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Toroidal orbifolds, destackification, and Kummer blowings up

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With an appendix by David Rydh

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Toroidal orbifolds, destackification, and Kummer blowings up

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We show that any toroidal DM stack X with finite diagonalizable inertia possesses a maximal toroidal coarsening X_{tcs} such that the morphism $X \to X_{\mathrm{tcs}}$ is logarithmically smooth.

Further, we use torification results of Abramovich and Temkin (2017) to construct a destackification functor, a variant of the main result of Bergh (2017), on the category of such toroidal stacks X. Namely, we associate to X a sequence of blowings up of toroidal stacks $\widetilde{\mathcal{F}}_X : Y \to X$ such that Y_{tcs} coincides with the usual coarse moduli space Y_{cs}. In particular, this provides a toroidal resolution of the algebraic space X_{cs}.

Both X_{tcs} and $\widetilde{\mathcal{F}}_X$ are functorial with respect to strict inertia preserving morphisms $X' \to X$.

Finally, we use coarsening morphisms to introduce a class of nonrepresentable birational modifications of toroidal stacks called Kummer blowings up.

These modifications, as well as our version of destackification, are used in our work on functorial toroidal resolution of singularities.

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1. Introduction

We study the birational geometry of toroidal orbifolds, aiming towards applications in resolution of singularities and semistable reduction, as initiated in our paper [Abramovich et al. 2020].

Throughout this paper a noetherian logarithmically regular logarithmic DM stack X will be referred to as a *toroidal DM stack*, and if its inertia is finite and diagonalizable then we say that X is a *toroidal orbifold*. Finally, X is called *simple* if its inertia groups I_x act trivially on the sharpened stalks $\overline{M}_x$ of the

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logarithmic structure. The coarse moduli space is denoted X_{cs}. For such objects we prove the following destackification result:

Theorem 1 (see Theorem 4.1.5). *Let $\mathcal{C}$ be the category of simple toroidal orbifolds. Then to any object X in $\mathcal{C}$ one can associate a **destackifying blowing up** of toroidal stacks $\mathcal{F}_X : X' \to X$ along a nowhere zero ideal I_X and a **coarse destackifying blowing up** $\mathcal{F}^0_X : X_0 \to X_{\mathrm{cs}}$ along a nowhere zero ideal J_X so that*

(i) *$X_0 = (X')_{\mathrm{cs}}$ and X_0 inherits from X' a logarithmic structure making it a toroidal algebraic space such that the morphism $X' \to X_0$ is logarithmically smooth*;

(ii) *the blowings up are compatible with any surjective logarithmically smooth inert morphism $f : Y \to X$ from $\mathcal{C}$*:

$$I_X\mathcal{O}_Y = I_Y, \quad J_X\mathcal{O}_{Y_{\mathrm{cs}}} = J_Y, \quad Y' = X' \times_X Y, \quad Y'_0 = X'_0 \times_{X_{\mathrm{cs}}} Y_{\mathrm{cs}}.$$

Moreover, the last two isomorphisms hold even without assuming that f is surjective.

In addition, we remove the assumption on the triviality of the inertia action in Theorem 4.1.4. In this case, destackification is achieved by a sequence of blowings up, which is only compatible with strict inert morphisms.

The theorem above is a variant of the main result of [Bergh 2017]. It is tuned for different purposes and uses different methods. First, we restrict to diagonalizable inertia. In this case, Theorem 4.1.5 generalizes the main result of [Bergh 2017] in that we allow arbitrary toroidal singularities. Our method is also different from Bergh's, in that we use the *torific ideal* of [Abramovich and Temkin 2017] which produces the destackification result in one step. Unlike Bergh's result we do not describe the destackification in terms of a sequence of well-controlled operations such as blowings up and root stacks. In particular, applications to factorization of birational maps must use [Bergh 2017] rather than our theorems.

Our study of destackification requires understanding the degree to which one may remove stack structure while keeping logarithmic smoothness. For this purpose we introduce and study the properties of coarsening morphisms of Deligne–Mumford stacks in general in Section 2. A full classification of Deligne–Mumford coarsenings and in particular their existence, generalizing the Keel–Mori theorem, is a question we believe is of independence interest. This task, as well as a discussion of key cases, is provided in Appendix A written by David Rydh.

We then specialize to toroidal stacks in Section 3. We associate to a toroidal Deligne–Mumford stack X its *total toroidal coarsening* X_{tcs}, whose existence follows from Appendix A, and prove:

Theorem 2 (see Theorem 3.4.7). *Let $\widetilde{\mathcal{C}}$ be the 2-category of toroidal orbifolds and let X be an object of $\widetilde{\mathcal{C}}$. Then*

(i) *the total toroidal coarsening $X \to X_{\mathrm{tcs}}$ exists*;

(ii) *for any geometric point $x \to X$, we have $(I_{X/X_{\mathrm{tcs}}})_x = G^{\mathrm{tor}}_x$, where $(I_{X/X_{\mathrm{tcs}}})_x$ is the relative stabilizer and $G^{\mathrm{tor}}_x \subset G_x$ the maximal subgroup of inertia acting toroidally*;

(iii) *any logarithmically flat morphism $h : Y \to X$ in $\widetilde{\mathcal{C}}$ induces a morphism $h_{\mathrm{tcs}} : Y_{\mathrm{tcs}} \to X_{\mathrm{tcs}}$ with a* 2*-commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\phi_Y} & Y_{\mathrm{tcs}} \\ {\scriptstyle h}\downarrow & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle h_{\mathrm{tcs}}} \\ X & \xrightarrow[\phi_X]{} & X_{\mathrm{tcs}} \end{array}$$

and the pair $(h_{\mathrm{tcs}}, \alpha)$ is unique in the 2*-categorical sense*;

(iv) *assume in addition that Y is simple and h is logarithmically smooth and inert. Then the diagram in* (*iii*) *is* 2*-cartesian.*

We emphasize that in this paper the theorem above is only used in Theorems 4.1.4 and 4.1.5, and only tangentially. Our original treatment of Theorem 3 below used toroidal coarsenings, but our current formalism requires a relative coarsening over $B\mathbb{G}_m$.

Apart from destackification, our treatment of coarsening morphisms figures in our study of a collection of nonrepresentable birational modifications which is essential in our work [Abramovich et al. 2020] on resolution of singularities. This is detailed in Section 5, which is mostly independent of Sections 3 and 4. We define in Section 5.4.1 the notion of a *permissible Kummer center I* on a toroidal scheme, and in Section 5.4.4 we define its blowing up $[\mathrm{Bl}_I(X)] \to X$, which is in general a toroidal DM stack. Furthermore, in Section 5.5 we extend these notions to the case when X itself is a toroidal DM stack. The key properties of Kummer blowings up are as follows:

Theorem 3 (see Theorems 5.4.5 and 5.4.16, Lemmas 5.4.21, 5.4.19 and 5.4.18, and Section 5.5). *Let X be a toroidal DM stack and let I be a permissible Kummer ideal on X with the associated Kummer blowing up $f : [\mathrm{Bl}_I(X)] \to X$. Then*

(i) (*$V(I)$-modification*) *f is proper and an isomorphism over $X \smallsetminus V(I)$*;

(ii) (*principalization property*) *$f^{-1}(I)$ is an invertible ideal*;

(iii) (*universal property*) *f is the universal morphism of toroidal DM stacks $h : Z \to X$ such that $h^{-1}(I)$ is an invertible ideal*;

(iv) (*orbifold property*) *the relative inertia $I_{[\mathrm{Bl}_I(X)]/X}$ is finite diagonalizable, and it acts trivially on the monoids $\overline{M}_x$. If X is a simple toroidal orbifold then $[\mathrm{Bl}_I(X)]$ is a simple toroidal orbifold as well*;

(v) (*functoriality*) *let $f : Y \to X$ be a logarithmically smooth morphism of toroidal orbifolds and $J = I\mathcal{O}_Y$. Then $[\mathrm{Bl}_J(Y)] = [\mathrm{Bl}_I(X)] \times_X Y$, where the product is taken in the category of fs logarithmic stacks*;

(vi) (*coarse blowing up*) *assume $Z \hookrightarrow X$ is a strict closed logarithmic subscheme. Let $Z' \to Z$ be the strict transform* (*i.e., the closure of $Z \smallsetminus V(I)$ in $[\mathrm{Bl}_I(X)]$*). *Set $J_n = I^{n!} \cap \mathcal{O}_X$. Then the relative coarse moduli space $Z'_{\mathrm{cs}/X}$ is the blowing up of Z along the saturated ideal $((J_n)^m)^{\mathrm{nor}}\mathcal{O}_Z$ for large enough n and m*;

(vii) (*strict transform*) *assume further in* (vi) *that $J = I\mathcal{O}_Z$ is a permissible Kummer ideal on Z. Then the morphism $Z' \to Z$ factors through a unique isomorphism $Z' = [\mathrm{Bl}_J(Z)]$.*

Remark 4. We expect some of our statements to apply in greater generality: it is natural to allow X to be an Artin stack, where the stabilizer at any $x \in X$ acts discretely on the monoid $\overline{M}_x$, and where the kernel of this action is linearly reductive. With this generality, permissible Kummer centers (Section 5.4.1) may have index d divisible by the characteristic of the residue field at x.

2. Coarsening morphisms and inertia

2.1. *Inertia stack.*

2.1.1. *Basic properties of inertia.* Recall that the inertia stack $I_{X/Y}$ of a morphism $f : X \to Y$ of Artin stacks is the second diagonal stack $I_{X/Y} = X \times_{X\times_Y X} X$, where both structure arrows $X \to X \times_Y X$ are the diagonal. It is a representable group object over X.

The absolute inertia stack of X is $I_X = I_{X/\mathbb{Z}}$. Recall that by [Stacks, Tag 04Z6]

$$I_{X/Y} = I_X \times_{I_Y} X. \tag{1}$$

In other words, $I_{X/Y} = \mathrm{Ker}(I_X \to f^*(I_Y))$, where $f^*(I_Y) = I_Y \times_Y X$.

In fact, the inertia stack is a group functor in the following sense: given a morphism $f : X \to Y$ a natural morphism $I_f : I_X \to I_Y$ arises, and the induced morphism $I_X \to f^*(I_Y)$ is a homomorphism. In addition, the inertia functor is defined as a 2-limit and hence it respects 2-limits, including fiber products. So, given $T = X \times_Z Y$ with projections $f : T \to X$, $g : T \to Y$ and $h : T \to Z$, one has that

$$I_{X\times_Z Y} = I_X \times_{I_Z} I_Y = f^*(I_X) \times_{h^*(I_Z)} g^*(I_Y). \tag{2}$$

Similar facts hold for relative inertia over a fixed stack S.

2.1.2. *Inert morphisms.* We say that a morphism $f : X \to Y$ is *inert* or *inertia-preserving* if it respects the inertia in the sense that $I_X = f^*(I_Y)$. In particular, $I_{X/Y} = X$ and hence f is representable (see [Stacks, Tag 04SZ] for the absolute case, the relative case follows easily). Inert morphisms are preserved by base changes. Finally, inert morphisms have no nontrivial automorphisms.

2.1.3. *Inert groupoids.* In general, one runs into 2-categorical issues when trying to define groupoids in stacks or their quotients. This is addressed, using the theory of higher stacks and their truncations, in [Harper 2017, Definition 3.10, Proposition 3.11], where groupoids with representable projection arrows are considered. We sketch the situation here in the case of inert groupoids, suppressing the specification of a number of 2-arrows that the theory of higher stacks provides. The treatment here is thus a restatement of [Stacks, Tag 044U] in the situation of inert groupoids. By *an inert groupoid* in stacks we mean a usual datum $(p_{1,2} : X_1 \rightrightarrows X_0, m, i, \delta)$ as in [Stacks, Tag 0231], where X_i are stacks and all morphisms are inert.

Let $f : X_0 \to Y$ be a morphism. An isomorphism $\phi : f \circ p_1 \to f \circ p_2$ is said to *satisfy the cocycle condition* on

$$X_2 := X_1 \times_{p_2, X_0, p_1} X_1 \overset{\pi_{1,2}}{\rightrightarrows} X_1$$

if $\pi_2^*\phi \circ \pi_1^*\phi = m^*\phi$.

Lemma 2.1.4. *Assume that $p_{1,2} : X_1 \rightrightarrows X_0$ is a smooth inert groupoid in Artin stacks. Then there exists a representable smooth morphism of stacks $q : X_0 \to X$ such that $X_1 = X_0 \times_X X_0$, with a 2-isomorphism $q \circ p_1 \to q \circ p_2$ satisfying the cocycle condition on X_2, and moreover*

(1) *X is the quotient $[X_0/X_1]$ in the sense that any morphism $f : X_0 \to Y$ with a 2-isomorphism $f \circ p_1 \to f \circ p_2$ satisfying the cocycle condition on X_2 are induced by q from a morphism $X \to Y$, which is unique up to a unique 2-isomorphism*;

(2) *if $Z \to X$ is a morphism from an algebraic space, inducing a smooth inert groupoid in algebraic spaces $p^Z_{1,2} : Z_1 \rightrightarrows Z_0$, then $[Z_0/Z_1] \to Z$ is an isomorphism*;

(3) *if $Y_1 \rightrightarrows Y_0$ is another inert groupoid with quotient Y, and a given smooth morphism $X_0 \to Y_0$ extends to a cartesian morphism of groupoids, then there is a smooth morphism $X \to Y$, unique up to unique isomorphism, with $X_i = Y_i \times_Y X$.*

Sketch of proof. Let $U \to X_0$ be a smooth covering by a scheme and set

$$R = U \times_{X_0, p_1} X_1 \times_{p_2, X_0} U.$$

Since inert morphisms are representable, R is an algebraic space and we obtain a smooth groupoid $R \rightrightarrows U$ in algebraic spaces. So the quotient $X = [U/R]$ is an Artin stack, and a (mostly 1-categorical) diagram chase shows that X is as required and satisfies (1) and (2). The existence of a morphism $X \to Y$ in part (3) follows from (1), and its properties follow from (2) by taking compatible smooth covers $Z_X \to X$ and $Z_Y \to Y$. □

2.1.5. *Inertia of special types.* We say that a stack X has *finite inertia* if the morphism $I_X \to X$ is finite, and we say that X has *diagonalizable inertia* if the geometric fibers of $I_X \to X$ are diagonalizable groups. For example, both conditions are satisfied when X admits an étale inert covering of the form $[Z/G] \to X$, where Z is a separated scheme acted on by a finite diagonalizable group G.

2.2. *Coarse spaces.*

2.2.1. *Coarse moduli spaces and their basic properties.* Recall that by the Keel–Mori theorem, a stack X with finite inertia possesses a coarse moduli space X_{cs}; see [Keel and Mori 1997] and more generally [Rydh 2013, pp. 630–631]. Rydh's treatment removes all but necessary assumptions; here the morphism $\pi : X \to X_{\mathrm{cs}}$ is a separated universal homeomorphism with $\pi_* \mathcal{O}_X = \mathcal{O}_{X_{\mathrm{cs}}}$, but cannot be assumed proper unless X is of finite type over a scheme.

In the sequel, we will say that X_{cs} is the *coarse space* of X and $X \to X_{\mathrm{cs}}$ is the *total coarsening morphism* of X. Recall that for any flat morphism of algebraic spaces $Z \to X_{\mathrm{cs}}$, the base change morphism $Y = X \times_{X_{\mathrm{cs}}} Z \to Z$ is a total coarsening morphism and the projection $Y \to X$ is flat and inert. As a partial converse, a morphism $Y \to X$ which is either inert and étale [Rydh 2013, Theorem 6.10], or inert and flat with X tame [Rydh 2020] is the base change of $h_{\mathrm{cs}} : Y_{\mathrm{cs}} \to X_{\mathrm{cs}}$.

2.2.2. *The universal property.* The coarse space of X is initial among morphisms $X \to Z$ to algebraic spaces, and we will extend this, under appropriate assumptions, to morphisms $X \to Z$ of stacks. We say that an inertia map $I_X \to I_Z$ is *trivial* if it factors through the unit $Z \to I_Z$. This happens if and only if $I_{X/Z} = I_X$.

Theorem 2.2.3. *Assume that $\phi : X \to Z$ is a morphism of Artin stacks and the inertia of X is finite.*

(i) *Assume either X is tame or Z is a Deligne–Mumford stack. Then the inertia map $I_\phi : I_X \to I_Z$ is trivial if and only if ϕ factors through the coarse space $f : X \to X_{\mathrm{cs}}$: there exists $\psi : X_{\mathrm{cs}} \to Z$ and a 2-isomorphism $\alpha : \phi \xrightarrow{\sim} \psi \circ f$.*

(ii) *A factorization in* (i) *is unique in the sense of 2-categories: if ψ' and α' form another such datum then there exists a unique 2-isomorphism $\psi = \psi'$ making the whole diagram 2-commutative.*

Proof. If ϕ factors through f then I_ϕ factors through the inertia $I_{X_{\mathrm{cs}}}$, which is trivial. Conversely, assume that I_ϕ is trivial.

- Assume Z is Deligne–Mumford. Choose an étale covering of Z by a scheme Z_0 and set $Z_1 = Z_0 \times_Z Z_0$ and $X_i = X \times_Z Z_i$, as in the left part of following diagram, which is cartesian:

$$\begin{array}{ccccc} Z_1 & \longleftarrow & X_1 & \longrightarrow & Y_1 \\ \Downarrow & & \Downarrow & & \Downarrow \\ Z_0 & \longleftarrow & X_0 & \longrightarrow & Y_0 \\ \downarrow & & \downarrow & & \\ Z & \longleftarrow & X & & \end{array}$$

  Since I_{Z_i} and I_ϕ are trivial, equations (1) and (2) imply that $I_{X_i} = I_X \times_X X_i$, and we obtain that the étale surjective morphisms $X_i \to X$ are inert.

  It follows that each X_i has finite inertia, in particular, coarse spaces $Y_i = (X_i)_{\mathrm{cs}}$ are defined as in the right-hand side of the diagram above.

  Since the arrows $X_1 \to X_0$ are both étale and inert, [Rydh 2013, Theorem 6.10] applies (with $\mathcal{W} \to \mathcal{X}$ there replaced by $X_1 \to X_0$). Thus the left-hand diagram above is cartesian and the morphisms $Y_1 \to Y_0$ are étale. Now $Y_1 \rightrightarrows Y_0$ is an étale groupoid with quotient X_{cs}. For $i = 0, 1$ the map $X_i \to Z_i$ factors through Y_i uniquely, and the induced morphism of groupoids

$$(Y_1 \rightrightarrows Y_0) \to (Z_1 \rightrightarrows Z_0)$$

  gives rise to the unique morphism $\psi : X_{\mathrm{cs}} \to Z$ as required.

- Assume instead X is tame. The same argument as in the Deligne–Mumford case above holds, replacing the reference [Rydh 2013] with [Rydh 2020]. Here we present another argument valid when both X and Z are tame. By [Abramovich et al. 2011, Theorem 3.1] the morphism $X \to Z$ factors through its relative coarse moduli space $X_{\mathrm{cs}/Z}$, hence it suffices to replace Z by $X_{\mathrm{cs}/Z}$ and show that $X_{\mathrm{cs}/Z} \to X_{\mathrm{cs}}$ is an isomorphism. The problem is local in the étale topology of X_{cs}, hence we may assume $X = [V/G]$ with V a scheme and G finite linearly reductive, in which case the result follows from [Abramovich et al. 2011, Proposition 3.6].

For (ii), consider a diagram

$$X \longrightarrow X_{\mathrm{cs}} \underset{\psi'}{\overset{\psi}{\rightrightarrows}} Z$$

with isomorphisms $\alpha : \phi \xrightarrow{\sim} \psi \circ f$, $\alpha' : \phi \xrightarrow{\sim} \psi' \circ f$. Given a presentation $Z_0 \to Z$, the isomorphisms α, α' provide a commutative base change diagram

$$\begin{array}{ccccc} & & (X_{\mathrm{cs}})_0 & & \\ & \nearrow & \vdots & \searrow^{\psi_0} & \\ X_0 & & \downarrow & & Z_0. \\ & \searrow & & \nearrow_{\psi'_0} & \\ & & (X_{\mathrm{cs}})'_0 & & \end{array}$$

Since $(X_{\mathrm{cs}})_0$, $(X_{\mathrm{cs}})'_0 \to X_{\mathrm{cs}}$ are flat, both $X_0 \to (X_{\mathrm{cs}})_0$, $(X_{\mathrm{cs}})'_0$ are coarse moduli spaces, giving a unique $(X_{\mathrm{cs}})_0 \to (X_{\mathrm{cs}})'_0$ making the diagram commutative. The same holds with Z_0 replaced by $Z_1 = Z_0 \times_Z Z_0$, providing a unique isomorphism of ψ with ψ'. □

Remark 2.2.4. We note that further results are provided in [Abramovich and Temkin 2018; Romagny et al. 2018; Rydh 2020]. Part (i) does not hold without restrictions; see the example in Section A.2.3.

2.3. *General coarsening morphisms.*

2.3.1. *Coarsening morphisms.* We say that a morphism of stacks $\pi : X \to Y$ is a *coarsening morphism* if the inertia $I_{X/Y}$ is finite and for any flat morphism $Z \to Y$ with Z an algebraic space the base change $X \times_Y Z \to Z$ is a total coarsening morphism as discussed in Section 2.2. It follows, see Lemma 2.3.4, that these are separated universal homeomorphisms with $\pi_* \mathcal{O}_X = \mathcal{O}_Y$. It is easy to see that coarsening morphisms are preserved by composition and arbitrary flat base change, not necessarily representable. In addition, being a coarsening morphism is a flat-local property on the target. In fact, one can show that this is the smallest class of morphisms containing total coarsening morphisms and closed under flat base changes and descent.

Remark 2.3.2. We use a new terminology and definition, but the object is not new. We refer to [Abramovich et al. 2011, Section 3] for the definition of relative coarse moduli space $X_{\mathrm{cs}/S}$ of a morphism of stacks $X \to S$ with finite relative inertia. It is easy to see that $X \to X_{\mathrm{cs}/S}$ is a coarsening morphism and, conversely, for every coarsening morphism $X \to Y$ one has that $Y = X_{\mathrm{cs}/Y}$.

2.3.3. *Basic properties.* In view of Remark 2.3.2, the following lemma is essentially covered by [Abramovich et al. 2011, Theorem 3.2], but we provide a proof for completeness.

Lemma 2.3.4. *Let X be an Artin stack with finite inertia and let $f : X \to Y$ be a coarsening morphism. Then*

(i) *there exists a unique morphism $g : Y \to X_{\mathrm{cs}}$ such that $g \circ f$ is isomorphic to the total coarsening morphism $h : X \to X_{\mathrm{cs}}$;*

(ii) *f is a separated universal homeomorphism*;

(iii) *$Y_{\mathrm{cs}} = X_{\mathrm{cs}}$, i.e., g is the total coarsening morphism.*

Proof. (i) Choose an atlas $Y_1 \rightrightarrows Y_0$ of Y and set $X_i = Y_i \times_Y X$. Then $Y_i = (X_i)_{\mathrm{cs}}$ and hence the composed morphisms $X_i \to X \to X_{\mathrm{cs}}$ factor uniquely through morphisms $g_i : Y_i \to X_{\mathrm{cs}}$. The uniqueness implies that g_1 coincides with both pullbacks of g_0, hence f descends to a morphism $g : Y \to X_{\mathrm{cs}}$, which is unique.

(ii) Continuing with the notation above, since the projections $f_i : X_i \to Y_i$ are total coarsening morphisms (Section 2.2.1), they are separated universal homeomorphisms, and hence the same is true for f by descent.

(iii) We should prove that a morphism $Y \to T$ with T an algebraic space factors uniquely through X_{cs}. The composed morphism $X \to Y \to T$ factors through X_{cs} uniquely, hence the morphisms $X_i \to X \to T$ factor through X_{cs}. Since $Y_i = (X_i)_{\mathrm{cs}}$ we obtain that the morphisms $Y_i \to T$ factor through X_{cs} in a compatible way, and hence they descend to a morphism $Y \to X_{\mathrm{cs}}$ through which $Y \to T$ factors. □

2.3.5. *The universal property.* Similarly to coarse spaces, with appropriate assumptions, coarsening morphisms can be described by a universal property.

Theorem 2.3.6. *Let $\phi : X \to Z$ be a morphism of Artin stacks and let $f : X \to Y$ be a coarsening morphism.*

(i) *Assume either X is tame or Z is a Deligne–Mumford stack. Then the following conditions are equivalent*:
   (a) *ϕ factors through f.*
   (b) *$I_\phi : I_X \to \phi^*(I_Z)$ factors through $I_f : I_X \to f^*(I_Y)$.*
   (c) *The map $I_{X/Y} \to \phi^* I_Z$ is trivial.*
   (d) *$I_{X/Y} \subseteq I_{X/Z}$.*

(ii) *A factoring of ϕ through f in* (i) *is unique in the* 2*-categorical sense* (*see Theorem 2.2.3*(ii)). *In other words, f is a* 2*-categorical epimorphism.*

(iii) *In particular, the* 2*-category of coarsening morphisms of X is equivalent to a partially ordered set and the total coarsening morphism h is its final object.*

Proof. The implications (a)⇒(b)⇒(c)⇔(d) in (i) follow from the definitions and the base change property of inertia, see (1) in Section 2.1.1. So assume that the map $I_{X/Y} \to I_Z$ is trivial and let us prove (a). Consider a smooth covering of Y by a scheme Y_0 and set $Y_1 = Y_0 \times_Y Y_0$ and $X_i = Y_i \times_X Y$. Since $I_{X_i} = I_X \times_{I_Y} I_{Y_i}$ and I_{Y_i} is trivial, we obtain that I_{X_i} is the pullback of $I_{X/Y}$, and hence the morphisms $I_{X_i} \to I_Z$ are trivial. By Theorem 2.2.3, the morphisms $X_i \to Z$ factor through $Y_i = (X_i)_{\mathrm{cs}}$ uniquely. We obtain a morphism of groupoids $(Y_1 \rightrightarrows Y_0) \to Z$, which gives rise to a required morphism $Y \to Z$.

In the same way, part (ii) reduces to Theorem 2.2.3(ii) using that the question is smooth-local on Y. Part (iii) follows from part (ii). □

Remark 2.3.7. The implication (c)⇒(b) in the theorem is nontrivial. Informally, it indicates that $f^*(I_Y) = I_X/I_{X/Y}$. (To prove that this is indeed a group scheme quotient we should have tested it with all group schemes over X, while (b) only uses group schemes which are a pullback of some I_Z.)

Note that again the example in Section A.2.3 shows that part (i) does not hold without appropriate assumptions.

Remark 2.3.8. A full classification of Deligne–Mumford coarsenings, as well as a discussion of key cases, is provided in Appendix A.

3. Toroidal stacks and moduli spaces

3.1. *Toroidal schemes.*

3.1.1. *References.* We adopt the terminology of [Abramovich and Temkin 2017] concerning toroidal schemes and their morphisms with the only difference that we replace Zariski fine and saturated logarithmic structures by the étale fine and saturated logarithmic structures. In other words, in this paper we extend the class of toroidal schemes so that it contains "toroidal embeddings with self-intersections" in the terminology of [Kempf et al. 1973].

Note that when Kato [1994] introduced logarithmically regular logarithmic schemes, he worked with Zariski logarithmic schemes for simplicity. However, étale locally any fine logarithmic scheme is a Zariski logarithmic scheme, and this allows to easily extend all results about logarithmic regularity to general fs logarithmic schemes; see [Nizioł 2006].

We will make use of Kummer logarithmically étale morphisms; see [Nizioł 2008] and Section 5.3.5.

3.1.2. *Toroidal schemes.* Now, let us recall the main points quickly. In this paper, a *toroidal scheme* X is a logarithmically regular logarithmic scheme (X, M_X) in the sense of [Nizioł 2006]. Alternatively, one can represent X by a pair (X, U), where the open subscheme U is the locus where the logarithmic structure is trivial. One reconstructs the monoid by $M_X = \mathcal{O}_{X_{\text{ét}}} \cap i_*(\mathcal{O}^\times_{U_{\text{ét}}})$, where $i : U \hookrightarrow X$ is the open immersion. Usually, we will denote a toroidal scheme X or (X, U).

3.1.3. *Fans.* Recall that the logarithmic stratum $X(n)$ of a logarithmic scheme $(X, \mathcal{M}_X)$ consists of all points $x \in X$ with $\operatorname{rank}(\overline{M}_x) = n$. Here and in the sequel we use the convention that $\overline{M}_x$ denotes $\overline{M}_{\bar{x}}$ for a geometric point $\bar{x} \to X$ over x. In particular, $\overline{M}_x$ is defined up to an automorphism, but its rank is well defined.

If X is a toroidal scheme then, by logarithmic regularity, each stratum $X(n)$ is regular of pure codimension n. By the *fan* of a toroidal scheme X we mean the set $\operatorname{Fan}(X)$ of all generic points of the logarithmic strata of X. Also, let $\eta : X \to \operatorname{Fan}(X)$ denote the contraction map sending a point x to the generic point of the connected component of the logarithmic stratum containing x.

3.1.4. *Morphisms.* A morphism of toroidal schemes $(Y, V) \to (X, U)$ is a morphism of the associated logarithmic schemes. Equivalently one can describe it as a morphism $f : Y \to X$ such that $f(V) \subseteq U$. *Logarithmically smooth morphisms* form an important class of morphisms (called *toroidal morphisms* in [Abramovich and Temkin 2017]). *Strict morphisms* form another important class: these are the morphisms that induce an isomorphism $f^*\mathcal{M}_X \xrightarrow{\sim} \mathcal{M}_Y$.

3.2. *Toroidal actions.*

3.2.1. *Definitions.* A *diagonalizable group* G is a $\mathbb{Z}$-flat group scheme of the form $\boldsymbol{D}_L$ for a finitely generated group L; see [Abramovich and Temkin 2018, Section 3.2]. An action of G on a scheme X is

relatively affine if there is a scheme Z and an affine G-invariant morphism $X \to Z$; see [Abramovich and Temkin 2018, Section 5.1]. This will be a running assumption throughout. It implies the existence of schemes of fixed points and a good inertia stratification. We also assume that X is toroidal and G acts on it in the sense of [Abramovich and Temkin 2017, Section 3.1]: $p^*M_X \xrightarrow{\sim} m^*M_X$, where $p, m : X \times G \rightrightarrows X$ are the projection and the action morphisms, but in this paper M_X is an étale sheaf. In particular $G_{\eta(x)} \subseteq G_x$. The action is *simple* at a point $x \in X$ if the stabilizer G_x acts trivially on $\overline{M}_x$, and the action is *toroidal* at x if it is simple at x and $G_x = G_{\eta(x)}$. Note that the latter happens if and only if G_x acts trivially on the connected component of the logarithmic stratum through x; see [Abramovich and Temkin 2017, Sections 3.1.4, 3.1.7].

Remark 3.2.2. (i) By [Abramovich and Temkin 2017, Corollary 3.2.18], the set of points $x \in X$, at which the action is toroidal or simple, is open.

(ii) Let us temporary say that the action is quasitoroidal at x is $G_x = G_{\eta(x)}$. This notion is not so meaningful due to the following examples:

(1) The openness property fails for quasitoroidality. For example, let $G = \mathbb{Z}/2\mathbb{Z}$ act on $X = \mathrm{Spec}(k[x, y])$ by switching the coordinates. Then the action is quasitoroidal at the origin, but it is not quasitoroidal at other points of the line X^G, which is given by $x = y$. Note that this action is not simple at the origin, so the example is consistent with the openness result for the toroidal locus.

(2) Let $G = \mathbb{Z}/4\mathbb{Z}$ with a generator g act on $X = \mathrm{Spec}(k[x, y])$ by $gx = y$ and $gy = -x$. Then the action is quasitoroidal everywhere but is not simple at the origin.

(iii) We note, as in Remark 4 of the introduction, that while the restrictions imposed here are sufficient for the immediate applications we have in mind, we expect some of our statements to hold in greater and more natural generality.

3.2.3. *The groups G_x^{tor}.* Let $G_{\overline{M}_x}$ be the subgroup of G_x that stabilizes $\overline{M}_x$. By the *toroidal stabilizer* at x we mean the subgroup $G_x^{\mathrm{tor}} = G_{\eta(x)} \cap G_{\overline{M}_x}$ of the stabilizer G_x. Thus G_x^{tor} is the maximal subgroup of G_x that acts toroidally at x.

Lemma 3.2.4. *If a diagonalizable group G acts in a relatively affine manner on a toroidal scheme X then any point $x \in X$ possesses a neighborhood X' such that $G_x^{\mathrm{tor}} \cap G_{x'} = G_{x'}^{\mathrm{tor}}$ for any point $x' \in X'$.*

Proof. Let X' be obtained by removing from X the Zariski closures of all points $\varepsilon \in \mathrm{Fan}(X)$ which are not generizations of x. Thus, $\eta(x')$ is a generization of $\eta(x)$ for any $x' \in X'$. Note that $\overline{M}_{x'} = \overline{M}_{\eta(x')}$ since $\overline{M}_X$ is locally constant along logarithmic strata. Therefore $G_{x'}^{\mathrm{tor}} = G_{\eta(x')}^{\mathrm{tor}}$, and it suffices to deal with the case when $x, x' \in \mathrm{Fan}(X)$. Then x' specializes to x and our claim reduces to the check that $G_{\overline{M}_x} \cap G_{x'} = G_{\overline{M}_{x'}}$. Since any cospecialization $\phi : \overline{M}_x \to \overline{M}_{x'}$ is surjective, $G_{\overline{M}_x} \cap G_{x'} \subseteq G_{\overline{M}_{x'}}$. Conversely, we need to show $G_{\overline{M}_{x'}} \subset G_{\overline{M}_x}$.

Let $F \subset \overline{M}_x$ be a face associated to the closed stratum $Y = \overline{\{x'\}}$ and cospecialization ϕ, so that $\overline{M}_{x'} = \overline{M}_x/F$ and ϕ is the quotient homomorphism. The normalization Y^{nor} of Y is itself toroidal, having characteristic monoid F at a point x^{nor} over x (and trivial monoid at the generic point x'). Since $G_{\overline{M}_{x'}}$

fixes x' it acts trivially on Y^{nor} and hence on F. Since $G_{\overline{M}_{x'}}$ also acts trivially on $\overline{M}_{x'} = \overline{M}_x/F$ it acts trivially on $\overline{M}_x$, as needed. □

3.2.5. *The quotients.* Toroidal stabilizers can also be characterized in terms of the quotient morphisms. To obtain a nice picture we restrict to étale groups.

Lemma 3.2.6. *Assume that an **étale** diagonalizable group G acts in a relatively affine manner on a toroidal scheme (X, U) and $x \in X$ is a point. Then G^{tor}_x is the maximal subgroup H of the stabilizer G_x such that if $q : X \to X/H$ is the quotient morphism then the pair $(X/H, U/H)$ is toroidal at $q(x)$ and the morphism $(X, U) \to (X/H, U/H)$ is Kummer logarithmically étale at x.*

Proof. If $H \subseteq G^{\mathrm{tor}}_x$, that is H acts toroidally at x, then the quotient is as asserted by [Abramovich and Temkin 2017, Theorem 3.3.12]. Conversely, assume that H is such that q is Kummer logarithmically étale at x. Then $\overline{M}_{q(x)}$ contains $n\overline{M}_x$ for a large enough n, and since H acts trivially on $\overline{M}_{q(x)}$, it acts trivially on $\overline{M}_x$. So the action of H is simple in a neighborhood of x. Let C be the connected component of the logarithmic stratum containing x. If $H \nsubseteq G_\eta$ then the induced morphism $C \to q(C)$ is ramified at x because η is the generic point of C. But we assumed that q is logarithmically étale, and hence $C \to q(C)$ is étale at x. This shows that $H \subseteq G_\eta$, and hence $H \subseteq G_\eta \cap G_{\overline{M}_x} = G^{\mathrm{tor}}_x$, as required. □

3.2.7. *Functoriality.* Assume that toroidal schemes X and Y are provided with relatively affine actions of diagonalizable groups G and H, respectively, $\lambda : H \to G$ is a homomorphism, and $f : Y \to X$ is a λ-equivariant morphism. We want to study when the toroidal inertia groups are functorial in the sense that $H^{\mathrm{tor}}_y \hookrightarrow \lambda^{-1}(G^{\mathrm{tor}}_x)$ for any $y \in Y$ with $x = f(y)$. By [Abramovich and Temkin 2017, Lemma 3.1.6(i)], strict morphisms respect simplicity of the action. The toroidal property is more subtle: the functoriality of toroidal inertia may fail even for surjective fix-point reflecting strict morphisms.

Example 3.2.8. Let $X = \mathrm{Spec}(k[x, y])$ with the toroidal structure (x) and $G = \mathbb{Z}/2\mathbb{Z}$ acting by the sign both on x and y. Then the action is not toroidal at the origin O, so $G^{\mathrm{tor}}_{X,O} = 1$. Let Y be the x-axis $\mathrm{Spec}(k[x])$ with the toroidal structure (x). Then Y embeds G-equivariantly into X, but the action is toroidal on Y and hence $G^{\mathrm{tor}}_{Y,O} = G$ is not mapped into $G^{\mathrm{tor}}_{X,O}$. Furthermore, if $X_0 = X \smallsetminus \{O\}$ then $X_0 \coprod Y \to X$ is a surjective fix-point reflecting strict morphism which is not functorial for the toroidal inertia.

Remark 3.2.9. As this example shows, the statement in [Abramovich and Temkin 2017, Lemma 3.1.9(ii)] needs to be corrected to read "and the converse is true if f is *étale* and surjective", and the proof should read "Hence (ii) follows from (i), Lemma 3.1.6(i) and *étale descent*". This does not affect other results of that paper, since only the direct implication was used.

The problem in Example 3.2.8 is that O is in the fan of Y but not in the fan of X, and the stabilizer drops at $\eta_X(O)$. To avoid such examples we will restrict to logarithmically flat morphisms.

Lemma 3.2.10. *Assume that $f : Y \to X$ is a logarithmically flat morphism of toroidal schemes. Then for any point $y \in Y$ with $x = f(y)$ one has that $f(\eta_Y(y)) = \eta_X(x)$. In particular, $f(\mathrm{Fan}(Y)) \subseteq \mathrm{Fan}(X)$.*

Proof. It suffices to prove that each connected component C of a logarithmic stratum on Y goes to the same logarithmic stratum $X(n)$, and the induced morphism $f : C \to X(n)$ is flat. The claim is étale local, hence we can assume that f splits into a composition of a strict flat morphism $Y \to X_P[Q]$ and the projection $X_P[Q] \to X$, where $P \hookrightarrow Q$ and $X_P[Q] = X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q])$. The first case is clear, and in the second case the maps of the strata are easily seen to be flat. □

Lemma 3.2.11. *Let $f : Y \to X$ be a λ-equivariant morphism as in Section 3.2.7, and let $y \in Y$ be a point with $x = f(y)$ and the induced homomorphism $\lambda_y : H_y \to G_x$ such that f is logarithmically flat at y. Then*

(i) $\lambda_y(H_y^{\mathrm{tor}}) \subseteq G_x^{\mathrm{tor}}$;

(ii) *if, in addition, f is fix-point reflecting and either f is strict at y, or the action of H is simple at y, then $\lambda_y : H_y^{\mathrm{tor}} \xrightarrow{\sim} G_x^{\mathrm{tor}}$.*

Proof. Claim (i) follows from the following two observations: by logarithmic flatness $\overline{M}_x \subset \overline{M}_y$ so the inclusion $\lambda_y(H_{\overline{M}_y}) \subseteq G_{\overline{M}_x}$ holds, and the inclusion $\lambda_y(H_{\eta(y)}) \subseteq G_{\eta(x)}$ holds because $f(\eta(y)) = \eta(x)$ by Lemma 3.2.10.

In part (ii), strictness or simplicity assumption implies that $H_{\overline{M}_y} \xrightarrow{\sim} G_{\overline{M}_x}$. It remains to note that $H_{\eta(y)} \xrightarrow{\sim} G_{\eta(x)}$ because $f(\eta_Y(y)) = \eta_X(x)$ by Lemma 3.2.10 and f is fix-point reflecting. □

3.2.12. *Toroidal inertia.* For the sake of completeness we note that the groups G_x^{tor} glue to a *toroidal inertia* group scheme I_X^{tor} over the G-scheme X. Namely, if $\bar{\varepsilon}$ denotes the Zariski closure of ε then

$$I_X^{\mathrm{tor}} := \bigcup_{\varepsilon \in \mathrm{Fan}(X)} G_\varepsilon^{\mathrm{tor}} \times \bar{\varepsilon}$$

is a subgroup of $G \times X$, which is obviously contained in I_X. Since G is discrete there is no ambiguity about the scheme structure: $G \times X = \coprod_{g \in G} X$ and $I_X = \coprod_{g \in G} X^g$, where X^g is the closed subscheme fixed by g. The functoriality results of Lemma 3.2.11 extend to the toroidal inertia schemes in the obvious way.

3.3. *Toroidal stacks.* Using descent, the notions of toroidal schemes and morphisms can easily be extended to Artin stacks; see [Olsson 2003, Section 5]. We will stick to the case of DM stacks, since only they show up in our applications. A minor advantage of this restriction is that one can work with the étale topology instead of the lisse-étale topology.

3.3.1. *Logarithmic structures on stacks.* By a logarithmic structure on an DM stack X we mean a sheaf of monoids M_X on the étale site $X_{\text{ét}}$ and a homomorphism $\alpha_X : M_X \to \mathcal{O}_{X_{\text{ét}}}$ inducing an isomorphism $M_X^\times \xrightarrow{\sim} \mathcal{O}_{X_{\text{ét}}}^\times$. If $p_{1,2} : X_1 \rightrightarrows X_0$ is an atlas of X then giving a logarithmic structure M is equivalent to giving compatible logarithmic structures M_{X_i} in the sense that $p_i^{-1} M_{X_0} = M_{X_1}$ for $i = 1, 2$. We say that (X, M_X) is fine, saturated, etc., if (X_0, M_{X_0}) is so. We use here that these properties of M_{X_0} are étale local on X_0, and hence are independent of the choice of the atlas.

3.3.2. *Logarithmic stacks and atlases.* By a logarithmic stack (X, M_X) we mean a stack provided with a logarithmic structure. In this case, for any smooth atlas $X_1 \rightrightarrows X_0$ of X we provide X_0 and X_1 with the pullbacks of M_X and say that $(X_1, M_{X_1}) \rightrightarrows (X_0, M_{X_0})$ is an atlas of (X, M_X). Indeed, $\alpha_X : M_X \to \mathcal{O}_{X_{\text{ét}}}$ is uniquely determined by this datum.

3.3.3. *Toroidal stacks.* A logarithmic stack (X, M_X) is *logarithmically regular* or *toroidal* if it admits an atlas such that (X_0, M_{X_0}) is toroidal. In this case any atlas is toroidal because logarithmic regularity is a smooth-local property; see [Gabber and Ramero 2004, Proposition 12.5.46].

Furthermore, the triviality loci $U_i \subseteq X_i$ of M_{X_i} are compatible with respect to the strict morphisms $p_{1,2}$, hence U_0 descends to an open substack $i : U \hookrightarrow X$ that we call the triviality locus of M_X. Furthermore, when (X, M_X) is logarithmically regular, U determines the logarithmic structure by $M_X = \mathcal{O}_{X_{\text{ét}}} \cap i_*(\mathcal{O}^\times_{U_{\text{ét}}})$ because the same formulas reconstruct M_{X_i}. In the sequel, we will often view toroidal stacks as pairs (X, U). Again, a morphism $(Y, V) \to (X, U)$ of toroidal stacks is nothing else but a morphism $f : Y \to X$ of stacks such that $V \hookrightarrow f^{-1}(U)$.

3.4. ***Total toroidal coarsening.*** Let (X, U) be a toroidal DM stack.

3.4.1. *Toroidal coarsening morphisms.* Let $f : X \to Y$ be a coarsening morphism and $V \hookrightarrow Y$ the open substack corresponding to the open subset $f(|U|)$. We say that $f : X \to Y$ is *toroidal* if the pair (Y, V) is a toroidal stack, and the morphism $(X, U) \to (Y, V)$ is Kummer logarithmically étale. If it exists, the final object of the category of toroidal coarsening morphisms of X will be called the *total toroidal coarsening* of X and denoted $\phi_X : X \to X_{\text{tcs}}$.

Our next goal is to construct X_{tcs}. By Theorem A.1.3, ϕ_X is determined by the geometric points of its inertia, so our plan is as follows. First, we will extend the notion of toroidal stabilizers from Section 3.2.3 to geometric points of stacks, and then we will use them to construct ϕ_X so that, indeed, $(I_{\phi_X})_x$ is the toroidal stabilizer of x. In this context, I_{ϕ_X} is the generalization to toroidal stacks of the toroidal inertia I^{tor}_X from Section 3.2.12.

3.4.2. *Toroidal inertia.* Let $Z = X_{\text{cs}}$. By [Abramovich and Vistoli 2002, Lemma 2.2.3], a geometric point $x \to X$ possesses an étale neighborhood $X' = X \times_Z Z'$ of the form $[X'_0/G_x]$, in particular $X' \to X$ is inert. Pulling back the toroidal structure of X we obtain a G_x-equivariant toroidal structure on X'_0 and we take $G^{\text{tor}}_{X'_0,x}$ to be the maximal subgroup of G_x acting toroidally along x. By the following lemma, we can denote this group simply G^{tor}_x. It will be called the *toroidal stabilizer* at x. Note also that $\overline{M}_{X,x} = \overline{M}_{X'_0,x}$, and hence we obtain an action of G_x on $\overline{M}_x$. We say that X is *simple* if for any point $x \to X$ the group G_x acts on $\overline{M}_x$ trivially.

The toroidal stabilizer is related to the previous paragraph: by Lemma 3.2.6 a coarsening morphism $f : X \to Y$ is toroidal if and only if $\text{Ker}(G_x \to G_{f(x)}) \subset G^{\text{tor}}_x$.

Lemma 3.4.3. *With the above notation, the group $G^{\text{tor}}_{X'_0,x}$ and the action of G_x on $\overline{M}_x$ are independent of the choices of neighborhood X' and quotient presentation $X' = [X'_0/G_x]$.*

Proof. Given a finer étale neighborhood $Z'' \to Z'$ of the image of x in Z, set $X'' = X \times_Z Z''$ and $X_0'' = X_0' \times_{X'} X''$. In particular, $X'' = [X_0''/G_x]$. It suffices to check that $G^{\mathrm{tor}}_{X_0',x} = G^{\mathrm{tor}}_{X_0'',x}$. Being a base change of a morphism of algebraic spaces, the morphism $X'' \to X'$ is inert, and it follows that the strict étale G_x-equivariant morphism $X_0'' \to X_0'$ is inert. Therefore, $G^{\mathrm{tor}}_{X_0',x} = G^{\mathrm{tor}}_{X_0'',x}$ by [Abramovich and Temkin 2017, Lemma 3.1.9(ii)] and Remark 3.2.9. Also, it is clear that $\overline{M}_{X_0',x} = \overline{M}_{X_0'',x}$ as G_x-sets.

It remains to consider two different presentations $X' = [X_0'/G_x] \simeq [X_0''/G_x]$ over the same étale $Z' \to Z$. Write $Y = X_0' \times_{X'} X_0''$, so that $X' \simeq [Y/(G_x \times G_x)]$. One checks that $Y \to X_0'$ and $Y \to X_0''$ are inert. Lemma 3.2.11 implies $G^{\mathrm{tor}}_{X_0',x} = G^{\mathrm{tor}}_{X_0'',x}$, giving the result. □

Functoriality properties from Lemma 3.2.11 extend to stacks straightforwardly.

Lemma 3.4.4. *Let $f : Y \to X$ be a morphism of toroidal stacks, and $y \to Y$ a point with $x = f(y)$ and the induced homomorphism $\lambda_y : G_y \to G_x$.*

(i) *If f is étale, strict and inert, then $\lambda_y : G_y^{\mathrm{tor}} \xrightarrow{\sim} G_x^{\mathrm{tor}}$.*

(ii) *If f is logarithmically flat at y, then $\lambda_y(G_y^{\mathrm{tor}}) \subseteq G_x^{\mathrm{tor}}$. If, in addition, f is inert and Y is simple at y, then $\lambda_y : G_y^{\mathrm{tor}} \xrightarrow{\sim} G_x^{\mathrm{tor}}$.*

Proof. If $Y_0 \to X_0$ is a λ_y-equivariant morphism of affine schemes inducing $f : Y = [Y_0/G_y] \to X = [X_0/G_x]$ then the toroidal stabilizers equal to the toroidal stabilizers of the actions of G_y and G_x on Y_0 and X_0, respectively. In this case, (i) follows from [Abramovich and Temkin 2017, Lemma 3.1.9(ii)] and Remark 3.2.9, and (ii) follows from Lemma 3.2.11.

The general case is reduced to this by local work on the coarse moduli spaces: first we base change both stacks with respect to an étale morphism $Z' \to X_{\mathrm{cs}}$ such that we can present $X = [X_0/G_x]$. Then we replace Y further by an appropriate étale neighborhood of y induced from an étale neighborhood of its image in Y_{cs}, so that we can present $Y = [Y_0/G_y]$. Now the G_x-torsors associated to $Y \to BG_y \to BG_x$ and $Y \to X \to BG_x$ agree on the residual gerbe $BG_y \subset Y$, so that after further inert localization of Y they agree on Y. This provides a λ-equivariant morphism $Y_0 \to X_0$ as needed. □

3.4.5. *Toroidal orbifolds.* In the sequel, by a *toroidal orbifold* we mean a toroidal DM stack X with finite diagonalizable inertia (but note Remarks 4 and 3.2.2(iii)). We allow the generic stabilizer to be nontrivial.

3.4.6. *The construction.* Now we can construct the total toroidal coarsening.

Theorem 3.4.7. *Let $\widetilde{\mathcal{C}}$ be the 2-category of toroidal orbifolds with the subcategory $\mathcal{C}$ of simple objects. Then*

(i) *for any object X of $\widetilde{\mathcal{C}}$, the total toroidal coarsening X_{tcs} exists;*

(ii) *for any geometric point $x \to X$, we have $(I_{X/X_{\mathrm{tcs}}})_x = G_x^{\mathrm{tor}}$, where $(I_{X/X_{\mathrm{tcs}}})_x$ is the relative stabilizer and G_x^{tor} the toroidal inertia group;*

(iii) *any logarithmically flat morphism $h : Y \to X$ in $\widetilde{\mathcal{C}}$ induces a morphism $h_{\mathrm{tcs}} : Y_{\mathrm{tcs}} \to X_{\mathrm{tcs}}$ with a 2-commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\phi_Y} & Y_{\mathrm{tcs}} \\ {\scriptstyle h}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle h_{\mathrm{tcs}}} \\ X & \xrightarrow[\phi_X]{} & X_{\mathrm{tcs}} \end{array}$$

and the pair $(h_{\mathrm{tcs}}, \alpha)$ is unique in the 2-categorical sense: if $(h'_{\mathrm{tcs}}, \alpha')$ is another such pair then there exists a unique 2-isomorphism $h'_{\mathrm{tcs}} = h_{\mathrm{tcs}}$ making the whole diagram 2-commutative;

(iv) *assume that h is logarithmically smooth and inert, and Y is simple. Then the diagram in (iii) is 2-cartesian.*

The present proof of (i) and (ii) was suggested by David Rydh.

Proof. We first show that there is an open and closed subgroup $I_X^{\mathrm{tor}} \subset I_X$ with fibers G_x^{tor}.

Fix x and write $G = G_x$. By [Abramovich and Vistoli 2002, Lemma 2.3.3] there is a neighborhood $Z_0 \to Z := X_{\mathrm{cs}}$ and a G_x-scheme W_0 with isomorphism $X_0 := [W_0/G] \simeq X \times_Z Z_0$. By Lemma 3.4.3 we may replace X by X_0. Since $|X_0| = |Z_0|$, by Lemma 3.2.4 we can shrink Z_0 so that $G_w^{\mathrm{tor}} = G_x^{\mathrm{tor}} \cap G_w$ for any $w \in W_0$. Since $G_x^{\mathrm{tor}} \subset G$ are discrete groups this defines an open and closed subgroup $I_X^{\mathrm{tor}} \subset I_X$.

Theorem A.1.3 provides a coarsening morphism $X \to X_{\mathrm{tcs}}$ satisfying (i), (ii).

To prove (iii) we should prove that the morphism $Y \to X_{\mathrm{tcs}}$ factors through Y_{tcs} uniquely. So, by Theorem 2.3.6 we should prove that $I_{Y/Y_{\mathrm{tcs}}}$ is mapped to zero in $I_{X_{\mathrm{tcs}}}$. We claim that, moreover, the map $I_Y \to I_X$ takes $I_{Y/Y_{\mathrm{tcs}}}$ to $I_{X/X_{\mathrm{tcs}}}$. It suffices to check this on the geometric points, since the inertia are étale for DM stacks. But the latter is covered by Lemma 3.4.4(ii).

Let us prove (iv). Let Q denote the square diagram from (iii). Choose an étale covering $f : Z \to X_{\mathrm{tcs}}$ with Z a scheme. It suffices to show that the base change square $f^*(Q) := Q \times_{X_{\mathrm{tcs}}} Z$ is 2-cartesian. For any point $y \to Y$ with $x = h(y)$ we have that $G_y^{\mathrm{tor}} \xrightarrow{\sim} G_x^{\mathrm{tor}}$ by Lemma 3.4.4(ii). Hence $I_{\phi_Y(y)} = I_{\phi_X(x)}$, and we obtain that the morphism h_{tcs} is inert. It follows that $Z \times_{X_{\mathrm{tcs}}} Y_{\mathrm{tcs}}$ is an algebraic space. Thus, the morphisms $f^*(\phi_X)$ and $f^*(\phi_Y)$ are coarsening morphisms whose targets are algebraic spaces, and hence both are usual coarse spaces. We can now apply Lemma B.2.6 to conclude that the square $f^*(Q)$ is 2-cartesian. □

4. Destackification

4.1. *The main result.*

4.1.1. *Blowings up of toroidal stacks.* We say that a morphism $f : (X', U') \to (X, U)$ of toroidal stacks is the *blowing up along* a closed substack $Z \hookrightarrow X$ if $f : X' \to X$ is a blowing up along Z and $U' = f^{-1}(U) \smallsetminus f^{-1}(Z)$. For example, a blowing up of toroidal schemes is a blowing up of usual schemes $f : X' \to X$ such that the toroidal divisor $X' \smallsetminus U'$ of (X', U') is the union of the preimage of the toroidal divisor of (X, U) and the exceptional divisor of f. We use the same definition for normalized blowings up.

4.1.2. *Torification.* Our destackification results are based on and can be viewed as stack-theoretic enhancements of torification theorems of [Abramovich and Temkin 2017]. In Appendix B we recall these results and slightly upgrade them according to the needs of this paper.

4.1.3. *Destackification theorem.* Let us first formulate our main results on destackification. Their proof will occupy the rest of Section 4. Using the torification functors $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ we will construct two destackification functors: $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. The former one has stronger functoriality properties, but only applies to toroidal stacks with inertia acting *simply*.

Theorem 4.1.4. *Let $\widetilde{\mathcal{C}}$ be the category of toroidal orbifolds.*

(i) *For any object X of $\widetilde{\mathcal{C}}$ there exists a sequence of birational blowings up of toroidal stacks $\widetilde{\mathcal{F}}_X : X_n \to \cdots \to X$ such that $(X_n)_{\mathrm{tcs}} = (X_n)_{\mathrm{cs}}$.*

(ii) *In addition, one can choose $\widetilde{\mathcal{F}}$ compatible with surjective smooth strict inert morphisms $f : X' \to X$ from $\widetilde{\mathcal{C}}$ in the sense that for any such f the sequence $\widetilde{\mathcal{F}}_{X'}$ is the pullback of $\widetilde{\mathcal{F}}_X$. Compatibility on the level of morphisms holds even without assuming that f is surjective.*

Theorem 4.1.5. *Let $\mathcal{C}$ be the category of simple toroidal orbifolds. Then to any object X in $\mathcal{C}$ one can associate a birational blowing up of toroidal stacks $\mathcal{F}_X : X_1 \to X$ along an ideal I_X and a blowing up $\mathcal{F}^0_X : X_0 \to X_{\mathrm{cs}}$ along an ideal J_X so that*

(i) $(X_1)_{\mathrm{tcs}} = (X_1)_{\mathrm{cs}} = X_0$;

(ii) *if $f : X' \to X$ is a surjective logarithmically smooth inert morphism in $\mathcal{C}$, then $\mathcal{F}_{X'}$ and $\mathcal{F}^0_{X'}$ are the pullbacks of $\mathcal{F}_X$ and $\mathcal{F}^0_X$, respectively. Compatibility on the level of morphisms holds even without assuming that f is surjective.*

For the sake of completeness, we note that claim (ii) of the two theorems is also satisfied for strict morphisms f which are strongly equivariant in the sense that $f : X' \to X$ is the pullback of $f_{\mathrm{cs}} : X'_{\mathrm{cs}} \to X_{\mathrm{cs}}$. For these versions of Theorem 4.1.4(ii) (resp. Theorem 4.1.5(ii)) the proof is the same, but the reference to Corollary B.2.7 should be replaced by a reference to Theorem B.2.2 (resp. Theorem B.2.4). In both cases birationality follows from Proposition B.1.4.

4.2. ***The proof.*** We will work with Theorem 4.1.5 for concreteness. The proof of Theorem 4.1.4 is similar and involves less details; the main difference is that one should use Theorem B.2.2 as the torification input instead of Corollary B.2.7. (Recall that smooth inert morphisms are strongly equivariant by [Abramovich and Temkin 2018, Theorem 1.3.1(ii)(b)].)

We will construct the functor $\mathcal{F}$ by showing that the torification functor $\mathcal{T}$ descends to stacks. This will be done in two stages: first we will establish its descent to global quotients $[W/G]$ and then will use étale descent with respect to inert morphisms.

4.2.1. *Step 1: the global quotient case.* We will first prove the theorem for the subcategory $\mathcal{C}'$ of $\mathcal{C}$ whose objects X are of the form $[W/G]$, where G is an étale diagonalizable group acting on a toroidal quasiaffine scheme W.

Since the blowing up and the center of $\mathcal{T}'_{W,G}$ are G-equivariant, they descend to X. Namely, there exists a unique blowing up of toroidal stacks $\mathcal{F}_{X,W} : X_1 \to X$ whose pullback to W is $\mathcal{T}'_{W,G} : W_1 \to W$. Since $[W/G]_{\mathrm{cs}} = W/G$, we simply set $\mathcal{F}^0_{X,W} = \mathcal{T}'^0_{W,G}$. We claim that these $\mathcal{F}_{X,W}$ and $\mathcal{F}^0_{X,W}$ are independent of the choice of the covering W.

Suppose that $X = [W'/G']$ is another such representation. Note that $X = [W''/G'']$, where $W'' = W \times_X W'$ and $G'' = G \times G'$, and it suffices to compare the blowings up induced from W and W''. In this case the projection $W'' \to W$ is inert and λ-equivariant for the projection $\lambda : G'' \twoheadrightarrow G$, and hence $\mathcal{T}'_{W'',G''}$ and $\mathcal{T}'^0_{W'',G''}$ are the pullbacks of $\mathcal{T}'_{W,G}$ and $\mathcal{T}'^0_{W,G}$ by Corollary B.2.7. It follows that $\mathcal{F}_{X,W} = \mathcal{F}_{X,W''}$ and $\mathcal{F}^0_{X,W} = \mathcal{F}^0_{X,W''}$, and in the sequel we can safely write $\mathcal{F}_X$ and $\mathcal{F}^0_X$.

The properties of $\mathcal{F}$ and $\mathcal{F}^0$ are checked similarly, so we will only discuss $\mathcal{F}$. The action of G on W_1 is toroidal, hence $G_w = G^{\mathrm{tor}}_w$ for any $w \in W_1$. Since $X_1 = [W_1/G]$, the definition of toroidal stabilizers in Section 3.4.2 implies that $G_x = G^{\mathrm{tor}}_x$ for any geometric point $x \to X_1$. Therefore, $(X_1)_{\mathrm{tcs}} = (X_1)_{\mathrm{cs}}$ by Theorem 3.4.7. Assume that $f : X' \to X$ is a logarithmically smooth inert morphism in $\mathcal{C}'$. Choose presentations $X = [W/G]$ and $X' = [W'/G']$. Replacing the latter presentation by $[W' \times_X W/G \times G']$, we can assume that there is a homomorphism $\lambda : G' \to G$ such that f lifts to a λ-equivariant morphism $h : W' \to W$. Since f is inert, the same is true for h, and $\mathcal{T}'_{W,G}$ and $\mathcal{T}'_{W',G'}$ are compatible by Corollary B.2.7. By the definition of $\mathcal{F}$ on $\mathcal{C}'$, we obtain that $\mathcal{F}_X$ and $\mathcal{F}_{X'}$ are compatible too.

4.2.2. *Step 2: inert étale descent.* Assume now that X is an arbitrary toroidal orbifold. By [Abramovich and Vistoli 2002, Lemma 2.2.3], the coarse moduli space $Z = X_{\mathrm{cs}}$ possesses an étale covering

$$Z' = \coprod_{i=1}^{l} Z_i \to Z$$

such that each Z_i is affine and each $X_i = X \times_Z Z_i$ lies in $\mathcal{C}'$, say $X_i = [W_i/G_i]$. Note that $X' = \coprod_{i=1}^{l} X_i$ is also in $\mathcal{C}'$, for example, $X' = W'/G'$ for $W' = \coprod_i (X_i \times \prod_{j \neq i} G_j)$ and $G' = \prod_j G_j$. Furthermore, $X'' = X' \times_X X'$ is also in $\mathcal{C}'$ since $X'' = [W''/G'']$ for $W'' = W' \times_X W'$ and $G'' = G' \times G'$. (Although $I_X \to X$ is finite, X does not have to be separated, so W'' can be quasiaffine even though we started with an affine W'.)

By Section 4.2.1 $\mathcal{F}$ was defined for X' and X'' and $\mathcal{F}_{X''}$ is the pullback of $\mathcal{F}_{X'}$ with respect to either of the projections $X'' \rightrightarrows X'$. By étale descent, $\mathcal{F}_{X'}$ is the pullback of a blowing up $\mathcal{F}_{X,X'} : X_1 \to X$ of the toroidal stack X. In the same fashion, the blowings up $\mathcal{F}^0_{X'}$ and $\mathcal{F}^0_{X''}$ of Z' and $Z'' = Z' \times_Z Z'$ descend to a blowing up $\mathcal{F}^0_{X,X'} : Z_1 \to Z$, and by descent $(X_1)_{\mathrm{cs}} = Z_1$. Independence of the covering $X' \to X$ is proved as usual: given another such covering one passes to their fiber product, which is also a global quotient of a quasiaffine scheme, and then uses that $\mathcal{F}$ is compatible with inert morphisms.

We have now constructed $\mathcal{F}_X$ and $\mathcal{F}^0_X$ for an arbitrary object of $\mathcal{C}$. Their properties are established by étale descent via a covering $f : X' \to X$ as above. For example, for any geometric point $x \to X_1$ choose a lifting $x' \to X'_1$. Then $G_x = G_{x'}$ because f is inert, and hence $f_1 : X'_1 \to X_1$ is inert too. In addition, $G^{\mathrm{tor}}_x = G^{\mathrm{tor}}_{x'}$ by Lemma 3.4.4(i), and $G_{x'} = G^{\mathrm{tor}}_{x'}$ by Step 1. Thus, $G_x = G^{\mathrm{tor}}_x$, and hence $(X_1)_{\mathrm{tcs}} = (X_1)_{\mathrm{cs}}$.

5. Kummer blowings up

5.1. *Permissible centers.*

5.1.1. *Toroidal subschemes.* Let X be a toroidal scheme. We say that a closed subscheme Y of X is *toroidal* if $(Y, \mathcal{M}_X|_Y)$ is toroidal. Thus toroidal closed subschemes correspond to strict closed immersions of toroidal schemes. We stress that this differs from the terminology of [Abramovich and Temkin 2017, §2.3.12], in that toroidal subschemes are not defined by monomial ideals. Rather, they are locally described as follows:

Lemma 5.1.2. *Let X be a toroidal scheme and Y a closed subscheme of X. Then Y underlies a toroidal subscheme if and only if locally at any point $y \in Y$ there exist elements $t_1, \dots, t_n \in \mathcal{O}_{X,y}$ restricting to regular parameters on the stratum $X(d)$ of X through y, and $m \le n$ such that $Y = V(t_1, \dots, t_m)$ locally at y.*

Elements $t_1, \dots, t_n \in \mathcal{O}_{X,y}$ as in the statement will be called *regular coordinates*.

Proof. The inverse implication follows from the formal-local description of toroidal schemes; see [Kato 1994, Theorem 3.2]. Assume that Y is toroidal and let us construct required coordinates at y. We can assume that X and Y are local with closed point y. Let d be the rank of $\overline{M}_{X,y} = \overline{M}_{Y,y}$, and let n and $n-m$ be the dimensions of the closed logarithmic strata $X(d)$ and $Y(d)$. Since $X(d)$ and $Y(d)$ are regular, $\mathcal{O}_{X(d),y}$ possesses a regular family of parameters $t'_1, \dots, t'_n$ such that $V(t'_1, \dots, t'_m) = Y(d)$. Lift them to coordinates $t_1, \dots, t_n \in \mathcal{O}_{X,y}$. Since $Y(d) = X(d) \times_X Y$, we can also achieve that $t_1, \dots, t_m$ vanish on Y. The scheme $V(t_1, \dots, t_m)$ is integral (even toroidal) by the inverse implication, and $\dim(X) = d + n$ and $\dim(Y) = d + n - m$, hence the closed immersion $Y \hookrightarrow V(t_1, \dots, t_m)$ is an isomorphism. □

5.1.3. *Permissible centers.* Let X be a toroidal scheme. An ideal $J \subset \mathcal{O}_X$ is *monomial* if it is the image of a monoid ideal in M_X. A closed subscheme $Z = \mathrm{Spec}_X(\mathcal{O}_X/I)$ is called a *permissible center* if locally at any point $z \in Z$ it is the intersection of a toroidal subscheme and a monomial subscheme, that is, there exists a regular family of parameters $t_1, \dots, t_n$ and a monomial ideal J such that $I = (t_1, \dots, t_l, J)$ for $l \le n$.

5.1.4. *Playing with the toroidal structure.* A standard method used in toroidal geometry is to enlarge/decrease the toroidal structure by adding/removing components to/from $X \smallsetminus U$. For example, see [Abramovich and Temkin 2017, §§3.4, 3.5]. We will use this method, and here is a first step.

Lemma 5.1.5. *Assume that (X, U) is a local toroidal scheme, C is the closed logarithmic stratum and $t_1, \dots, t_n$ a regular family of parameters of $\mathcal{O}_{C,x}$. Let W be obtained from U by removing the divisors $V(t_1), \dots, V(t_l)$, where $0 \le l \le n$. Then (X, W) is toroidal and $\overline{M}_{(X,W),x} = \overline{M}_{(X,U),x} \oplus \mathbb{N}^l$.*

Proof. The equality of the monoids is clear. Since the intersection of C with $V(t_1, \dots, t_l)$ is regular of codimension l we obtain that (X, W) is toroidal at x and hence toroidal. □

Corollary 5.1.6. *Assume that (X, U) is a toroidal scheme and $Z \hookrightarrow X$ is a permissible center. Then locally on X one can enlarge the toroidal structure of X so that Z is a monomial subscheme of the new toroidal scheme (X, W).*

Proof. Locally at $x \in X$ the center is given by $(t_1, \dots, t_l, J)$, where J is monomial. Set $W = U \setminus \bigcup_{i=1}^{l} V(t_i)$ and use Lemma 5.1.5. □

5.1.7. *Functoriality.* Permissible centers are respected by logarithmically smooth morphisms.

Lemma 5.1.8. *Assume that $f : Y \to X$ is a logarithmically smooth morphism of toroidal schemes and $Z \hookrightarrow X$ is a permissible center (resp. a toroidal subscheme). Then $Z \times_X Y$ is a permissible center (resp. a toroidal subscheme) in Y.*

Proof. Note that f induces smooth morphisms between logarithmic strata of Y and X. It follows that if $t_1, \dots, t_n$ are regular coordinates at $x \in X$ then their pullbacks form a part of a family of regular coordinates at a point $y \in f^{-1}(x)$. In view of Lemma 5.1.2, this implies the claim about toroidal subschemes. Since pullback of a monomial subscheme is obviously monomial, we also obtain the claim about permissible centers. □

5.2. *Permissible blowings up.*

5.2.1. *The model case.* We will prove that permissible centers give rise to normalized blowings up of toroidal schemes in the sense of Section 4.1.1. This can be done very explicitly in the model case when $X = \boldsymbol{A}^n_M = \mathrm{Spec}(B[M, t_1, \dots, t_n])$, where B is an arbitrary regular ring, M is a toric monoid, and $I = (t_1, \dots, t_n, m_1, \dots, m_r)$ for $m_i \in M$. For the sake of illustration we consider this case separately. Let $X' = \mathrm{Bl}_I(X)^{\mathrm{nor}}$ be the normalized blowing up of X along I. We have two types of charts:

(1) The t_i-chart is $\boldsymbol{A}^{n-1}_N = \mathrm{Spec}(B[N, t_1/t_i, \dots, t_n/t_i])$, where N is the saturation of the submonoid of $M \oplus \mathbb{Z}t_i$ generated by M, t_i and the elements $m_1 - t_i, \dots, m_r - t_i$. In particular, for any point x' of the chart with image $x \in X$ one has that $\mathrm{rk}(\overline{M}_{x'}) \le \mathrm{rk}(\overline{M}_x) + 1$. The monoid N is still sharp.

(2) The m_j-chart is $\boldsymbol{A}^n_P = \mathrm{Spec}(B[P, t_1/m_j, \dots, t_n/m_j])$, where P is the saturation of the submonoid of M^{gp} generated by M and the elements $m_1 - m_j, \dots, m_r - m_j$. In particular, the rank does not increase on this chart: $\mathrm{rk}(\overline{M}_{x'}) \le \mathrm{rk}(\overline{M}_x)$ for any point x' sitting over $x \in X$. The monoid P need not be sharp.

5.2.2. *The general case.* One can deal with the general case similarly by reducing to formal charts, but this is slightly technical, especially in the mixed characteristic case. A faster way is to play with the toroidal structure, reducing to the known properties of toroidal blowings up.

Lemma 5.2.3. *Assume that (X, U) is a toroidal scheme and $f : X' \to X$ is the normalized blowing up along a permissible center $Z \hookrightarrow X$, and set $U' = f^{-1}(U \setminus Z)$. Then (X', U') is a toroidal scheme and hence f underlies a normalized blowing up of toroidal schemes.*

Proof. The question is étale local on X, so we can assume that $X = \mathrm{Spec}(A)$ is a strictly henselian scheme with closed point x. Then $Z = V(t_1, \dots, t_l, m_1, \dots, m_r)$, where m_i are monomials and $t_1, \dots, t_n$ is a family of regular parameters of the logarithmic stratum through x. Set $W = U \setminus \bigcup_{i=1}^{l} V(t_i)$. Then (X, W) is toroidal by Lemma 5.1.5 and Z is a monomial subscheme of (X, W). Set $W' = f^{-1}(W \setminus Z)$.

Then (X', W') is toroidal and the toroidal blowing up $(X', W') \to (X, W)$ is logarithmically smooth; see [Nizioł 2006, Section 4] for proofs or [Abramovich and Temkin 2017, Lemma 4.3.3] for a summary. Furthermore, $X' \smallsetminus U'$ is obtained from $X' \smallsetminus W'$ by removing the strict transforms D_i' of $D_i = V(t_i)$, so we should prove that this operation preserves the toroidal property. By [Abramovich and Temkin 2017, Theorem 2.3.15] it suffices to prove that each D_i' is a Cartier divisor.

Now choose $y \in \{t_1, \dots, t_l, m_1, \dots, m_r\}$ and let us study the situation on the y-chart X'_y. We claim that the inclusion $D_i'|_{X'_y} \hookrightarrow V(t_i/y)$ is an equality and hence D_i' is Cartier, as required. If $y = t_i$ there is nothing to prove, so assume that $y \neq t_i$. It suffices to show that $V(t_i/y)$ is integral. So, for any $x' \in X'_y$ it suffices to prove that $\overline{M}_{x'}$ splits as $Q \oplus (t_i - y)\mathbb{N}$. To compute $\overline{M}_{x'}$ we recall that toroidal blowings up are base changes of toric blowings up of the charts. In particular, $X' \to X$ is the base change of the blowing up of $\operatorname{Spec}(\mathbb{Z}[M, t_1, \dots, t_l])$ along the ideal generated by $(t_1, \dots, t_l, m_1, \dots, m_r)$. The latter was computed in Section 5.2.1, and we saw that, indeed, its charts are of the form $\operatorname{Spec}(\mathbb{Z}[Q, t_i/y])$. □

5.2.4. *Functoriality.* In the sequel, by a *permissible blowing up* we mean the normalized blowing up along a permissible center. To simplify the notation, we will omit the normalization and will simply write $\mathrm{Bl}_I(X)$ or $\mathrm{Bl}_Z(X)$. Naturally, permissible blowings up are compatible with logarithmically smooth morphisms.

Lemma 5.2.5. *Let X be a toroidal scheme and let $Z \hookrightarrow X$ be a permissible center. Then for any logarithmically smooth morphisms $f : Y \to X$ of toroidal schemes, the pullback $T = Z \times_X Y$ is a permissible center and $\mathrm{Bl}_T(Y) = \mathrm{Bl}_Z(X) \times_X Y$ in the category of fs logarithmic schemes.*

Proof. We know that T is permissible by Lemma 5.1.8. The problem is local on X hence we can assume that X is local. As in the proof of Lemma 5.2.3, $Z = V(t_1, \dots, t_l, m_1, \dots, m_r)$ and Z becomes monomial once we replace $U = X(0)$ by $U' = U \smallsetminus \bigcup_{i=1}^{l} V(t_i)$. Since the pullbacks of t_i form a subfamily of a regular family at any point of $f^{-1}(x)$, we also have that $V' = Y(0) \smallsetminus \bigcup_{i=1}^{l} f^{-1}(V(t_i))$ defines a toroidal structure and T is monomial on (Y, V'). We omit the easy check that the morphism $(Y, V') \to (X, U')$ is logarithmically smooth. The lemma now follows from the fact that toroidal blowings up are compatible with logarithmically smooth morphisms; see [Nizioł 2006, Corollary 4.8]. □

5.3. ***Kummer ideals.*** Let X be a logarithmic scheme. In [Abramovich et al. 2020] we also use a generalization of permissible blowings up that we are going to define now. Informally speaking, we will blow up "ideals" of the form $(t_1, \dots, t_n, m_1^{1/d}, \dots, m_r^{1/d})$. Our next aim is to formalize such objects, and the main task is to define "ideals" $(m^{1/d})$.

5.3.1. *Ideals $I^{[1/d]}$.* First, let us describe the best approximation to extracting roots on the logarithmic scheme itself. For any monomial ideal I and $d \geq 1$ let $I^{[1/d]}$ denote the monomial ideal J generated by monomials m with $m^d \in I$. Recall that monomial ideals are in a one-to-one correspondence with the ideals of $\overline{M}_X$. If I corresponds to $J \subseteq \overline{M}_X$ then $I^{[1/d]}$ corresponds to $(1/d)J \cap \overline{M}_X$. So, extracting the root is a purely monomial operation, and hence it is compatible with strict morphisms $f : Y \to X$ in the sense that

$$(f^{-1}(I))^{[1/d]} = f^{-1}(I_X^{[1/d]}).$$

Remark 5.3.2. It may happen that I is invertible but $I^{[1/d]}$ is not. On the level of monoids this can be constructed as follows: take $M \subset \mathbb{N}^2$ given by (x, y) with $x + y \in 3\mathbb{Z}$ and $I = (3, 3) + M$. Then $I^{[1/3]}$ is generated by $(1, 2)$ and $(2, 1)$ and it is not principal.

5.3.3. *Kummer monomials.* By a *Kummer monomial* on a logarithmic scheme X we mean a formal expression $m^{1/d}$ where m is a monomial on X and $d \geq 1$ is an integer which is invertible on X. In order to view $m^{1/d}$ as an actual function we should work locally with respect to a certain log-étale topology. For example, $X[m^{1/d}] := (X \otimes_{k[m]} k[m^{1/d}])^{\mathrm{sat}}$ is the universal fs logarithmic scheme over X on which $m^{1/d}$ is defined, and $X[m^{1/d}] \to X$ is logarithmically étale by our assumption on d.

Remark 5.3.4. One can also consider roots with a noninvertible d but then the morphism $X[m^{1/d}] \to X$ is only logarithmically syntomic, i.e., logarithmically flat and lci. We prefer to exclude such cases because we will later consider only toroidal schemes, and logarithmic regularity is not local with respect to the log-syntomic topology.

5.3.5. *Kummer topology.* In order to define operations on different monomials one has to pass to larger covers of X, and there are two ways to do this uniformly. The first one is to consider the pro-finite coverings and work with structure sheaves on nonnoetherian schemes; see [Talpo and Vistoli 2018]. Another possibility is to work with the structure sheaf of a topology generated by finite coverings. The two approaches are equivalent. We adopt the second one using the Kummer logarithmically étale topology defined by Nizioł [2008]. For brevity, it will be called the Kummer topology.

Recall that a logarithmically étale morphism $f : Y \to X$ is called *Kummer* if for any point $y \in Y$ with $x = f(y)$ the homomorphism $\overline{M}_x^{\mathrm{gp}} \to \overline{M}_y^{\mathrm{gp}}$ is injective with finite cokernel, and $\overline{M}_y$ is the saturation of $\overline{M}_x$ in $\overline{M}_y^{\mathrm{gp}}$. Setting surjective Kummer morphisms to be coverings, we obtain a *Kummer topology* on the category of fs logarithmic schemes. The site of Kummer logarithmic schemes over X will be denoted $X_{\mathrm{két}}$. The following lemma shows that when working with the Kummer topology it suffices to consider two special types of coverings. The proof is simple, and we refer to [Nizioł 2008, Corollary 2.17] for details.

Lemma 5.3.6. *The topology of $X_{két}$ is generated by two types of coverings*: *strict étale morphisms $Z \to Y$ and morphisms of the form $Y[m^{1/d}] \to Y$, with d invertible in $\mathcal{O}_Y$.*

5.3.7. *The structure sheaf.* The rule $Y \mapsto \Gamma(\mathcal{O}_Y)$ defines a presheaf of rings $\mathcal{O}_{X_{\mathrm{két}}}$ on $X_{\mathrm{két}}$.

Lemma 5.3.8. *The presheaf $\mathcal{O}_{X_{két}}$ is a sheaf.*

Proof. A more general claim is proved in [Nizioł 2008, Proposition 2.18]. Let us outline a simple argument that works in our case. It suffices to check the sheaf condition for the two coverings from Lemma 5.3.6. The first case is clear since $\mathcal{O}_{X_{\mathrm{ét}}}$ is a sheaf. In the second case we note that μ_d acts on $Y' = Y[m^{1/d}]$ and Y is the quotient, in particular, $\mathcal{O}_Y(Y')^{\mu_d} = \mathcal{O}_Y(Y)$. The saturated fiber product $Y'' = (Y' \times_Y Y')^{\mathrm{sat}}$ equals $\mu_d \times Y'$, and hence the equalizer of $\mathcal{O}_Y(Y') \rightrightarrows \mathcal{O}_Y(Y'')$ equals $\mathcal{O}_Y(Y')^{\mu_d}$, that is, $\mathcal{O}_Y$ satisfies the sheaf condition with respect to the covering $Y' \to Y$. □

5.3.9. *Kummer ideals.* By a Kummer ideal we mean an ideal $I \subseteq \mathcal{O}_{X_{\text{két}}}$ which is coherent in the following sense: there exists a Kummer covering $Y \to X$ and a coherent ideal $I_Y \subseteq \mathcal{O}_Y$ such that $I|_{Y_{\text{két}}}$ is generated by I_Y in the sense that $\Gamma(Z, I) = \Gamma(Z, I_Y\mathcal{O}_Z)$ for any Kummer morphism $Z \to Y$.

Example 5.3.10. (i) If I_X is a monomial ideal on X let I be the associated ideal on $X_{\text{két}}$ and for Y Kummer over X let I_Y denote restrictions of I onto Y. Given $d \geq 1$ define $J = I^{1/d}$ by $J_Y = (I_Y)^{[1/d]}$. Note that the projections $p_{1,2}$ of $Z = (Y \times_X Y)^{\text{sat}}$ onto Y are strict. Hence $p_i^{-1}(J_Y) = J_Z$ for $i = 1, 2$, and we obtain that the pullbacks are naturally isomorphic, that is, J is an ideal in $\mathcal{O}_{X_{\text{két}}}$. Moreover, J is coherent because one can construct a covering $Y \to X$ such that $I_Y = J_Y^d$ and then $J_Z = J_Y\mathcal{O}_Z$ for any Kummer morphism $Z \to Y$. For example, choose an étale covering $\bigcup_i X_i \to X$ such that the ideals $I|_{X_i} = (\{m_{ij}\})$ are globally generated by monomials, let $Y_i = (X_i[m_{i1}^{1/d}, m_{i2}^{1/d}, \ldots])^{\text{sat}}$, and take $Y = \coprod_i Y_i$.

(ii) One can produce more ideals using addition and multiplication, ideals coming from $\mathcal{O}_X$, and Kummer ideals from (i). For example, if $t_i \in \Gamma(\mathcal{O}_X)$ and m_j are global monomials then the ideal $J = (t_1, \ldots, t_n, m_1^{1/d}, \ldots, m_r^{1/d})$ is a well-defined coherent Kummer ideal, as well as its powers J^l.

Remark 5.3.11. (i) It is essential that we are working with saturated logarithmic schemes and the Kummer topology. For example, if $X = \text{Spec}(k[t])$ and X_{fl} denotes the small flat site of X then by the usual flat descent $\mathcal{O}_{X_{\text{fl}}}$ is a sheaf in which any coherent ideal comes from a coherent ideal of $\mathcal{O}_X$. In particular, the ideal $t\mathcal{O}_{X_{\text{fl}}}$ is not a square. This happens for the following reason: although $(t) = (y^2)$ on the double covering $Y = \text{Spec}(k[y]) \to X$ with $y^2 = t$, the fiber product $Z = Y \times_X Y$ equals to $\text{Spec}(k[y_1, y_2]/(y_1^2 - y_2^2))$ and the two pullbacks of (y) to Z are different: $(y_1) \neq (y_2)$. In other words, the root $(y) = \sqrt{(t)}$ is not unique locally on X_{fl} and hence does not give rise to an ideal.

(ii) The sheaf $\mathcal{O}_{X_{\text{két}}}$ also has noncoherent ideals. For example, for $X = \text{Spec}(k[m])$ the maximal monomial ideal $\sum_{d=1}^{\infty}(m^{1/d})$. In fact, it is not even quasicoherent because it is not generated by an ideal on a Kummer étale cover of X.

5.4. ***Blowings up of permissible Kummer ideals.*** This section provides the key construction of a Kummer blowing up of a toroidal scheme. It was pointed out by David Rydh that Kummer blowings up have an elegant construction using stack-theoretic $\mathcal{P}roj$ constructions and specifically stack-theoretic blowings up. Rydh's forthcoming foundational paper on these notions will simplify this entire section significantly.

5.4.1. *Permissible Kummer centers.* We restrict our consideration to toroidal schemes. Permissible centers extend to Kummer ideals straightforwardly: we say that a Kummer ideal I on a toroidal scheme X is *permissible* if it is generated by the ideal of a toroidal subscheme and a monomial Kummer ideal. In other words, for any geometric point $\bar{x} \to X$ one has that $I_{\bar{x}} = (t_1, \ldots, t_n, m_1^{1/d}, \ldots, m_r^{1/d})$, where $t_1, \ldots, t_n$ is a part of a regular sequence of parameters, and $m_1, \ldots, m_r$ are monomials. We impose the additional assumption that d is invertible on X, which is sufficient for our characteristic 0 applications but not optimal; see Remark 4. By $V(I)$ we denote the set of points of X where I is not the unit ideal; it is a closed subset of X.

5.4.2. *Kummer blowings up: global quotient case.* Let I be a permissible Kummer center on X. The idea of defining $\mathrm{Bl}_I(X)$ is to blow up a sufficiently fine Kummer covering of X and then descend it to a modification of X.

Assume first that there exists a G-Galois Kummer covering $Y \to X$ such that I is generated by I_Y. Note that $X = Y/G$. Locally, I_Y is generated by monomials and elements coming from I. Since G acts by characters on monomials and preserves elements coming from I, the ideal I_Y and the blowing up $Y' = \mathrm{Bl}_{I_Y}(Y) \to Y$ are G-equivariant. Moreover, using these generators we see that the blowing up Y' is covered by G-equivariant affine charts. In particular, the algebraic space Y'/G is a scheme, which we denote X'_{cs}, and $X'_{\mathrm{cs}} \to X$ is a W-modification, where $W = X \smallsetminus V(I)$. Here a *$W$-modification* $X'_{\mathrm{cs}} \to X$ is a modification restricting to the identity over the dense open $W \subset X$.

Note that X'_{cs} is the coarse space $[Y'/G]_{\mathrm{cs}}$ of the stack quotient $[Y'/G]$. We will show that X'_{cs} depends only on X and I, but it may happen that X'_{cs} with the quotient logarithmic structure is not toroidal: see Section 5.4.6 below for a general explanation and Example 5.4.12(ii) for a concrete example. On the other hand, $[Y'/G]$ is too close to Y': the morphism $Y' \to [Y'/G]$ is étale hence $[Y'/G]$ is toroidal, but it is ramified over the same points of X'_{cs} over which Y' is ramified, and hence depends on the choice of the covering $Y \to X$. Finally, we would like to ensure that the exceptional divisor E on $[Y'/G]$ remains Cartier, in other words, we would like the morphism $[Y'/G] \to B\mathbb{G}_m$ corresponding to the line bundle $\mathcal{O}(E)$ to descend to our modification. For these reasons the main player in the sequel will be the relative coarsening $[Y'/G]_{\mathrm{cs}/B\mathbb{G}_m}$ (see Section 2.3 and Remark 2.3.2). In particular, we will see that it is toroidal and independent of the choice of the covering $Y \to X$.

Lemma 5.4.3. *With the above notation, the X-stack $X' = [Y'/G]_{\mathrm{cs}/B\mathbb{G}_m}$ and its coarse space $X'_{\mathrm{cs}} = Y'/G$ depend on X and I only, but not on the Kummer covering $Y \to X$.*

Proof. It suffices to deal with X', since X'_{cs} is obtained from it. We should prove that if $Z \to X$ is another Kummer covering with Galois group H and $Z' = \mathrm{Bl}_{I_Z}(Z)$ then $[Z'/H]_{\mathrm{cs}/B\mathbb{G}_m} = X'$. The family of Kummer coverings is filtered, hence it suffices to consider the case when Z dominates Y. In this case, $Z/K = Y$ where K is a subgroup of H with $H/K = G$.

Since $I_Z = I_Y\mathcal{O}_Z$, the charts of both $\mathrm{Bl}_{I_Y}(Y)$ and $\mathrm{Bl}_{I_Z}(Z)$ can be given by the same elements. It follows that $Z' \to Y$ factors through a finite morphism $Z' \to Y'$. Since Y' is normal, this implies that $Z'/K = Y'$, and we obtain a coarsening morphism $h : [Z'/H] \to [Y'/G]$. Clearly, the exceptional divisor on $[Z'/H]$ is the pullback of the exceptional divisor on $[Y'/G]$. Therefore the morphism $[Z'/H] \to B\mathbb{G}_m$ factors through the morphism $[Y'/G] \to B\mathbb{G}_m$, and this implies that $[Z'/H]_{\mathrm{cs}/B\mathbb{G}_m} = [Y'/G]_{\mathrm{cs}/B\mathbb{G}_m}$, as required. □

5.4.4. *Kummer blowings up: the general case.* In the general case, the Kummer blowing up of X along I is defined by gluing. Namely, X has an étale covering $\sqcup X_i \to X$ such that $I_i = I|_{X_i}$ is generated by global functions and roots of global monomials, and then each X_i has a G_i-Kummer Galois covering $Y_i \to X_i$ such that $J_i = I_{Y_i}$ generates $I|_{Y_i}$. By Lemma 5.4.3 the stack $X'_i = [\mathrm{Bl}_{J_i}(Y_i)/G_i]_{\mathrm{cs}/B\mathbb{G}_m}$ and its coarse space $(X'_i)_{\mathrm{cs}} = \mathrm{Bl}_{J_i}(Y_i)/G_i$ depend on X_i and I_{X_i} only.

Over $X_{ij} := X_i \times_X X_j$ the stacks $(X'_i)_{X_{ij}}$ and $(X'_j)_{X_{ij}}$ are isomorphic by Lemma 5.4.3. Indeed the isomorphism over X is unique: the stacks are birational, normal, separated and Deligne–Mumford; hence [Fantechi et al. 2010, Proposition A.1] applies. This implies that X'_i glue uniquely over the intersections X_{ij}. Thus, we obtain morphisms $X' \to X$ and $X'_{\mathrm{cs}} \to X$ depending only on X and I. We say that $X'_{\mathrm{cs}} := \mathrm{Bl}_I(X)$ is the *coarse Kummer blowing up* of X along I and $X' = [\mathrm{Bl}_I(X)]$ is the *Kummer blowing up* of X along I. Here are two basic properties of this operation.

Theorem 5.4.5. *Assume that (X, U) is a toroidal scheme and I is a permissible Kummer center, and let $W = X \smallsetminus V(I)$. Then*

(i) *$f : [\mathrm{Bl}_I(X)] \to X$ and $\mathrm{Bl}_I(X) \to X$ are W-modifications of X;*

(ii) *$([\mathrm{Bl}_I(X)], f^{-1}(U))$ is a simple toroidal orbifold.*

Proof. The claims are local on X, so we can assume that X possesses a G-Galois Kummer covering Y such that I_Y generates $I|_{Y_{\text{két}}}$. Then $[\mathrm{Bl}_{I_Y}(Y)/G]$ is proper over X and the preimage of W is dense, and hence the same is true for the partial coarse spaces $[\mathrm{Bl}_I(X)]$ and $\mathrm{Bl}_I(X)$. Furthermore, the constructions are compatible with localizations and $I|_W = 1$, hence both are W-modifications of X.

The fact that $([\mathrm{Bl}_I(X)], f^{-1}(U))$ is a toroidal orbifold is shown in Lemma 5.4.7 below, using the explicit charts described in Section 5.4.6. Its simplicity follows from the observation that G acts simply on Y, and hence it also acts simply on $\mathrm{Bl}_{I_Y}(Y)$. □

5.4.6. *Charts of Kummer blowings up.* Next, let us describe explicit charts of Kummer blowings up. Assume that $X = \mathrm{Spec}(A)$ and $I = (t_1, \dots, t_n, m_1^{1/d}, \dots, m_r^{1/d})$ is a permissible Kummer ideal, where $(t_1, \dots, t_n)$ defines a toroidal subscheme and m_i are global monomials. Then $X' = [\mathrm{Bl}_I(X)]$ is of the form $[\mathrm{Bl}_J(Y)/G]_{\mathrm{cs}/B\mathbb{G}_m}$, where

$$B = A \otimes_{\mathbb{Z}[m_1,\dots,m_r]} \mathbb{Z}[m_1^{1/d}, \dots, m_r^{1/d}],$$

$Y = \mathrm{Spec}(B^{\mathrm{sat}})$, $G = (\boldsymbol{\mu}_d)^r$, and $J = I\mathcal{O}_Y$. Note that $\mathrm{Bl}_J(Y)$ is covered by the charts

$$Y'_y = \mathrm{Spec}(B[t'_1, \dots, t'_n, u'_1, \dots, u'_r]^{\mathrm{sat}}),$$

where $y \in \{t_1, \dots, t_n, m_1^{1/d}, \dots, m_r^{1/d}\}$, $t'_i = t_i/y$ and $u'_j = m_i^{1/d}/y$. Hence X' is covered by the charts $X'_y = [Y'_y/G]_{\mathrm{cs}/B\mathbb{G}_m}$.

Let us describe X'_y locally at the image of a point $q \in Y'_y$. The stabilizer G_q is the inertia group of $[Y'_y/G]$ at the image of q. Hence the morphism $[Y'_y/G] \to B\mathbb{G}_m$ induces a homomorphism $G_q \to \mathbb{G}_m$, whose kernel $G_{q/B\mathbb{G}_m}$ is the relative stabilizer of $[Y'_y/G]$ over $\mathbb{G}_m$ at the image of q. In particular, $X'_y = [(Y'_y/G_{q/B\mathbb{G}_m})/(G/G_{q/B\mathbb{G}_m})]$ locally at the image of q. To complete the picture it remains to observe that the relative stabilizer $G_{q/B\mathbb{G}_m}$ is the subgroup of G_q acting trivially on y, that is, G_q acts on y through its image in $\mathbb{G}_m$. To show this explicitly consider two cases:

(1) The t_i-chart. Since G acts trivially on t_i we have that $G_{q/B\mathbb{G}_m} = G_q$ and hence $X'_y = Y'_y/G$ is a scheme.

(2) The $m_i^{1/d}$-chart. In this case, $G_{q/B\mathbb{G}_m}$ contains $G_q \cap \boldsymbol{\mu}_d^{r-1}$ and $G_q/G_{q/B\mathbb{G}_m} = \boldsymbol{\mu}_e$, where e is the minimal divisor of d such that $m_i \in \mathcal{M}_x^{d/e}$, where $x \in X$ is the image of q; in particular, G_q acts through $\boldsymbol{\mu}_e$ on the image of $m_i^{1/d}$ in $\mathcal{M}_q$.

Lemma 5.4.7. *Keep the above notation. Then the group $G_{q/B\mathbb{G}_m}$ acts toroidally at q. In particular, the coarsening $[Y'/G] \to [\mathrm{Bl}_I(X)]$ is toroidal and $[\mathrm{Bl}_I(X)] = [Y'/G]_{\mathrm{cs}/\mathbb{G}_m} = [Y'/G]_{\mathrm{tcs}/B\mathbb{G}_m}$.*

Proof. The regular coordinates on Y'_y are of the form $t'_i = t_i/y$. Since $G_{q/B\mathbb{G}_m}$ acts trivially on t_i and y, it acts trivially on t'_i. Thus, its action at q is toroidal. □

We will not need the following remark, so its justification is left to the interested reader.

Remark 5.4.8. (i) The whole group G_q can act nontrivially on $m_i^{1/d}$-charts, see Example 5.4.12(ii) below. So, one may wonder what is the maximal toroidal coarsening $[Y'/G]_{\mathrm{tcs}}$. By the above lemma, we have a natural morphism $f : [\mathrm{Bl}_I(X)] \to [Y'/G]_{\mathrm{tcs}}$. It turns out that in the nonmonomial case (i.e., there exists at least one regular parameter t_1), f is an isomorphism. On the other hand, in the monomial case the action of the whole G_q is automatically toroidal, and hence $[Y'/G]_{\mathrm{tcs}} = Y'/G$. In this case, f can be a nontrivial coarsening; see Example 5.4.12(i).

(ii) In an early version of the paper, we defined $[\mathrm{Bl}_I(X)]$ to be equal to $[Y'/G]_{\mathrm{tcs}}$. This definition possesses worse functorial properties and often required to distinguish the monomial and nonmonomial cases. It seems that the new definition is the "right" one.

5.4.9. *The coarse blowing up.* The coarse blowing up can be computed directly.

Lemma 5.4.10. *Assume given a toroidal affine scheme $X = \mathrm{Spec}(A)$ with a positive number $e \in d\mathbb{Z}$ and a Kummer ideal $I = (t_1, \dots, t_n, m_1^{1/d}, \dots, m_r^{1/d})$. Then $\mathrm{Bl}_I(X)$ is the normalized blowing up of X along either of the following ideals: $J_e = (t_1^e, \dots, t_n^e, m_1^{e/d}, \dots, m_r^{e/d})$, $\widetilde{J}_e = I^e \cap \mathcal{O}_X$.*

Proof. Set $Y = \mathrm{Spec}(B)$ with $B = A[m_1^{1/d}, \dots, m_r^{1/d}]$. It suffices to check that $\mathrm{Bl}_{I_Y}(Y)$ is finite over both $\mathrm{Bl}_{J_e}(X)$ and $\mathrm{Bl}_{\widetilde{J}_e}(X)$. Indeed, in this case $\mathrm{Bl}_I(X) = \mathrm{Bl}_{I_Y}(Y)/\boldsymbol{\mu}_d^r$ is a finite modification of both $\mathrm{Bl}_{J_e}(X)^{\mathrm{nor}}$ and $\mathrm{Bl}_{\widetilde{J}_e}(X)^{\mathrm{nor}}$, and since the latter are normal we are done.

We will check the finiteness on charts. Let $y \in \{t_1, \dots, t_n, m_1^{1/d}, \dots, m_r^{1/d}\}$ and $x = y^e$. It suffices to show that $B[I/y]$ is finite over both $A[J_e/x]$ and $A[\widetilde{J}_e/x]$. But this is clear because $B[I/y]$ is integral over both $B[J_eB/x]$ and $B[\widetilde{J}_eB/x]$. □

5.4.11. *Examples.* Let us consider two basic examples of Kummer blowings up.

Example 5.4.12. (i) Let $X = \mathrm{Spec}(k[\pi])$ with the logarithmic structure given by π, and let $I = (\pi^{1/d})$. Then $[\mathrm{Bl}_I(X)] = [\mathrm{Spec}(k[\pi^{1/d}])/\mu_d]$ has stabilizer μ_d at the origin.

(ii) Let $X = \mathrm{Spec}(k[t, \pi])$ with the logarithmic structure given by π, and let $I = (t, \pi^{1/2})$. By Lemma 5.4.10, the coarse blow up $X'_{\mathrm{cs}} = \mathrm{Bl}_I(X)$ coincides with $\mathrm{Bl}_J(X)^{\mathrm{nor}}$, where $J = (t^2, \pi)$. In fact, $\mathrm{Bl}_J(X)$ is already normal and covered by two charts: $(X'_1)_{\mathrm{cs}} = \mathrm{Spec}(k[t, \pi, t^2/\pi])$ and $(X'_2)_{\mathrm{cs}} =$

$\mathrm{Spec}(k[t,\pi/t^2])$. The chart $(X'_2)_{\mathrm{cs}}$ is regular, but the chart $(X'_1)_{\mathrm{cs}}$ has an orbifold singularity O_X at the origin. Moreover, the natural logarithmic structure on $(X'_1)_{\mathrm{cs}}$ is generated by π only, and $(X'_1)_{\mathrm{cs}}$ is not toroidal with this logarithmic structure. (Though $(X'_1)_{\mathrm{cs}}$ can be made toroidal by increasing the toroidal structure, for example, by adding the divisor (t).)

Now let us consider the finer stack-theoretic picture. The Kummer blowing up $X' = [\mathrm{Bl}_I(X)]$ can be computed using the Kummer covering $Y = \mathrm{Spec}(k[t,\pi^{1/2}])$ with $G = \boldsymbol{\mu}_2$. This can be done directly, but for the sake of comparison we will first compute $X'' = [Y'/G]_{\mathrm{tcs}}$, where $Y' = \mathrm{Bl}_{(t,\pi^{1/2})}(Y)$. Cover Y' by two charts: $Y'_1 = \mathrm{Spec}(k[t/\pi^{1/2},\pi^{1/2}])$ and $Y'_2 = \mathrm{Spec}(k[t,\pi^{1/2}/t])$. Then X'' is covered by the charts $X''_i = [Y'_i/G]_{\mathrm{tcs}}$. The action of G on Y'_2 is toroidal, and hence $X''_2 = Y'_2/G = (X'_2)_{\mathrm{cs}}$. The action of G at the origin O_Y of Y'_1 is not toroidal because G acts via the nontrivial character on both parameters. Therefore the stabilizer at the image $O_{X''} \in X''$ of O_Y is G. In particular, the coarse moduli space $X'' \to X'_{\mathrm{cs}}$ is an isomorphism over $X'_{\mathrm{cs}} \smallsetminus \{O_{X'_{\mathrm{cs}}}\}$, and the preimage of $O_{X'_{\mathrm{cs}}}$ is the point $O_{X''}$ with a nontrivial stack structure. Furthermore, it is easy to see that the exceptional divisor is Cartier on X'', and hence the morphism $X' \to X''$ admits a section. Thus, $X' = X''$ is the cone orbifold.

5.4.13. *Enlarging the toroidal structure.* As in the proof of Lemma 5.2.3, enlarging the toroidal structure any Kummer blowing up can be made into a logarithmically smooth morphism.

Lemma 5.4.14. *Let $X = (X,U)$ be a toroidal scheme, I be a permissible Kummer ideal on X and $f : X' = [\mathrm{Bl}_I(X)] \to X$ be the associated Kummer blowing up. Assume that $X_1 = (X,U_1)$ is a toroidal scheme obtained by enlarging the toroidal structure so that I is monomial on X_1 (see Corollary 5.1.6). Then $X'_1 = (X', f^{-1}(U_1))$ is a toroidal orbifold and the morphism $X'_1 \to X_1$ is logarithmically smooth.*

Proof. The claim is local on X, hence we can assume that there exists a G-Galois Kummer covering $Y \to X$ such that $J = I\mathcal{O}_Y$ is a permissible ideal. Let $Y' = \mathrm{Bl}_J(Y)$ and let Y'_1 and Y_1 be the toroidal schemes with the toroidal structure induced from U_1. Since J is monomial on Y_1, we have that $Y'_1 \to Y_1$ is a toroidal blowing up. By Section 5.4.6 the action of G on Y'_1 is toroidal (it acts trivially on all regular coordinates). Therefore, any subgroup $H \subseteq G$ acts toroidally and hence the morphism $Y'_1/H \to X_1$ is logarithmically smooth. It follows that for any coarsening T of $[Y'_1/G]$ the morphism $T \to Y_1/G = X_1$ is logarithmically smooth. It remains to recall that, by definition, X' is a coarsening of $[Y'/G]$, namely the relative coarse space with respect to the morphism $[Y'/G] \to B\mathbb{G}_m$ induced by the exceptional divisor. □

5.4.15. *The universal property.* Kummer blowings up can be characterized by a universal property which extends the classical characterization of blowings up.

Theorem 5.4.16. *Let X be a toroidal scheme and let I be a permissible Kummer ideal with the associated Kummer blowing up $f : [\mathrm{Bl}_I(X)] \to X$. Then $f^{-1}(I)$ is an invertible ideal and f is the universal morphism of toroidal DM stacks $h : Z \to X$ such that $h^{-1}(I)$ is an invertible ideal.*

Proof. All claims are local on X, so we can use the description of charts from Section 5.4.6: choosing a G-Galois Kummer covering $Y \to X$, such that I_Y is an ordinary ideal, and setting $Y' = \mathrm{Bl}_{I_Y}(Y)$ we have that $[\mathrm{Bl}_I(X)] = [Y'/G]_{\mathrm{cs}/B\mathbb{G}_m}$. Now, the first claim is obtained by unraveling the definition of

$X' := [\mathrm{Bl}_I(X)]$. Indeed, the exceptional divisor on Y', and hence also on Y'/G, is Cartier. Furthermore, the induced morphism $[Y'/G] \to B\mathbb{G}_m$ factors through X', that is the exceptional divisor on X' is also Cartier.

Now, let us check the universal property. So, assume that $h : Z \to X$ is such that $h^{-1}(I)$ is an invertible ideal, and let us show that it factors through $[\mathrm{Bl}_I(X)]$ uniquely up to a unique 2-isomorphism. Set $T = Z \times_X Y$ as an fs logarithmic scheme. From the factorization $T \to Z \to X$, the pullback of I to T is an invertible Kummer ideal. From the factorization $T \to Y \to X$, the pullback of I to T is the usual ideal $I_Y\mathcal{O}_T$. Therefore $I_Y\mathcal{O}_T$ is an invertible ideal, and by the universal property of blowings up, $T \to Y$ factors through a morphism $T \xrightarrow{\phi} Y' = \mathrm{Bl}_{I_Y}(Y)$ in a unique way. The exceptional divisors on T and Y' are compatible, hence induce compatible morphisms to $B\mathbb{G}_m$.

Note that $T \to Z$ is Kummer étale with Galois group $G = \boldsymbol{\mu}_d^r$ equal to the Galois group of $Y \to X$. Taking the stack quotient by G, the exceptional divisors remain Cartier, hence morphisms $[T/G] \to [Y'/G] \to B\mathbb{G}_m$ arise. Passing to the relative coarse moduli spaces yields a morphism $[T/G]_{\mathrm{cs}/B\mathbb{G}_m} \to X'$. It remains to recall that the exceptional divisor on $Z = T/G$ is already Cartier, hence $[T/G]_{\mathrm{cs}/B\mathbb{G}_m} = Z$ and we obtain the required morphism $Z \to X'$. □

5.4.17. *Strict transforms.* By a classical observation, the universal property of blowings up implies that if $X' \to X$ is the blowing up along an ideal I then the strict transform Z' of a closed subscheme $Z \hookrightarrow X$ is the blowing up of Z along $I\mathcal{O}_Z$. The same reasoning applies to Kummer blowings up as well.

Lemma 5.4.18. *Assume that X is a toroidal scheme, $Z \hookrightarrow X$ is a closed toroidal subscheme, and $I \subseteq \mathcal{O}_X$ is a permissible Kummer ideal whose restriction $J = I\mathcal{O}_Z$ is a permissible Kummer ideal on Z. Let $X' \to X$ be the Kummer blowing up along I and let Z' be the strict transform of Z (i.e., the closure of $Z \setminus V(I)$ in X'). Then the morphism $Z' \to Z$ factors through a unique isomorphism $Z' = [\mathrm{Bl}_J(Z)]$.*

Proof. On the one hand, since $Z' \to X$ factors through X', the ideal $I\mathcal{O}_{Z'} = J\mathcal{O}_{Z'}$ is invertible. So, $Z' \to Z$ factors through a morphism $h : Z' \to Y = [\mathrm{Bl}_J(Z)]$ by Theorem 5.4.16. On the other hand, $J\mathcal{O}_Y$ is an invertible ideal, and since $J\mathcal{O}_Y = I\mathcal{O}_Y$, we obtain by Theorem 5.4.16 that the morphism $Y \to X$ factors through X'. Furthermore, $Y \to X$ factors through Z' because $Z \setminus V(J)$ is dense in Y. This provides a morphism $Y \to Z'$, which is easily seen to be the inverse of h by the uniqueness of the factorization in Theorem 5.4.16. □

Since Kummer blowings up were only defined for toroidal schemes, we cannot extend the above theorem to the case when Z is an arbitrary closed logarithmic subscheme of X. However, in this case we can at least describe the strict transform on the level of the coarse space.

Lemma 5.4.19. *Assume that X is a toroidal scheme, $Z \hookrightarrow X$ is a strict closed logarithmic subscheme, and $I \subseteq \mathcal{O}_X$ is a permissible Kummer ideal. Let $X' \to X$ be the Kummer blowing up along I and let $Z' \to Z$ be the strict transform. Set $J_n = I^{n!} \cap \mathcal{O}_X$. Then Z'_{cs} is the blowing up of Z along $((J_n)^m)^{\mathrm{nor}}\mathcal{O}_Z$ for large enough n and m.*

Proof. The claim is local on X, hence by Lemma 5.4.14 we can enlarge the logarithmic structure on X making I monomial. Recall that by Lemma 5.4.10, $X'_{\mathrm{cs}} \to X$ is the normalized blowing up along J_n for

a large enough n. Clearly J_n is monomial, hence by [Abramovich and Temkin 2017, Corollary 5.3.6] $X'_{\mathrm{cs}} \to X$ is the blowing up along $((J_n)^m)^{\mathrm{nor}}$ for a large enough m. Note that Z'_{cs} is the closed subscheme of X'_{cs} coinciding with the image of Z'. It follows that Z'_{cs} is the strict transform of Z and hence it is the blowing along $((J_n)^m)^{\mathrm{nor}}\mathcal{O}_Z$ by the usual theory of strict transforms. □

5.4.20. *Functoriality.* The universal property can also be used to show that, as most other constructions of this paper, Kummer blowings up are compatible with logarithmically smooth morphisms.

Lemma 5.4.21. *Let $f : Y \to X$ be a logarithmically smooth morphisms of toroidal schemes, I a permissible Kummer center on X, and $J = f^{-1}(I)$. Then $[\mathrm{Bl}_J(Y)] = [\mathrm{Bl}_I(X)] \times_X Y$, where the product is taken in the category of fs logarithmic schemes.*

Proof. Recall that J is permissible by Lemma 5.2.5. Set $X' = [\mathrm{Bl}_I(X)]$ and $Y' = [\mathrm{Bl}_J(Y)]$. Since $J\mathcal{O}_{Y'} = I\mathcal{O}_{Y'}$, the morphism $Y' \to X$ factors through X' by Theorem 5.4.16, and we obtain a morphism $Y' \to X' \times_X Y$. Conversely, since $X' \times_X Y$ is logarithmically smooth over X', the pullback of the invertible ideal $I\mathcal{O}_{X'}$ to $X' \times_X Y$ is also invertible. The latter coincides with the pullback of J to $X' \times_X Y$, and using Theorem 5.4.16 again we obtain a morphism $X' \times_X Y \to Y'$. It follows from the uniqueness of the factorizations that these two morphisms are inverse, implying the lemma. □

5.5. ***Kummer blowings up of stacks.*** It is also desirable to work with compositions of Kummer blowings up. For example, such sequences will be our main tool in constructing logarithmic desingularization in [Abramovich et al. 2020]. For this one should at least extend the construction to the case when X itself is a toroidal orbifold. We will see that, in fact, everything works fine when X is a toroidal DM stack.

5.5.1. *Kummer ideals.* The Kummer topology naturally extends to logarithmic stacks, giving rise to the notion of Kummer ideals. Permissibility of Kummer ideals is an étale-local notion and hence it extends to toroidal DM stacks too. Also, Lemma 5.2.3, which concerns usual coherent ideals, generalizes as follows:

> A permissible blowing up of a toroidal DM stack (resp. simple toroidal orbifold) is again a toroidal DM stack (resp. simple toroidal orbifold).

To combine the two notions and form the *Kummer* blowing up of a toroidal DM stack we must check that 2-categorical issues do not arise.

5.5.2. *Kummer blowings up.* Assume now that X is a toroidal DM stack and I is a permissible Kummer ideal on $X_{\text{két}}$. Find a strict étale covering of X by a toroidal scheme X_0 and set $X_1 = X_0 \times_X X_0$. The pullback I_i of I to X_i is a permissible Kummer ideal, and we set $Y_i = [\mathrm{Bl}_{I_i}(X_i)]$. Since $[X_1 \rightrightarrows X_0]$ is an étale groupoid whose projections and the multiplication morphism are strict, we obtain by Lemma 5.4.21 that $Y_1 \rightrightarrows Y_0$ is an étale groupoid *of stacks* whose projections are strict and *inert*. By Lemma 2.1.4 the quotient $Y = [Y_0/Y_1]$ exists as a toroidal DM stack and satisfies $Y_i = X_i \times_X Y$. We call Y the Kummer blowing up of X along I and denote it $[\mathrm{Bl}_I(X)] := Y$. A straightforward verification using Lemma 5.4.21 shows:

(1) The X-stack $Y = [\mathrm{Bl}_I(X)]$ is independent of the presentation $X = [X_0/X_1]$ and depends only on X and I. The uniqueness of Y is understood up to an isomorphism of X-stacks, which is unique up to a unique 2-isomorphism, again by [Fantechi et al. 2010, Proposition A.1]. If X is simple then Y is simple.

(2) If $f : X' \to X$ is a logarithmically smooth morphism and $I' = f^{-1}(I)$ then $[\mathrm{Bl}_{I'}(X')] = [\mathrm{Bl}_I(X)] \times_X X'$, the product taken in the fs category.

5.5.3. *Proof of Theorem 3.* If X is a toroidal scheme, then parts (i) and (iv) were proved in Theorem 5.4.5, parts (ii) and (iii) in Theorem 5.4.16, part (v) in Lemma 5.4.21, part (vi) in Lemma 5.4.19, and part (vii) in Lemma 5.4.18. In general, part (v) holds by (2) above, and this allows to reduce all other claims to the case of schemes. Namely, choose a strict étale covering $f : X' \to X$ of X by a toroidal scheme X', set $I' = f^{-1}(I)$, and consider the Kummer blowing up $Y' = [\mathrm{Bl}_{I'}(X')]$. Then $Y' = Y \times_X X'$, and all assertions for $Y \to X$ follow from the case of $Y' \to X'$ by étale descent. For example, $I_{Y/X} \times_X X' = I_{Y'/X'} = I_{Y'}$ is finite diagonalizable and acts trivially on the monoids $\overline{M}_{x'} = \overline{M}_{f(x')}$, hence the same is true for $I_{Y/X}$.

Appendix A: Existence of coarsenings

by David Rydh

A.1. *Classification of Deligne–Mumford coarsenings.*

A.1.1. *The category of coarsenings.* Recall that a *coarsening* is a morphism $f : X \to Y$ of Artin stacks such that Y is the coarse space of X relative to Y (Section 2.3.1). Equivalently, for any flat morphism $Y' \to Y$ from an algebraic space Y', the base change $f' : X' \to Y'$ is a coarse space. Equivalently, f is a universal homeomorphism with finite diagonal and $f_*\mathcal{O}_X = \mathcal{O}_Y$.

A priori, coarsenings $f : X \to Y$ of a fixed Artin stack X constitute a 2-category $\mathcal{C}_X$ where a 1-morphism from $f_1 : X \to Y_1$ to $f_2 : X \to Y_2$ is a 1-morphism $h : Y_1 \to Y_2$ together with a 2-morphism $\alpha : h \circ f_1 \Rightarrow f_2$; and a 2-morphism $(h_1, \alpha_1) \Rightarrow (h_2, \alpha_2)$ is a 2-morphism $\gamma : h_1 \Rightarrow h_2$ such that $\alpha_2 \circ \gamma = \alpha_1$. The 2-category $\mathcal{C}_X$ is, however, always equivalent to a partially ordered set (Theorem 2.3.6(iii)). The initial object of $\mathcal{C}_X$ is id_X. If X has finite inertia, then the final object of $\mathcal{C}_X$ is the usual coarse space, or *total coarsening*, $f : X \to X_{\mathrm{cs}}$ (Section 2.2.1).

A.1.2. *The main theorem.* Let $\mathcal{C}_X^{\mathrm{DM}} \subseteq \mathcal{C}_X$ denote the full 2-subcategory of DM-coarsenings, that is, coarsenings $X \to Y$ with Y a Deligne–Mumford stack. The purpose of this appendix is to prove the following classification result for DM-coarsenings.

Theorem A.1.3. *Let X be an Artin stack with finite inertia. The 2-category $\mathcal{C}_X^{\mathrm{DM}}$ is equivalent to the partially ordered set of open and closed subgroups $N \subseteq I_X$. A DM-coarsening $X \to Y$ corresponds to the subgroup $I_{X/Y} \subseteq I_X$.*

A morphism $\phi : X \to Z$, with Z Deligne–Mumford, factors uniquely through a given DM-coarsening $f : X \to Y$ if and only if the induced map on inertia $I_{X/Y} \to \phi^* I_Z$ is trivial (Theorem 2.3.6(i)). It follows that the map $(X \to Y) \mapsto I_{X/Y}$ is injective on DM-coarsenings.

If $f : X \to Y$ is a DM-coarsening, then $I_Y \to Y$ is finite and unramified so the unit section of I_Y is an open and closed immersion. Since $I_{X/Y} = \ker(I_X \to f^* I_Y)$ it follows that $I_{X/Y} \subseteq I_X$ is an open and closed subgroup.

It remains to prove that every open and closed subgroup N of I_X gives rise to a DM-coarsening. Note that any subgroup $N \subseteq I_X$ is necessarily normal: if T is a scheme, $\xi : T \to X$ is a morphism and s is a section of $\xi^* I_X \to T$, then s corresponds to a 2-morphism $u : \xi \Rightarrow \xi$ and the induced isomorphism $\xi^* N \to \xi^* N$ is conjugation by s (see the discussion in [Abramovich et al. 2008, Appendix A] right before Theorem A.1). The final object, corresponding to $N = I_X$, is the total coarsening morphism $X \to X_{\mathrm{cs}}$. Theorem A.1.3 is thus a generalization of the Keel–Mori theorem on the existence of total coarsenings.

A.1.4. *Étale neighborhoods with desired inertia.* The key step in the proof of the Keel–Mori theorem is the existence of a suitable étale neighborhood $h : W \to X$; see [Keel and Mori 1997, §4; Rydh 2013, Proposition 6.11]. Specifically, h should be inert, that is, $I_W = h^* I_X$, and W should admit a finite flat presentation by a scheme (this is the basic case where we know how to construct a coarse space). We give the following variant of this result.

Proposition A.1.5. *Let X be an Artin stack with finite inertia and let $N \subseteq I_X$ be an open and closed subgroup. Then there is a representable, separated, étale and surjective morphism $h : W \to X$ such that $I_W = h^* N$ as subgroups of $h^* I_X$.*

Proof. Let $p : U \to X$ be a locally quasifinite flat presentation [Rydh 2011, Theorem 7.1] (or [Stacks, Tag 04N0] if X is not quasiseparated). Note that p is separated. The relative Hilbert functor $\mathrm{Hilb}(U/X) \to X$ is thus representable, separated and locally of finite presentation. Indeed, if T is a scheme and $T \to X$ is a morphism, then $U \times_X T$ is an algebraic space, separated and locally of finite presentation over T, and hence so is $\mathrm{Hilb}(U/X) \times_X T = \mathrm{Hilb}(U \times_X T/T)$, by Artin's representability theorem [1969, Corollary 6.2].

Let $W' \subset \mathrm{Hilb}(U/X)$ be the open substack parametrizing open and closed subschemes along the fibers, namely, the restriction of the universal closed subscheme to W' is open in $\mathrm{Hilb}(U/X) \times_X U$. Let $h' : W' \to X$ be the structure map. It is representable, separated, étale and surjective, but allows for all possible open and closed subgroups of inertia. Over W' we have two open and closed subgroups $I_{W'} \subseteq h'^* I_X$ and $h'^* N \subseteq h'^* I_X$. The locus $W \subseteq W'$ where these coincide is open since $h'^* I_X \to W'$ is closed. It remains to verify that $h : W \to X$ is surjective which can be done on points.

Let $x : \operatorname{Spec} k \to X$ be a point with k algebraically closed. Then the stabilizer G_x acts freely on the finite k-scheme $x^* U$. Let $Z \subseteq x^* U$ be an open and closed subscheme such that $x^* N$ acts set-theoretically transitively on Z, that is, Z is the preimage of a connected component of $x^* U / x^* N$. Then the stabilizer of $[Z]$ in W' is $x^* N$ so $[Z]$ is a point in W lifting x. □

As in [Rydh 2013, Proposition 6.11], by construction the stacks W and W' admit finite flat presentations by AF-schemes.

A.1.6. *Proof of Theorem A.1.3.* Two Deligne–Mumford coarsenings $f_i : X \to Y_i$ with the same subgroups I_{X/Y_i} are uniquely isomorphic by Theorem 2.3.6. Given an open and closed subgroup $N \subseteq I_X$, take an

étale neighborhood $h : W \to X$ as in Proposition A.1.5. Note that $I_{W\times_X W} = I_W \times_N I_W = I_W \times_X W$, hence the étale projections $W \times_X W \to W$ are inert. It follows from [Rydh 2013, Theorem 6.10] that the two induced maps $(W \times_X W)_{\mathrm{cs}} \to W_{\mathrm{cs}}$ are also étale morphisms and give rise to an étale groupoid. The quotient stack Y thus admits a morphism $X \to Y$ and, tautologically, $W = X \times_Y W_{\mathrm{cs}}$ and $h^*I_{X/Y} = I_W = h^*N$. The morphism $X \to Y$ is thus a Deligne–Mumford coarsening with $I_{X/Y} = N$.

A.2. *Examples of coarsenings.*

A.2.1. *Characteristic zero.* In characteristic zero, every stack with finite inertia is Deligne–Mumford and Theorem A.1.3 gives a full classification of all coarsenings.

A.2.2. *Tame Deligne–Mumford stacks.* If X is tame and Deligne–Mumford, then every coarsening is Deligne–Mumford. This is an immediate consequence of Theorem 2.3.6(i). Thus we obtain a full classification of all coarsenings in this case as well.

A.2.3. *Wild Deligne–Mumford stacks.* When X is Deligne–Mumford but not tame, then there may exist coarsenings that are not Deligne–Mumford. The following example is given in [Romagny et al. 2018, §4.5].

Let $U = \operatorname{Spec} \mathbb{F}_p[\epsilon, x]/(\epsilon^2)$ and let $G = \mathbb{Z}/p\mathbb{Z}$ act via $t.(\epsilon, x) = (\epsilon, x + t\epsilon)$. Let $X = [U/G]$. There is a p-torsion line bundle $\mathcal{L}$ on X corresponding to the trivial line bundle $\mathcal{O}_U \cdot e$ on U with action $t.e = (1+t\epsilon)e$. The classifying map $\phi : X \to B\boldsymbol{\mu}_p$ induces a trivial map $I_X \to \boldsymbol{\mu}_p$ on inertia. Nevertheless, ϕ does not factor through the coarse space $f : X \to X_{\mathrm{cs}}$. If we let $Z = X_{\mathrm{cs}/B\boldsymbol{\mu}_p}$, then $X \to Z$ is a coarsening that is not Deligne–Mumford and $I_{X/Z} = I_X$.

A.2.4. *Tame Artin stacks.* When X is tame, then its coarsenings correspond to subgroups of inertia by Theorem 2.3.6(i). These subgroups are closed but not necessarily open as in the following example.

Let $U = \operatorname{Spec} \mathbb{F}_p[x]$ and let $G = \boldsymbol{\mu}_{2p} = \boldsymbol{\mu}_p \times \mathbb{Z}/2\mathbb{Z}$ act on U via $t.x = tx$. Let $X = [U/G]$ and $Y = [V/\boldsymbol{\mu}_p]$ where $V = \operatorname{Spec} \mathbb{F}_p[x^2]$ and the action is $t.x^2 = t^2x^2$. The inertia stack of X is trivial except for a $\boldsymbol{\mu}_{2p}$ over the origin. The natural map $f : X \to Y$ is a coarsening and the closed subgroup $I_{X/Y} \subset I_X$ is not open: it is trivial except for a $\mathbb{Z}/2\mathbb{Z}$ over the origin.

A.2.5. *Initial DM-coarsening.* There is always an initial DM-coarsening of X corresponding to the intersection of all open and closed subgroups of I_X. This initial DM-coarsening need not commute with restrictions to open substacks though. The reason is that the identity component $(I_X)^0$ need not be open. For example, this happens if $X = BG$ where G is a 1-parameter deformation of $\mathbb{Z}/p\mathbb{Z}$ to $\boldsymbol{\mu}_p$ in mixed characteristic p or from $\mathbb{Z}/p\mathbb{Z}$ to $\boldsymbol{\alpha}_p$ in equal characteristic p. One can, however, show that $(I_X)^0$ is open and closed if X is a tame Artin stack in equal characteristic.

A.2.6. *Rigidifications.* When X is any Artin stack and $N \subseteq I_X$ is a *flat* subgroup, then there is a *rigidification* $f : X \to X /\!\!/ N$ [Abramovich et al. 2008, Appendix A]. This is a coarsening that also is an fppf-gerbe. It has the universal property that for any Artin stack Z, a morphism $\phi : X \to Z$ factors through f if and only if the induced map $N \to \phi^*I_Z$ is trivial. The universal property does not require Z to be Deligne–Mumford or X to be tame.

Appendix B: Torification

B.1. *The torification functors.*

B.1.1. *The general case.* Let W be a toroidal scheme acted on by a diagonalizable group G in a relatively affine way. For example, any action of G on a quasiaffine scheme is relatively affine. The main results of [Abramovich and Temkin 2017] establish a so-called torification $\widetilde{\mathcal{T}}_{W,G}: W^{\mathrm{tor}} \to W$, which is a composition of two G-equivariant morphisms of toroidal schemes: the barycentric subdivision and the normalized blowing up of a so-called torifying ideal, see [op. cit., Theorem 4.6.5], such that the action on W^{tor} is toroidal. The barycentric subdivision is naturally a composition of blowings up, see [op. cit., §4.1.2]. The resulting sequence of normalized blowings up is compatible with strict strongly G-equivariant morphisms $f: W' \to W$ in the sense that $\widetilde{\mathcal{T}}_{W',G'}$ is the *contracted pullback* of $\widetilde{\mathcal{T}}_{W,G}$, i.e., $f^*(\widetilde{\mathcal{T}}_{W,G})$ with all empty blowings up removed. Furthermore, it is shown in [op. cit., Theorem 5.4.5] that the normalized blowing up of a torifying ideal I_W can also be realized as a blowing up of another ideal I'_W, in particular, $\widetilde{\mathcal{T}}_{W,G}$ is a projective modification even when W is not qe and it is not obvious a priori that normalizations are finite. However, the resulting realization of $W^{\mathrm{tor}} \to W$ as a sequence of blowings up, that we denote $\widetilde{\mathcal{T}}'_{W,G}$, is only compatible with surjective morphisms $f: W' \to W$ as above.

B.1.2. *Simple actions.* If the action is simple then slightly stronger results are available; see [Abramovich and Temkin 2017, Theorems 4.6.3 and 5.4.2]. In particular, torification is achieved by a single G-equivariant normalized blowing up $\mathcal{T}_{W,G}: W^{\mathrm{tor}} \to W$, and the quotient morphism $\mathcal{T}^0_{W,G}: W^{\mathrm{tor}} /\!\!/ G \to W /\!\!/ G$ has a natural structure of a normalized blowing up. This is compatible with strict strongly G-equivariant morphisms $f: W' \to W$. In addition, both morphisms can be enhanced to blowings up, that we denote $\mathcal{T}'_{W,G}$ and $\mathcal{T}'^0_{W,G}$. This involves the choice of a large enough threshold n — their centers are obtained from the centers of $\mathcal{T}_{W,G}$ and $\mathcal{T}^0_{W,G}$ by raising them to the n-th powers and applying the integral closure operation. As a result, $\mathcal{T}'_{W,G}$ and $\mathcal{T}'^0_{W,G}$ are only compatible with surjective morphisms.

B.1.3. *Birationality.* In [Abramovich and Temkin 2017, Theorems 4.6.3, 4.6.5, 5.4.2, and 5.4.5] it was shown that the torification functors used here are birational modifications only under a technical assumption that the action is *full*. For the purpose of this article we note the following:

Proposition B.1.4. *Assume G is finite. Then the torification morphisms are birational.*

Proof. For a point $w \in W$ write $\eta(w)$ for the generic point specializing to w — it is unique since W is normal. The subset $U_1 \subset W$ where the logarithmic structure is trivial and the subset $U_2 \subset W$ where $G_w = G_{\eta(w)}$ are both open, invariant, and dense, hence the same is true for $U = U_1 \cap U_2$. Since G is finite the strict embedding $U \hookrightarrow W$ is strongly equivariant, hence the torific ideal restricts to $\mathcal{O}_U$ and the torification morphisms are trivial on U. □

We note that, when G is infinite, some assumption on the action is necessary: the standard action of $\mathbb{G}_m$ on $\mathbb{A}^1$ has $\sigma_x = \{1\}$, which cannot be balanced since $\mathcal{I}_{-1} = 0$.

B.2. ***Stronger functoriality.*** Using the methods of [Abramovich and Temkin 2018] one can easily show that the functors $\widetilde{\mathcal{T}}$ and $\mathcal{T}$ possess stronger functoriality properties than asserted there. Let us discuss this strengthening.

B.2.1. *λ-equivariance.* We start with an aspect that holds for both algorithms. Recall that a G-morphism $f : W' \to W$ is *strongly equivariant* if f is the base change of the GIT quotient $f /\!\!/ G$. Some criteria of strong equivariance and related properties can be found in [Abramovich and Temkin 2018, Theorem 1.3.1 and Lemma 5.6.2; Rydh 2020]. More generally, assume that G' acts on W', G acts on W, and f is λ-equivariant for a homomorphism $\lambda : G' \to G$. We say that f is *strongly λ-equivariant* if it is fix-point reflecting and the G-morphism

$$W' \times^{G'} G = (W' \times G)/G' \to W$$

is strongly equivariant. Recall that the fixed-point reflecting condition means that f induces an isomorphism $G'_x = G_{f(x)}$ for any $x \in W'$, and hence G' acts freely on $W' \times G$.

Theorem B.2.2. *Assume that toroidal schemes W and W' are provided with relatively affine actions of diagonalizable groups G and G', respectively. Further assume that $\lambda : G' \to G$ is a homomorphism, and $f : W' \to W$ is a strict and strongly λ-equivariant morphism. Then $\widetilde{\mathcal{T}}_{W',G'}$ is the contracted pullback of $\widetilde{\mathcal{T}}_{W,G}$. In addition, $\widetilde{\mathcal{T}}'_{W',G'}$ is the contracted pullback of $\widetilde{\mathcal{T}}'_{W,G}$ if f is surjective.*

Proof. This happens because $\widetilde{\mathcal{T}}$ is defined in terms of local combinatorial data $(\overline{M}_x, G_x, \sigma_x)$, see [Abramovich and Temkin 2017, Section 3.6.8], and the latter only depends on G_x rather than on the entire G. □

B.2.3. *Weakening the strictness assumption.* A finer observation is that the strictness assumption is not so essential for the functoriality of $\mathcal{T}$. For comparison, note that $\widetilde{\mathcal{T}}$ is constructed using barycentric subdivisions which depend on the monoids $\overline{M}_x$, hence it is not functorial with respect to nonstrict morphisms.

Theorem B.2.4. *Assume that toroidal schemes W and W' are provided with relatively affine and simple actions of diagonalizable groups G and G', respectively, $\lambda : G' \to G$ is a homomorphism, and $f : W' \to W$ is a strongly λ-equivariant morphism. Further assume that for any point $x' \in W'$ with $x = f(x')$ the restriction $f_S : S' \to S$ of f to the logarithmic strata through x' and x is strongly λ-equivariant. Then the normalized blowings up $\mathcal{T}_{W',G'}$ and $\mathcal{T}^0_{W',G'}$ are the pullbacks of $\mathcal{T}_{W,G}$ and $\mathcal{T}^0_{W,G}$, respectively. If f is also surjective, then the same is true for the blowings up $\mathcal{T}'_{W',G'}$, $\mathcal{T}'^0_{W',G'}$ and $\mathcal{T}'_{W,G}$, $\mathcal{T}'^0_{W,G}$.*

Proof. Note that a reference to [Abramovich and Temkin 2017, Lemma 4.2.13(ii)] is the only place in the proof of [op. cit., Theorem 4.6.3], where one uses the assumption that f is strict. The lemma asserts that f respects the reduced signatures: $f^*(\sigma_x) = \sigma_{x'}$. Recall that the latter are defined as the multisets of nontrivial characters through which G_x acts on the cotangent spaces to S and S' at x and x', respectively. But we assume that f_S is strongly G_x-equivariant, hence $f^*(\sigma_x) = \sigma_{x'}$ by [op. cit., Lemma 3.6.4], and we avoid the use of [op. cit., Lemma 4.2.13(ii)]. □

B.2.5. *Logarithmically smooth morphisms.* The assumption that $f : W' \to W$ is strong can be omitted when f is logarithmically smooth. For this we need the following instance of Luna's fundamental lemma.

Lemma B.2.6. *Assume that Y and X are toroidal schemes provided with relatively affine actions of étale diagonalizable groups, the action on Y is simple, $\lambda : H \to G$ is a homomorphism, and $f : Y \to X$ is a logarithmically smooth λ-equivariant inert morphism. Then f is strongly λ-equivariant.*

Proof. Replacing Y by $Y \times^H G$ we can assume that $G = H$. In addition, it suffices to work locally on $Y /\!\!/ G$ and $X /\!\!/ G$, hence we can assume that these schemes are local and f is surjective. Since f is logarithmically smooth and inert, simplicity of the action on Y implies that the action on X is simple too.

In addition, let $\widetilde{G}$ denote the stabilizer of the closed orbits of Y and X. Then $f /\!\!/ \widetilde{G}$ is strongly $G/\widetilde{G}$-equivariant because $G/\widetilde{G}$ acts freely on $Y /\!\!/ \widetilde{G}$ and $X /\!\!/ \widetilde{G}$. Therefore, it suffices to prove that f is strongly $\widetilde{G}$-equivariant, and replacing G by $\widetilde{G}$ and localizing again, we can assume that $G = \widetilde{G}$.

Note that if f is strict, then it is a smooth morphism and the claim was proved in Luna's lemma [Abramovich and Temkin 2018, Theorem 1.3.1(2b)]. We will deduce the lemma from this particular case. In particular, using this claim we can replace X and Y by their equivariant étale covers, hence by [Abramovich and Temkin 2017, Proposition 3.2.10(i); Illusie and Temkin 2014, Proposition 1.2] we can assume that there exist an equivariant chart $P \to Q$, $X \to \boldsymbol{A}_P$, $Y \to \boldsymbol{A}_Q$ of f, where $\boldsymbol{A}_M = \mathrm{Spec}(\mathbb{Z}[M])$ and the actions are trivial on P and Q. Then the morphism $g : Y_P[Q] = Y \times_{\boldsymbol{A}_P} \boldsymbol{A}_Q \to Y$ is strong as both g and $g /\!\!/ G$ are pullbacks of $\boldsymbol{A}_Q \to \boldsymbol{A}_P$. In addition, $Y \to Y_P[Q]$ is strict and hence smooth. It remains to observe that $Y \to Y_P[Q]$ is also fix-points preserving, and hence it is strongly smooth by the above case. □

As an application we obtain:

Corollary B.2.7. *Assume that toroidal schemes W and W' are provided with relatively affine and simple actions of étale diagonalizable groups G and G', respectively, $\lambda : G' \to G$ is a homomorphism, and $f : W' \to W$ is a logarithmically smooth, fix-point reflecting, λ-equivariant morphism. Then the normalized blowings up $\mathcal{T}_{W',G'}$ and $\mathcal{T}^0_{W',G'}$ are the pullbacks of $\mathcal{T}_{W,G}$ and $\mathcal{T}^0_{W,G}$, respectively. If f is also surjective, then the same is true for the blowings up $\mathcal{T}'_{W',G'}$, $\mathcal{T}'^0_{W',G'}$ and $\mathcal{T}'_{W,G}$, $\mathcal{T}'^0_{W,G}$.*

Proof. Since f is strongly equivariant by Lemma B.2.6, the claim will follow from Theorem B.2.4 once we prove that the induced morphisms $f_S : S' \to S$ between the logarithmic strata are strongly equivariant. Since f_S is logarithmically smooth, f_S is smooth. Clearly, f_S is fix-point reflecting. Since the groups are finite, all orbits are special and hence f_S is inert [Abramovich and Temkin 2018, §5.1.8 and §5.5.3]. Thus, f_S is strongly equivariant (even strongly smooth) by [Abramovich and Temkin 2018, Theorem 1.1.3(ii)]. □

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