι gives rise to a bijection $\iota_*^+ : (\mathbb{Z}^{n,+})_c^{\mathrm{Hom}(F,\mathcal{K})} \cong (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F,\mathbb{R})}$, and the assignment $f \mapsto (h \mapsto \theta_{\boldsymbol{\omega}}(h_\ell . f(h)))$ provides isomorphisms of $\mathbb{C}H(\mathbb{A}_{F^+}^\infty)$-modules

$$\bigcup_U S_{\boldsymbol{\omega}}(U, \mathbb{C}) \cong \mathcal{S}_{\iota_*^+(\boldsymbol{\omega})^\vee} \quad \text{and} \quad S_{\boldsymbol{\omega}}(U, \mathbb{C}) \cong \mathcal{S}_{\iota_*^+(\boldsymbol{\omega})^\vee}(U). \tag{11}$$

(Here, $\mathbb{C}$ is understood as a $\mathcal{O}_\mathcal{K}$-algebra via ι and $\iota_*^+(\boldsymbol{\omega})^\vee$ is defined by $\iota_*^+(\boldsymbol{\omega})^\vee_{\tau,i} = -\iota_*^+(\boldsymbol{\omega})^\vee_{\tau,n+1-i}$.)

For a place w not dividing ℓ, the operators $T_{F_w}^{(j)}$ also act on $S_{\boldsymbol{\omega}}(U, \mathcal{O}_\mathcal{K}) \subset S_{\boldsymbol{\omega}}(U, \mathbb{C})$, and this action commutes with the isomorphism (11). This motivates the following definition: Let $\mathcal{T}'$ be a finite set of places of F^+ containing $\mathcal{T} \cup \Omega_\ell^{F^+}$ and let $\mathscr{R}$ be a subring of $\mathcal{O}_\mathcal{K}$, then we define the Hecke algebra

$${}^{\mathscr{R}}\mathbb{T}_{\boldsymbol{\omega}}^{\mathcal{T}'}(U) = \mathrm{im}\big(q : \mathscr{R}[T_{F_w}^{(j)} \mid j \in \{1,\dots,n\}, w \in \mathrm{Pl}_F^{\mathtt{split},\mathcal{T}'}] \to \mathrm{End}_{\mathcal{O}_\mathcal{K}}(S_{\boldsymbol{\omega}}(U, \mathcal{O}_\mathcal{K}))\big),$$

where we will often abbreviate $\mathbb{T}_{\boldsymbol{\omega}}^{\mathcal{T}'}(U) = {}^{\mathcal{O}_\mathcal{K}}\mathbb{T}_{\boldsymbol{\omega}}^{\mathcal{T}'}(U)$. If $f \in S_{\boldsymbol{\omega}}(U, \mathcal{O}_\mathcal{K})$ is an eigenform for this algebra, then we see, using the compatibility with the isomorphism (11), that the eigenvalue for a Hecke operator T is given by $\iota^{-1}(a_{\tilde{f}})$, where $\tilde{f} \in \mathcal{S}_{\iota_*^+(\boldsymbol{\omega})^\vee}(U)$ is the corresponding complex automorphic form. In other words, we can interpret the map $\varphi_{\tilde{f}}$ from above as

$$\varphi_f^\ell : {}^{\mathbb{Z}}\mathbb{T}_{\boldsymbol{\omega}}^{\mathcal{T}_\ell}(U) \to \iota(\mathcal{E}(U)) \cong \mathcal{E}(U).$$

Note that we use the bold symbol $\boldsymbol{T}$ for complex Hecke algebras and the blackboard bold symbol $\mathbb{T}$ for ℓ-adic Hecke algebras.

Fixed type Hecke algebras. Fix a finite set $\tilde{\Sigma} \subset (\mathcal{T}' - \Omega_\ell^F)$ of places of F together with a tuple $\underline{\sigma} = (\sigma_\nu)_{\nu \in \tilde{\Sigma}}$, where each σ_ν is a finite-dimensional complex representation of $\mathrm{GL}_n(\mathcal{O}_{F_\nu})$. Let ${}_{\underline{\sigma}}S_{\boldsymbol{\omega}}(U, \mathcal{O}_\mathcal{K}) \subset S_{\boldsymbol{\omega}}(U, \mathcal{O}_\mathcal{K})$ be the subspace generated by those forms f whose complex correspondents $\tilde{f}$ fulfill the following condition for all places $\nu \in \tilde{\Sigma}$: If π_ν denotes the local component of the automorphic representation $\pi = \langle \tilde{f} \rangle$ at ν, then $\pi_v \mid \mathrm{GL}_n(\mathcal{O}_{F_v})$ contains σ_v as a subrepresentation. Note that the $T^{(j)}_{F_w}$ (for w in $\mathrm{Pl}_F^{\mathtt{split},\mathcal{T}'}$) stabilize the subspace ${}_{\underline{\sigma}}S_{\boldsymbol{\omega}}(U, \mathcal{O}_\mathcal{K})$, so we can define

$$ {}^{\mathscr{R}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U) = \mathrm{im}\big({}_{\underline{\sigma}}q : \mathscr{R}[T^{(j)}_{F_w} \mid j \in \{1, \dots, n\}, w \in \mathrm{Pl}_F^{\mathtt{split},\mathcal{T}'}] \to \mathrm{End}_{\mathcal{O}_\mathcal{K}}({}_{\underline{\sigma}}S_{\boldsymbol{\omega}}(U, \mathcal{O}_\mathcal{K}))\big). $$

We easily see that the assignment $q(T^{(j)}_{F_w}) \mapsto {}_{\underline{\sigma}}\, q(T^{(j)}_{F_w})$ defines an $\mathscr{R}$-algebra surjection ${}_{\underline{\sigma}}\theta$ from ${}^{\mathscr{R}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)$ to ${}^{\mathscr{R}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)$. We note the following (for $\mathscr{R} = \mathcal{O}_\mathcal{K}$):

- In the same way as for ${}^{\mathcal{O}_\mathcal{K}}\boldsymbol{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)$, we can check that ${}^{\mathcal{O}_\mathcal{K}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)$ is free and finitely generated over $\mathcal{O}_\mathcal{K}$.
- Assume that ${}^{\mathcal{O}_\mathcal{K}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)_\mathfrak{m} \cong \mathcal{O}_\mathcal{K}$ holds for any maximal ideal $\mathfrak{m}$, then ${}^{\mathcal{O}_\mathcal{K}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)_\mathfrak{n}$ is a quotient of $\mathcal{O}_\mathcal{K}$ for any maximal ideal $\mathfrak{n}$. By the above bullet point, it thus follows that ${}^{\mathcal{O}_\mathcal{K}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)_\mathfrak{n}$ is isomorphic to $\mathcal{O}_\mathcal{K}$ for any maximal ideal $\mathfrak{n}$.

6C. *Attaching Galois representations to automorphic forms.* Retain all notation from above and let $\mathfrak{m} \subset {}^{\mathcal{O}_\mathcal{K}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}}(U)$ be a maximal ideal.

Proposition 6.7 [Guerberoff 2011, Proposition 3.1 and 3.2]. *There exists a representation*

$$ \rho_\mathfrak{m} : \mathrm{Gal}_F \to \mathrm{GL}_n({}^{\mathcal{O}_\mathcal{K}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}}(U)_\mathfrak{m}) $$

with the following properties, where the first two already characterize $\rho_\mathfrak{m}$ uniquely:

(1) *$\rho_\mathfrak{m}$ is unramified at all but finitely many places. If a place v of F^+ is inert and unramified in F and if U_v is a hyperspecial maximal compact subgroup of $H(F_v^+)$, then $\rho_\mathfrak{m}$ is unramified above v.*

(2) *If $v \in \mathrm{Pl}^{\mathrm{fin}}_{F^+} \setminus \mathcal{T}_\ell$ splits as $\tilde{v}\tilde{v}^c$ in F, then $\rho_\mathfrak{m}$ is unramified at $\tilde{v}$ and*

$$ \mathrm{charPoly}(\rho_\mathfrak{m}(\mathrm{Frob}_{\tilde{v}})) = X^n - T^{(1)}_{\tilde{v}} X^{n-1} + \dots + (-1)^j (N\tilde{v})^{j(j-1)/2} T^{(j)}_{\tilde{v}} X^{n-j} + \dots + (-1)^n (N\tilde{v})^{n(n-1)/2} T^{(n)}_{\tilde{v}}. $$

(3) $\bar{\rho}^c_\mathfrak{m} \cong \bar{\rho}_\mathfrak{m} \otimes \bar{\epsilon}_\ell^{1-n}$.

(4) *Fix a set $\widetilde{\Omega}^{F^+}_\ell$ of places of F such that $\widetilde{\Omega}^{F^+}_\ell \sqcup \widetilde{\Omega}^{F^+,c}_\ell = \Omega^F_\ell$ and denote by $\tilde{I}_\ell$ the set of embeddings $F \hookrightarrow \mathcal{K}$ which give rise to an element of $\widetilde{\Omega}^{F^+}_\ell$. Suppose that $w \in \widetilde{\Omega}^{F^+}_\ell$ is unramified over ℓ, that $U_{\overline{w}} = H(\mathcal{O}_{F^+,\overline{w}})$ (for $\overline{w} \in \mathrm{Pl}_{F^+}$ the place below w) and that for each $\tau \in \tilde{I}_\ell$ above w we have*

$$ \ell - 1 - n \geq \underline{\omega}_{\tau,1} \geq \dots \geq \underline{\omega}_{\tau,n} \geq 0. $$

Then, for each open ideal $I \subset {}^{\mathcal{O}_\mathcal{K}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}}(U)$ there is an object $M_{\mathfrak{m},I,w}$ of $\underline{\mathrm{MF}}_{\mathcal{O}_{F_w},\mathcal{O}_\mathcal{K}}$ such that

$$ (\rho_\mathfrak{m} \otimes_{\mathcal{O}_\mathcal{K}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}}(U)} {}^{\mathcal{O}_\mathcal{K}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}}(U)/I) \mid \mathrm{Gal}_{F_w} \cong \mathtt{G}_{F_w}(M_{\mathfrak{m},I,w}). $$

If $\mathfrak{m}$ is non-Eisenstein in the sense of [Clozel et al. 2008, Definition 3.4.3], *then $\rho_\mathfrak{m}$ and its reduction extend to*

$$ r_\mathfrak{m} : \mathrm{Gal}_{F^+} \to \mathcal{G}_n({}^{\mathcal{O}_\mathcal{K}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}}(U)_\mathfrak{m}) \quad \text{and} \quad \bar{r}_\mathfrak{m} : \mathrm{Gal}_{F^+} \to \mathcal{G}_n({}^{\mathcal{O}_\mathcal{K}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}}(U)/\mathfrak{m}). $$

Moreover, $\mathscr{m} \circ r_\mathfrak{m} = \epsilon_\ell^{1-n} \delta^{\mu_\mathfrak{m}}_{F \mid F^+}$ for a suitable $\mu_\mathfrak{m} \in \mathbb{Z}/2\mathbb{Z}$, where $\delta_{F \mid F^+}$ is the nontrivial character of $\mathrm{Gal}(F \mid F^+)$.

In this way we can associate to a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ and a finite place λ of $\mathcal{E}(U)$ a residual representation $\bar{r}_{\pi,\lambda} : \mathrm{Gal}_{F^+} \to \mathcal{G}_n(\overline{\mathbb{F}}_{\ell(\lambda)})$. Let us assume that $\bar{r}_{\pi,\lambda}$ is absolutely irreducible for all λ in a subset of $\mathrm{Pl}^{\mathrm{fin}}_{\mathcal{E}(U)}$ of Dirichlet density 1. Then the set

$$\Lambda^1_{\mathcal{E}(U)} = \{\lambda \mid \bar{\rho}_{\pi,\lambda'} \text{ is absolutely irreducible for all } \lambda' \text{ dividing } \ell(\lambda)\}$$

has also Dirichlet density 1. In this way, we get an association from π to the compatible systems of residual Galois representations $\mathcal{R}_\pi = (\bar{r}_{\pi,\lambda})_{\lambda\in\Lambda^1_{\mathcal{E}(U)}}$ and $\mathcal{R}'_\pi = (\bar{\rho}_{\pi,\lambda})_{\lambda\in\Lambda^1_{\mathcal{E}(U)}}$.

7. Consequences from modularity lifting theorems

Let us start with the following adaption of [Khare and Wintenberger 2009a, Lemma 3.6]:

Lemma 7.1. *Let k be a finite field of characteristic ℓ, G a profinite group satisfying the ℓ-finiteness condition and $\eta : G \to \mathcal{G}_n(k)$ be an absolutely irreducible continuous representation. Let $\mathcal{F}_n(G)$ be a subcategory of deformations of η in k-algebras which defines a deformation condition. Let $\eta_{\mathcal{F}} : G \to \mathrm{GL}_n(R_{\mathcal{F}})$ be the universal deformation of η in $\mathcal{F}_n(G)$. Then $R_{\mathcal{F}}$ is finite if and only if $\eta_{\mathcal{F}}(G)$ is finite.*

Proof. The proof of Lemma 3.6 of Khare–Wintenberger goes through verbatim, except that we have to refer to [Clozel et al. 2008, Lemma 2.1.12] instead of Carayol's lemma. □

7A. *A minimal $R = T$-theorem.* Our starting point is a RACSDC automorphic representation $\pi = \langle f \rangle \subset \mathcal{S}_\omega(U)$ (where $U = U_S$ for a finite set of places S of F) and a place $\lambda \in \Lambda^1_{\mathcal{E}(U)}$. Fix a finite F-big enough extension $\mathcal{K}$ of $\mathcal{E}(U)_\lambda$. We abbreviate $r, \bar{r}, \rho, \bar{\rho}$ for the associated Galois representations via Proposition 6.7 for the unique maximal ideal $\mathfrak{m}$ containing $\mathcal{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} \ker \varphi^\ell_f$. We assume furthermore the following:

- All places of S_ℓ split in the extension $F \mid F^+$.
- All ramification of ρ is unipotent (this can always be achieved by a finite solvable base change).
- ρ is a minimally ramified (at all places in S) and FL-crystalline (at all places dividing ℓ) lift of $\bar{\rho}$.
- $\bar{\rho}$ is absolutely irreducible.
- $\bar{\rho}(\mathrm{Gal}_{F(\zeta_\ell)})$ is adequate in the sense of Thorne [2012, Definition 2.20]:
  - $H^1(X_\ell, k_\lambda) = 0$ and $H^1(X_\ell, \mathfrak{gl}^0_n) = 0$.
  - For any simple $k_\lambda[X_\ell]$-submodule $W \subset \mathfrak{gl}_n$, there exists a semisimple element $\sigma \in X_\ell$ with eigenvalue $\alpha \in k_\lambda$ such that $\mathrm{tr}\, e_{\sigma,\alpha} W \neq 0$. (Here, $e_{\sigma,\alpha} \in \mathfrak{gl}_n$ denotes the unique idempotent in $k_\lambda[\sigma]$ with image equal to the α-eigenspace of σ.)

Let us abbreviate $R^{(\mathtt{min}),[\mathtt{crys}]} := R^{\chi,(\mathtt{min}),[\mathtt{crys}]}_{\mathcal{O}_{\mathcal{K}},S_\ell}(\bar{r})$ for the ring parametrizing fixed-determinant deformations of $\bar{r}$ which are unramified outside S_ℓ (minimally ramified in S) and [FL-crystalline at places dividing ℓ]. Moreover, let $\mathbb{T}$ (resp. $\mathbb{T}^{\mathtt{min}}$) denote the Hecke algebra ${}^{\mathcal{O}_{\mathcal{K}}}\mathbb{T}^{S_\ell}_\omega(U)_\mathfrak{m}$ (resp. ${}^{\mathcal{O}_{\mathcal{K}}}_{\underline{\sigma}}\mathbb{T}^{S_\ell}_\omega(U)_\mathfrak{m}$), where $\mathfrak{m}$ is the maximal ideal such that $\bar{r}_\mathfrak{m} \cong \bar{r}$ and where $\underline{\sigma} = (\sigma_v)_{v\in\tilde{S}}$ is defined as follows: For each $v \in \tilde{S}$ we can associate an inertial type τ_v in the same way as we did just before Theorem 5.10. To each

τ_v one can associate a representation $\sigma_v = \sigma(\tau_v)$ of $K = \mathrm{GL}_n(\mathcal{O}_{F_v})$ (which is then the K-type of the $\mathrm{GL}_n(F_v)$-representation associated to an extension of τ_v to Gal_{F_v}.) To construct $\sigma(\tau)$ (see also [Shotton 2015, Section 4.6; Bellaïche and Chenevier 2009, Section 6.5.2; Schneider and Zink 1999]):

- Consider the finite group $\mathfrak{G} = \mathrm{GL}_n(\ell(v))$ and its standard Borel subgroup $\mathfrak{B} \subset \mathfrak{G}$. Then the irreducible constituents of the (complex) representation $\mathrm{ind}_{\mathfrak{B}}^{\mathfrak{G}}(1)$ are called the unipotent representations of $\mathfrak{G}$. These representations can (canonically) be parametrized by the irreducible representations of the Weyl group $\mathcal{W}(\mathfrak{G}) \cong S_n$; see e.g., [Prasad 2014, Corollary 4.4]. The irreducible representations of S_n in turn can be parametrized by partitions of n in terms of Specht modules; see [James and Kerber 1981]. In other words, we get a canonical bijection $h : \mathcal{Y}_n \cong \mathrm{Rep}(\mathfrak{G})^{\mathtt{uni}}$, where $\mathrm{Rep}(\mathfrak{G})^{\mathtt{uni}}$ denotes the set of all unipotent representations of $\mathfrak{G}$ up to isomorphism. The map h can be explicitly described in terms of induction from Levi subgroups (see [Shotton 2015, Definition 4.34]) and sends $(1, \dots, 1)$ to the trivial representation and (n) to the Steinberg representation.
- Under the unipotent ramification assumption, the set of inertial types $\mathcal{I}^{\mathtt{uni}}$ is in bijection with the set $\mathcal{Y}_n$ of partitions of n via ∇ from Section 5D.

We have a decomposition

$$\mathrm{ind}_I^K(1) \cong \mathrm{infl}_{\mathfrak{G}}^K \, \mathrm{ind}_{\mathfrak{B}}^{\mathfrak{G}}(1) \cong \bigoplus_{\Pi \in \mathrm{Rep}(\mathfrak{G})^{\mathtt{uni}}} m_\Pi \, \mathrm{infl}_{\mathfrak{G}}^K(\Pi),$$

where $I \subset K$ denotes the Iwahori subgroup, $\mathrm{infl}_{\mathfrak{G}}^K$ denotes the inflation along the pro-$\ell(v)$-radical of K and where the $m_\Pi \geq 1$ are suitable multiplicities. Analogous to [Bellaïche and Chenevier 2009, Remark 6.5.2(iii)], one can thus check that the assignment $\tau \mapsto \sigma(\tau)$ is described in terms of partitions as $\tau \mapsto \mathrm{infl}_{\mathfrak{G}}^K(h \circ \nabla(\tau))$. Observe that the special case $n = 2$ is precisely [loc. cit., Remark 6.5.2(iii)] and [Shotton 2015, Example 2.17].

We stress that the notions $\mathbb{T}$ and $\mathbb{T}^{\mathtt{min}}$ depend on the choice of the place λ.

Proposition 7.2. *A map $h \colon \mathbb{T} \to \overline{\mathbb{Q}}_\ell$ factors through $\mathbb{T}^{\mathtt{min}}$ if and only if the concatenation*

$$h' \colon R^{\mathtt{crys}} \to \mathbb{T} \to \overline{\mathbb{Q}}_\ell$$

factors through $R^{\mathtt{min,crys}}$.

Proof. The map h corresponds to an automorphic form $g \in S_\omega(U, \mathcal{O}_\mathcal{K})$ such that $\bar{r}_{\langle g \rangle} \cong \bar{r}$. By Theorem 5.10 and the above, $\rho_{\langle g \rangle, v}$ (for $v \in S$) is a minimally ramified lift of $\bar{\rho}_v$ if and only if it is of type τ_v if and only if $\langle g \rangle_v$ is of type σ_v. (If $\langle g \rangle_v$ is of type σ_v, then $\rho_{\langle g \rangle, v}$ is at most as ramified as τ_v.) Thus, h factors through $\mathbb{T}^{\mathtt{min}}$ if and only if $g \in {}_{\underline{\sigma}}S_\omega(U, \mathcal{O}_\mathcal{K})$ if and only if $r_{\langle g \rangle}$ is (as a lift of $\bar{r}$) minimally ramified in S if and only if the associated map $h' : R^{\mathtt{crys}} \to \overline{\mathbb{Q}}_\ell$ factors through $R^{\mathtt{min,crys}}$. □

Theorem 7.3. *$R^{\mathtt{min,crys}}$ is finite flat over $\mathcal{O}_\mathcal{K}$, so in particular there exists a characteristic-*0 *point of* Spec $R^{\mathtt{min,crys}}$*. Moreover, we have isomorphisms*

$$R^{\mathtt{min,crys}} \cong R^{\mathtt{min,crys}}_{\mathtt{red}} \cong \mathbb{T}^{\mathtt{min}}.$$

Proof. We first remark that $R^{\mathtt{min,crys}}/(\ell)$ is of finite cardinality or, equivalently (by Nakayama's lemma), that $R^{\mathtt{min,crys}}$ is finitely generated as a $\mathcal{O}_{\mathcal{K}}$-module. This follows directly from [Barnet-Lamb et al. 2014, Theorem 2.3.2], as we know that the local deformation rings $R^{\square,\chi_v,\mathtt{crys}}(\bar{\rho}_v)$ and $R^{\square,\chi_v,\mathtt{min}}(\bar{\rho}_v)$ are smooth, hence correspond to irreducible components of Spec $R^{\square,\chi_v}(\bar{\rho}_v)$ on which the local lifts ρ_v live.

Next, we remark that by Corollary 3.21 (together with the smoothness-results Lemma 5.4 and Propositions 5.6 and 3.22, the identity $\dim(\mathfrak{gl}_n^{c_v=-1}) = n(n+1)/2$ and Remark 7.5 below) there exists a presentation

$$R^{\mathtt{min,crys}} \cong \mathcal{O}_{\mathcal{K}}[\![X_1, \ldots, X_m]\!]/(f_1, \ldots, f_m)$$

for some $m \in \mathbb{N}_0$.

Using this presentation and the finiteness of $R^{\mathtt{min,crys}}/(\ell)$, it follows as in the proof of Theorem 3.7 of [Khare and Wintenberger 2009a] or of Lemma 2 in Böckle's appendix to [Khare 2003] that $R^{\mathtt{min,crys}}$ is finite flat over $\mathcal{O}_{\mathcal{K}}$, hence free and finitely generated over $\mathcal{O}_{\mathcal{K}}$. This proves the first claim.

As a second step, we remark that any morphism $f: R^{\mathtt{min,crys}} \to \overline{\mathbb{Q}}_\ell$ factors over $\mathbb{T}^{\mathtt{min}}$: By [Barnet-Lamb et al. 2014, Theorem 2.3.1], such an f factors over the nonminimal Hecke algebra $\mathbb{T}$. Therefore Proposition 7.2 applies.

Now, $R^{\mathtt{min,crys}}\left[\frac{1}{\ell}\right] = R^{\mathtt{min,crys}} \otimes_{\mathcal{O}_{\mathcal{K}}} \mathcal{K}$ is a finite $\mathcal{K}$-algebra, hence $R^{\mathtt{min,crys}}\left[\frac{1}{\ell}\right]$ is Artinian; see e.g., [Atiyah and Macdonald 1969, Exercise 8.3]. Therefore, $R^{\mathtt{min,crys}}\left[\frac{1}{\ell}\right]$ can be decomposed into a product of finitely many local Artinian rings

$$R^{\mathtt{min,crys}}\left[\frac{1}{\ell}\right] \cong \oplus_{\mathfrak{p}} R^{\mathtt{min,crys}}\left[\frac{1}{\ell}\right]_{\mathfrak{p}}$$

and by [Allen 2016, Theorem 3.1.3] the tangent space $\mathfrak{p}/\mathfrak{p}^2$ of each $R^{\mathtt{min,crys}}\left[\frac{1}{\ell}\right]_{\mathfrak{p}}$ vanishes. Hence, $\mathfrak{p} = \mathfrak{p}^2$, and it follows from Nakayama's lemma that $\mathfrak{p} = 0$, i.e., that $R^{\mathtt{min,crys}}\left[\frac{1}{\ell}\right]_{\mathfrak{p}}$ is a field. Thus, $R^{\mathtt{min,crys}}\left[\frac{1}{\ell}\right]$ is a finite product of fields. The same is true for $\mathbb{T}^{\mathtt{min}}\left[\frac{1}{\ell}\right]$: As $\mathbb{T}^{\mathtt{min}}\left[\frac{1}{\ell}\right]$ is finitely generated, its Jacobson radical equals its nilradical, which vanishes because $\mathbb{T}^{\mathtt{min}}$ is reduced. Hence $\mathbb{T}^{\mathtt{min}}\left[\frac{1}{\ell}\right]$ is semisimple, i.e., a product of finitely many fields.

Consider the exact sequence

$$0 \to \ker(\varphi) \to R^{\mathtt{min,crys}} \xrightarrow{\varphi} \mathbb{T}^{\mathtt{min}} \to 0, \tag{12}$$

where φ denotes the canonical projection. It follows from the above observation about $R^{\mathtt{min,crys}}\left[\frac{1}{\ell}\right]$ and $\mathbb{T}^{\mathtt{min}}\left[\frac{1}{\ell}\right]$ together with Proposition 7.2 that $\varphi\left[\frac{1}{\ell}\right]$ is an isomorphism. Moreover, as both $R^{\mathtt{min,crys}}$ and $\mathbb{T}^{\mathtt{min}}$ are finite free over $\mathcal{O}_{\mathcal{K}}$, (12) is a split exact sequence of free $\mathcal{O}_{\mathcal{K}}$-modules. Hence, $\ker(\varphi) = 0$ since $\varphi\left[\frac{1}{\ell}\right]$ is an isomorphism. This completes the proof of the theorem. □

Corollary 7.4. *The following is a pushout diagram*:

$$\begin{array}{ccc} R^{\mathtt{crys}} & \longrightarrow & R^{\mathtt{min,crys}} \\ \downarrow & & \downarrow \\ \mathbb{T} & \longrightarrow & \mathbb{T}^{\mathtt{min}} \end{array}$$

Remark 7.5. We remark that for each $v \in \Omega_\infty$ the local deformation ring $R^{\square,\chi_v}_{W(k_\lambda)}(\bar r_{\lambda,v})$ is formally smooth of relative dimension $d_v^\square = \dim(\mathfrak{b}_n^{\mathtt{der}})$: We get from Proposition 3.22 that $R^{\square,\chi_v}_{W(k_\lambda)}(\bar r_{\lambda,v})$ is formally smooth of relative dimension $\dim((\mathfrak{g}_n)^{c_v=-1}) = \dim((\mathfrak{gl}_n)^{c_v=-1})$, where c_v is the nontrivial element of the decomposition group at v. By construction (see Lemma 2.1.4 and Proposition 3.4.4 of [Clozel et al. 2008]), the image of $\bar r_\lambda(c_v)$ is not contained in $\mathrm{GL}_n \times \mathrm{GL}_1$. Moreover,

$$\mathtt{m} \circ \bar r_\lambda(c_v) = \bar\epsilon_\ell^{1-n}(c_v)\delta^{\mu_{\mathtt{m}}}_{F\,|\,F^+}(c_v) = (-1)^{\mu_{\mathtt{m}}+p},$$

where $p = n+1 \pmod 2) \in \mathbb{Z}/2\mathbb{Z}$, where ϵ_ℓ denotes the cyclotomic character (sending c_v to -1), where $\delta_{F\,|\,F^+}$ denotes the nontrivial character of $\mathrm{Gal}(F \mid F^+)$ and where $\mu_{\mathtt{m}}$ is a suitable element of $\mathbb{Z}/2\mathbb{Z}$. As in [Thorne 2012, Corollary 6.9], we get $\mu_{\mathtt{m}} \equiv n \pmod 2$, so we have $\mathtt{m}\circ\bar r_\lambda(c_v) = -1$, independent of the parity of n. Using [Clozel et al. 2008, Lemma 2.1.3], this implies $\dim((\mathfrak{gl})_n^{c_v=-1}) = n(n+1)/2 = \dim(\mathfrak{b}_n^{\mathtt{der}})$.

7B. *A $T = O$-theorem.* Let $\mathcal{E} \supset \mathcal{E}(U)$ be a number field in $\mathbb{C}$ with ring of integers $\mathcal{O}_\mathcal{E}$. For each $\lambda \in \mathrm{Pl}^{\mathrm{fin}}_\mathcal{E}$ such that $\ell := \ell(\lambda) \gg 0$, let us fix an F-big enough extension $\mathcal{K}_\lambda$ of $\mathcal{E}_\lambda$ and let us abbreviate

$$\boldsymbol{T} := {}^{\mathcal{O}_\mathcal{E}}\boldsymbol{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U) \text{ and } \boldsymbol{T}_\lambda := \mathcal{O}_{\mathcal{K}_\lambda} \otimes_{\mathcal{O}_\mathcal{E}} \boldsymbol{T} \cong {}^{\mathcal{O}_{\mathcal{K}_\lambda}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\ell}(U).$$

Observe the following about the isomorphism on the right-hand side: Using that $\mathcal{S}_{\boldsymbol{\omega}}(U)$ admits a basis of eigenforms, we can embed $\boldsymbol{T}$ into a product of finitely many $\mathcal{O}_{\mathcal{E}(U)}$. Hence, $\boldsymbol{T}$ is finitely generated as a $\mathbb{Z}$-module, hence as a $\mathbb{Z}$-algebra. It follows that there exists a Sturm-like bound $C \in \mathbb{N}$ such that $\boldsymbol{T}$ is already generated by those $T^{(j)}_{F_w}$ with $\ell(w) \le C$. Hence, using the compatibility from (11), we get

$$\mathcal{O}_{\mathcal{K}_\lambda} \otimes_{\mathcal{O}_\mathcal{E}} \boldsymbol{T} \cong \mathcal{O}_{\mathcal{K}_\lambda} \otimes_{\mathcal{O}_\mathcal{E}} {}^{\mathcal{O}_\mathcal{E}}\boldsymbol{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}}(U) \cong {}^{\mathcal{O}_{\mathcal{K}_\lambda}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\ell}(U)$$

as long as $\ell > C$. Then we have:

Lemma 7.6. *For almost all λ (the failure set depending only on $\boldsymbol{T}$), $\boldsymbol{T}_\lambda$ decomposes as a product of finitely many complete discrete valuation rings, finite over $\mathbb{Z}_\ell$.*

Proof. First, we see that $\boldsymbol{T}$ is an order in $\boldsymbol{T} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{E} \cong k_1 \times \cdots \times k_m$, where the k_i denote suitable number fields (containing E) and the decomposition follows because $\boldsymbol{T} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{E}$ is reduced (as already remarked). Hence, $\boldsymbol{T}$ is contained in the maximal order $\oplus_{i=1}^m \mathcal{O}_{k_i}$. It follows that there exists a suitable $N \in \mathbb{N}$ such that:

- $\boldsymbol{T}\big[\frac{1}{N}\big] \cong \oplus_{i=1}^m \mathcal{O}_{k_i}\big[\frac{1}{N}\big]$.
- for any λ with $\ell(\lambda) \nmid N$, we have $\boldsymbol{T}_\lambda \cong \boldsymbol{T}\big[\frac{1}{N}\big]_\lambda := \mathcal{O}_{\mathcal{K}_\lambda} \otimes_{\mathcal{O}_\mathcal{E}} \boldsymbol{T}\big[\frac{1}{N}\big]$.

Thus, for those λ we get an isomorphism $\boldsymbol{T}_\lambda \cong \oplus_{i=1}^m \mathcal{O}_{\mathcal{K}_\lambda} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{O}_{k_i}$. As each factor itself is a product of complete discrete valuation rings (see, e.g., [Serre 1979, Chapter 2, Section 3, Theorem 1(ii)]), the lemma follows. □

Because we assumed that $\mathcal{E}$ contains all Hecke eigenvalues, in fact all the fields k_i in the above proof are equal to $\mathcal{E}$. Hence, for almost all ℓ, the above lemma implies that $\boldsymbol{T}_\lambda$ is isomorphic to a product of finitely many copies of $\mathcal{O}_{\mathcal{K}_\lambda}$. Thus, we get:

Corollary 7.7. *For almost all λ and all maximal ideals $\mathfrak{m} \subset \boldsymbol{T}_\lambda$, we have an isomorphism $\boldsymbol{T}_{\lambda,\mathfrak{m}} \cong \mathcal{O}_{\mathcal{K}_\lambda}$.*

7C. *An $R = R^{\mathrm{min}}$-theorem.* We retain all notation from the above and start with a preparatory corollary (to Corollary 5.14):

Corollary 7.8. *For almost all λ for which $\bar{\rho}_\lambda$ is absolutely irreducible, we have $R^{\mathtt{crys},(1,\dots,1)} = R^{\mathtt{crys}}$.*

Proof. Let $m := \max\{p \in \mathbb{N} \mid p \text{ prime}, \nu \mid p \text{ for some } \nu \in S\}$. Then, for all λ with $\ell(\lambda) > m^{n!}$, the claim follows directly from Corollary 5.14. □

Moreover, we need a congruence argument: First, recall that the Hecke algebra ${}^{\mathcal{O}_\mathcal{K}}\mathbb{T}_{\boldsymbol{\omega}}^{S_\ell}(U)$ acts semisimply on $\mathcal{S}_{\boldsymbol{\omega}}(U)$, so the space $\mathcal{S}_{\boldsymbol{\omega}}(U)$ decomposes into finitely many eigenspaces. For the following, let us consider *congruences*, by what we mean triples (H_1, H_2, ℓ), where $H_1 \neq H_2$ are two Hecke eigenspaces and where ℓ is a rational prime such that there exists an isomorphism $\bar{\rho}_{f_1,\lambda_1} \otimes \overline{\mathbb{F}}_\ell \cong \bar{\rho}_{f_2,\lambda_2} \otimes \overline{\mathbb{F}}_\ell$ for some choice of forms $f_i \in H_i$ and of places λ_i of the corresponding coefficient fields fulfilling $\ell(\lambda_i) = \ell$.

Proposition 7.9. *There exist only finitely many such congruences in $\mathcal{S}_{\boldsymbol{\omega}}(U)$.*

Proof. We easily see that a congruence (H_1, H_2, ℓ) corresponds to two distinct minimal prime ideals $\mathfrak{p}_{f_1}, \mathfrak{p}_{f_2}$ of $\boldsymbol{T}$ for which there exists a maximal ideal $\mathfrak{m} \subset \boldsymbol{T}$ which contains ℓ, $\mathfrak{p}_{f_1}$ and $\mathfrak{p}_{f_2}$. It follows from the finite flatness of $\boldsymbol{T}$ over $\mathbb{Z}$ that for given eigenforms f_1, f_2 there exist only finitely many maximal ideals containing $\mathfrak{p}_{f_1}$ and $\mathfrak{p}_{f_2}$. Thus, the claim follows immediately from the finite-dimensionality of the space of automorphic representations of given level and weight. □

Theorem 7.10. *For almost all λ for which $\bar{\rho}_\lambda$ is absolutely irreducible, we have*

$$R^{\mathtt{crys}} \cong R^{\mathtt{min,crys}}.$$

Proof. We apply the proof of Theorem 7.3, where we replace $R^{\mathtt{min,crys}}$ by $R^{\mathtt{crys}}$ and $\mathbb{T}^{\mathtt{min}}$ by $R^{\mathtt{min,crys}}$:

Let us first show that $R^{\mathtt{crys}}/(\ell)$ is of finite cardinality for almost all ℓ: By Nakayama's Lemma, this is equivalent to $R^{\mathtt{crys}}$ being finitely generated as a W-module. So consider the exact sequence

$$\mathrm{Nil}(R^{\mathtt{crys}})/(\ell) \to R^{\mathtt{crys}}/(\ell) \to R^{\mathtt{crys}}_{\mathtt{red}}/(\ell) \to 0.$$

We can assume that the nilradical $\mathrm{Nil}(R^{\mathtt{crys}})$ is finitely generated as a W-module: The filtration quotients $\mathrm{Nil}(R^{\mathtt{crys}})^i / \mathrm{Nil}(R^{\mathtt{crys}})^{i+1}$ are finitely generated $R^{\mathtt{crys}}$-modules (by Noetherianness) on which $\mathrm{Nil}(R^{\mathtt{crys}})$ operates trivially, hence finitely generated $R^{\mathtt{crys}}_{\mathtt{red}}$-modules. Assuming that $R^{\mathtt{crys}}_{\mathtt{red}}$ is finitely generated as a W-module, it follows that each filtration quotient is a finitely generated W-module. But, again by Noetherianness, $\mathrm{Nil}(R^{\mathtt{crys}})^i$ vanishes for $i \gg 0$. Hence, $\mathrm{Nil}(R^{\mathtt{crys}})$ is a finitely generated W-module. Thus, it remains to show that $R^{\mathtt{crys}}_{\mathtt{red}}$ is a finitely generated W-module. By Corollary 7.8, we can apply [Guerberoff 2011, Theorem 4.1] which yields the existence of a suitable finite extension $L^+ \mid F^+$, unramified at all places above ℓ, such that the ring $R^{(1,\dots,1),\mathtt{crys}}_{L^+,\mathtt{red}} := R^{\chi_{L^+},(1,\dots,1),\mathtt{crys}}_{W,\tilde{S}_\ell,\mathtt{red}}(\bar{r}_{L^+})$ is isomorphic to a suitable Hecke algebra $\mathbb{T}$ (acting on automorphic forms on a unitary group over $\mathbb{A}_{F^+}$), hence that $R^{(1,\dots,1),\mathtt{crys}}_{L^+,\mathtt{red}}$ is finitely generated over W. In order to use this result, we apply the approach of

[Khare and Wintenberger 2009a, proof of Proposition 3.8]: First, we remark that it is sufficient to show that the mod-ℓ reduction $\bar{r}^{(1,\dots,1),\mathtt{crys}}$ of the universal deformation

$$r^{(1,\dots,1),\mathtt{crys}}\colon G_{F^+} \to \mathcal{G}_n(R^{(1,\dots,1),\mathtt{crys}})$$

has finite image in order to deduce finiteness of $R^{(1,\dots,1),\mathtt{crys}}/(\ell)$, using Lemma 7.1. On the other hand, the image of the reduction of the universal deformation $r_{L^+}^{(1,\dots,1),\mathtt{crys}}$, parametrizing crystalline deformations of $\bar{r} \mid \mathrm{Gal}_{L^+}$ fulfilling the Taylor condition at $\tilde{S}$, is finite (as shown above). As $r^{(1,\dots,1),\mathtt{crys}} \mid G_{L^+}$ is a crystalline deformation of $\bar{r} \mid G_{L^+}$ (see Lemma 5.5 and Proposition 5.12), we get a morphism

$$\varphi\colon R_{L^+}^{(1,\dots,1),\mathtt{crys}}/(\ell) \to R^{(1,\dots,1),\mathtt{crys}}/(\ell)$$

such that $\bar{r}^{(1,\dots,1),\mathtt{crys}} \mid G_{L^+} = \varphi \circ \bar{r}_{L^+}^{(1,\dots,1),\mathtt{crys}}$. It follows that $\bar{r}^{(1,\dots,1),\mathtt{crys}} \mid G_{L^+}$ has finite image, hence (as $[L:F] < \infty$) that $\bar{r}^{(1,\dots,1),\mathtt{crys}}$ has finite image. As $R^{\mathtt{crys}}/(\ell)$ is a quotient of $R^{(1,\dots,1),\mathtt{crys}}/(\ell)$, the former is finitely generated, as claimed.

Next, we remark that by Corollary 3.21 (together with the smoothness-results Lemma 5.4, Theorem 5.1, Proposition 3.22, the identity $\dim(\mathfrak{gl}_n^{c_v=-1}) = n(n+1)/2$ and Remark 7.5 above) there exists a presentation

$$R^{\mathtt{crys}} \cong \mathcal{O}_{\mathcal{K}}[\![X_1, \dots, X_m]\!]/(f_1, \dots, f_m).$$

for some $m \in \mathbb{N}_0$.

Using this presentation and the finiteness of $R^{\mathtt{crys}}/(\ell)$, it follows as in the proof of Theorem 3.7 of [Khare and Wintenberger 2009a] or of Lemma 2 in Böckle's appendix to [Khare 2003] that $R^{\mathtt{crys}}$ is finite flat over $\mathcal{O}_{\mathcal{K}}$, hence free and finitely generated over $\mathcal{O}_{\mathcal{K}}$. This proves the first claim.

Moreover, we claim that for almost all λ, any morphism $R^{\mathtt{crys}} \to \overline{\mathbb{Q}}_\ell$ factors over $R^{\mathtt{min},\mathtt{crys}}$. Using automorphy lifting, this claim can be restated as follows: For almost all λ, the following holds: For any automorphic form g whose associated Galois representation $\rho_{g,\lambda}$ reduces to $\bar{\rho}_\lambda$, $\rho_{g,\lambda}$ is a minimally ramified lift of $\bar{\rho}_\lambda$. Now, let λ be a place such that this statement fails. Then, as there always exists a minimally ramified lift of $\bar{\rho}_{g,\lambda}$ with a corresponding automorphic form f (see Theorem 7.3), we get a congruence $({}^{\mathcal{O}_{\mathcal{K}}}\mathbb{T}_{\omega}^{S_\ell}(U).f, {}^{\mathcal{O}_{\mathcal{K}}}\mathbb{T}_{\omega}^{S_\ell}(U).g, \ell(\lambda))$. Thus, the claim follows from Proposition 7.9.

This completes the proof of the theorem. ☐

Let us close with the following corollary (to Theorem 7.10), giving a local $R = R^{\mathtt{min}}$ result:

Corollary 7.11. *For almost all λ, $R^{\square,\chi_v,\mathtt{min}}(\bar{\rho}_{\lambda,v}) \cong R^{\square,\chi_v}(\bar{\rho}_{\lambda,v})$ holds for any $v \in S$.*

Proof. Otherwise, there would be a nonminimal component of $R^{\square,\chi_v,\mathtt{min}}(\bar{\rho}_{\lambda,v})$. By [Barnet-Lamb et al. 2014, Theorem 4.3.1], there would be a lift of $\bar{\rho}_{\lambda,v}$ whose local representation at v lies on this component, in contradiction to Theorem 7.10. ☐

8. Unobstructedness for RACSDC automorphic representations

We are now in a position to state and prove our main result. For this, let π be a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with ramification set S. To π we can attach a compatible system $\mathcal{R}_\pi =$

$(\bar{r}_\lambda)_{\lambda \in \Lambda^1_{\mathcal{E}_\Pi}}$ where $\Pi \subset \mathcal{S}_{\boldsymbol{\omega}}(U)$ (for a suitable weight $\boldsymbol{\omega}$ and level $U = U_S$). Here, $\mathcal{E}_\Pi$ denotes the number field generated by all Hecke eigenvalues of Π, $\Lambda^1_{\mathcal{E}_\Pi} \subset \mathrm{Pl}_{\mathcal{E}_\Pi}$ denotes the set of places for which $\bar{\rho}_\lambda$ is absolutely irreducible and we assume the following:

Assumption 8.1. (Irreducibility): The set $\Lambda^1_{\mathcal{E}_\Pi} \subset \mathrm{Pl}_{\mathcal{E}_\Pi}$ has Dirichlet density 1.

(No consecutive weights): The multisets of Hodge–Tate weights HT_τ of the system $\mathcal{R}_\pi$ fulfill (for all embeddings τ) the condition from Theorem 5.2: If two numbers a, b occur in HT_τ, then $|a - b| \neq 1$.

We stress that we understand the first part as a general conjecture on Galois representations attached to RACSDC representations (so, in particular, we assume that this is correct independently of the choice of F or π), while the second part puts a constraint on our choice of π. We also have the following:

Remark 8.2. The first part of Assumption 8.1 is known to hold e.g., if π is extremely regular [Barnet-Lamb et al. 2014]. Results in this direction are also contained in [Patrikis and Taylor 2015], but they are not directly applicable to our situation. We also remark that all entries in the ℓ-adic system $(\rho_{\pi,\lambda})_{\lambda \in \mathrm{Pl}_{\mathcal{E}(U)}}$ are expected (by cuspidality of π) to be absolutely irreducible and that this, using suitable modularity lifting theorems, is expected to imply absolute irreducibility of $\bar{\rho}_{\pi,\lambda}$ for almost all λ. An established result in this direction is that absolute irreducibility of the ℓ-adic system implies absolute irreducibility of $\bar{\rho}_{\pi,\lambda}$ except for a failure set of Dirichlet density 0, see [Patrikis et al. 2018].

Our main result is now as follows:

Theorem 8.3. *Presuming Assumption 8.1, there exists a subset $\Lambda^0_{\mathcal{E}_\Pi} \subset \Lambda^1_{\mathcal{E}_\Pi}$ of Dirichlet density 1 such that the functor $D^{\square_{S_\ell},\chi}_{S_\ell, W(k_\lambda)}(\bar{r}_\lambda)$ is globally unobstructed whenever $\lambda \in \Lambda^0_{\mathcal{E}_\Pi}$.*

As a first step towards the proof, let us consider the following simplifying assumption:

Assumption 8.4. (1) $F \mid F^+$ is unramified at all finite places and, in case n is even, then also $\frac{n}{2}[F^+ : \mathbb{Q}]$ is even.

(2) Each place v of F^+ which lies below S splits in $F \mid F^+$ as, say, $\tilde{v}\tilde{v}^c$. (For archimedean places, this condition is automatically fulfilled, so we can replace S by $S \sqcup \Omega_\infty$ without loss of generality.)

(3) For each place v of F^+ which lies below S, the Weil–Deligne representation (r_v, W_v) attached to Π has unramified underlying Weil-representation r_v.

Remark that the third part can be characterized as follows: The ℓ-adic representation $r_{\Pi,\lambda}$ is at v a minimally ramified deformation of $\bar{r}_{\Pi,\lambda}$. (As the system associated to Π is compatible, this does not depend on the choice of $\lambda \in \Lambda^1_{\mathcal{E}_\Pi}$.) Now, consider the following (seemingly weaker) variation of Theorem 8.3:

Theorem 8.5. *Presuming Assumptions 8.1 and 8.4, there exists a subset $\Lambda^0_{\mathcal{E}_\Pi} \subset \Lambda^1_{\mathcal{E}_\Pi}$ of Dirichlet density 1 such that the functor $D^{\square_{S_\ell},\chi}_{S_\ell, W(k_\lambda)}(\bar{r}_\lambda)$ is globally unobstructed whenever $\lambda \in \Lambda^0_{\mathcal{E}_\Pi}$.*

Proof that Theorem 8.5 implies Theorem 8.3. It is a standard argument (see, e.g., the proof of [Clozel et al. 2008, Theorem 4.4.2]) that there exists a finite solvable extension $F_1^+ \mid F^+$ of totally real fields such that the extension $F_1 = F_1^+.F \mid F_1^+$ and the compatible family associated to the base change Π_{F_1} of Π to F_1 fulfill Assumption 8.4. Thus, referring to Lemma 4.8 and eliminating the finitely many places λ which divide the index $[F_1^+ : F^+]$, we see that Assumption 8.4 can be included in the statement of Theorem 8.3 without causing loss of generality. □

Consequently, the remainder of this section is devoted to the proof of Theorem 8.5. For better comprehension, let us give an overview of the strategy of the proof: We want to arrange for a situation where the framework of Section 4 is applicable, i.e., we want to consider suitable field extensions $L^+_{(\lambda)}$ for as many λ as possible such that Theorem 4.2 implies the vanishing of the dual Selmer groups of the base changed functors $D^{\square_{S_\ell},\chi}_{S_\ell,W(k_\lambda)}(\bar{r}_\lambda \mid G_{L^+_{(\lambda)}})$. This application of Theorem 4.2 happens in Theorem 8.12 below. By a careful choice of the extensions $L^+_{(\lambda)}$, we ensure that the potential unobstructedness arguments of Section 4 apply and yield the vanishing of the dual Selmer groups of the non-base changed functor. The local parts of the unobstructedness-condition then follow directly from the material in Section 5B, allowing us to conclude the statement of Theorem 8.5. The crucial property we have to impose on the extension $L^+_{(\lambda)}$ is *procurability* (Definition 8.7), i.e., that the deformation ring $R^{\chi\,|\,\mathrm{Gal}_{L^+},\mathtt{crys}}_{S_\ell,\mathcal{O}_{\mathcal{K}_\lambda}}(\bar{r}_\lambda \mid \mathrm{Gal}_{L^+})$ is isomorphic to $\mathcal{O}_{\mathcal{K}_\lambda}$ (corresponding to condition $(R = T)$ in Section 4). It is the content of Theorem 8.8 that for a set of places of Dirichlet density 1 we can find such suitable procurable extensions. This, in turn, is established by studying the seemingly weaker condition of $\star$-procurability (see the list $(\star_1)$–$(\star_5)$ below), which is proved to imply procurability almost everywhere (see Claim 1 below). By an argument based on Chebotarev's density theorem (and postponed to the Appendix), we can conclude that for a density-1 set we can find such $\star$-procurable extensions of 2-power degree.

8A. *Proof of Theorem 8.5.*

Let us begin with some preparatory definitions.

Definition 8.6. A totally real, finite extension L^+ of F^+ is called *preadmissible* if the extension $L^+ \mid F^+$ is Galois and solvable and if $L := F.L^+$ is unramified over L^+ at every finite place.

We remark that these conditions are designed to capture the following: If L^+ is preadmissible, then there exists a unitary group H over L^+ (as considered in Section 6B) and a unitary avatar Π_L on $H(\mathbb{A}_{L^+})$ of the base change π_L of π to L.

For the following, let $\mathcal{E}$ be a number field containing $\mathcal{E}(U)$ and let L^+ be preadmissible.

Definition 8.7. A place $\lambda \in \Lambda^1_{\mathcal{E}}$ is L^+*-procurable* if the following two conditions are fulfilled:

(P.1) The restriction of $\bar{\rho}_\lambda$ to Gal_L remains absolutely irreducible.

(P.2) There exists an L-big enough extension $\mathcal{K}_\lambda$ of $\mathcal{E}_\lambda$ such that there is an isomorphism

$$R^{\chi,\mathtt{crys}}_{S_\ell,\mathcal{O}_{\mathcal{K}_\lambda}}(\bar{r}_\lambda \mid G_{L^+}) := R^{\chi\,|\,\mathrm{Gal}_{L^+},\mathtt{crys}}_{S_\ell,\mathcal{O}_{\mathcal{K}_\lambda}}(\bar{r}_\lambda \mid \mathrm{Gal}_{L^+}) \cong \mathcal{O}_{\mathcal{K}_\lambda}. \tag{13}$$

We remark that the first condition is rather harmless and affects only a failure set of Dirichlet density 0; see Assumption 8.1. We also remark that in the second condition, we consider $\bar{r}_\lambda$ as a representation with values in the residue field $k_{\mathcal{O}_{\mathcal{K}_\lambda}}$ of $\mathcal{O}_{\mathcal{K}_\lambda}$ instead of k_λ.

With respect to a preadmissible extension L^+, define $\mathrm{Proc}(L^+) \subset \Lambda^1_{\mathcal{E}}$ as the subset of those λ which are L^+-procurable.

Since there is a lack of $R = T$-theorems for mod-ℓ representations where the unitary group is associated to an extension F/F^+ in which places above ℓ do not split, we need to work around it by the following chain of extensions of F^+:

Theorem 8.8. *There exists a nested sequence $F^+ = L^+_0 \subset L^+_1 \subset \cdots$ of preadmissible extensions of F^+ such that*

$$\lim_{i\to\infty} \delta\biggl(\bigcup_{j=1}^{i} \mathrm{Proc}(L^+_j)\biggr) = 1, \tag{14}$$

where $\delta(\Delta)$ denotes the density of those rational primes q for which each $\lambda \in \mathrm{Pl}_{\mathcal{E}}$ above q fulfills $\lambda \in \Delta$.

Proof. Let us first introduce another notation: With respect to a preadmissible extension L^+, we say that $\lambda \in \Lambda^1_{\mathcal{E}}$ is L^+-$\star$-*procurable*, if the following list is met (where, as usual, we abbreviate $\ell = \ell(\lambda)$):

($\star_1$) ℓ is not divisible by any element of S.

($\star_2$) ℓ is unramified in the extension $L \mid \mathbb{Q}$.

($\star_3$) All places of L above S_ℓ are split in the extension $L \mid L^+$.

($\star_4$) The base change π_L of π to L remains cuspidal.

($\star_5$) If $\nu \in \mathrm{Pl}_A$ lies above S, then π_L admits a nontrivial Iwahori fix-vector.

Of particular difficulty will be proving that there are sufficiently many λ that fulfill condition ($\star_3$); this will be postponed to the Appendix.

As above, this defines a subset $\mathrm{Proc}^\star(L^+) \subset \Lambda^1_{\mathcal{E}}$. (Observe that condition ($\star_4$) does not depend on λ, but we intentionally include it in the list. So, if Π_L fails to be cuspidal, we have $\mathrm{Proc}^\star(L^+) = \varnothing$.)

Claim 1: $\mathrm{Proc}^\star(L^+) - \mathrm{Proc}(L^+)$ is finite.

Proof of Claim 1. We can suppose that $\mathrm{Proc}^\star(L^+)$ is not empty (otherwise the claim is trivially true), so in particular that π_L is a RACSDC representation and there exists a unitary group and an avatar Π_L over L. Now, for each $\lambda \in \mathrm{Proc}^\star(L^+)$ we pick an L-big enough field extension $\mathcal{K}_\lambda$ of $\mathcal{E}_\lambda$. We consider the complex Hecke algebra ${}^{\mathcal{O}_\mathcal{E}}\boldsymbol{T}^S_{\boldsymbol{\omega}}(U)$ and the ℓ-adic model $\mathbb{T} := {}^{\mathcal{O}_\mathcal{E}}\mathbb{T}^{S_\ell}_{\boldsymbol{\omega}}(U)$.

Write $\Pi_L = \langle f \rangle$ for the unitary avatar of the base change of π to L and for a suitable choice $f \in \mathcal{S}_{\boldsymbol{\omega}}(U)$. We see that $\bar{\rho}_\lambda \mid \mathrm{Gal}_L$ equals the reduction of the representation attached to the maximal ideal $\mathfrak{m} = \ker(\varphi_{f^{(\lambda)}}) \subset \mathbb{T}$ by Proposition 6.7, where $f^{(\lambda)}$ is the ℓ-adic model of f.

Recalling that we presume Assumption 8.1, we see that the conditions at the beginning of Section 7A hold for almost all of these choice of $L \mid L^+$, $\ell = \ell(\lambda)$, U, $\boldsymbol{\omega}$, $\mathcal{E}(U)$, $\mathcal{K}_\lambda$ and $\mathfrak{m}$. The main issue is the adequateness of $\bar{\rho}(\mathrm{Gal}_{F(\zeta_\ell)})$, which follows from [Barnet-Lamb et al. 2014, Proposition 2.1.2] as long as $\ell > 2(n+1)$. Thus, the desired isomorphism (13) follows for almost all λ in $\mathrm{Proc}^\star(L^+)$ by Corollary 7.7. This completes the proof of the claim. $\square$

Consequently, it suffices to show that there exists a nested sequence $F^+ = L_0^+ \subset L_1^+ \subset \cdots$ of preadmissible extensions of F^+ such that (14) holds with $\mathrm{Proc}^\star$ instead of Proc. For the construction of these extensions, we define the set

$$\Theta_F := \{d \in \mathbb{N} \mid \sqrt{d} \notin F, \text{ the base change } \pi \rightsquigarrow \pi_{F(\sqrt{d})} \text{ remains cuspidal}\}.$$

By [Arthur and Clozel 1989, Theorem 4.2] there exists a finite extension E of F such that for any extension K' of F we have the following implication: If $E \cap K' = F$, then the base change of Π to K' remains cuspidal. This implication remains true after replacing E by its Galois closure, so we can assume that $E \mid F$ is Galois. Therefore this set is not empty, so choose a $d_1 \in \Theta_F$ and take $L_1^+ = F^+(\sqrt{d_1})$.

Claim 2: L_1^+ is preadmissible.

Proof of Claim 2. The extension $L_1^+ \mid F^+$ is automatically Galois and solvable because $[L_1^+ : F^+] = 2$. Thus we are left to check that $L_1 \mid L_1^+$ is unramified everywhere. This follows from e.g., [Marcus 1977, Chapter 4, Exercise 10]. □

Claim 3: $\delta(\mathrm{Proc}^\star(F_1^+)) \geq \frac{1}{2}$.

Proof of Claim 3. We check which λ fail the list ($\star_1$)–($\star_5$):

- Concerning ($\star_1$) and ($\star_2$), we have to exclude the finitely many places λ for which $\ell(\lambda)$ is not coprime to S or ramifies in $L_1^+ \mid \mathbb{Q}$.
- By an estimation based on Chebotarev's density theorem (postponed as Lemma A to the appendix), the density of those ℓ which fulfill the condition that all primes of L_1^+ above ℓ are split in the extension $L_1 \mid L_1^+$ is at least $\frac{1}{2}$.
- Condition ($\star_4$) is universally fulfilled by our choice of L_1^+.
- Concerning condition ($\star_5$), we remark that by local-global compatibility (see [Chenevier and Harris 2013, Theorem 1.4] and the references therein) π_L admits an Iwahori-fixed vector if $\rho \mid \mathrm{Gal}_L$ has unipotent ramification at v [Wedhorn 2008, (4.3.6) Proposition]. Thus, condition ($\star_5$) follows immediately from Assumption 8.4. □

For the next tower step we take $F_2^+ = F_1^+(\sqrt{d_2})$ for some $d_2 \in \Theta_{F_2}$. It is again easy to check that $\Theta_{F_2} \neq \varnothing$ and that F_2^+ is preadmissible. As in the proof of Claim 3, the statement of Lemma A implies $\delta(\mathrm{Proc}^\star(F_2^+)) \geq \frac{3}{4}$. Iterating this construction of quadratic extensions we end up with a nested sequence of preadmissible fields F_j^+ such that

$$\delta\Big(\bigcup_{j=1}^{i} \mathrm{Proc}^\star(L_j^+)\Big) \geq \delta(\mathrm{Proc}^\star(L_i^+)) \geq 1 - \frac{1}{2^i} \xrightarrow[i\to\infty]{} 1.$$

Together with Claim 1, this concludes the proof of Theorem 8.8. □

We now give a slight variant of the above:

Definition 8.9. With regard to a preadmissible extension L^+ of F^+, we say that $\lambda \in \Lambda^1_{\mathcal{E}}$ is L^+-$\sharp$-*procurable* if the restriction of $\bar{\rho}_\lambda$ to Gal_L (with $L = F.L^+$) remains absolutely irreducible and if there is an isomorphism

$$R_\lambda^{\square,\chi,\mathtt{crys}}(L^+) \cong W(k_\lambda)[\![x_1,\dots,x_u]\!], \tag{15}$$

where $R_\lambda^{\square,\chi,\mathtt{crys}}(L^+) = R_{S_\ell,W(k_\lambda)}^{\square,\chi,\mathtt{crys}}(\bar{r}_\lambda \mid G_{L^+})$ and $u = \dim(\mathfrak{g}_n^{\mathtt{der}}) = n^2$. The set of all λ which are L^+-$\sharp$-procurable is denoted by $\mathrm{Proc}^\sharp(L^+)$.

Corollary 8.10. *There exists a nested sequence $F^+ = L_0^+ \subset L_1^+ \subset \cdots$ of preadmissible extensions of F^+ such that*

$$\lim_{i\to\infty} \delta\Big(\bigcup_{j=1}^{i} \mathrm{Proc}^\sharp(L_j^+)\Big) = 1.$$

Proof. For $i \in \mathbb{N}$, denote $\Delta_i = \bigcup_{j\le i} \mathrm{Proc}(L_j^+)$. Also fix for each $\lambda \in \Delta_i$ some $j \le i$ such that $\lambda \in \mathrm{Proc}(L_j^+)$. Denote the corresponding field extension from the proof of Theorem 8.8 by $L_{(\lambda)} = L_{(\lambda)}^+.F$. By Theorem 8.8, for such a $\lambda \in \Delta_i$ we have the identity (13) for a suitable extension $\mathcal{O}_{\mathcal{K}_\lambda}$ of $W(k_\lambda)$. The third part of Proposition 3.18 then yields

$$R_{S_\ell,\mathcal{O}_{\mathcal{K}_\lambda}}^{\square,\chi,\mathtt{crys}}(\bar{r}_\lambda \mid G_{L_{(\lambda)}^+}) \cong \mathcal{O}_{\mathcal{K}_\lambda}[\![x_1,\dots,x_u]\!].$$

Thus, we can use Lemma 3.7 (and, if necessary, Remark 3.11) to deduce the desired isomorphism (15). □

Corollary 8.11. *There exists a subset $\Lambda^2_{\mathcal{E}} \subset \Lambda^1_{\mathcal{E}}$ of Dirichlet density 1 such that for each $\lambda \in \Lambda^2_{\mathcal{E}}$ there exists a finite, totally real extension $L_{(\lambda)}^+$ of F and an isomorphism*

$$R_{S_\ell,W(k_\lambda)}^{\square_{S_\ell},\chi,\mathtt{crys}}(\bar{r}_\lambda \mid G_{L_{(\lambda)}^+}) \cong W(k_\lambda)[\![x_1,\dots,x_{w(\lambda)}]\!]$$

with $w(\lambda) = (n^2+1).\#S_\ell - 1$.

Proof. This follows directly from Proposition 3.18. □

Next, we will apply the framework of Section 4 to the attained λ.

Theorem 8.12. *There exists a cofinite subset $\Lambda^3_{\mathcal{E}} \subset \Lambda^2_{\mathcal{E}}$ such that the following holds: Let $\lambda \in \Lambda^3_{\mathcal{E}}$ and $L_{(\lambda)}^+$ the corresponding extension from Corollary 8.11. Then the functors*

$$D_{S_\ell,W(k_\lambda)}^{\square_{S_\ell},\chi,\mathtt{min}}(\bar{r}_\lambda \mid G_{L_{(\lambda)}^+}) \quad \text{and} \quad D_{S_\ell,W(k_\lambda)}^{\square_{S_\ell},\chi}(\bar{r}_\lambda \mid G_{L_{(\lambda)}^+})$$

have vanishing dual Selmer group.

Proof. We start with the min-case. When applying the framework, we take for sm the condition parametrizing arbitrary deformations, for crys the condition parametrizing FL-crystalline deformations (see Section 5C) and for min the condition parametrizing minimally ramified deformations (see Section 5D). Moreover, we take $\chi = \epsilon_\ell^{1-n}\delta_{F\mid F^+}^{n \pmod 2}$. Let us now check the following list of conditions (and we abbreviate $L^+ = L_{(\lambda)}^+$ as we check this for a fixed $\lambda \in \Lambda^2_{\mathcal{E}}$):

(**sm**/k): As we took for $\mathtt{sm}$ the unrestricted deformation condition, we have to check that for each $v \in \Omega_\ell$ the functor $D^{\square,\chi_v}_{W(k_\lambda)}(\bar{r}_{\lambda,v} \mid G_{L_v^+})$ is representable and that the representing object is formally smooth of relative dimension

$$d_v^{\square,\mathtt{sm}} = \dim(\mathfrak{g}_n^{\mathtt{der}})([L_{\tilde{v}} : \mathbb{Q}_\ell] + 1) = n^2([L_{\tilde{v}} : \mathbb{Q}_\ell] + 1) = n^2([L_v^+ : \mathbb{Q}_\ell] + 1).$$

(This also amounts to the vanishing of the error terms δ_v in Theorem 4.2.)

Check: Representability was already remarked in Section 3. For the remaining claim, we first refer to Proposition 6.1 in order to get an isomorphism

$$R^{\square,\chi_v}_{W(k_\lambda)}(\bar{r}_{\lambda,v} \mid G_{L_v^+}) \cong D^{\square,\chi_v}_{W(k_\lambda)}(\bar{\rho}_{\lambda,v} \mid G_{L_{\tilde{v}}}).$$

Now the claim follows from Theorem 5.2.

(**crys**): For each $v \in \Omega_\ell$, the subfunctor

$$D^{\square,\chi_v,\mathtt{crys}}_{W(k_\lambda)}(\bar{r}_{\lambda,v} \mid G_{L_v^+}) \hookrightarrow D^{\square,\chi_v}_{W(k_\lambda)}(\bar{r}_{\lambda,v} \mid G_{L_v^+})$$

is relatively representable and the representing object is formally smooth of relative dimension $d_v^{\square,\mathtt{crys}} = \dim(\mathfrak{g}_n^{\mathtt{der}}) + (\dim(\mathfrak{g}_n^{\mathtt{der}}) - \dim(\mathfrak{b}_n^{\mathtt{der}}))[L_v^+ : \mathbb{Q}_\ell]$, where $\mathfrak{b}_n$ denotes the Lie algebra of a Borel subgroup of $\mathcal{G}_n$.

Check: By definition, we have

$$R^{\square,\chi_v,\mathtt{crys}}_{W(k_\lambda)}(\bar{r}_{\lambda,v} \mid G_{L_v^+}) \cong D^{\square,\chi_v,\mathtt{crys}}_{W(k_\lambda)}(\bar{\rho}_{\lambda,v} \mid G_{L_{\tilde{v}}^+}).$$

Thus, the claim follows from Lemma 5.4.

(**min**): For each $v \in S$, the subfunctor

$$D^{\square,\chi_v,\mathtt{min}}_{W(k_\lambda)}(\bar{r}_{\lambda,v} \mid G_{L_v^+}) \hookrightarrow D^{\square,\chi_v}_{W(k_\lambda)}(\bar{r}_{\lambda,v} \mid G_{L_v^+})$$

is relatively representable and the representing object is formally smooth of relative dimension $d_v^{\square,\mathtt{min}} = \dim(\mathfrak{g}_n^{\mathtt{der}})$.

Check: Again, by definition, we have

$$R^{\square,\chi_v,\mathtt{min}}_{W(k_\lambda)}(\bar{r}_{\lambda,v} \mid G_{L_v^+}) \cong D^{\square,\chi_v,\mathtt{min}}_{W(k_\lambda)}(\bar{\rho}_{\lambda,v} \mid G_{L_{\tilde{v}}^+}).$$

Thus, the claim follows from Proposition 5.6.

(∞): For each $v \in \Omega_\infty$, the local deformation ring $R^{\square,\chi_v}_{W(k_\lambda)}(\bar{r}_{\lambda,v} \mid G_{L_v^+})$ is formally smooth of relative dimension $d_v^{\square} = \dim(\mathfrak{b}_n^{\mathtt{der}})$.

Check: This was already used, see Remark 7.5.

(**Presentability**): Consider the ring

$$R^{\mathtt{loc},\mathtt{min}}(L^+) := \widehat{\bigotimes}_{v \in S_\ell} \tilde{R}_v(L^+)$$

with $\tilde{R}_v(L^+) = D^{\square,\chi_v,\mathtt{min}}_{W(k_\lambda)}(\bar{r}_{\lambda,v} \mid G_{L_v^+}))$ if $v \in S$ and $D^{\square,\chi_v}_{W(k_\lambda)}(\bar{r}_{\lambda,v} \mid G_{L_v^+}))$ otherwise. Then, there exists a presentation

$$R^{\square_{S_\ell},\chi}_{S_\ell,W(k_\ell)}(\bar{r}_\lambda \mid G_{L^+}) \cong R^{\mathtt{loc},\mathtt{min}}(L^+)[\![X_1, \dots, X_a]\!]/(f_1, \dots, f_b)$$

with $a - b = (\#S_\ell - 1).\dim(\mathfrak{g}_n^{\mathtt{ab}})$.

Check: This is the content of Proposition 3.20, but we have to check Assumption 3.19. As $\mathfrak{g}_n^{\mathtt{der}} = \mathfrak{gl}_n$, this condition holds by Corollary 3.25 for almost all λ.

($\boldsymbol{R = T}$): The ring $R^{\square_{S_\ell},\chi,\mathtt{min},\mathtt{crys}}_{S_\ell,W(k_\ell)}(\bar r_\lambda \mid G_{L^+})$ is formally smooth of relative dimension $r_0 = \dim(\mathfrak{g}).\#S_\ell - \dim(\mathfrak{g}^{\mathrm{ab}})$.

Check: This follows from Corollary 8.11.

We see that the general requirements of Theorem 4.2 are met, so let us check the additional requirements of part 2 of Theorem 4.2:

- The condition $\ell \gg 0$ can be achieved by leaving out finitely many λ.
- The vanishing of $H^0(\mathrm{Gal}_{L^+}, \mathfrak{g}_n^{\mathtt{der},\vee})$ can be checked by observing
$$H^0(\mathrm{Gal}_{L^+}, \mathfrak{g}_n^{\mathtt{der},\vee}) \subset H^0(\mathrm{Gal}_L, \mathfrak{g}_n^{\mathtt{der},\vee}) \cong H^0(\mathrm{Gal}_L, \mathfrak{gl}_n^{\vee}),$$
as the adjoint representation of Gal_L on $\mathfrak{g}_n^{\mathtt{der}}$ (via $\bar r_\lambda$) corresponds to the adjoint representation of Gal_L on $\mathfrak{gl}_n$ (via $\bar\rho_\lambda$); see [Clozel et al. 2008, Section 2.1]. Thus, the desired vanishing follows for almost all λ by Corollary 3.25.
- For $v \in S$, $\dim(L_{\lambda,v}) = h^0(\mathrm{Gal}_{L_v^+}, \mathfrak{g}_n^{\mathtt{der}})$: As v is split, Proposition 6.1 yields $h^0(\mathrm{Gal}_{L_v^+}, \mathfrak{g}_n^{\mathtt{der}}) = h^0(\mathrm{Gal}_{L_{\tilde v}}, \mathfrak{gl}_n^{\mathtt{der}})$, where the action on $\mathfrak{gl}_n$ is via $\bar\rho_{\lambda,\tilde v}$. The claim thus follows from [Clozel et al. 2008, Corollary 2.4.21].

The finitely many exclusions which occurred in the above items are now the places we must exclude from $\Lambda^2_{\mathcal{E}}$ to get $\Lambda^3_{\mathcal{E}}$. This finishes the first part, i.e., that $D^{\square_{S_\ell},\chi,\mathtt{min}}_{S_\ell,W(k_\lambda)}(\bar r_\lambda \mid G_{L^+_{(\lambda)}})$ has vanishing dual Selmer group. Concerning the second statement (i.e., the claimed vanishing of the non-minimal dual Selmer group) we first note that on each level $L^+_{(\lambda)}$ we can apply the $R = R^{\mathtt{min}}$-result of Corollary 7.11, yielding the desired vanishing except for a finite failure set. In other words, fix a place λ', then we have
$$D^{\square_{S_\ell},\chi,\mathtt{min}}_{S_\ell,W(k_\lambda)}(\bar r_\lambda \mid G_{L^+_{(\lambda)}}) = D^{\square_{S_\ell},\chi}_{S_\ell,W(k_\lambda)}(\bar r_\lambda \mid G_{L^+_{(\lambda)}})$$
for all λ with $L^+_{(\lambda)} = L^+_{(\lambda')}$, except for a finite failure set $\mathfrak{F}_{\lambda'}$. We should check that the occurrence of these failure sets at each step in the tower of field extensions does not disturb the desired result. For this, recall that the $L^+_{(\lambda)}$ show up in the tower $F^+ = L^+_0 \subset L^+_1 \subset \cdots$ and that, by the first statement, we have
$$\lim_{i\to\infty} \delta\{\lambda \mid D^{\square_{S_\ell},\chi,\mathtt{min}}_{S_\ell,W(k_\lambda)}(\bar r_\lambda \mid G_{L^+_{(\lambda)}}) \text{ has vanishing dual Selmer group}, L^+_{(\lambda)} \subset L^+_i\} = 1.$$
But this clearly implies
$$\lim_{i\to\infty} \delta\{\lambda \mid D^{\square_{S_\ell},\chi}_{S_\ell,W(k_\lambda)}(\bar r_\lambda \mid G_{L^+_{(\lambda)}}) \text{ has vanishing dual Selmer group}, L^+_{(\lambda)} \subset L^+_i, \lambda \notin \mathfrak{F}_\lambda\} = 1,$$
completing the proof. □

Proof of Theorem 8.5. The " has vanishing dual Selmer group" part of Theorem 8.5 follows immediately from Theorem 8.12 and the potential unobstructedness result of Lemma 4.8, as each $[L^+_{(\lambda)} : F^+]$ is a power of 2 (and k_λ has odd characteristic for $\lambda \in \Lambda^2_{\mathcal{E}}$). It remains to show that for all $v \in \Omega_\ell^{F^+}$ the local lifting ring $R^{\square,\chi_\nu}_{W(k_\lambda)}(\bar{r}_{\lambda,\nu})$ is relatively smooth. By Theorem 5.2 we know that

$$\lim_{i\to\infty}\delta\{\lambda \mid R^{\square,\chi_\nu}_{W(k_\lambda)}(\bar{r}_\lambda \mid \mathrm{Gal}_{L^+_{(\lambda)},\nu'}) \text{ is unobstructed for all } \nu' \in \Omega^{L^+_{(\lambda)}}_{\ell(\lambda)},\ L^+_{(\lambda)} \subset L^+_i\}$$
$$= \lim_{i\to\infty}\delta\{\lambda \mid \text{any } \nu' \in \Omega^{L^+_{(\lambda)}}_{\ell(\lambda)} \text{ is split in the extension } L_{(\lambda)} \mid L^+_{(\lambda)},\ L^+_{(\lambda)} \subset L^+_i\} = 1.$$

Using Proposition 4.5 and Corollary 4.4, the claim follows. □

Appendix: A lemma on prime densities in non-Galois extensions

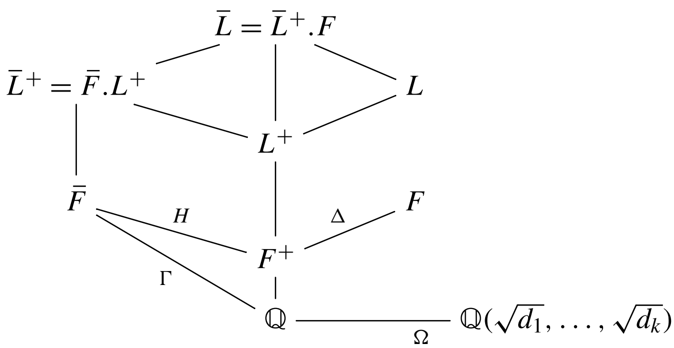
Let us consider a CM-field F with totally real subfield F^+ and we denote by $L^+ = F^+(\sqrt{d_1}, \dots, \sqrt{d_k})$ the totally real extension of F^+ of degree 2^k, obtained by adjoining the square roots of some choice of elements $d_1, \dots, d_k \in \mathbb{N}$ such that each d_i is a nonsquare in the Galois closure $\bar{F}^+$ of F^+. Set $L = L^+.F$. Then we have:

Lemma A. *Let $\Xi_{\mathbb{Q}}$ be the set of all those rational primes ℓ with the following property: For any place $\wp$ of L^+,*

$$[\wp \text{ divides } \ell] \Rightarrow [\wp \text{ splits in } L \mid L^+].$$

Then the density $\delta(\Xi_{\mathbb{Q}})$ of $\Xi_{\mathbb{Q}}$ in the set of all rational primes is at least $1 - 1/2^k$.

Proof. Consider the following diagram of fields:



with corresponding Galois groups $\Delta = \mathbb{Z}/2\mathbb{Z}$, $\Omega = (\mathbb{Z}/2\mathbb{Z})^k$ and Γ, H (for which we don't make an assumption). By our initial assumption that the d_i are not squares we have

$$\mathrm{Gal}(\bar{L}^+ \mid \mathbb{Q}) \cong \Gamma \times \Omega \quad \text{and, hence,} \quad \mathrm{Gal}(\bar{L} \mid \mathbb{Q}) \cong \Gamma \times \Omega \times \Delta.$$

Now, let $\mathfrak{P}$ be a place of $\bar{L}$ with corresponding Frobenius element $(\gamma, \omega, \delta) \in \mathrm{Gal}(\bar{L} \mid \mathbb{Q})$. As Ω and Δ are abelian, the conjugacy class of $\mathfrak{P}$ can be written as $\{(u\gamma u^{-1}, \omega, \delta) \mid u \in \Gamma\}$ and consists precisely of the Frobenii of the places of L lying over the same rational prime p as $\mathfrak{P}$. Let $\wp$ be the place of L below $\mathfrak{P}$. Its Frobenius element is given by

$$(\gamma, \omega, \delta)^{e_{\gamma,\omega,\delta}} \in H \times \{1\} \times \Delta = \mathrm{Gal}(\bar{L} \mid L^+)$$

for $e_{\gamma,\omega,\delta}$ minimal such that $(\gamma, \omega, \delta)^{e_{\gamma,\omega,\delta}} \in H \times \{1\} \times \Delta$. The condition that $\wp$ splits in $L \mid L^+$ then amounts precisely to $(\gamma, \omega, \delta)^{e_{\gamma,\omega,\delta}} \in H \times \{1\} \times \{1\}$, or, written in a more sophisticated way, that $q((\gamma, \omega, \delta)^{e_{\gamma,\omega,\delta}}) = 1$, where

$$q : \mathrm{Gal}(\bar{L} \mid L^+) \to \mathrm{Gal}(\bar{L} \mid L^+) / \mathrm{Gal}(\bar{L} \mid \bar{L}^+)$$

is the quotient map. If $\omega \neq 1$, we clearly must have $2 \mid e_{\gamma,\omega,\delta}$, which implies that $\wp$ splits in $L \mid L^+$. It is also important to note that the condition $\omega \neq 1$ is kept intact by conjugation inside $\mathrm{Gal}(\bar{L} \mid \mathbb{Q})$. Now, set

$$\Xi^* = \{(\gamma, \omega, \delta) \in \mathrm{Gal}(\bar{L} \mid \mathbb{Q}) \mid q((\gamma, \omega, \delta)^{e_{\gamma,\omega,\delta}}) = 1\}$$

and consider the subset $\Xi \subset \Xi^*$ which consists of those $g \in \Xi^*$ for which the complete conjugacy class is contained in Ξ^*, i.e., $\Xi = \{g \in \Xi^* \mid \langle g \rangle \subset \Xi^*\}$. We can give another characterization of this set, Ξ is the union of all conjugacy classes $\langle g \rangle \subset \mathrm{Gal}(\bar{L} \mid \mathbb{Q})$ with the following property: If $\boldsymbol{P}_g$ denotes the set of all places $\mathfrak{P}$ of $\bar{L}$ such that $\mathrm{Frob}_{\mathfrak{P}} \in \langle g \rangle$, then for any place $\wp$ of L^+ the following hold:

$$[\exists \mathfrak{P} \in \mathbf{P}_g \text{ such that } \mathfrak{P} \text{ divides } \wp] \Rightarrow [\wp \text{ splits in } L \mid L^+].$$

Then we have

$$\#\Xi \geq \#\{(\gamma, \omega, \delta) \in \mathrm{Gal}(\bar{L} \mid \mathbb{Q}) \mid \omega \neq 1\} = (2^k - 1).2.\#\Gamma.$$

As $\Xi_{\mathbb{Q}} = \{\ell \in \mathrm{Pl}_{\mathbb{Q}} \mid \exists g \in \Xi \text{ such that } \mathfrak{P} \mid \ell \text{ for all } \mathfrak{P} \in \mathbf{P}_g\}$, it follows from Chebotarev's density theorem that

$$\delta(\Xi_{\mathbb{Q}}) \geq \frac{(2^k - 1).2.\#\Gamma}{\mathrm{Gal}(\bar{L} \mid \mathbb{Q})} = 1 - \frac{1}{2^k}. \qquad \square$$

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