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We determine the set of geometric endomorphism algebras of geometrically split abelian surfaces defined over \mathbb{Q} . In particular we find that this set has cardinality 92. The essential part of the classification consists in determining the set of quadratic imaginary fields M with class group $C_2 \times C_2$ for which there exists an abelian surface A defined over \mathbb{Q} which is geometrically isogenous to the square of an elliptic curve with CM by M . We first study the interplay between the field of definition of the geometric endomorphisms of A and the field M . This reduces the problem to the situation in which E is a \mathbb{Q} -curve in the sense of Gross. We can then conclude our analysis by employing Nakamura's method to compute the endomorphism algebra of the restriction of scalars of a Gross \mathbb{Q} -curve.

1. Introduction

Let A be an abelian variety of dimension $g \geq 1$ defined over a number field k of degree d . Let us denote by $A_{\overline{\mathbb{Q}}}$ its base change to $\overline{\mathbb{Q}}$. We refer to $\text{End}(A_{\overline{\mathbb{Q}}})$, the \mathbb{Q} -algebra spanned by the endomorphisms of A defined over $\overline{\mathbb{Q}}$, as the $\overline{\mathbb{Q}}$ -endomorphism algebra of A . For a fixed choice of g and d , it is conjectured that the set of possibilities for $\text{End}(A_{\overline{\mathbb{Q}}})$ is finite. A slightly stronger form of this conjecture, applying to endomorphism rings of abelian varieties over number fields, has been attributed to Coleman in [Bruin et al. 2006].

Hereafter, let A denote an abelian surface defined over \mathbb{Q} . In the case that A is geometrically simple (that is, $A_{\overline{\mathbb{Q}}}$ is simple), the previous conjecture stands widely open. If A is principally polarized and has CM it has been shown by Murabayashi and Umegaki [2001] that $\text{End}(A_{\overline{\mathbb{Q}}})$ is one of 19 possible quartic CM fields. However, narrowing down to a finite set the possible quadratic real fields and quaternion division algebras over \mathbb{Q} which occur as $\text{End}(A_{\overline{\mathbb{Q}}})$ for some A has escaped all attempts of proof. See also [Orr and Skorobogatov 2018] for recent more general results which prove Coleman's conjecture for CM abelian varieties.

In the present paper, we focus on the case that A is geometrically split, that is, the case in which $A_{\overline{\mathbb{Q}}}$ is isogenous to a product of elliptic curves, which we will assume from now on. Let \mathcal{A} be the set of possibilities for $\text{End}(A_{\overline{\mathbb{Q}}})$, where A is a geometrically split abelian surface over \mathbb{Q} .

Let us briefly recall how scattered results in the literature ensure the finiteness of \mathcal{A} (we will detail the arguments in Section 4). Indeed, if $A_{\overline{\mathbb{Q}}}$ is isogenous to the product of two nonisogenous elliptic curves, then the finiteness (and in fact the precise description) of the set of possibilities for $\text{End}(A_{\overline{\mathbb{Q}}})$ follows

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from [Fité et al. 2012, Proposition 4.5]. If, on the contrary, $A_{\overline{\mathbb{Q}}}$ is isogenous to the square of an elliptic curve, then the finiteness of the set of possibilities for $\text{End}(A_{\overline{\mathbb{Q}}})$ was established by Shafarevich [1996] (see also [González 2011] for the determination of the precise subset corresponding to modular abelian surfaces). In the present work, we aim at an effective version of Shafarevich's result. Our starting point is [Fité and Guitart 2018a, Theorem 1.4], which we recall in our particular setting.

Theorem 1.1 [Fité and Guitart 2018a]. *If A is an abelian surface defined over \mathbb{Q} such that $A_{\overline{\mathbb{Q}}}$ is isogenous to the square of an elliptic curve $E/\overline{\mathbb{Q}}$ with complex multiplication (CM) by a quadratic imaginary field M , then the class group of M is 1, C_2 , or $C_2 \times C_2$.*

It should be noted that several other works can be used to see that, in the situation of the theorem, the exponent of the class group of M divides 2 (see [Schütt 2007; Kani 2011], for example).

While it is an easy observation that an abelian surface A as in the theorem can be found for each quadratic imaginary field M with class group 1 or C_2 (see [Fité and Guitart 2018a, Remark 2.20] and also Section 4), the question whether such an A exists for each of the fields M with class group $C_2 \times C_2$ is far from trivial. The aforementioned results are thus not sufficient for the determination of the set \mathcal{A} . The main contribution of this article is the following theorem.

Theorem 1.2. *Let M be a quadratic imaginary field with class group $C_2 \times C_2$. There exists an abelian surface defined over \mathbb{Q} such that $A_{\overline{\mathbb{Q}}}$ is isogenous to the square of an elliptic curve $E/\overline{\mathbb{Q}}$ with CM by M if and only if the discriminant of M belongs to the set*

$$\begin{aligned} &\{-84, -120, -132, -168, -228, -280, -372, -408, -435, \\ &-483, -520, -532, -595, -627, -708, -795, -1012, -1435\}. \end{aligned} \quad (1-1)$$

The only imaginary quadratic fields with class group $C_2 \times C_2$ whose discriminant does not belong to (1-1) are

$$\mathbb{Q}(\sqrt{-195}), \quad \mathbb{Q}(\sqrt{-312}), \quad \mathbb{Q}(\sqrt{-340}), \quad \mathbb{Q}(\sqrt{-555}), \quad \mathbb{Q}(\sqrt{-715}), \quad \mathbb{Q}(\sqrt{-760}). \quad (1-2)$$

With Theorem 1.2 at hand, the determination of the set \mathcal{A} follows as a mere corollary (see Section 4 for the proof).

Corollary 1.3. *The set \mathcal{A} of $\overline{\mathbb{Q}}$ -endomorphism algebras of geometrically split abelian surfaces over \mathbb{Q} is made of:*

- (i) $\mathbb{Q} \times \mathbb{Q}, \mathbb{Q} \times M, M_1 \times M_2$, where M, M_1 and M_2 are quadratic imaginary fields of class number 1;
- (ii) $M_2(\mathbb{Q}), M_2(M)$, where M is a quadratic imaginary field with class group 1, C_2 , or $C_2 \times C_2$ and distinct from those listed in (1-2).

In particular, the set \mathcal{A} has cardinality 92.

The paper is organized in the following manner. In Section 2 we attach a c -representation ϱ_V of degree 2 to an abelian surface A defined over \mathbb{Q} such that $A_{\overline{\mathbb{Q}}}$ is isogenous to the square of an elliptic curve $E/\overline{\mathbb{Q}}$ with CM by M . It is well known that E is a \mathbb{Q} -curve and that one can associate a 2-cocycle c_E to E .

A c -representation is essentially a representation up to scalar and it is thus a notion closely related to that of projective representation. In the case of the c -representation ϱ_V attached to A , the scalar that measures the failure of ϱ_V to be a proper representation is precisely the 2-cocycle c_E . Choosing the language of c -representations instead of that of projective representations has an unexpected payoff: the tensor product of a c -representation ϱ and its contragredient c -representation ϱ^* is again a proper representation. We show that $\varrho_V \otimes \varrho_V^*$ coincides with the representation of $G_{\mathbb{Q}}$ on the 4-dimensional M -vector space $\text{End}(A_{\overline{\mathbb{Q}}})$. This representation has been studied in detail in [Fité and Sutherland 2014] and the tensor decomposition of $\text{End}(A_{\overline{\mathbb{Q}}})$ is exploited in Theorems 2.20 and 2.27 to obtain obstructions on the existence of A . These obstructions extend to the general case those obtained in [Fité and Guitart 2018a, §3.1, §3.2] under very restrictive hypotheses. The c -representation point of view also allows us to understand in a unified manner what we called *group theoretic* and *cohomological* obstructions in [Fité and Guitart 2018a]. It should be noted that one can define analogues of ϱ_V in other more general situations. For example, a parallel construction in the context of geometrically isotypic abelian varieties potentially of GL_2 -type has been exploited in [Fité and Guitart 2019] to determine a tensor factorization of their Tate modules. This can be used to deduce the validity of the Sato–Tate conjecture for them in certain cases.

In Section 3, we describe a method of Nakamura to compute the endomorphism algebra of the restriction of scalars of certain Gross \mathbb{Q} -curves (see Definition 2.9 below for the precise definition of these curves). Then we apply this method to all Gross \mathbb{Q} -curves with CM by a field M of class group $C_2 \times C_2$. This computation plays a key role in the proof of Theorem 1.2, both in proving the existence of the abelian surfaces for the fields M different from those listed in (1-2), and in proving the nonexistence for the fields of (1-2).

In Section 4 we culminate the proofs of Theorem 1.2 and Corollary 1.3 by assembling together the obstructions and existence results from Sections 2 and 3. We essentially show that we can use the results of Section 2 to reduce to the case of Gross \mathbb{Q} -curves, and then deal with this case using the results of Section 3.

Notations and terminology. For k a number field, we will work in the category of abelian varieties up to isogeny over k . Note that isogenies become invertible in this category. Given an abelian variety A defined over k , the set of endomorphisms $\text{End}(A)$ of A defined over k is endowed with a \mathbb{Q} -algebra structure. More generally, if B is an abelian variety defined over k , we will denote by $\text{Hom}(A, B)$ the \mathbb{Q} -vector space of homomorphisms from A to B that are defined over k . We note that for us $\text{End}(A)$ and $\text{Hom}(A, B)$ denote what some other authors call $\text{End}^0(A)$ and $\text{Hom}^0(A, B)$. We will write $A \sim B$ to mean that A and B are isogenous over k . If L/k is a field extension, then A_L will denote the base change of A from k to L . In particular, we will write $A_L \sim B_L$ if A and B become isogenous over L , and we will write $\text{Hom}(A_L, B_L)$ to refer to what some authors write as $\text{Hom}_L(A, B)$.

2. c -representations and k -curves

The goal of this section is to obtain obstructions to the existence of abelian surfaces defined over \mathbb{Q} such that $\text{End}(A_{\overline{\mathbb{Q}}}) \simeq M_2(M)$, where M is a quadratic imaginary field. To this purpose, we analyze the interplay between the k -curves and c -representations that arise from them.

2A. c -representations: general definitions. Let V be a vector space of finite dimension over a field k and let G be a finite group. We say that a map

$$\varrho_V : G \rightarrow \text{GL}(V)$$

is a c -representation (of the group G) if $\varrho_V(1) = 1$ and there exists a map

$$c_V : G \times G \rightarrow k^\times$$

such that for every $\sigma, \tau \in G$ one has

$$\varrho_V(\sigma)\varrho_V(\tau) = \varrho_V(\sigma\tau)c_V(\sigma, \tau). \quad (2-1)$$

Remark 2.1. The following properties follow easily from the definition:

(i) We have

$$\varrho_V(\sigma^{-1}) = \varrho_V(\sigma)^{-1}c_V(\sigma^{-1}, \sigma) \quad \text{and} \quad \varrho_V(\sigma^{-1}) = \varrho_V(\sigma)^{-1}c_V(\sigma, \sigma^{-1}).$$

In particular, $c_V(\sigma, \sigma^{-1}) = c_V(\sigma^{-1}, \sigma)$.

(ii) If $c_V(\cdot, \cdot) = 1$, the notion of c -representation corresponds to the usual notion of representation.

Let V and W be c -representations of the group G . Let $T = \text{Hom}(V, W)$ denote the space of k -linear maps from V to W . A homomorphism of c -representations from V to W is a k -linear map $f \in T$ such that

$$f(v) = \varrho_W(\sigma)(f(\varrho_V(\sigma)^{-1}v))$$

for every $v \in V$ and $\sigma \in G$.

Consider now the map

$$\varrho_T : G \rightarrow \text{GL}(\text{Hom}(V, W)),$$

defined by

$$(\varrho_T(\sigma)f)(v) = \varrho_W(\sigma)(f(\varrho_V(\sigma)^{-1}v)).$$

Proposition 2.2. *The map ϱ_T together with the map $c_T : G \times G \rightarrow k^\times$ defined by $c_T = c_V^{-1} \cdot c_W$ equip T with the structure of a c -representation.*

Before proving the proposition we show a particular case. In the case that W is k equipped with the trivial action of G , let us write $V^* = T$ and $\varrho^* = \varrho_T$. In this case, $\varrho^*(\sigma)$ is the inverse transpose of $\varrho_V(\sigma)$. The assertion of the proposition is then immediate from (2-1).

The following two lemmas, whose proof is straightforward, imply the proposition.

Lemma 2.3. *The maps*

$$\varrho_\otimes : G \rightarrow \text{GL}(V \otimes W),$$

defined by $\varrho_\otimes(\sigma)(v \otimes w) = \varrho_V(\sigma)(v) \otimes \varrho_W(\sigma)(w)$ and $c_\otimes = c_V \cdot c_W$ endow $V \otimes W$ with a structure of c -representation.

Lemma 2.4. *The map*

$$\phi : W \otimes V^* \rightarrow T$$

defined by $\phi(w \otimes f)(v) = f(v)w$ is an isomorphism of c -representations.

Corollary 2.5. *When $V = W$, the c -representation T is in fact a representation.*

2B. k -curves: general definitions. We briefly recall some definitions and results regarding \mathbb{Q} -curves and, more generally, k -curves with complex multiplication. More details can be found in [Fité and Guitart 2018a, §2.1] and the references therein (especially [Quer 2000; Ribet 1992; Nakamura 2004]).

Let $E/\overline{\mathbb{Q}}$ be an elliptic curve and let k be a number field, whose absolute Galois group we denote by G_k .

Definition 2.6. We say that E is a k -curve if for every $\sigma \in G_k$ there exists an isogeny $\mu_\sigma : \sigma E \rightarrow E$.

Definition 2.7. We say that E is a Ribet k -curve if E is a k -curve and the isogenies μ_σ can be taken to be compatible with the endomorphisms of E , in the sense that the diagram

$$\begin{array}{ccc}
 \sigma E & \xrightarrow{\mu_\sigma} & E \\
 \downarrow \sigma\varphi & & \downarrow \varphi \\
 \sigma E & \xrightarrow{\mu_\sigma} & E
 \end{array} \tag{2-2}$$

commutes for all $\sigma \in G_k$ and all $\varphi \in \text{End}(E)$.

Remark 2.8. (i) Observe that if E does not have CM, then E is a k -curve if and only if it is a Ribet k -curve. If E has CM (say by a quadratic imaginary field M), it is well known that E is isogenous to all of its Galois conjugates and hence it is always a k -curve; it is a Ribet k -curve if and only if $M \subseteq k$; see [Silverman 1994, Theorem 2.2].

(ii) We warn the reader that in the present paper we are using a slightly different terminology from that of [Fité and Guitart 2018a]: as in [Fité and Guitart 2018a] the only relevant notion was that of a Ribet k -curve, we called Ribet k -curves simply k -curves.

Let K be a number field containing k . We say that an elliptic curve E/K is a k -curve defined over K (resp. a Ribet k -curve defined over K) if $E_{\overline{\mathbb{Q}}}$ is a k -curve (resp. a Ribet k -curve). We will say that E is completely defined over K if, in addition, all the isogenies $\mu_\sigma : \sigma E \rightarrow E$ can be taken to be defined over K .

Definition 2.9. Let H denote the Hilbert class field of M and let E/H be an elliptic curve with CM by M . We say that E is a Gross \mathbb{Q} -curve if E is completely defined over H .

The next proposition characterizes the existence of Gross \mathbb{Q} -curves and Ribet M -curves with CM by M defined over the Hilbert class field H .

Proposition 2.10. *Let M be a quadratic imaginary field and let D denote its discriminant. Then:*

- (i) *There exists a Ribet M -curve E^* with CM by M and completely defined over H .*
- (ii) *There exists a Gross \mathbb{Q} -curve E^* with CM by M (and completely defined over H) if and only if D is not of the form*

$$D = -4p_1 \dots p_{t-1}, \tag{2-3}$$

where $t \geq 2$ and p_1, \dots, p_{t-1} are primes congruent to 1 modulo 4.

The first part of the previous proposition is a weaker form of [Shimura 1971, Proposition 5, p. 521] (see also [Nakamura 2001, Remark 1]). For the second part, we refer to [Gross 1980, §11; Nakamura 2004, Proposition 5]. Discriminants of the form (2-3) are called *exceptional*.

Suppose from now on that E is a k -curve defined over K with CM by an imaginary quadratic field M . Fix a system of isogenies $\{\mu_\sigma : {}^\sigma E \rightarrow E\}_{\sigma \in G_k}$. By enlarging K if necessary, we can always assume that K/k is Galois and that E is completely defined over K . We will equip $\text{End}(E)$ with the following action. For $\sigma \in \text{Gal}(K/k)$ and $\varphi \in \text{End}(E)$ define

$$\sigma \star \varphi = \mu_\sigma \circ {}^\sigma \varphi \circ \mu_\sigma^{-1}.$$

Note that if E is a Ribet k -curve, then this action is trivial. If we regard M as a $\text{Gal}(K/k)$ -module by means of the natural Galois action (which is actually the trivial action when k contains M) and $\text{End}(E)$ endowed with the action defined above, then the identification of $\text{End}(E)$ with M becomes a $\text{Gal}(K/k)$ -equivariant isomorphism. The map

$$c_E^K : \text{Gal}(K/k) \times \text{Gal}(K/k) \rightarrow M^\times, \quad (\sigma, \tau) \mapsto \mu_{\sigma\tau} \circ {}^\sigma \mu_\tau^{-1} \circ \mu_\sigma^{-1}$$

satisfies the condition

$$(\varrho \star c_E^K(\sigma, \tau)) \cdot c_E^K(\varrho\sigma, \tau)^{-1} \cdot c_E^K(\varrho, \sigma\tau) \cdot c_E^K(\varrho, \sigma)^{-1} = 1, \tag{2-4}$$

for $\varrho, \sigma, \tau \in \text{Gal}(K/k)$, and is then a 2-cocycle.¹ Denote the cohomology class in $H^2(\text{Gal}(K/k), M^\times)$ corresponding to c_E^K by γ_E^K . The class γ_E^K only depends on the K -isogeny class of E .

The next result is a consequence of Weil’s descent criterion, extended to varieties up to isogeny by Ribet [1992, §8].

Theorem 2.11 (Ribet–Weil). *Suppose that E is a Ribet k -curve completely defined over K (and hence $M \subseteq k$). Let L be a number field with $k \subseteq L \subseteq K$, and consider the restriction map*

$$\text{res} : H^2(\text{Gal}(K/k), M^\times) \rightarrow H^2(\text{Gal}(K/L), M^\times).$$

If $\text{res}(\gamma_E^K) = 1$, there exists an elliptic curve C/L such that $E \sim C_K$.

2C. M -curves from squares of CM elliptic curves. Let M be a quadratic imaginary field. Let A be an abelian surface defined over \mathbb{Q} such that $A_{\overline{\mathbb{Q}}}$ is isogenous to E^2 , where E is an elliptic curve defined over $\overline{\mathbb{Q}}$ with CM by M . Let K/\mathbb{Q} denote the minimal extension over which

$$\text{End}(A_{\overline{\mathbb{Q}}}) \simeq \text{End}(A_K).$$

By the theory of complex multiplication, K contains the Hilbert class field H of M . Note also that K/\mathbb{Q} is Galois and the possibilities for $\text{Gal}(K/\mathbb{Q})$ can be read from [Fité et al. 2012, Table 8]. For our purposes,

¹Actually, this is the inverse of the cocycle considered in [Fité and Guitart 2018a], but this does not affect any of the results that we will use.

it is enough to recall that

$$\text{Gal}(K/M) \simeq \begin{cases} C_r & \text{for } r \in \{1, 2, 3, 4, 6\}, \\ D_r & \text{for } r \in \{2, 3, 4, 6\}, \\ A_4, S_4. & \end{cases} \tag{2-5}$$

Here, C_r denotes the cyclic group of r elements, D_r denotes the dihedral group of $2r$ elements, and A_4 (resp. S_4) stands for the alternating (resp. symmetric) group on 4 letters.

We can (and do) assume that E is in fact defined over K , and then we have that $A_K \sim E^2$. For $\sigma \in \text{Gal}(K/\mathbb{Q})$ we have that $(\sigma E)^2 \sim \sigma A_K = A_K \sim E^2$. Therefore, Poincaré’s decomposition theorem implies that E is a \mathbb{Q} -curve completely defined over K .

For the purposes of this article, we need to consider the following (slightly more general) situation: Let N/M be a Galois subextension of K/M , and let E^* be a Ribet M -curve which is completely defined over N and such that $E_{\overline{\mathbb{Q}}} \sim E_{\overline{\mathbb{Q}}}^*$. Observe that there always exist N and E^* satisfying these conditions, for example by taking $N = K$ and $E^* = E$; but in Sections 2D and 2E we will exploit certain situations where $N \subsetneq K$ and $E^* \neq E$.

Then we can consider two cohomology classes: the class γ_E^K attached to E , and the class $\gamma_{E^*}^N$ attached to E^* . We recall the following key result about γ_E^K , which is a particular case of [Fité and Guitart 2018a, Corollary 2.4].

Theorem 2.12. *The cohomology class γ_E^K is 2-torsion.*

Denote by U the set of roots of unity of M and put $P = M^\times/U$. The same argument of [Fité and Guitart 2018a, Proof of Theorem 2.14] proves the following decomposition of the 2-torsion of $H^2(\text{Gal}(K/M), M^\times)$:

$$H^2(\text{Gal}(K/M), M^\times)[2] \simeq H^2(\text{Gal}(K/M), U)[2] \times \text{Hom}(\text{Gal}(K/M), P/P^2). \tag{2-6}$$

If $M \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$ this particularizes to

$$H^2(\text{Gal}(K/M), M^\times)[2] \simeq H^2(\text{Gal}(K/M), \{\pm 1\}) \times \text{Hom}(\text{Gal}(K/M), P/P^2). \tag{2-7}$$

For $\gamma \in H^2(\text{Gal}(K/M), M^\times)[2]$ we will denote by $(\gamma_\pm, \bar{\gamma})$ its components under the isomorphism (2-7); we will refer to γ_\pm as the sign component and to $\bar{\gamma}$ as the degree component.

In order to study the relation between γ_E^K and $\gamma_{E^*}^N$, define L/K to be the smallest extension such that E_L^* and E_L are isogenous. Since all the endomorphisms of E are defined over K , this is also the smallest extension L/K such that $\text{Hom}(E_L^*, E_L) = \text{Hom}(E_{\overline{\mathbb{Q}}}^*, E_{\overline{\mathbb{Q}}})$. The extension L/\mathbb{Q} is Galois. Indeed, for $\sigma \in G_{\mathbb{Q}}$ put $L' = \sigma L$ and let $\beta_\sigma : \sigma E^* \rightarrow E^*$ and $\mu_\sigma : \sigma E \rightarrow E$ be isogenies defined over N and over K respectively; then, if $\phi : E_L^* \rightarrow E_L$ is an isogeny defined over L we find that $\mu_\sigma \circ \sigma \phi \circ \beta_\sigma^{-1}$ is an isogeny defined over L' between $E_{L'}^*$ and $E_{L'}$, so that $L \subseteq L'$ and therefore $L = L'$.

One can also characterize L/K as the minimal extension such that

$$\text{Hom}(E_{\overline{\mathbb{Q}}}^*, A_{\overline{\mathbb{Q}}}) \simeq \text{Hom}(E_L^*, A_L).$$

Denote by

$$\text{inf}_N^K : H^2(\text{Gal}(N/M), M^\times) \rightarrow H^2(\text{Gal}(K/M), M^\times)$$

the inflation map in Galois cohomology.

Lemma 2.13. *Suppose that $M \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$. Then*

$$\text{inf}_N^K(\gamma_{E^*}^N) = w \cdot \gamma_E^K,$$

for some $w \in H^2(\text{Gal}(K/M), \{\pm 1\})$.

Proof. Since $E_L \sim (E_*)_L$ we have that

$$\text{inf}_N^L(\gamma_{E^*}^N) = \text{inf}_K^L(\gamma_E^K). \tag{2-8}$$

Now consider the following piece of the inflation–restriction exact sequence

$$H^1(\text{Gal}(L/K), M^\times) \xrightarrow{t} H^2(\text{Gal}(K/M), M^\times) \xrightarrow{\text{inf}_K^L} H^2(\text{Gal}(L/M), M^\times). \tag{2-9}$$

Equality (2-8) implies that $\text{inf}_N^K(\gamma_{E^*}^N)$ and γ_E^K have the same image under the inflation map inf_K^L , and thus

$$\text{inf}_N^K(\gamma_{E^*}^N) = t(v) \cdot \gamma_E^K$$

for some $v \in H^1(\text{Gal}(L/K), M^\times)$. If $M \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$ we have that

$$H^1(\text{Gal}(L/K), M^\times) \simeq \text{Hom}(\text{Gal}(L/K), \{\pm 1\})$$

and therefore $t(v)$ belongs to $H^2(\text{Gal}(K/M), \{\pm 1\})$. □

Observe that from Theorem 2.12 one cannot deduce that the class $\gamma_{E^*}^N$ is 2-torsion, since A_N is not isogenous to $(E^*)^2$ in general. By Lemma 2.13, what we do deduce is that $\text{inf}_N^K(\gamma_{E^*}^N)^2 = 1$. Therefore, once again by the inflation–restriction exact sequence

$$H^1(\text{Gal}(K/N), M^\times) \xrightarrow{t} H^2(\text{Gal}(N/M), M^\times) \xrightarrow{\text{inf}_N^K} H^2(\text{Gal}(K/M), M^\times) \tag{2-10}$$

we have that

$$(\gamma_{E^*}^N)^2 = t(\mu) \quad \text{for some } \mu \in H^1(\text{Gal}(K/N), M^\times). \tag{2-11}$$

The following technical lemma will be used in Section 2E below.

Lemma 2.14. *Suppose that N/M is abelian and that $M \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$. Let $c_{E^*}^N$ be a cocycle representing the class $\gamma_{E^*}^N$. Then $c_{E^*}^N(\sigma, \tau) = \pm c_{E^*}^N(\tau, \sigma)$ for all $\sigma, \tau \in \text{Gal}(N/M)$.*

Proof. Since $M \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$ we have that

$$H^1(\text{Gal}(K/N), M^\times) = \text{Hom}(\text{Gal}(K/N), \{\pm 1\}). \tag{2-12}$$

By (2-11) and (2-12) we can suppose that there exists a map $d : \text{Gal}(N/M) \rightarrow M^\times$ such that

$$c_{E^*}^N(\sigma, \tau)^2 = d(\sigma)d(\tau)d(\sigma\tau)^{-1} \cdot t(\mu)(\sigma, \tau),$$

where $t(\mu)(\sigma, \tau) \in \{\pm 1\}$. Therefore

$$c_{E^*}^N(\sigma, \tau)^2 = \pm d(\sigma)d(\tau)d(\sigma\tau)^{-1} = \pm d(\sigma)d(\tau)d(\tau\sigma)^{-1} = \pm c_{E^*}^N(\tau, \sigma)^2.$$

We see that $c_{E^*}^N(\sigma, \tau)/c_{E^*}^N(\tau, \sigma)$ is a root of unity in M , and hence is equal to ± 1 . \square

2D. c -representations from squares of CM elliptic curves. Keep the notations from Section 2C. We will denote by V the M -module $\text{Hom}(E_L^*, A_L)$. Fix a system of isogenies $\{\mu_\sigma : {}^\sigma E^* \rightarrow E^*\}_{\sigma \in \text{Gal}(L/M)}$. We do not have a natural action of $\text{Gal}(L/M)$ on V , but the next lemma says that we can use the chosen system of isogenies to define a c -action on V .

Lemma 2.15. *The map*

$$\varrho_V : \text{Gal}(L/M) \rightarrow \text{GL}(V)$$

defined by

$$\varrho_V(f) = {}^\sigma f \circ \mu_\sigma^{-1} \quad \text{for } \sigma \in \text{Gal}(L/M), f \in V$$

and the 2-cocycle $c_{E^*}^L$ endow the module V with a structure of a c -representation.

Proof. This is tautological:

$$\varrho_V(\sigma)\varrho_V(\tau)(f) = {}^{\sigma\tau} f \circ {}^\sigma \mu_\tau^{-1} \circ \mu_\sigma^{-1} = {}^{\sigma\tau} f \circ \mu_{\sigma\tau}^{-1} \cdot c_{E^*}^L(\sigma, \tau) = \varrho_V(\sigma\tau)(f)c_{E^*}^L(\sigma, \tau). \quad \square$$

Let now R denote the M -module $\text{End}(A_K)$. It is equipped with the natural Galois conjugation action of $\text{Gal}(L/M)$, which factors through $\text{Gal}(K/M)$ and which we sometimes will write as $\varrho_R(\sigma)(\psi) = {}^\sigma \psi$. Let T denote $\text{Hom}(V, V)$, equipped with the c -representation structure given by Lemma 2.15 and Proposition 2.2. Note that by Corollary 2.5, we know that T is actually a $M[\text{Gal}(L/M)]$ -module.

Lemma 2.16. *The map*

$$\Phi : R \rightarrow T \simeq V \otimes V^*, \quad \Phi(\psi)(f) = \psi \circ f, \quad \text{for } f \in V, \psi \in \text{End}(A_K)$$

is an isomorphism of c -representations (and thus of $M[\text{Gal}(L/M)]$ -modules).

Proof. The fact that Φ is a morphism of c -representations is straightforward:

$$\begin{aligned} \varrho_T(\sigma)(\Phi({}^{\sigma^{-1}}\psi))(f) &= \varrho_V(\sigma)(\Phi({}^{\sigma^{-1}}\psi)(\varrho_V(\sigma)^{-1}(f))) \\ &= \varrho_V(\sigma)({}^{\sigma^{-1}}\psi \circ \varrho_V(\sigma^{-1})(f)c_{E^*}^L(\sigma^{-1}, \sigma)^{-1}) \\ &= \psi \circ f \circ {}^\sigma \mu_{\sigma^{-1}}^{-1} \mu_\sigma^{-1} c_{E^*}^L(\sigma^{-1}, \sigma)^{-1} \\ &= \Phi(\psi)(f), \end{aligned}$$

where we have used Remark 2.1 in the second and last equalities. The lemma follows by noting that Φ is clearly injective and that both R and T have dimension 4 over M . \square

We now describe the $M[\text{Gal}(K/M)]$ -module structure of R . It follows from (2-5) that the order r of an element $\sigma \in \text{Gal}(K/M)$ is 1, 2, 3, 4, or 6.

Lemma 2.17. $\text{Tr } \varrho_R(\sigma) = 2 + \zeta_r + \bar{\zeta}_r$, where ζ_r is a primitive r -th root of unity.

Remark 2.18. This lemma is proven in [Fité and Sutherland 2014, Proposition 3.4] under the strong running hypothesis of that paper: in our setting that hypothesis would say that there exists E^* defined over M such that $A_{\overline{\mathbb{Q}}} \sim E_{\overline{\mathbb{Q}}}^{*2}$ (i.e., that N can be taken to be M , in the notation of the previous section).

Proof. We claim that $\text{Tr}(\varrho_R) \in M$ is in fact rational. Let us postpone the proof of this claim until the end of the proof of the lemma. Assuming it, we have that

$$\text{Tr}_{M/\mathbb{Q}}(\text{Tr}(\varrho_R(\sigma))) = 2 \text{Tr}(\varrho_R)(\sigma). \quad (2-13)$$

But if $\varrho_{R_{\mathbb{Q}}}$ is the representation afforded by R regarded as an 8-dimensional module over \mathbb{Q} , we have

$$\text{Tr}_{M/\mathbb{Q}}(\text{Tr}(\varrho_R(\sigma))) = \text{Tr}(\varrho_{R_{\mathbb{Q}}})(\sigma) = 2(2 + \zeta_r + \bar{\zeta}_r), \quad (2-14)$$

where the last equality is [Fité et al. 2012, Proposition 4.9]. The comparison of (2-13) and (2-14) concludes the proof of the lemma.

We turn now to prove the rationality of $\text{Tr} \varrho_R$. We first recall the aforementioned proof (that of [Fité and Sutherland 2014, Proposition 3.4]) which uses the fact that we can choose E^* to be defined over M . In this case, we have that V is an $M[\text{Gal}(L/M)]$ -module, that $\text{Tr}(\varrho_{V^*})$ is a sum of roots of unity so that $\text{Tr}(\varrho_{V^*}) = \overline{\text{Tr}(\varrho_V)}$, and hence that $\text{Tr}(\varrho_R) = \text{Tr}(\varrho_V) \cdot \overline{\text{Tr} \varrho_V}$ belongs to \mathbb{Q} .

For the general case, assume that $\text{Tr} \varrho_R$ does not belong to \mathbb{Q} . Since it is a sum of roots of unity of orders dividing either 4 or 6, then M would be $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, but then we could take a model of E^* defined over M , and by the above paragraph, the trace $\text{Tr} \varrho_R$ would be rational, which is a contradiction. \square

2E. Obstructions. Keep the notations from Sections 2C and 2D. Let S denote the normal subgroup of $\text{Gal}(K/M)$ generated by the square elements. In this section, we make the following hypotheses.

Hypothesis 2.19. (i) *There exists a Ribet M -curve E^* with CM by M completely defined over N , where N/M is the subextension of K/M fixed by S .*

(ii) $M \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$.

Let $\sigma \in \text{Gal}(K/M)$ be an element of order $r \in \{4, 6\}$. Let

$$\bar{\cdot} : \text{Gal}(K/M) \rightarrow \text{Gal}(N/M) \simeq \text{Gal}(K/M)/S \quad (2-15)$$

denote the natural projection map. Note that $\text{Gal}(N/M)$ is a group of exponent dividing 2.

Theorem 2.20. *Under Hypothesis 2.19, we have:*

(i) *If $r = 4$, then $2c_{E^*}^N(\bar{\sigma}, \bar{\sigma})$ belongs to $\pm(M^\times)^2$.*

(ii) *If $r = 6$, then $3c_{E^*}^N(\bar{\sigma}, \bar{\sigma})$ belongs to $\pm(M^\times)^2$.*

Proof. First of all, note that E^* is completely defined over N . Thus we can, and do, assume that $c_{E^*}^L$ is the inflation of $c_{E^*}^N$. Let $s \in \text{Gal}(L/M)$ be a lift of σ . By Hypothesis 2.19(ii), we have that $[L : K] \leq 2$.

Therefore, the order of s divides $2r$. We then have

$$\varrho_V(s)^{2r} = \varrho_V(s^2)^r c_{E^*}^N(\bar{\sigma}, \bar{\sigma})^r = \varrho_V(s^{2r})c_{E^*}^N(\bar{\sigma}, \bar{\sigma})^r = c_{E^*}^N(\bar{\sigma}, \bar{\sigma})^r, \tag{2-16}$$

where we have used that $c_{E^*}^N(\bar{\sigma}^{2e}, \bar{\sigma}^{2e'}) = 1$ for any pair of integers e, e' . Let α and β be the eigenvalues of $\varrho_V(s)$. By (2-16), we have that $\alpha^{2r} = c_{E^*}^N(\bar{\sigma}, \bar{\sigma})^r$, from which we deduce that $\omega_r \alpha^2 = c_{E^*}^N(\bar{\sigma}, \bar{\sigma}) \in M^\times$, where ω_r is a (not necessarily primitive) r -th root of unity.

Since the eigenvalues of $\varrho_{V^*}(s)$ are $1/\alpha$ and $1/\beta$, by Lemmas 2.17 and 2.16 we have that

$$2 + \zeta_r + \bar{\zeta}_r = (\alpha + \beta) \left(\frac{1}{\alpha} + \frac{1}{\beta} \right); \text{ equivalently, } \alpha^2 + \beta^2 = (\zeta_r + \bar{\zeta}_r)\alpha\beta. \tag{2-17}$$

This means that α/β satisfies the r -th cyclotomic polynomial and thus, by reordering α and β if necessary, we have that $\alpha = \beta\zeta_r$.

Combining this with (2-17), we get

$$(2 + \zeta_r + \bar{\zeta}_r)c_{E^*}^N(\bar{\sigma}, \bar{\sigma}) = (2 + \zeta_r + \bar{\zeta}_r)\omega_r\alpha^2 = (2 + \zeta_r + \bar{\zeta}_r)\alpha\beta\omega_r\zeta_r = (\alpha + \beta)^2\omega_r\zeta_r.$$

Since the left-hand side is in M^\times , the fact that $\alpha + \beta \in M^\times$ tells us that $\omega_r\zeta_r \in M^\times$. If $\omega_r\zeta_r$ is not rational, then $M = \mathbb{Q}(\zeta_r)$, which contradicts Hypothesis 2.19(ii). If $\omega_r\zeta_r \in \mathbb{Q}$, since it is a root of unity, it must be equal to ± 1 and thus we get

$$\pm(2 + \zeta_r + \bar{\zeta}_r)c_{E^*}^N(\bar{\sigma}, \bar{\sigma}) = (\alpha + \beta)^2.$$

Therefore, $(2 + \zeta_r + \bar{\zeta}_r)c_{E^*}^N(\bar{\sigma}, \bar{\sigma})$ belongs to $\pm(M^\times)^2$. □

Remark 2.21. It follows from the above proof that if $r = 4$, then any lift $s \in \text{Gal}(L/M)$ of σ has order $2r = 8$. Indeed, if the order of s was r , then arguing as in (2-16), we would obtain $\varrho_V(s)^r = c_{E^*}^N(\bar{\sigma}, \bar{\sigma})^{r/2}$, from which we would infer $\omega_{r/2}\alpha^2 = c_{E^*}^N(\bar{\sigma}, \bar{\sigma})$, for some (not necessarily primitive) $r/2$ -th root of unity. We could then run the same argument as above, but since $\omega_{r/2}\zeta_r$ is never rational, we would deduce now that $M = \mathbb{Q}(i)$. Note that if $r = 6$ it can certainly happen that $\omega_{r/2}\zeta_r \in \mathbb{Q}$.

Until the end of this section, we make the following additional assumption on M .

Hypothesis 2.22. (i) $\text{Gal}(K/M) \simeq D_4$ or D_6 .

(ii) $M \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$.

Hypothesis 2.22(i) implies that N/M is a biquadratic extension. By Proposition 2.10(i), there exists a Ribet M -curve E^* with CM by M completely defined over the Hilbert class field H of M . Using [Fité and Guitart 2018a, Theorem 2.14], it is immediate to see that $H \subseteq N$, so that Hypothesis 2.22 implies Hypothesis 2.19.

The next two propositions describe the structure of the group $\text{Gal}(L/M)$.

Proposition 2.23. *If $\text{Gal}(K/M) \simeq D_4$, then $\text{Gal}(L/M)$ is isomorphic to either the dihedral group D_8 ; the generalized dihedral group QD_8 of order 16; or the generalized quaternion group Q_{16} .²*

²The gap identification numbers of QD_8 and Q_{16} are $\langle 16, 8 \rangle$ and $\langle 16, 9 \rangle$, respectively.

Proof. If $\text{Gal}(K/M) \simeq D_4$, then by Remark 2.21 we have that any element of $\text{Gal}(L/M)$ projecting onto an element of $\text{Gal}(K/M)$ of order 4 must have order 8. The groups of order 16 with a quotient isomorphic to D_4 satisfying the previous property are those in the statement of the proposition. \square

Proposition 2.24. *If $\text{Gal}(K/M) \simeq D_6$, there exists a Ribet M -curve E^* completely defined over N with CM by M such that $E \sim E_K^*$ and hence $L = K$ and $\text{Gal}(L/M) \simeq D_6$.*

Proof. Recall the cohomology class $\gamma_E^K \in H^2(\text{Gal}(K/M), M^\times)[2]$ attached to E and consider the restriction map

$$\text{res} : H^2(\text{Gal}(K/M), M^\times) \rightarrow H^2(\text{Gal}(K/N), M^\times).$$

We will first see that $\gamma = \text{res}\gamma_E^K$ is trivial. Recall the decomposition (2-7) of the 2-torsion cohomology classes into degree and sign components

$$H^2(\text{Gal}(K/N), M^\times)[2] \simeq H^2(\text{Gal}(K/N), \{\pm 1\}) \times \text{Hom}(\text{Gal}(K/N), P/P^2),$$

and the notation γ_\pm (resp. $\bar{\gamma}$) for the sign component (resp. degree component) of γ . Since $\text{Gal}(K/N) \simeq C_3$ is the subgroup of $\text{Gal}(K/M)$ generated by the squares, we have that $\bar{\gamma}$ is trivial. Since

$$H^2(\text{Gal}(K/N), \{\pm 1\}) \simeq H^2(C_3, \{\pm 1\}) = 0,$$

we see that γ_\pm is also trivial. By Theorem 2.11, there exists an elliptic curve E^* defined over N such that $E_K^* \sim E$. To see that E^* is completely defined over N , on the one hand, note that since $M \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$, then E^* and any Galois conjugate ${}^\sigma E^*$ of it are isogenous over a quadratic extension of N . On the other hand, since $E_K^* \sim E$ and E is completely defined over K , we have that the smallest field of definition of $\text{Hom}(E_{\mathbb{Q}}^*, {}^\sigma E_{\mathbb{Q}}^*)$ is contained in K . Since K/N is a cubic extension, we deduce that E^* and ${}^\sigma E^*$ are in fact isogenous over N . \square

Corollary 2.25. *If $\text{Gal}(K/M) \simeq D_r$ for $r = 4$ or 6 , there exists a Ribet M -curve E^* with CM by M completely defined over N for which $\text{Gal}(L/M)$ contains*

- (i) *an element s of order 8 if $r = 4$ and of order 6 if $r = 6$;*
- (ii) *an element t such that $tst^{-1} = t^a$ for $1 \leq a \leq 2r$ such that $a \equiv -1 \pmod{r}$.*

Proof. This is obvious when $\text{Gal}(L/M)$ is dihedral. For the other options allowed by Proposition 2.23, recall that

$$\text{QD}_8 \simeq \langle s, t \mid s^8, t^2, tst^5 \rangle, \quad \text{Q}_{16} \simeq \langle s, t \mid s^8, t^2s^4, tst^{-1}s \rangle. \quad \square$$

Remark 2.26. It is clear from the proof of Proposition 2.24 that, in the case that $N = H$ and H is not exceptional, we can choose E^* in the above corollary to be a Gross \mathbb{Q} -curve.

Until the end of this section, we will assume that E^* is as in the previous corollary. Let s and t be also as in the corollary, and let σ and τ be the images of s and t under the projection map

$$\text{Gal}(L/M) \rightarrow \text{Gal}(K/M).$$

Recall also the projection map $\bar{\cdot} : \text{Gal}(K/M) \rightarrow \text{Gal}(N/M)$ and note that $\bar{\sigma}$ and $\bar{\tau}$ are nontrivial elements of $\text{Gal}(N/M)$.

Theorem 2.27. *Under Hypothesis 2.22, we have $c_{E^*}^N(\bar{\tau}, \bar{\tau}) = \pm 1$.*

Proof. By Lemma 2.14, we have that $c_{E^*}^N(g, g') = \pm c_{E^*}^N(g', g)$ for every $g, g' \in \text{Gal}(N/M)$. Moreover, the 2-cocycle condition (2-4) asserts that

$$c_{E^*}^N(\bar{\tau}, \bar{\tau}) = c_{E^*}^N(\bar{\tau}, \bar{\tau})c_{E^*}^N(\bar{\sigma}, 1) = c_{E^*}^N(\bar{\sigma}\bar{\tau}, \bar{\tau})c_{E^*}^N(\bar{\sigma}, \bar{\tau}).$$

Then, we have

$$\begin{aligned} \varrho_V(t)\varrho_V(s)\varrho_V(t)^{-1} &= \varrho_V(t)\varrho_V(s)\varrho_V(t^{-1})c_{E^*}^N(\bar{\tau}, \bar{\tau}) = \varrho_V(ts)\varrho_V(t^{-1})c_{E^*}^N(\bar{\tau}, \bar{\sigma})c_{E^*}^N(\bar{\tau}, \bar{\tau}) \\ &= \varrho_V(tst^{-1})c_{E^*}^N(\bar{\tau}\bar{\sigma}, \bar{\tau})c_{E^*}^N(\bar{\tau}, \bar{\sigma})c_{E^*}^N(\bar{\tau}, \bar{\tau}) = \pm\varrho_V(s^a)c_{E^*}^N(\bar{\tau}, \bar{\tau})^2. \end{aligned} \tag{2-18}$$

It is easy to observe that

$$\varrho_V(s)^a = \varrho_V(s^a)c_{E^*}^N(\bar{\sigma}, \bar{\sigma})^{(a-1)/2}. \tag{2-19}$$

Letting α and β be the eigenvalues of $\varrho_V(s)$, taking traces of (2-18), and applying (2-19), we obtain

$$(\alpha + \beta) = \pm(\alpha^a + \beta^a)c_{E^*}^N(\bar{\sigma}, \bar{\sigma})^{-(a-1)/2}c_{E^*}^N(\bar{\tau}, \bar{\tau})^2.$$

But as in the proof of Theorem 2.20, we have $\beta = \zeta_r\alpha$ and $c_{E^*}^N(\bar{\sigma}, \bar{\sigma}) = \omega_r\alpha^2$, where ζ_r and ω_r are r -th roots of unity and ζ_r is primitive. This, together with the fact that $a \equiv -1 \pmod{r}$, permits to write the above equation as

$$\pm \frac{1 + \zeta_r}{\omega_r^{-(a-1)/2}(1 + \bar{\zeta}_r)} = c_{E^*}^N(\bar{\tau}, \bar{\tau})^2 \in (M^\times)^2.$$

One easily verifies that $(1 + \zeta_r)/(1 + \bar{\zeta}_r)$ is an r -th root of unity. Therefore, the left-hand side of the above equation is a root of unity in M^\times , and hence it must be ± 1 . \square

3. Restriction of scalars of Gross \mathbb{Q} -curves

For the convenience of the reader, in this section we review some results of [Nakamura 2004] on Gross \mathbb{Q} -curves, to which we refer for more details and proofs.

Let M be an imaginary quadratic field. Throughout this section, we make the following hypothesis.

Hypothesis 3.1. (i) M is nonexceptional.

(ii) M has class group isomorphic to $C_2 \times C_2$.

Remark 3.2. If M has class group isomorphic to $C_2 \times C_2$, then the discriminant D of M belongs to the set

$$\{-84, -120, -132, -168, -195, -228, -280, -312, -340, -372, -408, -435, -483, -520, -532, -555, -595, -627, -708, -715, -760, -795, -1012, -1435\}.$$

This list can be easily obtained from [Watkins 2004], for example. Among them, only -340 is exceptional.

Then, by Proposition 2.10, there exists a Gross \mathbb{Q} -curve E with CM by M , which is thus completely defined over the Hilbert class field H of M . The aim of this section is to describe Nakamura’s method for computing the endomorphism algebra of the restriction of scalars of a Gross \mathbb{Q} -curve, which we will then apply to all Gross \mathbb{Q} -curves attached to M satisfying Hypothesis 3.1. Our account of Nakamura’s method will be only in the particular case where M has class group $C_2 \times C_2$, which is the case of interest to us.

As seen in Section 2B, one can associate a cohomology class $\gamma_E := \gamma_E^H$ in the group $H^2(\text{Gal}(H/\mathbb{Q}), M^\times)$ to E . The set of cohomology classes arising from Gross \mathbb{Q} -curves over H has cardinality 8 (see [Nakamura 2004, Proposition 4]), and we regard the set of Gross \mathbb{Q} -curves over H as partitioned into 8 equivalence classes according to their cohomology class.

Let $\text{Res}_{H/M}(E)$ denote Weil’s restriction of scalars of E . This variety is a priori defined over M , but it can be defined over \mathbb{Q} , in the sense that $\text{Res}_{H/M}(E) \simeq (B_E)_M$ for some variety B_E/\mathbb{Q} . It turns out that the endomorphism algebra $\mathcal{D}_E = \text{End}(B_E)$ only depends on the cohomology class γ_E [Nakamura 2004, Proposition 6]. Nakamura devised a method for computing \mathcal{D}_E in terms of the Hecke character attached to E , which he applied to compute all the endomorphism algebras arising in this way from Gross \mathbb{Q} -curves in the cases where $D = -84$ and $D = -195$. We extend his computation to the remaining 21 nonexceptional discriminants of Remark 3.2.

3A. Hecke characters of Gross \mathbb{Q} -curves. The first step is to compute a set of Hecke characters whose associated elliptic curves represent all the equivalence classes of Gross \mathbb{Q} -curves.

Local characters. We begin by defining certain local characters that will be used to describe the Hecke characters. Let $\mathbb{1}_M$ be the group of ideles of M . If \mathfrak{p} is a prime of M , we denote by $U_{\mathfrak{p}} = \mathcal{O}_{M,\mathfrak{p}}^\times$ the group of local units. Also, for a rational prime p put $U_p = \prod_{\mathfrak{p}|p} U_{\mathfrak{p}}$.

Suppose that p is odd and inert in M . Then define η_p as the unique character $\eta_p : U_p \rightarrow \{\pm 1\}$ such that $\eta_p(-1) = (-1)^{\frac{1}{2}(p-1)}$.

Suppose now that 2 is ramified in M and write $D = 4m$. If m is odd, then

$$U_2/U_2^2 \simeq (\mathbb{Z}/2\mathbb{Z})^3 \simeq \langle \sqrt{m}, 3 - 2\sqrt{m}, 5 \rangle.$$

Define $\eta_{-4} : U_2 \rightarrow \{\pm 1\}$ to be the character with kernel $\langle 3 - 2\sqrt{m}, 5 \rangle$. If m is even then

$$U_2/U_2^2 \simeq (\mathbb{Z}/2\mathbb{Z})^3 \simeq \langle 1 + \sqrt{m}, -1, 5 \rangle.$$

Define η_8 to be the character with kernel $\langle 1 + \sqrt{m}, -1 \rangle$ and η_{-8} the character with kernel $\langle 1 + \sqrt{m}, -5 \rangle$.

Hecke characters. Let $U_M = \prod_{\mathfrak{p}} U_{\mathfrak{p}}$ be the maximal compact subgroup of $\mathbb{1}_M$. Let S be a finite set of primes of M and put $U_S = \prod_{\mathfrak{p} \in S} U_{\mathfrak{p}}$. Suppose that $\eta : U_S \rightarrow \{\pm 1\}$ is a continuous homomorphism such that $\eta(-1) = -1$. Next, we explain how to construct from η a Hecke character $\phi : \mathbb{1}_M \rightarrow \mathbb{C}^\times$ (not uniquely determined) that gives rise, in certain cases, to a Gross \mathbb{Q} -curve.

First of all, extend η to a character that we denote by the same name $\eta : U_M \rightarrow \{\pm 1\}$ by composing with the projection $U_M \rightarrow U_S$. Now this character η can be extended to a character $\tilde{\eta} : U_M M^\times M_\infty^\times \rightarrow \mathbb{C}^\times$ by imposing that

$$\tilde{\eta}(M^\times) = 1, \quad \tilde{\eta}(z) = z^{-1} \quad \text{for } z \in M_\infty^\times. \tag{3-1}$$

Let $\phi : \mathbb{I}_M \rightarrow \mathbb{C}^\times$ be a Hecke character that extends $\tilde{\eta}$ (there are $[H : M] = 4$ such extensions; see [Shimura 1971, p. 523]). For future reference, it will be useful to have the following formula for ϕ evaluated at certain principal ideals.

Lemma 3.3. *Suppose that (α) is a principal ideal of M such that $v_p(\alpha) = 0$ for all $p \in S$, and denote by $\alpha_S \in U_S$ the natural image of α in U_S . Then*

$$\phi((\alpha)) = \eta(\alpha_S)\alpha_\infty, \tag{3-2}$$

where α_∞ denotes the image of α in $M_\infty = \mathbb{C}$.

Proof. If we write $(\alpha) = \prod_{q \in T} q^{v_q(\alpha)}$, where T denotes the support of (α) , then

$$\phi((\alpha)) = \prod_{q \in T} \phi_q(\alpha_q),$$

where ϕ_q denotes the restriction of ϕ to M_q and α_q the image of α in M_q . Observe that by hypothesis $S \cap T = \emptyset$, and that if $q \notin S \cup T$, then $\phi_q(\alpha_q) = 1$, since α_q belongs to U_q and $\phi|_{U_q} = \tilde{\eta}|_{U_q} = 1$. Therefore, we can write

$$\phi((\alpha)) = \prod_{q \in T} \phi_q(\alpha_q) \prod_{q \notin T} \phi_q(\alpha_q) \prod_{q \in S} \phi_q^{-1}(\alpha_q) = \left(\prod_q \phi_q(\alpha_q) \right) \eta(\alpha_S),$$

where we have used that η has order 2. Then, by (3-1) we have that

$$\phi((\alpha)) = \left(\phi_\infty(\alpha_\infty) \prod_q \phi_q(\alpha_q) \right) \phi_\infty(\alpha_\infty)^{-1} \eta(\alpha_S) = \phi(\alpha)\alpha_\infty \eta(\alpha_S) = \alpha_\infty \eta(\alpha_S). \quad \square$$

Define now a Hecke character of H by means of $\psi = \phi \circ N_{H/M}$, where

$$N_{H/M} : \mathbb{I}_H \rightarrow \mathbb{I}_M$$

denotes the norm on ideles. By a result of Shimura [1971, Proposition 9], the Hecke character ψ is attached to a Gross \mathbb{Q} -curve if and only if $\bar{\phi} = \phi$, where the bar denotes the action of complex conjugation.

For example, if D has some prime factor $q \equiv 3 \pmod{4}$, put $\eta_0 = \eta_q$. If all the odd primes dividing D are congruent to 1 modulo 4, then $D = 8m$ for some odd m and we define η_0 to be η_{-8} . If we denote by $\phi_0 : \mathbb{I}_M \rightarrow \mathbb{C}^\times$ a Hecke character attached to η_0 by the above construction, then the Hecke character $\psi_0 = \phi_0 \circ N_{H/M}$ is the Hecke character attached to a Gross \mathbb{Q} -curve over H .

Let W be the set of characters $\theta : U_M \rightarrow \{\pm 1\}$ such that $\theta(-1) = 1$ and $\bar{\theta} = \theta$. Denote also by W_0 the set of $\theta \in W$ such that $\theta = \kappa \circ N_{M/\mathbb{Q}}$ for some Dirichlet character κ . By [Nakamura 2004, Proposition 3], the group W/W_0 is generated by two characters that can be described explicitly in terms of the characters $\eta_p, \eta_{-4}, \eta_{-8}$, and η_8 . More precisely:

- (1) If $D = -pqr$ with p, q , and r primes congruent to 3 modulo 4, then $W/W_0 = \langle \eta_p \eta_q, \eta_p \eta_r \rangle$.
- (2) If $D = -pqr$ with p and q primes congruent to 1 modulo 4, and r congruent to 3 modulo 4, then $W/W_0 = \langle \eta_p, \eta_q \rangle$.

- (3) If $D = -4pq$ with p and q congruent to 3 modulo 4, then $W/W_0 = \langle \eta_{-4}, \eta_p \eta_q \rangle$.
- (4) If $D = -8pq$ with p and q congruent to 3 modulo 4, then $W/W_0 = \langle \eta_{-8} \eta_p, \eta_{-8} \eta_q \rangle$.
- (5) If $D = -8pq$ with p congruent to 1 modulo 4 and q congruent to 3 modulo 4, then $W/W_0 = \langle \eta_8, \eta_p \rangle$.
- (6) If $D = -8pq$ with p and q congruent to 1 modulo 4, then $W/W_0 = \langle \eta_p, \eta_q \rangle$.

Denote by $\tilde{\omega}_1, \tilde{\omega}_2$ the generators of W/W_0 , and define $\omega_i = \tilde{\omega}_i \circ N_{H/M}$.

Now let k/H be a quadratic extension such that k/\mathbb{Q} is Galois and k/M is nonabelian. Such quadratic extensions exist by [Nakamura 2004, Theorem 1]. Denote by $\chi : \mathbb{I}_H \rightarrow \{\pm 1\}$ the Hecke character attached to k/H .

By [Nakamura 2004, Theorem 2], the eight equivalence classes of \mathbb{Q} -curves over H are represented by the Hecke characters $\psi_0 \cdot \omega$ with $\omega \in \langle \omega_1, \omega_2, \chi \rangle$. Observe that, in particular, this set of Hecke characters does not depend on the choice of k (any k which is Galois over \mathbb{Q} and nonabelian over M will produce the same set of Hecke characters).

3B. Method for computing the endomorphism algebra. Let \mathfrak{p}_1 and \mathfrak{p}_2 be prime ideals of M that generate the class group and that are coprime to the conductors of $\psi_0, \omega_1, \omega_2$, and χ . Let L_i be the decomposition field of \mathfrak{p}_i in H , and F_i the maximal totally real subfield of L_i .

Suppose that E is a Gross \mathbb{Q} -curve over H with Hecke character of the form $\psi = \psi_0 \omega_1^a \omega_2^b$ for some $a, b \in \{0, 1\}$. We can write $\psi = \phi \circ N_{H/M}$, where $\phi = \phi_0 \tilde{\omega}_1^a \tilde{\omega}_2^b$. Then $\phi(\mathfrak{p}_i) + \phi(\bar{\mathfrak{p}}_i)$ generates a quadratic number field $\mathbb{Q}(\sqrt{n_i})$, and the endomorphism algebra $\mathcal{D}_E = \text{End}(B_E)$ is isomorphic to the biquadratic field $\mathbb{Q}(\sqrt{n_1}, \sqrt{n_2})$; see [Nakamura 2004, Proposition 7, Theorem 3].

Remark 3.4. Observe that $\phi(\mathfrak{p}_i) + \phi(\bar{\mathfrak{p}}_i)$ can be computed if one knows the two quantities $\phi(\mathfrak{p}_i^2)$ and $\phi(\mathfrak{p}_i \bar{\mathfrak{p}}_i)$. Since \mathfrak{p}_i^2 and $\mathfrak{p}_i \bar{\mathfrak{p}}_i$ are principal, one can compute $\phi(\mathfrak{p}_i^2)$ and $\phi(\mathfrak{p}_i \bar{\mathfrak{p}}_i)$ by means of (3-2).

Suppose now that the Hecke character of E is of the form $\psi = \psi_0 \chi \omega_1^a \omega_2^b$. Then \mathcal{D}_E is a quaternion algebra over \mathbb{Q} , say

$$\mathcal{D}_E \simeq \left(\frac{t_1, t_2}{\mathbb{Q}} \right).$$

The t_i can be computed as follows; see [Nakamura 2004, Proposition 7]. First of all, let n_1 and n_2 be the rational numbers defined as in the previous paragraph for the character $\psi/\chi = \psi_0 \omega_1^a \omega_2^b$.

(1) Suppose that $\text{Gal}(k/L_i) \simeq C_2 \times C_2$. Then:

- (a) If k/F_i is abelian then $t_i = n_i$.
- (a) If k/F_i is nonabelian, then $t_i = D/n_i$.

(2) Suppose that $\text{Gal}(k/L_i) \simeq C_4$. Then:

- (a) If k/F_i is abelian, then $t_i = -n_i$.
- (b) If k/F_i is nonabelian, then $t_i = -D/n_i$.

3C. Computations and tables. For each of the 23 nonexceptional imaginary quadratic fields of class group $C_2 \times C_2$, we have computed the 8 endomorphism algebras arising from restriction of scalars of Gross \mathbb{Q} -curves. The results are displayed in Table 1. The notation is as follows: for the biquadratic fields, the notation (a, b) indicates the field $\mathbb{Q}(\sqrt{a}, \sqrt{b})$; for the quaternion algebras, we write the discriminant of the algebra.

For a Gross \mathbb{Q} -curve E , recall that B_E denotes the abelian variety over \mathbb{Q} such that $\text{Res}_{H/M} E \sim (B_E)_M$. Since B_E is isogenous to its quadratic twist over M , this implies that

$$\text{Res}_{H/\mathbb{Q}} E \sim (B_E)^2.$$

We observe in Table 1 that for all discriminants except -195 , -312 , -555 , -715 , and -760 , at least one of the quaternion algebras is the split algebra $M_2(\mathbb{Q})$ of discriminant 1. This implies that for the corresponding Gross \mathbb{Q} -curve E the variety B_E decomposes as

$$B_E \sim A^2,$$

with A/\mathbb{Q} an abelian surface. Therefore, $\text{Res}_{H/\mathbb{Q}} E$ decomposes as the fourth power of an abelian surface.

On the other hand, for the discriminants -195 , -312 , -555 , -715 , and -760 we see that B_E is always simple: its endomorphism algebra is either a biquadratic field or a quaternion division algebra over \mathbb{Q} . Therefore, $\text{Res}_{H/\mathbb{Q}} E \sim W^2$ for some simple variety W of dimension 4. We record these findings in the following statement.

Theorem 3.5. *Let M be an imaginary quadratic field of discriminant D and Hilbert class field H . Suppose that D is nonexceptional and that $\text{Gal}(H/M) \simeq C_2 \times C_2$. If $D \neq -195, -312, -555, -715, -760$, there exists a Gross \mathbb{Q} -curve E/H such that*

$$\text{Res}_{H/\mathbb{Q}} E \sim A^4, \quad \text{for some simple abelian surface } A/\mathbb{Q}.$$

If $D = -195, -312, -555, -715, -760$, then for every Gross \mathbb{Q} -curve E/H we have that

$$\text{Res}_{H/\mathbb{Q}} E \sim W^2, \quad \text{for some simple abelian variety } W/\mathbb{Q} \text{ of dimension 4.}$$

Remark 3.6. As mentioned above, the cases of $D = -84$ and $D = -195$ were already computed by Nakamura [2004, §6]. We note what appears to be a typo in Nakamura's table in page 647: the last biquadratic field should be $\mathbb{Q}(\sqrt{-14}, \sqrt{42})$, instead of $\mathbb{Q}(\sqrt{-14}, \sqrt{-42})$.

We have used the software [Sage] and [Magma] to perform the computations of Table 1. The interested reader can find the code we used in [Fité and Guitart 2018b].

4. Proof of the main theorems

We begin with a lemma that will be used in the proof of Theorem 1.2.

Lemma 4.1. *Let E be a Gross \mathbb{Q} -curve with CM by a field M of discriminant D , and suppose that $\text{Gal}(H/M)$ is isomorphic to $C_2 \times C_2$. Denote by γ_E^H the class in $H^2(\text{Gal}(H/M), M^\times)$ attached to E ,*

and by c_E a cocycle representing γ_E^H . If $\sigma \in \text{Gal}(H/M)$ is nontrivial, then $\pm d \cdot c_E(\sigma, \sigma) \in (M^\times)^2$ for some divisor d of D such that d is not a square in M^\times .

Proof. Let \mathcal{O}_M denote the ring of integers of M . Denote by p_1, p_2, p_3 the primes dividing D . Observe that $p_i \mathcal{O}_M = \mathfrak{p}_i^2$, with \mathfrak{p}_i a nonprincipal prime ideal of \mathcal{O}_M . Clearly, we can always find p_i, p_j such that $\pm p_i p_j$ is not a square in M^\times , and therefore $\mathfrak{p}_i \mathfrak{p}_j$ is not principal. Thus $\mathfrak{p}_i, \mathfrak{p}_j$ generate the class group. Therefore, we can assume that any nontrivial element of $\text{Gal}(H/K)$ is of the form σ_q for some unramified prime q which is equivalent to either $\mathfrak{p}_i, \mathfrak{p}_j$ or $\mathfrak{p}_i \cdot \mathfrak{p}_j$. Here σ_q stands for the Frobenius automorphism of H/K at q .

Now we argue (and use the same notation) as in [Nakamura 2004, Proof of Theorem 3]. Namely, denote by $u(q)$ the q -multiplication isogenies

$$u(q) : {}^\sigma q E \rightarrow E,$$

and denote by c the 2-cocycle associated to E using the system of isogenies $u(q)$ (together with the identity isogeny for $1 \in \text{Gal}(H/M)$). Note that c_E is any cocycle representing γ_E^H , and it may be different from c . But in any case they are cohomologous, which in particular implies that

$$c(\sigma_q, \sigma_q) = b_q^2 \cdot c_E(\sigma_q, \sigma_q) \quad \text{for some } b_q \in M^\times. \tag{4-1}$$

From [loc. cit., Equation (6) and the following display], since the order n of σ_q is 2 in our case, we see that

$$c(\sigma_q, \sigma_q) \mathcal{O}_M = \mathfrak{q}^2.$$

The proof is finished by observing that $\mathfrak{q}^2 = \alpha \mathcal{O}_M$, where $\alpha \in M^\times$ is, up to an element of $(M^\times)^2$, equal to $\pm p_i, \pm p_j$, or $\pm p_i \cdot p_j$. □

Proof of Theorem 1.2. For all the quadratic imaginary fields not listed in (1-2), we have constructed in the first part of Theorem 3.5 abelian surfaces defined over \mathbb{Q} satisfying the hypothesis of the theorem. To rule out the remaining 6 fields, we proceed in the following way.

Let M be one of the fields in the list (1-2) and suppose that an abelian surface A satisfying the hypothesis of the theorem exists for M . Resume the notations from Section 2D. As $\text{Gal}(H/M) \simeq C_2 \times C_2$ and $H \subseteq K$ (by [Fité and Guitart 2018a, Theorem 2.14]), the only possibilities for $\text{Gal}(K/M)$ are $C_2 \times C_2, D_4$, and D_6 .

Suppose that $\text{Gal}(K/M)$ is $C_2 \times C_2$. Then $K = H$ and thus E is a Gross \mathbb{Q} -curve. By Proposition 2.10, we have that M is not exceptional and thus we cannot have $M = \mathbb{Q}(\sqrt{-340})$. For the other possibilities for M , we have seen in the second part of Theorem 3.5 that $\text{Res}_{H/\mathbb{Q}} E$ does not have any simple factor of dimension 2, but this is a contradiction with the fact that A should be a factor of $\text{Res}_{H/\mathbb{Q}} E$ (indeed, the universal property of Weil’s restriction of scalars implies that $\text{Hom}(A, \text{Res}_{H/\mathbb{Q}} E) = \text{Hom}(A_H, E) \simeq M^2$, and thus $\text{Hom}(A, \text{Res}_{H/\mathbb{Q}} E) \neq 0$).

Suppose that $\text{Gal}(K/M)$ is D_4 or D_6 . Resume the notations of Section 2E. Let E^* be a Ribet M -curve completely defined over H with CM by M which we chose as in Corollary 2.25 (and which exists because of Proposition 2.10). Note that Hypothesis 2.22 is satisfied. Then, by Theorem 2.27, there is a nontrivial element $\bar{\tau} \in \text{Gal}(N/M) = \text{Gal}(H/N)$ such that

$$c_{E^*}^H(\bar{\tau}, \bar{\tau}) = \pm 1. \tag{4-2}$$

D	Biquadratic fields	Quaternion algebras
-84	$(-14, -2), (-6, 2), (-6, -42), (-14, 42)$	2, 1, 2, 1
-120	$(-5, 10), (5, -10), (-5, -10), (5, 10)$	1, 6, 3, 1
-132	$(22, -2), (-6, -2), (6, -66), (-22, -66)$	1, 2, 1, 2
-168	$(-14, -2), (3, -21), (14, 21), (-3, 2)$	2, 1, 1, 1
-195	$(13, -5), (-13, -5), (-13, 5), (13, 5)$	13, 39, 26, 39
-228	$(-38, -2), (6, -2), (-6, -114), (38, -114)$	2, 1, 2, 1
-280	$(-10, -5), (-10, 5), (10, -5), (10, 5)$	2, 1, 14, 14
-312	$(13, -26), (-13, 26), (-13, -26), (13, 26)$	13, 39, 26, 39
-372	$(-62, 31), (-6, -3), (-6, 31), (-62, -3)$	2, 1, 2, 1
-408	$(-17, 34), (-17, -34), (17, -34), (17, 34)$	2, 1, 1, 1
-435	$(-29, -5), (-29, 5), (29, -5), (29, 5)$	2, 1, 1, 1
-483	$(-23, 7), (23, -69), (-21, -7), (21, 69)$	2, 1, 1, 1
-520	$(-13, -5), (13, -5), (-13, 5), (13, 5)$	1, 1, 1, 2
-532	$(-38, -19), (-14, 7), (-14, -19), (-38, 7)$	1, 2, 1, 2
-555	$(37, -5), (-37, -5), (-37, 5), (37, 5)$	37, 111, 74, 111
-595	$(-17, 85), (17, -85), (-17, -85), (17, 85)$	7, 1, 1, 14
-627	$(19, -11), (-19, -57), (-33, 11), (33, 57)$	1, 2, 1, 1
-708	$(118, -59), (-6, 3), (6, -59), (-118, 3)$	1, 2, 1, 2
-715	$(-13, -65), (13, -65), (-13, 65), (13, 65)$	5, 10, 55, 55
-760	$(-10, 5), (10, -5), (-10, -5), (10, 5)$	5, 95, 10, 95
-795	$(-53, -5), (53, -5), (-53, 5), (53, 5)$	6, 1, 1, 3
-1012	$(-46, 23), (-22, -11), (-22, 23), (-46, -11)$	2, 1, 2, 1
-1435	$(-41, 205), (-41, -205), (41, -205), (41, 205)$	2, 1, 1, 1

Table 1. Endomorphism algebras of the restriction of scalars of Gross \mathbb{Q} -curves. For the biquadratic fields, the notation (a, b) indicates the field $\mathbb{Q}(\sqrt{a}, \sqrt{b})$; for the quaternion algebras, we write the discriminant of the algebra

If M is nonexceptional, as noted in Remark 2.26, we can suppose that E^* is in fact a Gross \mathbb{Q} -curve. Then (4-2) is a contradiction with Lemma 4.1.

It remains to show that (4-2) also brings a contradiction if $M = \mathbb{Q}(\sqrt{-340})$ is the exceptional field. Put $T = H^{\langle \bar{\tau} \rangle}$, the fixed field by $\bar{\tau}$. Observe that $M \subsetneq T \subsetneq H$. If $c_{E^*}^H(\bar{\tau}, \bar{\tau}) = 1$ then by Theorem 2.11 the curve E^* is isogenous to a curve defined over T , and this is a contradiction with the fact that $M(j_{E^*}) = H$.

Suppose now that $c_{E^*}^H(\bar{\tau}, \bar{\tau}) = -1$. We will see that we can apply the above argument to an appropriate quadratic twist of E^* .

Claim 4.2. *There exists a quadratic extension S/H such that S/M is Galois with $\text{Gal}(S/M) \simeq D_4$ and such that $\bar{\tau}$ lifts to an element of order 4 of $\text{Gal}(S/M)$.*

We now show how this claim allows us to produce the appropriate twisted curve (and we will prove the claim later on). Define C to be the S/H quadratic twist of E^* . By [Fité and Guitart 2018a, Lemma 3.13], the curve C is an M -curve completely defined over H and the cohomology classes of E^* and C are related by

$$\gamma_C^H = \gamma_{E^*}^H \cdot \gamma_S,$$

where $\gamma_S \in H^2(\text{Gal}(H/M), \{\pm 1\})$ is the cohomology class attached to the exact sequence

$$1 \rightarrow \text{Gal}(S/H) \simeq \{\pm 1\} \rightarrow \text{Gal}(S/M) \simeq D_4 \rightarrow \text{Gal}(H/M) \rightarrow 1. \tag{4-3}$$

If we identify $\text{Gal}(S/M) \simeq \langle s, t | s^4, t^2, stst \rangle$, then $\text{Gal}(S/H)$ can be identified with the subgroup generated by s^2 and we can assume that $\bar{\tau}$ lifts to s . Let c_S be a cocycle representing γ_S . The usual construction that associates a cohomology class to (4-3) gives that $c_S(\bar{\tau}, \bar{\tau}) = s \cdot s$. Since s^2 is the nontrivial element of $\text{Gal}(S/H)$, it corresponds to -1 under the isomorphism $\text{Gal}(S/H) \simeq \{\pm 1\}$, so that $c_S(\bar{\tau}, \bar{\tau}) = -1$.

We conclude that $c_C^H(\bar{\tau}, \bar{\tau}) = c_{E^*}^H(\bar{\tau}, \bar{\tau})c_S(\bar{\tau}, \bar{\tau}) = 1$, and as before this implies that C can be defined over T , which is a contradiction.

Proof of Claim 4.2. The Hilbert class field of M is $H = \mathbb{Q}(i, \sqrt{5}, \sqrt{17})$. If we write $H = M(\sqrt{a}, \sqrt{b})$ and suppose that $\bar{\tau}(\sqrt{b}) = \sqrt{b}$, it is well known (see, e.g., [Ledet 2001, §0.4]) that the obstruction to the existence of S is given by the quaternion algebra

$$\left(\frac{a, ab}{M} \right)$$

being nonsplit. There are 3 possibilities for T , namely $T = M(\sqrt{5})$, $T = M(\sqrt{17})$, or $T = M(\sqrt{5 \cdot 17})$, each one giving a different obstruction. The resulting quaternion algebras giving the obstruction are

$$\left(\frac{17 \cdot 5, 5}{M} \right), \left(\frac{17 \cdot 5, 17}{M} \right), \left(\frac{17, 5}{M} \right).$$

Since they are all the split, the field S does exist in all three cases. □

Remark 4.3. As a byproduct of the above proof, we see that there do not exist abelian surfaces over \mathbb{Q} such that $\text{End}(A_{\overline{\mathbb{Q}}}) \simeq M_2(M)$ with M a quadratic imaginary field with class group $C_2 \times C_2$ and $\text{Gal}(K/M) \simeq D_4$ or D_6 . As shown by the table of [Cardona Juanals 2001, p. 112], there do exist abelian surfaces over \mathbb{Q} such that $\text{End}(A_{\overline{\mathbb{Q}}}) \simeq M_2(M)$ with M a quadratic imaginary field with class group C_2 and $\text{Gal}(K/M) \simeq D_4$ (resp. D_6). If M is not exceptional, Theorem 2.20 and Lemma 4.1 imply that 2 (resp. 3) divide the discriminant of M is a necessary condition for the existence of such an A . The examples of the table of [Cardona Juanals 2001, p. 112] show that this is actually a necessary and sufficient condition.

Proof of Corollary 1.3. Suppose that A is an abelian surface defined over \mathbb{Q} such that $A_{\overline{\mathbb{Q}}} \sim E \times E'$, where E and E' are elliptic curves defined over $\overline{\mathbb{Q}}$. If E and E' are not isogenous, then $\text{End}(A_{\overline{\mathbb{Q}}})$ is

$$\mathbb{Q} \times \mathbb{Q}, \quad M \times \mathbb{Q} \quad \text{or} \quad M_1 \times M_2,$$

where $M, M_1 \not\cong M_2$ are quadratic imaginary fields, depending on whether none of E and E' has CM, only one of E and E' has CM, or both of E and E' have CM. In any case, note that by [Fité et al. 2012, Proposition 4.5], both E and E' can be defined over \mathbb{Q} , whereby the class number of M, M_1 , and M_2 must be 1. Recalling that there are 9 quadratic imaginary fields of class number 1, this accounts for 46 distinct $\overline{\mathbb{Q}}$ -endomorphism algebras.

If E and E' are isogenous, we have that $\text{End}(A_{\overline{\mathbb{Q}}})$ is $M_2(M)$ or $M_2(\mathbb{Q})$, where M is a quadratic imaginary field, depending on whether E has CM or not. Assume that we are in the former case. By Theorem 1.1, we have that M has class group 1, C_2 , or $C_2 \times C_2$. As explained in [Fité and Guitart 2018a, Remark 2.20], for all fields M with class group 1 (resp. C_2), abelian surfaces A over \mathbb{Q} with $\text{End}(A_{\overline{\mathbb{Q}}}) \simeq M_2(M)$ can be easily found. Indeed, let E be an elliptic curve with CM by the maximal order of M and defined over \mathbb{Q} (resp. $\mathbb{Q}(j_E)$). Then consider the square (resp. the restriction of scalars from $\mathbb{Q}(j_E)$ down to \mathbb{Q}) of E . If M has class group $C_2 \times C_2$, invoke Theorem 1.2 to obtain 18 possibilities for M . Taking into account that there are 18 quadratic imaginary fields of class group C_2 (see [Watkins 2004] for example), we obtain 46 possibilities for the endomorphism algebra of a geometrically split abelian surface over \mathbb{Q} with $\overline{\mathbb{Q}}$ -isogenous factors.

An open problem. We wish to conclude the article with an open question.

Question 4.4. Which is the subset of \mathcal{A} made of the $\overline{\mathbb{Q}}$ -endomorphism algebras $\text{End}(\text{Jac}(C)_{\overline{\mathbb{Q}}})$ of geometrically split Jacobians of genus 2 curves C defined over \mathbb{Q} ?

Again the most intriguing case is to determine how many of the 45 possibilities for $M_2(M)$, with M a quadratic imaginary field, allowed by Theorem 1.2 for geometrically split abelian surfaces defined over \mathbb{Q} still occur among geometrically split Jacobians of genus 2 curves C defined over \mathbb{Q} . Looking at the more restrictive setting that requires $\text{Jac}(C)$ to be *isomorphic* to the square of an elliptic curve with CM by the *maximal order* of M , G elin, Howe, and Ritzenthaler [G elin et al. 2019] have shown that there are 13 possibilities for such an M (all with class number ≤ 2).

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