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Galois action on the principal block and cyclic Sylow subgroups

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We characterize finite groups G having a cyclic Sylow p-subgroup in terms of the action of a specific Galois automorphism on the principal p-block of G, for $p = 2, 3$. We show that the analog statement for blocks with arbitrary defect group would follow from the blockwise McKay–Navarro conjecture.

Introduction

One of the most prevalent questions in the representation theory of finite groups is to determine what relationships hold between the set $\mathrm{Irr}(G)$ of irreducible complex characters of a finite group G and its local structure, such as the structure of a Sylow p-subgroup P of G. There is, of course, the more sophisticated question of relating the set $\mathrm{Irr}(B)$ of irreducible characters belonging to a given Brauer p-block B of G with the structure of a defect group D of B.

G. Navarro and P. H. Tiep [2019] conjecture that for a prime p, one can determine the exponent of the abelianization of P in terms of the action of certain Galois automorphisms on $\mathrm{Irr}(G)$. To be more precise, for a fixed prime p and an integer $e \geq 1$, let $\sigma_e \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) = \mathcal{G}$ be such that σ_e fixes p'-roots of unity and sends any root of unity of order a power of p to its (p^e+1)-st power. In [Navarro and Tiep 2019] it is proven that the exponent of P/P' is less than or equal to p^e whenever all of the irreducible characters of p'-degree of G are σ_e-fixed, and the converse is reduced to a question on finite simple groups. (Thanks to Malle [2019] we know that the converse holds for $p = 2$.)

In the present work, we show that one can determine whether P is cyclic (for small primes) by just counting the number of certain σ_1-invariant elements of $\mathrm{Irr}(B_0)$, where B_0 is the principal p-block of G. This is the main result of our paper.

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Theorem A. *Let G be a finite group of order divisible by p, where $p \in \{2, 3\}$. Let $P \in \mathrm{Syl}_p(G)$ and let B_0 be the principal p-block of G. Then*

$$|\mathrm{Irr}_{p'}(B_0)^{\sigma_1}| = p \quad \textit{if, and only if,} \quad P \textit{ is cyclic,}$$

where $\mathrm{Irr}_{p'}(B_0)^{\sigma_1}$ is the set of irreducible characters in B_0 with degree relatively prime to p that are fixed under the action of σ_1.

With the definition above, σ_1 is an element of the subgroup $\mathcal{H} \leq \mathcal{G}$ consisting of all $\sigma \in \mathcal{G}$ for which there exists some integer f such that $\sigma(\xi) = \xi^{p^f}$ whenever ξ is a root of unity of order not divisible by p. Navarro predicted [2004, Conjecture A] the existence of bijections for the McKay conjecture commuting with the action of $\mathcal{H}$ on characters. This is the celebrated McKay–Navarro conjecture (sometimes also referred to as the Galois–McKay conjecture), which has been recently reduced to a question on finite simple groups in [Navarro et al. 2019]. The McKay–Navarro conjecture admits a blockwise version [Navarro 2004, Conjecture B], which remains unreduced at the present moment and which we will refer to as the Alperin–McKay–Navarro conjecture, as it can also be seen as a refined version of the celebrated Alperin–McKay conjecture. In this context, it is natural to wonder the extent to which Theorem A holds for arbitrary blocks. We propose the following.

Conjecture B. *Let $p \in \{2, 3\}$. Let G be a finite group and let B be a p-block of G with nontrivial defect group D. Then*

$$|\mathrm{Irr}_0(B)^{\sigma_1}| = p \quad \textit{if, and only if,} \quad D \textit{ is cyclic,}$$

where $\mathrm{Irr}_0(B)^{\sigma_1}$ is the set of height zero irreducible characters in the block B that are fixed under the action of σ_1.

We prove that Conjecture B follows from the Alperin–McKay–Navarro conjecture. In this sense, Theorem A provides more evidence of the elusive Alperin–McKay–Navarro conjecture. Since the latter holds whenever D is cyclic, by work of Navarro [2004], it follows that the "if" direction of Conjecture B (and of Theorem A) holds. For many consequences of the (Alperin–)McKay–Navarro conjecture, the statements take different forms depending on the prime (see, for instance, [Navarro et al. 2007; Schaeffer Fry 2019]). This might well be the case here, however, we are not yet aware of such a statement for $p > 3$.

To prove Theorem A, we use the classification of finite simple groups. In particular, we contribute to the problem of understanding Galois action on the characters in blocks of nonabelian simple groups in the following way.

Theorem C. *Let S be a nonabelian simple group of order divisible by $p \leq 3$, $P \in \mathrm{Syl}_p(S)$ and $X \in \mathrm{Syl}_p(\mathrm{Aut}(S))$. Let B_0 be the principal p-block of S.*

(a) *If P is cyclic, then $p = 3$ and $\mathrm{Irr}_{p'}(B_0)^{\sigma_1} = \{1_S, \phi_1, \phi_2\}$, where the ϕ_i are nontrivial and not $\mathrm{Aut}(S)$-conjugate, and some ϕ_i is X-invariant.*

(b) *If P is not cyclic, then $\mathrm{Irr}_{p'}(B_0)^{\sigma_1} \supseteq \{1_S, \phi_1, \dots, \phi_p\}$, where the nontrivial ϕ_i are pairwise not $\mathrm{Aut}(S)$-conjugate, and some ϕ_i is X-invariant.*

This paper is structured as follows. In Section 1 we prove that Conjecture B follows from the Alperin–McKay–Navarro conjecture. To do so, we study the action of σ_1 on the irreducible characters of blocks with normal defect group. The rest of the paper is devoted to proving Theorem A. In Section 2, we reduce Theorem A to statements on finite simple groups, and in Section 3 we prove Theorem C thus completing the proof of Theorem A.

1. Blocks with normal defect group

The aim of this section is to prove that Conjecture B follows from the Alperin–McKay–Navarro conjecture, stated below.

For a fixed prime p, consider the set $\mathrm{Bl}(G)$ of Brauer (p-)blocks of G as in [Navarro 1998], so that $\mathrm{Bl}(G)$ is a partition of $\mathrm{Irr}(G) \cup \mathrm{IBr}(G)$ (recall that p-Brauer characters are defined on p-regular elements of G). Write $\mathrm{Irr}(B) = B \cap \mathrm{Irr}(G)$ and $\mathrm{IBr}(B) = B \cap \mathrm{IBr}(G)$ for any $B \in \mathrm{Bl}(G)$. Every block B has associated a uniquely defined conjugacy class of p-subgroups of G, namely its defect groups. Given a block B of G with defect group D, we write $B \in \mathrm{Bl}(G \mid D)$ and we let $b \in \mathrm{Bl}(\boldsymbol{N}_G(D) \mid D)$ denote its Brauer first main correspondent. Finally, $\chi \in \mathrm{Irr}(B)$ has height zero in B if $\chi(1)_p = |G : D|_p$, and we write $\mathrm{Irr}_0(B)$ to denote the subset of height zero characters in $\mathrm{Irr}(B)$.

Assuming the notation of the introduction, we have that the group $\mathcal{G}$ acts on $\{\mathrm{Irr}(B) \mid B \in \mathrm{Bl}(G)\}$ by [Navarro 1998, Theorem 3.19]. The group $\mathcal{H}$ further acts on the set $\mathrm{Bl}(G)$ by [Navarro 2004, Theorem 2.1]. While the action of $\mathcal{G}$ on characters is not natural enough in global-local contexts, Navarro [2004] conjectured the following.

Conjecture (Alperin–McKay–Navarro conjecture). *Let $B \in \mathrm{Bl}(G \mid D)$ and let $b \in \mathrm{Bl}(\boldsymbol{N}_G(D) \mid D)$ be its Brauer first main correspondent. If $\sigma \in \mathcal{H}$, then*

$$|\mathrm{Irr}_0(B)^\sigma| = |\mathrm{Irr}_0(b)^\sigma|.$$

Here we are only concerned with the action of a specific element of $\mathcal{H}$, namely σ_1. Recall that $\sigma_1 \in \mathcal{H}$ fixes p'-roots of unity and sends any root of unity of order a power of p to its $(p+1)$-st power. If G is a finite group of order dividing some integer n and ξ_n is a primitive n-th root of unity, then by elementary number theory, the restriction ω of σ_1 to the n-th cyclotomic field $\mathbb{Q}(\xi_n)$ has order a power of p, and ω acts as σ_1 on the ordinary characters of every subgroup of G. Abusing notation, we will also write σ_1 for any such restriction. In particular, σ_1 fixes the elements of $\mathrm{IBr}(G)$, and hence acts trivially on $\mathrm{Bl}(G)$. (Note that in general $\mathcal{G}$ does not act on $\mathrm{IBr}(G)$, but $\mathcal{H}$ does by Theorem 2.1 of [Navarro 2004].)

In order to prove that Conjecture B follows from the Alperin–McKay–Navarro conjecture, we need to study blocks with a normal defect group. We follow the notation in Chapter 9 of [Navarro 1998]. Let $B \in \mathrm{Bl}(G \mid D)$ and assume that $D \triangleleft G$. Write $C = \boldsymbol{C}_G(D)$. We will denote by $b \in \mathrm{Bl}(CD \mid D)$ a *root* of B, and we will let $\theta \in \mathrm{Irr}(b)$ be the *canonical character* associated with B, which is unique up to G-conjugacy (see [Navarro 1998, Theorem 9.12] and the subsequent discussion). Recall that $D \subseteq \ker\theta$ and θ has p-defect zero when viewed as a character of CD/D (that is, $\theta(1)_p = |CD : D|_p$), the stabilizer of the block b is $G_b = G_\theta$, and the inertial index $|G_\theta : CD|$ is not divisible by p. In this situation, $\mathrm{Irr}(b) = \{\theta_\lambda \mid \lambda \in \mathrm{Irr}(D)\}$, where the irreducible characters $\theta_\lambda \in \mathrm{Irr}(CD)$ are defined for $x \in CD$ as follows: $\theta_\lambda(x) = \lambda(x_p)\theta(x_{p'})$ if $x_p \in D$ and $\theta_\lambda(x) = 0$ otherwise. One can see that

$$G_{\theta_\lambda} = G_\theta \cap G_\lambda.$$

Let $c \in \mathrm{Bl}(G_b \mid D)$ be the Fong–Reynolds correspondent of b and B as in [Navarro 1998, Theorem 9.14]. Then the induction map $\mathrm{Irr}(c) \to \mathrm{Irr}(B)$ defines a height-preserving bijection. By [loc. cit., Theorems 9.21 and 9.22] $c = b^{G_b}$ is the only block of G_b that covers b and

$$\mathrm{Irr}(B) = \bigcup_{\lambda \in \mathrm{Irr}(D)} \mathrm{Irr}(G \mid \theta_\lambda). \tag{1}$$

It is not difficult to see that height zero characters of B further lie over characters parametrized by linear characters of D, so that

$$\mathrm{Irr}_0(B) = \bigcup_{\lambda \in \mathrm{Irr}(D/D')} \mathrm{Irr}(G \mid \theta_\lambda). \tag{2}$$

In order to explicitly describe the set $\mathrm{Irr}_0(B)^{\sigma_1}$ when the defect group of B is normal we will use the following technical lemma.

Lemma 1.1. *Let G be a finite group and let p be a prime. Suppose that B is a block of G with normal defect group D. Let b be a root of B with canonical character θ. Write $A = \langle \sigma_1 \rangle \leq \mathrm{Gal}(\mathbb{Q}(\xi_{|G|})/\mathbb{Q})$. If λ is a linear character of D, then let $G_{\theta_\lambda^A} = \{g \in G \mid (\theta_\lambda)^g = (\theta_\lambda)^a$ for some $a \in A\}$. With this definition*

$$G_{\theta_\lambda^A} = G_{\theta_\lambda} = G_\theta \cap G_\lambda.$$

Proof. Write $C = \mathbf{C}_G(D)$. Recall that b is a block of CD of defect D and $\theta \in \mathrm{Irr}(CD)$ has defect zero as a character of CD/D. Note that θ is A-fixed since $b^a = b$ for every $a \in A$. Let $g \in G_{\theta_\lambda^A}$. We start by proving that $g \in G_\theta$. Since θ is A-fixed, by the definition of θ_λ we have $(\theta_\lambda)^g = (\theta_\lambda)^a = \theta_{\lambda^a}$ for some $a \in A$. Evaluating on D we see that

$$\theta(1)\lambda^a(x) = \theta_{\lambda^a}(x) = \theta_\lambda^g(x) = \theta(1)\lambda^g(x),$$

for every $x \in D$. Hence $\lambda^g = \lambda^a$. Let $x \in CD$ be such that $xD \in (CD/D)^0$, the set of p-regular elements of CD/D, and notice that $x_p \in D$. (Otherwise $\theta(x) = 0$.) Then

$$\lambda^g(x_p)\theta^g(x_{p'}) = \theta_\lambda^g(x) = \theta_\lambda^a(x) = \lambda^a(x_p)\theta(x'_p) = \lambda^g(x_p)\theta(x'_p).$$

This implies $\theta^g(x_{p'}) = \theta(x_{p'})$. Since $xD = x_{p'}D$, then $\theta^g = \theta$ and $g \in G_\theta$.

Next we prove that $g \in G_\lambda$. We know that $\lambda^g = \lambda^a$ for some $a \in A$, and that $g \in G_\theta$. Since G_θ/CD is a p'-group, then $\lambda^{g^m} = \lambda$ for some integer m relatively prime to p. In particular, $\lambda^{a^m} = \lambda$ and the order of a as an element of $\mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q}) = \mathrm{Gal}(\mathbb{Q}(\xi_{o(\lambda)})/\mathbb{Q})$ divides m, which forces $a = 1$ and $\lambda^g = \lambda$, as wanted. ∎

Lemma 1.2. *Let G be a finite group and let p be a prime. Suppose that B is a block of G with a normal defect group D. Let b be a root of B with canonical character θ. Then*

$$\mathrm{Irr}_0(B)^{\sigma_1} = \bigcup_{\lambda\in \mathrm{Irr}(D/\Phi(D))} \mathrm{Irr}(G \mid \theta_\lambda),$$

where $\Phi(D)$ is the Frattini subgroup of D. Moreover, if $c \in \mathrm{Bl}(G_b \mid D)$ is the Fong–Reynolds correspondent of B, then

$$|\mathrm{Irr}_0(B)^{\sigma_1}| = |\mathrm{Irr}_0(c)^{\sigma_1}|.$$

Proof. First notice that as a p-group, D has a unique block, the principal one, and $\mathrm{Irr}_0(B_0(D)) = \mathrm{Irr}_{p'}(D) = \mathrm{Irr}(D/D')$. Then $\mathrm{Irr}_{p'}(D)^{\sigma_1} = \mathrm{Irr}(D/\Phi(D))$. Since $D/\Phi(D)$ is p-elementary abelian, one inclusion is straight-forward. To see that $\mathrm{Irr}_{p'}(D)^{\sigma_1} \subseteq \mathrm{Irr}(D/\Phi(D))$ notice that if $\lambda \in \mathrm{Irr}(D/D')$ is σ_1-fixed, then $\lambda^{\sigma_1} = \lambda^{p+1} = \lambda$, and hence $|D/\ker\lambda| \le p$, implying $\Phi(D) \subseteq \ker\lambda$.

Write $A = \langle\sigma_1\rangle$ and let $G_{\theta_\lambda^A}$ be as in Lemma 1.1. By (2), we know that

$$\mathrm{Irr}_0(B) = \bigcup_{\lambda\in \mathrm{Irr}(D/D')} \mathrm{Irr}(G \mid \theta_\lambda).$$

If $\chi \in \mathrm{Irr}_0(B)^{\sigma_1}$ lies over θ_λ, then $(\theta_\lambda)^{\sigma_1} = (\theta_\lambda)^g$, for some $g \in G$. In particular, $g \in G_{\theta_\lambda^A} = G_{\theta_\lambda} = G_\theta \cap G_\lambda$ by Lemma 1.1. Then $\lambda^{\sigma_1} = \lambda^g = \lambda$. Hence $\Phi(D) \subseteq \ker\lambda$ and $\lambda \in \mathrm{Irr}(D/\Phi(D))$.

Conversely, let $\chi \in \mathrm{Irr}(G \mid \theta_\lambda)$, where $\lambda \in \mathrm{Irr}(D/\Phi(D))$. Then $\lambda^{\sigma_1} = \lambda$. As $b^{\sigma_1} = b$, we see σ_1 fixes θ too. Then $(\theta_\lambda)^{\sigma_1} = \theta_\lambda$. Let $\psi \in \mathrm{Irr}(G_{\theta_\lambda})$ be the Clifford correspondent of χ over θ_λ. Since $G_{\theta_\lambda} \subseteq G_\theta$, we know that p does not divide the order of G_{θ_λ}/CD. By [Navarro and Tiep 2019, Lemma 5.1], ψ is σ_1-invariant and so is χ.

To prove the last part of the statement, recall that the Fong–Reynolds correspondence states that the induction map $\psi \mapsto \psi^G$ provides a bijection $\mathrm{Irr}_0(c) \to \mathrm{Irr}_0(B)$. In particular, $|\mathrm{Irr}_0(c)^{\sigma_1}| \le |\mathrm{Irr}_0(B)^{\sigma_1}|$. Now let $\chi \in \mathrm{Irr}_0(B)^{\sigma_1}$ lie over θ_λ, for some $\lambda \in \mathrm{Irr}(D/\Phi(D))$ by the first part of this proof. Then $(\theta_\lambda)^{\sigma_1} = (\theta_\lambda)^g$ for some $g \in G$. In particular, $g \in G_{\theta_\lambda^A}$. Since $G_{\theta_\lambda^A} = G_{\theta_\lambda}$ by Lemma 1.1, θ_λ is σ_1-fixed. Let $\xi \in \mathrm{Irr}(G_{\theta_\lambda} \mid \theta_\lambda)$ be the Clifford correspondent of χ. Since both χ and θ_λ are σ_1-fixed then so is ξ. We have that ξ^{G_b} is the Fong–Reynolds correspondent of χ by the transitivity of block induction (see [Navarro 1998, Problem 4.2]), which is σ_1-fixed. □

The Alperin–McKay–Navarro conjecture holds for blocks with cyclic defect groups by [Navarro 2004, Theorem 3.4]. We obtain the following as a consequence of this fact.

Lemma 1.3. *Let G be a finite group and let B be a block of G with cyclic defect group D. Then*

$$1 \le |\mathrm{Irr}_0(B)^{\sigma_1}| \le p.$$

The set $\mathrm{Irr}_0(B)^{\sigma_1}$ has minimal size 1 if, and only if, D is trivial. Furthermore, if $p \in \{2, 3\}$ and D is nontrivial, then

$$|\mathrm{Irr}_0(B)^{\sigma_1}| = p.$$

Proof. By [Navarro 2004, Theorem 3.4], we may assume that $D \lhd G$. Write $C = \boldsymbol{C}_G(D) \supseteq D$. Let $b \in \mathrm{Bl}(C \mid D)$ be a root of B with canonical character θ. By Lemma 1.2, we may assume that θ is G-invariant (in particular, G/C is a p'-group) and

$$\mathrm{Irr}_0(B)^{\sigma_1} = \bigcup_{\lambda \in \mathrm{Irr}(D/\Phi(D))} \mathrm{Irr}(G \mid \theta_\lambda) \subseteq \mathrm{Irr}(G/\Phi(D)).$$

Write $\overline{G} = G/\Phi(D)$ and use the bar convention. Let $\overline{F} = \boldsymbol{C}_{\overline{G}}(\overline{D})$, where $\Phi(D) \subseteq F \leq G$. We claim that $F = C$. Clearly $C \subseteq F$. Note that $\overline{F}$ acts trivially on $\overline{D}$ and coprimely on D. By [Isaacs 2008, Theorem 3.29] we have that $\overline{F}$ acts trivially on D as well. Thus $F = C$ as claimed.

Notice that since D is cyclic and G/C is a p'-group, then G/C is isomorphic to a subgroup of C_{p-1}. Say $|G/C| = m$ and let $\{\lambda_i\}_{i=1}^t$ be a complete set of representatives of the G/C-orbits on $\mathrm{Irr}(\overline{D}) \setminus \{1_D\}$, where here we view $\mathrm{Irr}(\overline{D}) \subseteq \mathrm{Irr}(D)$, and with this identification $\mathrm{Irr}(\overline{D})$ are exactly the elements of $\mathrm{Irr}(D)$ with order dividing p. Note that $\ker \lambda_i = \Phi(D)$ for all $1 \leq i \leq t$, hence $G_{\lambda_i} = C$ for every $1 \leq i \leq t$, and all the orbits of the action of G/C on $\mathrm{Irr}(\overline{D}) \setminus \{1_D\}$ have the same size m. In particular, $t = (p-1)/m$. Since θ is G-invariant, for every $1 \leq i \leq t$ we have that $G_{\theta_{\lambda_i}} = G_{\lambda_i} = C$, and by the Clifford correspondence, $|\mathrm{Irr}(G \mid \theta_{\lambda_i})| = |\mathrm{Irr}(C \mid \theta_{\lambda_i})| = 1$. Also, since G/C is cyclic, θ extends to G and therefore by Gallagher theory $|\mathrm{Irr}(G \mid \theta)| = m$. Then

$$|\mathrm{Irr}_0(B)^{\sigma_1}| = |\mathrm{Irr}(G \mid \theta)| + \sum_{i=1}^{t} |\mathrm{Irr}(G \mid \theta_{\lambda_i})| = m + t = m + \frac{p-1}{m} \leq p.$$

Note that if $p = 2, 3$ then $m + (p-1)/m = p$, whenever m divides $p-1$. Also notice that $|\mathrm{Irr}_0(B)^{\sigma_1}| = 1$ if, and only if, $D = 1$. □

The upper bound in Lemma 1.3 is not generally attained if $p > 3$, as shown by the dihedral group D_{2p}, which satisfies $|\mathrm{Irr}(B_0(\mathsf{D}_{2p}))^{\sigma_1}| < p$. We care to remark that the numerical condition $|\mathrm{Irr}_0(B)^{\sigma_1}| \leq p$ does not generally imply that a defect group D of B is cyclic. For instance, for $p = 11$, the semidirect product $H = \mathbb{F}_{11}^2 \rtimes \mathrm{SL}_2(5)$ satisfies $|\mathrm{Irr}_{11'}(B_0(H))^{\sigma_1}| = |\mathrm{Irr}(H)| = 10$. (We would like to thank Gabriel Navarro for providing us with this example.)

We will need the following divisibility result, which we obtain by adapting the proof of [Gow 1979, Theorem 5.2].

Lemma 1.4. *Let G be a finite group, let $p \in \{2, 3\}$, and let B be a block of G with nontrivial defect group D. Then p divides $|\mathrm{Irr}_0(B)^{\sigma_1}|$.*

Proof. Write

$$\psi = \sum_{\chi \in \mathrm{Irr}(B)} \chi(1)\chi, \tag{3}$$

and notice that ψ is a character of G that vanishes on p-singular elements by the weak block orthogonality relation (see [Navarro 1998, Corollary 3.7]). In particular, $\psi_P = f\rho_P$ for some natural number f, where ρ_P denotes the regular character of P.

Let $\mathrm{Irr}(B) = \{\chi_1, \dots, \chi_t\}$ and write $\chi_i(1) = p^{a-d+h_i} b_i$, where $|P| = p^a$, $|D| = p^d$, $h_i \geq 0$ is the height of χ_i and p does not divide b_i, for $1 \leq i \leq t$. Arrange the elements in $\mathrm{Irr}(B)$ in such a way that $\mathrm{Irr}_0(B) = \{\chi_1, \dots, \chi_k\}$, so that $h_j \geq 1$ for all $k+1 \leq j \leq t$. By [Navarro 1998, Theorem 3.28] we have that $\psi(1) = p^{2a-d}c$, where c is a nonnegative integer relatively prime to p. Thus, evaluating (3) at $1 \in G$ we obtain

$$p^d c = \sum_{i=1}^{k} b_i^2 + \sum_{j=k+1}^{t} p^{2h_j} b_j^2.$$

As $d \geq 1$, we get $\sum_{i=1}^{k} b_i^2 \equiv 0 \bmod p$. Since $p \in \{2, 3\}$, we have that $b_i^2 \equiv 1 \bmod p$ for every $1 \leq i \leq k$, and hence k is divisible by p.

Recall that the group $A = \langle \sigma_1 \rangle$ acts on $\mathrm{Irr}_0(B)$, and as such, we may view A as having order a power of p. Since $|\mathrm{Irr}_0(B)^{\sigma_1}| = |\mathrm{Irr}_0(B)^A|$, we obtain that p divides $|\mathrm{Irr}_0(B)^{\sigma_1}|$ by the class equation for group actions. □

The conclusion of the result above does not hold if $p > 3$, as the dihedral group D_{2p} provides a counterexample. Indeed, D_{2p} has a unique p-block and every irreducible character has p'-degree and is σ_1-fixed. Hence $|\mathrm{Irr}_{p'}(B_0(\mathsf{D}_{2p}))^{\sigma_1}| = |\mathrm{Irr}(\mathsf{D}_{2p})| = 2 + (p-1)/2 < p$.

Finally, we prove the main result of this section.

Theorem 1.5. *Let $p \in \{2, 3\}$. Let G be a finite group and let B be a p-block of G with a nontrivial normal defect group D. Then*

$$|\mathrm{Irr}_0(B)^{\sigma_1}| = p \quad \textit{if, and only if,} \quad D \textit{ is cyclic.}$$

In particular, Conjecture B follows from the Alperin–McKay–Navarro conjecture.

Proof. By Lemma 1.3 we know that the "if" implication holds. We now assume that $|\mathrm{Irr}_0(B)^{\sigma_1}| = p$ and we work to show that D is cyclic.

Write $C = \boldsymbol{C}_G(D)$ and let $\theta \in \mathrm{Irr}(CD)$ be the canonical character of B. Let $\{\lambda_i\}_{i=1}^t$ be a complete set of representatives of the G/CD-orbits on $\mathrm{Irr}(D/\Phi(D)) \setminus \{1_D\}$. By Lemma 1.2 we may assume that $G_\theta = G$ and

$$\mathrm{Irr}_0(B)^{\sigma_1} = \bigcup_{i=1}^{t} \mathrm{Irr}(G \mid \theta_{\lambda_i})$$

is a disjoint union. If $p = 2$, then

$$2 = |\mathrm{Irr}_0(B)^{\sigma_1}| = |\mathrm{Irr}(G \mid \theta)| + \sum_{i=1}^{t} |\mathrm{Irr}(G \mid \theta_{\lambda_i})|.$$

Since D is nontrivial by hypothesis, we have that $t \geq 1$. Thus $t = 1$ and the characters θ and θ_{λ_1} are fully ramified with respect to their inertia subgroups. In particular, there are positive integers e and e_1 such that $|G : C| = e^2$ and $|G_{\theta_{\lambda_1}} : C| = e_1^2$. Suppose that $|D| = 2^n$. Since G/CD acts transitively on the nontrivial elements of $D/\Phi(D)$, we have that $2^n - 1 = |G : G_{\lambda_1}| = (e/e_1)^2 = f^2$. The equality $f^2 + 1 = 2^n$ forces

f to be odd, then $f^2 \equiv 1 \mod 8$, and so $f^2+1 \equiv 2 \mod 8$ leaves as the only possibility $n = 1 = f$, that is, $D = \mathsf{C}_2$, as wanted. These techniques do not totally suffice to prove the case where $p = 3$. We first need to show that we may assume $\Phi(D) = 1$. Indeed, write $\overline{G} = G/\Phi(D)$, $\overline{D} = D/\Phi(D)$ and let $\overline{B}$ be a block of $\overline{G}$ contained in B such that $\overline{D}$ is the defect group of $\overline{B}$ by [Navarro 1998, Theorem 9.9]. Then $\mathrm{Irr}_0(\overline{B})^{\sigma_1} \subseteq \mathrm{Irr}_0(B)^{\sigma_1}$. By Lemma 1.2 we have that $\mathrm{Irr}_0(\overline{B})^{\sigma_1} = \mathrm{Irr}(\overline{B})$ is nonempty. Hence by Lemma 1.4, we have that p divides $|\mathrm{Irr}_0(\overline{B})^{\sigma_1}| \leq |\mathrm{Irr}_0(B)^{\sigma_1}| = p$, that forces $|\mathrm{Irr}_0(\overline{B})^{\sigma_1}| = p$. If $\Phi(D) \neq 1$ we can apply induction to obtain that $\overline{D}$ is cyclic, and thus D is cyclic. Hence we may assume that $\Phi(D) = 1$. Since D is p-elementary abelian, then $\mathrm{Irr}_0(B)^{\sigma_1} = \mathrm{Irr}(B)$ by the description of these sets in (1) and Lemma 1.2. By [Sambale 2014, Proposition 15.2], if $p = 3$ then $|\mathrm{Irr}(B)| = p$ implies $|D| = p$ and the proof is finished. □

2. Reducing to simple groups

The aim of this section is to reduce the statement of Theorem A to a problem on simple groups that we will solve in Section 3.

2A. ***Preliminaries.*** We start these preliminaries with results concerning the action of Galois automorphisms on characters belonging to principal blocks. Recall that $\chi \in \mathrm{Irr}(B_0(G))$ if, and only if,

$$\sum_{x \in G^0} \chi(x) \neq 0,$$

where G^0 is the subset of elements of G of order not divisible by p. Some properties of characters in the principal block are listed below.

Lemma 2.1. *Let G be a finite group, and let $N \triangleleft G$.*

(a) *We have that* $\mathrm{Irr}(B_0(G/N)) \subseteq \mathrm{Irr}(B_0(G))$, *with equality whenever N is a p'-group.*

(b) *If H_i are finite groups and $\gamma_i \in \mathrm{Irr}(B_0(H_i))$, for $i = 1, \ldots, t$, then $\gamma_1 \times \cdots \times \gamma_t \in \mathrm{Irr}(B_0(H_1 \times \cdots \times H_t))$.*

Proof. The first part of (a) and (b) follow directly from the definition of principal block [Navarro 1998, Definition 3.1]. The second part of (a) is [loc. cit., Theorem 9.9.(c)]. □

We summarize below some results obtained in Section 1, here stated with respect to the principal block. The first part was first observed by G. Navarro (in private communication).

Lemma 2.2. *Let G be a finite group and let P be a Sylow p-subgroup of G.*

(a) *If P is normal in G, then $\mathrm{Irr}_{p'}(B_0(G))^{\sigma_1} = \mathrm{Irr}(G/\boldsymbol{O}_{p'}(G)\Phi(P))$.*

(b) *If P is cyclic, then $1 \leq |\mathrm{Irr}_{p'}(B_0(G))^{\sigma_1}| \leq p$.*

(c) *If P is nontrivial and $p \in \{2, 3\}$, then $|\mathrm{Irr}_{p'}(B_0(G))^{\sigma_1}| \neq 0$ is divisible by p.*

Proof. To prove part (a), assume that $P \lhd G$. Then G is p-solvable and by Fong's theorem [Navarro 1998, Theorem 10.20] $\mathrm{Irr}(B_0(G)) = \mathrm{Irr}(G/\boldsymbol{O}_{p'}(G))$. Hence we may assume that $\boldsymbol{O}_{p'}(G) = 1$ and, in particular, $\boldsymbol{C}_G(P) \subseteq P$. By Lemma 1.2

$$\mathrm{Irr}_{p'}(B_0(G))^{\sigma_1} = \mathrm{Irr}_{p'}(G)^{\sigma_1} = \bigcup_{\lambda \in \mathrm{Irr}(P/\Phi(P))} \mathrm{Irr}(G \mid \lambda) = \mathrm{Irr}(G/\Phi(P)).$$

Part (b) is a straightforward application of Lemma 1.3. Part (c) is a direct consequence of Lemma 1.4. □

Next is a classical result by J. L. Alperin and E. C. Dade.

Theorem 2.3. *Suppose that N is a normal subgroup of G and G/N is a p'-group. Let $P \in \mathrm{Syl}_p(G)$ and assume that $G = N\boldsymbol{C}_G(P)$. Then restriction of characters defines a bijection* $\mathrm{Irr}(B_0(G)) \to \mathrm{Irr}(B_0(N))$. *In particular*, $|\mathrm{Irr}_{p'}(B_0(G))^{\sigma_1}| = |\mathrm{Irr}_{p'}(B_0(N))^{\sigma_1}|$.

Proof. The case where G/N is solvable was proved in [Alperin 1976, Lemma 1.1]. The general case in the main result of [Dade 1977]. The latter statement follows since σ_1 acts on $\mathrm{Irr}_{p'}(B_0(G))$. □

We will also use the following.

Lemma 2.4. *Suppose that G is a finite group, $P \in \mathrm{Syl}_p(G)$ and $P\boldsymbol{C}_G(P) \leq H \leq G$. If $\theta \in \mathrm{Irr}_{p'}(B_0(H))^{\sigma_1}$, then there exists a some $\chi \in \mathrm{Irr}_{p'}(B_0(G))^{\sigma_1}$ lying over θ.*

Proof. Note that $B_0(H)^G = B_0(G)$ by the comments before [Navarro 1998, Theorem 9.24] and Brauer's third main theorem [loc. cit., Theorem 6.7]. Write

$$\Psi = \sum_{\chi \in \mathrm{Irr}(B_0(G))} [\theta^G, \chi]\chi,$$

so that Ψ has p'-degree by [loc. cit., Theorem 6.4]. (Note that Ψ is exactly $(\theta^G)_B$ where $B = B_0(G)$ in the notation of [loc. cit.].) Let $A = \langle \sigma_1 \rangle$, where here we view σ_1 as an element of $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ for $o(\xi) = |G|$. For every $a \in A$ we have that $\Psi^a = \Psi$ as A acts on $\mathrm{Irr}(B_0(G))$ and fixes θ. By [Navarro and Tiep 2019, Lemma 2.1(ii)] there is some $\chi \in \mathrm{Irr}_{p'}(G)^{\sigma_1}$ appearing with p'-multiplicity in Ψ. The statement now follows since every irreducible constituent of Ψ lies in the principal block and its multiplicity in Ψ is exactly the multiplicity of θ in its restriction to H. □

We end the preliminaries with a technical result.

Lemma 2.5. *Let G be a finite group and let $N \lhd G$ be a direct product of t copies of a simple nonabelian group S transitively permuted by G. Let $P \in \mathrm{Syl}_p(G)$. If some $1_S \neq \phi \in \mathrm{Irr}_{p'}(B_0(S))^{\sigma_1}$ is X-invariant, where $X \in \mathrm{Syl}_p(\mathrm{Aut}(S))$, then there exists some P-invariant $1_N \neq \theta \in \mathrm{Irr}_{p'}(B_0(N))^{\sigma_1}$. In particular, if N is a minimal normal subgroup of G, then θ extends to a σ_1-invariant irreducible character of PN.*

Proof. Let $1_N \neq \theta \in \mathrm{Irr}(N)$ the character of N corresponding to $\phi \times \cdots \times \phi \in \mathrm{Irr}_{p'}(B_0(S)^t)^{\sigma_1}$, then $\theta \in \mathrm{Irr}_{p'}(B_0(N))^{\sigma_1}$ by Lemma 2.1(b). By [Navarro et al. 2007, Lemma 4.1(ii)], we may assume that θ is P-invariant.

For the second part of the statement, notice that since PN/N is a p-group and N is perfect, θ has a canonical extension $\hat{\theta} \in \mathrm{Irr}_{p'}(PN)$ by [Isaacs 1976, Corollary 6.28]. In particular, $\hat{\theta}$ is σ_1-invariant. □

2B. ***The reduction.*** Here we reduce Theorem A to a problem on simple groups, which is done in Theorem 2.6 below. Theorem C collects the properties of simple groups that will be key for performing such reduction. We would like to remark that the conditions in Theorem C related to the conjugation by group automorphisms are not needed in this context, but may be of independent interest.

Theorem 2.6. *Let G be a finite group of order divisible by p where $p \in \{2, 3\}$. Let $P \in \mathrm{Syl}_p(G)$. Then*

$$|\mathrm{Irr}_{p'}(B_0(G))^{\sigma_1}| = p \quad \textit{if, and only if,} \quad P \textit{ is cyclic.}$$

Proof. If P is cyclic, then $|\mathrm{Irr}_{p'}(B_0(G))^{\sigma_1}| = p$ by Lemma 1.3.

We assume now that $|\mathrm{Irr}_{p'}(B_0(G))^{\sigma_1}| = p$ and we work to prove that P is cyclic by induction on the order of G.

First, notice that we may assume that G is not simple, by Theorem C(a), and $\boldsymbol{N}_G(P) < G$ by Theorem 1.5.

Step 1. *We may assume $\boldsymbol{O}_{p'}(G) = 1$.* This follows by Lemma 2.1(a) and induction.

Step 2. *We may assume that $\boldsymbol{O}^{p'}(G) = G$.* Otherwise, let $M \lhd G$ with $|G/M|$ not divisible by p and $G/M > 1$ simple. Then $P \subseteq M$ and by the Frattini argument $M\boldsymbol{N}_G(P) = G$. Hence $M\boldsymbol{C}_G(P) \lhd G$ and therefore $G = \boldsymbol{C}_G(P)M$ or $\boldsymbol{C}_G(P) \subseteq M$. Suppose $G = M\boldsymbol{C}_G(P)$, then restriction defines a bijection $\mathrm{Irr}_{p'}(B_0(G))^{\sigma_1} \to \mathrm{Irr}_{p'}(B_0(M))^{\sigma_1}$ by Theorem 2.3. In this case we are done by induction. Therefore we may assume that $\boldsymbol{C}_G(P) \subseteq M$. We claim that $B_0(G)$ is the only block of G covering $B_0(M)$. Indeed, let B be a block of G covering $B_0(M)$. By [Navarro 1998, Theorem 9.26], we have that P is a defect group of B. By [loc. cit., Lemma 9.20], B is regular with respect to M and hence by [loc. cit., Theorem 9.19], $B_0(M)^G = B$. By Brauer's third main theorem we have that $B_0(M)^G = B_0(G)$ and hence $B = B_0(G)$ and the claim is proven. In particular, $\mathrm{Irr}(G/M) \subseteq \mathrm{Irr}_{p'}(B_0(G))^{\sigma_1}$ as every character in $\mathrm{Irr}(G/M)$ has p'-degree and is σ_1-invariant (for G/M is a p'-group). By hypothesis $|\mathrm{Irr}(G/M)| \leq p$. As G/M is a nontrivial p'-group, we immediately get a contradiction if $p = 2$. If $p = 3$, then $|\mathrm{Irr}(G/M)| \leq 3$ forces $G/M = \mathsf{C}_2$. Write $\mathrm{Irr}_{p'}(B_0(G))^{\sigma_1} = \{1, \lambda, \theta\}$ with $M \subseteq \ker\lambda$, for instance. Let $\tau \in \mathrm{Irr}_{p'}(B_0(M))^{\sigma_1}$ be nontrivial by Lemma 1.4. Let $\chi \in \mathrm{Irr}(B_0(G))$ be over τ. Since $|G/M|$ is not divisible by p, we have that $\chi \in \mathrm{Irr}_{p'}(B_0(G))^{\sigma_1}$ by [Navarro and Tiep 2019, Lemma 5.1]. Thus necessarily $\chi = \theta$. Since θ_M has at most two irreducible constituents, we have that $|\mathrm{Irr}_{p'}(B_0(M))^{\sigma_1}| = 3$ and we are done by induction in this case.

Step 3. *If $1 \neq M \lhd G$, then every $\chi \in \mathrm{Irr}_{p'}(B_0(G))^{\sigma_1}$ satisfies $M \subseteq \ker\chi$ and $PM/M > 1$ is cyclic.* By Step 2, we have that p divides $|G/M|$ and hence $|\mathrm{Irr}_{p'}(B_0(G/M))^{\sigma_1}| = p$ follows from Lemma 1.4. The claim of the step now follows from Lemma 2.1(a) and by induction.

Step 4. *If $1 \neq M \lhd G$ and $\gamma \in \mathrm{Irr}_{p'}(B_0(MP))^{\sigma_1}$, then there is some $\chi \in \mathrm{Irr}_{p'}(B_0(G))^{\sigma_1}$ lying over γ.* Write $H = MP\boldsymbol{C}_G(P)$, so that $MP \lhd H$. By Theorem 2.3, restriction defines a bijection $\mathrm{Irr}_{p'}(B_0(H))^{\sigma_1} \to \mathrm{Irr}_{p'}(B_0(MP))^{\sigma_1}$, and hence some $\theta \in \mathrm{Irr}_{p'}(B_0(H))^{\sigma_1}$ extends γ. By Lemma 2.4, the claim of the step follows.

Step 5. *Let N be a minimal normal subgroup of G. We may assume $PN < G$.* Suppose the contrary. By Step 1 and the fact that $\mathbf{N}_G(P) < G$ (so G is not a p-group), we have that N is the direct product of t copies of a nonabelian simple group S of order divisible by p (which are transitively permuted by G). By Theorem C there exist $1_S \neq \phi \in \mathrm{Irr}_{p'}(B_0(S))^{\sigma_1}$ X-invariant for some $X \in \mathrm{Syl}_p(\mathrm{Aut}(S))$. By Lemma 2.5, there is some $1_N \neq \theta \in \mathrm{Irr}_{p'}(B_0(N))^{\sigma_1}$ that extends to a σ_1-invariant character $\chi \in \mathrm{Irr}(G)$. Since $B_0(G)$ is the only block covering $B_0(N)$ by [Navarro 1998, Corollary 9.6], we have that $\chi \in \mathrm{Irr}_{p'}(B_0(G))^{\sigma_1}$ contradicting Step 3.

Final Step. Since $NP < G$ by Step 5, if $|\mathrm{Irr}_{p'}(B_0(NP))^{\sigma_1}| = p$, then we are done by induction. Hence we may assume that $|\mathrm{Irr}_{p'}(B_0(NP))^{\sigma_1}| > p$ by Lemma 1.4. By Step 3, we have that PN/N is cyclic, and hence $|\mathrm{Irr}_{p'}(B_0(PN/N))^{\sigma_1}| = p$. Therefore there exists some $\theta \in \mathrm{Irr}_{p'}(B_0(NP))^{\sigma_1}$ such that $N \nsubseteq \ker(\theta)$ (here we are using that NP/N has just one p-block). By Step 4, some $\chi \in \mathrm{Irr}_{p'}(B_0(G))^{\sigma_1}$ lies over θ. In particular $N \nsubseteq \ker \chi$, a contradiction with Step 3. □

3. Simple groups

In this Section we prove Theorem C, which will complete the proof of Theorem A.

3A. ***Some generalities on groups of Lie type.*** Since the groups of Lie type play a large role in what follows, we begin by recalling some essentials about their blocks and characters.

Let q be a power of a prime. When $G = \mathbf{G}^F$ is the group of fixed points of a connected reductive algebraic group $\mathbf{G}$ defined over $\overline{\mathbb{F}}_q$ under a Steinberg map F, the set of irreducible characters $\mathrm{Irr}(G)$ can be written as a disjoint union $\bigsqcup \mathcal{E}(G, s)$ of so-called rational Lusztig series corresponding to G^*-conjugacy classes of semisimple elements $s \in G^*$ (i.e., elements of order relatively prime to q). Here $G^* = (\mathbf{G}^*)^{F^*}$, where $(\mathbf{G}^*, F^*)$ is dual to $(\mathbf{G}, F)$.

With this notation, we record the following lemma, proved in [Schaeffer Fry and Taylor 2018, Lemma 3.4], which describes the action of $\mathcal{H}$ on the set of rational Lusztig series and will be useful throughout this section.

Lemma 3.1. *Let p be a prime and let $s \in G^*$ be a semisimple element. Let f and b be integers and let $\sigma \in \mathcal{H}$ be such that $\sigma(\xi) = \xi^{p^f}$ for all p'-roots of unity ξ and $\sigma(\zeta) = \zeta^b$ for all p-power roots of unity ζ. If $\chi \in \mathcal{E}(G, s)$, then $\chi^\sigma \in \mathcal{E}(G, s_{p'}^{p^f} s_p^b)$.*

The characters in the series $\mathcal{E}(G, 1)$ are called *unipotent* characters, and there is a bijection $\mathcal{E}(G, s) \to \mathcal{E}(\mathbf{C}_{G^*}(s), 1)$. Hence, characters of $\mathrm{Irr}(G)$ may be indexed by pairs (s, ψ), where $s \in G^*$ is a semisimple element, up to G^*-conjugacy, and $\psi \in \mathrm{Irr}(\mathbf{C}_{G^*}(s))$ is a unipotent character of $\mathbf{C}_{G^*}(s)$. We remark that $\mathbf{C}_{G^*}(s)$ may fail to be connected, in which case unipotent characters of $\mathbf{C}_{G^*}(s)$ are taken to be those lying over a unipotent character of $(\mathbf{C}_{G^*}(s)^\circ)^F$. In particular, we will denote by χ_s the character indexed by $(s, 1_{\mathbf{C}_{G^*}(s)})$, which are *semisimple*, and they have degree $|G^* : \mathbf{C}_{G^*}(s)|_{q'}$.

Using [Cabanes and Enguehard 2004, Theorem 9.12], it follows that when $p \nmid q$, the set $\mathcal{E}_p(G, 1) := \bigcup \mathcal{E}(G, s)$, where s ranges over elements of p-power order in G^*, is a union of p-blocks (first shown in [Broué and Michel 1989]) and that each such block intersects $\mathcal{E}(G, 1)$ nontrivially. Such blocks are called *unipotent blocks*.

3A1. *A general set-up.* We will often be interested in the following situation: Let S be a simple group such that there exist $\boldsymbol{G}$ a simple, simply connected algebraic group over $\overline{\mathbb{F}}_q$ and F a Steinberg morphism satisfying $S = G/\boldsymbol{Z}(G)$ with $G := \boldsymbol{G}^F$ perfect. Let $(\boldsymbol{G}^*, F^*)$ be dual to $(\boldsymbol{G}, F)$.

If $\boldsymbol{Z}(G)$ is trivial, we define $\tilde{\boldsymbol{G}} := \boldsymbol{G}$. Otherwise, we further let $\iota\colon \boldsymbol{G} \hookrightarrow \tilde{\boldsymbol{G}}$ be a regular embedding as in [Cabanes and Enguehard 2004, 15.1] and let $\iota^*\colon \tilde{\boldsymbol{G}}^* \to \boldsymbol{G}^*$ be the corresponding surjection of dual groups. Write $\tilde{G} := \tilde{\boldsymbol{G}}^F$, $G^* := (\boldsymbol{G}^*)^{F^*}$, and $\tilde{G}^* := (\tilde{\boldsymbol{G}}^*)^{F^*}$. We may then find F-stable maximally split tori $\boldsymbol{T}$ and $\tilde{\boldsymbol{T}}$ for $\boldsymbol{G}$ and $\tilde{\boldsymbol{G}}$, respectively, such that $\boldsymbol{T} \leq \tilde{\boldsymbol{T}}$. Write $T := \boldsymbol{T}^F$ and $\tilde{T} := \tilde{\boldsymbol{T}}^F$. Then $\boldsymbol{Z}(\tilde{\boldsymbol{G}})$ is connected, $G \triangleleft \tilde{G}$, and $\boldsymbol{Z}(\tilde{G}) \cap G = \boldsymbol{Z}(G)$. We will write $\tilde{S} := \tilde{G}/\boldsymbol{Z}(\tilde{G})$, and note that $\mathrm{Aut}(S)$ is generated by $\tilde{S}$ and the graph-field automorphisms. Further, the (linear) characters of $\tilde{G}/G$ are in bijection with elements of $\boldsymbol{Z}(\tilde{G}^*)$, and we have $\tilde{\chi}_s \otimes \hat{z} = \tilde{\chi}_{sz}$, where $z \in \boldsymbol{Z}(\tilde{G}^*)$ corresponds to $\hat{z} \in \mathrm{Irr}(\tilde{G}/G)$ and for semisimple $s \in \tilde{G}^*$, $\tilde{\chi}_s$ denotes the semisimple character of $\tilde{G}$ corresponding to s. (See [Digne and Michel 1991, 13.30].) It will also be useful in what follows to note that if $s \in [\tilde{G}^*, \tilde{G}^*]$ is semisimple, then the semisimple character of $\tilde{G}$ corresponding to s is trivial on $\boldsymbol{Z}(\tilde{G})$ by [Navarro and Tiep 2013, Lemma 4.4].

When q is a power of p, we note that $\mathrm{Irr}_{p'}(B_0(S)) = \mathrm{Irr}_{p'}(S)$, which can be seen using [Cabanes and Enguehard 2004, 6.14, 6.15, and 6.18] and the facts that $p \nmid |\boldsymbol{Z}(G)|$ and S is a group with a strongly split BN pair as in [loc. cit., 2.20].

In the case of types A_{n-1} and ${}^2A_{n-1}$, we have S is $\mathrm{PSL}_n^\epsilon(q)$ with $\epsilon \in \{\pm 1\}$; $G = \mathrm{SL}_n^\epsilon(q)$; $\tilde{G} = \mathrm{GL}_n^\epsilon(q)$; and $\tilde{S} = \mathrm{PGL}_n^\epsilon(q)$. Here $\epsilon = 1$ means $S = \mathrm{PSL}_n(q)$, $\epsilon = -1$ means $S = \mathrm{PSU}_n(q)$, and similarly for G and $\tilde{G}$. We use similar notation for other twisted types. For example, $E_6^\epsilon(q)$ will denote $E_6(q)$ for $\epsilon = +$ and ${}^2E_6(q)$ for $\epsilon = -$.

3B. ***The case $p = 2$.*** Here we prove Theorem C in the case $p = 2$. The following, found in [Navarro et al. 2018, Lemma 3.1], will be useful in what follows.

Lemma 3.2 (Navarro, Sambale and Tiep). *Let G be a finite group. If $\chi \in \mathrm{Irr}_{2'}(G)$ is real-valued, then χ belongs to $B_0(G)$.*

In particular, note that an odd character degree of G with multiplicity one must necessarily come from a character fixed by all automorphisms and $\mathcal{G}$, which is therefore an X-invariant member of $\mathrm{Irr}_{2'}(B_0(G))^{\sigma_1}$.

Lemma 3.3. *Let S be a simple sporadic group, alternating group $\mathfrak{A}_n$ with $n \geq 5$, or one of the simple groups* $\mathrm{PSL}_2(4)$, $\mathrm{PSL}_3(2)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_4(3)$, $\mathrm{PSL}_6^\epsilon(2)$, ${}^2B_2(8)$, $B_3(2)$, $B_3(3)$, $D_4(2)$, $F_4(2)$, ${}^2F_4(2)'$, $E_6(2)$, ${}^2E_6(2)$, $G_2(2)'$, $G_2(3)$, *or* $G_2(4)$. *Then Theorem C holds for S and the prime $p = 2$.*

Proof. For $n \geq 7$, the automorphism group of $\mathfrak{A}_n$ is the symmetric group $\mathfrak{S}_n$. Recall that every irreducible character of $\mathfrak{S}_n$ is rational-valued and that an odd-degree character of $\mathfrak{S}_n$ must restrict irreducibly to $\mathfrak{A}_n$ since it has index 2. In this case, if $n = 2^{n_1} + \cdots + 2^{n_t}$ with $n_1 < n_2 < \cdots < n_t$ is the 2-adic decomposition of n, then [Macdonald 1971, Corollary 1.3] yields that there are $2^{n_1 + \cdots + n_t} \geq 8$ odd-degree characters of $\mathfrak{S}_n$, whose restrictions therefore yield at least 3 nontrivial members of $\mathrm{Irr}_{2'}(B_0(\mathfrak{A}_n))^{\sigma_1}$ invariant under $\mathrm{Aut}(\mathfrak{A}_n)$.

For the remaining groups listed, the statement can be seen using [GAP 2004] and the GAP character table library. In fact, we see that for the sporadic groups other than the Janko groups, there exist at least two nontrivial odd character degrees with multiplicity 1. □

Proposition 3.4. *Let S be a simple group of Lie type defined over $\mathbb{F}_q$ with q a power of an odd prime ℓ. Then Theorem C holds for S and the prime $p = 2$.*

Proof. We may assume that S is not isomorphic to any of the groups in Lemma 3.3, so is as in Section 3A1. In this case, the Steinberg character is rational-valued and $\mathrm{Aut}(S)$-invariant, and therefore it suffices to show that there is another nontrivial member of $\mathrm{Irr}_{2'}(B_0(S))^{\sigma_1}$. Further, we note that if S is not a Suzuki or Ree group, then unipotent characters of odd degree are rational-valued (see, e.g., [Schaeffer Fry 2019, Lemma 4.4]). Hence in these cases, applying Lemma 3.2, it suffices to find another nontrivial unipotent character of odd degree, when possible. By observing the explicit list of unipotent character degrees in [Carter 1985, Section 13.9], we see that there is at least one other nontrivial odd-degree unipotent character for the exceptional groups $G_2(q)$, ${}^3D_4(q)$, $F_4(q)$, $E_6^\epsilon(q)$, $E_7(q)$, and $E_8(q)$. For ${}^2G_2(q)$, we see from the generic character table in [Geck et al. 1996] that there is another odd degree with multiplicity one.

For the classical groups $A_{n-1}(q)$, ${}^2A_{n-1}(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$, or ${}^2D_n(q)$, we know by [Malle and Späth 2016, Proposition 7.4] that all unipotent characters of G with odd degree lie in the principal series, and hence are in bijection with the odd-degree irreducible characters of the Weyl group W of G. In these cases, W contains a quotient isomorphic to $\mathfrak{S}_n$, which has at least 4 odd-degree characters for $n \geq 4$, again using [Macdonald 1971, Corollary 1.3]. We also see, for example using the GAP, that there are at least 4 odd-degree characters of W in the case of B_2, B_3, and C_3. Using the well-known character table for $\mathrm{PSL}_2(q)$, we see that all four odd-degree characters are fixed by σ_1. Further, in this case, $\mathrm{Irr}_{2'}(S) = \mathrm{Irr}_{2'}(B_0(S))$. We see from part (iii) of the proof of [Navarro et al. 2018, Theorem 3.3] that if $S = \mathrm{PSL}_3^\epsilon(q)$, then the Weil character $\zeta_{3,q}^{(q-\epsilon)/2}$ is a member of $\mathrm{Irr}_{2'}(B_0(S))$ and is real-valued. □

Proposition 3.5. *Let S be a simple group of Lie type defined in characteristic 2. Then Theorem C holds for S for the prime $p = 2$.*

Proof. Again, we may assume that S is not as in Lemma 3.3. In particular, we may keep the notation as in Section 3A1 and we have $\mathrm{Irr}_{2'}(B_0(S)) = \mathrm{Irr}_{2'}(S)$. If S is ${}^2B_2(q)$ or ${}^2F_4(q)$, then the generic character tables available in CHEVIE yield the result, since $|\mathrm{Out}(S)|$ is odd and there are at least two distinct degrees of nontrivial odd-degree characters whose values are fixed by σ_1.

Otherwise, we may take the Steinberg endomorphism on $\boldsymbol{G}$ to be $F = F_q \circ \tau$, where F_q is the standard Frobenius induced by the map $x \mapsto x^q$ and τ is some graph automorphism. Write $\bar{q} := q^{|\tau|} = 2^{2^b m}$ with m odd and let $X \leq \mathrm{Aut}(S)$ such that $X/S \in \mathrm{Syl}_2(\mathrm{Out}(S))$.

Since q is a power of 2, we have $\boldsymbol{Z}(G) = 1$ and $\tilde{G} = G$ unless S is one of $\mathrm{PSL}_n^\epsilon(q)$ or $E_6^\epsilon(q)$. In the latter cases, $G = [\tilde{G}, \tilde{G}]$. In any case, since $\tilde{G}/G$ has odd order, we may view X/S as generated by F_2^m and graph automorphisms.

Now, if $m > 1$, then the proof of [Schaeffer Fry and Taylor 2018, Lemma 6.4] (and taking into account the remark after [loc. cit., Proposition 6.5]) yields a member of $\mathrm{Irr}_{2'}(S)$ invariant under X which is the restriction to G of a semisimple character of $\tilde{G}$ trivial on $\boldsymbol{Z}(G)$. Since semisimple elements have odd order and σ_1 fixes odd roots of unity, Lemma 3.1 shows that this character is also fixed by σ_1. If $m = 1$, we may similarly obtain an X-invariant member of $\mathrm{Irr}_{2'}(S)$ fixed by σ_1 by arguing as in [loc. cit., Lemma 6.4] and the remark after [loc. cit., Proposition 6.5] but using an element of $\mathbb{F}_4^\times$ of order 3 rather than an element of $\bar{\mathbb{F}}_q^\times$ of order 5.

For $S = G_2(q)$, $F_4(q)$, ${}^3D_4(q)$, $E_7(q)$, or $E_8(q)$, the list of character degrees at [Lübeck 2007] yields at least one more distinct nontrivial odd character degree, completing the proof in these cases, since by [Malle 2007, Theorem 6.8], odd-degree characters are semisimple (recall that we may assume $q \neq 2$ when $S = G_2(q)$ or $F_4(q)$), and hence fixed by σ_1 using Lemma 3.1.

Now, in the remaining cases, S is a classical group or $E_6^\epsilon(q)$. Here $\tilde{G}^* \cong \tilde{G}$. In the case $S = \mathrm{PSL}_2(q)$ or $\mathrm{PSL}_3^\epsilon(q)$, we see that there is at least one more odd-degree character with a different degree that is fixed by σ_1, using the generic character tables available in [Geck et al. 1996]. If $\tilde{G} = \mathrm{GL}_n^\epsilon(q)$, $\mathrm{Sp}_n(q)$, or $\Omega_n^\epsilon(q)$ with $n \geq 4$ and n even in the latter two cases, let s_1 and s_2 be elements of $\tilde{G}$ with eigenvalues $\{\delta, \delta^{-1}, 1, \dots, 1\}$ and $\{\delta, \delta, \delta^{-1}, \delta^{-1}, 1, \dots, 1\}$, respectively, where $1 \neq \delta \in \mathbb{F}_{\bar{q}}^\times$.

Then s_1 and s_2 are not $\mathrm{Aut}(S)$-conjugate, and hence the corresponding semisimple characters of $\tilde{G}$ have odd degree, are not $\mathrm{Aut}(S)$-conjugate, and are fixed by σ_1 by Lemma 3.1. Further, if $\tilde{G} = \mathrm{GL}_n^\epsilon(q)$, semisimple classes are determined by the eigenvalues, and $\boldsymbol{Z}(\tilde{G})$ is comprised of scalar matrices, so we see for $i = 1, 2$, s_i is not conjugate to $s_i z$ for any $z \in \boldsymbol{Z}(\tilde{G})$ unless possibly if $n = 6$. In this case, we may assume $q \neq 2$ using Lemma 3.3 and instead take $\delta \in \mathbb{F}_{q^2}^\times$ to have order at least 5, again yielding s_i is not conjugate to $s_i z$ for any $z \in \boldsymbol{Z}(\tilde{G})$. In any case, the corresponding semisimple characters therefore restrict irreducibly to G and are trivial on $\boldsymbol{Z}(\tilde{G})$ since $s_i \in [\tilde{G}, \tilde{G}] = G$. Finally, let S be $E_6^\epsilon(q)$ with $q > 2$. Then we may argue analogously to [Giannelli et al. 2020, Proposition 4.3] to find elements s_1 and s_2 in $\tilde{G}$ with $|\boldsymbol{C}_{\tilde{G}}(s_1)|_2 \neq |\boldsymbol{C}_{\tilde{G}}(s_2)|_2$ such that the corresponding semisimple characters (which again must be fixed by σ_1) are irreducible on G and trivial on $\boldsymbol{Z}(\tilde{G})$. (Indeed, we may replace the δ used there with a $\delta \in \mathbb{F}_{q^2}^\times$ such that $3 \nmid |\delta|$). In all cases, this yields at least one more nontrivial member of $\mathrm{Irr}_{2'}(B_0(S))^{\sigma_1}$ that is not $\mathrm{Aut}(S)$-conjugate to the X-invariant one from above. □

Theorem C for $p = 2$ now follows by combining Propositions 3.4 and 3.5 with Lemma 3.3.

3C. *The case p = 3.* Here we prove Theorem C in the case $p = 3$. We begin by stating the following classification of simple groups with cyclic Sylow 3-subgroups.

Proposition 3.6. *Let S be a finite nonabelian simple group with order divisible by* 3*. Then S has a cyclic Sylow* 3*-subgroup if and only if S is one of: the alternating group $\mathfrak{A}_5$; the sporadic simple group J_1;* $\mathrm{PSL}_2(q)$ *for* $3 \nmid q$; *or* $\mathrm{PSL}_3^\epsilon(q)$ *for* $3 \mid (q + \epsilon)$.

Proof. The main result of [Shen and Zhou 2016] yields a classification of simple groups S and primes p such that S has an abelian Sylow p-subgroup. In particular, if $p = 3$, then such a simple group must be of the form $\mathfrak{A}_n$ with $n < 9$, one of a short list of sporadic simple groups, $\mathrm{PSL}_2(q)$, $\mathrm{PSL}_n^\epsilon(q)$ for $3 \mid (q + \epsilon)$ and $n = 3, 4, 5$, or $\mathrm{PSp}_4(q)$ with $3 \nmid q$.

Using the atlas [Conway et al. 1985] and since $\mathfrak{A}_6$ has a noncyclic Sylow 3-subgroup and can be viewed as a subgroup of $\mathfrak{A}_n$ for $n \geq 7$, we see that the only simple alternating or sporadic groups with cyclic Sylow 3-subgroups are $\mathfrak{A}_5$ and the Janko group J_1. The remaining possibilities are of the form $G/\boldsymbol{Z}(G)$ for G a classical group $\mathrm{SL}_n^\epsilon(q)$ with $n < 6$, or $\mathrm{Sp}_4(q)$. Further, except in the cases of $\mathrm{PSL}_3^\epsilon(q)$ listed in the statement, $|\boldsymbol{Z}(G)|$ is relatively prime to 3, and hence S has a cyclic Sylow 3-subgroup if and only if G does. Further, for the cases $G = \mathrm{SL}_n^\epsilon(q)$ with $n = 3, 4, 5$ under consideration, we may view the Sylow subgroup as a Sylow subgroup of $\tilde{G} = \mathrm{GL}_n^\epsilon(q)$, since $[\tilde{G} : G]$ is not divisible by 3.

Now, using the description of the Sylow subgroups of classical groups in [Carter and Fong 1964; Weir 1955], we see that the Sylow subgroups of $\mathrm{GL}_4^\epsilon(q)$, $\mathrm{GL}_5^\epsilon(q)$, and $\mathrm{Sp}_4(q)$ are direct products of Sylow subgroups of at least two lower-rank groups, and hence the Sylow 3-subgroup of G is not cyclic. In the case $\mathrm{PSL}_3^\epsilon(q)$ with $3 \mid (q+\epsilon)$ or $\mathrm{PSL}_2(q)$ with $3 \nmid q$, we may explicitly construct a cyclic Sylow 3-subgroup. Finally, if $S = \mathrm{PSL}_2(q)$ with $3 \mid q$, the Sylow 3-subgroup can be identified with the unipotent radical of $\mathrm{SL}_2(q)$, which is not cyclic unless $q = 3$, contradicting that S is simple. □

Our goal in the remainder of this section is to prove the following, from which we obtain Theorem C for $p = 3$ as a corollary.

Theorem 3.7. *Let S be a nonabelian simple group with order divisible by* 3.

(i) *If S has a cyclic Sylow* 3*-subgroup, then there exist $1_S \neq \chi_1, \chi_2 \in \mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ such that χ_1 extends to* $\mathrm{Aut}(S)$.

(ii) *If S does not have a cyclic Sylow* 3*-subgroup and is not a group of Lie type defined in characteristic* 3*, then there exist nontrivial $\chi_1, \chi_2, \chi_3 \in \mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ such that χ_1 and χ_2 extend to* $\mathrm{Aut}(S)$. *In this case, if S is further not one of $\mathfrak{A}_6$, $\mathfrak{A}_7$, ${}^2F_4(2)'$, $\mathrm{PSL}_n(q)$ with $n \leq 4$, or $\mathrm{PSp}_4(2^{2m+1})$, then there exist nontrivial $\chi_1, \chi_2, \chi_3 \in \mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ such that χ_i each extend to* $\mathrm{Aut}(S)$.

(iii) *If S is a group of Lie type in characteristic* 3*, then there exist nontrivial $\chi_1, \chi_2, \chi_3 \in \mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ that are pairwise not $\mathrm{Aut}(S)$-conjugate and such that χ_1 is invariant under X, where $X/S \in \mathrm{Syl}_3(\mathrm{Aut}(S)/S)$.*

We first consider Theorem 3.7 for sporadic and alternating groups, as well as some "small" groups of Lie type. For two positive integers n and m, we will use $n \parallel m$ to mean that $n \mid m$ and $\gcd(n, m/n) = 1$.

Proposition 3.8. *Theorem 3.7 holds for the sporadic simple groups, $G_2(3)$, ${}^2F_4(2)'$, $B_3(3)$, $G_2(2)' = \mathrm{PSU}_3(3)$, $\mathrm{PSU}_4(3)$, and the alternating groups $\mathfrak{A}_n$ with $n \geq 5$.*

Proof. Since the result can be seen directly using GAP for the other cases, we may assume $S = \mathfrak{A}_n$ with $n > 10$. In this case, S does not have a cyclic Sylow 3-subgroup and satisfies $\mathrm{Aut}(S) = \mathfrak{S}_n$, where $\mathfrak{S}_n$ denotes the corresponding symmetric group.

The characters of $\mathfrak{S}_n$ are rational-valued and parametrized by partitions of n, with their degrees given by the hook formula. Further, two characters lie in the same 3-block if and only if they have the same 3-core. We also know that $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$ corresponding to the partition λ restricts irreducibly to $\mathfrak{A}_n$ if and only if

Condition on n	Partition	$\chi(1)$
$3 \mid n$	$(1, n-1)$	$n-1$
$3 \mid n$	$(1, 1, n-2)$	$(n-1)(n-2)/2$
$3 \parallel n$, $3^2 \mid (n-2)$, or $3 \parallel (n-1)$	$(3, n-3)$	$n(n-1)(n-5)/6$
$3^2 \mid n$, $3 \parallel (n-2)$, or $3 \parallel (n-1)$	$(1^3, n-3)$	$(n-1)(n-2)(n-3)/6$
$3 \mid (n-1)$	$(2, n-2)$	$n(n-3)/2$
$3^2 \mid (n-1)$	$(1, 2, n-3)$	$n(n-2)(n-4)/3$
$3^2 \mid (n-1)$	$(1^3, 2, n-5)$	$n(n-2)(n-3)(n-4)(n-6)/30$
$3 \mid (n-2)$	$(1^{n-4}, 2, 2)$	$n(n-3)/2$
$3 \mid (n-2)$	$(1^{n-2}, 2)$	$n-1$

Table 1. Some members of $\mathrm{Irr}_{3'}(B_0(\mathfrak{S}_n))$ irreducible on $\mathfrak{A}_n$, $n > 10$.

the partition is not self-conjugate. Table 1 lists the partitions and character degrees for three characters in $\mathrm{Irr}_{3'}(\mathfrak{S}_n)$ that restrict irreducibly to $\mathfrak{A}_n$, completing the proof. □

3C1. *Lie type in cross-characteristic for $p = 3$.* In this section, we prove Theorem 3.7 for groups of Lie type in nondefining characteristic. That is, we deal with the case S is of the form $G/\boldsymbol{Z}(G)$ for G a finite group of Lie type of simply connected type defined over a field $\mathbb{F}_q$ with $3 \nmid q$. (Given Proposition 3.8, this will complete the proof of parts (i) and (ii) of Theorem 3.7.)

We will use Φ_m to denote the m-th cyclotomic polynomial in the variable q. Note that using e.g., [Malle 2007, Lemma 5.2], 3 divides Φ_m if and only if $m = 3^i d$ for some $i \geq 0$, where d is the order of q modulo 3, and in this case $3 \parallel \Phi_m$ unless $m = d$.

Proposition 3.9. *Let S be a simple group of Lie type defined over $\mathbb{F}_q$ with $3 \nmid q$ and assume S is not one of the groups $\mathrm{PSL}_n^\epsilon(q)$ with $n \leq 3$. Then there exist three nontrivial characters $\chi_1, \chi_2, \chi_3 \in \mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ such that χ_1 and χ_2 extend to $\mathrm{Aut}(S)$.*

Further, if S is not $\mathrm{PSL}_4^\epsilon(q)$ nor $\mathrm{PSp}_4(2^a)$ with a odd, then χ_1, χ_2, and χ_3 may be chosen to extend to $\mathrm{Aut}(S)$.

Proof. We may assume that S is not isomorphic to one of the groups considered in Proposition 3.8. Keep the notation and considerations for G, $\tilde{G}$, $\tilde{T}$, and $\tilde{S}$ from Section 3A. By the work of Lusztig [1988], the unipotent characters of $\tilde{G}$ are trivial on $\boldsymbol{Z}(\tilde{G})$ and restrict irreducibly to G. Further, when viewed as characters of $\tilde{S}$, they are extendible to $\mathrm{Aut}(S)$, by [Malle 2008, Theorems 2.4 and 2.5], aside from some specific exceptions. The only unipotent characters which take irrational values occur for exceptional groups and have values in $\mathbb{Q}(\sqrt{-1}, \zeta_3, \zeta_5, \sqrt{q})$, where ζ_3 and ζ_5 are third and fifth roots of unity, respectively, by [Geck 2003, Proposition 5.6 and Table 1]. In any case, the unipotent characters are σ_1-invariant, since $\sqrt{q}$ is a sum of roots of unity of order relatively prime to 3.

Let d be the order of q modulo 3. In particular, we have $d = 1$ or 2. If $d = 1$, unipotent characters of degree relatively prime to 3 are constituents of the Harish-Chandra induced character $R_{\tilde{T}}^{\tilde{G}}(1)$ using [Malle

Type	Condition on d	Character (notation from [Carter 1985, 13.9])	$\chi(1)$
$G_2(q)$	$d=1$	$\phi_{2,2}$	$\frac{1}{2}q\Phi_2^2\Phi_6$
		$\phi_{1,3'}$	$\frac{1}{3}q\Phi_3\Phi_6$
	$d=2$	$G_2[1]$	$\frac{1}{6}q\Phi_1^2\Phi_6$
		$\phi_{1,3'}$	$\frac{1}{3}q\Phi_3\Phi_6$
${}^3D_4(q)$	$d=1,2$	$\phi_{1,3'}$	$q\Phi_{12}$
		$\phi_{1,3''}$	$q^7\Phi_{12}$
$F_4(q)$	$d=1$	$\phi_{4,1}$	$\frac{1}{2}q\Phi_2^2\Phi_6^2\Phi_8$
		$\phi_{8,3'}$	$q^3\Phi_4^2\Phi_8\Phi_{12}$
	$d=2$	$B_{2,\epsilon}$	$\frac{1}{2}q^{13}\Phi_1^2\Phi_3^2\Phi_8$
		$B_{2,1}$	$\frac{1}{2}q\Phi_1^2\Phi_3^2\Phi_8$
$E_6(q)$	$d=1,2$	$\phi_{20,2}$	$q^2\Phi_4\Phi_5\Phi_8\Phi_{12}$
		$\phi_{20,20}$	$q^{20}\Phi_4\Phi_5\Phi_8\Phi_{12}$
${}^2E_6(q)$	$d=1,2$	$\phi_{4,1}$	$q^2\Phi_4\Phi_8\Phi_{10}\Phi_{12}$
		$\phi_{4,13}$	$q^{20}\Phi_4\Phi_8\Phi_{10}\Phi_{12}$
$E_7(q)$	$d=1,2$	$\phi_{7,1}$	$q\Phi_7\Phi_{12}\Phi_{14}$
		$\phi_{7,46}$	$q^{46}\Phi_7\Phi_{12}\Phi_{14}$
$E_8(q)$	$d=1,2$	$\phi_{8,1}$	$q\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}$
		$\phi_{35,2}$	$q^2\Phi_5\Phi_7\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{30}$
${}^2B_2(q)$, $q^2=2^{2m+1}$	Note: $3\nmid \lvert S\rvert$	${}^2B_2[a]$	$\frac{1}{\sqrt{2}}q(q^2-1)$
		${}^2B_2[b]$	$\frac{1}{\sqrt{2}}q(q^2-1)$
${}^2F_4(q)$, $q^2=2^{2m+1}$	Note: $3\mid(q^2+1)$	cusp	$\frac{1}{12}q^4\Phi_1^2\Phi_2^2(\Phi_8')^2\Phi_{12}(\Phi_{24}')^2$
		cusp	$\frac{1}{12}q^4\Phi_1^2\Phi_2^2(\Phi_8'')^2\Phi_{12}(\Phi_{24}'')^2$

Table 2. Some unipotent characters in $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ for exceptional types with $3\nmid q$.

2007, Corollary 6.6]. Further, by [Enguehard 2000, Theorem A], all members of $R_{\tilde{T}}^{\tilde{G}}(1)$ lie in the same block, namely $B_0(\tilde{G})$.

If $d=2$, then the centralizer of a Sylow d-torus is a maximal torus, using e.g., [Malle and Späth 2016, Lemma 3.2]. Unipotent blocks of $\tilde{G}$ are parametrized by certain $\tilde{G}$-conjugacy classes of pairs $(\tilde{L},\lambda)$ where $\tilde{L}$ is a d-split Levi subgroup of $\tilde{G}$ and λ is a d-cuspidal unipotent character of $\tilde{L}$, by [Enguehard 2000, Theorem A]. Further, a unipotent character in the block parametrized by $(\tilde{L},\lambda)$ can have $3'$-degree

n	Additional Condition on n, e	Partition	$\chi(1)_{q'}$
$n \geq 6$	$e = 2$ and n even; or $e = 1$ and $3 \nmid (n-1)$	$(1, n-1)$ $(1^{n-2}, 2)$	$\frac{q^{n-1} - \epsilon^{n-1}}{q - \epsilon}$
$n \geq 6$	$e \mid n$ and $3 \nmid n$	$(2, n-2)$ $(1^{n-4}, 2, 2)$	$\frac{(q^n - \epsilon^n)(q^{n-3} - \epsilon^{n-3})}{(q-\epsilon)(q^2-1)}$
$n \geq 6$	$3 \mid n$; or $e = 2$ and n odd and $3 \mid (n-2)$	$(1, 1, n-2)$ $(1^{n-3}, 3)$	$\frac{(q^{n-1} - \epsilon^{n-1})(q^{n-2} - \epsilon^{n-2})}{(q-\epsilon)(q^2-1)}$
$n \geq 6$	$e = 2$ and n odd and $3 \mid (n-1)$	$(1, 1, 2, n-4)$ $(2, n-2)$	$\frac{(q^n-\epsilon^n)(q^{n-2}-\epsilon^{n-2})(q^{n-3}-\epsilon^{n-3})(q^{n-5}-\epsilon^{n-5})}{(q-\epsilon)^2(q^2-1)^2(q^2+1)}$ $\frac{(q^n-\epsilon^n)(q^{n-3}-\epsilon^{n-3})}{(q-\epsilon)(q^2-1)}$
$n = 5$	$e = 1$	$(1, 4)$ $(1, 1, 1, 2)$	$(q+\epsilon)(q^2+1)$
$n = 5$	$e = 2$	$(2, 3)$ $(1, 1, 3)$	$\frac{q^5 - \epsilon}{q - \epsilon}$ $(q^2+1)(q^2+\epsilon q+1)$
$n = 4$	$e = 2$	$(1, 3)$ $(1, 1, 2)$	$q^2 + \epsilon q + 1$
$n = 4$	$e = 1$	$(2, 2)$	$q^2 + 1$

Table 3. Some unipotent characters in $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ for type $A^{\epsilon}_{n-1}(q)$ with $n \geq 4$ and $3 \nmid q$.

only when $\tilde{L}$ is the centralizer of a Sylow d-torus, using [Malle 2007, Corollary 6.6]. This yields that again in the case $d = 2$, there is a unique block of $\tilde{G}$ containing unipotent characters of $3'$-degree.

Hence when $3 \nmid q$, every unipotent character in $\mathrm{Irr}_{3'}(\tilde{G})$ is a member of $\mathrm{Irr}_{3'}(B_0(\tilde{G}))^{\sigma_1}$, and restricts to a member of $\mathrm{Irr}_{3'}(B_0(G))^{\sigma_1}$ trivial on the center. Then this restriction may be viewed as an element of $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$, using e.g., [Cabanes and Enguehard 2004, Lemma 17.2].

In particular, since the Steinberg character has degree a power of q, it suffices to find two more unipotent characters of $3'$-degree that are not one of the exceptional cases in [Malle 2008, Theorem 2.5]. In what follows, we will use the notation and degrees for unipotent characters as in [Carter 1985, Sections 13.8 and 13.9].

Exceptional Types. In the case that S is an exceptional group of Lie type defined over $\mathbb{F}_q$ with $3 \nmid q$, we list in Table 2 two unipotent characters invariant under $\mathrm{Aut}(S)$ that have degree relatively prime to 3, completing the proof in this case.

Types A_{n-1} and $^2A_{n-1}$, $n \geq 4$. In this case, let S be $\mathrm{PSL}^{\epsilon}_n(q)$ with $n \geq 4$ and $\epsilon \in \{\pm 1\}$. Write $e \in \{1, 2\}$ for the number such that $q \equiv \epsilon e \pmod 3$. That is, e is the order of ϵq modulo 3. Two unipotent characters are in the same 3-block of $\tilde{G} = \mathrm{GL}^{\epsilon}_n(q)$ if and only if they have the same e-core (see [Fong and Srinivasan

1982]). For $n \geq 5$ or for $(n, e) = (4, 2)$, the unipotent characters described in Table 3 are $\mathrm{Aut}(S)$-invariant members of $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$.

Now assume $n = 4$ and $e = 1$. Then the unipotent character in the last line of Table 3 is an $\mathrm{Aut}(S)$-invariant member of $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$. In this case, 1_S, St_S, and the character listed are the only unipotent characters in $\mathrm{Irr}_{3'}(S)$. However, since $e = 1$, we see that every unipotent character is a member of the principal block of $\tilde{G}$, which means that $\mathcal{E}_3(\tilde{G}, 1)$ is comprised of only one block. Let $\zeta \in \mathbb{F}_{q^2}^{\times}$ with order 3. Then taking s to be the element $\mathrm{diag}(\zeta, \zeta, \zeta, 1)$ of $\tilde{G}^* \cong \mathrm{GL}_4^{\epsilon}(q)$, the semisimple character $\chi_s \in \mathcal{E}(\tilde{G}, s)$ lies in the principal block of $\tilde{G}$ and is trivial on $\boldsymbol{Z}(\tilde{G})$ since $s \in \mathrm{SL}_4^{\epsilon}(q) \cong [\tilde{G}^*, \tilde{G}^*]$. Further, we see using Lemma 3.1 that χ_s is fixed by σ_1.

Since $\boldsymbol{C}_{\tilde{G}^*}(s) \cong \mathrm{GL}_1^{\epsilon}(q) \times \mathrm{GL}_3^{\epsilon}(q)$, we see

$$\chi_s(1) = (q + \epsilon)(q^2 + 1).$$

Further, since the semisimple classes of $\tilde{G}$ are determined by their eigenvalues and $\boldsymbol{Z}(\tilde{G})$ is comprised of scalar matrices, we see that s is not conjugate to sz for any nontrivial $z \in \boldsymbol{Z}(\tilde{G})$. Hence $\chi_s|_G$ is irreducible, by the second-to-last paragraph of Section 3A, and is therefore a member of $\mathrm{Irr}_{3'}(B_0(G))$, since the principal block of G is the only block covered by the principal block of $\tilde{G}$. But since $\boldsymbol{Z}(G) \leq \boldsymbol{Z}(\tilde{G})$ is in the kernel of χ_s, this character is therefore a member of $\mathrm{Irr}_{3'}(B_0(G/\boldsymbol{Z}(G)))^{\sigma_1} = \mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$, again using [Cabanes and Enguehard 2004, Lemma 17.2]. Note that this character is not $\mathrm{Aut}(S)$-conjugate to 1_S, St_S, nor the unipotent character labeled by $(2, 2)$, which completes the proof for $S = \mathrm{PSL}_n^{\epsilon}(q)$ with $n \geq 4$.

Types B_n and C_n, $n \geq 2$. When S is type B_n or C_n with $n \geq 2$ defined in characteristic different than 3, Table 4 exhibits at least two distinct unipotent characters in $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ that are $\mathrm{Aut}(S)$-invariant, with the exception of the case $S = \mathrm{PSp}_4(2^a)$ with a odd. In the latter situation, we may instead consider the characters indexed by $\binom{1\ 2}{0}$ and $\binom{0\ 1\ 2}{\varnothing}$ with degrees $\frac{q}{2}(q^2+1)$ and $\frac{q}{2}(q-1)^2$, respectively. (Note that we do not consider $\mathrm{PSp}_4(2) \cong \mathfrak{S}_6$.) These characters lie in $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ and the latter character extends to $\mathrm{Aut}(S)$. (However, we remark that the first character is not $\mathrm{Aut}(S)$-invariant, as in this case it is switched with $\binom{0\ 1}{2}$ under the action of the graph automorphism, by [Malle 2008, Theorem 2.5]).

Type D_n and ${}^2D_n$, $n \geq 4$. In this case, if S is not $D_4(q)$, Tables 5 and 6 list at least two distinct unipotent characters that are $\mathrm{Aut}(S)$-invariant members of $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$. If S is $D_4(q)$ and $3 \mid (q-1)$, we may instead take the unipotent characters labeled by symbols $\binom{1\ 3}{0\ 2}$ and $\binom{3}{1}$ with $\chi(1)_{q'} = \frac{1}{2}(q+1)^3(q^3+1)$ and $(q^2+1)^2$, respectively. When $3 \mid (q+1)$, we may take the characters index by $\binom{3}{1}$ and $\binom{0\ 1\ 2\ 3}{\varnothing}$, the latter of which satisfies $\chi(1)_{q'} = \frac{1}{2}(q-1)^3(q^3-1)$. □

We next establish Theorem 3.7 for the case that S has cyclic Sylow 3-subgroups, which we recall from Proposition 3.6 occurs when $S = \mathrm{PSL}_2(q)$ for $3 \nmid q$ and when $S = \mathrm{PSL}_3^{\epsilon}(q)$ for $3 \mid (q + \epsilon)$.

Proposition 3.10. *Let $S = \mathrm{PSL}_2(q)$ with $3 \nmid q$ or $\mathrm{PSL}_3^{\epsilon}(q)$ with $3 \mid (q+\epsilon)$. Then there exist nontrivial $\chi_1, \chi_2 \in \mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ such that χ_1 extends to $\mathrm{Aut}(S)$.*

Conditions on q, n	Symbol	$\chi(1)_{q'}$ (possibly excluding factors of $\frac{1}{2}$)
$3 \mid (q-1)$ or $3 \mid (q+1)$; n even; $3 \nmid (n-1)$ or $3 \mid (q+1)$; n odd; $3 \nmid n$	$\binom{0\ 1}{n}$	$\dfrac{(q^{n-1}+1)(q^n+1)}{q+1}$
$3 \mid (q-1)$; $3 \nmid n$	$\binom{0\ 2}{n-1}$	$\dfrac{(q^{2n}-1)(q^{n-3}+1)(q^{n-1}+1)}{q^4-1}$
$3 \mid (q-1)$; $3 \nmid (n-1)$ or $3 \mid (q+1)$; n even	$\binom{1\ n}{0}$	$\dfrac{(q^{n-1}-1)(q^n+1)}{q-1}$
$3 \mid (n-1)$	$\binom{1\ n-1}{1}$	$\dfrac{(q^{2n}-1)(q^{2(n-2)}-1)}{(q^2-1)^2}$
$3 \mid (q+1)$; n odd or $3 \mid (q-1)$; $3 \nmid n$	$\binom{0\ n}{1}$	$\dfrac{(q^{n-1}+1)(q^n-1)}{q-1}$
$3 \mid (q+1)$; n odd; $3 \mid n$ or $3 \mid (q+1)$; n even; $3 \mid (n-2)$ or $3 \mid (q-1)$; $3 \mid (n-2)$	$\binom{0\ 1\ n}{1\ 2}$	$\dfrac{(q^{2(n-1)}-1)(q^n-1)(q^{n-2}+1)}{(q^2-1)^2}$

Table 4. Some unipotent characters in $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ for type $B_n(q)$ and $C_n(q)$ with $n \geq 2$ and $3 \nmid q$.

Proof. First let $S = \mathrm{PSL}_2(q)$ with $3 \nmid q$. In this case, every character of S is either 3-defect zero or has degree prime to 3. As before, the Steinberg character is a member of $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ and extends to $\mathrm{Aut}(S)$. Further, the only two unipotent characters, $1_{\tilde{G}}$ and $\mathrm{St}_{\tilde{G}}$, both lie in the principal block of $\tilde{G} = \mathrm{GL}_2(q)$, and hence there is a unique unipotent block of $\tilde{G}$. We may take $\chi_1 = \mathrm{St}_S$ as before.

Now, let $s \in \tilde{G} = \mathrm{GL}_2(q)$ have eigenvalues ζ, ζ^{-1}, where $\zeta \in \mathbb{F}_{q^2}^{\times}$ has order 3. Then the semisimple character $\chi_s \in \mathcal{E}(\tilde{G}, s) \subseteq \mathcal{E}_3(\tilde{G}, 1)$ lies in the principal block of $\tilde{G}$ and is trivial on $\boldsymbol{Z}(\tilde{G})$ since $s \in \mathrm{SL}_2(q) \cong [\tilde{G}^*, \tilde{G}^*]$. Since sz is not conjugate to s for $1 \neq z \in \boldsymbol{Z}(\tilde{G}^*)$, we also see χ_s is irreducible on restriction to G. Further, Lemma 3.1 yields that χ_s is fixed by σ_1. Then the restriction $(\chi_s)_G$ lies in $B_0(G)$ since the principal block of $\tilde{G}$ covers a unique block of G. Finally, in this case $\chi_s(1) = q + \eta$, where $\eta \in \{\pm 1\}$ is such that $3 \mid q - \eta$. Hence this character may be viewed as a member of $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$, arguing as before.

Now let $S = \mathrm{PSL}_3^{\epsilon}(q)$ with $3 \mid (q + \epsilon)$. Then

$$S = G = \mathrm{SL}_3^{\epsilon}(q) \quad \text{and} \quad \tilde{G}^* \cong \tilde{G} = G \times \boldsymbol{Z}(\tilde{G}).$$

Since the unipotent characters of G are 1_G, St_G, and a character of degree $q(q+\epsilon)$, we see that again $B_0(G)$ is the only unipotent block of maximal defect (as the other has defect zero). Then every character of $\mathcal{E}_3(G, 1)$ with $3'$-degree is a member of $B_0(G)$. We may again take $\chi_1 = \mathrm{St}_G$. Taking $s \in \tilde{G}^*$ to have

Conditions on q, n	Symbol	$\chi(1)_{q'}$ (possibly excluding factors of $\frac{1}{2}$)
$3\nmid(n-1)$	$\binom{1\ n}{0\ 1}$	$\dfrac{q^{2(n-1)}-1}{q^2-1}$
$3 \mid n$	$\binom{1\ 2\ n}{0\ 1\ 2}$	$\dfrac{(q^{2(n-2)}-1)(q^{2(n-1)}-1)}{(q^2-1)^2(q^2+1)}$
$3 \mid (n-2)$ or $3 \mid (q+1)$; $3 \mid n$; n odd	$\binom{n-2}{2}$	$\dfrac{(q^{2(n-1)}-1)(q^n-1)(q^{n-4}+1)}{(q^2-1)^2(q^2+1)}$
$3 \mid (q-1)$; $3 \mid (n-1)$ or $3 \mid (q+1)$; n odd	$\binom{1\ n-1}{0\ 2}$	$\dfrac{(q^n-1)(q^{n-2}-1)(q^{n-1}+1)(q^{n-3}+1)}{(q-1)^2(q^2+1)}$
$3 \mid (q-1)$; $3 \mid (n-2)$ or $3 \mid (q+1)$; $3\nmid n$; n even or $3 \mid (q+1)$; $3 \mid n$; n odd	$\binom{0\ n-1}{1\ 2}$	$\dfrac{(q^n-1)(q^{n-2}+1)(q^{n-1}-1)(q^{n-3}+1)}{(q^2-1)^2}$
$3 \mid (q-1)$; $3\nmid n$ or $3 \mid (q+1)$; $3 \mid (n-1)$ or $3 \mid (q+1)$; $3 \mid (n-2)$; n even or $3 \mid (q+1)$; $3 \mid n$; n odd	$\binom{n-1}{1}$	$\dfrac{(q^n-1)(q^{n-2}+1)}{q^2-1}$

Table 5. Some unipotent characters in $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ for type $D_n(q)$ with $n \geq 5$ and $3\nmid q$.

eigenvalues $\{\zeta, \zeta^{-1}, 1\}$, where $\zeta \in \mathbb{F}_{q^2}^{\times}$ has order 3, the corresponding character of G has degree $q^3-\epsilon$, and we may again view $(\chi_s)_G$ as a character of $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$. □

Proposition 3.11. *Let $S = \mathrm{PSL}_3^{\epsilon}(q)$ with $3 \mid (q-\epsilon)$. Then there exist nontrivial $\chi_1, \chi_2, \chi_3 \in \mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ such that χ_1 and χ_2 extend to $\mathrm{Aut}(S)$.*

Proof. In this case, we see that all three unipotent characters are members of $\mathrm{Irr}_{3'}(\tilde{G})$ and that there is a unique unipotent block $B_0(\tilde{G})$. Further, the unipotent characters are rational-valued, and therefore are members of $\mathrm{Irr}_{3'}(B_0(\tilde{G}))^{\sigma_1}$. Then we may take χ_1 and χ_2 to be the restrictions to G (viewed as a character of S) of the two nontrivial unipotent characters.

The semisimple element $s \in \tilde{G}^*$ with eigenvalues $\{\zeta, \zeta^{-1}, 1\}$, where $\zeta \in \mathbb{F}_{q^2}^{\times}$ has order 3, is now conjugate to sz where $z = \zeta \cdot I_3 \in \boldsymbol{Z}(\tilde{G}^*)$. The corresponding semisimple character has degree $\chi_s(1) = (q+\epsilon)(q^2+\epsilon q+1)$, so $\chi_s \in \mathrm{Irr}(B_0(\tilde{G}))^{\sigma_1}$ satisfies $3 \parallel \chi_s(1)$ and is not irreducible on restriction to G. Then the constituents of the restriction to G are members of $\mathrm{Irr}_{3'}(B_0(G))$, and are trivial on $\boldsymbol{Z}(G)$ since $s \in [\tilde{G}^*, \tilde{G}^*] \cong G$, so it suffices to see that they are also σ_1-invariant, using the character table available in CHEVIE. □

Together, Propositions 3.9 through 3.11 yield Theorem 3.7 for the simple groups of Lie type in nondefining characteristic, completing parts (i) and (ii).

Conditions on q, n	Symbol	$\chi(1)_{q'}$ (possibly excluding factors of $\frac{1}{2}$)
$3 \nmid (n-1)$	$\binom{0\ 1\ n}{1}$	$\frac{q^{2(n-1)}-1}{q^2-1}$
$3 \mid (n-1)$ or $3 \mid (q-1)$; $3 \mid n$ or $3 \mid (q+1)$; $3 \mid n$; n even or $3 \mid (q+1)$; $3 \mid (n-2)$; n odd	$\binom{1\ n-1}{\varnothing}$	$\frac{(q^n+1)(q^{n-2}-1)}{q^2-1}$
$3 \mid (q-1)$; $3 \nmid (n-1)$ or $3 \mid (q+1)$; $3 \mid (n-2)$ or $3 \mid (q+1)$; $3 \mid n$; n even	$\binom{2\ n-2}{\varnothing}$	$\frac{(q^{2(n-1)}-1)(q^n+1)(q^{n-4}-1)}{(q^2-1)^2(q^2+1)}$
$3 \mid n$	$\binom{0\ 1\ 2\ n}{1\ 2}$	$\frac{(q^{2(n-1)}-1)(q^{2(n-2)}-1)}{(q^2-1)^2(q^2+1)}$
$3 \mid (q-1)$; $3 \mid (n-1)$ $3 \mid (q+1)$; $3 \nmid n$; n odd $3 \mid (q+1)$; $3 \mid n$; n even	$\binom{1\ 2\ n-1}{0}$	$\frac{(q^n+1)(q^{n-1}+1)(q^{n-2}-1)(q^{n-3}-1)}{(q^2-1)^2}$
$3 \mid (q-1)$; $3 \mid n$ $3 \mid (q+1)$; $3 \mid (n-1)$; n even $3 \mid (q+1)$; $3 \mid (n-2)$; n odd	$\binom{0\ 1\ n-1}{2}$	$\frac{(q^n+1)(q^{n-1}-1)(q^{n-2}-1)(q^{n-3}+1)}{(q^2-1)^2}$

Table 6. Some unipotent characters in $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ for type ${}^2D_n(q)$ with $n \geq 4$ and $3 \nmid q$.

3C2. *Lie type in defining characteristic for $p = 3$.* We now consider the case S is as in Section 3A1 with $\boldsymbol{G}$ of simply connected type defined in characteristic 3. Let $(\boldsymbol{G}^*, F^*)$ be dual to $(\boldsymbol{G}, F)$. Keep in mind the notations and considerations of Section 3A, where now q is a power of 3. Note that $|\tilde{S}/S| = |\boldsymbol{Z}(G)|$, and this is 1 unless G is of classical type or $G = E_{7,sc}(q)$.

Since $\tilde{G}/G$ has size prime to 3, it follows that any irreducible character of G lying under $\mathrm{Irr}_{3'}(\tilde{G})$ is a member of $\mathrm{Irr}_{3'}(G)$. Since $|\sigma_1|$ is a power of 3, we further see that for any $\tilde{\chi} \in \mathrm{Irr}_{3'}(\tilde{G})^{\sigma_1}$, there is a member of $\mathrm{Irr}_{3'}(G)^{\sigma_1}$ lying under $\tilde{\chi}$. We also have $\mathrm{Irr}_{3'}(B_0(S)) = \mathrm{Irr}_{3'}(S)$, so any member of $\mathrm{Irr}_{3'}(G)$ with $\boldsymbol{Z}(G)$ in its kernel may be viewed as a member of $\mathrm{Irr}_{3'}(B_0(S))$.

Now, given a semisimple element $s \in \tilde{G}^*$, we have $|s|$ is prime to 3, and hence the Lusztig series $\mathcal{E}(\tilde{G}, s)$ is fixed by σ_1 using Lemma 3.1. Then in particular, the unique semisimple character $\tilde{\chi}_s \in \mathrm{Irr}_{3'}(\tilde{G})$ in this series must be fixed by σ_1. To illustrate three nontrivial characters of $\mathrm{Irr}_{3'}(G)^{\sigma_1}$ that are not $\mathrm{Aut}(S)$-conjugate, it therefore suffices to show that there are semisimple elements $1 \neq s_1, s_2, s_3 \in \tilde{G}^*$ such that

(1) s_i is not $\tilde{G}^*$-conjugate to $s_j^{\varphi}z$ for $i \neq j$, $z \in \boldsymbol{Z}(\tilde{G}^*)$, and φ any (possibly trivial) graph-field automorphism.

In most cases, we further ensure that one of these characters is $\mathrm{Aut}(S)$-invariant, by choosing s_1 so that

(2) the class of s_1 is invariant under graph-field automorphisms and

(3) s_1 is not $\tilde{G}^*$-conjugate to $s_1 z$ for any $1 \neq z \in \boldsymbol{Z}(\tilde{G}^*)$.

Property (2) will ensure that $\tilde{\chi}_{s_1}$ is invariant under graph-field automorphisms, using [Navarro et al. 2008, Corollary 2.4], and property (3) will imply that $\tilde{\chi}_{s_1}$ restricts irreducibly to G, so the resulting character of G is $\mathrm{Aut}(S)$- and σ_1-invariant. Finally, we will choose s_1, s_2, and s_3 such that

(4) $s_i \in [\tilde{G}^*, \tilde{G}^*]$ for $i = 1, 2, 3$,

so that the $\tilde{\chi}_{s_i}$ are trivial on $\boldsymbol{Z}(\tilde{G})$, ensuring that all three characters of G may be viewed as characters of $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ from the above discussion.

Proposition 3.12. *Let $S = G_2(q)$, ${}^3D_4(q)$, ${}^2G_2(q)$, $E_6^{\pm}(q)$, $E_7(q)$, $F_4(q)$ or $E_8(q)$ be simple with q a power of 3. Then there exist nontrivial $\chi_1, \chi_2, \chi_3 \in \mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ that are pairwise not $\mathrm{Aut}(S)$-conjugate and such that χ_1 is $\mathrm{Aut}(S)$-invariant.*

Proof. Note that we may assume S is not one of the groups from Proposition 3.8. The character degrees in these cases are available at [Lübeck 2007]. If S is $G_2(q)$, ${}^3D_4(q)$, ${}^2G_2(q)$, $E_6(q)$, ${}^2E_6(q)$, $F_4(q)$ or $E_8(q)$, then $\tilde{G} = G = S$ and there is a nontrivial odd character degree of multiplicity one, which therefore must be σ_1- and $\mathrm{Aut}(S)$-invariant. Similarly, $E_7(q)$ has a unique character of degree $\Phi_3\Phi_6\Phi_7\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$, which restricts irreducibly from a character of $\tilde{S} = E_7(q)_{ad}$. Finally, in each case there are at least two more semisimple characters with different degrees, which must yield members of $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ by the above discussion. □

Proposition 3.13. *Let $S = \mathrm{PSL}_n^{\epsilon}(q)$ be simple with q a power of 3 and $n \geq 2$ and let $X \leq \mathrm{Aut}(S)$ such that X/S is a Sylow 3-subgroup of $\mathrm{Aut}(S)/S$. Then there exist nontrivial $\chi_1, \chi_2, \chi_3 \in \mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ that are pairwise not $\mathrm{Aut}(S)$-conjugate and such that χ_1 is X-invariant. Further, if $n \geq 3$, then χ_1 may be chosen to be $\mathrm{Aut}(S)$-invariant, and if $n \geq 5$, then χ_1, χ_2, χ_3 may all be chosen to be $\mathrm{Aut}(S)$-invariant.*

Proof. Throughout, let $\delta \in \mathbb{F}_{q^2}^{\times}$ have order 4 and assume S is not isomorphic to one of the groups in Proposition 3.8. Recall that the conjugacy classes of semisimple elements in $\tilde{G}^* = \mathrm{GL}_n^{\epsilon}(q)$ are determined by their eigenvalues and that $\boldsymbol{Z}(\tilde{G}^*)$ is comprised of scalar matrices.

If $n = 2$, then $|\tilde{S}/S| = 2$ and $\mathrm{Aut}(S)/\tilde{S}$ is generated by a field automorphism. The semisimple elements s_1, s_2, and s_3 with eigenvalues $\{\delta, \delta^{-1}\}$, $\{\zeta_1, \zeta_1^{-1}\}$, and $\{\xi_1, \xi_1^{-1}\}$ with $\zeta_1 \in \mathbb{F}_q^{\times}$ and $\xi_1 \in \mathbb{F}_{q^2}^{\times} \setminus \mathbb{F}_q^{\times}$ and $|\xi| \neq 4 \neq |\zeta|$ satisfy properties (1), (2), and (4). Now, since $\tilde{\chi}_{s_1}$ is fixed by field automorphisms, and hence by X, and since $|\tilde{S}/S|$ is relatively prime to 3, we see that the irreducible constituents of the restriction $(\tilde{\chi}_{s_1})_G$ are still fixed by X and by σ_1. If $n = 3$ or 4, then s_1, s_2, s_3 satisfy (1)–(4) if chosen to have eigenvalues $\{\delta, \delta^{-1}\}$, $\{-1, -1\}$, and $\{\xi, \xi^{-1}\}$ with remaining eigenvalues 1, where $|\xi| > 2$ divides $q + \eta$ if $4 \mid q - \eta$.

Now suppose that $n \geq 5$. Consider semisimple elements s_1, s_2, and s_3 of $\tilde{G}^* = \mathrm{GL}_n^\epsilon(q)$ with eigenvalues $(\delta, \delta^{-1}, 1, \dots, 1)$, $(-1, -1, 1, \dots, 1)$, and $(\delta, \delta^{-1}, \delta, \delta^{-1}, 1, \dots, 1)$, respectively. If $n = 6$, instead define s_3 to have eigenvalues $(-1, -1, -1, -1, 1, 1)$. Then these satisfy (1)–(4), and in fact properties (2) and (3) are held by all three elements. Hence the corresponding semisimple characters $\tilde{\chi}_{s_i}$ of $\tilde{G}$ are invariant under graph-field automorphisms and restrict irreducibly to members of $\mathrm{Irr}_{3'}(B_0(G))^{\sigma_1}$ that are trivial on $Z(G)$. Hence these restrictions are members of $\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ invariant under $\mathrm{Aut}(S)$. □

Proposition 3.14. *Let q be a power of 3. Let $S = \mathrm{PSp}_{2n}(q)$, $P\Omega_{2n+1}(q)$, or $P\Omega_{2n}^{\pm}(q)$ be simple with $n \geq 2, 3, 4$ respectively. Then there exist nontrivial $\chi_1, \chi_2, \chi_3 \in \mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ that are pairwise not $\mathrm{Aut}(S)$-conjugate and such that χ_1 is invariant under $\mathrm{Aut}(S)$.*

Proof. We may again assume S is not one of the groups in Proposition 3.8. Let $\delta \in \mathbb{F}_{q^2}^\times$ with $|\delta| = 4$, and we keep the notation from Section 3A. Let Φ and $\Delta := \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a system of roots and simple roots, respectively, for $\tilde{\boldsymbol{G}}^*$ with respect to a maximal torus $\tilde{\boldsymbol{T}}^*$, following the standard model described in [Gorenstein et al. 2002, Remark 1.8.8]. Then Φ is type B_n, C_n, or D_n in the case $S = \mathrm{PSp}_{2n}(q)$, $P\Omega_{2n+1}(q)$, or $P\Omega_{2n}^{\pm}(q)$, respectively. Further, Φ has no nontrivial graph automorphism unless we are in the case of D_n, in which case all members of Δ have the same length and that automorphism has order 2 unless $n = 4$.

We use the notation as in [Gorenstein et al. 2002] for the Chevalley generators. In particular, given $\alpha \in \Phi$, let h_α denote the corresponding coroot. Let $\boldsymbol{K} := [\tilde{\boldsymbol{G}}^*, \tilde{\boldsymbol{G}}^*]$, so we have $h_\alpha(t) \in \boldsymbol{K}$ for $t \in \overline{\mathbb{F}}_q^\times$ by [loc. cit., Theorem 1.10.1(a)] and $\tilde{\boldsymbol{G}}^* = \boldsymbol{K}.\boldsymbol{Z}(\tilde{\boldsymbol{G}}^*)$. Notice that for $s, s' \in \boldsymbol{K}$ (not necessarily distinct), we have s is $\tilde{\boldsymbol{G}}^*$-conjugate to $s'z$ for $z \in \boldsymbol{Z}(\tilde{\boldsymbol{G}}^*)$ if and only if $z \in \boldsymbol{Z}(\boldsymbol{K})$ and the conjugating element can be chosen in $\boldsymbol{K}$.

By [Gorenstein et al. 2002, Theorem 1.12.4] and [Cabanes and Enguehard 2004, 15.1], $\boldsymbol{K}$ is isomorphic as an abstract group to the simply connected simple algebraic group $(\tilde{\boldsymbol{G}}^*)_{sc}$ associated to $\tilde{\boldsymbol{G}}^*$, and the Chevalley relations and generators of $(\tilde{\boldsymbol{G}}^*)_{sc}$ and $\boldsymbol{K}$ may be identified. We will make this identification. In particular, choosing s_1, s_2, and s_3 in $\boldsymbol{K}$, the properties (1)–(3) may be verified by computation in $\boldsymbol{K}$ rather than $\tilde{\boldsymbol{G}}^*$.

Let $\boldsymbol{T}$ denote a maximal torus of $\boldsymbol{K}$ under this identification, and note that

$$\boldsymbol{T} = \langle h_\alpha(t) \mid t \in \overline{\mathbb{F}}_q^\times, \alpha \in \Phi\rangle \quad \text{and} \quad N_{\boldsymbol{K}}(\boldsymbol{T}) = \langle \boldsymbol{T}, n_\alpha(1) \mid \alpha \in \Phi\rangle.$$

Further, note that

$$\boldsymbol{W} := N_{\tilde{\boldsymbol{G}}^*}(\tilde{\boldsymbol{T}}^*)/\tilde{\boldsymbol{T}}^* \cong N_{\boldsymbol{K}}(\boldsymbol{T})/\boldsymbol{T}.$$

By [Digne and Michel 1991, Corollary 0.12], we know that $N_{\boldsymbol{K}}(\boldsymbol{T})$ controls fusion in $\boldsymbol{T}$, so two elements of $\boldsymbol{T}$ are conjugate if and only if there is a conjugating element in $\boldsymbol{W}$. Further, we have an isomorphism $(\overline{\mathbb{F}}_q^\times)^n \to \boldsymbol{T}$ given by $(t_1, \dots, t_n) \mapsto \prod_{i=1}^n h_{\alpha_i}(t_i)$.

Now using the standard model for Φ and Δ as in [Gorenstein et al. 2002], since Φ is type B_n, C_n, or D_n, we have $\alpha_i := e_i - e_{i+1}$ for $1 \leq i \leq n-1$, where $\{e_1, \dots, e_n\}$ is an orthonormal basis for the n-dimensional Euclidean space. Here $\boldsymbol{W} \leq C_2 \wr \mathfrak{S}_n$ where the generators of the base subgroup C_2^n act via negation on the e_i and the copy of $\mathfrak{S}_n$ permutes the e_i.

Using this information and the description of $\boldsymbol{Z}(\boldsymbol{K})$ in [Gorenstein et al. 2002, Table 1.12.6], computation with the Chevalley relations yields that the element $s_1' := h_{\alpha_1}(\delta)$ is not $\tilde{\boldsymbol{G}}^*$-conjugate to $s_1'z$ for any $1 \neq z \in \boldsymbol{Z}(\tilde{\boldsymbol{G}}^*)$. If $\delta \in \mathbb{F}_q^\times$, we see that s_1' is F^*-fixed, and we write $s_1 := s_1'$. Otherwise, let $\dot{s}_{\alpha_1} \in \boldsymbol{W}$ induce the reflection corresponding to α_1. Then $s_1 := s_1'^g$ is F^*-fixed, where $g \in \tilde{\boldsymbol{G}}^*$ satisfies $g^{-1}F^*(g) = \dot{s}_\alpha$. (Note that such a g exists by the Lang–Steinberg theorem.)

Let F_3 denote a generating field automorphism such that $F_3(h_\alpha(t)) = h_\alpha(t^3)$ for $\alpha \in \Phi$ and $t \in \overline{\mathbb{F}}_q^\times$. Then s_1' is $\tilde{\boldsymbol{G}}^*$-conjugate to $F_3(s_1')$, taking for example $\dot{s}_{\alpha_1}$ as the conjugating element. Hence s_1 is also $\tilde{\boldsymbol{G}}^*$-conjugate to $F_3(s_1)$. Since the $\boldsymbol{C}_{\boldsymbol{G}^*}(\iota^*(s_1))$ is connected, using [Bonnafé 2005, Corollary 2.8(a)], this yields that the $\tilde{G}^*$-conjugacy class of s_1 is fixed by field automorphisms, using [Digne and Michel 1991, (3.25)]. Further, by construction, the $\tilde{G}^*$-conjugacy class of s_1 is fixed by graph automorphisms unless Φ is type D_4. In the latter case, we may make similar considerations using $s_1' := h_{\alpha_2}(\delta)$.

Now, further taking $s_2 := h_{\alpha_1}(-1)$ and $s_3 \in \boldsymbol{K}^{F^*}$ an element of order larger than 4, we obtain properties (1)–(4). □

Theorem 3.7 now follows from Propositions 3.8–3.14, completing the proof of Theorem A.

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