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Invertible functions on nonarchimedean symmetric spaces

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Let u be a nowhere vanishing holomorphic function on the Drinfeld space Ω^r of dimension $r-1$, where $r \geq 2$. The logarithm $\log_q |u|$ of its absolute value may be regarded as an affine function on the attached Bruhat–Tits building $\mathcal{BT}^r$. Generalizing a construction of van der Put in case $r=2$, we relate the group $\mathcal{O}(\Omega^r)^*$ of such u with the group $\mathbf{H}(\mathcal{BT}^r, \mathbb{Z})$ of integer-valued harmonic 1-cochains on $\mathcal{BT}^r$. This also gives rise to a natural $\mathbb{Z}$-structure on the first (ℓ-adic or de Rham) cohomology of Ω^r.

0. Introduction

The nonarchimedean symmetric spaces $\Omega = \Omega^r$ introduced by Drinfeld [1974] have shown great importance in the theories of modular and automorphic forms and of Shimura varieties, in the analytic uniformization of algebraic varieties, in the representation theory of $\mathrm{GL}(r, K)$, in the local Langlands correspondence, and in several other topics of the arithmetic of nonarchimedean local fields K. An incomplete list of a few references is [Manin and Drinfeld 1973; Mustafin 1978; Gerritzen and van der Put 1980; Schneider and Stuhler 1991; Laumon 1996; de Shalit 2001].

For a complete nonarchimedean local field K with finite residue class field $\mathbb{F}$ and completed algebraic closure C, the space Ω is defined as the complement of the K-rational hyperplanes in $\mathbb{P}^{r-1}(C)$. It carries a natural structure as a rigid-analytic space defined over K, and is supplied with an action of the group $\mathrm{PGL}(r, K)$. In contrast with the case of real symmetric spaces, it fails to be simply connected (in the étale topology, see [Fresnel and van der Put 2004, pages 160–161], but has a rich cohomological structure. Its cohomology (for cohomology theories satisfying some natural axioms) has been calculated by Schneider and Stuhler [1991]; see also [de Shalit 2001] and [Iovita and Spiess 2001].

Suppose for the moment that $r=2$. In this case, $\Omega = \Omega^2$ has dimension 1, and a coarse combinatorial picture is provided by the Bruhat–Tits tree $\mathcal{T}$ of $\mathrm{PGL}(2, K)$, a $(q+1)$-regular tree, where $q = \#(\mathbb{F})$ is the residue class cardinality of K. A map φ from the set $\boldsymbol{A}(\mathcal{T})$ of oriented 1-simplices ("arrows") of $\mathcal{T}$ to $\mathbb{Z}$ that satisfies

(A) $\varphi(e) + \varphi(\bar{e}) = 0$ for each $e \in \boldsymbol{A}(\mathcal{T})$ with inverse $\bar{e}$, and

(B) $\sum \varphi(e) = 0$ for each vertex v of $\mathcal{T}$, where e runs through the arrows emanating from v,

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is called a ($\mathbb{Z}$-valued) *harmonic cochain* on $\mathcal{T}$. The group $\mathbf{H}(\mathcal{T}, \mathbb{Z})$ of all such yields upon tensoring with $\mathbb{Z}_\ell$ (ℓ a prime coprime with q) the first étale cohomology group $H^1_{ét}(\Omega^2, \mathbb{Z}_\ell)$ of Ω^2 [Drinfeld 1974, Proposition 10.2]. Marius van der Put [1983] (see also [Fresnel and van der Put 1981, I.8.9]) established a short exact sequence

$$1 \to C^* \to \mathcal{O}(\Omega^2)^* \xrightarrow{P} \mathbf{H}(\mathcal{T}, \mathbb{Z}) \to 0 \tag{0.1}$$

of $\mathrm{PGL}(2, K)$-modules, where $\mathcal{O}(\Omega^2)$ is the C-algebra of holomorphic functions on Ω^2 with multiplicative group $\mathcal{O}(\Omega^2)^*$. The van der Put transform $P(u)$ of an invertible function u is a substitute for the logarithmic derivative u'/u, and (0.1) provides the starting point for a study of the "Riemann surface" $\Gamma \setminus \Omega^2$, where $\Gamma \subset \mathrm{PGL}(2, K)$ is a discrete subgroup [Gerritzen and van der Put 1980; Gekeler and Reversat 1996].

It is the aim of the present paper to develop a higher-rank (i.e., $r > 2$) analogue of (0.1). In [Gekeler 2019] it was shown that the absolute value $|u|$ of $u \in \mathcal{O}(\Omega^r)^*$ factors over the building map

$$\lambda\colon \Omega^r \to \mathcal{BT}^r$$

and that its logarithm $\log_q |u|$ defines an affine map on $\mathcal{BT}^r(\mathbb{Q})$. Here $\mathcal{BT}^r$ is the Bruhat–Tits building of $\mathrm{PGL}(r, K)$ (the higher-dimensional analogue of $\mathcal{BT}^2 = \mathcal{T}$) and $\mathcal{BT}^r(\mathbb{Q})$ is the set of $\mathbb{Q}$-points of its realization $\mathcal{BT}^r(\mathbb{R})$. This makes it feasible that $u \mapsto \log_q |u|$ gives rise to a construction of P generalizing van der Put's in the case $r = 2$. The transform $P(u)$ of u will be a $\mathbb{Z}$-valued function on the set of arrows $\boldsymbol{A}(\mathcal{BT}^r)$ of $\mathcal{BT}^r$ subject to (obvious generalizations of) the conditions (A) and (B) above.

Our first result, Proposition 3.1, is that $P(u)$ satisfies one more relation (condition (C) in Corollary 2.12) not visible if $r = 2$. We then define $\mathbf{H}(\mathcal{BT}^r, \mathbb{Z})$ as the group of those $\varphi\colon \boldsymbol{A}(\mathcal{BT}^r) \to \mathbb{Z}$ which satisfy (A), (B) and (C).

The principal result of the present paper is the fact that the set of these relations is complete:

Theorem 3.11. *The map $P\colon \mathcal{O}(\Omega^r)^* \to \mathbf{H}(\mathcal{BT}^r, \mathbb{Z})$ is surjective, and the van der Put sequence*

$$1 \to C^* \to \mathcal{O}(\Omega^r)^* \to \mathbf{H}(\mathcal{BT}^r, \mathbb{Z}) \to 0 \tag{0.2}$$

is an exact sequence of $\mathrm{PGL}(r, K)$*-modules.*

The proof requires the construction of certain functions $u = f_{H,H',n}$ whose transforms $P(u)$ have a prescribed behavior on the finite subcomplex $\mathcal{BT}^r(n)$ of $\mathcal{BT}^r$, and a crucial technical result (Proposition 3.10), which solely refers to the geometry of $\mathcal{BT}^r$.

Still, $\mathbf{H}(\mathcal{BT}^r, \mathbb{Z})$ is a torsion-free abelian group of complicated appearance. However, as a further consequence of Proposition 3.10, we are able to describe it in Theorem 4.16

- either as $\mathbf{H}(\mathcal{T}_{v_0}, \mathbb{Z})$, where $\mathcal{T}_{v_0}$ is a subcomplex of dimension 1 of $\mathcal{BT}^r$ (in fact, a tree, which for $r = 2$ agrees with the Bruhat–Tits tree $\mathcal{T} = \mathcal{BT}^2$), and where only conditions (A) and (B) are involved,
- or as the group $\mathbf{D}^0(\mathbb{P}(V), \mathbb{Z})$ of $\mathbb{Z}$-valued distributions of total mass 0 on the projective space $\mathbb{P}(V)$, or by duality, on the compact space $\mathbb{P}(V^\wedge)$ of hyperplanes of the K-vector space $V = K^r$.

As the corresponding group $\mathbf{D}^0(\mathbb{P}(V^\wedge), A)$ with coefficients in some ring A depending on the cohomology theory used (e.g., $A = \mathbb{Z}_\ell$ for étale cohomology, or, in characteristic zero, $A = K$ for de Rham cohomology) has been shown to agree with the first cohomology $H^1(\Omega^r, A)$ [Schneider and Stuhler 1991, Section 3, Theorem 1], we get in some cases a natural integral structure on $H^1(\Omega^r, A)$ along with a concrete arithmetic interpretation.

1. Background

1.1. Throughout, K denotes a nonarchimedean local field with ring O of integers, a fixed uniformizer π, and finite residue class field $O/(\pi) = \mathbb{F} = \mathbb{F}_q$ of cardinality q. Hence K is a finite extension of either a p-adic field $\mathbb{Q}_p$ or of a Laurent series field $\mathbb{F}_p((X))$. We normalize its absolute value $|\cdot|$ by $|\pi| = q^{-1}$, and let $C = \hat{\overline{K}}$ be its completed algebraic closure with respect to the unique extension of $|\cdot|$ to $\overline{K}$. The ring of integers of C and its maximal ideal are denoted by O_C and $\mathfrak{m}_C$. Note that the residue class field $O_C/\mathfrak{m}_C$ is an algebraic closure $\bar{\mathbb{F}}$ of $\mathbb{F}$. Further, $\log\colon C^* \to \mathbb{Q}$ is the map $z \mapsto \log_q |z|$.

1.2. Given a natural number $r \geq 2$, the Drinfeld symmetric space $\Omega = \Omega^r$ of dimension $r - 1$ is the complement $\Omega = \mathbb{P}^{r-1} \setminus \bigcup H$ of the K-rational hyperplanes H in projective space $\mathbb{P}^{r-1}$. Hence the set of C-valued points of Ω (for which we briefly write Ω) is

$$\Omega = \{(\omega_1 : \cdots : \omega_r) \in \mathbb{P}^{r-1}(C) \mid \text{the } \omega_i \text{ are } K\text{-linearly independent}\}.$$

If not indicated otherwise, we always suppose that projective coordinates $(\omega_1 : \cdots : \omega_r)$ are *unimodular*, that is $\max_i |\omega_i| = 1$. The set Ω carries a natural structure as a rigid-analytic space defined over K (see [Drinfeld 1974; Deligne and Husemoller 1987; Schneider and Stuhler 1991]); in fact, it is an admissible open subspace of $\mathbb{P}^{r-1}$, and even a Stein domain [Schneider and Stuhler 1991, Section 1, Proposition 14]; see [Kiehl 1967] for the notion of nonarchimedean Stein domain.

1.3. Let G be the group scheme $\mathrm{GL}(r)$ with center Z; hence $G(K) = \mathrm{GL}(r, K)$, $Z(K) \cong K^*$, etc. The Bruhat–Tits building [Bruhat and Tits 1972] $\mathcal{BT} = \mathcal{BT}^r$ of $G(K)/Z(K) = \mathrm{PGL}(r, K)$ is a contractible simplicial complex with set of vertices

$$V(\mathcal{BT}) = \{[L] \mid L \text{ an } O\text{-lattice in } V\}, \tag{1.3.1}$$

where L runs through the set of O-lattices in the K-vector space $V = K^r$ and $[L]$ is the similarity class of L. (An *O-lattice* is a free O-submodule of rank r of V, two such, L and L', are *similar* if there exists $0 \neq c \in K$ such that $L' = cL$.) The classes $[L_0], \ldots, [L_s]$ form an s-simplex if and only if they are represented by lattices L_i such that

$$L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_s \supsetneq \pi L_0. \tag{1.3.2}$$

The *combinatorial distance* $d(v, v')$ of two vertices $v, v' \in \boldsymbol{V}(\mathcal{BT})$ is the length of a shortest path connecting them in the 1-skeleton of $\mathcal{BT}$. It is easily verified that

$$d(v, v') = \min\{n \mid \exists \text{ representatives } L, L' \text{ for } v, v' \text{ such that } L \supset L' \supset \pi^n L\}. \tag{1.3.3}$$

The *star* $\mathrm{st}(v)$ of $v \in \boldsymbol{V}(\mathcal{BT})$ will always denote the full subcomplex of $\mathcal{BT}$ with set of vertices

$$\boldsymbol{V}(\mathrm{st}(v)) = \{w \in \boldsymbol{V}(\mathcal{BT}) \mid d(v, w) \leq 1\}. \tag{1.3.4}$$

We regard V as a space of row vectors, on which $G(K)$ acts as a matrix group from the right. Hence $G(K)$ acts also from the right on $\mathcal{BT}$. If the syntax requires a left action, we shift this action to the left by the usual formula $\gamma x := x\gamma^{-1}$.

1.4. The relationship between Ω and $\mathcal{BT}$ is as follows: By the Goldman–Iwahori theorem [Goldman and Iwahori 1963], the realization $\mathcal{BT}(\mathbb{R})$ of $\mathcal{BT}$ is in a natural one-to-one correspondence with the set of similarity classes of real-valued nonarchimedean norms on V, where a vertex $v = [L] \in \boldsymbol{V}(\mathcal{BT}) = \mathcal{BT}(\mathbb{Z})$ corresponds to the class of a norm with unit ball $L \subset V$. Now the *building map*

$$\begin{aligned} \lambda\colon \Omega &\to \mathcal{BT}(\mathbb{R}) \\ \boldsymbol{\omega} = (\omega_1 : \ldots : \omega_r) &\mapsto [\nu_{\boldsymbol{\omega}}] \end{aligned} \tag{1.4.1}$$

is well-defined, where the norm $\nu_{\boldsymbol{\omega}}$ maps $\boldsymbol{x} = (x_1, \ldots, x_r) \in V$ to

$$\nu_{\boldsymbol{\omega}}(\boldsymbol{x}) = \Big|\sum_{1 \leq i \leq r} x_i \omega_i\Big|,$$

and $[\nu_{\boldsymbol{\omega}}]$ is its similarity class. Since the value group is $|C^*| = q^{\mathbb{Q}}$, λ maps to $\mathcal{BT}(\mathbb{Q})$, and is in fact onto $\mathcal{BT}(\mathbb{Q})$, the set of points of $\mathcal{BT}(\mathbb{R})$ with rational barycentric coordinates. $G(K)$ acts from the left on the set of norms via

$$\gamma\nu(\boldsymbol{x}) := \nu(\boldsymbol{x}\gamma) \tag{1.4.2}$$

for $\boldsymbol{x} \in V$, a norm ν, and $\gamma \in G(K)$; the reader may verify that λ is $G(K)$-equivariant, where the action on Ω is the standard one through left matrix multiplication. The preimages under λ of simplices of $\mathcal{BT}$ yield an admissible covering of Ω; see e.g., [de Shalit 2001, (6.2) and (6.3)]. We therefore consider $\mathcal{BT}$ as a combinatorial picture of Ω.

We cite the following results from [Gekeler 2017; 2019].

Theorem 1.5 [Gekeler 2019, Theorem 2.4]. *Let u be an invertible holomorphic function on Ω. Then $|u(\boldsymbol{\omega})|$ depends only on the image $\lambda(\boldsymbol{\omega})$ of $\boldsymbol{\omega} \in \Omega$ in $\mathcal{BT}(\mathbb{Q})$.*

1.5.1. We thus define the *spectral norm* $\|u\|_x$ as the common absolute value $|u(\boldsymbol{\omega})|$ for all $\omega \in \lambda^{-1}(x)$, where $x \in \mathcal{BT}(\mathbb{Q})$.

Theorem 1.6 [Gekeler 2019, Theorem 2.6]. *Let u be an invertible holomorphic function on Ω. Then $\log u = \log_q |u|$ regarded as a function on $\mathcal{BT}(\mathbb{Q})$ is affine, that is, interpolates linearly in simplices.*

1.7. Let $\boldsymbol{A}(\mathcal{BT})$ be the set of *arrows*, i.e., of oriented 1-simplices of $\mathcal{BT}$. For each arrow $e = (v, v') = ([L], [L'])$ we write

$$o(e) = \text{origin of } e := v, \quad t(e) = \text{terminus of } e := v', \quad \text{and} \quad \text{type}(e) := \dim_{\mathbb{F}}(L'/\pi L),$$

where L, L' are representatives with $L \supset L' \supset \pi L$. Then $1 \leq \text{type}(e) \leq r-1$ and $\text{type}(e) + \text{type}(\bar{e}) = r$, where $\bar{e} = (v', v)$ is e with reverse orientation. We let

$$\boldsymbol{A}_v = \dot{\bigcup_{1 \leq t \leq r-1}} \boldsymbol{A}_{v,t} \tag{1.7.1}$$

be the arrows e with $o(e) = v$, grouped according to their types t. For an invertible function u on Ω and an arrow $e = (v, w)$, define the *van der Put value* $P(u)(e)$ of u on e as

$$P(u)(e) = \log_q \|u\|_w - \log_q \|u\|_v \tag{1.7.2}$$

with the spectral norm of Section 1.5.1.

Proposition 1.8 [Gekeler 2017, Proposition 2.9]. *The* van der Put transform

$$P(u)\colon \boldsymbol{A}(\mathcal{BT}) \to \mathbb{Q}$$
$$e \mapsto P(u)(e)$$

of u has in fact values in $\mathbb{Z}$ and satisfies

$$\sum_{e \in \boldsymbol{A}_{v,1}} P(u)(e) = 0 \tag{1.8.1}$$

for all $v \in \boldsymbol{V}(\mathcal{BT})$. Here the sum is over the arrows e with $o(e) = v$ and $\text{type}(e) = 1$.

Actually, in [Gekeler 2017] the condition $\sum_{e \in \boldsymbol{A}_{v,r-1}} P(u)(e) = 0$ is given instead of (1.8.1), due to another choice of orientation. We will discuss this in more detail in Section 2.6 and Remarks 3.3(i), which will also show that both conditions are equivalent in our framework.

Remarks 1.9. (i) In the case $r = 2$, Theorems 1.5, 1.6 and Proposition 1.8 have been known for quite some time; see [van der Put 1983] and e.g., [Fresnel and van der Put 1981, I.8.9]. For general r, they are shown in [Gekeler 2017; 2019] in the framework of these papers, where $\text{char}(K) = \text{char}(\mathbb{F}) = p$. However, the proofs make no use of this assumption, and are therefore valid for $\text{char}(K) = 0$, too.

(ii) The three cited results are local in the sense that they do not require u to be a global unit. If, e.g., u is a holomorphic function without zeroes on the affinoid $\lambda^{-1}(x)$ with $x \in \mathcal{BT}(\mathbb{Q})$, then $|u(\boldsymbol{\omega})|$ is constant on $\lambda^{-1}(x)$; if u is invertible on $\lambda^{-1}(\sigma)$ with a closed simplex σ of $\mathcal{BT}$, then $\log u$ is affine there, and if u is invertible on $\lambda^{-1}(\text{st}(v))$, where $\text{st}(v)$ is the star of $v \in \boldsymbol{V}(\mathcal{BT})$ (see (1.3.4)), then $P(u)(e)$ is defined for all $e \in \boldsymbol{A}_v$ and satisfies (1.8.1).

(iii) It is immediate from the definitions that for invertible functions u, u' and arrows e,

$$P(u)(e) + P(u)(\bar{e}) = 0, \tag{1.9.1}$$

and more generally

$$\sum P(u)(e) = 0, \quad \text{if } e \text{ runs through the arrows of a closed path in } \mathcal{BT}, \tag{1.9.2}$$

as well as

$$P(uu') = P(u) + P(u'). \tag{1.9.3}$$

Hence the van der Put transform $P: u \mapsto P(u)$ is a homomorphism from the multiplicative group $\mathcal{O}(\Omega)^*$ of invertible holomorphic functions on Ω to the additive group of maps $\varphi\colon \boldsymbol{A}(\mathcal{BT}) \to \mathbb{Z}$ that satisfy (1.9.1), (1.9.2) and (1.8.1). Moreover, for $\gamma \in G(K)$,

$$P(u)(e\gamma) = P(u \circ \gamma^{-1})(e), \tag{1.9.4}$$

i.e., $\gamma(P(u)) = P(\gamma u) := P(u \circ \gamma^{-1})$ holds; whence P is $G(K)$-equivariant.

In Theorem 3.11 we will find exact conditions that characterize the image of P. This will yield the exact sequence (0.2) of $G(K)$-modules that generalizes (0.1).

2. Evaluation of P on elementary rational functions

2.1. Let U be a subspace of $V = K^r$ of dimension t, where $1 \le t \le r-1$. We define the shift toward U on $\boldsymbol{V}(\mathcal{BT})$ by

$$\begin{aligned} \tau_U\colon \boldsymbol{V}(\mathcal{BT}) &\to \boldsymbol{V}(\mathcal{BT}), \\ v = [L] &\mapsto [L'] \end{aligned} \tag{2.1.1}$$

where $L' = (L \cap U) + \pi L$. Obviously, $e = (v, \tau_U(v))$ is a well-defined arrow of type type$(e) = \dim U = t$. We say that *e points to U*.

2.1.2. For a local ring R (in practice: $R = K$, or O, or a finite quotient $O_n := O/(\pi^n)$) and a free R-module F of finite rank, let $\mathrm{Gr}_{R,t}(F)$ be the Grassmannian of direct summands F' such that $\mathrm{rank}_R F' = t$. Fixing $v = [L] \in \boldsymbol{V}(\mathcal{BT})$, there is a natural surjective map

$$\begin{aligned} \mathrm{Gr}_{K,t}(V) &\to \boldsymbol{A}_{v,t} \\ U &\mapsto (v, \tau_U(v)) \end{aligned} \tag{2.1.3}$$

and a canonical bijection

$$\boldsymbol{A}_{v,t} \xrightarrow{\cong} \mathrm{Gr}_{\mathbb{F},t}(L/\pi L) \tag{2.1.4}$$

given by $e = (v, w) = ([L], [M]) \mapsto \overline{M} := M/\pi L$, where $L \supset M \supset \pi L$. We denote the image of e by $\overline{M}_e$ and the preimage of $\overline{M}$ in $\boldsymbol{A}_{v,t}$ by $e_{\overline{M}}$.

2.1.5. For two arrows $e = e_{\overline{M}}$ and $e' = e_{\overline{M}'}$ with the same origin, we write $e \prec e'$ (e' *dominates* e) if and only if $\overline{M} \subset \overline{M}'$.

2.1.6. Fix $n \in \mathbb{N}$, let O_n be the ring $O/(\pi^n)$ and let $t \in \{1, r-1\}$. Then, as a generalization of the above, $U \mapsto (v, \tau_U(v), \dots, \tau_U^n(v))$ is surjective from $\mathrm{Gr}_{K,t}(V)$ onto the set $\boldsymbol{A}_{v,t,n}$ of paths of length n in $\mathcal{BT}$ which emanate from v, are composed of arrows of type t, and whose endpoints w have distance $d(v, w) = n$ (e.g., $\boldsymbol{A}_{v,t,1} = \boldsymbol{A}_{v,t}$). The set $\boldsymbol{A}_{v,t,n}$ corresponds one-to-one to $\mathrm{Gr}_{O_n,t}(L/\pi^n L)$, where the composite map from $\mathrm{Gr}_{K,t}(V)$ to $\mathrm{Gr}_{O_n,t}(L/\pi^n L)$ is given by $U \mapsto ((L \cap U) + \pi^n L)/\pi^n L$. This yields in the limit the canonical bijections

$$\mathrm{Gr}_{K,t}(V) \xrightarrow{\cong} \varprojlim_n \boldsymbol{A}_{v,t,n} = \varprojlim_n \mathrm{Gr}_{O_n,t}(L/\pi^n L) = \mathrm{Gr}_{O,t}(L), \tag{2.1.7}$$

whose composition is simply $U \mapsto U \cap L$. Let e be an arrow of type t. Then

$$\mathrm{Gr}_{K,t}(e) := \{U \in \mathrm{Gr}_{K,t}(V) \mid e \text{ points to } U\} \tag{2.1.8}$$

is compact and open in the compact space $\mathrm{Gr}_{K,t}(V)$, and it follows from the considerations above that the set of all $\mathrm{Gr}_{K,t}(e)$, where v is fixed and e belongs to $\boldsymbol{A}_{v,t,n}$ for some $n \in \mathbb{N}$, forms a basis for the topology on $\mathrm{Gr}_{K,t}(V)$.

2.2. Given a hyperplane H in V, we let $\ell_H : V \to K$ be a linear form with kernel H. We denote by the same symbol its extension $\ell_H : V \otimes_K C = C^r \to C$. The quotients

$$\ell_{H,H'} := \ell_H / \ell_{H'} \tag{2.2.1}$$

of two such are rational functions on $\mathbb{P}^{r-1}(C)$ without zeroes or poles on $\Omega \hookrightarrow \mathbb{P}^{r-1}(C)$. Note that ℓ_H is determined up to multiplication by a nonzero scalar in K; hence $P(\ell_{H,H'})$ depends only on H and H', but not on the scaling of ℓ_H and $\ell_{H'}$. Our first task will be to describe $P(\ell_{H,H'})$.

2.3. We start with a closer look to the building map λ. Let $v_0 = [L_0]$ be the standard vertex, where L_0 is the standard lattice O^r in V. Let us first recall the easily verified fact (where the unimodularity normalization of $\boldsymbol{\omega} \in \Omega$ is used):

$$\begin{aligned}\Omega_{v_0} := \lambda^{-1}(v_0) &= \{\boldsymbol{\omega} \in \Omega \mid \nu_{\boldsymbol{\omega}} \text{ has unit ball } L_0\} \\ &= \{\boldsymbol{\omega} \in \Omega \mid \text{ the } \omega_i \text{ are orthogonal and } |\omega_i| = 1 \text{ for } 1 \le i \le r\}.\end{aligned} \tag{2.3.1}$$

($z_1, \dots, z_n \in C$ are *orthogonal* if and only if $|\sum_{1 \le i \le r} a_i z_i| = \max_i |a_i z_i|$ for arbitrary coefficients $a_i \in K$.) Hence the canonical reduction of Ω_{v_0} equals

$$\overline{\Omega}_{v_0} = (\mathbb{P}^{r-1}/\mathbb{F}) \setminus \bigcup \overline{H}, \tag{2.3.2}$$

where $\overline{H}$ runs through the hyperplanes defined over $O/(\pi) = \mathbb{F}$. A similar description holds for $\overline{\lambda^{-1}(v)}$ if v is an arbitrary vertex, but we need some preparations.

2.4. Write $\langle \cdot, \cdot \rangle$ for the standard bilinear form on V given by

$$\langle \boldsymbol{x}', \boldsymbol{x} \rangle = \sum_{1 \le i \le r} x_i' x_i,$$

which we extend to a form $\langle \cdot, \cdot \rangle$ on C^r. It identifies $V = K^r$ with its dual $V^\wedge$. For each K-subspace U of V, let

$$U^\perp := \{\boldsymbol{x} \in V \mid \langle \boldsymbol{x}, U \rangle = 0\} \tag{2.4.1}$$

be its orthogonal with respect to $\langle \cdot, \cdot \rangle$. For an O-lattice L in V,

$$L^\wedge := \{\boldsymbol{x} \in V \mid \langle \boldsymbol{x}, L \rangle \subset O\} \tag{2.4.2}$$

is the dual lattice. We put $\tilde{\Omega}$ for the preimage of Ω in C^r. Then $L^\wedge \otimes_O O_C$ embeds into C^r, and by (1.4.1) we find:

2.4.3. The image of $(L^\wedge \otimes O_C) \cap \tilde{\Omega}$ in Ω equals $\Omega_v := \lambda^{-1}(v)$, where $v = [L]$ is the vertex of $\mathcal{BT}$ corresponding to L.

Similarly,

$$(L^\wedge \otimes O_C) \otimes O_C/\mathfrak{m}_C \xrightarrow{\cong} (L^\wedge/\pi L^\wedge) \otimes_{\mathbb{F}} \bar{\mathbb{F}} = (L/\pi L)^\wedge \otimes_{\mathbb{F}} \bar{\mathbb{F}}. \tag{2.4.4}$$

2.5. Let $\mathbb{P}(L^\wedge/\pi L^\wedge)/\mathbb{F}$ be the projective space attached to the r-dimensional vector space $L^\wedge/\pi L^\wedge$, regarded as a scheme over $O/(\pi) = \mathbb{F}$. Its $\mathbb{F}$-rational hyperplanes correspond to those of the vector space $(L^\wedge/\pi L^\wedge) \otimes_{\mathbb{F}} \bar{\mathbb{F}}$, or, by duality, to the $\mathbb{F}$-lines (one-dimensional $\mathbb{F}$-subspaces) $\overline{G}$ in $L/\pi L$. Therefore, the canonical reduction $\overline{\Omega}_v$ of Ω_v is

$$\overline{\Omega}_v = (\mathbb{P}(L^\wedge/\pi L^\wedge)/\mathbb{F}) \setminus \bigcup \overline{H}, \tag{2.5.1}$$

where $\overline{H}$ runs through the hyperplanes defined over $\mathbb{F}$. The set of these is in canonical bijection with the set of $\mathbb{F}$-lines in $L/\pi L$, that is, with $\boldsymbol{A}_{v,1}$. For each $e \in \boldsymbol{A}_{v,1}$ let $\overline{H}_e$ be the corresponding $\mathbb{F}$-hyperplane in (2.5.1).

2.5.2. The object $\overline{H}_e$ (an $\mathbb{F}$-subspace of $L^\wedge/\pi L^\wedge$ or the corresponding hyperplane in $\mathbb{P}(L^\wedge/\pi L^\wedge)/\mathbb{F}$, described through the same symbol) mustn't be confused with the $\overline{M}_e$ of (2.1.4), which is an $\mathbb{F}$-subspace of $L/\pi L$. The relationship is as follows. The form $\langle \cdot, \cdot \rangle$ induces an $\mathbb{F}$-bilinear form

$$\overline{\langle \cdot, \cdot \rangle} \colon L/\pi L \times L^\wedge/\pi L^\wedge \to \mathbb{F}.$$

For an $\mathbb{F}$-subspace $\overline{M}$ of $L/\pi L$, $\overline{M}^\perp$ denotes its orthogonal with respect to $\overline{\langle \cdot, \cdot \rangle}$ in $L^\wedge/\pi L^\wedge$. Let $\overline{G}_e \subset L/\pi L$ be the line defined by $e \in \boldsymbol{A}_{v,1}$ as in (2.1.4). Then $\overline{H}_e = (\overline{G}_e)^\perp$.

2.6. Let u be an invertible holomorphic function on Ω, scaled such that $\|u\|_v = 1$. Its reduction $\bar{u}$ at v is a rational function on $\overline{\Omega}_v$ without zeroes or poles. For each $e \in \boldsymbol{A}_{v,1}$ let m_e be the vanishing order of $\bar{u}$ along $\overline{H}_e$ (negative, if $\bar{u}$ has a pole along $\overline{H}_e$), and let ℓ_e be a linear form on $\mathbb{P}(L^\wedge/\pi L^\wedge)/\mathbb{F}$ with vanishing locus $\overline{H}_e$. Up to a multiplicative constant, $\bar{u}$ equals $\prod_{e \in \boldsymbol{A}_{v,1}} \ell_e^{m_e}$, and so

$$\sum_{e \in \boldsymbol{A}_{v,1}} m_e = \text{weight of the form } \bar{u} = 0. \tag{2.6.1}$$

Now the value of the van der Put transform on $e \in \boldsymbol{A}_{v,1}$ is (with notation above)

$$P(u)(e) = -m_e. \tag{2.6.2}$$

To see this, we may assume (by (1.9.4), and since the action of $G(K)$ on arrows of type 1 is transitive) that $e = (v_0, v_1)$ with $v_0 = [L_0]$, $v_1 = [L_1]$, $L_0 = O^r$, $L_1 = (\pi) \times \cdots \times (\pi) \times O$. Then $L_0^\wedge = L_0$, $L_1^\wedge = (\pi^{-1}) \times \cdots \times (\pi^{-1}) \times O$, $\overline{H}_e$ is the hyperplane $\{(* : \cdots : * : 0)\}$ in $\mathbb{P}(L_0^\wedge/\pi L_0^\wedge)/\mathbb{F} = \mathbb{P}^{r-1}/\mathbb{F}$, and we may choose $\ell_e \colon \boldsymbol{x} = (x_1 : \cdots : x_r) \mapsto x_r$, which is the reduction of the global form $\tilde{\ell} \colon \boldsymbol{\omega} = (\omega_1 : \cdots : \omega_r) \mapsto \omega_r$ on Ω. In order to get "functions" instead of "forms", we work with ℓ_e/ℓ_1 (resp. $\tilde{\ell}/\tilde{\ell}_1$), where $\ell_1 \colon \boldsymbol{x} \mapsto x_1$ with lift $\tilde{\ell}_1 \colon \boldsymbol{\omega} \mapsto \omega_1$. If $\bar{u}$ has a zero of order m along $\overline{H}_e$ (i.e., $\bar{u} = \bar{u}_0(\ell_e/\ell_1)^m$, where $\bar{u}_0$ has neither zeroes nor poles on $\overline{\Omega}_{v_0}$ and along $\overline{H}_e$), then u grows like $(\tilde{\ell}/\tilde{\ell}_1)^m$ when moving from Ω_{v_0} to Ω_{v_1}. But the absolute value of $\tilde{\ell}/\tilde{\ell}_1$ on Ω_{v_0} is 1, while it is $|\pi| = q^{-1}$ on Ω_{v_1}, which shows (2.6.2).

Finally, combining the above yields

$$\sum_{e \in \boldsymbol{A}_{v,1}} P(u)(e) = 0, \tag{2.6.3}$$

that is, the assertion of (1.8.1).

2.7. Each hyperplane H of V is given as the kernel of a linear form

$$\ell_H = \ell_{\boldsymbol{y}} \colon \boldsymbol{x} \mapsto \langle \boldsymbol{y}, \boldsymbol{x} \rangle \tag{2.7.1}$$

with some $\boldsymbol{y} \in L_0 \setminus \pi L_0$. Let $G = K\boldsymbol{y} = H^\perp$ be the line spanned by $\boldsymbol{y}$. The arrow $(v_0, \tau_G(v_0)) \in \boldsymbol{A}_{v_0,1}$ equals $e_{\overline{G}}$ with

$$\overline{G} = (O\boldsymbol{y} + \pi L_0)/\pi L_0.$$

Two such vectors $\boldsymbol{y}$, $\boldsymbol{y}'$ give rise to the same $e_{\overline{G}}$ if and only if $\boldsymbol{y}' \equiv c \cdot \boldsymbol{y} \pmod{\pi}$ with some unit $c \in O^*$. More generally, $\boldsymbol{y}$ and $\boldsymbol{y}'$ give rise to the same path $(v_0, \tau_G(v_0), \ldots, \tau_G^n(v_0)) \in \boldsymbol{A}_{v_0,1,n}$ if and only if

$$\boldsymbol{y}' \equiv c \cdot \boldsymbol{y} \pmod{\pi^n} \tag{2.7.2}$$

with $c \in O^*$. In this case we call $\boldsymbol{y}$ and $\boldsymbol{y}'$ *n-equivalent*; the respective equivalence classes are briefly the *n-classes* of $\boldsymbol{y}$, $\boldsymbol{y}'$.

2.8. Let now hyperplanes H, H' of V be given by $\boldsymbol{y}$, $\boldsymbol{y}'$ as above. Put $G = H^\perp = K\boldsymbol{y}$, $G' = K\boldsymbol{y}'$. The function $\ell_{H,H'} = \ell_{\boldsymbol{y}}/\ell_{\boldsymbol{y}'}$ has constant absolute value 1 on Ω_{v_0} and therefore, by reduction, gives a rational function $\bar{\ell}_{H,H'}$ without zeroes or poles on $\overline{\Omega}_{v_0} \hookrightarrow \mathbb{P}^{r-1}/\mathbb{F}$. Put

$$\overline{H} = ((L_0 \cap H) + \pi L_0)/\pi L_0,$$

and ditto $\overline{H}'$. By definition, it is an $\mathbb{F}$-subvector space of $L_0/\pi L_0 \xrightarrow{\cong} \mathbb{F}^r$. As usual, we denote by the same symbol the corresponding $\mathbb{F}$-rational linear subvariety of $\mathbb{P}^{r-1}/\mathbb{F}$ that appears e.g., in (2.3.2). Suppose that $\overline{H}$ differs from $\overline{H}'$. Then $\bar{\ell}_{H,H'}$ has vanishing order 1 along $\overline{H}$, vanishing order -1 along $\overline{H}'$, and vanishing order 0 along the other hyperplanes in the boundary of $\overline{\Omega}_{v_0}$ (see (2.3.2)). If however $\overline{H} = \overline{H}'$, then $\bar{\ell}_{H,H'}$ has neither zeroes nor poles along the boundary (and is therefore constant). According to the recipe discussed in Section 2.6, we find the following description.

Proposition 2.9. *Let e be an arrow in $\boldsymbol{A}_{v_0,1}$. Then*

$$P(\ell_{H,H'})(e) = \begin{cases} -1 & e = (v_0, \tau_G(v_0)) \neq (v_0, \tau_{G'}(v_0)), \\ +1 & e = (v_0, \tau_{G'}(v_0)) \neq (v_0, \tau_G(v_0)), \\ 0 & \textit{otherwise.} \end{cases} \qquad \square$$

Again by the transitivity of the action of $G(K)$, we may transfer Proposition 2.9 to arbitrary arrows of type 1, and thus get:

Corollary 2.10. *Let $e \in \boldsymbol{A}_{v,1}$ be an arrow of type 1 with arbitrary origin $v \in \boldsymbol{V}(\mathcal{BT})$. Write $e_H^\perp$ (resp. $e_{H'}^\perp$) for the arrow $(v, \tau_{H^\perp}(v))$ (resp. $(v, \tau_{H'^\perp}(v))$). Then*

$$P(\ell_{H,H'})(e) = \begin{cases} -1 & e = e_H^\perp \neq e_{H'}^\perp, \\ +1 & e = e_{H'}^\perp \neq e_H^\perp, \\ 0 & \textit{otherwise.} \end{cases} \qquad \square$$

Next, we deal with arrows of arbitrary type.

Proposition 2.11. *Given hyperplanes H, H' of V and an arrow e of $\mathcal{BT}$ with origin $v \in \boldsymbol{V}(\mathcal{BT})$, let $e_H^\perp$ (resp. $e_{H'}^\perp$) be the arrow with origin v pointing to $G = H^\perp$ (resp. to $G' = H'^\perp$). The transform $P(\ell_{H,H'})$ evaluates on e as follows*:

$$P(\ell_{H,H'})(e) = \begin{cases} -1 & e_H^\perp \prec e, e_{H'}^\perp \not\prec e, \\ +1 & e_{H'}^\perp \prec e, e_H^\perp \not\prec e, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let L be a lattice with $[L] = v$ and $e = e_{\overline{M}}$, where $\overline{M}$ is a subspace of $L/\pi L$ of dimension $t = \text{type}(e)$. Since the case $t = 1$ is given by the last corollary, we may assume that $t \geq 2$. Suppose that $e_H^\perp \prec e$, i.e.,

$$\overline{G} = ((L \cap G) + \pi L)/\pi L \subset \overline{M} \subset L/\pi L.$$

Let $\overline{M}_0 = 0 \subsetneq \overline{M}_1 = \overline{G} \subsetneq \cdots \subsetneq \overline{M}_t = \overline{M}$ be a complete flag connecting 0 to $\overline{M}$, where $\dim \overline{M}_i = i$ for $0 \leq i \leq t$. It corresponds to a path $(v_0, v_1, \ldots, v_t)$ in $\mathcal{BT}$, where $v_0 = v = [L]$, $v_t = t(e_{\overline{M}})$, and all the arrows $e_1 = (v_0, v_1), \ldots, e_t = (v_{t-1}, v_t)$ of type 1. As $\{v_0, \ldots, v_t\}$ is a t-simplex, $d(v_0, v_i) = 1$ for $1 \leq i \leq t$, and therefore no e_i different from $e_1 = e_{\overline{G}}$ points to G.

Suppose that moreover $e_{H'}^\perp \not\prec e$, that is,

$$((L \cap G') + \pi L)/\pi L \not\subset \overline{M}.$$

Then none of the e_i $(1 \leq i \leq t)$ points to G', so

$$P(\ell_{H,H'})(e) = \sum_{1 \leq i \leq t} P(\ell_{H,H'})(e_i) = P(\ell_{H,H'})(e_1) = -1$$

by (1.9.2) and Corollary 2.10. If $e_H^\perp \neq e_{H'}^\perp \prec e$, then we can arrange the flag $\overline{M}_0 \subsetneq \cdots \subsetneq \overline{M}_t$ such that as before e_1 points to G, e_2 points to G', and no e_i $(3 \leq i \leq t)$ points to G or G'. In this case

$$P(\ell_{H,H'})(e) = P(\ell_{H,H'})(e_1) + P(\ell_{H,H'})(e_2) = -1 + 1 = 0.$$

If $e_H^\perp = e_{H'}^\perp \prec e$, then

$$P(\ell_{H,H'})(e) = P(\ell_{H,H'})(e_1) = 0 \quad \text{by Corollary 2.10.}$$

If neither $e_H^\perp \prec e$ nor $e_{H'}^\perp \prec e$, neither of the arrows e_i $(1 \leq i \leq t)$ corresponding to a flag $\overline{M}_0 = 0 \subsetneq \cdots \subsetneq \overline{M}_t = \overline{M}$ points to G or to G', and so $P(\ell_{H,H'})(e) = 0$ results. The case $e_{H'}^\perp \prec e$, $e_H^\perp \not\prec e$ comes out by symmetry. □

Corollary 2.12. *Let $H_1, \ldots, H_n$ be finitely many hyperplanes of V with corresponding linear forms $\ell_i = \ell_{H_i}$, $\ker(\ell_i) = H_i$, and multiplicities $m_i \in \mathbb{Z}$ such that $\sum_{1 \leq i \leq n} m_i = 0$. The function*

$$u := \prod_{1 \leq i \leq n} \ell_i^{m_i}$$

is a unit on Ω, whose van der Put transform $P(u)$ satisfies the condition:

(C) *For each arrow $e \in \boldsymbol{A}(\mathcal{BT})$ with $o(e) = v \in \boldsymbol{V}(\mathcal{BT})$,*

$$P(u)(e) = \sum_{\substack{e' \in \boldsymbol{A}_{v,1} \\ e' \prec e}} P(u)(e').$$

Proof. (C) is satisfied for $u = \ell_{H,H'} = \ell_H/\ell_{H'}$ by Proposition 2.11. The general case follows as condition (C) is linear (it holds for $u \cdot u'$ if it holds for u and u') and $\prod \ell_i^{m_i}$ is a product of functions of type $\ell_{H,H'}$. □

3. The van der Put sequence

Proposition 3.1. *Let u be an invertible holomorphic function on Ω. Then its van der Put transform $P(u)$ satisfies condition* (C) *from Corollary 2.12.*

Proof. Again by (1.9.4) we may suppose that the origin $o(e)$ of the arrow in question is equal to $v_0 = [L_0]$. So $e = e_{\overline{M}}$ with some nontrivial $\mathbb{F}$-subspace $\overline{M}$ of $L_0/\pi L_0$. As in Section 2.8 we use the same letter $\overline{M}$ for the corresponding linear subvariety of $\mathbb{P}^{r-1}/\mathbb{F}$ of dimension $t-1$, where $t = \text{type}(e) = \dim \overline{M}$.

Multiplying u by suitable functions of type $\ell_{H,H'}$ (which doesn't alter the (non)validity of (C) for u), we may assume that $P(u)(e') = 0$ for all $e' \in \boldsymbol{A}_{v_0,1}$ dominated by e. Then we must show that $P(u)(e) = 0$ too. Let u be normalized such that $\|u\|_{v_0} = 1$, and let $\bar{u}$ be its reduction as a rational function on $\mathbb{P}^{r-1}/\mathbb{F}$, see (2.3.2).

If $P(u)(e) < 0$ then $|u|$ decays along $e = e_{\overline{M}}$ and $\bar{u}$ vanishes along $\overline{M}$. Correspondingly, if $P(u)(e) > 0$ then $(\bar{u})^{-1} = \overline{(u^{-1})}$ vanishes along $\overline{M}$. Hence it suffices to show that, under our assumptions, $\bar{u}$ restricts to a well-defined rational function on $\overline{M}$, i.e., $\overline{M}$ is neither contained in the vanishing locus $V(\bar{u})$ nor in $V(\bar{u}^{-1})$. But the latter is obvious: With a suitable constant $c \neq 0$ we have

$$\bar{u} = c \cdot \prod \ell_{\overline{H}}^{m(\overline{H})},$$

where $\overline{H}$ runs through the boundary components of $\overline{\Omega}_{v_0}$ as in (2.3.2), $\ell_{\overline{H}}$ is a linear form vanishing on $\overline{H}$, $\sum m(\overline{H}) = 0$, and $m(\overline{H}) = -P(u)(e_{\overline{H}}^{\perp}) = 0$ if $\overline{H}^{\perp} \subset \overline{M}$. Hence neither the rational function $\bar{u}$ nor its reciprocal vanishes identically on $\overline{M}$. □

3.2. The proposition motivates the following definition. Let A be any additively written abelian group. The group of *A-valued harmonic 1-cochains* $\mathbf{H}(\mathcal{BT}, A)$ is the group of maps $\varphi\colon \boldsymbol{A}(\mathcal{BT}) \to A$ that satisfy

(A) $\sum \varphi(e) = 0$, whenever e ranges through the arrows of a closed path in $\mathcal{BT}$;

(B) for each type t, $1 \leq t \leq r-1$, and each $v \in V(\mathcal{BT})$, the condition

$$\sum_{e \in \boldsymbol{A}_{v,t}} \varphi(e) = 0 \quad \text{holds;} \tag{B$_t$}$$

(C) for each $v \in \boldsymbol{V}(\mathcal{BT})$ and each $e \in \boldsymbol{A}_v$,

$$\sum_{e' \in \boldsymbol{A}_{v,1},\ e' \prec e} \varphi(e') = \varphi(e).$$

Remarks 3.3. (i) In the case where the coefficient group A equals $\mathbb{Z}$, condition (A) is (1.9.2), (B_1) is (1.8.1), and (C) is the condition dealt with in Corollary 2.12 and Proposition 3.1. (A) in particular implies that φ is alternating, i.e., $\varphi(\bar{e}) = -\varphi(e)$. Further, ($\mathrm{B}_1$) together with (C) implies (B_t) for all types t, as

$$\sum_{e \in \boldsymbol{A}_{v,t}} \varphi(e) = \sum_{e' \in \boldsymbol{A}_{v,1}} \varphi(e') \#\{e \in \boldsymbol{A}_{v,t} \mid e' \prec e\},$$

where $\#\{\cdots\}$, the cardinality of some finite Grassmannian, is independent of e'.

(ii) Note that the current $\mathbf{H}(\mathcal{BT}, \mathbb{Z})$ differs from the group defined in [Gekeler 2017], as condition (C) is absent there.

(iii) Proposition 3.1 together with the preceding considerations shows that

$$P\colon \mathcal{O}(\Omega)^* \to \mathbf{H}(\mathcal{BT}, \mathbb{Z})$$
$$u \mapsto P(u)$$

is well-defined. Its kernel consists of the invertible holomorphic functions f on Ω with constant absolute value, which equals the constants C^*, as will be shown in Proposition 3.4. Hence, by (1.9.4), we have the exact sequence of $G(K)$-modules

$$1 \to C^* \to \mathcal{O}(\Omega)^* \xrightarrow{P} \mathbf{H}(\mathcal{BT}, \mathbb{Z}).$$

In fact, we will show that P is also surjective, which yields our principal result Theorem 3.11.

(iv) Beyond the natural coefficient domains $A = \mathbb{Z}$ or $\mathbb{Q}$ for $\mathbf{H}(\mathcal{BT}, A)$, at least the torsion groups $A = \mathbb{Z}/(N)$ deserve interest. For example, in the case $r = 2$ and $\mathrm{char}(C) = \mathrm{char}(\mathbb{F}) = p$, the invariants $\mathbf{H}(\mathcal{BT}, \mathbb{F}_p)^\Gamma$ under an arithmetic subgroup $\Gamma \subset G(K)$ differ in general from $\mathbf{H}(\mathcal{BT}, \mathbb{Z})^\Gamma \otimes \mathbb{F}_p$; see [Gekeler and Reversat 1996, Section 6]. The coefficient rings $A = \mathbb{Z}_\ell$ (ℓ a prime number) and $A = K$ come into the game by relating $\mathbf{H}(\mathcal{BT}, \mathbb{Z})$ with the first cohomology of Ω; see Section 5.5.

Proposition 3.4. *Each bounded holomorphic function f on Ω is constant. In particular, the kernel of the map P equals the constants C^*.*

Proof. Let $\boldsymbol{\omega} = (\omega_1 : \cdots : \omega_r)$ be an element of Ω. We are going to show that f as a function in ω_i, where $\omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_r$ are fixed, is constant for each i, which will give the result. Let

$$\alpha_{\boldsymbol{\omega}}^{(i)} : \mathbb{P}^{r-1}(K) \to \mathbb{P}^1(C) \quad \text{be the map} \quad (x_1 : \cdots : x_r) \mapsto \Bigg(\sum_{\substack{1 \le j \le r \\ i \ne j}} x_j \omega_j : x_i \Bigg).$$

It is well-defined (since the ω_j are K-linearly independent) and continuous with respect to the nonarchimedean topologies on both sides, whence its image $\mathrm{im}(\alpha_{\boldsymbol{\omega}}^{(i)})$ is compact. Moreover, the complement $\Omega_{\boldsymbol{\omega}}^{(i)} := \mathbb{P}^1(C) \setminus \mathrm{im}(\alpha_{\boldsymbol{\omega}}^{(i)})$ in $\mathbb{P}^1(C)$ equals the set of those $\omega \in C = \{(\omega : 1)\} \hookrightarrow \mathbb{P}^1(C)$ which are eligible for $(\omega_1 : \cdots : \omega_{i-1} : \omega : \omega_{i+1} : \cdots : \omega_r)$ to lie in Ω. Analytic spaces of this shape are extensively discussed in [Fresnel and van der Put 2004, Chapter II]. Notably, their Proposition 2.7.9 states that bounded functions on $\Omega_{\boldsymbol{\omega}}^{(i)}$ are constant as wanted. □

3.5. The strategy of proof of the surjectivity of P will be to approximate a given $\varphi \in \mathbf{H}(\mathcal{BT}, \mathbb{Z})$ by linear combinations of elements $P(u)$, where u is a function of type $\ell_{H,H'}$, or a relative of it.

Given two hyperplanes $H \ne H'$ of V and $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, define

$$f_{H,H',n} := 1 + \pi^n \ell_{H,H'}. \tag{3.5.1}$$

Here $\ell_{H,H'} = \ell_H / \ell_{H'} = \ell_{\boldsymbol{y}} / \ell_{\boldsymbol{y}'}$, where $\boldsymbol{y}, \boldsymbol{y}' \in L_0 \setminus \pi L_0$, $H = \ker(\ell_{\boldsymbol{y}})$, $H' = \ker(\ell_{\boldsymbol{y}'})$. Like $\ell_{H,H'}$, $f_{H,H',n}$ is a unit on Ω. We denote by

$$\mathcal{BT}(n) \subset \mathcal{BT} \tag{3.5.2}$$

the full subcomplex with vertices $\boldsymbol{V}(\mathcal{BT}(n)) = \{v \in \boldsymbol{V}(\mathcal{BT}) \mid d(v_0, v) \le n\}$. Hence $\mathcal{BT}(0) = \{v_0\}$, $\mathcal{BT}(1) = \mathrm{st}(v_0)$, etc. Further,

$$\Omega(n) := \lambda^{-1}(\mathcal{BT}(n)). \tag{3.5.3}$$

Then $\Omega(n)$ is an admissible affinoid subspace of Ω and $\Omega = \bigcup_{n \ge 0} \Omega(n)$. (In [Schneider and Stuhler 1991, Section 1, Proposition 4] $\Omega(n)$ is called $\overline{\Omega}_n$, and a system of generators of its affinoid algebra is constructed.)

Lemma 3.6. *For $n \in \mathbb{N}_0$, the following hold on $\Omega(n)$:*

(i) $\log \ell_{H,H'} \le n$.

(ii) $|f_{H,H',n}| = 1$.

Proof. (i) By our normalization, $|\ell_{H,H'}(\boldsymbol{\omega})| = 1$ for $\boldsymbol{\omega} \in \lambda^{-1}(v_0)$. Then by Proposition 2.11, $\|\ell_{H,H'}(\boldsymbol{\omega})\|_v \leq q^n$ for $v \in \boldsymbol{V}(\mathcal{BT})$ whenever $d(v_0, v) \leq n$, which gives the assertion.

(ii) $|f_{H,H',n}(\boldsymbol{\omega})| = |1 + \pi^n \ell_{H,H'}(\boldsymbol{\omega})| \leq 1$ on $\Omega(n)$ by (i), with equality at least if $n = 0$ or $\boldsymbol{\omega}$ doesn't belong to $\lambda^{-1}(v)$, where v is a vertex with $d(v_0, v) = n$, since in this case $\log \ell_{H,H'}(\boldsymbol{\omega}) < n$. But the equality must also hold for $\boldsymbol{\omega}$ with $\lambda(\boldsymbol{\omega}) =$ such a v, due to the linear interpolation property Theorem 1.6 of $\log_q \|f_{H,H',n}\|_x$ for x belonging to an arrow $e = (v', v)$ with $d(v_0, v') = n - 1$. □

Definition 3.7. A vertex $v \in \boldsymbol{V}(\mathcal{BT})$ is called *n-special* ($n \in \mathbb{N}_0$) if there exists a (necessarily uniquely determined) path $(v_0, v_1, \ldots, v_n = v) \in \boldsymbol{A}_{v_0,1,n}$, i.e., the arrows $e_i = (v_{i-1}, v_i)$, $i = 1, 2, \ldots, n$ all have type 1, and $d(v_0, v) = n$. (By definition, v_0 is 0-special.) An arrow $e \in \boldsymbol{A}(\mathcal{BT})$ is *n-special* ($n \in \mathbb{N}$) if $o(e)$ is $(n-1)$-special and $t(e)$ is n-special, that is, if it appears as some e_n as above. Also, the path $(v_0, \ldots, v_n) = (e_1, \ldots, e_n)$ is called *n-special.* An arrow e with $d(v_0, o(e)) = n$ is *inbound* (of level n) if it belongs to $\mathcal{BT}(n)$, and *outbound* otherwise. That is, e is inbound $\Leftrightarrow d(v_0, t(e)) \leq n$.

3.8. Next, we describe the restriction of $P(f_{H,H',n})$ to $(n+1)$-special arrows e. Let $n \in \mathbb{N}_0$, and choose hyperplanes H, H' of V, given as $H = \ker(\ell_{\boldsymbol{y}})$, $H' = \ker(\ell_{\boldsymbol{y}'})$ as in (3.5.1), $G = H^\perp = K\boldsymbol{y}$, $G' = K\boldsymbol{y}'$. Assume that $\boldsymbol{y}$ and $\boldsymbol{y}'$ are not 1-equivalent (2.7.2), that is, $\tau_G(v_0) \neq \tau_{G'}(v_0)$.

(i) According to Corollary 2.10, $\ell_{H,H'} = \ell_{\boldsymbol{y}}/\ell_{\boldsymbol{y}'}$ has the property that $\log \ell_{H,H'}$ grows by 1 in each step of the $(n+1)$-special path

$$(v_0, v_1, \ldots, v_n, v_{n+1}) = (e_1, e_2, \ldots, e_{n+1}) \tag{3.8.1}$$

from v_0 toward G'. Together with Lemma 3.6(ii), this implies that $P(f_{H,H',n})(e_{n+1}) = 1$.

(ii) On the other hand, again by Corollary 2.10, $\log \ell_{H,H'} < n$ on $\lambda^{-1}(v)$ for each n-special v different from v_n. By a variation of the linear interpolation argument in the proof of Lemma 3.6(ii), $P(f_{H,H',n})(e) = 0$ for each $(n+1)$-special arrow e with $o(e) \neq v_n$.

(iii) The function $u := f_{H,H',n} = (\ell_{\boldsymbol{y}'} + \pi^n \ell_{\boldsymbol{y}})/\ell_{\boldsymbol{y}'}$ satisfies $\|u\|_{v_n} = 1$. Its reduction $\bar{u}$ as a rational function on the reduction

$$\overline{\Omega}_{v_n} = (\mathbb{P}(L^\wedge/\pi L^\wedge)/\mathbb{F}) \setminus \bigcup \overline{H} \quad \text{(see (2.5.1), here } v_n = [L]) \tag{3.8.2}$$

of $\Omega_{v_n} = \lambda^{-1}(v_n)$ has a simple pole along the hyperplane $\overline{H}_{e_{n+1}}$ corresponding to the arrow e_{n+1}, a simple zero along a unique $\overline{H}_e$, where $e = (v_n, w)$, and neither zeroes nor poles along the other hyperplanes that appear in (3.8.2). The hyperplane $\overline{H}_e$ is the vanishing locus in $\mathbb{P}(L^\wedge/\pi L^\wedge)/\mathbb{F}$ of the reduction of the form $\ell_{\boldsymbol{y}'} + \pi^n \ell_{\boldsymbol{y}} = \ell_{\boldsymbol{y}''}$; accordingly, $w = \tau_{G''}(v_n)$, where $G'' = K\boldsymbol{y}''$ and

$$\boldsymbol{y}'' = \boldsymbol{y}' + \pi^n \boldsymbol{y}. \tag{3.8.3}$$

(iv) If $\boldsymbol{y}'$ is fixed and $\boldsymbol{y}$ runs through the elements of $L_0 \setminus \pi L_0$ not 1-equivalent with $\boldsymbol{y}'$, then the corresponding $\boldsymbol{y}''$ are n-equivalent but not $(n+1)$-equivalent with $\boldsymbol{y}'$ (see (2.7.2)). In this way we get all the $(n+1)$-classes with this property, that is, all the $(n+1)$-special paths $(e_1, e_2, \dots, e_n, e)$ which agree with the path $(e_1, \dots, e_n, e_{n+1})$ of (3.8.1) except for the last arrow. We collect what has been shown.

Proposition 3.9. (i) *Let H, H' be two hyperplanes in V, $G = H^{\perp}$, $G' = H'^{\perp}$, with $\tau_G(v_0) \neq \tau_{G'}(v_0)$ and $n \in \mathbb{N}_0$. Put $v_i := (\tau_{G'})^i(v_0)$. If e is an $(n+1)$-special arrow then*

$$P(f_{H,H',n})(e) = \begin{cases} +1 & \text{if } e = (v_n, v_{n+1}), \\ -1 & \text{if } e = (v_n, w), \\ 0 & \text{otherwise.} \end{cases} \tag{3.9.1}$$

Here $w = \tau_{G''}(v_n) \neq v_{n+1}$, where $G'' = K\mathbf{y}''$ with $\mathbf{y}'' = \mathbf{y}' + \pi^n \mathbf{y}$ as described in Section 3.8, notably in (3.8.3)*.*

(ii) *If H' is fixed, each $(n+1)$-special arrow $e \neq (v_n, v_{n+1})$ with $o(e) = v_n$ occurs through a suitable choice of H as the arrow $e = (v, w)$ where $P(f_{H,H',n})$ evaluates to -1.* □

The next result, technical in nature, is crucial for the proof of Theorem 3.11. Its proof is postponed to the next section.

Proposition 3.10. *Let $n \in \mathbb{N}_0$ and $\varphi \in \mathbf{H}(\mathcal{BT}, \mathbb{Z})$ be such that $\varphi(e) = 0$ for arrows e that either belong to $\mathcal{BT}(n)$ or are $(n+1)$-special. Then $\varphi(e) = 0$ for all arrows e of $\mathcal{BT}(n+1)$.*

Now we are able to show (modulo Proposition 3.10) the principal result.

Theorem 3.11. *The van der Put map $P : \mathcal{O}(\Omega)^* \to \mathbf{H}(\mathcal{BT}, \mathbb{Z})$ is surjective, and so the sequence*

$$1 \to C^* \to \mathcal{O}(\Omega)^* \to \mathbf{H}(\mathcal{BT}, \mathbb{Z}) \to 0 \tag{0.2}$$

is a short exact sequence of $G(K)$-modules.

Proof. (i) Let $\varphi \in \mathbf{H}(\mathcal{BT}, \mathbb{Z})$ be given. By successively subtracting $P(u_n)$ from φ, where $(u_n)_{n\in\mathbb{N}}$ is a suitable series of functions in $\mathcal{O}(\Omega)^*$ with $u_n \to 1$ locally uniformly (i.e., uniformly on affinoids) we will achieve that

$$\varphi - P\Bigl(\prod_{1\le i\le n} u_i\Bigr) \equiv 0 \quad \text{on } \mathcal{BT}(n).$$

Then $\varphi = P(u)$, where $u = \lim_{n\to\infty} \prod_{1\le i\le n} u_i$ is the limit function.

(ii) From condition (B_1) for φ and Proposition 2.9 we find a function u_1, namely a suitable finite product of functions of type $\ell_{H,H'}$, such that $(\varphi - P(u_1))(e) = 0$ for each $e \in \boldsymbol{A}_{v_0,1}$. By condition (C), $\varphi - P(u_1)$ vanishes on all $e \in \boldsymbol{A}_{v_0}$, and thus by (A) on all e that belong to $\mathcal{BT}(1) = \mathrm{st}(v_0)$.

(iii) Suppose that $u_1, \dots, u_n \in \mathcal{O}(\Omega)^*$ are constructed ($n \in \mathbb{N}$) such that for $1 \le i \le n$

(a) $P(u_i) \equiv 0$ on $\mathcal{BT}(i-1)$,

(b) $u_i \equiv 1 \pmod{\pi^{[(i-1)/2]}}$ on $\mathcal{BT}([(i-1)/2])$, here $[\cdot]$ is the Gauss bracket,

(c) $\varphi - P(\prod_{1\le i\le n} u_i) \equiv 0$ on $\mathcal{BT}(n)$

hold. (Condition (a) is empty for $i = 1$ and therefore trivially fulfilled.) We are going to construct u_{n+1} such that $u_1, \ldots, u_{n+1}$ fulfill the conditions on level $n+1$.

(iv) From (c) and (B_1) we have for n-special vertices v and $\psi := \varphi - P(\prod_{1 \le i \le n} u_i) \in \mathbf{H}(\mathcal{BT}, \mathbb{Z})$:

$$\sum_{e \in \boldsymbol{A}_{v,1} \text{ outbound}} \psi(e) = \sum_{e \in \boldsymbol{A}_{v,1}} \psi(e) = 0.$$

(v) According to Proposition 3.9, we find u_{n+1}, viz, a suitable product of functions $f_{H,H',n}$, such that

$$(\psi - P(u_{n+1}))(e) = \Big(\varphi - P\Big(\prod_{1 \le i \le n+1} u_i\Big)\Big)(e) = 0$$

on all $(n+1)$-special arrows e. Furthermore, that u_{n+1} (like the functions $f_{H,H',n}$, see Lemma 3.6(ii)) satisfies $P(u_{n+1}) \equiv 0$ on $\mathcal{BT}(n)$, i.e., condition (a), and condition (b): $u_{n+1} \equiv 1 \pmod{\pi^{[n/2]}}$ on $\mathcal{BT}([n/2])$. Hence $\varphi - P(\prod_{1 \le i \le n+1} u_i)$ vanishes on arrows which belong to $\mathcal{BT}(n)$ or are $(n+1)$-special. Using Proposition 3.10, we see that $\varphi - P(\prod_{1 \le i \le n+1} u_i)$ vanishes on $\mathcal{BT}(n+1)$. That is, conditions (a), (b), (c) hold for $u_1, \ldots, u_{n+1}$, and we have inductively constructed an infinite series $u_1, u_2, \ldots$ with (a), (b) and (c) for all n.

(vi) It follows from (b) that the infinite product

$$u = \prod_{i \in \mathbb{N}} u_i$$

is normally convergent on each $\Omega(n)$ and thus defines a holomorphic invertible function u on Ω. Its van der Put transform $P(u)$ restricted to $\mathcal{BT}(n)$ depends only on $u_1, \ldots, u_n$, due to (c), and thus agrees with φ reduced to $\mathcal{BT}(n)$. Therefore, $\varphi = P(u)$, and the result is shown. □

4. The group $\mathbf{H}(\mathcal{BT}, \mathbb{Z})$

4.1.

Proof of Proposition 3.10. (i) The requirements of Proposition 3.10 for $\varphi \in \mathbf{H}(\mathcal{BT}, \mathbb{Z})$ on level $n \in \mathbb{N}_0$ will be labeled by $R(n)$.

(ii) Suppose that $R(n)$ holds for φ. Then φ vanishes on all arrows $\boldsymbol{A}_{v,1}$ whenever v is n-special, since such an e is either $(n+1)$-special or belongs to $\mathcal{BT}(n)$. Hence by conditions (C) and (A) of Section 3.2, $\varphi(e) = 0$ whenever e is contiguous with v, i.e., if e belongs to $\mathrm{st}(v)$. This shows, in particular, that Proposition 3.10 holds for $n = 0$.

(iii) Let $v \in \boldsymbol{V}(\mathcal{BT})$ have distance $d(v_0, v) = n$, but be not necessarily n-special. For the same reason as in (ii), φ vanishes identically on $\mathrm{st}(v)$ if it vanishes on all outbound arrows $e \in \mathrm{A}_{v,1}$. Hence it suffices to show

$$\varphi(e) = 0 \quad \text{for outbound arrows } e \text{ of type 1 and level } n. \tag{O}$$

(iv) For a vertex v with $d(v_0, v) = n$, we let $s(v)$ be the distance to the next $w \in \boldsymbol{V}(\mathcal{BT})$ which is n-special. We are going to show assertion (O) by induction on $s(o(e))$.

(v) By R(n), (O) holds if $s = s(o(e)) = 0$, i.e., if $o(e)$ is n-special. Therefore, suppose that $s > 0$. By the preceding we are reduced to showing:

$$\begin{aligned}&\text{Let } e \text{ be an outbound arrow of type 1, level } n\text{, and with } s = s(o(e)) > 0.\\ &\text{Then } e \text{ belongs to } \mathrm{st}(\tilde{v})\text{, where } d(v_0, \tilde{v}) = n \text{ and } s(\tilde{v}) < s.\end{aligned} \tag{P}$$

(vi) We reformulate (P) in lattice terms. Representing $v_0 = [L_0]$ through $L_0 = O^r$, the vertices $v \in \boldsymbol{V}(\mathcal{BT})$ correspond one-to-one to sublattices L of full rank r which satisfy $L \subset L_0$, $L \not\subset \pi L_0$. For such a vertex v or its lattice L, we let $(n_1, n_2, \dots, n_r)$ with $n_1 \geq n_2 \geq \cdots \geq n_r = 0$ be the sequence of elementary divisors (*sed*) of L_0/L ($n_r = 0$ as $L \not\subset \pi L_0$). That is,

$$L_0/L \cong O/(\pi^{n_1}) \times \cdots \times O/(\pi^{n_r}).$$

Then $n_1 = d(v_0, v)$, and v is n-special if and only if its sed is $(n, \dots, n, 0)$.

(vii) Let $e = (v, v')$ be given as required for (P), $v = [L]$, $v' = [L']$, where $\pi^{n+1}L_0 \subset L' \subset L \subset L_0$. Let $(n_1 = n, n_2, \dots, n_r)$ be the sed of L_0/L. Then, as $\dim_{\mathbb{F}}(L/L') = r - 1$ and $d(v_0, v') = n + 1$, $(n'_1 = n+1, \dots, n'_{r-1} = n_{r-1}+1, n_r)$ is the sed of L_0/L'. This means that L_0 has an ordered O-basis $\{x_1, \dots, x_r\}$ such that $\{\pi^{n+1}x_1, \pi^{n_2+1}x_2, \dots, \pi^{n_{r-1}+1}x_{r-1}, \pi^{n_r}x_r\}$ is a basis of L' and $\{\pi^n x_1, \pi^{n_2}x_2, \dots, \pi^{n_r}x_r\}$ is a basis of L. Assume that k with $1 \leq k \leq r-1$ is minimal with $n_{r-1} = n_k$. Let M be the sublattice of L_0 with basis $\{\pi^n x_1, \dots, \pi^n x_{r-1}, x_r\}$. Then $w = [M]$ is n-special and $s(v) = d(v, w) = n - n_{r-1}$, which by assumption is positive. Put $\tilde{L}$ for the lattice with basis

$$\{\pi^n x_1, \pi^{n_2}x_2, \dots, \pi^{n_{k-1}}x_{k-1}, \pi^{n_k+1}x_k, \pi^{n_{k+1}+1}x_{k+1}, \dots, \pi^{n_{r-1}+1}x_{r-1}, \pi^{n_r}x_r\}.$$

The vertex $\tilde{v} := [\tilde{L}]$ satisfies

$$d(v_0, \tilde{v}) = n, \quad d(v, \tilde{v}) = 1 = d(v', \tilde{v}) \quad \text{and} \quad s(\tilde{v}) = d(w, \tilde{v}) = n - n_{r-1} - 1 = s(v) - 1. \tag{4.1.1}$$

Hence $e = (v, v')$ belongs to $\mathrm{st}(\tilde{v})$, where $\tilde{v}$ is as wanted for assertion (P).

This finishes the proof of Proposition 3.10. □

Corollary 4.2. *Let $\varphi \in \mathbf{H}(\mathcal{BT}, \mathbb{Z})$ be such that $\varphi(e) = 0$ for all i-special arrows e, where $1 \leq i \leq n$. Then $\varphi \equiv 0$ on $\mathcal{BT}(n)$.*

Proof. This follows by induction from Proposition 3.10. □

4.3. Next we give a different description of $\mathbf{H}(\mathcal{BT}, \mathbb{Z})$, see Theorem 4.16. Let v be an n-special vertex ($n \geq 1$), v^* its predecessor on the uniquely determined n-special path $(v_0, v_1, \dots, v_{n-1} = v^*, v)$ from v_0 to v, and e^* the n-special arrow (v^*, v). Its inverse $\overline{e^*} = (v, v^*)$ belongs to $\boldsymbol{A}_{v,r-1}$.

Lemma 4.4. *In the given situation, $e \in \boldsymbol{A}_{v,1}$ is inbound if and only if $e \prec \overline{e^*}$.*

Proof. As the stabilizer $\mathrm{GL}(r, O)$ of $L_0 = O^r$ acts transitively on n-special vertices or arrows, we may suppose that $v = [L_n]$, where L_n is the O-lattice with basis $\{\pi^n x_1, \ldots, \pi^n x_{r-1}, x_r\}$, and thus $v^* = [L_{n-1}]$. (Here $\{x_1, \ldots, x_r\}$ is the standard basis of L_0.) Under (2.1.4), $\overline{e^*}$ corresponds to the $(r-1)$-dimensional subspace $\pi L_{n-1}/\pi L_n$ of the r-dimensional $\mathbb{F}$-space $L_n/\pi L_n$, which has the $(\overline{\pi^n x_i}) = \pi^n x_i \pmod{\pi L_n}$, $1 \leq i < r$, as a basis. Let $\overline{G}$ be a line in $L_n/\pi L_n$ with preimage G in L_n, and let $e_{\overline{G}} = (v, v_{\overline{G}})$ be the arrow of type 1 determined by $\overline{G}$. Then $v_{\overline{G}} = [G]$ and

$$e_{\overline{G}} \prec e^* \Leftrightarrow \overline{G} \subsetneq \pi L_{n-1}/\pi L_n \Leftrightarrow G \subset \pi L_{n-1}.$$

If this is the case, $\pi^n L_0 \subset L_n \subset \pi^{-1} G \subset L_{n-1} \subset L_0$, that is, $d(v_0, [G]) \leq n$, and $e_{\overline{G}}$ is inbound. On the other hand, if $G \not\subset \pi L_{n-1}$, then $\pi^{-1} G \not\subset L_0$. Since $\pi^n L_0 \not\subset G$, we then have $d(v_0, [G]) = n+1$, and $e_{\overline{G}}$ is outbound. □

4.5. We may now reformulate condition (B_1) for $\varphi \in \mathbf{H}(\mathcal{BT}, \mathbb{Z})$ at the n-special vertex v of level $n \geq 1$ as follows: Splitting

$$\boldsymbol{A}_{v,1} = \boldsymbol{A}_{v,1,\mathrm{in}} \cup\!\!\!\!\cdot\;\, \boldsymbol{A}_{v,1,\mathrm{out}} \tag{4.5.1}$$

into the subsets of inbound / outbound arrows (note that $e \in \boldsymbol{A}_{v,1}$ is outbound if and only if it is $(n+1)$-special), (B_1) reads

$$0 = \sum_{e \in \boldsymbol{A}_{v,1}} \varphi(e) = \sum_{e \in \boldsymbol{A}_{v,1,\mathrm{in}}} \varphi(e) + \sum_{e \in \boldsymbol{A}_{v,1,\mathrm{out}}} \varphi(e) = \varphi(\overline{e^*}) + \sum_{e \in \boldsymbol{A}_{v,1,\mathrm{out}}} \varphi(e)$$

(where we used Lemma 4.4 and condition (C) for $\varphi(\overline{e^*})$), i.e., as the flow condition

$$\varphi(e^*) = \sum_{e \in \boldsymbol{A}_{v,1,\mathrm{out}}} \varphi(e). \tag{4.5.2}$$

The number of terms in the sum is

$$\#\boldsymbol{A}_{v,1,\mathrm{out}} = \#\boldsymbol{A}_{v,1} - \#\boldsymbol{A}_{v,1,\mathrm{in}} = \#\mathbb{P}^{r-1}(\mathbb{F}) - \#\mathbb{P}^{r-2}(\mathbb{F}) = q^{r-1}. \tag{4.5.3}$$

4.6. Let $\mathcal{T}_{v_0}$ be the full subcomplex of $\mathcal{BT}$ composed of the n-special vertices ($n \in \mathbb{N}_0$) along with the 1-simplices connecting them. In other words, $\mathcal{T}_{v_0}$ is the union of the paths $\boldsymbol{A}_{v_0,1,n}$, where $n \in \mathbb{N}$, see Section 2.1.6. It is connected, one-dimensional and cycle-free, hence a tree. The valence (= number of neighbors) of v_0 is $\#\mathbb{P}^{r-1}(\mathbb{F}) = (q^r - 1)/(q-1)$, the valence of each other vertex $v \neq v_0$ is $q^{r-1} + 1$, as we read off from (4.5.3). Let further $\mathcal{T}_{v_0}(n) := \mathcal{T}_{v_0} \cap \mathcal{BT}(n)$.

4.6.1. We define $\mathbf{H}(n)$ as the image of $\mathbf{H}(\mathcal{BT}, \mathbb{Z})$ in $\{\varphi\colon \boldsymbol{A}(\mathcal{BT}(n)) \to \mathbb{Z}\}$ obtained by restriction. Hence

$$\mathbf{H}(\mathcal{BT}, \mathbb{Z}) = \varprojlim_{n \in \mathbb{N}} \mathbf{H}(n). \tag{4.6.2}$$

Put further

$$\mathbf{H}'(n) := \{\varphi\colon \boldsymbol{A}(\mathcal{T}_{v_0}(n)) \to \mathbb{Z} \mid \varphi \text{ is subject to (4.6.4) and (4.6.5)}(v) \text{ for each } i\text{-special } v,\ 0 \le i < n\}. \tag{4.6.3}$$

Here $\boldsymbol{A}(\mathcal{S})$ is the set of arrows (oriented 1-simplices) of the simplicial complex $\mathcal{S}$, and the conditions are

$$\varphi(e) + \varphi(\bar{e}) = 0 \quad \text{for each arrow } e \text{ with inverse } \bar{e}; \tag{4.6.4}$$

$$\sum_{\substack{e\in \boldsymbol{A}(\mathcal{T}_{v_0})\\ o(e)=v}} \varphi(e) = 0. \tag{4.6.5)(v}$$

4.7. Equality (4.5.2) together with the condition (B_1) at v_0 states that the restriction of $\varphi \in \mathbf{H}(\mathcal{BT}, \mathbb{Z})$ to $\mathcal{T}_{v_0}(n)$ is an element of $\mathbf{H}'(n)$. Therefore, restriction defines homomorphisms $r_n\colon \mathbf{H}(n) \to \mathbf{H}'(n)$, which make the diagram (with natural maps q_n, q'_n)

$$\begin{array}{ccc} \mathbf{H}(n+1) & \xrightarrow{r_{n+1}} & \mathbf{H}'(n+1) \\ \downarrow{\scriptstyle q_n} & & \downarrow{\scriptstyle q'_n} \\ \mathbf{H}(n) & \xrightarrow{r_n} & \mathbf{H}'(n) \end{array} \tag{4.7.1}$$

commutative. Note that both q_n and q'_n are surjective, the first by definition, the second one since $\mathcal{T}_{v_0}$ is a tree. Corollary 4.2 may be rephrased as

Proposition 4.8. *r_n is injective for $n \in \mathbb{N}$.* □

Lemma 4.9. *r_n is also surjective.*

Proof. For $n = 1$, this is implicit in the proof of Theorem 3.11 (i.e., one may arbitrarily prescribe the value of $\varphi \in \mathbf{H}(\mathcal{BT}, \mathbb{Z})$ on $e \in \mathbf{A}_{v_0,1}$, subject only to (B_1) at v_0).

For $n \ge 1$, let Q_{n+1} (respectively Q'_{n+1}) be the kernel of q_n (respectively q'_n). Then $r_{n+1}(Q_{n+1}) \subset Q'_{n+1}$, and we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_{n+1} & \longrightarrow & \mathbf{H}(n+1) & \longrightarrow & \mathbf{H}(n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle r_{n+1}} & & \downarrow{\scriptstyle r_n} & & \\ 0 & \longrightarrow & Q'_{n+1} & \longrightarrow & \mathbf{H}'(n+1) & \longrightarrow & \mathbf{H}'(n) & \longrightarrow & 0. \end{array}$$

By induction hypothesis, r_n is surjective, so the surjectivity of r_{n+1} is implied by

$$r_{n+1}(Q_{n+1}) = Q'_{n+1}. \tag{$*$}$$

But

$$Q_{n+1} = \{\varphi \in \mathbf{H}(n+1) \mid \varphi \equiv 0 \text{ on } \mathcal{BT}(n)\} \quad \text{and} \quad Q'_{n+1} = \{\varphi \in \mathbf{H}'(n+1) \mid \varphi \equiv 0 \text{ on } \mathcal{T}_{v_0}(n)\},$$

so ($*$) follows from the existence of sufficiently many elements of Q_{n+1} (e.g., the classes in $\mathbf{H}(n+1)$ of the $P(f_{H,H',n})$) which have sufficiently independent values on the arrows in $\mathcal{T}_{v_0}(n+1)$ not in $\mathcal{T}_{v_0}(n)$. See also the proof of Theorem 3.11, steps (iv) and (v). □

4.10. Let $\mathbf{H}(\mathcal{T}_{v_0}, \mathbb{Z}) = \varprojlim_{n\in\mathbb{N}} \mathbf{H}'(n)$ be the group of functions $\varphi\colon \boldsymbol{A}(\mathcal{T}_{v_0}) \to \mathbb{Z}$ which satisfy (4.6.4) and (4.6.5)(v) for all vertices v of $\mathcal{T}_{v_0}$. Similarly, we define $\mathbf{H}(\mathcal{T}_{v_0}, A)$ for an arbitrary abelian group A instead of $\mathbb{Z}$. That is, elements of $\mathbf{H}(\mathcal{T}_{v_0}, A)$ are characterized by conditions analogous with (A) and (B) of Section 3.2, while (C) is not applicable. Putting together the considerations of Section 4.5 with Proposition 4.8 and Lemma 4.9, we find

$$\mathbf{H}(\mathcal{BT}, \mathbb{Z}) \xrightarrow{\cong} \mathbf{H}(\mathcal{T}_{v_0}, \mathbb{Z}), \tag{4.11}$$

where the canonical isomorphism is given by restricting $\varphi \in \mathbf{H}(\mathcal{BT}, \mathbb{Z})$, $\varphi\colon \boldsymbol{A}(\mathcal{BT}) \to \mathbb{Z}$ to the subset $\boldsymbol{A}(\mathcal{T}_{v_0})$ of $\boldsymbol{A}(\mathcal{BT})$.

In what follows, A is an arbitrary abelian group. The next result is a consequence of the above.

Proposition 4.12. *Restriction to the arrows of $\mathcal{T}_{v_0}$ yields an isomorphism*

$$\mathbf{H}(\mathcal{BT}, A) \xrightarrow{\cong} \mathbf{H}(\mathcal{T}_{v_0}, A). \tag{4.12.1}$$

Proof. It suffices to observe that the preceding results Proposition 3.10, Corollary 4.2, Proposition 4.8, and Lemma 4.9 remain valid — with identical proofs — for A-valued functions instead of $\mathbb{Z}$-valued functions. □

4.13. Recall that an A-valued distribution on a compact totally disconnected topological space X is a map $\delta\colon U \mapsto \delta(U) \in A$ from the set of compact-open subspaces U of X to A which is additive in finite disjoint unions. We call $\delta(U)$ the *volume* of U *with respect to* δ. The *total mass* (or volume) of δ is $\delta(X)$.

We apply this to the situation (see Section 2.1.6, (2.1.7) and (2.1.8)) where

$$X = \mathrm{Gr}_{K,1}(V) = \{\text{lines } G \text{ of the } K\text{-space } V\} = \mathbb{P}(V). \tag{4.13.1}$$

As we have identified V with its dual $V^\wedge$ through the bilinear form $\langle\cdot,\cdot\rangle$, we also have an identification

$$\mathrm{Gr}_{K,1}(V) = \mathbb{P}(V) \xrightarrow{\cong} \mathbb{P}(V^\wedge) = \mathrm{Gr}_{K,r-1}(V)$$

given by $G \mapsto G^\perp$. Hence we could state the following assertions concerning distributions on $\mathbb{P}(V)$ for distributions on $\mathbb{P}(V^\wedge)$.

4.13.2. Let $\mathbf{D}(\mathbb{P}(V), A)$ be the group of A-valued distributions on $\mathbb{P}(V)$ with subgroup $\mathbf{D}^0(\mathbb{P}(V), A)$ of distributions with total mass 0. By (2.1.8), the sets $\mathbb{P}(V)(e) = \mathrm{Gr}_{K,1}(e)$, where e runs through the outbound arrows of $\boldsymbol{A}_{v_0,1,n}$ ($n \in \mathbb{N}$), i.e., through the set

$$\boldsymbol{A}^+(\mathcal{T}_{v_0}) = \{e \in \boldsymbol{A}(\mathcal{T}_{v_0}) \mid e \text{ oriented away from } v_0\}, \tag{4.13.3}$$

form a basis for the topology on $\mathbb{P}(V)$. Therefore, an element δ of $\mathbf{D}(\mathbb{P}(V), A)$ is an assignment

$$\delta\colon \boldsymbol{A}^+(\mathcal{T}_{v_0}) \to A$$

(where we interpret $\delta(e)$ as the volume of $\mathbb{P}(V)(e)$ with respect to δ) subject to the requirement

$$\delta(e^*) = \sum_{\substack{e \in \boldsymbol{A}^+(\mathcal{T}_{v_0}) \\ o(e)=t(e^*)}} \delta(e) \tag{4.13.4}$$

for each $e^* \in \boldsymbol{A}^+(\mathcal{T}_{v_0})$. The total mass of δ is

$$\delta(\mathbb{P}(V)) = \sum_{\substack{e \in \boldsymbol{A}^+(\mathcal{T}_{v_0}) \\ o(e)=v_0}} \delta(e) = \sum_{e \in \boldsymbol{A}_{v_0,1}} \delta(e). \tag{4.13.5}$$

In view of (4.5.2) and (4.6.5)(v), we find that

$$\mathbf{D}^0(\mathbb{P}(V), A) \xrightarrow{\cong} \mathbf{H}(\mathcal{T}_{v_0}, A), \tag{4.14}$$

where some $\delta\colon \boldsymbol{A}^+(\mathcal{T}_{v_0}) \to A$ in the left hand side is completed to a map on $\boldsymbol{A}(\mathcal{T}_{v_0})$ by (4.6.4), i.e., by $\varphi(\bar{e}) = -\varphi(e)$.

While both isomorphisms in (4.11) (or (4.12.1)) and (4.14) fail to be $G(K)$-equivariant (as $G(K)$ fixes neither v_0 nor $\mathcal{T}_{v_0}$), the resulting isomorphism

$$\begin{aligned} \mathbf{H}(\mathcal{BT}, \boldsymbol{A}) &\xrightarrow{\cong} \mathbf{D}^0(\mathbb{P}(V), A) \\ \varphi &\longmapsto \tilde{\varphi} \end{aligned} \tag{4.15}$$

is. Here the distribution $\tilde{\varphi}$ evaluates on $\mathbb{P}(V)(e)$ as $\varphi(e)$ whenever e is an arrow of $\mathcal{BT}$ of type 1 and $\mathbb{P}(V)(e)$ is the compact-open subset of lines G of V such that e points to G.

We summarize what has been shown.

Theorem 4.16. *Let A be an arbitrary abelian group. Restricting the evaluation of $\varphi \in \mathbf{H}(\mathcal{BT}, A)$ to arrows of $\mathcal{T}_{v_0}$ (resp. arrows of type 1 of $\mathcal{BT}$) yields canonical isomorphisms*

$$\mathbf{H}(\mathcal{BT}, A) \xrightarrow{\cong} \mathbf{H}(\mathcal{T}_{v_0}, A) \quad \textit{resp.} \quad \mathbf{H}(\mathcal{BT}, A) \xrightarrow{\cong} \mathbf{D}^0(\mathbb{P}(V), A).$$

The second of these is equivariant for the natural actions of $G(K) = \mathrm{GL}(r, K)$ on both sides, while the first isomorphism is equivariant for the actions of the stabilizer $G(O)Z(K)$ of $v_0 \in G(K)$.

As a direct consequence of the first isomorphism, i.e., of (4.12.1), we find the following corollary, which is in keeping with the fact that bounded holomorphic functions on Ω are constant (see Proposition 3.4).

Corollary 4.17. *If $\varphi \in \mathbf{H}(\mathcal{BT}, A)$ has finite support, it vanishes identically.*

Proof. Suppose that φ has support in $\mathcal{BT}(n)$ with $n \in \mathbb{N}$. Then its restriction to $\mathcal{T}_{v_0}(n+1)$ satisfies (4.6.4) and (4.6.5)(v) at all vertices v of $\mathcal{T}_{v_0}(n+1)$. As $\mathcal{T}_{v_0}(n+1)$ is a finite tree, this forces φ to vanish identically on $\mathcal{T}_{v_0}(n+1)$, thus on $\mathcal{BT}$. □

5. Concluding remarks

5.1. Ehud de Shalit [2001, Section 3.1] postulated four conditions $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ for what he calls harmonic k-cochains on $\mathcal{BT}$. These conditions specialized to $k = 1$ are essentially our conditions (A), (B), (C) from Section 3.2. Grosso modo, de Shalit's $\mathfrak{B}$ corresponds to (B), $\mathfrak{C}$ to (C) and $\mathfrak{D}$ to (A), while $\mathfrak{A}$ is a special case of (A).

5.2. In fact, the relationship with de Shalit's work is as follows. Suppose that $\operatorname{char}(K) = 0$, and consider the diagram

$$\begin{array}{ccccc} u & & \mathcal{O}(\Omega)^* & \xrightarrow{\quad P \quad} & \mathbf{H}(\mathcal{BT}, \mathbb{Z}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ d\log u = u^{-1}du & & \{\text{closed 1-forms on } \Omega\} & \xrightarrow{\text{res}} & \mathbf{H}(\mathcal{BT}, K)(= C^1_{\text{har}} \text{ of [de Shalit 2001]}), \end{array} \tag{5.2.1}$$

where "res" is de Shalit's residue mapping. Its commutativity follows for $u = \ell_{H,H'}$ from Corollary 7.6 and Theorem 8.2 of [de Shalit 2001] (along with the explanations given there, and our description of $P(u)$), and may be verified for general u by approximating. Hence the van der Put transform P yields a concrete description of the residue mapping on logarithmic 1-forms. On the other hand, in characteristic p the van der Put transform is finer than "$d\log$", as the latter kills all p-powers.

5.3. Now suppose that $\operatorname{char}(K) = p > 0$, and that moreover $r = 2$. Then $\mathcal{BT}$ is the Bruhat–Tits tree $\mathcal{T}$, and the residue mapping

$$\text{res}\colon \{\text{1-forms on } \Omega = \Omega^2\} \to \mathbf{H}(\mathcal{T}, C)$$

(see [Gekeler and Reversat 1996, 1.8]) is such that the diagram analogous with (5.2.1)

$$\begin{array}{ccccc} u & & \mathcal{O}(\Omega)^* & \xrightarrow{P} & \mathbf{H}(\mathcal{T}, \mathbb{Z}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ d\log u & & \{\text{1-forms on } \Omega\} & \xrightarrow{\text{res}} & \mathbf{H}(\mathcal{T}, C) \end{array} \tag{5.3.1}$$

commutes, with remarkable arithmetic consequences [loc. cit., Sections 6 and 7]. A similar residue map for $r > 2$ unfortunately lacks so far. In any case, we should regard P as a substitute for the logarithmic derivation operator

$$u \mapsto d\log u = u^{-1}du$$

in characteristic 0.

5.4. Peter Schneider and Ulrich Stuhler [1991] described the cohomology $H^*(\Omega, A)$ of $\Omega = \Omega^r$ with respect to an abstract cohomology theory, where $A = H^0(\operatorname{Sp}(K))$. That theory is required to satisfy four natural axioms, [loc. cit., Section 2]. As they explain, these axioms are fulfilled at least

- for the étale ℓ-adic cohomology of rigid-analytic spaces over K, where ℓ is a prime different from $p = \operatorname{char}(\mathbb{F})$, and $A = \mathbb{Z}_\ell$, and
- for the de Rham cohomology (where one must moreover assume that $\operatorname{char}(K) = 0$); here $A = K$.

Their result is stated [loc. cit., Section 3, Theorem 1], which in dimension 1 is (in our notation)

$$H^1(\Omega^r, A) \xrightarrow{\cong} \mathbf{D}^0(\mathbb{P}(V^\wedge), A). \tag{5.4.1}$$

Theorem 8.2 in [de Shalit 2001] gives that (in the case where $\mathrm{char}(K) = 0$ and $H^* = H^*_{\mathrm{dR}}$ is the de Rham cohomology)

$$H^k_{\mathrm{dR}}(\Omega^r) \xrightarrow{\cong} C^k_{\mathrm{har}}, \tag{5.4.2}$$

where C^1_{har} is our $\mathbf{H}(\mathcal{BT}, K)$. Hence our Theorems 3.11 and 4.16 refine the above in the case $k = 1$. In [Alon and de Shalit 2002], the authors relate the approaches of [Schneider and Stuhler 1991; de Shalit 2001; Iovita and Spiess 2001] to the de Rham cohomology of Ω. Specialized to $k = 1$, this gives some more insight into our situation. In particular, it is possible to derive the surjectivity of the map P in Theorem 3.11 also with the methods of [Alon and de Shalit 2002], at least if $\mathrm{char}(K) = 0$.

5.5. Let now Γ be a discrete subgroup of $G(K)$. The most interesting cases are those where the image of Γ in $G(K)/Z(K) = \mathrm{PGL}(r, K)$ has finite covolume with respect to Haar measure, or is even cocompact. Examples are given as Schottky groups in $\mathrm{PGL}(2, K)$ [Gerritzen and van der Put 1980] or as arithmetic subgroups of $G(K)$ of different types, when K is the completion k_∞ of a global field k at a nonarchimedean place ∞ [Drinfeld 1974; Reiner 1975]. Then often the quotient analytic space $\Gamma \setminus \Omega$ is the set of C-points of an algebraic variety [Goldman and Iwahori 1963; Drinfeld 1974; Mustafin 1978], which may be studied via a spectral sequence relating the cohomologies of Ω and Γ with that of $\Gamma \setminus \Omega$ [Schneider and Stuhler 1991, Section 5]. For $r = 2$, this essentially boils down to a study of the Γ-cohomology sequence of (0.2) [Gekeler and Reversat 1996, Section 5]. But also for $r > 2$, (0.2) with its Γ-action will be useful, which is the topic of ongoing work.

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