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Breuil–Mézard conjectures for central division algebras

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We formulate an analogue of the Breuil–Mézard conjecture for the group of units of a central division algebra over a p-adic local field, and we prove that it follows from the conjecture for GL_n. To do so we construct a transfer of inertial types and Serre weights between the maximal compact subgroups of these two groups, in terms of Deligne–Lusztig theory, and we prove its compatibility with mod p reduction, via the inertial Jacquet–Langlands correspondence and certain explicit character formulas. We also prove analogous statements for ℓ-adic coefficients.

1. Introduction

Let $F/\mathbb{Q}_p$ be a finite extension. The Breuil–Mézard conjecture, as originally formulated in [Breuil and Mézard 2002] and generalized in [Kisin 2010; Emerton and Gee 2014], provides a description of the singularities of potentially semistable deformation rings for $G_F = \mathrm{Gal}(\bar{F}/F)$ in terms of the representation theory of maximal compact subgroups of $\mathrm{GL}_n(F)$. Gee and Geraghty [2015] raised the question of whether an analogous statement holds for the unit groups of central division algebras, and answered it affirmatively for quaternion algebras, proving that it would follow from the truth of the conjecture for $\mathrm{GL}_2(F)$. This acquires particular relevance in light of the work of Scholze [2018] and Chojecki and Knight [2017] on p-adic Jacquet–Langlands correspondences.

In this paper we prove similar results for an arbitrary central division F-algebra D. Recall that the Jacquet–Langlands correspondence is a bijection from the irreducible smooth representations of $D^\times$ to the essentially square-integrable representations of $\mathrm{GL}_n(F)$, characterized by an equality of characters on matching regular elliptic elements. It is compatible with unramified twists; hence it induces a map on inertial equivalence classes. Under the local Langlands correspondence for $\mathrm{GL}_n(F)$, the inertial classes

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correspond to *inertial types*, which are smooth representations of the inertia group extending to the Weil group. The correspondence is such that two Weil–Deligne representations are Langlands parameters of inertially equivalent representations if and only if their underlying W_F-representations have isomorphic restrictions to inertia. If $\bar{\rho} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is a continuous representation, the choice of an n-dimensional inertial type τ and a dominant cocharacter λ for $\mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_{n,F}$ defines a quotient of the universal lifting ring $R^{\square}_{\bar{\rho}}$ of $\bar{\rho}$, whose points in characteristic zero correspond to potentially semistable lifts of $\bar{\rho}$ with Hodge type λ and inertial type τ. The Breuil–Mézard conjecture is concerned with the cycles that the mod p fibres of these rings define on Spec $R^{\square}_{\bar{\rho}}$.

To be more precise, recall that work of Henniart (appendix to [Breuil and Mézard 2002]) and Schneider and Zink [1999] associates to τ certain smooth representations $\sigma_{\mathfrak{P}}(\tau)$ of $\mathrm{GL}_n(\mathcal{O}_F)$, which refine the Bushnell–Kutzko theory of types by taking into account the monodromy operator on Langlands parameters. On the side of $\mathrm{GL}_n(F)$, these types compute the shape of a partition $\mathfrak{P}(\pi)$ attached to a representation π by Bernstein and Zelevinsky. We will only be concerned with the case of τ corresponding to an inertial class of the form $\mathfrak{s}(\tau) = \left[\prod_{i=1}^{r} \mathrm{GL}_{n/r}(F), \pi_0^{\otimes r}\right]$, where π_0 is a supercuspidal representation (these are precisely the inertial classes containing discrete series representations). In this case, $\sigma_{\mathfrak{P}}(\tau)$ has the property that, for a generic representation π of $\mathrm{GL}_n(F)$, the space $\mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\sigma_{\mathfrak{P}}(\tau), \pi)$ is not zero if and only if π is supported in $\mathfrak{s}(\tau)$ and $\mathfrak{P}(\pi) \geq \mathfrak{P}$ in the reverse of the dominance partial order on partitions of r. The maximal partition $\mathfrak{P}_{\max}$ is $r = 1 + \cdots + 1$, and for a generic π the partition $\mathfrak{P}(\pi)$ is maximal if and only if the monodromy operator on the Langlands parameter $\mathrm{rec}(\pi)$ equals zero.

In line with this, [Emerton and Gee 2014] asks for the existence of a map

$$R_{\overline{\mathbb{F}}_p}(\mathrm{GL}_n(\mathcal{O}_F)) \to Z(R^{\square}_{\bar{\rho}}/\pi)$$

from the Grothendieck group of finite length $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_n(\mathcal{O}_F)$ to the group of cycles on $R^{\square}_{\bar{\rho}}/\pi$, such that the image of the semisimplified mod p reduction $\bar{\sigma}_{\mathfrak{P}_{\max}}(\tau)$ of $\sigma_{\mathfrak{P}_{\max}}(\tau)$ is $Z(R^{\square}_{\bar{\rho}}(\tau, 0)_{\mathrm{cris}}/\pi)$, the cycle attached to the mod p fibre of the potentially *crystalline* deformation ring with inertial type τ and $\lambda = 0$ (one should work with coefficients in a large finite extension $E/\mathbb{Q}_p$, and we do so in the paper, so that π is a uniformizer of E). There is a similar statement for arbitrary λ, by tensoring $\sigma_{\mathfrak{P}_{\max}}(\tau)$ with the corresponding algebraic representation.

One might guess that the extension of this to semistable representations will relate $\bar{\sigma}_{\mathfrak{P}}(\tau)$ to the mod p fibre of the strata $R^{\square}_{\bar{\rho}}(\tau, \lambda)_{\mathfrak{P}}$ induced by the monodromy operator on the universal φ, N-module on $R^{\square}_{\bar{\rho}}(\tau, \lambda)$, which are again classified by partitions. However, we have found that one needs to be slightly careful in formulating this, and work instead with a virtual representation $\sigma^{+}_{\mathfrak{P}}(\tau)$ closely related to the Schneider–Zink types. It has the property that, for a generic representation π of $\mathrm{GL}_n(F)$,

$$\dim \mathrm{Hom}_K(\sigma^{+}_{\mathfrak{P}}(\tau), \pi) = \begin{cases} 1 & \text{if } \pi \text{ has inertial class } \mathfrak{s}(\tau) \text{ and } \mathfrak{P}(\pi) = \mathfrak{P}, \\ 0 & \text{otherwise.} \end{cases}$$

That these representations appear is consistent with the work of Shotton [2018] in the case of ℓ-adic coefficients for $\ell \neq p$.

Main results. With the above discussion in place, we can state our main results. The characteristic zero points of the stratum $R^{\square}_{\bar\rho}(\tau, \lambda)_{\mathfrak{P}_{\min}}$ indexed by the minimal partition $\mathfrak{P}_{\min}$ of r correspond to potentially semistable lifts of the representation $\bar\rho$ whose Weil–Deligne representation is the Langlands parameter of an essentially square-integrable representation, and these can be transferred to $D^\times$. Indeed, let τ be a discrete series inertial type of dimension n, corresponding to an inertial class $\mathfrak{s}(\tau)$ of $\mathrm{GL}_n(F)$-representations. The Jacquet–Langlands correspondence provides an inertial class $\mathfrak{s}_D(\tau) = \mathrm{JL}^{-1}\mathfrak{s}(\tau)$ of representations of $D^\times$, which admits types on the maximal compact subgroup $\mathcal{O}_D^\times$. In contrast with the case of $\mathrm{GL}_n(F)$, they are not uniquely determined, and we write $\sigma_D(\tau)$ for an arbitrarily chosen one: our results apply to all possible choices of $\sigma_D(\tau)$. The weight λ also determines a representation of $\mathcal{O}_D^\times$, and we write $\sigma_D(\tau, \lambda)$ for the tensor product of the two.

Theorem (Breuil–Mézard conjecture for $D^\times$; see Section 6). *If the geometric Breuil–Mézard conjecture holds for* $\mathrm{GL}_n(F)$, *then there exists a group homomorphism*

$$R_{\overline{\mathbb{F}}_p}(\mathcal{O}_D^\times) \to Z(R^{\square}_{\bar\rho}/\pi)$$

which for any (τ, λ) *sends the semisimplified mod p reduction* $\bar\sigma_D(\tau, \lambda)$ *of* $\sigma_D(\tau, \lambda)$ *to* $Z(R^{\square}_{\bar\rho}(\tau, \lambda)_{\mathfrak{P}_{\min}}/\pi)$.

The theorem is proved following the same strategy as [Gee and Geraghty 2015], but the techniques we use are different. We begin by constructing a group homomorphism

$$\mathrm{JL}_p : R_{\overline{\mathbb{F}}_p}(\mathcal{O}_D^\times) \to R_{\overline{\mathbb{F}}_p}(\mathrm{GL}_n(\mathcal{O}_F))$$

via Deligne–Lusztig induction, and describing it in terms of the combinatorics of parabolic induction. Our main technical result is Theorem 5.3, stating the equality $\mathrm{JL}_p(\bar\sigma_D(\tau, \lambda)) = \bar\sigma^{+}_{\mathfrak{P}_{\min}}(\tau, \lambda)$. Granting this, one transfers the result from $\mathrm{GL}_n(F)$ to $D^\times$ by composing with JL_p. In order to prove Theorem 5.3 we need a complete description of the Jacquet–Langlands correspondence in terms of type theory, which was obtained in [Dotto 2022]. We deduce our result from this description, a base change procedure to unramified extensions of F originating in [Bushnell and Henniart 1996], and explicit computations with a number of character formulas.

A Jacquet–Langlands transfer on maximal compact subgroups. Since $F^\times\mathcal{O}_D^\times$ is a normal subgroup of $D^\times$ with finite cyclic quotient, one proves that every smooth irreducible representation of $\mathcal{O}_D^\times$ with complex coefficients is a type for a Bernstein component of $D^\times$. It follows that our constructions in type theory give rise to a group homomorphism

$$\mathrm{JL}_{\boldsymbol{K}} : R_{\overline{\mathbb{Q}}_p}(\mathcal{O}_D^\times) \to R_{\overline{\mathbb{Q}}_p}(\mathrm{GL}_n(\mathcal{O}_F))$$

and our main results imply that the following diagram commutes. See Section 5 for details.

$$\begin{array}{ccc} R_{\overline{\mathbb{Q}}_p}(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{JL}_{\boldsymbol{K}}} & R_{\overline{\mathbb{Q}}_p}(\mathrm{GL}_n(\mathcal{O}_F)) \\ \big\downarrow r_p & & \big\downarrow r_p \\ R_{\overline{\mathbb{F}}_p}(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{JL}_p} & R_{\overline{\mathbb{F}}_p}(\mathrm{GL}_n(\mathcal{O}_F)) \end{array} \tag{1-1}$$

After a first version of this paper was written, we have been notified of work in preparation of Zijian Yao that makes the following equivalent construction. Consider the abelian group $\bigoplus_{(\tau,N)} \mathbb{Z}$ where the sum is indexed by Galois inertial types τ with monodromy operator N. There is a map $R_{\overline{\mathbb{Q}}_p}(\mathrm{GL}_n(\mathcal{O}_F)) \to \bigoplus_{(\tau,N)} \mathbb{Z}$, sending a representation σ to

$$\big(\dim_{\overline{\mathbb{Q}}_p} \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\sigma, \pi_{\tau,N})\big)_{(\tau,N)}$$

for any generic irreducible representation $\pi_{\tau,N}$ such that $\mathrm{rec}(\pi_{\tau,N})$ has inertial type τ and monodromy operator N. By definition, our representations $\sigma^+_{\mathfrak{P}}(\tau)$ yield a section of this map. There is an analogous map defined for $\mathcal{O}_D^\times$, whose image is contained in the direct sum of the factors indexed by discrete series inertial types. Yao defines $\mathrm{JL}_{\boldsymbol{K}}$ as the map making the diagram

$$\begin{array}{ccc} R_{\overline{\mathbb{Q}}_p}(\mathcal{O}_D^\times) & \xrightarrow{\ \mathrm{JL}_{\boldsymbol{K}}\ } & R_{\overline{\mathbb{Q}}_p}(\mathrm{GL}_n(\mathcal{O}_F)) \\ & \searrow \qquad \nearrow_{\sigma^+_{\mathfrak{P}}(\tau)} & \\ & \bigoplus_{(\tau,N)} \mathbb{Z} & \end{array} \tag{1-2}$$

commute, and goes on to conjecture the existence of a map JL_p making diagram (1-1) commute. Our results therefore provide a proof of this.

Yao makes similar conjectures in the case of more general inner forms, as this definition of $\mathrm{JL}_{\boldsymbol{K}}$ makes sense for $\mathrm{GL}_r(D)$ when formulated for those inertial types (τ, N) extending to a Langlands parameter for $\mathrm{GL}_r(D)$. At least in the case of discrete series parameters, it seems that our methods extend to this situation without too much trouble: the inertial Jacquet–Langlands correspondence is proved in full generality in [Dotto 2022], and there is a natural candidate for the JL_p map, namely Lusztig induction for the twisted Levi subgroup $\mathrm{GL}_r(\boldsymbol{d})$ of $\mathrm{GL}_n(\boldsymbol{f})$. We have chosen to focus on the simpler case of $D^\times$: amongst other reasons for this choice, we remark that from the viewpoint of a Jacquet–Langlands correspondence for maximal compact subgroups one expects weaker results for $\mathrm{GL}_r(D)$ than for $D^\times$. For instance, not every irreducible representation of $\mathrm{GL}_r(\mathcal{O}_D)$ is a type for $\mathrm{GL}_r(D)$, and $\mathrm{JL}_{\boldsymbol{K}}$ does not see any information about nontypical representations of $\mathrm{GL}_r(\mathcal{O}_D)$, except their multiplicities in restrictions of $\mathrm{GL}_r(D)$-representations.

We have the following parallel statement for ℓ-adic coefficients when $\ell \neq p$.

Theorem 5.5. *There exists a* (*necessarily unique*) *morphism* JL_ℓ *making the following diagram commute*:

$$\begin{array}{ccc} R_{\overline{\mathbb{Q}}_\ell}(\mathcal{O}_D^\times) & \xrightarrow{\ \mathrm{JL}_{\boldsymbol{K}}\ } & R_{\overline{\mathbb{Q}}_\ell}(\mathrm{GL}_n(\mathcal{O}_F)) \\ \downarrow r_\ell & & \downarrow r_\ell \\ R_{\overline{\mathbb{F}}_\ell}(\mathcal{O}_D^\times) & \xrightarrow{\ \mathrm{JL}_\ell\ } & R_{\overline{\mathbb{F}}_\ell}(\mathrm{GL}_n(\mathcal{O}_F)) \end{array} \tag{1-3}$$

The uniqueness statement follows from the fact that the reduction mod ℓ map for $\mathcal{O}_D^\times$ is surjective. In fact we can give an explicit description of JL_ℓ in terms of $\mathrm{JL}_{\boldsymbol{K}}$, and (as usual) the theorem has no new content when ℓ does not divide the pro-order of $\mathrm{GL}_n(\mathcal{O}_F)$, since in this case both vertical arrows are isomorphisms. It is worth stating explicitly a difference with the case $\ell = p$. The mod p irreducible representations of $\mathcal{O}_D^\times$ are characters, and they lift to level zero types for $D^\times$. Hence compatibility with the Jacquet–Langlands transfer of level zero types already determines the JL_p map uniquely, and compatibility for all types imposes a strong constraint on their mod p reductions. When $\ell \neq p$, there are a lot more irreducible $\overline{\mathbb{F}}_\ell$-representations of $\mathcal{O}_D^\times$, and the only congruences arise between types with the same endo-class. This allows us to construct JL_ℓ by fixing the endo-class and studying the mod ℓ reduction of the level zero part, which is what is done in the proof of Theorem 5.5.

From our theorem together with [Shotton 2018, Theorem 4.6] (which requires the assumption that $p \neq 2$), we deduce that a form of the geometric Breuil–Mézard conjecture holds for $D^\times$ and ℓ-adic coefficients, expressing the fact that congruences between the special fibres of discrete series deformation rings are described by mod ℓ congruences between types on the maximal compact subgroup of $D^\times$. See Theorem 6.3.

Structure of the article. The paper is organized as follows. Section 2 is about Deligne–Lusztig theory and begins with definitions and some simple results that are certainly well known but we could not find in the literature in the exact form we needed (although for instance [Lusztig 1976, 1.18] is closely related). Then we specialize to $\mathrm{GL}(n)$: we study the structure of parabolic induction, give a character formula (Proposition 2.6) and construct the representations $\sigma_{\mathfrak{P}}^+$ (Theorem 2.10). Section 3 recalls the results of [Schneider and Zink 1999] and proves analogues for $D^\times$. We repeat the Schneider–Zink construction for $\sigma_{\mathfrak{P}}^+$ and construct our virtual representations $\sigma_{\mathfrak{P}}^+(\tau)$. We end with two formulas for the trace of $\sigma_{\mathfrak{P}_{\min}}^+(\tau)$ and $\sigma_D(\tau)$ on pro-p-regular conjugacy classes of $\mathrm{GL}_n(\mathcal{O}_F)$ and $\mathcal{O}_D^\times$, and relate them via a computation of formal degrees. Section 4 recalls the monodromy stratification (see also [Pyvovarov 2021]) and states the geometric Breuil–Mézard conjecture for potentially semistable deformation rings. The connection with [Shotton 2018] is made explicit. Finally, we define our Jacquet–Langlands transfers of weights and types in Section 5 and prove our main theorems in Sections 5 and 6.

Notation and conventions. We use the same notation as [Dotto 2022], so that if F is a local field we write $\boldsymbol{f}$ for its residue field and μ_F for the group of Teichmüller (i.e., prime-to-p) roots of unity in $F^\times$. We fix an algebraic closure $\overline{F}$ of F, and write F_n for the unramified extension of F in $\overline{F}$ of degree n and $\boldsymbol{f}_n$ for its residue field. In general, $\Bbbk_n$ denotes an extension of the finite field $\Bbbk$ of degree n. A character of $\Bbbk_n^\times$ is called $\Bbbk$-regular if its orbit under the action of $\mathrm{Gal}(\Bbbk_n/\Bbbk)$ has n distinct elements. For an endo-class Θ_F over F we write $\delta(\Theta_F)$ for the degree over F of a parameter field of Θ_F, $e(\Theta_F)$ for its ramification index and $f(\Theta_F)$ for its residue field degree. We write $\boldsymbol{K}$ for the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}_F)$ of $\mathrm{GL}_n(F)$.

We consider partitions of a positive integer n as functions $\mathfrak{P} : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$ with finite support, such that $\sum_{i \in \mathbb{Z}_{>0}} i\mathfrak{P}(i) = n$. Whenever n is an integer and p a prime number, we write n_p for the highest power of p dividing n and $n_{p'} = n/n_p$.

Parabolic induction from a block-diagonal Levi subgroup of $\mathrm{GL}(n)$ is always taken with respect to the corresponding upper-triangular parabolic subgroup. We consider normalized induction for $\mathrm{GL}_n(F)$ unless stated otherwise. From Section 3, whenever dealing with a finite general linear group $\mathrm{GL}_n(\mathbb{F}_q)$ we will write R_w for the Deligne–Lusztig induction from an elliptic maximal torus (the type of such a torus consists of the n-cycles, and its group of rational points is isomorphic to $\mathbb{F}_{q^n}^\times$).

Unless stated otherwise, representations will have complex coefficients and representations of locally profinite groups will be smooth. The local Langlands correspondence for $\mathrm{GL}_n(F)$ is denoted by rec. If p is a prime number, any choice of an isomorphism $\iota_p : \overline{\mathbb{Q}}_p \to \mathbb{C}$ gives rise to a local Langlands correspondence $\mathrm{rec}_{\overline{\mathbb{Q}}_p}$ for smooth representations with $\overline{\mathbb{Q}}_p$-coefficients. This depends on the choice of ι_p only up to an unramified twist; hence its behaviour on inertial classes of representations is independent of the choice of ι_p.

2. Representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$

Fix a prime number p and let q be a power of p. In this section we recall the combinatorial classification, in terms of partitions, of the complex irreducible representations of $G = \mathrm{GL}_n(\mathbb{F}_q)$ with simple supercuspidal support, following [Schneider and Zink 1999, Sections 3, 4]. We give a construction, in terms of Deligne–Lusztig theory, of a certain virtual representation with special properties with respect to this classification, which will appear in the construction of the element of $R_{\overline{\mathbb{F}}_p}(\mathrm{GL}_n(\mathcal{O}_F))$ corresponding to the minimal stratum of a Galois deformation ring.

Harish-Chandra series. Every irreducible representation π of G has a supercuspidal support, which is unique up to conjugacy. The *simple* supercuspidal supports are those conjugate to

$$r\pi_0 = \Bigl(\prod_{i=1}^{r} \mathrm{GL}_{n/r}(\mathbb{F}_q), \pi_0^{\otimes r}\Bigr)$$

for some positive divisor r of n and some supercuspidal representation π_0 of $\mathrm{GL}_{n/r}(\mathbb{F}_q)$. There exists a unique nondegenerate representation supported in $r\pi_0$, denoted by $\mathrm{St}(\pi_0, r)$. To classify the others, we consider partitions $\mathfrak{P}$ of r, and to each $\mathfrak{P}$ we associate a block-diagonal Levi subgroup

$$L_{\mathfrak{P}}(\pi_0) = \prod_{i \in \mathbb{Z}_{>0}} \mathrm{GL}_{ni/r}(\mathbb{F}_q)^{\times \mathfrak{P}(i)}$$

and a parabolically induced representation of $\mathrm{GL}_n(\mathbb{F}_q)$

$$\pi_{\mathfrak{P}}(\pi_0) = \times_{i \in \mathbb{Z}_{>0}} \mathrm{St}(\pi_0, i)^{\times \mathfrak{P}(i)}.$$

The partition $\mathfrak{P}_{\max}$ sending 1 to r and every other positive integer to 0 corresponds to writing r as a sum of 1. The representation $\pi_{\mathfrak{P}_{\max}}(\pi_0)$ is the full parabolic induction $\pi_0^{\times r}$. The Harish-Chandra series corresponding to $r\pi_0$ is the set of irreducible representations of G with supercuspidal support $r\pi_0$. It coincides with the set of Jordan–Hölder factors of $\pi_{\mathfrak{P}_{\max}}(\pi_0)$.

Write $\mathfrak{P}' \le \mathfrak{P}$ for the reverse of the dominance partial order on partitions, as in [Schneider and Zink 1999]. Then $\mathfrak{P}_{\max}$ is the maximal element amongst partitions of r. There is a bijection $\mathfrak{P} \mapsto \sigma_{\mathfrak{P}}(\pi_0)$ from the set of partitions of r to the Harish-Chandra series for $r\pi_0$, characterized by the fact that $\sigma_{\mathfrak{P}}(\pi_0)$ occurs in $\pi_{\mathfrak{P}'}(\pi_0)$ if and only if $\mathfrak{P} \le \mathfrak{P}'$, and it occurs in $\pi_{\mathfrak{P}}(\pi_0)$ with multiplicity one. The smallest element amongst partitions of r is denoted by $\mathfrak{P}_{\min}$ and sends r to 1 and every other positive integer to 0. We have $\sigma_{\mathfrak{P}_{\min}}(\pi_0) = \pi_{\mathfrak{P}_{\min}}(\pi_0) = \mathrm{St}(\pi_0, r)$.

When $\mathfrak{P} \leq \mathfrak{P}'$, the multiplicity of $\sigma_{\mathfrak{P}}(\pi_0)$ in $\pi_{\mathfrak{P}'}(\pi_0)$ is by definition the *Kostka number* $K_{\mathfrak{P},\mathfrak{P}'}$. It depends only on the two partitions $\mathfrak{P}, \mathfrak{P}'$, and not on the representation π_0. More precisely, the standard definition of the Kostka numbers is formulated in terms of the representation theory of symmetric groups, as in [Shotton 2018, Section 6.1], and it is related to the representation theory of finite general linear groups in [Shotton 2018, Corollary 6.10]. Our normalizations coincide with [Shotton 2018, Definition 6.2], since the partial order that appears there is the reverse of $\leq$.

Lusztig induction. We follow the presentation of Deligne–Lusztig theory in [Digne and Michel 1991]. The material in this paragraph is mostly standard, but we need to fix notations and to provide certain results about products and Weil restriction of scalars that are probably well known but we could not find in the literature (although [Lusztig 1976, 1.18] is closely related). So we have decided to provide the proofs.

Let $\boldsymbol{G}_0$ be a connected reductive group over $\Bbbk = \mathbb{F}_q$, fix an algebraic closure $\bar{\Bbbk}$ of $\Bbbk$, and write $\boldsymbol{G} = \boldsymbol{G}_0 \times_{\Bbbk} \bar{\Bbbk}$. The rational structure $\boldsymbol{G}_0$ gives rise to a $\bar{\Bbbk}$-linear Frobenius endomorphism F of $\boldsymbol{G}$, the pullback of the absolute q-th power Frobenius morphism of $\boldsymbol{G}_0$. The Galois group $\mathrm{Gal}(\bar{\Bbbk}/\Bbbk)$ acts to the right on $\boldsymbol{G}$, via $\mathbb{F}_q$-linear automorphisms. We write φ for the geometric Frobenius element of the Galois group, acting as $x \mapsto x^{1/q}$. If $\boldsymbol{H}$ is a subgroup of $\boldsymbol{G}$ we will write $F\boldsymbol{H}$ for the parabolic subgroup $\varphi(\boldsymbol{H})$ of $\boldsymbol{G}$, whose group of $\bar{\Bbbk}$-points is $F(\boldsymbol{H}(\bar{\Bbbk}))$. We will say that $\boldsymbol{H}$ is *F-stable*, or *rational*, if $F\boldsymbol{H} = \boldsymbol{H}$. Recall from [Digne and Michel 1991, Definition 8.3] the invariant $\epsilon_{\boldsymbol{G}_0} = (-1)^{\eta(\boldsymbol{G}_0)}$, where the $\mathbb{F}_q$-rank $\eta(\boldsymbol{G}_0)$ is the dimension of the maximal split subtorus of any quasisplit rational maximal torus in $\boldsymbol{G}_0$ (the quasisplit rational maximal tori are those contained in a rational Borel subgroup).

Fix a parabolic subgroup $\boldsymbol{P}$ of $\boldsymbol{G}$, with unipotent radical $\boldsymbol{U}$ and F-stable Levi factor $\boldsymbol{L}$ (without assuming that $\boldsymbol{P}$ is F-stable). The associated Deligne–Lusztig varieties can be defined in terms of the Lang isogeny

$$\mathcal{L} : \boldsymbol{G} \to \boldsymbol{G}, \quad x \mapsto x^{-1}F(x)$$

by setting

$$\begin{aligned} \boldsymbol{X}^{\boldsymbol{G}}_{\boldsymbol{L}\subset\boldsymbol{P}} &= \mathcal{L}^{-1}(F\boldsymbol{P})/(\boldsymbol{P} \cap F\boldsymbol{P}), \\ \boldsymbol{Y}^{\boldsymbol{G}}_{\boldsymbol{L}\subset\boldsymbol{P}} &= \mathcal{L}^{-1}(F\boldsymbol{U})/(\boldsymbol{U} \cap F\boldsymbol{U}). \end{aligned}$$

Both varieties have an action of $\boldsymbol{G}^F \cong \boldsymbol{G}(\Bbbk)$ by left multiplication, and $\boldsymbol{Y}^{\boldsymbol{G}}_{\boldsymbol{L}\subset\boldsymbol{P}}$ has an action of $\boldsymbol{L}^F \cong \boldsymbol{L}(\Bbbk)$ by right multiplication. We write $H^*_c(\boldsymbol{Y}^{\boldsymbol{G}}_{\boldsymbol{L}\subset\boldsymbol{P}})$ for the alternating sum $\sum_{i\in\mathbb{Z}}(-1)^i[H^i_c(\boldsymbol{Y}^{\boldsymbol{G}}_{\boldsymbol{L}\subset\boldsymbol{P}}, \overline{\mathbb{Q}}_\ell)]$ of compactly supported ℓ-adic cohomology groups, for a prime number $\ell \neq p$. Each cohomology group carries a left action of $\boldsymbol{G}^F$ and a right action of $\boldsymbol{L}^F$. The associated Lusztig induction functor is

$$R^{\boldsymbol{G}}_{\boldsymbol{L}\subset\boldsymbol{P}} : R_{\overline{\mathbb{Q}}_\ell}(\boldsymbol{L}^F) \to R_{\overline{\mathbb{Q}}_\ell}(\boldsymbol{G}^F), \quad [V] \mapsto H^*_c(\boldsymbol{Y}^{\boldsymbol{G}}_{\boldsymbol{L}\subset\boldsymbol{P}}) \otimes_{\overline{\mathbb{Q}}_\ell[\boldsymbol{L}^F]} V.$$

On characters, we have the formula (see [Digne and Michel 1991, Proposition 4.5])

$$R^{\boldsymbol{G}}_{\boldsymbol{L}\subset\boldsymbol{P}}(\theta)(g) = |\boldsymbol{L}^F|^{-1} \sum_{l\in\boldsymbol{L}^F} \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{tr}\big((g,l) | H^i_c(\boldsymbol{Y}^{\boldsymbol{G}}_{\boldsymbol{L}\subset\boldsymbol{P}}, \overline{\mathbb{Q}}_\ell)\big)\theta(l^{-1}).$$

Remark 2.1. Since $\boldsymbol{U} \cap F\boldsymbol{U}$ is an affine space we obtain the same induction functor via the bimodule $H^*_c(\mathcal{L}^{-1}(F\boldsymbol{U}))$. This is the functor denoted by $R^{\boldsymbol{G}}_{\boldsymbol{L}\subset F\boldsymbol{P}}$ in [Digne and Michel 1991], since their $R^{\boldsymbol{G}}_{\boldsymbol{L}\subset\boldsymbol{P}}$ is constructed via $H^*_c(\mathcal{L}^{-1}(\boldsymbol{U}))$.

When $\boldsymbol{L}$ is a maximal torus, there is another description of Lusztig induction via the Bruhat decomposition of $\boldsymbol{G}$. Fix a pair $(\boldsymbol{B}, \boldsymbol{T})$ consisting of an F-stable maximal torus and an F-stable Borel subgroup of $\boldsymbol{G}$ containing $\boldsymbol{T}$. By [Deligne and Lusztig 1976, Lemma 1.13] there is a bijection between the $\boldsymbol{G}^F$-conjugacy classes of pairs $(\boldsymbol{B}', \boldsymbol{T}')$ consisting of a Borel subgroup of $\boldsymbol{G}$ and a rational maximal torus of $\boldsymbol{B}'$, and the Weyl group $W(\boldsymbol{T})$, given by the map $(g\boldsymbol{B}g^{-1}, g\boldsymbol{T}g^{-1}) \mapsto g^{-1}F(g)$ (here $g \in \boldsymbol{G}(\bar{\Bbbk})$). The F-conjugacy classes in $W(\boldsymbol{T})$ are the equivalence classes for $x \sim gxF(g^{-1})$, and they classify $\boldsymbol{G}^F$-conjugacy classes of F-stable maximal tori in $\boldsymbol{G}$ by [Deligne and Lusztig 1976, Corollary 1.14]. For w in $W(\boldsymbol{T})$, we write $\boldsymbol{T}_w$ for an F-stable maximal torus in $\boldsymbol{G}$ classified by the F-conjugacy class of w, and we say that w is the *type* of $\boldsymbol{T}_w$.

The Bruhat decomposition for $\boldsymbol{G}$ is $\boldsymbol{G} = \bigsqcup_{w\in W(\boldsymbol{T})} \boldsymbol{B}\dot{w}\boldsymbol{B}$ for any choice of representatives $\dot{w}$ of $W(\boldsymbol{T})$ in $\boldsymbol{G}$ (it is independent of the choice of $\dot{w}$). The quotient $\boldsymbol{B}w\boldsymbol{B}/\boldsymbol{B}$ is a Schubert cell in the flag variety $\boldsymbol{G}/\boldsymbol{B}$, and there is an associated Deligne–Lusztig variety

$$\boldsymbol{X}(w) = (\mathcal{L}^{-1}(\boldsymbol{B}w\boldsymbol{B}))/\boldsymbol{B}$$

together with a covering

$$\boldsymbol{Y}(\dot{w}) = (\mathcal{L}^{-1}(\boldsymbol{U}\dot{w}\boldsymbol{U}))/\boldsymbol{U}$$

induced by the canonical surjection $\boldsymbol{G}/\boldsymbol{U} \to \boldsymbol{G}/\boldsymbol{B}$. Both varieties have a left multiplication action by $\boldsymbol{G}^F$. If we equip $\boldsymbol{T}$ with the twisted Frobenius endomorphism $wF : t \mapsto wF(x)w^{-1}$, then the group of fixed points $\boldsymbol{T}^{wF}$ acts by right multiplication on $\boldsymbol{Y}(\dot{w})$. One checks as in [Deligne and Lusztig 1976, 1.8] that the isomorphism class of this covering, together with the action of $\boldsymbol{T}^{wF}$ and $\boldsymbol{G}^F$, is independent of the choice of $\dot{w}$.

Now consider a pair $(\boldsymbol{B}', \boldsymbol{T}')$ consisting of a Borel subgroup of $\boldsymbol{G}$ and a rational maximal torus of $\boldsymbol{B}'$, classified as in the above by some $w \in W(\boldsymbol{T})$. By [Deligne and Lusztig 1976, Proposition 1.19] whenever we have $x \in \boldsymbol{G}$ with $(\boldsymbol{B}', \boldsymbol{T}') = x(\boldsymbol{B}, \boldsymbol{T})x^{-1}$ and $\mathcal{L}(x) = \dot{w}$, the map $g \mapsto gx^{-1}$ induces an isomorphism $\boldsymbol{Y}(\dot{w}) \to \boldsymbol{Y}^{\boldsymbol{G}}_{\boldsymbol{T}'\subset\boldsymbol{B}}$ that is equivariant for the isomorphism $\mathrm{ad}(x) : \boldsymbol{T}^{wF} \to (\boldsymbol{T}')^F$, and $\boldsymbol{G}^F$-equivariant.

It follows that we can attach to each element $w \in W(\boldsymbol{T})$ an induction map

$$R_w : R_{\overline{\mathbb{Q}}_\ell}(\boldsymbol{T}^{wF}) \to R_{\overline{\mathbb{Q}}_\ell}(\boldsymbol{G}^F)$$

via the cohomology $H^*_c(\boldsymbol{Y}(\dot{w}))$ for any representative $\dot{w}$ of w.

We need to study the behaviour of the maps R_w with respect to Weil restriction of scalars and products. Define $\boldsymbol{G}_n = \boldsymbol{G}_0 \times_{\Bbbk} \Bbbk_n$ and

$$\boldsymbol{G}_0^+ = \mathrm{Res}_{\Bbbk_n/\Bbbk}(\boldsymbol{G}_0 \times_{\Bbbk} \Bbbk_n).$$

The base change $\boldsymbol{G}^+ = \boldsymbol{G}_0^+ \times_{\Bbbk} \bar{\Bbbk}$ is isomorphic to a product $\prod_{i=1}^n \boldsymbol{G}$, and its Frobenius endomorphism acts (on R-points, for any $\bar{\Bbbk}$-algebra R) by

$$(g_1, \ldots, g_m) \mapsto (F(g_m), F(g_1), \ldots, F(g_{m-1})),$$

where the map $F : \boldsymbol{G}(R) \to \boldsymbol{G}(R)$ is the Frobenius endomorphism for the $\Bbbk$-structure $\boldsymbol{G}_0$ (so that the one for the $\Bbbk_n$-structure $\boldsymbol{G}_n$ is F^n). Notice that projection on the first factor $(\boldsymbol{G}^+)^F \to \boldsymbol{G}^{F^n}$ is an isomorphism.

We fix an F^n-stable pair $(\boldsymbol{B}, \boldsymbol{T})$ in $\boldsymbol{G}$ and work with the F-stable pair $(\boldsymbol{B}^+, \boldsymbol{T}^+) = \left(\prod_{i=1}^n \boldsymbol{B}, \prod_{i=1}^n \boldsymbol{T}\right)$ in $\boldsymbol{G}^+$. Then there is an inclusion $\iota : W(\boldsymbol{T}) \to W(\boldsymbol{T}^+)$, $w \mapsto (w, 1, \dots, 1)$, inducing a bijection on F-conjugacy classes. Indeed, we see that $(w, 1, \dots, 1)$ and $(xwF^n(x^{-1}), 1, \dots, 1)$ are F-conjugates by $(x, F(x), \dots, F^{n-1}(x))$, and given an arbitrary $x = (x_1, \dots, x_n)$ we can always find $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $\alpha x F(\alpha)^{-1}$ is in the image of ι: it suffices to choose α_1 arbitrarily and to solve the equations $\alpha_i x_i F(\alpha_{i-1})^{-1} = 1$ recursively, for $2 \le i \le n$.

Lemma 2.2. *Let $w \in W(\boldsymbol{T})$. There is an isomorphism $(\boldsymbol{T}^+)^{\iota(w)F} \to \boldsymbol{T}^{wF^n}$ identifying R_w and $R_{\iota w}$.*

Proof. The isomorphism is again projection on the first factor. Indeed, the fixed points in the target are given by $(t, F(t), \dots, F^{n-1}(t))$ with the property that $t = wF^n(t)w^{-1}$. For the identification of Lusztig functors, we have that the cell $\boldsymbol{B}^+\iota(w)\boldsymbol{B}^+$ decomposes as a product $\boldsymbol{B}w\boldsymbol{B} \times \boldsymbol{B} \times \cdots \times \boldsymbol{B}$, and so the preimage $\mathcal{L}^{-1}(\boldsymbol{B}^+\iota(w)\boldsymbol{B}^+)$ is given on $\bar{\Bbbk}$-points by

$$(g, F(g)b_1, \dots, F^{m-1}(g)b_{m-1})$$

for arbitrary $b_i \in \boldsymbol{B}(\bar{\Bbbk})$ and $g \in \boldsymbol{G}(\bar{\Bbbk})$ such that $g^{-1}F^m(g) \in \boldsymbol{B}w\boldsymbol{B}$. A similar calculation works for the unipotent groups, after choosing a representative $\dot{w}$ of w and the corresponding representative $(\dot{w}, 1 \dots, 1)$ of $\iota(w)$. It follows that projection onto the first component induces a bijection $\boldsymbol{Y}(\dot{w}, 1, \dots, 1) \to \boldsymbol{Y}(\dot{w})$, which is equivariant with respect to our isomorphisms $(\boldsymbol{G}^+)^F \to \boldsymbol{G}^{F^n}$ and $(\boldsymbol{T}^+)^{\iota(w)F} \to \boldsymbol{T}^{wF^n}$. □

Lemma 2.3. *For $i \in \{1, \dots, n\}$, fix connected reductive groups $\boldsymbol{G}_{0,i}$ over $\Bbbk$, pairs $(\boldsymbol{B}_i, \boldsymbol{T}_i)$ in $\boldsymbol{G}_i$, and elements $w_i \in W(\boldsymbol{T}_i)$. Let $\boldsymbol{G}_0 = \prod_i \boldsymbol{G}_{0,i}$ with $(\boldsymbol{B}, \boldsymbol{T}) = \left(\prod_i \boldsymbol{B}_i, \prod_i, \boldsymbol{T}_i\right)$, and $\dot{w} = (\dot{w}_1, \dots, \dot{w}_n)$. Then $R_w : R_{\overline{\mathbb{Q}}_\ell}\left(\prod_i \boldsymbol{T}_i^F\right) \to R_{\overline{\mathbb{Q}}_\ell}\left(\prod_i \boldsymbol{G}_i^F\right)$ sends a one-dimensional character $\chi_1 \cdots \chi_n$ to $R_{w_1}(\chi_1) \cdots R_{w_n}(\chi_n)$.*

Proof. As in the proof of Lemma 2.2 we have an equivariant bijection $\boldsymbol{Y}(\dot{w}) \to \prod_i \boldsymbol{Y}(\dot{w}_i)$, and the claim follows from the Künneth formula for the cohomology of $\boldsymbol{Y}(\dot{w})$. □

A character formula. We now specialize to the case of $\boldsymbol{G}_0 = \mathrm{GL}_{n,\Bbbk}$, with $\boldsymbol{B}$ the upper triangular Borel subgroup and $\boldsymbol{T}$ the diagonal torus. The Weyl group $W(\boldsymbol{T})$ identifies with the symmetric group S_n, the F-conjugacy classes coincide with the conjugacy classes, and we normalize the lifts $\dot{w}$ via permutation matrices. We give a formula for the Lusztig induction map corresponding to the Weyl group element $w = (1, 2, \dots, n)$, on semisimple conjugacy classes. The group $\boldsymbol{T}^{wF}$ is isomorphic to $\Bbbk_n^\times$. Choosing a basis of $\Bbbk_n$ as a $\Bbbk$-vector space yields an inclusion $\mathrm{Res}_{\Bbbk_n/\Bbbk}\mathbb{G}_m \to \mathrm{GL}_{n,\Bbbk}$ contained in the $\boldsymbol{G}^F$-conjugacy class of rational maximal tori classified by w. These tori all have the same ϵ-invariant, which we will denote by ϵ_w. Notice that in our case the signs $\epsilon_{\boldsymbol{G}_0} = (-1)^n$ and $\epsilon_w = -1$, but it will sometimes be convenient not to make them explicit. The following proposition is a very special case of the Lusztig classification we will discuss later, and reformulates the Green parametrization of cuspidal representations in terms of Deligne–Lusztig induction.

Proposition 2.4. *Let $\chi : \Bbbk_n^\times \to \overline{\mathbb{Q}}_l^\times$ be a $\mathrm{Gal}(\Bbbk_n/\Bbbk)$-regular character. Then the function $(-1)^{n-1}R_w(\chi)$ is the character of an irreducible cuspidal representation of $\mathrm{GL}_n(\Bbbk)$. The map $\chi \mapsto (-1)^{n-1}R_w(\chi)$ induces a bijection from the set of orbits of $\mathrm{Gal}(\Bbbk_n/\Bbbk)$ on the $\Bbbk$-regular characters of $\Bbbk_n^\times$, to the irreducible cuspidal representations of $\mathrm{GL}_n(\Bbbk)$ over $\overline{\mathbb{Q}}_l$.*

Remark 2.5. If χ is not regular then it is not always the case that $(-1)^{n-1}R_w(\chi)$ is effective. However, these virtual representations will be important for us, since they will give rise to the Breuil–Mézard cycles of discrete series deformation rings.

In the next proposition, we compute the character $(-1)^{n-1}R_w(\chi)$ on semisimple classes, generalizing a well-known calculation in the case of $\Bbbk$-regular χ.

Proposition 2.6. *Let $\chi : \Bbbk_n^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a character, and let w be the conjugacy class of n-cycles in the symmetric group S_n. Then $R_w(\chi)$ vanishes on semisimple conjugacy classes of $\mathrm{GL}_n(\Bbbk)$ not represented in $\Bbbk_n^\times$, and for $x \in \Bbbk_n^\times$ we have*

$$\epsilon_{\boldsymbol{G}_0}\epsilon_w R_w(\chi)(x) = (-1)^{n+n/\deg(x)}(\mathrm{GL}_{n/\deg(x)}(\Bbbk_{\deg(x)}) : \Bbbk_n^\times)_{p'} \sum_{\gamma\in\mathrm{Gal}(\Bbbk_{\deg(x)}/\Bbbk)} \chi(\gamma x),$$

where $\deg(x)$ *is the degree of x over* $\Bbbk$.

Proof. By [Carter 1985, Proposition 7.5.4], we have the equality

$$\epsilon_{\boldsymbol{G}_0}\epsilon_w R_w(\chi)\mathrm{St}_{\boldsymbol{G}_0} = \mathrm{Ind}_{\Bbbk_n^\times}^{\mathrm{GL}_n(\Bbbk)}(\chi),$$

where $\mathrm{St}_{\boldsymbol{G}_0}$ is the Steinberg character and the induction is taken with respect to the embedding of $\Bbbk_n^\times$ in $\mathrm{GL}_n(\Bbbk)$ corresponding to some $\Bbbk$-basis of $\Bbbk_n$. By [Digne and Michel 1991, 9.3 Corollary], the Steinberg character vanishes away from semisimple classes, and if x is a semisimple element of $\mathrm{GL}_n(\Bbbk)$ then

$$\mathrm{St}_{\boldsymbol{G}_0}(x) = \epsilon_{\boldsymbol{G}_0}\epsilon_{Z^+_{\boldsymbol{G}}(x)}|Z^+_{\boldsymbol{G}^F}(x)|_p,$$

where $Z^+_{\boldsymbol{G}}(x)$ is the centralizer of x, a connected reductive group over $\Bbbk$.

Hence $R_w(\chi)(x) = 0$ if x is a semisimple element with no conjugates in $\Bbbk_n^\times$. When $x \in \Bbbk_n^\times$, we compute the character of the induction as

$$\mathrm{Ind}_{\Bbbk_n^\times}^{\mathrm{GL}_n(\Bbbk)}(\chi)(x) = |Z^+_{\boldsymbol{G}^F}(x)|\,|\Bbbk_n^\times|^{-1} \sum_{\gamma\in\mathrm{Gal}(\Bbbk_{\deg(x)}/\Bbbk)} \chi(\gamma x)$$

since the $\boldsymbol{G}^F$-conjugates of x in $\Bbbk_n^\times$ are precisely its Galois conjugates. The centralizer is isomorphic to $\mathrm{GL}_{n/\deg(x)}(\Bbbk_{\deg(x)})$. Then the claim follows since $\mathrm{Res}_{\Bbbk_{\deg(x)}/\Bbbk}\mathbb{G}_m^{\times n/\deg(x)}$ is a quasisplit maximal torus in $\mathrm{Res}_{\Bbbk_{\deg(x)}/\Bbbk}\mathrm{GL}_{n/\deg(x)}$ of rational rank $n/\deg(x)$. ☐

Remark 2.7. For any field R of characteristic zero containing all roots of unity of order dividing the exponent of $\mathrm{GL}_n(\Bbbk)$ there exists a unique map $\chi \mapsto (-1)^{n-1}R_w(\chi)$, from R-characters of $\Bbbk_n^\times$ to virtual R-representations of $\mathrm{GL}_n(\Bbbk)$, that satisfies the same character identity as [Deligne and Lusztig 1976, Theorem 4.2]. It induces a bijection from regular R-characters to irreducible supercuspidal R-representations, which is already characterized by the formula in Proposition 2.6, because of [Silberger and Zink 2000, Theorem 1.1]. If χ is an R-character of $\mathrm{GL}_n(\Bbbk)$, we will sometimes abuse notation and refer to $(-1)^{n-1}R_w(\chi)$ as the Deligne–Lusztig induction of χ, even if strictly speaking we are not repeating the same construction using cohomology with R-coefficients. (As a special case, this applies to $R = \overline{\mathbb{Q}}_p$.)

Unipotent characters. Let $\chi : W(\boldsymbol{T}) \to \overline{\mathbb{Q}}_l$ be the character of an irreducible representation. By [Digne and Michel 1991, Theorem 15.8], the unipotent characters of $\boldsymbol{G}^F$ are the functions

$$A_\chi = |W(\boldsymbol{T})|^{-1} \sum_{w \in W(\boldsymbol{T})} \chi(w) R_w(1_{\boldsymbol{T}^{wF}})$$

for varying χ. Notice that the maps R_{w_i} for $w_2 = w w_1 w^{-1}$ are intertwined by the isomorphism $\mathrm{ad}(\dot{w}) : \boldsymbol{T}^{w_1 F} \to \boldsymbol{T}^{w_2 F}$, since for an arbitrary F-stable maximal torus $\boldsymbol{T}$ the map $R^{\boldsymbol{G}}_{\boldsymbol{T} \subset \boldsymbol{B}}$ does not depend on the choice of Borel subgroup containing $\boldsymbol{T}$ (see [Digne and Michel 1991, Corollary 11.15]). By orthogonality of Deligne–Lusztig characters we deduce that

$$(R_w(1_{\boldsymbol{T}^{wF}}), A_\chi)_{\boldsymbol{G}^F} = \chi(w), \tag{2-1}$$

and so

$$R_w(1_{\boldsymbol{T}^{wF}}) = \sum_{\chi \in \mathrm{Irr}(W(\boldsymbol{T}))} \chi(w) A_\chi$$

since the unipotent characters form an orthonormal family. Since R_{id_W} coincides with the parabolic induction from $\boldsymbol{T}^F$, we see that the unipotent characters are the characters of the irreducible representations with supercuspidal support $n \cdot 1_{\Bbbk^\times}$. Further, by [Digne and Michel 1991, Proposition 12.13] we have that A_{triv} is the trivial character of $\boldsymbol{G}^F$. It follows from our discussion of Harish-Chandra series that $\sigma_{\mathfrak{P}_{\min}}(1) = \mathrm{St}(1, n)$ is the only other factor of $R_{\mathrm{id}}(\mathrm{triv})$ with multiplicity one. This is A_{sgn}, where $\mathrm{sgn} : W(\boldsymbol{T}) \to \overline{\mathbb{Q}}_l^\times$ is the sign character.

Lusztig series. Recall that two pairs $(\boldsymbol{T}_i, \theta_i)$ consisting of a rational maximal torus in $\boldsymbol{G}$ and a character of $\boldsymbol{T}_i^F$ are said to be *geometrically conjugate* if there exists $g \in \boldsymbol{G}(\bar{\Bbbk})$ such that $\boldsymbol{T}_2 = \mathrm{ad}(g)\boldsymbol{T}_1$ and, for all n such that $F^n(g) = g$, we have

$$\theta_1 \circ N_{\Bbbk_n/\Bbbk} = \theta_2 \circ N_{\Bbbk_n/\Bbbk} \circ \mathrm{ad}(g).$$

Here, the norm of an F-stable torus $\boldsymbol{S}$ is defined to be the morphism

$$N_{\Bbbk_n/\Bbbk} : \boldsymbol{S} \to \boldsymbol{S}, \quad t \mapsto tF(t) \cdots F^{n-1}(t),$$

and we are asking for equality to hold on $\boldsymbol{T}_i^{F^n}$. For $w \in S_n$, we write N_w for the $\Bbbk_n/\Bbbk$-norm of the diagonal torus with Frobenius endomorphism wF.

By Langlands duality, one can construct a bijection between geometric conjugacy classes of pairs $(\boldsymbol{S}, \theta)$ in $\boldsymbol{G}$ and semisimple conjugacy classes in $\boldsymbol{G}^F \cong \mathrm{GL}_n(\Bbbk)$. The details are in [Digne and Michel 1991, Chapter 13]. Here we just remark that the construction depends on a choice of norm-compatible generators ζ_n of every $\Bbbk_n^\times$, and an embedding $\bar{\Bbbk}^\times \to \overline{\mathbb{Q}}_l^\times$.

Example 2.8. The geometric conjugacy class of a pair $(\boldsymbol{S}, \theta)$ such that $\boldsymbol{S}$ has type $w = (1, 2, \ldots, n)$ corresponds to the semisimple conjugacy class in $\mathrm{GL}_n(\Bbbk)$ whose characteristic polynomial is a power of the minimal polynomial of $\theta(\zeta_n)$ over $\Bbbk$.

By [Digne and Michel 1991, Proposition 13.3, Theorem 14.51], two virtual characters $R_{w_i}(\theta_i)$ admit a common constituent if and only if the $(\boldsymbol{T}_{w_i}, \theta_i)$ are geometrically conjugate; and furthermore, by [Digne and Michel 1991, Proposition 13.1], every irreducible character of $\boldsymbol{G}^F$ is a constituent of some $R_w(\theta)$. It follows that the geometric conjugacy classes partition the set of irreducible characters of $\boldsymbol{G}^F$. An equivalence class in this partition is called the *Lusztig series* of the corresponding semisimple conjugacy class s in $\boldsymbol{G}^F$, and it is denoted by $\mathcal{E}(\boldsymbol{G}^F, s)$. The unipotent characters form the Lusztig series $\mathcal{E}(\boldsymbol{G}^F, [1])$. We record the following theorem, which implies that in certain cases Lusztig induction preserves irreducibility. We will apply it in the next paragraph.

Theorem 2.9 [Digne and Michel 1991, Theorem 13.25]. *Let s be a semisimple element of $\mathrm{GL}_n(\Bbbk)$, and let $\boldsymbol{L}$ be a rational Levi subgroup of $\boldsymbol{G}$ containing the centralizer $Z_{\boldsymbol{G}}(s)$. Then the map $\epsilon_{\boldsymbol{G}}\epsilon_{\boldsymbol{L}} R_{\boldsymbol{L}}^{\boldsymbol{G}}$ (taken with respect to any parabolic $\boldsymbol{P}$ with Levi factor $\boldsymbol{L}$) induces a bijection $\mathcal{E}(\boldsymbol{L}^F, [s]_{\boldsymbol{L}^F}) \to \mathcal{E}(\boldsymbol{G}^F, [s]_{\boldsymbol{G}^F})$.*

Virtual representations. Let m be a positive divisor of n and let π_0 be an irreducible supercuspidal representation of $\mathrm{GL}_m(\Bbbk)$. Since the matrix of Kostka numbers is upper unitriangular, it follows from the structure of the $\pi_{\mathfrak{P}}(\pi_0)$ that they form a basis for the Grothendieck group of finite length representations of $\mathrm{GL}_n(\Bbbk)$ all of whose factors have supercuspidal support $(n/m)\pi_0$. Then for any partition $\mathfrak{P}$ of n/m there exists an element $\sigma_{\mathfrak{P}}^+(\pi_0)$ of this Grothendieck group such that

$$(\sigma_{\mathfrak{P}}^+(\pi_0), \pi_{\mathfrak{P}'}(\pi_0))_{\mathrm{GL}_n(\Bbbk)} = \begin{cases} 1 & \text{if } \mathfrak{P} = \mathfrak{P}', \\ 0 & \text{otherwise.} \end{cases}$$

We now give an explicit construction of $\sigma_{\mathfrak{P}_{\min}}^+(\pi_0)$ in terms of Deligne–Lusztig theory.

Theorem 2.10. *Let w be an n-cycle in S_n, let w_m be an m-cycle in S_m, and assume $\pi_0 \cong (-1)^{m+1} R_{w_m}(\chi)$ for a $\Bbbk$-regular character $\theta_0 : \Bbbk_m^\times \to \overline{\mathbb{Q}}_l^\times$. Let $\theta = N_{\Bbbk_n/\Bbbk_m}^*(\theta_0)$. Then*

$$\sigma_{\mathfrak{P}_{\min}}^+(\pi_0) \cong (-1)^{n+1} R_w(\theta).$$

Proof. First observe that $R_w(\theta)$ is orthogonal to each of the $\pi_{\mathfrak{P}}(\pi_0)$ for $\mathfrak{P} \neq \mathfrak{P}_{\min}$, because these are full parabolic inductions, and the torus $\boldsymbol{T}_w$ has no conjugates in any proper split Levi subgroup of $\boldsymbol{G}$. So we have to prove that all irreducible constituents of $R_w(\theta)$ have supercuspidal support $(n/m)\pi_0$, and that

$$(R_w(\theta), \mathrm{St}(\pi_0, n/m))_{\boldsymbol{G}^F} = (-1)^{n+1},$$

since $\pi_{\mathfrak{P}_{\min}}(\pi_0) = \sigma_{\mathfrak{P}_{\min}}(\pi_0) = \mathrm{St}(\pi_0, n/m)$.

Write $z = \theta_0(\zeta_m)$, so that the geometric conjugacy class of $(\boldsymbol{T}_w, \theta)$ corresponds to the minimal polynomial of z over $\Bbbk$ (a degree m polynomial) to the n/m-th power, as in Example 2.8. The centralizer in $\mathrm{GL}_n(\Bbbk)$ of any rational element in this conjugacy class is isomorphic to $\mathrm{GL}_{n/m}(\Bbbk_m)$, and it is the group of rational points of a Levi subgroup $\boldsymbol{L}_0$ of $\boldsymbol{G}_0$, isomorphic to $\mathrm{Res}_{\Bbbk_m/\Bbbk}\mathrm{GL}_{n/m,\Bbbk_m}$. By the discussion preceding Lemma 2.2, the conjugacy classes of rational maximal tori in $\boldsymbol{L}_0 \times_{\Bbbk} \overline{\Bbbk}$ are in bijection with those in $\mathrm{GL}_{n/m,\Bbbk_m} \times_{\Bbbk_m} \overline{\Bbbk}$. Under this bijection, the torus $\boldsymbol{T}_w$ has type corresponding to the n/m-cycles, which we write as $w_{n/m}$.

By Lemma 2.2, the unipotent characters $\mathcal{E}(\boldsymbol{L}^F, [1])$ coincide with the unipotent characters of $\mathrm{GL}_{n/m}(\Bbbk_m)$ viewed as the group of $\Bbbk_m$-points of $\mathrm{GL}_{n/m,\Bbbk_m}$. Hence they are parametrized by $\chi \in \mathrm{Irr}(S_{n/m})$ as in our previous discussion: we write $\chi \mapsto A_\chi$ for this parametrization.

Lemma 2.11. *The Lusztig series of the geometric conjugacy class of $(\boldsymbol{T}_w, \theta)$ is*

$$\{(-1)^{n+n/m} R_{\boldsymbol{L}}^{\boldsymbol{G}}(\theta_0 A_\chi) : \chi \in \mathrm{Irr}(S_{n/m})\}.$$

Proof. The character θ_0 can be inflated to $\mathrm{GL}_{n/m}(\Bbbk_m)$ via the determinant, and its restriction to $\Bbbk_n^\times$ is θ. By [Digne and Michel 1991, Proposition 13.30], the Lusztig series of $\boldsymbol{L}^F$ attached to the geometric conjugacy class of $(\boldsymbol{T}_w, \theta)$ consists of the twists by θ of the unipotent characters of $\boldsymbol{L}^F$. Then the lemma follows from Theorem 2.9. □

Theorem 2.10 will follow from Lemma 2.11, Lemma 2.12, and the following two equations:

$$R_w(\theta) = \sum_{\chi \in \mathrm{Irr}(S_{n/m})} \chi(w_{n/m}) R_{\boldsymbol{L}}^{\boldsymbol{G}}(\theta_0 A_\chi), \tag{2-2}$$

$$\mathrm{St}(\pi_0, n/m) = (-1)^{n+n/m} R_{\boldsymbol{L}}^{\boldsymbol{G}}(\theta_0 A_{\mathrm{sgn}}). \tag{2-3}$$

Indeed, they imply that

$$\begin{aligned}(R_w(\theta), \sigma_{\mathfrak{P}_{\min}}(\pi_0))_{\boldsymbol{G}^F} &= \Big(\sum_{\chi \in \mathrm{Irr}(S_{n/m})} \chi(w_{n/m}) R_{\boldsymbol{L}}^{\boldsymbol{G}}(\theta_0 A_\chi), (-1)^{n+n/m} R_{\boldsymbol{L}}^{\boldsymbol{G}}(\theta_0 A_{\mathrm{sgn}})\Big)_{\boldsymbol{G}^F} \\ &= (-1)^{n+n/m}\mathrm{sgn}(w_{n/m}) = (-1)^{n+1}\end{aligned}$$

since $\mathrm{sgn}(w_{n/m}) = (-1)^{n/m+1}$.

Proof of equation (2-2). Transitivity of Lusztig induction (see [Digne and Michel 1991, 11.5]) implies that $R_w(\theta) = R_{\boldsymbol{L}}^{\boldsymbol{G}}(R_{\boldsymbol{T}_w}^{\boldsymbol{L}}(\theta))$, where we have chosen an arbitrary parabolic subgroup $\boldsymbol{P} \subseteq \boldsymbol{G}$ with Levi factor $\boldsymbol{L} = \boldsymbol{L}_0 \times_{\Bbbk} \bar{\Bbbk}$. By [Deligne and Lusztig 1976, Corollary 1.27], we have an equality $R_{\boldsymbol{T}_w}^{\boldsymbol{L}}(\theta) = \theta_0 R_{\boldsymbol{T}_w}^{\boldsymbol{L}}(1_{\boldsymbol{T}_w})$. By Lemma 2.2, the functor $R_{\boldsymbol{T}_w}^{\boldsymbol{L}}$ coincides with $R_{w_{n/m}}$ taken with respect to $\mathrm{GL}_{n/m,\Bbbk_m}$. But we have seen in (2-1) that

$$R_{w_{n/m}}(1) = \sum_{\chi \in \mathrm{Irr}(S_{n/m})} \chi(w_{n/m}) A_\chi, \tag{2-4}$$

and so (2-2) holds. □

Lemma 2.12. *The Lusztig series of $(\boldsymbol{T}_w, \theta)$ coincides with the Harish-Chandra series of $n/m \cdot \pi_0$.*

Proof. Let $w_{m,n/m} \in S_n$ be the product of n/m disjoint m-cycles and let $w_m \in S_m$ be an m-cycle. Then the group of rational points of the torus $\boldsymbol{T}_{w_{m,n/m}}$ is isomorphic to $(\Bbbk_m)^{\times n/m}$. We are going to prove the lemma by computing

$$(-1)^{n+n/m} R_{\boldsymbol{T}_{w_{m,n/m}}}^{\boldsymbol{G}}(\theta_0^{\otimes n/m})$$

in two ways. The first one is based on the fact that $\boldsymbol{T}_{w_{m,n/m}}$ is a rational maximal torus in $\boldsymbol{L}$: notice that $\boldsymbol{T}_{w_{m,n/m}}(\Bbbk)$ is the diagonal torus of $\mathrm{GL}_{n/m}(\Bbbk_m)$. Then similarly to (2-4), we have

$$(-1)^{n+n/m} R_{\boldsymbol{T}_{w_{m,n/m}}}^{\boldsymbol{G}}(\theta_0^{\otimes n/m}) = (-1)^{n+n/m} R_{\boldsymbol{L}}^{\boldsymbol{G}}(\theta_0 R_{\mathrm{id}}(1)) = (-1)^{n+n/m} \sum_{\chi \in \mathrm{Irr}(S_{n/m})} \chi(\mathrm{id}) R_{\boldsymbol{L}}^{\boldsymbol{G}}(\theta_0 A_\chi) \tag{2-5}$$

and so, by Lemma 2.11, the constituents of this character coincide with the Lusztig series of $(\boldsymbol{T}_w, \theta)$.

On the other hand, let $\boldsymbol{M} = \mathrm{GL}_{m,\Bbbk}^{\times n/m}$, a split Levi subgroup of $\boldsymbol{G}$. Notice that $\boldsymbol{T}_{w_{m,n/m}}$ is a maximal torus in $\boldsymbol{M}$ (indeed, $\Bbbk_m$ is a maximal torus in $\mathrm{GL}_m(\Bbbk)$). By transitivity, $R^{\boldsymbol{G}}_{\boldsymbol{T}_{w_{m,n/m}}}(\theta_0^{\otimes n/m})$ is the character of the parabolic induction of $R^{\boldsymbol{M}}_{\boldsymbol{T}_{w_{m,n/m}}}(\theta_0^{\otimes n/m})$, because Lusztig induction from a split Levi subgroup coincides with parabolic induction [Digne and Michel 1991, 11.1]. Now we can apply Lemma 2.3 to find that $R^{\boldsymbol{M}}_{\boldsymbol{T}_{w_{m,n/m}}}(\theta_0^{\otimes n/m})$ equals $R^{\mathrm{GL}_{m,\Bbbk}}_{\boldsymbol{T}_{w_m}}(\theta_0)^{\otimes n/m}$. Finally, we deduce that

$$\begin{aligned}(-1)^{n+n/m} R^{\boldsymbol{G}}_{\boldsymbol{T}_{w_{m,n/m}}}(\theta_0^{\otimes n/m}) &= \mathrm{PInd}^{\boldsymbol{G}^F}_{\boldsymbol{M}^F}\big((-1)^{m+1} R_{w_m}(\theta_0)\big)^{\otimes n/m} \\ &= \mathrm{PInd}^{\mathrm{GL}_n(\Bbbk)}_{\prod_{i=1}^{n/m}\mathrm{GL}_m(\Bbbk)}(\pi_0^{\otimes n/m}) \\ &= \pi_{\mathfrak{P}_{\max}}(\pi_0).\end{aligned}$$

Then the lemma follows from (2-5) and the fact that the Harish-Chandra series of $n/m \cdot \pi_0$ coincides with the set of constituents of $\pi_{\mathfrak{P}_{\max}}(\pi_0)$. □

Proof of equation (2-3). The character A_{sgn} is the Steinberg character of $\mathrm{GL}_{n/m}(\Bbbk_m)$, and the Lusztig induction of a nondegenerate character is nondegenerate, by [Digne and Michel 1983, Théorème 4.4] (the nondegenerate irreducible characters are the constituents of a Gelfand–Graev representation). Since nondegeneracy is preserved under twisting by one-dimensional characters (because unipotent elements have determinant one), we see that $(-1)^{n+n/m} R^{\boldsymbol{G}}_{\boldsymbol{L}}(\theta_0 A_{\mathrm{sgn}})$ is the character of a nondegenerate representation in the Lusztig series of $(\boldsymbol{T}_w, \theta)$. By Lemma 2.12, this representation is $\mathrm{St}(\pi_0, n/m)$. □

This completes the proof of Theorem 2.10. □

3. Type theory

In this section we recall the structure of maximal simple types for the inner forms of $\mathrm{GL}_n(F)$ and the results of Schneider and Zink about $\boldsymbol{K}$-types. We establish their analogues for types on the maximal compact subgroup of $D^\times$. We then prove some formulas for the trace of a $\boldsymbol{K}$-type in terms of its level zero part and its base change to unramified extensions, and begin studying their behaviour under the Jacquet–Langlands correspondence. From now on, whenever dealing with a finite general linear group $\mathrm{GL}_n(\mathbb{F}_q)$ we will write $R_w(-)$ for the Deligne–Lusztig induction from a maximal torus whose type consists of the n-cycles. If χ is an $\mathbb{F}_q$-regular character of $\mathbb{F}_{q^n}^\times$ then $(-1)^{n-1} R_w(\chi)$ is an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$. (See Remark 2.7.) We will also fix the maximal compact subgroup $\boldsymbol{K} = \mathrm{GL}_n(\mathcal{O}_F)$ of $\mathrm{GL}_n(F)$ for the rest of the paper.

Maximal simple types. In this paragraph we let $G = \mathrm{GL}_m(D)$ be an inner form of $\mathrm{GL}_n(F)$, for D a central division algebra over F of reduced degree d. We write $A = M_m(D)$. We summarize the parametrization of simple inertial classes of representations of G from the point of view of [Dotto 2022], which builds upon the work of Bushnell and Kutzko [1993] and Broussous, Sécherre and Stevens in a series of papers (see for instance [Broussous et al. 2012; Sécherre and Stevens 2019]). Recall that the simple inertial classes of representations of G are those whose supercuspidal support is inertially equivalent to $r\pi_0$ for some positive divisor $r|m$ and some supercuspidal representation π_0 of $\mathrm{GL}_{m/r}(D)$.

The supercuspidal Bernstein components of G admit types constructed as follows. One starts with a *maximal simple character*, which is a character of a compact open subgroup H^1_θ of G. As m and D vary, the maximal simple characters of $\mathrm{GL}_m(D)$ can be classified according to their endo-class (usually denoted by Θ_F). Two maximal simple characters in G have the same endo-class if and only if they are G-conjugate.

Attached to a maximal simple character θ there are subgroups $H^1_\theta \subseteq J^1_\theta \subseteq J_\theta$, of G, each normal in the next. There corresponds to θ an irreducible representation η_θ of J^1_θ, whose restriction to H^1_θ is a multiple of θ. One can extend η_θ to a representation of J_θ, and a class of *β-extensions* is singled out. They are all twists of each other by characters of J_θ/J^1_θ, which is a finite general linear group. There exists a unique β-extension κ_p, called the p-primary β-extension, with the property that the order of the character $\det(\kappa_p)$ is a power of p. However, the main result of [Dotto 2022] suggests that we work instead with a certain quadratic twist $\kappa_\theta = \epsilon^1_\theta \kappa_p$. The character ϵ^1_θ is one of the "symplectic invariants" of θ, for which see [Dotto 2022, Propositions 2.11, 2.13] (note that ϵ^1_θ is denoted by $\epsilon^1(-, V_\theta)$ in [Dotto 2022]).

To go further in the construction, let θ be a maximal simple character with endo-class $\mathrm{cl}(\theta) = \Theta_F$. We identify the group J_θ/J^1_θ with a certain finite general linear group. As in [Dotto 2022], we let E/F be the unramified parameter field of Θ_F in $\bar{F}$. If $[\mathfrak{A}, \beta]$ is a simple stratum for θ, and $Z_A(F[\beta]) \cong M_{m'}(D')$ for a central division algebra D' over $F[\beta]$ of reduced degree d', then $m'd' = n/\delta(\Theta_F)$, and $J_\theta/J^1_\theta \cong \mathrm{GL}_{m'}(\boldsymbol{e}_{d'})$ (we recall that $\boldsymbol{e}_{d'}$ is the residue field of the unramified extension of E in $\bar{F}$ of degree d'). More precisely, in [Dotto 2022, Section 3.1] there is constructed an injection from the set of lifts of Θ_F to an endo-class Θ_E over E to the set of conjugacy classes of isomorphisms

$$J_\theta/J^1_\theta \to \mathrm{GL}_{m'}(\boldsymbol{e}_{d'})$$

under the group $\mathrm{GL}_{m'}(\boldsymbol{e}_{d'}) \rtimes \mathrm{Gal}(\boldsymbol{e}_{d'}/\boldsymbol{e})$. (The notion of lift of an endo-class over F to a finite tamely ramified extension of F is defined in [Bushnell and Henniart 1996, Section 9], see especially [Bushnell and Henniart 1996, Corollary 9.13].)

Let $\chi : \boldsymbol{e}^\times_{n/\delta(\Theta_F)} \to \mathbb{C}^\times$ be a $\boldsymbol{e}_{d'}$-regular character, that is to say a character with trivial stabilizer in $\mathrm{Gal}(\boldsymbol{e}_{n/\delta(\Theta_F)}/\boldsymbol{e}_{d'})$. Then the Deligne–Lusztig induction $(-1)^{m'+1}R_w(\chi)$ is a supercuspidal representation of $\mathrm{GL}_{m'}(\boldsymbol{e}_{d'})$, depending only on the $\mathrm{Gal}(\boldsymbol{e}_{n/\delta(\Theta_F)}/\boldsymbol{e}_{d'})$-conjugacy class of χ. The representation $(J_\theta, \kappa_\theta \otimes (-1)^{m'+1}R_w(\chi))$ is a type for a supercuspidal Bernstein component of G, and all such components admit types of this kind. Furthermore, two types $(J_\theta, \kappa_\theta \otimes (-1)^{m'+1}R_w(\chi_i))$ determine the same component if and only if the χ_i are conjugate under $\mathrm{Gal}(\boldsymbol{e}_{n/\delta(\Theta_F)}/\boldsymbol{e})$.

The choice of $\Theta_E \to \Theta_F$ and of the β-extension κ_θ determines a *level zero map*, denoted by $\Lambda(-, \Theta_E, \kappa_\theta)$ in [Dotto 2022]. To shorten notation we will denote it $\Lambda_{\kappa_\theta}(-)$ or simply Λ, since most of the time we will be working with κ_θ and with a fixed lift $\Theta_E \to \Theta_F$. It goes from the set of irreducible smooth representations of $\mathrm{GL}_m(D)$ whose supercuspidal support is simple of endo-class Θ_F to the set of $\mathrm{Gal}(\boldsymbol{e}_{n/\delta(\Theta_F)}/\boldsymbol{e})$-orbits of characters of $\boldsymbol{e}^\times_{n/\delta(\Theta_F)}$. It only depends on the inertial class of a representation, and it sends a supercuspidal representation π to the orbit $[\chi]$ determined by the maximal simple types it contains. To describe its effect on simple, nonsupercuspidal representations, let π be an irreducible representation with supercuspidal support $r\pi_0$. Recall from [Mínguez and Sécherre 2014; Dotto 2022] that there exists a unique conjugacy class of maximal β-extensions in $\mathrm{GL}_{m/r}(D)$ that is *compatible* with κ_θ in the sense explained in these references. We denote it κ_θ^0. Then $\Lambda(\pi, \Theta_E, \kappa_\theta)$ is the inflation to $\boldsymbol{e}^\times_{n/\delta(\Theta_F)}$ of $\Lambda(\pi_0, \Theta_E, \kappa_\theta^0)$.

In summary, if we fix a lift $\Theta_E \to \Theta_F$ for all endo-classes Θ_F over F we find a bijection from the set of simple inertial classes of G to the set of pairs $(\Theta_F, [\chi])$ consisting of an endo-class over F of degree dividing n and an orbit of $\mathrm{Gal}(\boldsymbol{e}_{n/\delta(\Theta_F)}/\boldsymbol{e})$ on the set of characters of $\boldsymbol{e}^\times_{n/\delta(\Theta_F)}$.

Remark 3.1. We treat the case of level zero representations by letting J_θ be a maximal compact subgroup with principal congruence subgroup J_θ^1, letting $\kappa_\theta = 1$, and working with an arbitrary choice of isomorphism $J_\theta/J_\theta^1 \to \mathrm{GL}_m(\boldsymbol{f}_d)$ extending to an $\boldsymbol{f}$-algebra isomorphism. This introduces no ambiguity as here $\boldsymbol{e} = \boldsymbol{f}$ and so the action of $\mathrm{Gal}(\boldsymbol{f}_d/\boldsymbol{f})$ does not change the inertial class.

Finally, we recall a special case of the interior lifting construction of [Bushnell and Henniart 1996; Broussous et al. 2012]. If θ is a maximal simple character in $\mathrm{GL}_m(D)$ and $[\mathfrak{A}, \beta]$ is a simple stratum for θ, there exists a maximal unramified extension $K^+/F[\beta]$ in $Z_A(F[\beta])$ normalizing the order $\mathfrak{A}$, as in the proof of [Dotto 2022, Proposition 2.5]. Let L be any unramified extension of F in K^+. As in [Dotto 2022, Proposition 2.8, Lemma 2.12], the restriction $\theta_L = \theta | H_\theta^1 \cap Z_A(L)$ is a maximal simple character with corresponding groups $H^1_{\theta_L} = H^1_\theta \cap Z_A(L)$ and $J^i_{\theta_L} = J^i_\theta \cap Z_A(L)$. The character θ_L is called the interior lift of θ to L.

***K**-types for* $\mathrm{GL}_n(F)$. Let $A = M_n(F)$ and $G = A^\times = \mathrm{GL}_n(F)$. We recall some results from [Schneider and Zink 1999] and translate them in the form we will need later on. Let $F[\beta]$ be a field extension of F in $A = M_n(F)$, and let $B = Z_A(F[\beta])$. Choose a pair $\mathfrak{B}_{\min} \subseteq \mathfrak{B}_{\max}$ of hereditary $\mathcal{O}_{F[\beta]}$-orders in B, such that $\mathfrak{B}_{\min}$ is minimal and $\mathfrak{B}_{\max}$ is maximal. Recall that hereditary $\mathcal{O}_{F[\beta]}$-orders $\mathfrak{B}$ in B are in bijection with $\mathcal{O}_{F[\beta]}$-lattice chains in $V = F^n$ viewed as an $F[\beta]$-vector space via the inclusion $F[\beta] \subset A$. Since these are also $\mathcal{O}_F$-lattice chains, there corresponds to $\mathfrak{B}$ a unique hereditary $\mathcal{O}_F$-order $\mathfrak{A} = \mathfrak{A}(\mathfrak{B})$ of A, called the continuation of $\mathfrak{B}$ to A. It satisfies $\mathfrak{A}(\mathfrak{B}) \cap B = \mathfrak{B}$. Following [Schneider and Zink 1999, Section 5], we associate to $\mathfrak{B}$ a subgroup $J = J(\mathfrak{B}) = J(\beta, \mathfrak{A}(\mathfrak{B}))$ of the unit group $\mathfrak{A}(\mathfrak{B})^\times$ such that $J = J^1\mathfrak{B}^\times$ for $J^1 = J^1(\mathfrak{B}) = J \cap U^1(\mathfrak{A}(\mathfrak{B}))$. We write $J_{\max}$ and $J^1_{\max}$ for the groups corresponding to $\mathfrak{B}_{\max}$.

Remark 3.2. From now on, we make the assumption that the group $J_{\max}$ is contained in our fixed maximal compact subgroup $\boldsymbol{K}$. This can always be achieved after possibly replacing $F[\beta]$ with a conjugate.

We let θ be a simple character of the stratum $[\mathfrak{A}_{\max}, \beta]$ (so θ is maximal) and write $J_{\max} = J_\theta$ and $J^1_{\max} = J^1_\theta$. We write $\kappa_{\max}$ or $\kappa(\mathfrak{B}_{\max})$ for the β-extension κ_θ ([Schneider and Zink 1999] works with an arbitrary β-extension). There is a corresponding family of representations $\kappa(\mathfrak{B})$ of $J(\beta)$, one for each for any hereditary $\mathcal{O}_{F[\beta]}$-order $\mathfrak{B}_{\min} \subseteq \mathfrak{B} \subseteq \mathfrak{B}_{\max}$, satisfying a coherence property as in [Schneider and Zink 1999, Lemma 5.1].

Writing Θ_F for the endo-class of θ, and E/F for the unramified parameter field of Θ_F in $\overline{F}$, we have attached an inner conjugacy class of isomorphisms

$$J_{\max}/J^1_{\max} \to \mathrm{GL}_{n/\delta(\Theta_F)}(\boldsymbol{e})$$

to every lift of Θ_F to an endo-class Θ_E over E. We have previously fixed such a lift $\Theta_E \to \Theta_F$, and we now let ψ be a representative of the corresponding conjugacy class such that ψ identifies $\mathfrak{B}_{\min}^{\times} J_{\max}^1 / J_{\max}^1$ with the upper-triangular Borel subgroup (compare the discussion after [Schneider and Zink 1999, Lemma 5.5]).

There is a functor $V \mapsto V(\kappa_{\max}) = \mathrm{Hom}_{J_{\max}^1}(\kappa_{\max}, V)$, from the category of smooth representations of $\mathrm{GL}_n(F)$ to the category of representations of $J_{\max}/J_{\max}^1$, sending admissible representations to finite-dimensional ones, which we denoted $\boldsymbol{K}_{\kappa_{\max}}$ in [Dotto 2022]. We will compose it with our isomorphism ψ, and denote the resulting functor still by $V \mapsto V(\kappa_{\max})$.

For any positive divisor r of $n/\delta(\Theta_F)$ we have a standard parabolic subgroup of $\mathrm{GL}_{n/\delta(\Theta_F)}(\boldsymbol{e})$, with Levi factor isomorphic to $\prod_{i=1}^r \mathrm{GL}_{n/r\delta(\Theta_F)}(\boldsymbol{e})$, and consisting of block upper triangular matrices. It coincides with the image under ψ of $\mathfrak{B}^{\times} J_{\max}^1 / J_{\max}^1$ for some principal order $\mathfrak{B}_{\min} \subseteq \mathfrak{B} \subseteq \mathfrak{B}_{\max}$ that we fix. If σ_0 is a cuspidal representation of $\mathrm{GL}_{n/r\delta(\Theta_F)}(\boldsymbol{e})$ attached to the character χ of $\boldsymbol{e}_{n/r\delta(\Theta_F)}^{\times}$, and $\sigma = \sigma_0^{\otimes r}$ is inflated to $J(\mathfrak{B})/J^1(\mathfrak{B})$, then the pair $(J(\mathfrak{B}), \kappa(\mathfrak{B}) \otimes \sigma)$ is a simple type in $\mathrm{GL}_n(F)$. It is a maximal simple type precisely when $r = 1$. The next lemma connects this construction with our parametrization of simple inertial classes.

Lemma 3.3. *Let $\mathfrak{s}$ be the simple inertial class with invariants* $\mathrm{cl}(\mathfrak{s}) = \Theta_F$ *and* $\Lambda(\mathfrak{s}, \Theta_E, \kappa_{\max}) = [\chi]$. *With the notation of the previous paragraph, the pair $(J(\mathfrak{B}), \kappa(\mathfrak{B}) \otimes \sigma)$ is a type for $\mathfrak{s}$.*

Proof. Let V be an irreducible simple representation of $\mathrm{GL}_n(F)$ containing the simple type $(J(\mathfrak{B}), \kappa(\mathfrak{B}) \otimes \sigma)$. Let the supercuspidal support of V be $V_0^{\otimes s}$. Let θ_0 be a maximal simple character in $\mathrm{GL}_{n/s}(F)$ of endo-class Θ_F. Let $\kappa_{\max}^0$ be the β-extension of θ_0 compatible with $\kappa_{\max}$. We need to prove that $r = s$ and that $\Lambda(V_0, \Theta_E, \kappa_{\max}^0) = [\chi]$.

By definition of compatibility, the supercuspidal support of $V(\kappa_{\max})$ is a product of representations corresponding to $\Lambda(V_0, \Theta_E, \kappa_{\max}^0)$ under Deligne–Lusztig induction. On the other hand, by [Schneider and Zink 1999, Proposition 5.3], the supercuspidal support of $V(\kappa_{\max})$ is $\left[\prod_{i=1}^r \mathrm{GL}_{n/r\delta(\Theta_F)}(\boldsymbol{e}), \sigma_0^{\otimes r}\right]$. The lemma follows by uniqueness of supercuspidal support. □

Write $\mathfrak{s}$ for the inertial class in Lemma 3.3. We are going to define two classes of virtual representations of $\boldsymbol{K}$ attached to $\mathfrak{s}$, depending only on the maximal simple character θ. Recall that we assume $J_\theta \subset \boldsymbol{K}$. If $\mathfrak{P}$ is a partition of r, the constructions of Section 2 provide us a representation $\kappa_{\max} \otimes \sigma_{\mathfrak{P}}(\sigma_0)$ of J_θ, via $J_\theta / J_\theta^1 \cong \mathrm{GL}_{n/\delta(\Theta_F)}(\boldsymbol{e})$.

Definition 3.4. Write $\sigma_{\mathfrak{P}}(\mathfrak{s}) = \mathrm{Ind}_{J_\theta}^{\boldsymbol{K}}(\kappa_{\max} \otimes \sigma_{\mathfrak{P}}(\sigma_0))$, which by the discussion at the end of [Schneider and Zink 1999, Section 5] is an irreducible smooth representation of $\boldsymbol{K}$. Write $\sigma_{\mathfrak{P}}^{+}(\mathfrak{s})$ for the virtual representation $\mathrm{Ind}_{J_\theta}^{\boldsymbol{K}}(\kappa_{\max} \otimes \sigma_{\mathfrak{P}}^{+}(\sigma_0))$ of $\boldsymbol{K}$. We will refer to these representations as *$\boldsymbol{K}$-types* for $\mathfrak{s}$.

Next we show how the $\boldsymbol{K}$-types provide a refinement of Bushnell–Kutzko type theory for generic representations. Via the Bernstein–Zelevinsky classification, we can attach to each irreducible representation $V \in \mathrm{Irr}(\mathfrak{s})$ a partition $\mathfrak{P}(V)$ of r, in the following way.

Definition 3.5. Let $V \in \mathrm{Irr}(\mathfrak{s})$. We define $\mathfrak{P}(V)(i)$ to be the number of times a segment of length i appears in the multiset corresponding to V. We will sometimes shorten notation to $\mathfrak{P} = \mathfrak{P}(V)$.

Proposition 3.6. *Let $V \in \operatorname{Irr} \mathrm{GL}_n(F)$ be generic. Let $\mathfrak{P}$ be a partition of r. Then*

$$\operatorname{Hom}_{J_\theta}(\kappa_{\max} \otimes \sigma_{\mathfrak{P}}(\sigma_0), V) \neq 0$$

if and only if $V \in \mathfrak{s}$ and its partition $\mathfrak{P}(V)$ satisfies $\mathfrak{P} \leq \mathfrak{P}(V)$.

Proof. By [Schneider and Zink 1999, Lemma 5.2], the nonvanishing implies that $V \in \operatorname{Irr}(\mathfrak{s})$. By [Schneider and Zink 1999, Proposition 5.9] we have that $V(\kappa_{\max}) \cong \pi_{\mathfrak{P}(V)}(\sigma_0)$ whenever $V \in \operatorname{Irr}(\mathfrak{s})$ is generic. Then the claim follows from the existence of an isomorphism

$$\operatorname{Hom}_{J_\theta}(\kappa_{\max} \otimes \sigma_{\mathfrak{P}}(\sigma_0), V) \xrightarrow{\sim} \operatorname{Hom}_{\mathrm{GL}_{n/\delta(\Theta_F)}(\boldsymbol{e})}(\sigma_{\mathfrak{P}}(\sigma_0), V(\kappa_{\max})). \qquad \square$$

Remark 3.7. By [Schneider and Zink 1999, Lemma 5.2], the fact that

$$V \in \operatorname{Irr}(\mathfrak{s}) \quad \text{if } \operatorname{Hom}_{J_\theta}(\kappa_{\max} \otimes \sigma_{\mathfrak{P}}(\sigma_0), V) \neq 0$$

holds for any $V \in \operatorname{Irr} \mathrm{GL}_n(F)$, with no genericity assumptions.

Example 3.8. Let V be irreducible and generic. We have $(\sigma^+_{\mathfrak{P}_{\min}}(\mathfrak{s}), V)_{\boldsymbol{K}} \neq 0$ if and only if $V \in \operatorname{Irr}(\mathfrak{s})$ and $V(\kappa_{\max}) \cong \pi_{\mathfrak{P}_{\min}}(\sigma_0)$, in which case it equals one. This happens if and only if $\mathfrak{P}(V) = \mathfrak{P}_{\min}$, that is the multiset of V has only one segment (because $\mathfrak{P}_{\min}$ is the partition with only one summand). Equivalently, V is an essentially square-integrable representation in $\mathfrak{s}$.

Finally, we remove the dependence of the $\boldsymbol{K}$-types on θ.

Proposition 3.9. *The representations $\sigma_{\mathfrak{P}}(\mathfrak{s})$ and $\sigma^+_{\mathfrak{P}}(\mathfrak{s})$ are independent of the choice of θ.*

Proof. Let θ_1 and θ_2 be conjugate maximal simple characters in $\mathrm{GL}_n(F)$, with $J_{\theta_i} \subseteq \boldsymbol{K}$. The orders $\mathfrak{A}_i$ attached to the θ_i are principal orders with the same ramification, corresponding to lattice chains that contain the lattice chain defined by $\boldsymbol{K}$ (because $\mathfrak{A}_i$ is the continuation of $\mathfrak{A}_i \cap B_i$ and $(\mathfrak{A}_i \cap B_i)^\times \subseteq J_{\theta_i} \subseteq \boldsymbol{K}$). Hence the $\mathfrak{A}_i$ are $\boldsymbol{K}$-conjugate. Since intertwining maximal simple characters defined on the same order are conjugate under the group of units of that order (see [Bushnell and Kutzko 1993, Theorem 3.5.11]), we see that the θ_i are conjugate under $\boldsymbol{K}$; hence so are the J_{θ_i}. Write $J_{\theta_2} = \operatorname{ad}(g) J_{\theta_1}$. Since the lift $\Theta_E \to \Theta_F$ is fixed, by the proof of [Dotto 2022, Proposition 3.9] the inner conjugacy classes $[\psi_i] : J_{\theta_i}/J^1_{\theta_i} \to \mathrm{GL}_{n/\delta(\Theta_F)}(\boldsymbol{e})$ satisfy $[\psi_1] = \operatorname{ad}(g)^*[\psi_2]$. It follows that we get isomorphic representations when inducing. $\square$

So there are well-defined representations $\sigma_{\mathfrak{P}}(\mathfrak{s})$ and $\sigma^+_{\mathfrak{P}}(\mathfrak{s})$ of $\boldsymbol{K}$ for every simple Bernstein component $\mathfrak{s}$ of $\mathrm{GL}_n(F)$. By Remark 3.7, these are *typical* representations: each of them determines the Bernstein component of an irreducible representations of $\mathrm{GL}_n(F)$ that contains it. We do not claim that these are the only typical representations of $\boldsymbol{K}$, although some variant of this statement (assuming $p > n$) is expected to hold, as in [Emerton and Gee 2014, Conjecture 4.1.3]. This is closely related to the problem of "uniqueness of types", for which see [Paskunas 2005; Breuil and Mézard 2002, Annexe A].

***K*-types for $D^\times$.** The group $D^\times$ has a unique maximal compact subgroup $\mathcal{O}_D^\times$. Let $(J_\theta, \lambda = \kappa_\theta \otimes \chi)$ be a maximal simple type in $D^\times$, so that $J_\theta \subseteq \mathcal{O}_D^\times$. Fix a simple stratum $[\mathcal{O}_D, \beta]$ for θ and a uniformizer $\pi_{D'}$ of the central division algebra $D' = Z_D(F[\beta])$ over $F[\beta]$. Then the normalizer $\boldsymbol{J}(\theta)$ of θ in $D^\times$ is $\pi_{D'}^{\mathbb{Z}} \ltimes J_\theta = (D')^\times J_\theta^1$, and the normalizer $\boldsymbol{J}(\lambda)$ of λ in $D^\times$ has index in $\boldsymbol{J}(\theta)$ equal to the size $b(\chi)$ of the orbit of χ under $\mathrm{Gal}(\boldsymbol{e}_{n/\delta(\Theta_F)}/\boldsymbol{e})$ (see for instance [Mínguez and Sécherre 2014, 3.4]).

By [Bushnell and Henniart 2011, Proposition 2.6.1], the $D^\times$-intertwining set of (J_θ, λ) coincides with $\boldsymbol{J}(\lambda)$, which intersects $\mathcal{O}_D^\times$ in J_θ. It follows that the intertwining set of λ in $\mathcal{O}_D^\times$ is J_θ and that $\mathrm{Ind}_{J_\theta}^{\mathcal{O}_D^\times} \lambda$ is irreducible. By Frobenius reciprocity, it is a type for the Bernstein component corresponding to (J_θ, λ). We will refer to these representations as ***K*-types** for $D^\times$.

Another construction of $\boldsymbol{K}$-types in this context can be given as follows. A smooth irreducible representation π of D restricts to a semisimple representation of $\mathcal{O}_D^\times$, whose irreducible constituents form a unique orbit under conjugation by a uniformizer Π_D of $D^\times$. By [Roche 2009, Remark 1.6.1.3], each constituent occurs with multiplicity one. If τ is another smooth irreducible representation of $D^\times$, it follows that $\mathrm{Hom}_{\mathcal{O}_D^\times}(\pi, \tau)$ is nonzero if and only if $\pi|_{\mathcal{O}_{D^\times}}$ and $\tau|_{\mathcal{O}_{D^\times}}$ are isomorphic, and this is equivalent to π and τ being unramified twists of each other. It follows that any irreducible constituent of $\pi|_{\mathcal{O}_D^\times}$ is a type for the inertial class of π. This is the construction used in [Gee and Geraghty 2015] when $n = 2$.

In contrast with the case of $\mathrm{GL}_n(F)$ (see [Paskunas 2005]), the $\boldsymbol{K}$-type of a supercuspidal representation need not be unique: by [Roche 2009, Lemma 1.6.3.1], the number of constituents of $\pi|_{\mathcal{O}_D^\times}$ equals the *torsion number* of π, which is the number of unramified characters χ of $D^\times$ such that $\chi\pi \cong \pi$.

*Trace formulas for **K**-types.* A conjugacy class in a profinite group G is *pro-p-regular* if its elements are p-regular in all finite discrete quotients of G (that is, their order is coprime to p).

Lemma 3.10. *If G is a profinite group, H is a finite group, and $\pi : G \to H$ is a continuous surjection with pro-p kernel, then π induces a bijection from the pro-p-regular classes of G to the p-regular classes of H.*

Proof. Assume that the claim is true when G is a finite group. Then every p-regular element $h \in H$ has a p-regular lift in every finite discrete quotient of G surjecting onto H. Since a directed inverse limit of nonempty finite sets is nonempty, we find that h has a pro-p-regular lift in G, and so the map induced by π is surjective. To prove it is injective, let g_1 and g_2 be pro-p-regular elements of G that are conjugate in every finite discrete quotient $\overline{G}$ of G. Then the finite set of elements of $\overline{G}$ conjugating $\bar{g}_1$ to $\bar{g}_2$ is not empty. Taking the inverse limit as $\overline{G}$ varies of these finite sets, we find an element of G that conjugates g_1 to g_2. So the map induced by π is injective.

It follows that it suffices to prove the claim for G a finite group. In this case, the surjectivity of π on p-regular classes follows because every p-regular element $\pi(x)$ of H admits a p-regular lift, since if $x = x_{(p)} x^{(p)}$ then the images of these under π commute; hence $\pi(x_{(p)}) = \pi(x)_{(p)} = 1$ and $\pi(x^{(p)}) = \pi(x)$. Then the claim follows because G and H have the same number of p-regular classes, since every irreducible $\overline{\mathbb{F}}_p$-representation of G is trivial on $\ker(\pi)$, and the number of irreducible $\overline{\mathbb{F}}_p$-representations of a finite group equals the number of its p-regular conjugacy classes [Serre 1977, Section 18.2, Corollary 3]. □

We now go back to the situation where $G = \mathrm{GL}_n(F)$ for a finite extension $F/\mathbb{Q}_p$. Fix a maximal simple character θ of endo-class Θ_F and a character $\chi : \boldsymbol{e}^\times_{n/\delta(\Theta_F)} \to \mathbb{C}^\times$, possibly not $\boldsymbol{e}$-regular. Let $[\mathfrak{A}, \beta]$ be a simple stratum for θ and write $B = Z_A(F[\beta])$ and $\mathfrak{B} = \mathfrak{A} \cap B$. Let $\mathfrak{s}_G$ be the unique simple inertial class in G with invariants

$$\mathrm{cl}(\mathfrak{s}_G) = \Theta_F \quad \text{and} \quad \Lambda(\mathfrak{s}_G, \Theta_E, \kappa_\theta) = [\chi].$$

As in the above, we assume that $J_\theta \subseteq \boldsymbol{K}$, we fix a maximal unramified extension K^+ of $F[\beta]$ in $Z_A(F[\beta])$ normalizing $\mathfrak{A}$, and we let K be the maximal unramified extension of F in K^+. We remark that the unit group $K^\times$ normalizes $\boldsymbol{K}$: this is because $\mathcal{O}_K \subseteq \mathcal{O}_{K^+} \subseteq \mathfrak{B}$, and $K^\times = \pi_F^{\mathbb{Z}} \times \mathcal{O}_K^\times$ for a uniformizer π_F of F, which is central in G.

By Theorem 2.10, the $\boldsymbol{K}$-type $\sigma^+_{\mathfrak{P}_{\min}}(\mathfrak{s}_G)$ is $\mathrm{Ind}_{J_\theta}^{\boldsymbol{K}}(\kappa_\theta \otimes (-1)^{n/\delta(\Theta_F)+1} R_w(\chi))$, which is a virtual representation of $\boldsymbol{K}$ if χ is not $\boldsymbol{e}$-regular. When χ is $\boldsymbol{e}$-regular, this is a maximal simple type and $\mathfrak{s}_G$ is a supercuspidal inertial class. We will shorten notation to $\lambda = \kappa_\theta \otimes (-1)^{n/\delta(\Theta_F)+1} R_w(\chi)$ and $\sigma^+ = \sigma^+_{\mathfrak{P}_{\min}}(\mathfrak{s}_G)$.

Proposition 3.11. *If $x \in \boldsymbol{K}$ is a pro-p-regular element that is not $\boldsymbol{K}$-conjugate to an element of μ_K then* $\mathrm{tr}\,\sigma^+(x) = 0$. (*Recall that μ_K is the group of prime-to-p roots of unity in $K^\times$.*)

Proof. By the Frobenius formula for an induced character we have

$$\mathrm{tr}\,\sigma^+(x) = \sum_{y \in J_\theta \backslash \boldsymbol{K}} \mathrm{tr}\,\lambda(yxy^{-1}).$$

By Lemma 3.10, the pro-p-regular conjugacy classes of J_θ are in bijection with the semisimple conjugacy classes of $\mathrm{GL}_{n/\delta(\Theta_F)}(\boldsymbol{e})$, via our isomorphism $J_\theta/J^1_\theta \to \mathrm{GL}_{n/\delta(\Theta_F)}(\boldsymbol{e})$. Now the claim follows by Proposition 2.6, as $R_w(\chi)$ vanishes on semisimple conjugacy classes that are not represented in a maximal elliptic torus, and $\mu_K = \mu_{K^+}$ maps isomorphically to such a torus. □

We now give a formula for $\mathrm{tr}\,\sigma^+(x)$ when $x \in \mu_K$ generates an unramified extension L/F (in the sense that $L = F[x]$). For this, we take the interior lift θ_L, and notice the decomposition

$$J_{\theta_L} = \mathrm{GL}_{n/[L[\beta]:F]}(\mathcal{O}_{L[\beta]}) J^1_{\theta_L}.$$

We write $G_L = Z_G(L)$ and notice the equality $Z_{\boldsymbol{K}}(L) = \boldsymbol{K} \cap G_L$. Since $K^\times$ normalizes $\boldsymbol{K}$, this is a maximal compact subgroup of G_L that we denote $\boldsymbol{K}_L$.

We are going to apply the Glauberman correspondence in the form stated in [Bushnell and Henniart 2010; Dotto 2022]. Let $\tilde{\eta}(\theta)$ be the only extension of η_θ to $\mu_K \ltimes J^1_\theta$ whose determinant is trivial on μ_K. Then $\tilde{\eta}(\theta)$ is isomorphic to the restriction of $\epsilon^1_\theta \kappa_\theta$ to $\mu_K \ltimes J^1_\theta$, since $\epsilon^1_\theta \kappa_\theta$ is the p-primary β-extension of θ. It follows from [Dotto 2022, Proposition 2.13(2, 3)] that for a certain function $\epsilon_\theta : \mu_K \to \{\pm 1\}$ one has

$$\mathrm{tr}\,\kappa_\theta(x) = \epsilon^1_\theta(x)\epsilon_\theta(x) \dim \eta_{\theta_L}.$$

It follows from this and [Dotto 2022, Proposition 2.14] that if $x \in \mu_K$ and $L = F[x]$ then

$$\mathrm{tr}\,\kappa_\theta(x) = \epsilon^0_{\mu_K}(V_{\theta_L})\epsilon^0_{\mu_K}(V_\theta) \dim \eta_{\theta_L}, \tag{3-1}$$

where the ϵ^0 are signs, and the ϵ^1 are quadratic characters of μ_K. (We remark that by definition $\epsilon^1_\theta = \epsilon^1_{\mu_K}(-, V_\theta)$ in the notation of [Dotto 2022]: compare the statement of [Dotto 2022, Theorem 4.10].)

Now consider the pair $(J_{\theta_L}, \lambda_L = \kappa_{\theta_L} \otimes (-1)^{n/[L[\beta]:F]+1} R_w(\chi))$ where the Lusztig induction is taken from $\boldsymbol{e}^{\times}_{n/\delta(\Theta_F)}$ to the centralizer of the image of x in $\mathrm{GL}_{n/[F[\beta]:F]}(\boldsymbol{e})$, which is the group $\mathrm{GL}_{n/[L[\beta]:F]}(\boldsymbol{e}[x])$. When χ is $\boldsymbol{e}[x]$-regular, this is a maximal simple type in G_L. The corresponding $\boldsymbol{K}_L$-type $\sigma_L^+ = \mathrm{Ind}_{J_{\theta_L}}^{\boldsymbol{K}_L} \lambda_L$ has dimension equal to

$$\dim(\sigma_L^+) = \dim \eta_{\theta_L} |J_{\theta_L} \backslash \boldsymbol{K}_L| (\mathrm{GL}_{n/[L[\beta]:F]}(\boldsymbol{e}[x]) : \boldsymbol{e}^{\times}_{n/\delta(\Theta_F)})_{p'}. \tag{3-2}$$

Remark 3.12. The dimension of a virtual representation is the value of its character at the identity. In our case, it is independent of χ.

Proposition 3.13. *Let $x \in \mu_K$ and let $L = F[x]$. Then*

$$\mathrm{tr}\, \sigma^+(x) = (-1)^{n/[F[\beta]:F]+n/[L[\beta]:F]} \epsilon^0_{\mu_K}(V_{\theta_L}) \epsilon^0_{\mu_K}(V_\theta) \dim(\sigma_L^+) \sum_{\gamma \in \mathrm{Gal}(L/F)} \chi(\gamma x).$$

Proof. We begin with the Frobenius formula

$$\mathrm{tr}\, \sigma^+(x) = \sum_{J_\theta \backslash \boldsymbol{K}} \mathrm{tr}\, \lambda(yxy^{-1}) = \sum_{J_\theta \backslash \boldsymbol{K}} (-1)^{n/[F[\beta]:F]+1} R_w(\chi)(yxy^{-1})\, \mathrm{tr}\, \kappa_\theta(yxy^{-1})$$

and the remark that if $y \in \boldsymbol{K}$ and $\mathrm{tr}\, \lambda(yxy^{-1}) \neq 0$ then there exists an element of J_θ conjugating yxy^{-1} to an element of μ_K. Indeed, by Lemma 3.10 the pro-p-regular classes of J_θ are in bijection with those of $\mathrm{GL}_{n/[F[\beta]:F]}(\boldsymbol{e})$, and by Proposition 3.11 the only ones on which $R_w(\chi)$ is nonzero are those represented in μ_K. It follows that $J_\theta y = J_\theta \tilde{y}$ for some $\tilde{y} \in N_{\boldsymbol{K}}(L)$ (this is the normalizer of L for the conjugation action of $\boldsymbol{K}$). We have an isomorphism $N_{\boldsymbol{K}}(L)/\boldsymbol{K}_L \to \mathrm{Gal}(L/F)$, and the intersection $N_{\boldsymbol{K}}(L) \cap J_\theta$ maps onto $\mathrm{Gal}(L/L \cap F[\beta])$. (To see this, notice that if $x \in J_\theta = \mathfrak{B}^{\times}_{\max} J^1_\theta$ normalizes μ_L then its image $\bar{x}$ in $\mathrm{GL}_{n/[F[\beta]:F]}(\boldsymbol{e})$ normalizes the image of μ_L, and the automorphism of μ_L induced by $\mathrm{ad}\, \bar{x}$ determines that induced by $\mathrm{ad}\, x$, and hence is also induced by an element of $\mathfrak{B}^{\times}_{\max} \cong \mathrm{GL}_{n/[F[\beta]:F]}(\mathcal{O}_{F[\beta]})$.)

It follows that the space $J_\theta N_{\boldsymbol{K}}(L)$ decomposes into double cosets

$$J_\theta N_{\boldsymbol{K}}(L) = \bigcup_{\sigma \in \mathrm{Gal}(L/F)} J_\theta t_\sigma \boldsymbol{K}_L,$$

where $t_\sigma \in \boldsymbol{K}$ induces σ on L by conjugation, and $J_\theta t_\sigma \boldsymbol{K}_L = J_\theta t_\tau \boldsymbol{K}_L$ if and only if $\tau\sigma^{-1} \in \mathrm{Gal}(L/L \cap F[\beta])$. Since $J_\theta t_\sigma \boldsymbol{K}_L = J_\theta \boldsymbol{K}_L t_\sigma$, we deduce that

$$\mathrm{tr}\, \sigma^+(x) = [L : F[\beta] \cap L]^{-1} \sum_{\gamma \in \mathrm{Gal}(L/F)} |J_{\theta_L} \backslash \boldsymbol{K}_L|\, \mathrm{tr}\, \lambda(\gamma x).$$

Recalling (3-1), this is equal to

$$\mathrm{tr}\, \sigma^+(x) = [L : F[\beta] \cap L]^{-1} \epsilon^0_{\mu_K}(V_{\theta_L}) \epsilon^0_{\mu_K}(V_\theta) \dim \eta_{\theta_L} \sum_{\gamma \in \mathrm{Gal}(L/F)} |J_{\theta_L} \backslash \boldsymbol{K}_L| (-1)^{n/[F[\beta]:F]+1} R_w(\chi)(\gamma x).$$

By Proposition 2.6, and the fact that $[F[\beta] : F] = \delta(\Theta_F)$, we have

$$(-1)^{n/[F[\beta]:F]+1} R_w(\chi)(x) = (-1)^{n/[F[\beta]:F]+n/[L[\beta]:F]} ((\mathrm{GL}_{n/[L[\beta]:F]}(\boldsymbol{e}[x])) : \boldsymbol{e}^{\times}_{n/\delta(\Theta_F)})_{p'} \sum_{\alpha \in \mathrm{Gal}(\boldsymbol{e}[x]/\boldsymbol{e})} \chi(\alpha x).$$

Recall that $\boldsymbol{e}$ is isomorphic to the residue field of $F[\beta]$, and since $L[\beta]/F[\beta]$ is an unramified extension generated by x, $\boldsymbol{e}[x]$ is isomorphic to the residue field of $L[\beta]$. The restriction map is an isomorphism

$$\mathrm{res}:\mathrm{Gal}(L[\beta]/F[\beta])\to\mathrm{Gal}(L/F[\beta]\cap L)$$

and it follows that

$$[L:F[\beta]\cap L]^{-1}\sum_{\gamma\in\mathrm{Gal}(L/F)}\sum_{\alpha\in\mathrm{Gal}(\boldsymbol{e}[x]/\boldsymbol{e})}\chi(\alpha\gamma x)=\sum_{\gamma\in\mathrm{Gal}(L/F)}\chi(\gamma x).$$

The claim now follows by (3-2). □

We end this section by proving an analogous result for $D^\times$. Let $(J_\theta,\lambda=\kappa_\theta\otimes\chi)$ be a maximal simple type in $D^\times$ and let $\sigma_D^+=\mathrm{Ind}_{J_\theta}^{\mathcal{O}_D^\times}\lambda$ be the associated $\boldsymbol{K}$-type. Fix a simple stratum $[\mathcal{O}_D,\beta]$ defining θ and fix a maximal unramified extension K^+ of $F[\beta]$ in $D'=Z_D(F[\beta])$. Let $x\in\mu_{K^+}$ generate an extension L/F. Let θ_L be the interior lift of θ to L, and write λ_L for any maximal simple type in $Z_{D^\times}(L)$ with maximal simple character θ_L. Write $\sigma_L^+=\mathrm{Ind}_{J_{\theta_L}}^{Z_{\mathcal{O}_D^\times}(L)}\lambda_L$ for the corresponding $\boldsymbol{K}_L$-type.

Proposition 3.14. *We have an equality*

$$\mathrm{tr}\,\sigma_D^+(x)=\epsilon^0_{\mu_K}(V_\theta)\epsilon^0_{\mu_K}(V_{\theta_L})\dim(\sigma_L^+)\chi(x).$$

Proof. If $y\in\mathcal{O}_D^\times$ and $yxy^{-1}\in J_\theta=\mathcal{O}_{D'}^\times J_\theta^1$, then yxy^{-1} is J_θ-conjugate to an element of μ_{K^+}, because μ_{K^+} represents the pro-p-regular conjugacy classes in J_θ by Lemma 3.10. Again by Lemma 3.10, elements of μ_{K^+} are pairwise nonconjugate in $\mathcal{O}_D^\times$. So $J_\theta y=J_\theta\tilde{y}$ for some $\tilde{y}\in Z_{\mathcal{O}_D^\times}(L)$, and we deduce that

$$\mathrm{tr}\,\sigma_D^+(x)=|J_{\theta_L}\backslash Z_{\mathcal{O}_D^\times}(L)|\,\mathrm{tr}\,\lambda(x).$$

By the same argument as for $\mathrm{GL}_n(F)$, we know that

$$\epsilon^1_\theta(x)\,\mathrm{tr}\,\kappa_\theta(x)=\epsilon_\theta(x)\dim\eta_{\theta_L}=\epsilon_\theta(x)\dim(\lambda_L);$$

hence the claim follows from [Dotto 2022, Proposition 2.14]. □

The formal degree formula. Let $\mathfrak{s}$ be a supercuspidal inertial class for $\mathrm{GL}_n(F)$, and let $\mathfrak{s}_D=\mathrm{JL}^{-1}(\mathfrak{s})$ be its Jacquet–Langlands transfer to $D^\times$. We give a relation between the dimension of a $\boldsymbol{K}$-type σ_D^+ for $\mathfrak{s}_D$ and the dimension of a $\boldsymbol{K}$-type σ^+ for $\mathfrak{s}$. We assume that σ_D and σ have been constructed as in the above. Write $q=|\boldsymbol{f}|$ and $t(\mathfrak{s})=t(\mathfrak{s}_D)$ for the torsion numbers of the inertial classes. Normalize the formal degrees for $\mathrm{GL}_n(F)$ so that the Steinberg representation has formal degree one, and let $\mathrm{Iw}\subseteq\boldsymbol{K}$ be an Iwahori subgroup.

Theorem 3.15 [Bushnell and Henniart 2004, (1.4.1)]. *The formal degree of any irreducible representation containing a maximal simple type (J_θ,λ) corresponding to $\mathfrak{s}$ is*

$$d(\pi)=t(\mathfrak{s})\dim(\lambda)\frac{q^n-1}{(q-1)^n}\frac{\mu_G(\mathrm{Iw})}{\mu_G(J_\theta)}$$

for any Haar measure μ_G on G.

Proposition 3.16. *We have an equality* $\dim(\sigma^+) = (\mathrm{GL}_n(\boldsymbol{f}) : \boldsymbol{f}_n^\times)_{p'} \dim(\sigma_D^+)$.

Proof. Multiplying numerator and denominator of the equation in Theorem 3.15 by $\mu_G(\boldsymbol{K})^{-1}$ yields

$$d(\pi) = t(\mathfrak{s}) \dim(\sigma^+)(\mathrm{GL}_n(\boldsymbol{f}) : \boldsymbol{f}_n^\times)_{p'}^{-1}$$

because

$$\dim(\sigma^+) = \dim(\lambda)(\boldsymbol{K} : J_\theta) \quad \text{and} \quad (\boldsymbol{K} : \mathrm{Iw}) = \frac{(q^n - 1) \cdots (q^n - (q^{n-1}))}{q^{n(n-1)/2}(q-1)^n}.$$

We have seen that any irreducible representation in $\mathfrak{s}_D$ restricts to $\mathcal{O}_D^\times$ to a sum of $t(\mathfrak{s}_D)$ representations each appearing with multiplicity one and all conjugate under $D^\times$, which are precisely the $\boldsymbol{K}$-types for $\mathfrak{s}_D$. Since $d(\pi) = \dim(\mathrm{JL}^{-1}\pi)$, we deduce that

$$t(\mathfrak{s}_D) \dim(\sigma_D^+) = t(\mathfrak{s}) \dim(\sigma^+)(\mathrm{GL}_n(\boldsymbol{f}) : \boldsymbol{f}_n^\times)_{p'}^{-1}$$

and the claim follows since $t(\mathfrak{s}_D) = t(\mathfrak{s})$, as the Jacquet–Langlands correspondence commutes with character twists. □

4. Galois deformation theory

Working in the framework of [Emerton and Gee 2014, Section 4], we recall the definition of potentially semistable deformation rings of fixed Hodge type and discrete series Galois type, and prove some properties of the monodromy stratification. We state a form of the geometric Breuil–Mézard conjecture for the mod p fibres of these rings, and deduce a description of the cycle corresponding to discrete series lifts. In this section, we fix p-adic coefficients consisting of a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$, uniformizer π_E, and residue field $\boldsymbol{e}$. We let $\bar{\rho} : G_F \to \mathrm{GL}_n(\boldsymbol{e})$ be a continuous representation, and we assume that E is sufficiently large (so that, for instance, it contains all $[F : \mathbb{Q}_p]$ embeddings of F).

Weights and algebraic representations. Write $\mathbb{Z}_+^n$ for the set of n-tuples $(\lambda_1, \ldots, \lambda_n)$ of integers such that $\lambda_1 \geq \ldots \geq \lambda_n$. This defines a dominant character $\mathrm{diag}(t_1, \ldots, t_n) \mapsto \prod_{i=1}^n t_i^{\lambda_i}$ of the diagonal torus in $\mathrm{GL}_{n,F}$. There is an associated algebraic $\mathcal{O}_F$-representation $M_\lambda' = \mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_{\max}\lambda)_{/\mathcal{O}_F}$ of $\mathrm{GL}_{n,\mathcal{O}_F}$ with highest weight λ, for the upper-triangular Borel subgroup B_n and the longest element $w_{\max}$ of the Weyl group. We write M_λ for the $\mathcal{O}_F$-points of this representation. Then fix $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}_{\mathbb{Q}_p}(F,E)}$ and define an $\mathcal{O}_E$-representation of $\mathrm{GL}_n(\mathcal{O}_F)$ by

$$L_\lambda = \bigotimes_{\tau : F \to E} (M_{\lambda_\tau} \otimes_{\mathcal{O}_F, \tau} \mathcal{O}_E).$$

Next we recall some mod p representations. Given $a \in \mathbb{Z}_+^n$ with $p - 1 \geq a_i - a_{i+1}$ for all $1 \leq i \leq n-1$, define

$$P_a = \mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_{\max}a)_{/\boldsymbol{f}}(\boldsymbol{f})$$

and let N_a be the irreducible subrepresentation of P_a generated by a highest weight vector. The *Serre weights* of $\mathrm{GL}_n(\boldsymbol{f})$ are the elements $a \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(\boldsymbol{f},\boldsymbol{e})}$ such that for all $\sigma : \boldsymbol{f} \to \boldsymbol{e}$ we have $p-1 \geq a_{\sigma,i} - a_{\sigma,i+1}$

for $1 \le i \le n-1$, and $0 \le a_{\sigma,n} \le p-1$. We furthermore require that not all $a_{\sigma,n} = p-1$. To a Serre weight there corresponds an irreducible $\boldsymbol{e}$-representation of $\mathrm{GL}_n(\boldsymbol{f})$, defined by

$$F_a = \bigotimes_{\tau \in \mathrm{Hom}(\boldsymbol{f},\boldsymbol{e})} (N_{a_\tau} \otimes_{\boldsymbol{f},\tau} \boldsymbol{e}).$$

These are absolutely irreducible and pairwise nonisomorphic, and every irreducible $\boldsymbol{e}$-representation of $\mathrm{GL}_n(\boldsymbol{f})$ has this form.

Finally, we introduce analogues for $D^\times$. For every $\mathbb{Q}_p$-linear embedding $\tau : F \to E$, fix an embedding $\tau^+ : F_n \to E$ lifting τ and write M_λ^+ for the $\mathcal{O}_{F_n}$-points of M'_λ (so that $M_\lambda^+|_{\mathrm{GL}_n(\mathcal{O}_F)}$ is isomorphic to $M_\lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{F_n}$). Then we introduce

$$L_\lambda^+ = \bigotimes_{\tau : F \to E} (M_{\lambda_\tau}^+ \otimes_{\mathcal{O}_{F_n},\tau^+} \mathcal{O}_E),$$

which has an action of $\mathcal{O}_D^\times$ via a choice of F_n-linear isomorphism $j : D \otimes_F F_n \to M_n(F_n)$ mapping the order $\mathcal{O}_D \otimes_{\mathcal{O}_F} \mathcal{O}_{F_n}$ into $M_n(\mathcal{O}_{F_n})$, and the inclusion $D \to D \otimes_F F_n$, $d \mapsto d \otimes 1$. We have the following lemma.

Lemma 4.1. *If $z_D \in \mathcal{O}_D^\times$ corresponds to $z \in \mathrm{GL}_n(\mathcal{O}_F)$, in the sense that it is a semisimple element with the same characteristic polynomial as z, then $\mathrm{tr}_{L_\lambda^+}(z_D) = \mathrm{tr}_{L_\lambda}(z)$.*

Proof. This is because L_λ^+ is a lattice in a $\mathrm{GL}_n(F_n)$-representation over E, $\mathrm{tr}_{L_\lambda}(z) = \mathrm{tr}_{L_\lambda^+}(z)$, and z and z_D are conjugate in $\mathrm{GL}_n(F_n)$ under any choice of j. □

Inertial types and monodromy. An *inertial type* for F is a smooth finite-dimensional representation of I_F that extends to a representation of W_F. The type is a *supercuspidal type* if it extends to an irreducible representation of W_F, and a *discrete series type* if it is a multiple of a supercuspidal inertial type. Two n-dimensional Weil–Deligne representations have the same restriction to inertia if and only if they are the Langlands parameters of irreducible representations of $\mathrm{GL}_n(F)$ in the same inertial class. It follows that if $\tau = \tau_0^{\oplus r}$ for a supercuspidal inertial type τ_0, there are a corresponding simple inertial class $\mathfrak{s}$ for $\mathrm{GL}_n(F)$ and representations $\sigma_{\mathfrak{P}}(\tau) = \sigma_{\mathfrak{P}}(\mathfrak{s})$ of $\boldsymbol{K}$ indexed by partitions $\mathfrak{P}$ of r. There are also virtual representations $\sigma_{\mathfrak{P}}^+(\tau) = \sigma_{\mathfrak{P}}^+(\mathfrak{s})$. Similarly, we define representations of $\mathcal{O}_D^\times$ by letting $\sigma_D(\tau)$ be an arbitrary choice of $\boldsymbol{K}$-type for $\mathrm{JL}^{-1}(\mathfrak{s})$ (we will prove our results for all possible choices). For $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}_{\mathbb{Q}_p}(F,E)}$ we put $\sigma_{\mathfrak{P}}(\tau,\lambda) = \sigma_{\mathfrak{P}}(\tau) \otimes L_\lambda$, $\sigma_{\mathfrak{P}}^+(\tau,\lambda) = \sigma_{\mathfrak{P}}^+(\tau) \otimes L_\lambda$, and $\sigma_D(\tau,\lambda) = \sigma_D(\tau) \otimes L_\lambda^+$.

Let τ be a discrete series inertial type. Our results will relate $\sigma_{\mathfrak{P}_{\min}}^+(\tau)$ to the locus in the deformation space of $\bar{\rho}$ consisting of discrete series lift of inertial type τ, that is to say Galois representation lifting $\bar{\rho}$ whose associated Weil–Deligne representation is the Langlands parameter of an essentially square integrable representation in $\mathfrak{s}$. Making this precise requires an account of the monodromy operator on the universal deformation ring.

To start with, we recall some commutative algebra. Let A be a commutative ring with 1 and let M be a finite projective A-module of rank n with a nilpotent endomorphism $N : A \to A$. To each prime ideal $x \in \mathrm{Spec}(A)$ we attach a partition $\mathfrak{P}_x$ of n by considering the Jordan canonical form of the nilpotent endomorphism $N(x)$ on $M \otimes_A k(x)$, where $k(x)$ is the residue field at x.

Lemma 4.2. *Each partition* $\mathfrak{P}$ *of* n *defines a closed subset of* $\mathrm{Spec}(A)$

$$\mathrm{Spec}(A)_{\geq \mathfrak{P}} = \{x \in \mathrm{Spec}(A) : \mathfrak{P}_x \geq \mathfrak{P}\}.$$

Proof. See for instance [Pyvovarov 2021, Section 4]. By our definition of $\mathfrak{P}_x \geq \mathfrak{P}$ as the reverse of the dominance partial order on partitions, we find that $\mathfrak{P}_x \geq \mathfrak{P}$ if and only if $\dim(\ker N(x)^i) \geq \dim(\ker N(\mathfrak{P})^i)$ for all i, where $N(\mathfrak{P})$ has Jordan canonical form given by $\mathfrak{P}$. Since $\dim(\ker N(x)^i) = \dim(\mathrm{coker} N(x)^i)$ and $\mathrm{coker} N(x)^i \cong (\mathrm{coker} N^i) \otimes_A k(x)$, the claim follows since the set

$$\{x \in \mathrm{Spec}(A) : \dim_{k(x)}((\mathrm{coker} N^i) \otimes_A k(x)) \geq m\}.$$

is closed for all $m \in \mathbb{Z}$. □

Remark 4.3. It follows that if $\mathrm{Spec}(A)$ is irreducible then the function $x \mapsto \mathfrak{P}_x$ is constant on a dense open subset of $\mathrm{Spec}(A)$, where it attains its minimal value. So we can define subsets $\mathrm{Spec}(A)_{\mathfrak{P}}$ as the union of irreducible components of $\mathrm{Spec}(A)$ where the minimal value of $\mathfrak{P}_x$ is $\mathfrak{P}$ — equivalently, where the monodromy is generically $\mathfrak{P}$.

Potentially semistable deformation rings. Let $\tau : I_F \to \mathrm{GL}_n(E)$ be a discrete series inertial type and $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}_{\mathbb{Q}_p}(F,E)}$. Let L/F be a finite Galois extension such that τ is trivial on I_L. By [Kisin 2008, Theorem 2.7.6] there is a quotient $(R^{\square}_{\bar{\rho}}[1/p])(\tau, \lambda)$ of the generic fibre of the universal lifting $\mathcal{O}_E$-algebra $R^{\square}_{\bar{\rho}}$ whose points in a finite extension E'/E correspond to potentially semistable lifts of $\bar{\rho}$ with Hodge type λ and inertial type τ. By [Kisin 2008, Theorem 2.5.5], there is a finite projective $L_0 \otimes_{\mathbb{Q}_p} (R^{\square}_{\bar{\rho}}[1/p])(\tau, \lambda)$-module $D_{\bar{\rho}}(\tau, \lambda)[1/p]$ with an automorphism φ, semilinear with respect to $\sigma \otimes 1$, and a $L_0 \otimes_{\mathbb{Q}_p} (R^{\square}_{\bar{\rho}}[1/p])(\tau, \lambda)$-linear nilpotent endomorphism N, specializing to $D^*_{\mathrm{st}}(r^{\mathrm{univ}}_x|_{G_L})$ for any $\mathcal{O}_E$-linear ring homomorphism $x : (R^{\square}_{\bar{\rho}}[1/p])(\tau, \lambda) \to E'$. Since $D_{\bar{\rho}}(\tau, \lambda)[1/p]$ is a direct factor of a free $L_0 \otimes_{\mathbb{Q}_p} (R^{\square}_{\bar{\rho}}[1/p])(\tau, \lambda)$-module, it is also projective over $(R^{\square}_{\bar{\rho}}[1/p])(\tau, \lambda)$. By Lemma 4.2 we have a stratification $\mathrm{Spec}(R^{\square}_{\bar{\rho}}[1/p])(\tau, \lambda)_{\geq \mathfrak{P}}$, and $\mathrm{Spec}(R^{\square}_{\bar{\rho}}[1/p])(\tau, \lambda)_{\geq \mathfrak{P}_{\max}}$ corresponds to the vanishing of the monodromy operator, and hence to potentially crystalline deformations of $\bar{\rho}$ (recall that $\mathfrak{P}_{\max}$ is the partition $n = 1 + \cdots + 1$).

We write $(R^{\square}_{\bar{\rho}}[1/p])(\tau, \lambda)_{\mathfrak{P}}$ for the reduced quotient corresponding to the set $\mathrm{Spec}(R^{\square}_{\bar{\rho}}[1/p])(\tau, \lambda)_{\mathfrak{P}}$, and we let $R_{\bar{\rho}}(\tau, \lambda)_{\mathfrak{P}}$ be the image of $R^{\square}_{\bar{\rho}} \to (R^{\square}_{\bar{\rho}}[1/p])(\tau, \lambda)_{\mathfrak{P}}$. This is a reduced π_E-torsion free $\mathcal{O}_E$-algebra whose generic fibre is isomorphic to $(R^{\square}_{\bar{\rho}}[1/p])(\tau, \lambda)_{\mathfrak{P}}$, and its minimal primes have characteristic zero (by $\mathcal{O}_E$-flatness); hence they are in bijection with those of the generic fibre, which are the components where the monodromy is generically $\mathfrak{P}$. (By definition $R\bar{\rho}(\tau, \lambda)_{\mathfrak{P}}$ is the Zariski closure in $R\bar{\rho}^{\square}$ of the set of these components of the generic fibre.) We define $R_{\bar{\rho}}(\tau, \lambda)_{\geq \mathfrak{P}}$ similarly. By [Kisin 2008, Theorem 3.3.4], these rings are equidimensional of the same dimension d.

Cycles. Since the rings $R_{\bar{\rho}}(\tau, \lambda)_{\mathfrak{P}}$ are equidimensional and π_E-torsion free, their special fibres $R_{\bar{\rho}}(\tau, \lambda)_{\mathfrak{P}}/\pi_E$ are also equidimensional, and define a $(d-1)$-cycle on $R^{\square}_{\bar{\rho}}$ by [Breuil and Mézard 2014, Lemma 2.1]. The geometric conjecture in [Emerton and Gee 2014, Section 4.2] states that for each Serre weight a for $\mathrm{GL}_n(f)$ there exists a cycle $\mathcal{C}_a$ on $R^{\square}_{\bar{\rho}}$ such that

$$Z(R_{\bar{\rho}}(\tau, \lambda)_{\mathfrak{P}_{\max}}/\pi_E) = \sum_a n_a \mathcal{C}_a,$$

where the multiplicity n_a is equal to the multiplicity of the representation F_a in $\bar{\sigma}_{\mathfrak{P}_{\max}}(\tau,\lambda)$, the semisimplified mod π_E reduction of $\sigma_{\mathfrak{P}_{\max}}(\tau,\lambda)$. Notice that $R_{\bar{\rho}}(\tau,\lambda)_{\mathfrak{P}_{\max}}$ is a potentially crystalline deformation ring of $\bar{\rho}$. This can be reformulated by defining a group homomorphism

$$\overline{\mathrm{cyc}} : R_{\boldsymbol{e}}(\mathrm{GL}_n(\boldsymbol{f})) \to Z^{d-1}(R_{\bar{\rho}}^{\square}), \quad F_a \mapsto \mathcal{C}_a$$

and one can generalize the statement of the conjecture, and ask whether

$$Z(R_{\bar{\rho}}(\tau,\lambda)_{\geq\mathfrak{P}}/\pi_E) = \overline{\mathrm{cyc}}(\bar{\sigma}_{\mathfrak{P}}(\tau,\lambda)).$$

This is motivated by the fact that $\sigma(\tau)_{\mathfrak{P}}$ is contained in a generic irreducible representation π of $\mathrm{GL}_n(F)$ if and only if the inertial class of π corresponds to τ and the partition $\mathfrak{P}(\pi)$ attached to π satisfies $\mathfrak{P}(\pi) \geq \mathfrak{P}$, that is,

$$\mathrm{Hom}_{\boldsymbol{K}}(\sigma_{\mathfrak{P}}(\tau),\pi) \neq 0 \quad \text{if and only if} \quad \mathrm{rec}(\pi)|_{I_K} \cong \tau \text{ and } \mathfrak{P}(\pi) \geq \mathfrak{P}.$$

Under some assumptions on $\bar{\rho}$, this is true when $F = \mathbb{Q}_p$ and $n = 2$ by [Kisin 2009] or when $n = 2$ and $\lambda = 0$ by [Gee and Kisin 2014]. However, we expect that this statement has to be modified for $n \geq 3$ to account for multiplicities: it is not true in general that $\mathrm{Hom}_{\boldsymbol{K}}(\sigma_{\mathfrak{P}}(\tau),\pi)$ is one-dimensional when it is nonzero. The general statement should be

$$Z(R_{\bar{\rho}}(\tau,\lambda)_{\mathfrak{P}}/\pi_E) = \overline{\mathrm{cyc}}(\bar{\sigma}^{+}_{\mathfrak{P}}(\tau,\lambda)),$$

because of the multiplicities

$$\dim \mathrm{Hom}_{\boldsymbol{K}}(\sigma^{+}_{\mathfrak{P}}(\tau),\pi) = \begin{cases} 1 & \text{if } \mathrm{rec}(\pi)|_{I_K} \cong \tau \text{ and } \mathfrak{P}(\pi) = \mathfrak{P}, \\ 0 & \text{otherwise.} \end{cases}$$

We offer two pieces of evidence towards this. The first is our main result, concerning the case $\mathfrak{P}_{\min}$, which gives a compatibility with the analogous statement on central division algebras. Second, observe that $\dim_E \mathrm{Hom}_{\boldsymbol{K}}(\sigma_{\mathfrak{P}}(\tau),\pi_{\mathfrak{P}'}(\tau))$ equals the Kostka number $K_{\mathfrak{P},\mathfrak{P}'}$, and so we have an equality in the Grothendieck group

$$\bar{\sigma}_{\mathfrak{P}}(\tau,\lambda) = \sum_{\deg\mathfrak{P}'=\deg\mathfrak{P}} K_{\mathfrak{P},\mathfrak{P}'}\bar{\sigma}^{+}_{\mathfrak{P}'}(\tau,\lambda)$$

and

$$\bar{\sigma}^{+}_{\mathfrak{P}}(\tau,\lambda) = \sum_{\deg\mathfrak{P}'=\deg\mathfrak{P}} K^{+}_{\mathfrak{P},\mathfrak{P}'}\bar{\sigma}_{\mathfrak{P}'}(\tau,\lambda),$$

where $(K^{+}_{\mathfrak{P},\mathfrak{P}'})$ is the inverse of the matrix $(K_{\mathfrak{P},\mathfrak{P}'})$ of Kostka numbers. Now [Shotton 2018, Corollary 4.9] says that the direct analogues of our formulas give the right answer for deformation rings with ℓ-adic coefficients, where $\ell \neq p$ is a prime number. This is also consistent with the work of Yao described in the introduction.

5. Jacquet–Langlands transfers

In this section we construct a Jacquet–Langlands transfer of Serre weights from $D^{\times}$ to $\mathrm{GL}_n(F)$, and prove its compatibility with the inertial Jacquet–Langlands correspondence. We also consider analogues for ℓ-adic coefficients when $\ell \neq p$, so we begin by fixing a prime number ℓ (allowing, of course, the case $\ell = p$). We will mostly be interested in proving that our transfer preserves congruences of types. However, we point out that in the case of trivial regular weight we can interpret our result as describing a Jacquet–Langlands correspondence for representations of maximal compact subgroups of $D^\times$ and $\mathrm{GL}_n(F)$. This is because of the following lemma.

Lemma 5.1. *Let R be an algebraically closed field of any characteristic* (*including* char $R = p$) *and let τ be an irreducible smooth R-linear representation of $\mathcal{O}_D^\times$. Then τ occurs in the restriction to $\mathcal{O}_D^\times$ of an irreducible smooth representation of $D^\times$.*

Proof. We regard τ as a representation of $F^\times \mathcal{O}_D^\times$ with π_F acting trivially. As in [Vignéras 2001, Section 4], τ extends to a representation τ' of its normalizer $N = N_{D^\times}(\tau)$, and the induction $\mathrm{Ind}_N^{D^\times}(\tau')$ is an irreducible representation of $D^\times$ containing τ. □

Choosing an isomorphism $\iota_\ell : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, one gets a Jacquet–Langlands transfer from inertial classes of $\overline{\mathbb{Q}}_\ell$-representations of $D^\times$ to inertial classes of $\overline{\mathbb{Q}}_\ell$-representations of $\mathrm{GL}_n(F)$. Because the Harish-Chandra character is compatible with automorphisms of the coefficient field, this transfer is independent of the choice of ι_ℓ [Mínguez and Sécherre 2017, 10.1].

Definition 5.2. Define a map $\mathrm{JL}_{\boldsymbol{K}} : R_{\overline{\mathbb{Q}}_\ell}(\mathcal{O}_D^\times) \to R_{\overline{\mathbb{Q}}_\ell}(\mathrm{GL}_n(\mathcal{O}_F))$ as follows. Let σ_D be an irreducible representation of $\mathcal{O}_D^\times$. Then σ_D is a type for some Bernstein component $\mathfrak{s}_D$ of $D^\times$, by Lemma 5.1, and we let $\mathfrak{s} = \mathrm{JL}(\mathfrak{s}_D)$. We define $\mathrm{JL}_{\boldsymbol{K}}(\sigma_D) = \sigma^+_{\mathfrak{P}_{\min}}(\mathfrak{s})$.

Mod p reduction. Set $\ell = p$. We construct a map

$$\mathrm{JL}_p : R_{\overline{\mathbb{F}}_p}(\mathcal{O}_D^\times) \to R_{\overline{\mathbb{F}}_p}(\mathrm{GL}_n(\mathcal{O}_F))$$

and prove our main result, namely that $\mathrm{JL}_p(\bar{\sigma}_D(\tau, \lambda)) = \bar{\sigma}^+_{\mathfrak{P}_{\min}}(\tau, \lambda)$. Since every irreducible smooth $\overline{\mathbb{F}}_p$-representation of a pro-p group is trivial, it is enough to define a map

$$\mathrm{JL}_p : R_{\overline{\mathbb{F}}_p}(\boldsymbol{d}^\times) \to R_{\overline{\mathbb{F}}_p}(\mathrm{GL}_n(\boldsymbol{f})).$$

We choose any $\boldsymbol{f}$-linear isomorphism $\iota : \boldsymbol{d} \to \boldsymbol{f}_n$ and we define JL_p to be the semisimplified mod p reduction of $\chi \mapsto (-1)^{n+1} R_w(\chi)$, composed with the isomorphism $R_{\overline{\mathbb{F}}_p}(\boldsymbol{f}_n^\times) \to R_{\overline{\mathbb{Q}}_p}(\boldsymbol{f}_n^\times)$. Since R_w is constant on $\mathrm{Gal}(\boldsymbol{f}_n/\boldsymbol{f})$-orbits, this is independent of the choice of ι. Recall the explicit formula in Proposition 2.6, and observe that JL_p is a direct generalization of the construction in Section 2 of [Gee and Geraghty 2015].

For any profinite group G, one defines the Brauer character of a finite-dimensional representation V of G over a finite field $\mathbb{F}_q$ as in the finite group case, obtaining a function $\chi(V)$ on the set of pro-p-regular conjugacy classes of G valued in $\overline{\mathbb{Q}}_p$. From Lemma 3.10, and the corresponding assertion for finite groups, we find that whenever G has an open normal pro-p subgroup the Brauer character induces an isomorphism $R_{\overline{\mathbb{F}}_p}(G) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_p \to \mathcal{C}^{(p)}(G, \overline{\mathbb{Q}}_p)$, where $R_{\overline{\mathbb{F}}_p}(G)$ is the Grothendieck group of finite length smooth representations of G over $\overline{\mathbb{F}}_p$, and the target denotes the space of functions from the set of pro-p-regular classes of G to $\overline{\mathbb{Q}}_p$. We get an induced map

$$\mathrm{JL}_p : \mathcal{C}^{(p)}(\boldsymbol{f}_n^\times, \overline{\mathbb{Q}}_p) \to \mathcal{C}^{(p)}(\mathrm{GL}_n(\boldsymbol{f}), \overline{\mathbb{Q}}_p)$$

such that if $x \in \mathrm{GL}_n(\boldsymbol{f})$ has a conjugate in $\boldsymbol{f}_n^\times$ with degree $\deg(x)$ over $\boldsymbol{f}$ then

$$\mathrm{JL}_p(f)(x) = (-1)^{n+n/\deg(x)} (\mathrm{GL}_{n/\deg(x)}(\boldsymbol{f}_{\deg(x)}) : \boldsymbol{f}_n^\times)_{p'} \sum_{\gamma \in \mathrm{Gal}(\boldsymbol{f}_{\deg(x)}/\boldsymbol{f})} f(\gamma x) \tag{5-1}$$

by Proposition 2.6.

Theorem 5.3. *Let τ be a discrete series inertial type for I_F and $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}_{\mathbb{Q}_p}(F,E)}$. Then*

$$\mathrm{JL}_p(\bar{\sigma}_D(\tau, \lambda)) = \bar{\sigma}^+_{\mathfrak{P}_{\min}}(\tau, \lambda).$$

Proof. We have an equality of Brauer characters

$$\chi(\bar{\sigma}^+_{\mathfrak{P}_{\min}}(\tau, \lambda)) = \chi(\bar{L}_\lambda)\chi(\bar{\sigma}^+_{\mathfrak{P}_{\min}}(\tau)),$$

and similarly

$$\chi(\bar{\sigma}_D(\tau, \lambda)) = \chi(\bar{L}^+_\lambda)\chi(\bar{\sigma}_D(\tau)).$$

The representation $\sigma^+_{\mathfrak{P}_{\min}}(\tau)$ is smooth and defined over a finite extension $E/\mathbb{Q}_p$, so we can compute $\chi(\bar{\sigma}^+_{\mathfrak{P}_{\min}}(\tau))$ as the restriction of the trace of $\sigma^+_{\mathfrak{P}_{\min}}(\tau)$ to p-regular conjugacy classes: this follows from the corresponding statement in the finite group case, via Lemma 3.10. By Proposition 3.11, both $\chi(\bar{\sigma}_D(\tau))$ and $\chi(\bar{\sigma}^+_{\mathfrak{P}_{\min}}(\tau))$ vanish away from certain conjugacy classes represented by roots of unity. If z and z_D are matching p-regular roots of unity, then Lemma 4.1 actually implies that $\chi(\bar{L}_\lambda)(z) = \chi(\bar{L}^+_\lambda)(z_D)$, because the Brauer character of a representation of the finite groups generated by z and z_D can be computed on a lift to characteristic zero. Hence it is enough to prove that $\mathrm{JL}_p(\chi(\bar{\sigma}_D(\tau))) = \chi(\bar{\sigma}^+_{\mathfrak{P}_{\min}}(\tau))$.

Let $\mathfrak{s}(\tau)$ be the simple inertial class corresponding to τ. Let $\Theta_F = \mathrm{cl}(\mathfrak{s}(\tau))$, and recall that we have fixed a lift $\Theta_E \to \Theta_F$ to the unramified parameter field. If θ is a maximal simple character in $\mathrm{GL}_n(F)$ with endo-class Θ_F, this gives rise to a conjugacy class of isomorphisms

$$J_\theta / J^1_\theta \xrightarrow{\sim} \mathrm{GL}_{n/\delta(\Theta_F)}(\boldsymbol{e}).$$

We fix a simple stratum $[\mathfrak{A}, \beta]$ for θ, and a maximal unramified extension $K^+/F[\beta]$ in $Z_A(F[\beta])$ such that $J_\theta \subseteq \boldsymbol{K}$ and the maximal unramified extension K of F in K^+ normalizes the group $\boldsymbol{K}$. Let $[\chi] = \Lambda(\mathfrak{s}(\tau), \Theta_E, \kappa_\theta)$, so that by Proposition 3.9 we have

$$\sigma^+_{\mathfrak{P}_{\min}}(\tau) \cong \mathrm{Ind}_{J_\theta}^{\boldsymbol{K}}(\kappa_\theta \otimes (-1)^{n/\delta(\Theta_F)+1} R_w(\chi)).$$

By the main results of [Dotto 2022], the invariants of $\mathfrak{s}_D(\tau) = \mathrm{JL}^{-1}(\mathfrak{s}(\tau))$ are

$$\mathrm{cl}(\mathfrak{s}_D(\tau)) = \Theta_F \quad \text{and} \quad \Lambda(\mathfrak{s}_D(\tau), \Theta_E, \kappa_{\theta_D}) = [\chi].$$

It follows that we can choose a maximal simple character θ_D in $D^\times$ with $\mathrm{cl}(\theta) = \mathrm{cl}(\theta_D)$ and a simple stratum $[\mathcal{O}_D, \beta_D]$ for θ_D such that $\sigma_D(\tau)$ is isomorphic to the induction $\mathrm{Ind}_{J_{\theta_D}}^{\mathcal{O}_D^\times}(\kappa_{\theta_D} \otimes \chi)$. We fix a maximal unramified extension $K^+_D/F[\beta_D]$ in $Z_D(F[\beta_D])$ and write K_D for the maximal unramified extension of F in K^+_D. Since the Jacquet–Langlands correspondence preserves torsion numbers, we have $[K : F] = [K_D : F]$, and there exists a unique isomorphism $\iota : K_D \to K$ such that the equality of endo-classes $\mathrm{cl}(\theta_{D,K}) = \iota^* \mathrm{cl}(\theta_K)$ holds.

Let $z \in \mu_K$ and $z_D \in \mu_{K_D}$ generate isomorphic extensions of F, which we identify via ι with an unramified extension L/F. By Propositions 3.13 and 3.14 we have equalities

$$\operatorname{tr}\sigma^+_{\mathfrak{P}_{\min}}(\tau)(z) = (-1)^{n/[F[\beta]:F]+n/[L[\beta]:F]}\epsilon^0_{\mu_K}(V_{\theta_L})\epsilon^0_{\mu_K}(V_\theta)\dim(\sigma^+_L)\sum_{\gamma\in\mathrm{Gal}(L/F)}\chi(\gamma z) \tag{5-2}$$

and

$$\operatorname{tr}\sigma_D(\tau)(z_D) = \epsilon^0_{\mu_K}(V_{\theta_{D,L}})\epsilon^0_{\mu_K}(V_{\theta_D})\dim(\sigma^+_{D,L})\chi(x). \tag{5-3}$$

These compute the Brauer characters of the mod p reductions $\bar\sigma^+_{\mathfrak{P}_{\min}}(\tau)$ and $\bar\sigma_D(\tau)$ at z and z_D. It follows that

$$\begin{aligned}\mathrm{JL}_p(\bar\sigma_D(\tau)(z))\\ &= (-1)^{n+n/[L:F]}(\mathrm{GL}_{n/[L:F]}(\boldsymbol{f}_{[L:F]}) : \boldsymbol{f}_n^\times)_{p'}\epsilon^0_{\mu_K}(V_{\theta_{D,L}})\epsilon^0_{\mu_K}(V_{\theta_D})\dim(\sigma^+_{D,L})\sum_{\gamma\in\mathrm{Gal}(L/F)}\chi(\gamma z_D)\end{aligned}$$

by (5-1), and we have to compare this to (5-2).

Recall from Remark 3.12 that $\dim(\sigma^+_L)$ and $\dim(\sigma^+_{D,L})$ are equal to the dimensions of the $\boldsymbol{K}$-types corresponding to an arbitrary choice of *maximal* simple types with maximal simple characters θ_L and $\theta_{D,L}$, respectively. By our choice of $\iota : F[z_D] \to F[z]$, these characters have the same endo-class; hence we can choose maximal simple types with simple characters θ_L and $\theta_{D,L}$ that determine inertial classes corresponding to each other under the Jacquet–Langlands correspondence between $Z_{D^\times}(F[z_D])$ and $Z_{\mathrm{GL}_n(F)}(F[z])$ (identified with groups over L via ι). By Proposition 3.16, this implies that

$$\dim(\sigma^+_{D,L})(\mathrm{GL}_{n/[L:F]}(\boldsymbol{f}_{[L:F]}) : \boldsymbol{f}_n^\times)_{p'} = \dim(\sigma^+_L)$$

since $\boldsymbol{f}_{[L:F]}$ is isomorphic to the residue field of L and $\boldsymbol{f}_n^\times \cong ((\boldsymbol{f}_{[L:F]})_{n/[L:F]})^\times$. Finally, the computations at the end of the proof of [Dotto 2022, Theorem 4.10] show that[1]

$$(-1)^{n+n/[K:F]+n/[F[\beta]:F]}\epsilon^0_{\mu_K}(V_\theta) = -\epsilon^0_{\mu_K}(V_{\theta_D})$$

and

$$(-1)^{n/[L:F]+n/[K:F]+n/[L[\beta]:F]}\epsilon^0_{\mu_K}(V_{\theta_L}) = -\epsilon^0_{\mu_K}(V_{\theta_{D,L}}). \qquad \square$$

We remark that when the weight $\lambda = 0$, Theorem 5.3 implies that the diagram

$$\begin{array}{ccc} R_{\overline{\mathbb{Q}}_p}(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{JL}_{\boldsymbol{K}}} & R_{\overline{\mathbb{Q}}_p}(\mathrm{GL}_n(\mathcal{O}_F)) \\ \downarrow r_p & & \downarrow r_p \\ R_{\overline{\mathbb{F}}_p}(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{JL}_p} & R_{\overline{\mathbb{F}}_p}(\mathrm{GL}_n(\mathcal{O}_F)) \end{array} \tag{5-4}$$

commutes.

[1]The integers there denoted by m are all equal to one, since D is a division algebra.

Mod ℓ reduction. Now assume $\ell \neq p$. By our discussion of $\boldsymbol{K}$-types for $D^\times$, Lemma 5.1 implies that every irreducible smooth $\overline{\mathbb{Q}}_\ell$-representation τ of $\mathcal{O}_D^\times$ is a $\boldsymbol{K}$-type for a Bernstein component of $D^\times$. As such, there exists a simple character θ such that $\tau \cong \operatorname{Ind}_{J_\theta}^{\mathcal{O}_D^\times}(\kappa_\theta \otimes \chi)$ for some character χ of J_θ/J_θ^1, and the $\mathcal{O}_D^\times$-conjugacy class of the maximal simple type $(J_\theta, \kappa_\theta \otimes \chi)$ is uniquely determined by τ.

Lemma 5.4. *Every irreducible $\overline{\mathbb{F}}_\ell$-representation of $D^\times$ is the mod ℓ reduction of a $\overline{\mathbb{Q}}_\ell$-representation of $\mathcal{O}_D^\times$. The mod ℓ reduction of an irreducible $\overline{\mathbb{Q}}_\ell$-representation τ of $\mathcal{O}_D^\times$ is irreducible.*

Proof. Since $\mathcal{O}_D^\times$ is a solvable group, the first claim is a consequence of the Fong–Swan theorem [Serre 1977, Theorem 38]. For the second claim, observe first that $\tau|_{1+\mathfrak{p}_D}$ is a direct sum with multiplicity one of representations forming a unique $\mathcal{O}_D^\times$-orbit. Indeed, by [Vignéras 2001, Proposition 4.1] the group $\operatorname{Hom}_{1+\mathfrak{p}_D}(\sigma, \tau)$ is a simple module for the Hecke algebra $\mathcal{H}(\mathcal{O}_D^\times, \sigma)$, for every representation σ of $1+\mathfrak{p}_D$. Since the quotient $\mathcal{O}_D^\times/1+\mathfrak{p}_D$ is cyclic, by [Vignéras 2001, Proposition 4.2] this Hecke algebra is commutative; hence its simple $\overline{\mathbb{Q}}_\ell$-modules are one-dimensional, proving the claim of multiplicity one. Now, if τ^0 is any $\overline{\mathbb{Z}}_\ell$-lattice in τ then the reduction $\bar{\tau}^0$ will again be a direct sum with multiplicity one of irreducible $\overline{\mathbb{F}}_\ell$-representations of $1+\mathfrak{p}_D$, because $1+\mathfrak{p}_D$ is a pro-p group, and $\mathcal{O}_D^\times$ will act transitively on the summands. Hence every irreducible $\mathcal{O}_D^\times$-subrepresentation of $\bar{\tau}^0$ has to coincide with $\bar{\tau}^0$. □

Theorem 5.5. *There exists a unique map* JL_ℓ *making the diagram*

$$\begin{array}{ccc} R_{\overline{\mathbb{Q}}_\ell}(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{JL}_{\boldsymbol{K}}} & R_{\overline{\mathbb{Q}}_\ell}(\mathrm{GL}_n(\mathcal{O}_F)) \\ \big\downarrow{\scriptstyle \boldsymbol{r}_\ell} & & \big\downarrow{\scriptstyle \boldsymbol{r}_\ell} \\ R_{\overline{\mathbb{F}}_\ell}(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{JL}_\ell} & R_{\overline{\mathbb{F}}_\ell}(\mathrm{GL}_n(\mathcal{O}_F)) \end{array} \tag{5-5}$$

commute.

Proof. The mod ℓ reduction map for $\overline{\mathbb{Q}}_\ell$-representations is defined as the direct limit of the reduction maps over finite extensions of $\mathbb{Q}_\ell$. That JL_ℓ is unique follows from the first claim in Lemma 5.4, since the left vertical arrow is surjective. For the existence, by Lemma 5.4 it suffices to prove that if τ_1 and τ_2 are irreducible representations of $\mathcal{O}_D^\times$ with the same mod ℓ reduction, then $\boldsymbol{r}_\ell(\mathrm{JL}_{\boldsymbol{K}}(\tau_1)) = \boldsymbol{r}_\ell(\mathrm{JL}_{\boldsymbol{K}}(\tau_2))$. Indeed, this allows us to define $\mathrm{JL}_\ell(\bar{\sigma})$ as $\boldsymbol{r}_\ell\,\mathrm{JL}_{\boldsymbol{K}}(\sigma)$ for any irreducible lift σ of $\bar{\sigma}$, and then commutativity of the diagram holds by definition and the second part of Lemma 5.4.

Since $\boldsymbol{r}_\ell(\tau_1) = \boldsymbol{r}_\ell(\tau_2)$, we have $\tau_1 \cong \tau_2 \otimes \psi$ for some character $\psi : \mathcal{O}_D^\times/1+\mathfrak{p}_D \to \overline{\mathbb{Q}}_\ell^\times$, because the restrictions $\tau_i|_{1+\mathfrak{p}_D}$ are isomorphic modulo ℓ; hence they are isomorphic over $\overline{\mathbb{Q}}_\ell$ as $1+\mathfrak{p}_D$ is a pro-p group. Hence there exists a simple character θ_D with endo-class Θ_F such that $\tau_i = \operatorname{Ind}_{J_{\theta_D}}^{\mathcal{O}_D^\times}(\kappa_{\theta_D} \otimes \chi_i)$ (where the χ_i are computed with respect to a lift $\Theta_E \to \Theta_F$). By assumption, the representations $\boldsymbol{r}_\ell(\kappa_{\theta_D} \otimes \chi_i)$ intertwine in $\mathcal{O}_D^\times$, as they have isomorphic inductions to $\mathcal{O}_D^\times$. Since κ_{θ_D} is a β-extension, the intertwining set of κ_{θ_D} in $D^\times$ coincides with that of θ_D, which is also equal to its normalizer $\pi_{D'}^{\mathbb{Z}} \ltimes J_{\theta_D}$ (where we have fixed a parameter field $F[\beta]$ for θ_D, and $D' = Z_D(F[\beta])$). Hence we see that $\boldsymbol{r}_\ell[\chi_1] = \boldsymbol{r}_\ell[\chi_2]$, where $[\chi_i]$ denotes the orbit under $\mathrm{Gal}(\boldsymbol{e}_{n/\delta(\Theta_F)}/\boldsymbol{e})$.

There exists a maximal simple character θ in $\mathrm{GL}_n(F)$ with the same endo-class as θ_D, together with a conjugacy class of isomorphisms $J_\theta/J_\theta^1 \to \mathrm{GL}_{n/\delta(\Theta_F)}(\boldsymbol{e})$ induced by $\Theta_E \to \Theta_F$. We assume that the subgroup J_θ is contained in $\boldsymbol{K}$, so that the virtual representation $\mathrm{JL}_{\boldsymbol{K}}(\tau_i)$ is the induction $\mathrm{Ind}_{J_\theta}^{\boldsymbol{K}}(\kappa_\theta \otimes (-1)^{n/\delta(\Theta_F)+1} R_w(\chi_i))$.

To conclude, it suffices to prove that $\boldsymbol{r}_\ell R_w(\chi_1) = \boldsymbol{r}_\ell R_w(\chi_2)$, or that the ℓ-Brauer characters of the $R_w(\chi_i)$ coincide. These are the restrictions to ℓ-regular classes in $\mathrm{GL}_{n/\delta(\Theta_F)}(\boldsymbol{e})$ of the characters of the $R_w(\chi_i)$. An element of $\mathrm{GL}_{n/\delta(\Theta_F)}(\boldsymbol{e})$ is ℓ-regular if and only if its semisimple part is ℓ-regular, because the unipotent elements of this group have order a power of p. The character formula of Deligne and Lusztig [1976, Theorem 4.2] expresses the value of $R_w(\chi_i)$ at $g \in \mathrm{GL}_{n/\delta(\Theta_F)}(\boldsymbol{e})$ with Jordan decomposition $g = su$ in terms of a Green function evaluated at u (this is independent of χ_i) and the value of χ_i at those conjugates of s contained in the inducing torus. If s is an ℓ-regular element these character values for χ_1 and χ_2 coincide since we have seen that $[\chi_1^{(\ell)}] = [\chi_2^{(\ell)}]$ (because their mod ℓ reductions are the same). $\square$

6. Breuil–Mézard conjectures

Fix a prime number ℓ, possibly equal to p, and a finite extension $E/\mathbb{Q}_\ell$. Let $\bar\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\boldsymbol{e})$ be a continuous representation. Let $R_{\bar\rho}^{\square}$ be the framed deformation ring of $\bar\rho$ over $\mathcal{O}_E$.

Case $\ell = p$. We prove the following more precise form of the first theorem in the introduction.

Theorem 6.1. *Assume that E is large enough that all irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_n(\mathcal{O}_F)$ and $\mathcal{O}_D^\times$ are defined over $\boldsymbol{e}$. Assume the geometric Breuil–Mézard conjecture for $\mathrm{GL}_n(F)$, i.e., the existence of a homomorphism*

$$\overline{\mathrm{cyc}} : R_{\boldsymbol{e}}(\mathrm{GL}_n(\mathcal{O}_F)) \to Z^{d-1}(R_{\bar\rho}^{\square}/\pi_E)$$

such that $\overline{\mathrm{cyc}}(\bar\sigma^{+}_{\mathfrak{P}_{\min}}(\tau,\lambda)) = Z(R_{\bar\rho}^{\square}(\tau,\lambda)_{\mathfrak{P}_{\min}}/\pi_E)$ whenever τ is defined over E. Then there exists a homomorphism

$$\overline{\mathrm{cyc}}_{D^\times} : R_{\boldsymbol{e}}(\mathcal{O}_D^\times) \to Z^{d-1}(R_{\bar\rho}^{\square}/\pi_E)$$

such that $\overline{\mathrm{cyc}}_{D^\times}(\bar\sigma_D(\tau,\lambda)) = Z(R_{\bar\rho}^{\square}(\tau,\lambda)_{\mathfrak{P}_{\min}}/\pi_E)$ whenever τ is defined over E.

Proof. By Theorem 5.3, it suffices to define

$$\overline{\mathrm{cyc}}_{D^\times} = \overline{\mathrm{cyc}} \circ \mathrm{JL}_p. \qquad \square$$

Remark 6.2. The statement of the Breuil–Mézard conjecture for $\mathrm{GL}_n(F)$ that we assume in the previous theorem is given in [Emerton and Gee 2014, Conjecture 4.2.1] in the crystalline case and [Le et al. 2023, Conjecture 1.5.1] in the semistable case. Given the Breuil–Mézard conjecture for $\mathrm{GL}_n(F)$, our result gives a description of the mod p fibres of discrete series lifting rings in terms of the representation theory of $\mathcal{O}_D^\times$ and the type theory of $D^\times$.

Case $\ell \neq p$. In this case, we may not find a finite extension $E/\mathbb{Q}_l$ such that all irreducible $\boldsymbol{e}$-representations of $\mathcal{O}_D^\times$ are absolutely irreducible. We assume that E is large enough that whenever $\bar\rho$ has a lift of inertial type τ to some finite extension of $\mathbb{Q}_l$, then τ and all the corresponding $\boldsymbol{K}$-types for $\mathrm{GL}_n(F)$ and $D^\times$ are defined over E. We also assume that E and k_E are large enough that all irreducible components of $\mathrm{Spec}(R^{\square}_{\bar\rho}[1/p])$ and $\mathrm{Spec}(R^{\square}_{\bar\rho}/\pi_E)$ are geometrically irreducible. For any pair (τ, N) consisting of an inertial type and a monodromy operator, write $R^{\square}_{\bar\rho}(\tau, N)$ for the corresponding quotient of the $\mathcal{O}_E$-deformation ring $R^{\square}_{\bar\rho}$, as in [Shotton 2018]. The characteristic zero points of $R^{\square}_{\bar\rho}(\tau, N)$ correspond generically to lifts of $\bar\rho$ whose attached Weil–Deligne representation has inertial type τ, N. Define a map

$$\mathrm{cyc} : R_E(\mathrm{GL}_n(\mathcal{O}_F)) \to Z^d(R^{\square}_{\bar\rho}), \quad \sigma \mapsto \sum_{\tau,N} \dim_{\overline{\mathbb{Q}}_\ell} \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[\mathrm{GL}_n(\mathcal{O}_F)]}(\sigma^\vee \otimes_E \overline{\mathbb{Q}}_\ell, \pi_{\tau,N})[R^{\square}_{\bar\rho}(\tau, N)],$$

where $\pi_{\tau,N}$ is any irreducible generic $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_n(F)$ such that $\mathrm{rec}_{\overline{\mathbb{Q}}_\ell}(\pi_{\tau,N})$ has inertial type τ, N. The map $\mathrm{rec}_{\overline{\mathbb{Q}}_\ell}$ is only well-defined up to the choice of a square root of q in $\overline{\mathbb{Q}}_\ell$, but this plays no role when considering the inertial type. Similarly, we introduce a map

$$\mathrm{cyc}_{D^\times} : R_E(\mathcal{O}_D^\times) \to Z^d(R^{\square}_{\bar\rho}), \quad \sigma \mapsto \sum_{\tau,N} \dim \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[\mathrm{GL}_n(\mathcal{O}_F)]}(\sigma^\vee \otimes_E \overline{\mathbb{Q}}_\ell, \mathrm{JL}^{-1}(\pi_{\tau,N}))[R^{\square}_{\bar\rho}(\tau, N)].$$

In this formula we set $\mathrm{JL}^{-1}(\pi) = 0$ when π is a generic representation that is not essentially square-integrable (this is consistent with the fact that the Langlands–Jacquet transfer is nonzero on elliptic representations only, and the only generic elliptic representations are the essentially square-integrable representations. See [Dat 2007]).

Theorem 6.3 (Breuil–Mézard conjecture for $D^\times$, case $\ell \neq p$). *Assume $p \neq 2$. There exists a unique map $\overline{\mathrm{cyc}}_{D^\times,\ell}$ making the following diagram commute*:

$$\begin{array}{ccc} R_E(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{cyc}_{D^\times}} & Z^d(R^{\square}_{\bar\rho}) \\ \big\downarrow{\scriptstyle \boldsymbol{r}_\ell} & & \big\downarrow{\scriptstyle \mathrm{red}} \\ R_{k_E}(\mathcal{O}_D^\times) & \xrightarrow{\overline{\mathrm{cyc}}_{D^\times,\ell}} & Z^{d-1}(R^{\square}_{\bar\rho}/\pi_E) \end{array} \tag{6-1}$$

Proof. Since the map $\boldsymbol{r}_\ell$ is surjective for $\mathcal{O}_D^\times$, it suffices to prove that if $x \in \ker(\boldsymbol{r}_\ell)$ then $x \in \ker(\mathrm{red} \circ \mathrm{cyc}_{D^\times})$. This says that every congruence between $\boldsymbol{K}$-types gives rise to a congruence between deformation rings: it is not a formal statement.

By [Shotton 2018, Theorem 4.6], there exists a commutative diagram

$$\begin{array}{ccc} R_E(\mathrm{GL}_n(\mathcal{O}_F)) & \xrightarrow{\mathrm{cyc}} & Z^d(R^{\square}_{\bar\rho}) \\ \big\downarrow{\scriptstyle \boldsymbol{r}_\ell} & & \big\downarrow{\scriptstyle \mathrm{red}} \\ R_{k_E}(\mathrm{GL}_n(\mathcal{O}_F)) & \xrightarrow{\overline{\mathrm{cyc}}_\ell} & Z^{d-1}(R^{\square}_{\bar\rho}/\pi_E) \end{array} \tag{6-2}$$

Let $x_{\overline{\mathbb{Q}}_\ell}$ be the image of x in $R_{\overline{\mathbb{Q}}_\ell}(\mathcal{O}_D^\times)$. Fix a finite extension L/E large enough that all irreducible summands of $x_{\overline{\mathbb{Q}}_\ell}$ and $\mathrm{JL}_{\boldsymbol{K}}(x_{\overline{\mathbb{Q}}_\ell})$ are defined over L. Then $\mathrm{cyc}_{D^\times}(x^\vee_{\overline{\mathbb{Q}}_\ell}) = \mathrm{cyc}(\mathrm{JL}_{\boldsymbol{K}}(x_{\overline{\mathbb{Q}}_\ell})^\vee)$, where we regard $\mathrm{JL}_{\boldsymbol{K}}(x_{\overline{\mathbb{Q}}_\ell})$ as an element of $R_L(\mathrm{GL}_n(\mathcal{O}_F))$ and the two sides as cycles on the deformation ring with $\mathcal{O}_L$-coefficients. Indeed, if σ is an L-representation of $\mathcal{O}_D^\times$ then we have by construction the equality

$$\dim \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[\mathcal{O}_D^\times]}(\sigma_{\overline{\mathbb{Q}}_\ell}, \mathrm{JL}^{-1}(\pi_{\tau,N})) = \dim \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[\mathrm{GL}_n(\mathcal{O}_F)]}(\mathrm{JL}_{\boldsymbol{K}}(\sigma_{\overline{\mathbb{Q}}_\ell}), \pi_{\tau,N})$$

because this equality holds on $\boldsymbol{K}$-types for $\mathcal{O}_D^\times$, and by Lemma 5.1 the $\boldsymbol{K}$-types span $R_{\overline{\mathbb{Q}}_\ell}(\mathcal{O}_D^\times)$. Because of our assumptions on E, the natural maps $Z^d(R_{\bar\rho}^\square) \to Z^d(R_{\bar\rho}^\square \otimes_{\mathcal{O}_E} \mathcal{O}_L)$ and $Z^{d-1}(R_{\bar\rho}^\square/\pi_E) \to Z^{d-1}(R_{\bar\rho}^\square \otimes_{\mathcal{O}_E} k_L)$ are isomorphisms; hence it suffices to prove that

$$\operatorname{red}\operatorname{cyc}\mathrm{JL}_{\boldsymbol{K}}(x_{\overline{\mathbb{Q}}_\ell})^\vee = 0.$$

Since diagram (6-2) commutes (working with L-coefficients in the diagram), we have that

$$\operatorname{red}\ \operatorname{cyc}\ \mathrm{JL}_{\boldsymbol{K}}(x_{\overline{\mathbb{Q}}_\ell})^\vee = \overline{\operatorname{cyc}}_\ell\, \boldsymbol{r}_\ell\, \mathrm{JL}_{\boldsymbol{K}}(x_{\overline{\mathbb{Q}}_\ell})^\vee.$$

By Theorem 5.5, we have $\boldsymbol{r}_\ell\,\mathrm{JL}_{\boldsymbol{K}}(x_{\overline{\mathbb{Q}}_\ell})^\vee = (\boldsymbol{r}_\ell\,\mathrm{JL}_{\boldsymbol{K}}(x_{\overline{\mathbb{Q}}_\ell}))^\vee = (\mathrm{JL}_\ell\,\boldsymbol{r}_\ell(x_{\overline{\mathbb{Q}}_\ell}))^\vee = 0$, and the claim follows. □

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Canonical integral models for Shimura varieties of toral type

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We prove the Pappas–Rapoport conjecture on the existence of canonical integral models of Shimura varieties with parahoric level structure in the case where the Shimura variety is defined by a torus. As an important ingredient, we show, using the Bhatt–Scholze theory of prismatic F-crystals, that there is a fully faithful functor from $\mathcal{G}$-valued crystalline representations of $\mathrm{Gal}(\overline{K}/K)$ to $\mathcal{G}$-shtukas over $\mathrm{Spd}(\mathcal{O}_K)$, where $\mathcal{G}$ is a parahoric group scheme over $\mathbb{Z}_p$ and $\mathcal{O}_K$ is the ring of integers in a p-adic field K.

1. Introduction

In a recent breakthrough, Pappas and Rapoport [2024] have given conditions which uniquely characterize integral models of Shimura varieties when the level subgroup at p is parahoric. The aim of this work is to show that integral models satisfying these conditions exist for Shimura varieties of toral type, i.e., those defined by a torus.

Before we go into detail, let us give some background on the study of integral models of Shimura varieties. When the level subgroup at p of a Shimura variety is hyperspecial, it is expected that integral models with smooth reduction at p exist; see [Langlands 1976], for example. In that case, Milne [1992] observed that the collection of smooth integral models as the level away from p varies can be uniquely characterized by an extension property, similar in nature to the Néron mapping property. Milne's conjecture was later refined by Moonen [1998], who corrected the class of test schemes to be used in the extension property. For Shimura varieties of abelian type, smooth integral models satisfying the extension property have been constructed by Kisin [2010]; see also work of Vasiu [1999].

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For many applications, one is primarily interested in the mod p points of a given integral model of a Shimura variety, and it is from this interest that the necessity of a unique characterization of integral models arises. Indeed, different integral models may, a priori, give rise to different reductions modulo p. In order to avoid complications of this nature, one must start with a well chosen (or canonical) system of integral models.

When the level at p is more general than hyperspecial, however, the integral models can no longer always be expected to be smooth. Indeed, the singularities should be controlled by the associated local models, and these are known to be smooth only when the level structure is hyperspecial, and in certain exceptional cases, see [He et al. 2020]. This type of pathology even occurs in some simple cases, such as the case of GL_2 with Iwahori level structure, see, e.g., [Rapoport 2005, Section 1]. Without smoothness, a characterization of integral models is not straightforward.

Pappas and Rapoport's [2024] key observation is that Scholze's theory of p-adic shtukas over perfectoid spaces [Scholze and Weinstein 2020] can be extended to arbitrary $\mathbb{Z}_p$-schemes. When the level at p is parahoric, the problem of characterizing integral models can be solved by requiring the existence of a shtuka which lives over the integral model and satisfies certain properties. In the case of Hodge type Shimura data, Pappas and Rapoport construct integral models which admit such a shtuka.

To be more precise, let (G, X) be a Shimura datum. Associated to every neat compact open subgroup $\mathsf{K} \subset \mathsf{G}(\mathbb{A}_f)$ is the Shimura variety $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)$, which admits a canonical model over a number field E. Fix a parahoric subgroup $\mathsf{K}_p \subset \mathsf{G}(\mathbb{Q}_p)$ with associated $\mathbb{Z}_p$-group scheme $\mathcal{G}$. For the remainder of the introduction, we will consider only subgroups $\mathsf{K} \subset \mathsf{G}(\mathbb{A}_f)$ which can be written $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p$ for some neat $\mathsf{K}^p \subset \mathbb{G}(\mathbb{A}_f^p)$. Fix a place v above p, and let E be the completion of E at v. We let k denote an algebraic closure of the residue field of E.

To simplify notation, throughout the introduction we assume the $\mathbb{Q}$-split rank and the $\mathbb{R}$-split rank of G are equal; we work more generally in the body of the text. Using the work of Liu and Zhu [2017], Pappas and Rapoport show that there exists a $\mathcal{G}$-shtuka $\mathscr{P}_{\mathsf{K},E}$ over $(\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)_E)^\lozenge$ defined by the pro-étale $\mathcal{G}(\mathbb{Z}_p)$-cover

$$\mathrm{Sh}_{\mathsf{K}'_p\mathsf{K}^p}(\mathsf{G}, X)_E \to \mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p}(\mathsf{G}, X)_E \tag{1-1}$$

as K'_p varies over compact open subgroups of K_p. In this case, a $\mathcal{G}$-shtuka $\mathscr{P}$ over $(\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)_E)^\lozenge$ is a functorial rule which assigns to each untilt $S^\sharp$ of S and map $f : S^\sharp \to \mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)_E^{\mathrm{ad}}$ a $\mathcal{G}$-shtuka $(\mathscr{P}_S, \phi_{\mathscr{P}_S})$ over S with one leg at $S^\sharp$ in the sense of [Scholze and Weinstein 2020].

A collection of integral models $(\mathscr{S}_{\mathsf{K}_p\mathsf{K}^p})_{\mathsf{K}^p}$ as K^p varies over neat compact open subgroups of $\mathsf{G}(\mathbb{A}_f^p)$ is *canonical* in the sense of Pappas and Rapoport if:

(a) For every discrete valuation ring R of mixed characteristic $(0, p)$,

$$(\varprojlim_{\mathsf{K}^p} \mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)_E)(R[1/p]) = (\varprojlim_{\mathsf{K}^p} \mathscr{S}_{\mathsf{K}})(R).$$

(b) $\mathscr{P}_{\mathsf{K},E}$ extends to a shtuka $\mathscr{P}_{\mathsf{K}}$ over $\mathscr{S}_{\mathsf{K}}$ for every K.

(c) For every point $x \in \mathscr{S}_{\mathsf{K}}(k)$, there exists an isomorphism

$$\Theta_x : \widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu}}_{/x_0} \xrightarrow{\sim} (\widehat{\mathscr{S}}_{\mathsf{K}/x})^{\lozenge}, \tag{1-2}$$

where $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu}$ denotes the corresponding integral local Shimura variety, such that $\mathscr{P}_{\mathsf{K}}$ pulls back along Θ_x to the universal shtuka on $\widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu}}_{/x_0}$ given by its definition as a moduli space of shtukas.

See Section 4.3 for a detailed explanation of these conditions and of the notation above. Pappas and Rapoport show that a system of normal and flat integral models with finite étale transition maps satisfying these conditions is uniquely determined, following a similar argument in Pappas's earlier work [2023].

Our focus in this paper is on the toral case, so suppose $\mathsf{G} = \mathsf{T}$ is a torus, let $T = \mathsf{T}_{\mathbb{Q}_p}$, and let $\mathsf{K}_p = \mathcal{T}(\mathbb{Z}_p)$ be the unique parahoric subgroup of $T(\mathbb{Q}_p)$ with corresponding $\mathbb{Z}_p$-model $\mathcal{T}$. In this case, the corresponding Shimura varieties are zero dimensional, that is,

$$\mathrm{Sh}_{\mathsf{K}}(\mathsf{T}, X)_E = \coprod \mathrm{Spec}(E_i), \tag{1-3}$$

where each E_i is a finite extension of the local reflex field E. Then $\mathrm{Sh}_{\mathsf{K}}(\mathsf{T}, X)_E$ has an obvious integral model over $\mathcal{O}_E$,

$$\mathscr{S}_{\mathsf{K}} = \coprod \mathrm{Spec}(\mathcal{O}_{E_i}),$$

where $\mathcal{O}_{E_i}$ is the ring of integers in E_i. The following is Theorem 4.10 below.

Theorem A. *The collection of integral models $(\mathscr{S}_{\mathsf{K}})_{\mathsf{K}^p}$ defined above is canonical in the sense of Pappas and Rapoport.*

Surprisingly, although the Shimura varieties and their integral models are quite simple, the proof of Theorem A is not trivial. The key point is that p-adic shtukas over the v-sheaves associated to $\mathbb{Z}_p$-schemes are complicated objects, and to define such an object requires some machinery.

In the case of Shimura varieties of Hodge-type, the machinery is provided by the close connection between Shimura varieties of Hodge type and the Siegel moduli space, see [Pappas and Rapoport 2024]. In particular, the shtuka can roughly be constructed out of the Tate module of the pull-back of the universal abelian scheme over the Siegel space. In the toral case, we do not have access to any such moduli description, and instead, the machinery we need is provided by the following theorem (Theorem 3.13 below), which is the key technical result of this work.

Theorem B. *Let K be a complete discretely valued extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, and let $\mathcal{G}$ be a parahoric group scheme over $\mathbb{Z}_p$. Then there is a fully faithful functor*

$$(\mathcal{G}\textit{-valued crystalline representations of } \mathrm{Gal}(\overline{K}/K)) \to (\mathcal{G}\textit{-shtukas over } \mathrm{Spd}(\mathcal{O}_K)).$$

Before we explain what goes into the proof of Theorem B, let us explain how it can be used to construct the required shtuka. To construct a shtuka over $(\mathscr{S}_{\mathsf{K}})^{\lozenge}$, it is enough to do so over $\mathrm{Spd}(\mathcal{O}_{E_i})$ for each i. But in our case, the $\mathcal{T}$-valued representation of Γ_{E_i} obtained by from the pro-étale $\mathcal{T}(\mathbb{Z}_p)$-local system (1-1) is crystalline, by class field theory and the definition of the canonical model (1-3) over E, so Theorem B provides us with such a shtuka.

Theorem B itself has its roots in the Bhatt–Scholze theory of prismatic F-crystals [Bhatt and Scholze 2023]. The main theorem of [loc. cit.] states that, for K as in Theorem B, there is an equivalence of categories

$$\begin{pmatrix} \text{prismatic } F\text{-crystals} \\ \text{over } \mathrm{Spf}(\mathcal{O}_K)_{\mathbb{\Delta}} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \mathbb{Z}_p\text{-lattices in crystalline} \\ \text{representations of } \mathrm{Gal}(\overline{K}/K) \end{pmatrix}. \tag{1-4}$$

One can obtain a shtuka over $\mathrm{Spd}(\mathcal{O}_K)$ from a prismatic F-crystal over $\mathrm{Spf}(\mathcal{O}_K)_{\mathbb{\Delta}}$; this process is outlined in [Pappas and Rapoport 2024, Section 4.4]. Our main innovations in this direction are twofold. First of all, we show that the resulting functor from prismatic F-crystals to shtukas is fully faithful, which provides a link between the two theories which may be interesting beyond the scope of this paper, see Theorem 3.4 and Corollary 3.7. Second, we provide the group theoretic analog stated in Theorem B, see Proposition 3.10 and Theorem 3.13.

We note that the existence of the isomorphisms (1-2) follows immediately from the theory in [Pappas and Rapoport 2022], so the difficult part is showing that the shtukas pull back in the desired manner. For this we exploit the functoriality in the Shimura datum to reduce to the case of the Lubin–Tate tower, where the result essentially follows from the theory of Lubin–Tate formal groups, see Section 4.4.

In fact, our construction establishes the existence of a "prismatic realization functor" in the toral case (using the terminology of [Imai et al. 2023]). This establishes the conjecture prismatic refinement of the universal $\mathcal{G}$-shtuka as described in [Pappas and Rapoport 2024, Section 4.4]. We remark that, subsequent to the appearance of a first draft of this work, a prismatic realization functor was constructed in [Imai et al. 2023] for all abelian-type Shimura varieties with hyperspecial level structure. This is used to prove the Pappas–Rapoport conjecture in those cases.

Finally, let us briefly mention that while there is some overlap between the toral-type and Hodge-type cases, the two situations are quite different in general. In the cases of Shimura varieties which are of both Hodge and toral type, it follows a posteriori from Theorem A that the integral models described here agree with those defined in [Kisin and Pappas 2018; Kisin and Zhou 2021; Pappas and Rapoport 2024], since all of these models are canonical in the sense of Pappas and Rapoport.

1.1. *Notation and conventions.*

- If X is a scheme over A, and B is an A-algebra, we abbreviate $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ by $X \otimes_A B$.
- If F is a p-adic local field, denote by F^{un} the maximal unramified extension of F inside of a fixed algebraic closure $\overline{F}$, and we denote by $\breve{F}$ the completion of F^{un}.
- If (R, R^+) is a Huber pair with $R^+ = R^\circ$, we often write $\mathrm{Spa}(R)$ instead of $\mathrm{Spa}(R, R^+)$.
- For any field K with fixed separable closure $\overline{K}$, we write Γ_K for the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$.
- If X is a scheme or adic space, we denote by $\mathrm{Vect}(X)$ the category of vector bundles on X. If $X = \mathrm{Spec}(A)$ is an affine scheme we write $\mathrm{Vect}(A)$ instead of $\mathrm{Vect}(\mathrm{Spec}(A))$.
- For a perfect field k in characteristic p, we denote by Perf_k the category of perfectoid spaces over k.

2. Preliminaries

2.1. *Recollections on shtukas.* We begin by establishing notation from the theory of perfectoid spaces and v-sheaves. We refer the reader to [Scholze and Weinstein 2020] and [Scholze 2017] for comprehensive background. Fix a perfect field k of characteristic p.

In what follows we will work with schemes, formal schemes, and adic spaces defined over $\mathbb{Z}_p$ and $\mathbb{Q}_p$. We remark that all of the constructions below work when $\mathbb{Z}_p$ is replaced by a complete discrete valuation ring with perfect residue field, and we will use these constructions freely in the text below. The main examples of interest will be the ring of integers in a finite extension of $\mathbb{Q}_p$ or $\breve{\mathbb{Q}}_p$.

We first recall the functor from [Scholze and Weinstein 2020, Section 18] associating a v-sheaf to any adic space over $\mathrm{Spa}(\mathbb{Z}_p)$. If X is an adic space over $\mathrm{Spa}(\mathbb{Z}_p)$, $X^\diamondsuit$ is the set-valued functor on Perf_k given by

$$X^\diamondsuit(S) = \{(S^\sharp, f)\}/\mathrm{isom}.$$

for any S in Perf_k, where $S^\sharp$ is an untilt of S and f is a morphism of adic spaces $f : S^\sharp \to X$. By [Scholze and Weinstein 2020, Lemma 18.1.1], the functor $X^\diamondsuit$ defines a v-sheaf on Perf_k. If $X = \mathrm{Spa}(A, A^+)$ for a Huber pair (A, A^+), we write $\mathrm{Spd}(A, A^+)$ for $X^\diamondsuit$, and when $A^+ = A^\circ$, we write $\mathrm{Spd}(A)$ instead of $\mathrm{Spd}(A, A^+)$. If $\mathfrak{X}$ is a formal scheme over $\mathrm{Spf}(\mathbb{Z}_p)$ which is locally formally of finite type, then we have a corresponding adic space $\mathfrak{X}^{\mathrm{ad}}$ defined over $\mathrm{Spa}(\mathbb{Z}_p)$, and we use the shorthand $\mathfrak{X}^\diamondsuit$ to mean $(\mathfrak{X}^{\mathrm{ad}})^\diamondsuit$.

If $\mathscr{X}$ is now a scheme over $\mathrm{Spec}(\mathbb{Z}_p)$, we can associate to $\mathscr{X}$ a v-sheaf using two different methods. For this we follow [Anschütz et al. 2022, Section 2.2]. First assume $\mathscr{X} = \mathrm{Spec}(A)$ is affine. Consider the following v-sheaves:

- Let $\mathscr{X}^\diamond$ be the v-sheaf over $\mathrm{Spd}(\mathbb{Z}_p)$ defined by assigning to $S = \mathrm{Spa}(R, R^+)$ the set of isomorphism classes of pairs $(S^\sharp, f)$, where $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ is an untilt of S over $\mathbb{Z}_p$ and $f : A \to R^{\sharp+}$ is a ring homomorphism.
- Let $\mathscr{X}^\diamondsuit$ be the v-sheaf over $\mathrm{Spd}(\mathbb{Z}_p)$ defined by assigning to $S = \mathrm{Spa}(R, R^+)$ the set of isomorphism classes of pairs $(S^\sharp, f)$ where $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ is an untilt of S over $\mathbb{Z}_p$ and $f : A \to R^\sharp$ is a ring homomorphism.

We remark that in [Pappas and Rapoport 2024], the notation $(-)^\blacklozenge$ is used in place of the small diamond symbol $(-)^\diamond$. Both $(-)^\diamondsuit$ and $(-)^\diamond$ glue to define functors from schemes over $\mathrm{Spec}(\mathbb{Z}_p)$ to v-sheaves over $\mathrm{Spd}(\mathbb{Z}_p)$. These functors are referred to as the “small diamond” and “big diamond” functors, respectively. For any scheme $\mathscr{X}$ over $\mathrm{Spec}(\mathbb{Z}_p)$ there is a natural transformation

$$j_{\mathscr{X}} : \mathscr{X}^\diamond \to \mathscr{X}^\diamondsuit. \tag{2-1}$$

The morphism $j_{\mathscr{X}}$ is a monomorphism of $\mathscr{X}$ is separated over $\mathbb{Z}_p$, an open immersion if $\mathscr{X}$ is separated and of finite type over $\mathbb{Z}_p$, and an isomorphism if $\mathscr{X}$ is proper over $\mathbb{Z}_p$. There are analogous constructions for schemes over $\mathbb{Q}_p$, which we use freely in the text below.

When $\mathscr{X}$ is separated and of finite type over $\mathbb{Z}_p$, we have alternative derivations for each of the two diamond functors by passing first to one of two adic spaces over $\mathrm{Spa}(\mathbb{Z}_p)$ associated to $\mathscr{X}$, see [Anschütz et al. 2022, Remark 2.11]:

- Let $\widehat{\mathscr{X}}$ denote the p-adic completion of $\mathscr{X}$, which is a formal scheme over $\mathrm{Spf}(\mathbb{Z}_p)$. Then there is a natural isomorphism
$$\mathscr{X}^{\diamond} \xrightarrow{\sim} \widehat{\mathscr{X}}^{\lozenge}.$$
- Let X^{ad} denote the fiber product $X \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Z}_p)$ in the sense of [Huber 1994, Proposition 3.8]. Then there is a natural isomorphism
$$\mathscr{X}^{\lozenge} \xrightarrow{\sim} (\mathscr{X}^{\mathrm{ad}})^{\lozenge}.$$

For the benefit of the reader, let us spell out the construction $\mathscr{X}^{\mathrm{ad}}$ more explicitly. If
$$\mathscr{X} = \mathrm{Spec}(\mathbb{Z}_p[x_1, \dots, x_n]/(f_1, \dots, f_k)),$$
then
$$\mathscr{X}^{\mathrm{ad}} = \varinjlim_r \mathrm{Spa}(\mathbb{Z}_p\langle p^r x_1, \dots, p^r x_n\rangle/(f_1, \dots, f_k)).$$
This is in contrast to $\widehat{\mathscr{X}}^{\mathrm{ad}}$, which is given by
$$\widehat{\mathscr{X}}^{\mathrm{ad}} = \mathrm{Spa}(\mathbb{Z}_p\langle x_1, \dots, x_n\rangle/(f_1, \dots, f_k)).$$
In general, there is a morphism
$$\widehat{\mathscr{X}}^{\mathrm{ad}} \to \mathscr{X}^{\mathrm{ad}} \tag{2-2}$$
which is an open embedding when $\mathscr{X}$ is separated and is an isomorphism if $\mathscr{X} \to \mathrm{Spec}(\mathbb{Z}_p)$ is proper, see [Huber 1994, Remark 4.6(iv)]. We can also obtain the natural transformation (2-1) by applying the $(-)^{\lozenge}$-functor for adic spaces to (2-2).

When $\mathscr{X}$ is separated and of finite type over $\mathbb{Z}_p$, by applying base change along $\mathrm{Spd}(\mathbb{Z}_p) \to \mathrm{Spd}(\mathbb{Q}_p)$ to (2-1), we obtain an open immersion
$$\mathscr{X}^{\diamond} \times_{\mathrm{Spd}(\mathbb{Z}_p)} \mathrm{Spd}(\mathbb{Q}_p) \hookrightarrow (\mathscr{X})^{\lozenge} \times_{\mathrm{Spd}(\mathbb{Z}_p)} \mathrm{Spd}(\mathbb{Q}_p) = (\mathscr{X} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Q}_p))^{\lozenge},$$
with the last equality following from the compatibility of the functor $(-)^{\lozenge}$ with products. Following [Pappas and Rapoport 2024, Definition 2.1.9], we can then define $\mathscr{X}^{\lozenge/}$ to be the coproduct
$$\mathscr{X}^{\lozenge/} := \mathscr{X}^{\diamond} \sqcup_{(\mathscr{X}^{\diamond} \times_{\mathrm{Spd}(\mathbb{Z}_p)} \mathrm{Spd}(\mathbb{Q}_p))} (\mathscr{X} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Q}_p))^{\lozenge} \tag{2-3}$$
as a v-sheaf on $\mathrm{Spd}(\mathbb{Z}_p)$. We remark that there is a natural map
$$\mathscr{X}^{\lozenge/} \to \mathscr{X}^{\lozenge} \tag{2-4}$$
factoring $\mathscr{X}^{\diamond} \to \mathscr{X}^{\lozenge}$, which also becomes an isomorphism when $\mathscr{X} \to \mathrm{Spec}(\mathbb{Z}_p)$ is proper.

If S is a perfectoid space over k, we write $S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$ for the analytic adic space defined in [Scholze and Weinstein 2020, Proposition 11.2.1]. By the proof of [loc. cit.], $S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$ is sousperfectoid, i.e., it is covered by rational open subsets $\mathrm{Spa}(R, R^+)$ where R is sousperfectoid in the sense of [Scholze and Weinstein 2020, Definition 6.3.1]. When $S = \mathrm{Spa}(R, R^+)$ is affinoid, $S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$ is given by $S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p) = \mathrm{Spa}(W(R^+))\backslash\{[\varpi] = 0\}$, where ϖ is a fixed pseudouniformizer in R^+ and $[\varpi]$ denotes the canonical lift $(\varpi, 0, 0, \dots) \in W(R^+)$. Define furthermore

$$\mathcal{Y}(R, R^+) = \mathrm{Spa}(W(R^+))\backslash\{[\varpi] = 0,\, p = 0\},$$

which is also sousperfectoid [Kedlaya 2020, Proposition 3.6]. For any $S = \mathrm{Spa}(R, R^+)$ in Perf_k, we have the function

$$\kappa : |S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)| \to [0, \infty)$$

such that $\kappa(x) = (\log|[\varpi](\tilde{x})|)/(\log|p(\tilde{x})|)$, where $\tilde{x}$ denotes the maximal generalization of

$$x \in |S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)|,$$

see [Fargues and Scholze 2021, Proposition II.1.16]. For any interval $I = [a, b] \subset [0, \infty)$, where a and b are rational numbers, denote by $\mathcal{Y}_I(S)$ the open subset of $S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$ corresponding to the interior of $\kappa^{-1}(I)$. The Frobenius homomorphism $W(R^+) \to W(R^+)$ induces morphisms

$$\mathcal{Y}_{[r,\infty)}(S) \to \mathcal{Y}_{[\mathrm{pr},\infty)}(S),$$

which will be denoted Frob_S or, occasionally, ϕ.

Let $S = \mathrm{Spa}(R, R^+)$ is in Perf_k, and let $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ be an untilt over S. Associated with $S^\sharp$ are the de Rham period rings $B^+_{\mathrm{dR}}(R^\sharp)$ and $B_{\mathrm{dR}}(R^\sharp)$. Explicitly, if ξ is a generator of the surjective homomorphism $\theta : W(R^+) \to R^{\sharp+}$ coming from the choice of $S^\sharp$, then $B^+_{\mathrm{dR}}(R^\sharp)$ is the ξ-adic completion of $W(R^+)[1/p]$ and $B_{\mathrm{dR}}(R^\sharp) = B^+_{\mathrm{dR}}(R^\sharp)[1/\xi]$. We note for future reference that $B^+_{\mathrm{dR}}(R^\sharp)$ is then the completion $\widehat{\mathcal{O}}_{S \dot{\times} \mathbb{Z}_p, S^\sharp}$ of $S \dot{\times} \mathbb{Z}_p$ along the closed Cartier divisor of $S \dot{\times} \mathbb{Z}_p$ corresponding to the untilt $S^\sharp$ of S.

Now let G be a reductive group scheme over $\mathbb{Q}_p$, and suppose $\mathcal{G}$ is a parahoric $\mathbb{Z}_p$-model for G in the sense of [Bruhat and Tits 1984]. Let us briefly review the definitions of $\mathcal{G}$-shtukas, first over perfectoid spaces, and then over the v-sheaf associated to a locally Noetherian adic space over $\mathrm{Spa}(\mathbb{Z}_p)$. Let us note that all notions in this section have an obvious vector bundle analog, which we use freely in the text below. We refer the reader [Scholze and Weinstein 2020] and [Pappas and Rapoport 2024] for details.

Let $S = \mathrm{Spa}(R, R^+)$ be affinoid perfectoid space over k, and let $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ be an untilt of S over $\mathbb{Z}_p$. A *$\mathcal{G}$-shtuka over S with one leg at $S^\sharp$* is a pair $(\mathscr{P}, \phi_{\mathscr{P}})$, where $\mathscr{P}$ is a $\mathcal{G}$-torsor over $S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$, and $\phi_{\mathscr{P}}$ is a $\mathcal{G}$-torsor isomorphism

$$\phi_{\mathscr{P}} : \mathrm{Frob}^*_S(\mathscr{P})\big|_{S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)\backslash S^\sharp} \xrightarrow{\sim} \mathscr{P}\big|_{S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)\backslash S^\sharp},$$

which is meromorphic along the closed Cartier divisor $S^\sharp \subset S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$ in the sense of [Scholze and Weinstein 2020, Definition 5.3.5].

If $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ is a $G(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters with field of definition E, then we say a $\mathcal{G}$-shtuka $(\mathscr{P}, \phi_{\mathscr{P}})$ over S with one leg at $S^\sharp$ is *bounded by* μ if the relative position of $\mathrm{Frob}^*(\mathscr{P})$ and $\mathscr{P}$ at $S^\sharp$ with respect to $\phi_{\mathscr{P}}$ is bounded by the v-sheaf local model $\mathbb{M}^v_{\mathcal{G},\mu}$; see [Pappas and Rapoport 2024, Section 2.3.4] for a detailed explanation of this condition.

We will denote the pair $(\mathscr{P}, \phi_{\mathscr{P}})$ simply by $\mathscr{P}$ when it is clear that we are speaking of the $\mathcal{G}$-shtuka and not just the $\mathcal{G}$-torsor. Since untilts $S^\sharp$ of S correspond to morphisms $S \to \mathrm{Spd}(\mathbb{Z}_p)$, we occasionally refer to a $\mathcal{G}$-shtuka with one leg at $S^\sharp$ as a *$\mathcal{G}$-shtuka over $S/\,\mathrm{Spd}(\mathbb{Z}_p)$*, if the morphism $S \to \mathrm{Spd}(\mathbb{Z}_p)$ is understood. By [Scholze and Weinstein 2020, Proposition 19.5.3], the notion of a $\mathcal{G}$-shtuka can be defined over a (not necessarily affinoid) perfectoid space S equipped with a morphism to $\mathrm{Spd}(\mathbb{Z}_p)$, and $\mathcal{G}$-shtukas over $S/\,\mathrm{Spd}(\mathbb{Z}_p)$ form a stack for the v-topology.

Let R be an integral perfectoid ring in the sense of [Scholze and Weinstein 2020, Definition 17.5.1], and let ξ be a generator of $\ker(\theta : W(R^\flat) \to R)$. A *$\mathcal{G}$-Breuil–Kisin–Fargues-module* over R is a pair $(\mathscr{P}, \phi_{\mathscr{P}})$ consisting of a $\mathcal{G}$-torsor $\mathscr{P}$ over $\mathrm{Spec}(W(R^\flat))$ together with an isomorphism

$$\phi_{\mathscr{P}} : \phi^*\mathscr{P}[1/\xi] \xrightarrow{\sim} \mathscr{P}[1/\xi].$$

Hereafter we will refer to these as $\mathcal{G}$-BKF-modules. Following [Pappas and Rapoport 2024], if $S = \mathrm{Spa}(R, R^+)$ is perfectoid with untilt $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$, then we define a $\mathcal{G}$-BKF-module over S with one leg at $S^\sharp$ to be a $\mathcal{G}$-BKF-module over $R^{\sharp+}$. We note that $R^{\sharp+}$ is integral perfectoid in this case, see [loc. cit, Remark 2.2.3(i)].

If $S = \mathrm{Spa}(R, R^+)$ is perfectoid with untilt $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$, pulling back along the morphism $\mathcal{Y}_{[0,\infty)}(S) \to \mathrm{Spec}(W(R^+))$ of locally ringed spaces determines a functor

$$(\mathcal{G}\text{-BKF-modules over } S \text{ with one leg at } S^\sharp) \to (\mathcal{G}\text{-shtukas over } S \text{ with one leg at } S^\sharp).$$

We say a $\mathcal{G}$-BKF-module over S with one leg at $S^\sharp$ is bounded by μ if the corresponding $\mathcal{G}$-shtuka is bounded by μ.

Let $\mathcal{F}$ be a v-sheaf over $\mathrm{Spd}(\mathbb{Z}_p)$. We close this section by defining shtukas over $\mathcal{F}$, following [Pappas and Rapoport 2024, Section 2.3]. Consider the slice category $\mathrm{Perf}_{\mathbb{Z}_p}/\mathcal{F}$ consisting of perfectoid spaces S over $\mathrm{Spd}(\mathbb{Z}_p)$ together with a map $S \to \mathcal{F}$ over $\mathrm{Spd}(\mathbb{Z}_p)$. Then there is a category $\mathrm{Sht}_{\mathcal{G}}$ fibered over $\mathrm{Perf}_{\mathbb{Z}_p}/\mathcal{F}$ whose fiber over $S \to \mathcal{F}$ is the category of $\mathcal{G}$-shtukas over S with one leg at the untilt $S^\sharp$ corresponding to $S \to \mathrm{Spd}(\mathbb{Z}_p)$. A *$\mathcal{G}$-shtuka* over $\mathcal{F}/\,\mathrm{Spd}(\mathbb{Z}_p)$ is defined to be a Cartesian section of $\mathrm{Sht}_{\mathcal{G}} \to \mathrm{Perf}_{\mathbb{Z}_p}/\mathcal{F}$ (in the sense of [Stacks 2005–, Tag 07IV]).

In other words, a $\mathcal{G}$-shtuka over $\mathcal{F}/\,\mathrm{Spd}(\mathbb{Z}_p)$ is a compatible collection of functors

$$\mathcal{F}(S) \to (\mathcal{G}\text{-shtukas over } S), \quad (S^\sharp, f) \mapsto (\mathscr{P}_S, \phi_{\mathscr{P}_S}), \tag{2-5}$$

for every S in Perf_k such that $(\mathscr{P}, \phi_{\mathscr{P}_S})$ has one leg at $S^\sharp$. Here we view $\mathcal{F}(S)$ as a category whose only morphisms are the identity morphisms. To make the compatibility condition explicit, suppose $\mathscr{P}$ is a collection of functors as in (2-5). Let $a : T \to S$ be a morphism in Perf_k, and let $(S^\sharp, f)$ be an untilt of S over $\mathrm{Spa}(\mathcal{O}_{\breve{E}})$. By the tilting equivalence, $S^\sharp$ determines an untilt $(T^\sharp, g)$ of T and a morphism $a^\sharp : T^\sharp \to S^\sharp$. Then $\mathscr{P}$ associates to $(S^\sharp, f)$ a $\mathcal{G}$-shtuka $(\mathscr{P}_S, \phi_{\mathscr{P}_S})$ over S with one leg at $S^\sharp$ and to $(T^\sharp, g)$ a $\mathcal{G}$-shtuka $(\mathscr{P}_T, \phi_{\mathscr{P}_T})$ over T with one leg at $T^\sharp$. The collection of functors $\mathscr{P}$ is compatible if, for every T and S as above, there is a natural isomorphism $a^*(\mathscr{P}_S, \phi_{\mathscr{P}_S}) \xrightarrow{\sim} (\mathscr{P}_T, \phi_{\mathscr{P}_T})$ of $\mathcal{G}$-shtukas over T with one leg at $T^\sharp$. Moreover, this collection of isomorphisms should satisfy the obvious cocycle condition.

We denote by $\mathrm{Sht}_{\mathcal{G}}(\mathcal{F}/\mathrm{Spd}(\mathbb{Z}_p))$ the category of $\mathcal{G}$-shtukas over $\mathcal{F}/\mathrm{Spd}(\mathbb{Z}_p)$, and by $\mathrm{Sht}_{\mathcal{G},\mu}(\mathcal{F}/\mathrm{Spd}(\mathbb{Z}_p))$ the full subcategory of those bounded by μ, that is those collections $\mathscr{P}$ such that $(\mathscr{P}, \phi_{\mathscr{P}})$ is bounded by μ for every $S \in \mathrm{Perf}_k$. If the map $\mathcal{F} \to \mathrm{Spd}(\mathbb{Z}_p)$ is clear from context, we write simply $\mathrm{Sht}_{\mathcal{G}}(\mathcal{F})$. Moreover, we have the analogous category of vector bundle shtukas over $\mathcal{F}/\mathrm{Spd}(\mathbb{Z}_p)$, which we denote by $\mathrm{Sht}(\mathcal{F}/\mathrm{Spd}(\mathbb{Z}_p))$, or once again simply $\mathrm{Sht}(\mathcal{F})$ if $\mathcal{F} \to \mathrm{Spd}(\mathbb{Z}_p)$ is understood. Henceforth, we will usually refer to vector bundle shtukas simply as "shtukas".

We close this section with a lemma which helps us determine when a $\mathcal{G}$-shtuka is bounded by a minuscule cocharacter μ. For any small v-sheaf $\mathcal{F}$, let $|\mathcal{F}|$ denote the underlying topological space of $\mathcal{F}$ as in [Scholze 2017, Proposition 12.7]. If $\mathcal{F}$ is defined over $\mathrm{Spd}(\mathbb{Z}_p)$ then, following [Pappas and Rapoport 2024, Section 3.4], we say $\mathcal{F}$ is *topologically flat* if the topological space of its generic fiber $\mathcal{F}_\eta := \mathcal{F} \times_{\mathrm{Spd}(\mathbb{Z}_p)} \mathrm{Spd}(\mathbb{Q}_p)$ is dense in $|\mathcal{F}|$.

Lemma 2.1. *Let $\mu : \mathbb{G}_{m,E} \to G_E$ be a minuscule cocharacter defined over E. Let $\mathcal{F}$ be a small v-sheaf over* $\mathrm{Spd}(\mathcal{O}_E)$, *and let $\mathscr{P}$ be a $\mathcal{G}$-shtuka on $\mathcal{F}$. If $\mathcal{F}$ is topologically flat, then $\mathscr{P}$ is bounded by μ if and only if its restriction to $\mathcal{F}_\eta$ is bounded by μ.*

Proof. Let $\mathscr{P}_\eta$ denote the pullback of $\mathscr{P}$ along $\mathrm{Spd}(E) \to \mathrm{Spd}(\mathcal{O}_E)$. Then $\mathscr{P}_\eta$ is certainly bounded by μ if $\mathscr{P}$ is, so it remains to show the converse. We will do this by proving that "boundedness by μ" is a closed condition.

Let $\mathrm{Perf}_k / \mathrm{Spd}(\mathbb{Z}_p)$ denote the category of perfectoid spaces over k equipped with a morphism to $\mathrm{Spd}(\mathbb{Z}_p)$ (i.e., equipped with an untilt), and let $\mathrm{Gr}_{\mathcal{G}}$ denote the Beilinson–Drinfeld Grassmannian over $\mathrm{Spd}(\mathbb{Z}_p)$. If $S = \mathrm{Spa}(R, R^+)$ is in $\mathrm{Perf}_k / \mathrm{Spd}(\mathbb{Z}_p)$, then $\mathrm{Gr}_{\mathcal{G}}(S)$ parametrizes $\mathcal{G}$ torsors over $\mathrm{Spec}(B^+_{\mathrm{dR}}(R^\sharp))$ along with a trivialization over $\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp))$ [Scholze and Weinstein 2020, Proposition 20.3.2].

We define the integral local Hecke stack $\mathrm{Hk}_{\mathcal{G}}$ to be the stack on $\mathrm{Perf}_k / \mathrm{Spd}(\mathbb{Z}_p)$ which assigns to a perfectoid space S in $\mathrm{Perf}_k / \mathrm{Spd}(\mathbb{Z}_p)$ the groupoid of tuples $(\mathcal{E}_1, \mathcal{E}_2, \alpha)$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\mathcal{G}$-torsors on $\mathrm{Spec}(B^+_{\mathrm{dR}}(R^\sharp))$, and α is an isomorphism

$$\alpha : \mathcal{E}_1[1/\xi] \xrightarrow{\sim} \mathcal{E}_2[1/\xi].$$

It follows from [Scholze and Weinstein 2020, Proposition 20.3.2] that $\mathrm{Hk}_{\mathcal{G}}(S)$ is the quotient $L^+\mathcal{G} \backslash \mathrm{Gr}_{\mathcal{G}}$ for the étale topology, where $L^+\mathcal{G}$ is the group functor $(R, R^+) \mapsto \mathcal{G}(B^+_{\mathrm{dR}}(R^\sharp))$ on $\mathrm{Perf}_k / \mathrm{Spd}(\mathbb{Z}_p)$.

Recall the v-sheaf local model $\mathbb{M}^v_{\mathcal{G},\mu}$ [Scholze and Weinstein 2020, Section 21.4], which is the closure for the v-topology of the diamond associated to the affine Schubert cell $\mathrm{Gr}_{G,\mu}$ inside of $\mathrm{Gr}_{\mathcal{G}}$. Analogously, we denote by $\mathrm{Hk}_{\mathcal{G},\mu}$ the (left) quotient of $\mathbb{M}^v_{\mathcal{G},\mu}$ by $L^+\mathcal{G}$. Since $\mathbb{M}^v_{\mathcal{G},\mu} \to \mathrm{Gr}_{\mathcal{G}}$ is a closed immersion and $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Hk}_{\mathcal{G}}$ is a v-cover, it follows from [Scholze 2017, Proposition 10.11(i)] that $\mathrm{Hk}_{\mathcal{G},\mu} \to \mathrm{Hk}_{\mathcal{G}}$ is a closed immersion as well.

Associated to $\mathscr{P}$ we have a morphism $\mathcal{F} \to \mathrm{Hk}_{\mathcal{G}}$ of v-sheaves over $\mathrm{Spd}(\mathbb{Z}_p)$. Indeed, given a morphism $S = \mathrm{Spa}(R, R^+) \to \mathcal{F}$ over $\mathrm{Spd}(\mathbb{Z}_p)$, we obtain a $\mathcal{G}$-shtuka $\mathscr{P}_S$ over S with one leg at $S^\sharp$. We then obtain two $\mathcal{G}$-torsors on $\mathrm{Spec}(B^+_{\mathrm{dR}}(R^\sharp))$ by pulling back both $\mathscr{P}_S$ and $\mathrm{Frob}^*_S \mathscr{P}_S$ along $\mathrm{Spec}(B^+_{\mathrm{dR}}(S^\sharp)) \to S \dot{\times} \mathbb{Z}_p$. Since the composition $\mathrm{Spec}(B_{\mathrm{dR}}(S^\sharp)) \to \mathrm{Spec}(B^+_{\mathrm{dR}}(S^\sharp)) \to S \dot{\times} \mathbb{Z}_p$ factors through $S \dot{\times} \mathbb{Z}_p \setminus S^\sharp$, the two $\mathcal{G}$-torsors are additionally equipped with an isomorphism between their restrictions to $\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp))$, and hence we obtain an object in $\mathrm{Hk}_{\mathcal{G}}$.

Define $\mathcal{F}_\mu = \mathrm{Hk}_{\mathcal{G},\mu} \times_{\mathrm{Hk}_{\mathcal{G}}} \mathcal{F}$. Since $\mathrm{Hk}_{\mathcal{G},\mu} \to \mathrm{Hk}_{\mathcal{G}}$ is a closed immersion, the same is true of $\mathcal{F}_\mu \to \mathcal{F}$. Moreover, $\mathcal{F}_\mu(S)$ describes maps $S \to \mathcal{F}$ for which the corresponding shtuka $\mathscr{P}_S$ is bounded by μ. By assumption $\mathcal{F}_\eta \subset \mathcal{F}_\mu$, so $|\mathcal{F}_\mu| = |\mathcal{F}|$ by topological flatness of $\mathcal{F}$. Moreover, $\mathcal{F}_\mu \to \mathcal{F}$ is quasicompact, since it is a closed immersion; see, e.g., [Scholze 2017, Remark 18.2]. Hence by [loc. cit., Lemma 12.11], $\mathcal{F}_\mu \to \mathcal{F}$ is a surjective morphism of v-stacks, and is therefore an isomorphism. It follows that $\mathscr{P}$ is bounded by μ on all of $\mathcal{F}$. □

2.2. ***Shtukas and local systems.*** In this section we discuss the connection between shtukas and pro-étale $\mathbb{Z}_p$-local systems as in [Pappas and Rapoport 2024, Section 2]. Let S be a perfectoid space, and let $S_{\text{proét}}$ denote the pro-étale topology for S as in [Scholze and Weinstein 2020, Definition 8.2.6]. If H is a topological group, denote by $\underline{H}$ the sheaf for the pro-étale topology on Perf_k defined by

$$\underline{H}(S) = C^0(|S|, H), \tag{2-6}$$

where $|S|$ is the topological space underlying S and $C^0(|S|, H)$ denotes the set of continuous functions $|S| \to H$. In particular, for any finite free $\mathbb{Z}_p$-module M, we have an associated pro-étale sheaf $\underline{M}$ on Perf_k. A *pro-étale $\mathbb{Z}_p$-local system on S* is a sheaf $\mathbb{L}$ of $\underline{\mathbb{Z}}_p$-modules on $S_{\text{proét}}$ such that $\mathbb{L}$ is locally isomorphic to $\underline{M}$ for some finite free $\mathbb{Z}_p$-module M.

Suppose now X is a locally Noetherian adic space over $\mathbb{Z}_p$, and write $X_{\text{proét}}$ for the pro-étale site of X as in [Bhatt et al. 2018, Section 5.1]. We have also a notion of pro-étale $\mathbb{Z}_p$-local system on X. Denote by $\underline{\mathbb{Z}}_p$ the inverse limit $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$ as sheaves on $X_{\text{proét}}$. Then a *pro-étale $\mathbb{Z}_p$-local system on X* is a sheaf $\mathbb{L}$ of $\underline{\mathbb{Z}}_p$-modules on $X_{\text{proét}}$ such $\mathbb{L}$ is locally isomorphic to $\underline{\mathbb{Z}}_p \otimes M$ for some finitely generated $\mathbb{Z}_p$-module M. By [Scholze 2013, Proposition 8.2], this is equivalent to the standard notion of a lisse $\mathbb{Z}_p$-sheaf on the étale site $X_{\text{ét}}$ of X. We will denote by $\mathrm{Loc}_{\mathbb{Z}_p}(X)$ the resulting category of pro-étale $\mathbb{Z}_p$-local systems on X.

Remark 2.2. Using [Kedlaya and Liu 2015, Definition 9.1.4] we may extend the definition of the pro-étale site for locally Noetherian adic spaces given in [Bhatt et al. 2018, Section 5.1] to arbitrary preadic spaces (in the terminology of [Kedlaya and Liu 2015]), and in particular to perfectoid spaces. If S is a perfectoid space, the resulting pro-étale site differs from the site $S_{\text{proét}}$ defined in [Scholze and Weinstein 2020], but the two categories of $\mathbb{Z}_p$-local systems defined using the different sites are equivalent. Indeed, this follows from the fact that descent data for finite étale morphisms over S are effective in both pro-étale topologies. In particular, given a locally Noetherian adic space X over $\mathbb{Z}_p$, a pro-étale $\mathbb{Z}_p$-local system $\mathbb{L}$ on X, and a morphism $S \to X$ from a perfectoid space S to X, we may pull back $\mathbb{L}$ to a $\mathbb{Z}_p$-local system on S in the sense described above.

Following [Pappas and Rapoport 2024], we define a *pro-étale $\mathbb{Z}_p$-local system* on $X^\diamondsuit$ as a compatible system of functors

$$X^\diamondsuit(S) \to \mathrm{Loc}_{\mathbb{Z}_p}(S), \quad (S^\sharp, f) \mapsto \mathbb{L}_S,$$

for every S in Perf_k. We denote the resulting category of pro-étale $\mathbb{Z}_p$-local systems on $X^\diamondsuit$ by $\mathrm{Loc}_{\mathbb{Z}_p}(X^\diamondsuit)$. Let $\mathbb{L}$ be a pro-étale $\mathbb{Z}_p$-local system on X, and let S be a perfectoid space in Perf_k with untilt $S^\sharp$. Then any morphism of adic spaces $S^\sharp \to X$ allows us to define a pro-étale $\mathbb{Z}_p$-local system on S by first pulling back $\mathbb{L}$ to $S^\sharp$ (see Remark 2.2) and then applying the tilting equivalence. This defines a functor

$$\mathrm{Loc}_{\mathbb{Z}_p}(X) \to \mathrm{Loc}_{\mathbb{Z}_p}(X^\diamondsuit). \tag{2-7}$$

Lemma 2.3. *If X is an analytic adic space over $\mathbb{Z}_p$, the functor* (2-7) *is an equivalence of categories.*

Proof. We use pro-étale descent to construct a quasiinverse functor. Let $\mathbb{L}$ be a pro-étale $\mathbb{Z}_p$-local system over $X^\diamondsuit$. By [Scholze 2017, Lemma 15.3], there is a perfectoid space $\widetilde{X}$ which provides a pro-étale cover $\psi : \widetilde{X} \to X$. Then $(\widetilde{X}, \psi)$ defines a point in $X^\diamondsuit(\widetilde{X}^\flat)$, so $\mathbb{L}$ determines a pro-étale $\mathbb{Z}_p$-local system $\mathbb{L}_{\widetilde{X}^\flat}$ over $\widetilde{X}^\flat$, and by the tilting equivalence we obtain a pro-étale $\mathbb{Z}_p$-local system $\mathbb{L}_{\widetilde{X}}$ on $\widetilde{X}$.

The fiber product $\widetilde{X}^\flat \times_{X^\diamondsuit} \widetilde{X}^\flat$ is representable by a perfectoid space over $X^\diamondsuit$. Denote by p_1 and p_2 the two projection morphisms $\widetilde{X}^\flat \times_{X^\diamondsuit} \widetilde{X}^\flat \to \widetilde{X}^\flat$. Since $\psi \circ p_1 = \psi \circ p_2$, the transition isomorphisms for $\mathbb{L}$ induce an isomorphism $\alpha^\flat : p_1^*(\mathbb{L}_{\widetilde{X}^\flat}) \xrightarrow{\sim} p_2^*(\mathbb{L}_{\widetilde{X}^\flat})$. Applying the tilting equivalence again, we obtain an isomorphism $\alpha : p_1^*(\mathbb{L}_{\widetilde{X}}) \xrightarrow{\sim} p_2^*(\mathbb{L}_{\widetilde{X}})$ of pro-étale $\mathbb{Z}_p$-local systems on $\widetilde{X} \times_X \widetilde{X}$, which is a pro-étale descent datum by the cocycle condition for $\mathbb{L}$. Hence we obtain a pro-étale local system $\mathbb{L}_X$ on X via descent. We leave it to the reader to check that this construction defines a quasiinverse to (2-7). □

Remark 2.4. One could alternatively define $\mathbb{Z}_p$-local systems on $X^\diamondsuit$ using the étale or quasipro-étale site of $X^\diamondsuit$ in the sense of [Scholze 2017, Definition 14.1]. We choose the definition given above for consistency with [Pappas and Rapoport 2024] and because it is most easily seen to be compatible with the constructions we make below. In the end, the resulting category of $\mathbb{Z}_p$-local systems on $X^\diamondsuit$ is independent of this choice. This can be seen by combining Lemma 2.3 with [Scholze 2017, Lemma 15.6] and [Mann and Werner 2023, Proposition 3.7].

Remark 2.5. If K is a complete discretely valued extension of $\mathbb{Q}_p$, and $X = \mathrm{Spa}(K, \mathcal{O}_K)$, then $\widetilde{X}$ can be chosen as $\mathrm{Spa}(C, \mathcal{O}_C)$ for a complete algebraic closure C of K. In that case, $\mathrm{Loc}_{\mathbb{Z}_p}(X) = \mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_K)$, and, under the equivalence (2-7), evaluation on $\mathrm{Spa}(C, \mathcal{O}_C)$ corresponds to the forgetful functor $\mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_K) \to \mathrm{Vect}(\mathbb{Z}_p)$.

Let $\mathcal{G}$ is a smooth affine group scheme over $\mathbb{Z}_p$, and let $\underline{\mathcal{G}(\mathbb{Z}_p)}$ be the pro-étale sheaf on Perf_k associated to $\mathcal{G}(\mathbb{Z}_p)$ (see (2-6). Denote by $\mathrm{Tors}_{\underline{\mathcal{G}(\mathbb{Z}_p)}}(S)$ the category of pro-étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsors on S as in [Scholze and Weinstein 2020, Section 9.3].

A *pro-étale* $\underline{\mathcal{G}(\mathbb{Z}_p)}$*-torsor* on $X^\diamond$ is a compatible system of functors

$$X^\diamond(S) \to \mathrm{Tors}_{\underline{\mathcal{G}(\mathbb{Z}_p)}}(S), \quad (S^\sharp, f) \mapsto \mathbb{P}_S,$$

for every S in Perf_k. The category $\mathrm{Loc}_{\mathbb{Z}_p}(X^\diamond)$ inherits exact and tensor structures from the categories $\mathrm{Loc}_{\mathbb{Z}_p}(S)$ as S varies in Perf_k, hence we have a Tannakian interpretation of pro-étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsors as well. Let $\mathbb{P}$ be a pro-étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsor over $X^\diamond$, and let $\rho : \mathcal{G} \to \mathrm{GL}(\Lambda)$ be an algebraic representation of $\mathcal{G}$ on a finite $\mathbb{Z}_p$-module Λ. For every $(S^\sharp, f) \in X^\diamond(S)$, we obtain a pro-étale $\underline{\mathbb{Z}}_p$-local system

$$\mathbb{L}_\rho := \mathbb{P}_S \times^{\underline{\mathcal{G}(\mathbb{Z}_p)}} \underline{\Lambda}_S$$

on S. In other words, $\mathbb{L}_\rho$ is the quotient of $\mathbb{P}_S \times \underline{\Lambda}_S$ by the $\underline{\mathcal{G}(\mathbb{Z}_p)}$-action $g \cdot (p, \lambda) = (pg^{-1}, g\lambda)$.

Let $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G})$ denote the category of algebraic representations $\rho : \mathcal{G} \to \mathrm{GL}(\Lambda)$ of $\mathcal{G}$ on finite projective $\mathbb{Z}_p$-modules Λ. Thus, from a pro-étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsor $\mathbb{P}$ we obtain an exact $\mathbb{Z}_p$-linear tensor functor

$$\omega_{\mathbb{P}} : \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Loc}_{\mathbb{Z}_p}(X^\diamond), \quad (\Lambda, \rho) \mapsto \mathbb{L}_\rho. \tag{2-8}$$

Lemma 2.6. *The assignment* $\mathbb{P} \mapsto \omega_{\mathbb{P}}$ *determines an equivalence of categories between* $\mathrm{Tors}_{\underline{\mathcal{G}(\mathbb{Z}_p)}}(X^\diamond)$ *and the category of exact* $\mathbb{Z}_p$*-linear tensor functors* $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Loc}_{\mathbb{Z}_p}(X^\diamond)$.

Proof. By functoriality of the construction, it is enough to show that if S is in Perf_k, then the category of pro-étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsors on $S^\diamond$ is equivalent to the category of exact $\mathbb{Z}_p$-linear tensor functors $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Loc}_{\mathbb{Z}_p}(S^\diamond)$. For such an S, we have $\mathrm{Loc}_{\mathbb{Z}_p}(S^\diamond) = \mathrm{Loc}_{\mathbb{Z}_p}(S)$, and the result follows from the proof of [Scholze and Weinstein 2020, Proposition 22.6.1]. Let us explain.

Denote by ω_0 the exact $\mathbb{Z}_p$-linear tensor functor $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Loc}_{\mathbb{Z}_p}(S)$ given by $(\Lambda, \rho) \mapsto \underline{\Lambda}_S$. If ω is another exact $\mathbb{Z}_p$-linear tensor functor $\omega : \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Loc}_{\mathbb{Z}_p}(S)$, define a pro-étale sheaf $\mathbb{P}_\omega = \mathrm{Isom}^\otimes(\omega_0, \omega)$ classifying tensor isomorphisms between ω_0 and ω. That is, the points of $\mathbb{P}_\omega$ in $T \to S$ are given by isomorphisms of tensor functors $\omega_{0,T} \to \omega_T$, where the subscript $(-)_T$ indicates that we compose the given tensor functor with pullback $\mathrm{Loc}_{\mathbb{Z}_p}(S) \to \mathrm{Loc}_{\mathbb{Z}_p}(T)$. Similarly, define $\mathrm{Aut}^\otimes(\omega_0) = \mathrm{Isom}^\otimes(\omega_0, \omega_0)$.

The natural action of $\underline{\mathcal{G}(\mathbb{Z}_p)}$ on ω_0 determines morphism of pro-étale sheaves $\underline{\mathcal{G}(\mathbb{Z}_p)} \to \mathrm{Aut}^\otimes(\omega_0)$, which can be seen to be an isomorphism by reducing to the case where S is strictly totally disconnected and applying ordinary Tannakian duality. Thus we obtain an action of $\underline{\mathcal{G}(\mathbb{Z}_p)}$ on $\mathbb{P}_\omega$ by precomposition. The arguments of [Scholze and Weinstein 2020, Proposition 22.6.1] imply that ω is pro-étale locally isomorphic to ω_0, and therefore that $\mathbb{P}_\omega$ is a pro-étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsor. We conclude by noting that $\mathbb{P} \mapsto \omega_{\mathbb{P}}$ and $\omega \mapsto \mathbb{P}_\omega$ determine mutually quasiinverse functors. □

In the remainder of this section we explain how to assign a $\mathbb{Z}_p$-local system to a shtuka over a v-sheaf, following [Pappas and Rapoport 2024, Section 2.4] and [Scholze and Weinstein 2020, Chapter 22]. Let

$S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over k with an untilt $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ over $\mathcal{O}_{\breve{E}}$, and suppose $S^\sharp$ has the property that the corresponding morphism $S \to \mathrm{Spd}(\mathcal{O}_E)$ factors through $\mathrm{Spd}(E)$. Consider the integral Robba ring, as in [Kedlaya and Liu 2015, Definition 4.2.2] and [Scholze and Weinstein 2020, Section 22.3],

$$\widetilde{\mathcal{R}}_S^{\mathrm{int}} := \varinjlim_{r>0} H^0(\mathcal{Y}_{[0,r]}(S), \mathcal{O}_{\mathcal{Y}_{[0,r]}(S)}). \tag{2-9}$$

This ring carries a natural Frobenius morphism, which is compatible with that of $S \dot{\times} \mathbb{Z}_p$.

Now let $(\mathscr{V}, \phi_{\mathscr{V}})$ be a (vector bundle) shtuka over S with one leg at $S^\sharp$. For any $r > 0$ with the property that $\mathcal{Y}_{[0,r]}(S) \subset S \dot{\times} \mathbb{Z}_p \setminus S^\sharp$, we obtain by restriction of $(\mathscr{V}, \phi_{\mathscr{V}})$ a ϕ^{-1}-module on $\mathcal{Y}_{[0,r]}(S)$. Note that such an r is guaranteed to exist because we have taken the untilt $S^\sharp$ to live over E. Passing to the limit, we obtain a ϕ-module

$$\varinjlim_{r>0} H^0(\mathcal{Y}_{[0,r]}(S), \mathscr{V})$$

over $\widetilde{\mathcal{R}}_S^{\mathrm{int}}$. By [Kedlaya and Liu 2015, Theorems 8.5.3 and 9.3.7], such an object is equivalent to a $\mathbb{Z}_p$-local system on S. This construction defines a functor

$$(\text{shtukas over } S \text{ with one leg at } S^\sharp) \to \mathrm{Loc}_{\mathbb{Z}_p}(S). \tag{2-10}$$

Let us return to the notation of the previous section, so G is a reductive group scheme over $\mathbb{Q}_p$ and $\mathcal{G}$ is a parahoric $\mathbb{Z}_p$-model for G. Let $(\mathscr{P}, \phi_{\mathscr{P}})$ be a shtuka over S with one leg at $S^\sharp$. Applying the Tannakian formalism to (2-10), we obtain a functor

$$(\mathcal{G}\text{-shtukas on } S \text{ with one leg at } S^\sharp) \to \mathrm{Tors}_{\underline{\mathcal{G}(\mathbb{Z}_p)}}(S). \tag{2-11}$$

Since these constructions are functorial in S, they extend to functors for shtukas over v-sheaves associated to adic spaces.

2.3. *Crystalline representations and prismatic F-crystals.* Let K be a complete discretely valued extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and perfect residue field k. Fix an algebraic closure $\overline{K}$ of K, and let $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K. Denote by C the completion of $\overline{K}$, and by $\mathcal{O}_C$ its ring of integers.

Let $\mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_K)$ denote the category of finite free $\mathbb{Z}_p$-modules Λ equipped with a continuous action of Γ_K, and let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{crys}}(\Gamma_K)$ denote the full subcategory of $\mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_K)$ consisting of those representations such that $\Lambda[1/p]$ is crystalline. In this section we describe a method for obtaining a vector bundle shtuka over $\mathrm{Spd}(\mathcal{O}_K)$ from a representation in $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{crys}}(\Gamma_K)$. This method relies on the description of $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{crys}}(\Gamma_K)$ as the category of prismatic F-crystals on $\mathrm{Spf}(\mathcal{O}_K)_{\Delta}$ given in [Bhatt and Scholze 2023]. For background on prisms and prismatic F-crystals we refer the reader to [Bhatt and Scholze 2022; 2023].

Let $\mathcal{X}$ be a p-adic formal scheme. The *absolute prismatic site of* $\mathcal{X}$, denoted $\mathcal{X}_{\Delta}$, is the site whose underlying category consists of the opposite category of the category of pairs $((A, I), x)$, where (A, I) is a bounded prism and $x : \mathrm{Spf}(A/I) \to \mathcal{X}$ is a map of formal schemes, and whose topology is given by the flat topology on prisms. Here we say a morphism of prisms $(A, I) \to (B, IB)$ is *(faithfully) flat* if the ring homomorphism $A \to B$ is (p, I)-completely (faithfully) flat, that is, $B/(p, I)B$ is (faithfully) flat over $A/(p, I)$, and $\mathrm{Tor}_i^A(A/(p, I), B) = 0$ for all $i > 0$.

Remark 2.7. If $(A, I) \to (B, IB)$ is a (faithfully) flat morphism of prisms, the induced maps

$$A/(p, I)^n \to B/(p, I)^n B$$

are (faithfully) flat for every n. Indeed, flatness follows from [Stacks 2005–, Tag 051C], and then faithful flatness follows by induction since the maps $A/(p, I)^n \to A/(p, I)^{n-1}$ are nilpotent thickenings.

If $((A, I), x)$ is an object in the underlying category of $\mathcal{X}_{\Delta}$, we will sometimes say that (A, I) is a *prism over* $\mathcal{X}$. The site $\mathcal{X}_{\Delta}$ is naturally equipped with a structure sheaf $\mathcal{O}_{\Delta} : ((A, I), x) \mapsto A$ and an ideal sheaf $\mathcal{I}_{\Delta} : ((A, I), x) \mapsto I$.

Following [Bhatt and Scholze 2023], we write $\mathrm{Vect}^{\varphi}(\mathcal{X}_{\Delta}, \mathcal{O}_{\Delta})$ for the category of *prismatic F-crystals* on $\mathcal{X}_{\Delta}$. This is the category of pairs $(\mathcal{E}, \varphi_{\mathcal{E}})$ consisting of a vector bundle $\mathcal{E}$ on the ringed site $(\mathcal{X}_{\Delta}, \mathcal{O}_{\Delta})$ along with an isomorphism $\varphi_{\mathcal{E}} : \varphi^*\mathcal{E}[1/\mathcal{I}_{\Delta}] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_{\Delta}]$. By [Bhatt and Scholze 2023, Proposition 2.7], a prismatic F-crystal $(\mathcal{E}, \varphi_{\mathcal{E}})$ is given concretely by the following data: For every for every object $\tilde{x} = ((A, I), x)$ in $\mathcal{X}_{\Delta}$, $(\mathcal{E}, \varphi_{\mathcal{E}})$ determines a pair $(\mathcal{E}_A, \varphi_{\mathcal{E}_A})$ consisting of a vector bundle $\mathcal{E}_A$ on $\mathrm{Spec}(A)$ along with an isomorphism

$$\varphi_{\mathcal{E}_A} : \varphi_A^*(\mathcal{E}_A)\big|_{\mathrm{Spec}(A[1/I])} \xrightarrow{\sim} \mathcal{E}_A\big|_{\mathrm{Spec}(A[1/I])}.$$

These come equipped with transition isomorphisms

$$\theta_f : \mathcal{E}_A \otimes_A B \xrightarrow{\sim} \mathcal{E}_B,$$

which are compatible with Frobenius, for every morphism $f : ((A, I), x) \to ((B, J), y)$ in $\mathcal{X}_{\Delta}$. Moreover the collection $\{\theta_f\}$ satisfies the obvious cocycle condition.

As in [Bhatt and Scholze 2023], we write $\mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]$ for the sheaf of rings $(A, I) \mapsto A[1/I]$, and we define $\mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge}$ to be

$$\mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge} := \varprojlim_n \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]/p^n\mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}].$$

Let $\mathrm{Vect}(\mathcal{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge})^{\varphi=1}$ denote the category of *Laurent F-crystals* on $\mathcal{X}_{\Delta}$. This is the category of pairs (E, φ_E) consisting of a vector bundle E on the ringed site $(\mathcal{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge})$ along with an isomorphism $\varphi_E : \varphi^*E \xrightarrow{\sim} E$. These admit a similar concrete description as in the case of prismatic F-crystals. Let $\mathcal{X}_\eta$ be the adic space generic fiber of $\mathcal{X}$ (with respect to $\mathbb{Z}_p$), and let $\mathrm{Loc}_{\mathbb{Z}_p}(\mathcal{X}_\eta^{\diamond})$ be the category of $\mathbb{Z}_p$-local systems on $X_\eta^{\diamond}$. By Artin–Schreier theory (see [Bhatt and Scholze 2023, Corollary 3.8]), there is a natural equivalence of categories

$$\mathrm{Vect}(\mathcal{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge})^{\varphi=1} \xrightarrow{\sim} \mathrm{Loc}_{\mathbb{Z}_p}(\mathcal{X}_\eta) \tag{2-12}$$

for any bounded p-adic formal scheme $\mathcal{X}$.

Suppose now $\mathcal{X} = \mathrm{Spf}(\mathcal{O}_K)$. Then the right-hand side of (2-12) identifies with $\mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_K)$, and the functor (2-12) is given by

$$E \mapsto E(W(\mathcal{O}_C^\flat), \ker(\theta))^{\varphi=1}, \tag{2-13}$$

where C is the p-adic completion of $\overline{K}$, and the Γ_K-action on the right-hand side is induced by functoriality and the prismatic crystal property.

Via base change $\mathcal{O}_{\Delta} \to \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^\wedge$, we obtain a functor

$$T : \mathrm{Vect}^\varphi(\mathrm{Spf}(\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) \to \mathrm{Vect}(\mathrm{Spf}(\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^\wedge)^{\varphi=1} \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_K),$$

called the *étale realization functor*. By [Bhatt and Scholze 2023, Proposition 5.3], if $(\mathcal{E}, \varphi_{\mathcal{E}})$ is a prismatic F-crystal on $\mathrm{Spf}(\mathcal{O}_K)_{\Delta}$, then $T(\mathcal{E})[1/p]$ is a crystalline Γ_K-representation. The following is the main theorem of [loc. cit.].

Theorem 2.8 (Bhatt and Scholze). *The étale realization functor*

$$T : \mathrm{Vect}^\varphi(\mathrm{Spf}(\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) \to \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{crys}}(\Gamma_K)$$

is an equivalence of tensor categories.

Proof. That T is an equivalence is [loc. cit., Theorem 5.6]. Moreover, that T is compatible with tensor products follows from its definition and the description (2-13) of the equivalence (2-12). We conclude using the fact that a tensor functor which is an equivalence of categories is necessarily an equivalence of tensor categories [Saavedra Rivano 1972, I, 4.4]. □

We denote the quasiinverse tensor functor for T by

$$U : \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{crys}}(\Gamma_K) \xrightarrow{\sim} \mathrm{Vect}^\varphi(\mathrm{Spf}(\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}). \tag{2-14}$$

3. Prismatic F-crystals and shtukas

3.1. ***From prismatic F-crystals to shtukas.*** In this section we describe a process for obtaining shtukas from prismatic F-crystals, with the goal of defining a $\mathcal{G}$-shtuka over $\mathrm{Spd}(\mathcal{O}_K)$ from a $\mathcal{G}$-valued crystalline representation of Γ_K.

Let $\mathcal{X}$ be a formal scheme which is finite type over $\mathbb{Z}_p$. Following some ideas from [Pappas and Rapoport 2024, Section 4.4], we can associate to any prismatic F-crystal $(\mathcal{E}, \varphi_{\mathcal{E}})$ over $\mathcal{X}_{\Delta}$ a shtuka $(\mathscr{V}, \phi_{\mathscr{V}})$ over $\mathcal{X}^\diamondsuit$. For simplicity, suppose $\mathcal{X} = \mathrm{Spf}(A)$ is affine. Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over k, and let $(S^\sharp, x) \in \mathcal{X}^\diamondsuit(S)$ with $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$. Then x determines a ring homomorphism $A \to R^{\sharp+}$, and therefore a map of formal schemes $\mathrm{Spf}(R^{\sharp+}) \to \mathrm{Spf}(A)$, which we denote also by x. Since the map $\theta : W(R^+) \to R^{\sharp+}$ is surjective, and the pair $(W(R^+), \ker(\theta))$ is a bounded prism, we obtain an object

$$\tilde{x} = ((W(R^+), \ker(\theta)), x : \mathrm{Spf}(R^{\sharp+}) \to \mathrm{Spf}(A))$$

in $\mathcal{X}_{\Delta}$.

Evaluating $(\mathcal{E}, \varphi_{\mathcal{E}})$ on $\tilde{x}$, we obtain a BKF-module $(\mathcal{E}_{W(R^+)}, \varphi_{\mathcal{E}_{W(R^+)}})$ on S with one leg at $S^\sharp$. By pulling this back along the morphism of locally ringed spaces

$$\mathcal{Y}_{[0,\infty)}(S) \to \mathrm{Spa}(W(R^+)) \to \mathrm{Spec}(W(R^+)),$$

this BKF-module induces a shtuka $(\mathscr{V}_S, \phi_{\mathscr{V}_S})$ over S with one leg at $S^\sharp$. The compatibility isomorphisms coming from the prismatic F-crystal induce isomorphisms of the corresponding shtukas, hence the assignment

$$(S^\sharp, x) \in \mathcal{X}^\lozenge(S) \mapsto (\mathscr{V}_S, \phi_{\mathscr{V}_S}) \tag{3-1}$$

determines a shtuka $(\mathscr{V}, \phi_{\mathscr{V}})$ over $\mathcal{X}^\lozenge / \mathrm{Spd}(\mathbb{Z}_p)$.

Definition 3.1. The shtuka $(\mathscr{V}, \phi_{\mathscr{V}})$ over $\mathcal{X}^\lozenge / \mathrm{Spd}(\mathbb{Z}_p)$ defined by (3-1) is the *shtuka associated to the prismatic F-crystal* $(\mathcal{E}, \varphi_{\mathcal{E}})$.

Let us return to the notation of the previous section: Let K denote a complete discretely valued extension of $\mathbb{Q}_p$ with perfect residue field k, and let $\mathcal{O}_K$ be its ring of integers. Let C denote the completion of $\overline{K}$.

We have a functor

$$\mathrm{Sht}(\mathrm{Spd}(K)) \to \mathrm{Vect}(\mathrm{Spf}(\mathcal{O}_K)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}[1/\mathcal{I}_{\mathbb{\Delta}}]^\wedge_p)^{\varphi=1} \tag{3-2}$$

which factorizes the base change

$$\mathrm{Vect}^\varphi(\mathrm{Spf}(\mathcal{O}_K)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}) \to \mathrm{Vect}(\mathrm{Spf}(\mathcal{O}_K)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}[1/\mathcal{I}_{\mathbb{\Delta}}]^\wedge_p)^{\varphi=1}$$

along $\mathcal{O}_{\mathbb{\Delta}} \to \mathcal{O}_{\mathbb{\Delta}}[1/\mathcal{I}_{\mathbb{\Delta}}]^\wedge_p$. To construct this functor, first note that by [Scholze 2017, Example 11.12], there is a natural isomorphism

$$\mathrm{Spd}(C) \times_{\mathrm{Spd}(K)} \mathrm{Spd}(C) \xrightarrow{\sim} \mathrm{Spd}(C^0(\Gamma_K, C), C^0(\Gamma_K, \mathcal{O}_C)).$$

For brevity, let $\tilde{C} = C^0(\Gamma_K, C)$ and $\tilde{C}^+ = C^0(\Gamma_K, \mathcal{O}_C)$. Let $p_1, p_2 : C^\flat \to \tilde{C}^\flat$ be the two morphisms which induce the projections $\mathrm{Spd}(C) \times_{\mathrm{Spd}(K)} \mathrm{Spd}(C) \to \mathrm{Spd}(C)$. Then $\mathrm{Vect}((\mathrm{Spf}(\mathcal{O}_K)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}[1/\mathcal{I}_{\mathbb{\Delta}}]^\wedge_p)^{\varphi=1}$ can be interpreted as the category of $W(C^\flat)$-modules N equipped with a Frobenius $\varphi_N : \varphi^* N \xrightarrow{\sim} N$ and an isomorphism

$$\beta_N : N \otimes_{W(C^\flat), p_1} W(\tilde{C}^\flat) \xrightarrow{\sim} N \otimes_{W(C^\flat), p_2} W(\tilde{C}^\flat), \tag{3-3}$$

which is compatible with the Frobenius on both sides. For this description see, for example, [Wu 2021, Proof of Theorem 5.6].

Now let $\mathscr{V}$ be a shtuka over $\mathrm{Spd}(K)$. The evaluation of $\mathscr{V}$ on $\mathrm{Spa}(C, \mathcal{O}_C)$ can be pulled back along $\mathrm{Spa}(W(C^\flat)) \to \mathrm{Spa}(C^\flat, \mathcal{O}_C^\flat) \dot{\times} \mathbb{Z}_p$ to obtain a φ-module N over $W(C^\flat)$. Furthermore, since $\mathscr{V}$ is defined over $\mathrm{Spd}(K)$, it is equipped with a descent datum $\alpha_{\mathscr{V}} : p_1^* \mathscr{V}_{\mathrm{Spa}(C^\flat)} \xrightarrow{\sim} p_2^* \mathscr{V}_{\mathrm{Spa}(C^\flat)}$ over $\mathrm{Spd}(\tilde{C}, \tilde{C}^+)$. By pulling back this descent datum along $W(\tilde{C}^\flat) \to \mathrm{Spa}(\tilde{C}^\flat, \tilde{C}^{\flat+}) \dot{\times} \mathbb{Z}_p$, we obtain an isomorphism β_N as in (3-3).

Lemma 3.2. *The composition of functors*

$$\mathrm{Sht}(\mathrm{Spd}(K)) \xrightarrow{\text{(2-10)}} \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Spd}(K)) \xrightarrow{\text{(2-7)}} \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Spa}(K)) \tag{3-4}$$

is naturally isomorphic to the composition

$$\mathrm{Sht}(\mathrm{Spd}(K)) \xrightarrow{\text{(3-2)}} \mathrm{Vect}(\mathrm{Spf}(\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge})^{\varphi=1} \xrightarrow{\text{(2-12)}} \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Spa}(K)). \tag{3-5}$$

Proof. Let $\mathscr{V}$ be a shtuka on $\mathrm{Spd}(K)$. Denote by $\mathbb{L}$ and $\mathbb{L}'$ the $\mathbb{Z}_p$-local systems defined by (3-4) and (3-5), respectively. By pro-étale descent, it is enough to show that $\mathbb{L}$ and $\mathbb{L}'$ are isomorphic over $\mathrm{Spa}(C)$, compatibly with their descent data from $\mathrm{Spa}(\tilde{C}, \tilde{C}^+)$.

Following the notation above the statement of the lemma, let N denote the φ-$W(C^\flat)$-module obtained from pulling back the evaluation of $\mathscr{V}$ on $\mathrm{Spa}(C, \mathcal{O}_C)$ to $W(C^\flat)$. Then both $\mathbb{L}_{\mathrm{Spa}(C)}$ and $\mathbb{L}'_{\mathrm{Spa}(C)}$ are given by the finite free $\mathbb{Z}_p$-module $N^{\varphi=1}$. Indeed, that $\mathbb{L}_{\mathrm{Spa}(C)}$ is obtained in this way follows from the fact that the functor from ϕ-modules over the integral Robba ring $\widetilde{\mathcal{R}}^{\mathrm{int}}$ to $\mathbb{Z}_p$-local systems factors through the base extension functor $\widetilde{\mathcal{R}}^{\mathrm{int}} \to W(C^\flat)$, see [Kedlaya and Liu 2015, Theorem 8.5.3] and [Scholze and Weinstein 2020, Theorem 12.3.4]. That $\mathbb{L}'_{\mathrm{Spa}(C)}$ is obtained this way follows from the definitions of (3-2) and (2-12), see (2-13).

It remains to check that the pro-étale descent data are the same. First recall the isomorphism β_N as in (3-3), which defines the Laurent F-crystal associated to $\mathscr{V}$. This is an isomorphism in the category $\mathrm{Vect}(W(\tilde{C}^\flat))^{\varphi=1}$. By Artin–Schreier theory (see, e.g., [Kedlaya and Liu 2015, Proposition 3.2.7]) and tilting, $\mathrm{Vect}(W(\tilde{C}^\flat))^{\varphi=1}$ is equivalent to $\mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Spa}(\tilde{C}))$, and the descent datum for $\mathbb{L}'$ is obtained by passing β_N through this equivalence.

On the other hand, the descent datum for $\mathbb{L}$ is obtained by applying the functor $\mathrm{Sht}(\mathrm{Spd}(\tilde{C})) \to \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Spd}(\tilde{C}))$ from (2-10) to the descent datum of shtukas $\alpha_{\mathscr{V}}$ associated to $\mathscr{V}$. Explicitly, we first obtain from $\alpha_{\mathscr{V}}$ an isomorphism of ϕ-modules over $\widetilde{\mathcal{R}}^{\mathrm{int}}_{\tilde{C}^\flat}$, which induces an isomorphism of $\mathbb{Z}_p$-local systems on $\mathrm{Spa}(\tilde{C})$ by [Kedlaya and Liu 2015, Theorem 8.5.3] and tilting. But once again the equivalence in [loc. cit., Theorem 8.5.3] factors through base change along $\widetilde{\mathcal{R}}^{\mathrm{int}}_{\tilde{C}^\flat} \to W(\tilde{C}^\flat)$, and the isomorphism of ϕ-modules over $W(C^\flat)$ obtained from $\alpha_{\mathscr{V}}$ in this way is equal to β_N. Hence the descent datum for $\mathbb{L}$ is obtained from β_N as well, and the result follows. □

Consider now the composition of functors

$$\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K) \to \mathrm{Sht}(\mathrm{Spd}(\mathcal{O}_K)) \to \mathrm{Sht}(\mathrm{Spd}(K)) \to \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Spd}(K)) \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_K). \tag{3-6}$$

Here the first arrow is obtained by composing U (see (2-14)) with the functor from Definition 3.1, the second arrow is restriction, the third arrow is induced by (2-10), and the fourth arrow is the equivalence from Lemma 2.3.

Lemma 3.3. *The composition* (3-6) *is naturally isomorphic to the inclusion*

$$\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K) \hookrightarrow \mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_K).$$

Proof. By Lemma 3.2, (3-6) can be rewritten as

$$\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K) \to \mathrm{Sht}(\mathrm{Spd}(K)) \to \mathrm{Vect}(\mathrm{Spf}(\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p)^{\varphi=1} \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_K). \tag{3-7}$$

In turn, the composition

$$\mathrm{Vect}^{\varphi}(\mathrm{Spf}(\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) \to \mathrm{Sht}(\mathrm{Spd}(K)) \to \mathrm{Vect}(\mathrm{Spf}(\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p)^{\varphi=1} \tag{3-8}$$

is isomorphic to base change along $\mathcal{O}_{\Delta} \to \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p$. But this base change composed with the final arrow in (3-8) defines T, so (3-6) is given by $T \circ U$, which is naturally isomorphic to the identity. □

Theorem 3.4. *The functor*

$$\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K) \to \mathrm{Sht}(\mathrm{Spd}(\mathcal{O}_K)), \tag{3-9}$$

defined by composing the equivalence from Theorem 2.8 with the functor from Definition 3.1, is fully faithful.

Proof. Fix a uniformizer π of $\mathcal{O}_K$, and let $(\pi^{1/p^n})_n$ be a compatible system of p^n-th roots of π inside $\overline{K}$. Write K_∞ for the perfectoid field given by the p-adic completion of $\bigcup_n K(\pi^{1/p^n})$. Consider the functor

$$\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K) \to \mathrm{Sht}(\mathrm{Spd}(\mathcal{O}_K)) \to \mathrm{Sht}(\mathrm{Spa}(K^{\flat}_\infty, \mathcal{O}^{\flat}_{K_\infty})) \to \mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_{K_\infty})$$

given by postcomposing (3-9) with evaluation on K_∞ and the functor (2-10). By Lemma 3.3, this composition is isomorphic to the restriction functor $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K) \to \mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_{K_\infty})$, so the composition is fully faithful by [Kisin 2006, Corollary 2.1.14]. Hence it remains only to show the functor

$$\mathrm{Sht}(\mathrm{Spd}(\mathcal{O}_K)) \to \mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_{K_\infty}) \tag{3-10}$$

is faithful. In turn, it is enough to show the composition of (3-10) with the forgetful functor $\mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_{K_\infty}) \to \mathrm{Vect}(\mathbb{Z}_p)$ is faithful. But this composition factors as

$$\mathrm{Sht}(\mathrm{Spd}(\mathcal{O}_K)) \to \mathrm{Sht}(\mathrm{Spd}(\mathcal{O}_C)) \to \mathrm{Sht}(\mathrm{Spa}(C^{\flat}, \mathcal{O}^{\flat}_C)) \to \mathrm{Vect}(\mathbb{Z}_p), \tag{3-11}$$

where C is the completion of the algebraic closure $\overline{K}$ of K. The first arrow in (3-11) is faithful since $\mathrm{Spd}(\mathcal{O}_C) \to \mathrm{Spd}(\mathcal{O}_K)$ is a v-cover (see [Scholze and Weinstein 2020, Lemma 18.1.2]), the second is faithful by [Pappas and Rapoport 2024, Theorem 2.7.6], and the third is faithful by Fargues's theorem [Scholze and Weinstein 2020, Theorem 14.1.1]. □

Remark 3.5. The idea of the proof of Theorem 3.4 follows the ideas of [Pappas and Rapoport 2024, Proposition 2.2.17, arXiv v2]. Note, however, that the construction given in Example 2.2.16 of [loc. cit.] does not work as written, and the proposition was removed from subsequent versions.

Remark 3.6. It is not hard to see that the functor (3-9) is faithful, since U is fully faithful and the functor from Definition 3.1 is essentially given by restriction to perfectoid objects. The difficulty lies in proving it is full. Indeed, a priori one needs descent data to recover the morphism of prismatic F-crystals from the morphism of shtukas, but $\mathcal{O}_C \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C$ is only quasiregular semiperfectoid, not perfectoid.

Corollary 3.7. *The functor*

$$\mathrm{Vect}^{\varphi}((\mathrm{Spd}(\mathcal{O}_K))_{\Delta}, \mathcal{O}_{\Delta}) \to \mathrm{Sht}(\mathrm{Spd}(\mathcal{O}_K))$$

from Definition 3.1 is fully faithful. □

3.2. *$\mathcal{G}(\mathbb{Z}_p)$-valued crystalline representations and $\mathcal{G}$-shtukas.* In this section we define a Tannakian variant of the construction from the previous section. Let K and $\mathcal{O}_K$ be as in the previous section, and assume moreover that the residue field k of K is either finite or algebraically closed. Let $\mathcal{G}$ be a smooth group scheme over $\mathbb{Z}_p$ with connected fibers and generic fiber $G = \mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Definition 3.8. A *$\mathcal{G}$-valued crystalline representation of* Γ_K is an exact $\mathbb{Z}_p$-linear tensor functor

$$\alpha : \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K).$$

We denote the category of $\mathcal{G}$-valued crystalline representations of Γ_K by $\mathcal{G}\text{-}\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K)$.

Remark 3.9. Let α be a $\mathcal{G}$-valued crystalline representation of Γ_K. Then, by Lemma 2.6, the composition

$$\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \xrightarrow{\alpha} \mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K) \hookrightarrow \mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_K) \xrightarrow{\sim} \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Spd}\, K)$$

is pro-étale locally trivial. It follows from this and Remark 2.5 that the composition of α with the forgetful functor $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K) \to \mathrm{Vect}(\mathbb{Z}_p)$ yields the forgetful functor $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Vect}(\mathbb{Z}_p)$.

The following is the main result in this section. Let $D^{\times} = \mathrm{Spec}(W(k))[\![u]\!] \setminus \{(p, u)\}$. In the following proposition we will assume that all $\mathcal{G}$-torsors over $D^{\times}$ are trivial. This is the case when $\mathcal{G}$ is parahoric by Anschütz's theorem, [Anschütz 2022, Corollary 1.2].

Proposition 3.10. *Suppose $\mathcal{G}$ is a smooth group scheme over $\mathbb{Z}_p$ with connected fibers and with the property that every $\mathcal{G}$-torsor over $D^{\times}$ is trivial. Then for every $\mathcal{G}$-valued crystalline representation α of Γ_K, the functor*

$$U \circ \alpha : \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Vect}^{\varphi}(\mathrm{Spf}(\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta})$$

is an exact $\mathbb{Z}_p$-linear tensor functor.

Proof. Let α be a $\mathcal{G}$-valued crystalline representation of Γ_K, and suppose (A, I) is a prism over $\mathrm{Spf}(\mathcal{O}_K)$. Define ω^{α}_A to be the composition of functors

$$\omega^{\alpha}_A : \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \xrightarrow{\alpha} \mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K) \xrightarrow{U} \mathrm{Vect}^{\varphi}(\mathrm{Spf}(\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) \to \mathrm{Vect}(A), \tag{3-12}$$

where the final arrow is given by the composition of evaluation on $((A, I), \mathrm{Spf}(A/I) \to \mathrm{Spf}(\mathcal{O}_K))$ and the forgetful functor $\mathrm{Vect}^{\varphi}(A) \to \mathrm{Vect}(A)$. It is enough to prove that ω^{α}_A is an exact $\mathbb{Z}_p$-linear tensor functor for every prism (A, I) over $\mathrm{Spf}(\mathcal{O}_K)$.

By Theorem 2.8, to show that ω^{α}_A is a tensor functor, we need only show that evaluation

$$\mathrm{Vect}(\mathrm{Spf}(\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) \to \mathrm{Vect}(A)$$

is a tensor functor. In general, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are two prismatic crystals, then their tensor product $\mathcal{E}_1 \otimes_{\mathcal{O}_{\Delta}} \mathcal{E}_2$ is the sheaf associated to the presheaf $(B, J) \mapsto \mathcal{E}_1(B) \otimes_B \mathcal{E}_2(B)$. But this is already a sheaf for the faithfully flat topology on prisms by the crystal properties of $\mathcal{E}_1$ and $\mathcal{E}_2$ combined with the sheaf property of $\mathcal{O}_{\Delta}$ and the flatness of $\mathcal{E}_i(B)$.

It remains to show that ω^{α}_A is exact. We first show this for the Breuil–Kisin prism. Let $\mathfrak{S} = W(k)[\![u]\!]$. By choosing a uniformizer $\pi \in \mathcal{O}_K$ we obtain a surjection $\theta_K : \mathfrak{S} \to \mathcal{O}_K$ by sending u to π, and the kernel of this map is generated by an Eisenstein polynomial $E(u)$. The ring $\mathfrak{S}$ becomes a δ-ring in the sense of [Bhatt and Scholze 2022] when we extend the Witt vector Frobenius to $\mathfrak{S}$ by $u \mapsto u^p$, and $(\mathfrak{S}, (E(u)))$ defines a bounded prism (called the *Breuil–Kisin prism* in [Bhatt and Scholze 2023]). Moreover, the surjection θ_K endows $(\mathfrak{S}, (E(u)))$ with the structure of prism over $\mathrm{Spf}(\mathcal{O}_K)$. We claim the tensor functor $\omega^{\alpha}_{\mathfrak{S}} : \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Vect}(\mathfrak{S})$ is exact.

By [Bhatt and Scholze 2023, Remark 7.11], the functor

$$\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K) \to \mathrm{Vect}^{\varphi}(\mathrm{Spf}(\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) \to \mathrm{Vect}^{\varphi}(\mathfrak{S})$$

is naturally isomorphic to the functor $\mathfrak{M}$ defined by Kisin [2010, Theorem 1.2.1]. Since all $\mathcal{G}$-torsors on $D^{\times}$ are trivial, the composition

$$\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \xrightarrow{\alpha} \mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K) \xrightarrow{\mathfrak{M}} \mathrm{Vect}^{\varphi}(\mathfrak{S}) \to \mathrm{Vect}(\mathfrak{S}),$$

which is isomorphic to $\omega^{\alpha}_{\mathfrak{S}}$, is exact by [Kisin et al. 2021, Lemma 4.4.5] (note that the proof in [loc. cit.] is in the case where k is finite, but the same argument goes through when k is algebraically closed). Therefore $\omega^{\alpha}_{\mathfrak{S}}$ is exact.

Now let (A, I) be an arbitrary bounded prism over $\mathrm{Spf}(\mathcal{O}_K)$. By [Bhatt and Scholze 2023, Example 2.6(1)], there exists a faithfully flat map of prisms $(A, I) \to (B, IB)$ for which there is a map $(\mathfrak{S}, (E(u))) \to (B, IB)$ in $\mathrm{Spf}(\mathcal{O}_K)_{\Delta}$. By the crystal property, the functor ω^{α}_B is naturally isomorphic to the functor

$$\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Vect}(B), \quad (\Lambda, \rho) \mapsto \omega^{\alpha}_{\mathfrak{S}}(\Lambda) \otimes_{\mathfrak{S}} B,$$

and therefore ω^{α}_B is exact as well. Moreover, the crystal property applied again implies that ω^{α}_B is naturally isomorphic to the functor $(\Lambda, \rho) \mapsto \omega^{\alpha}_A(\Lambda) \otimes_A B$, and therefore we can conclude that ω^{α}_A is exact by adically flat descent; see, e.g., [Fujiwara and Kato 2018, Proposition 6.1.11, Chapter I]. □

For any tensor functors $\omega : \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Vect}(A)$, denote by $\omega \otimes_A B$ the functor given by postcomposition with the base change $\mathrm{Vect}(A) \to \mathrm{Vect}(B)$. If ω_1 and ω_2 are two tensor functors $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Vect}(A)$, let $\underline{\mathrm{Isom}}^{\otimes}(\omega_1, \omega_2)$ be the fpqc sheaf on $\mathrm{Spec}(A)$ which assigns to any A-algebra B the set of isomorphisms of tensor functors $\omega_1 \otimes_A B \xrightarrow{\sim} \omega_1 \otimes_A B$. Write $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ for $\underline{\mathrm{Isom}}^{\otimes}(\omega, \omega)$.

For any $\mathbb{Z}_p$-algebra A, define the standard fiber functor by

$$\mathbb{1}_A : \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Vect}(A), \quad (\Lambda, \rho) \mapsto \Lambda \otimes_{\mathbb{Z}_p} A.$$

By the reconstruction theorem in Tannakian duality (see [Wedhorn 2004, Theorem 5.17] for the statement in this situation), the canonical morphism $\mathcal{G} \to \underline{\mathrm{Aut}}^{\otimes}(\mathbb{1}_{\mathbb{Z}_p})$ is an isomorphism of fpqc sheaves. As a result, $\underline{\mathrm{Isom}}^{\otimes}(\mathbb{1}_A, \omega)$ carries a natural right $\mathcal{G}_A$-action for any tensor functor $\omega : \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Vect}(A)$. It follows from the Tannakian formalism [Scholze and Weinstein 2020, Theorem 19.5.1] and Proposition 3.10 that when $\omega = \omega^{\alpha}_A$ is defined as in (3-12), $\underline{\mathrm{Isom}}^{\otimes}(\mathbb{1}, \omega^{\alpha}_A)$ is a $\mathcal{G}$-torsor over $\mathrm{Spec}(A)$, which we denote by $\mathscr{P}^{\alpha}_A$.

The Frobenius morphisms for $(\alpha \circ U)(\Lambda, \rho)$ as (Λ, ρ) varies piece together to define an isomorphism of $\mathcal{G}$-torsors

$$\phi_{\mathscr{P}^{\alpha}_A} : \varphi^*_A(\mathscr{P}^{\alpha}_A)\big|_{\mathrm{Spec}(A)\setminus V(I)} \xrightarrow{\sim} \mathscr{P}^{\alpha}_A\big|_{\mathrm{Spec}(A)\setminus V(I)}.$$

over $\mathrm{Spec}(A) \setminus V(I)$. In particular, if $S = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space over k, with untilt $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ admitting a homomorphism $\mathcal{O}_K \to R^{\sharp+}$, this construction determines a $\mathcal{G}$-BKF-module $(\mathscr{P}^{\alpha}_{W(R^+)}, \phi_{\mathscr{P}^{\alpha}_{W(R^+)}})$ over S with one leg at $S^\sharp$. Pulling back along

$$S \dot{\times} \mathbb{Z}_p \to \mathrm{Spa}(W(R^+)) \to \mathrm{Spec}(W(R^+)),$$

we obtain a $\mathcal{G}$-shtuka over S with one leg at $S^\sharp$, which we denote by $(\mathscr{P}^{\alpha}_S, \phi_{\mathscr{P}^{\alpha}_S})$. It follows that the assignment

$$(S^\sharp, f) \in (\mathrm{Spd}(\mathcal{O}_K))(S) \mapsto (\mathscr{P}^{\alpha}_S, \phi_{\mathscr{P}^{\alpha}_S}) \tag{3-13}$$

defines a shtuka over $\mathrm{Spd}(\mathcal{O}_K)$.

Definition 3.11. Let α be a $\mathcal{G}$-valued crystalline representation of Γ_K. The shtuka $(\mathscr{P}^{\alpha}, \phi_{\mathscr{P}^{\alpha}})$ over $\mathrm{Spd}(\mathcal{O}_K)$ given by the assignment (3-13) is the *$\mathcal{G}$-shtuka associated* α.

The assignment $\alpha \mapsto (\mathscr{P}^{\alpha}, \phi_{\mathscr{P}^{\alpha}})$ satisfies the following obvious functorialities.

Lemma 3.12. *Let $\mathcal{G}$ be a parahoric group scheme over $\mathbb{Z}_p$, and let α be a $\mathcal{G}$-valued crystalline representation of Γ_K:*

(a) *Suppose $\rho : \mathcal{G} \to \mathcal{G}'$ is a morphism of parahoric group schemes over $\mathbb{Z}_p$. Then*

$$\mathscr{P}^{\rho^*(\alpha)} \cong \mathscr{P}^{\alpha} \times^{\mathcal{G}} \mathcal{G}',$$

where $\rho^*(\alpha)$ is the $\mathcal{G}'$-valued representation of $\mathcal{G}'$ induced via ρ by α.

(b) *Let K' be an extension of K inside of $\overline{K}$, and let α' be the restriction of α to $\Gamma_{K'}$. Then*

$$\mathscr{P}^{\alpha}\big|_{\mathrm{Spd}(\mathcal{O}_{K'})} \cong \mathscr{P}^{\alpha'}.$$

Proof. The proof is straightforward and left to the reader. □

We close this section with the group theoretic analog of Theorem 3.4.

Theorem 3.13. *The functor*

$$\mathcal{G}\text{-}\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K) \to \mathrm{Sht}_{\mathcal{G}}(\mathrm{Spd}(\mathcal{O}_K))$$

from Definition 3.11 is fully faithful.

Proof. This follows from Theorem 3.4 and the Tannakian formalism. □

3.3. ***The generic fiber.*** In the remainder of this section we show that the construction in Definition 3.11 is compatible with those of [Pappas and Rapoport 2024] over the generic fiber. To be more specific, suppose

$$\alpha : \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K)$$

is a $\mathcal{G}$-valued crystalline representation of Γ_K. Then to α we can attach two $\mathcal{G}$-shtukas over $\mathrm{Spd}(K)$. We have, on the one hand, the $\mathcal{G}$-shtuka $(\mathscr{P}^\alpha, \phi_{\mathscr{P}^\alpha})$ obtained by pulling back the $\mathcal{G}$-shtuka from Definition 3.11 along $\mathrm{Spd}(K) \to \mathrm{Spd}(\mathcal{O}_K)$. On the other hand, the representation $\alpha(\Lambda, \rho) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline for every (Λ, ρ), and therefore also de Rham. Hence we can attach to α a $\mathcal{G}$-shtuka $(\mathscr{P}^\alpha_{\mathrm{dR}}, \phi_{\mathscr{P}^\alpha_{\mathrm{dR}}})$ over $\mathrm{Spd}(K)$ using [Pappas and Rapoport 2024, Definition 2.6.6].

Lemma 3.14. *The restriction of the $\mathcal{G}$-shtuka $(\mathscr{P}^\alpha, \phi_{\mathscr{P}^\alpha})$ along* $\mathrm{Spd}(K) \to \mathrm{Spd}(\mathcal{O}_K)$ *is given by*

$$(\mathscr{P}^\alpha_{\mathrm{dR}}, \phi_{\mathscr{P}^\alpha_{\mathrm{dR}}}).$$

Proof. By [Pappas and Rapoport 2024, Proposition 2.5.1], it's enough to show that the two $\mathcal{G}$-shtukas determine the same pair $(\mathbb{P}, D)$, where $\mathbb{P}$ is a pro-étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsor on $\mathrm{Spd}(K)$ and D is a $\underline{\mathcal{G}(\mathbb{Z}_p)}$-equivariant morphism of v-sheaves $D : \mathbb{P} \to \mathrm{Gr}_{G,\mathrm{Spd}(K)}$ over $\mathrm{Spd}(K)$. Here, $\mathrm{Gr}_{G,\mathrm{Spd}(K)}$ denotes the Beilinson–Drinfeld Grassmannian over $\mathrm{Spd}(K)$, see [Scholze and Weinstein 2020, Definition 20.2.1].

Let us first address the pro-étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsors. By Lemma 2.6, it's enough to show the two determine the same tensor functor $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Spd}(K))$. We claim that for both $\mathscr{P}^\alpha$ and $\mathscr{P}^\alpha_{\mathrm{dR}}$, this functor is given by assigning to $(\Lambda, \rho) \in \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G})$ and $(S^\sharp, f) \in \mathrm{Spd}(K)(S)$ the local system obtained by applying the tilting equivalence to the pullback of $\alpha(\Lambda, \rho)$ along $f : S^\sharp \to \mathrm{Spa}(K)$. For $\mathscr{P}^\alpha_{\mathrm{dR}}$ this follows from [Pappas and Rapoport 2024, Definition 2.6.6].

Let $\mathbb{P}^\alpha$ denote the pro-étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsor corresponding to $\mathscr{P}^\alpha$. By the definition of (2-11), for every $(S^\sharp, f) \in \mathrm{Spd}(K)(S)$, the tensor functor corresponding to $\mathbb{P}^\alpha_S$ factors as

$$\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to (\text{shtukas over } S \text{ with one leg at } S^\sharp) \xrightarrow{\text{(2-10)}} \mathrm{Loc}_{\mathbb{Z}_p}(S),$$

where here the first arrow assigns to (Λ, ρ) the vector bundle shtuka $\mathscr{V}^\alpha_\rho$ over S with one leg at $S^\sharp$ obtained by pushing forward $\mathscr{P}^\alpha_S$ along ρ. By definition of $\mathscr{P}^\alpha$, the shtuka $\mathscr{V}^\alpha_{S,\rho}$ is the pullback to $\mathrm{Spd}(K)$ of the shtuka associated to the prismatic F-crystal $U(\alpha(\Lambda, \rho))$. Hence it follows from Lemma 3.3 that the $\mathbb{Z}_p$-local system on S corresponding to $\mathscr{V}^\alpha_{S,\rho}$ is obtained by tilting the pullback of $\alpha(\Lambda, \rho)$ to $S^\sharp$ along f, as desired.

It remains to show that the two morphisms $D : \mathbb{P} \to \mathrm{Gr}_{G,\mathrm{Spd}(K)}$ agree. Let C be the completion of $\overline{K}$. Since $\mathrm{Spd}(C) \to \mathrm{Spd}(K)$ is a pro-étale cover, it's enough to show the two morphisms agree after base change to $\mathrm{Spd}(C)$. Moreover, we can reduce to the case $\mathcal{G} = \mathrm{GL}_n$ by the Tannakian formalism. In this case pairs $(\mathbb{P}, D)$ correspond to pairs (T_0, Ξ), where T_0 is a finite free $\mathbb{Z}_p$-module and Ξ is a $B^+_{\mathrm{dR}}(C)$-lattice in $T_0 \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}(C)$.

Suppose T is a crystalline representation of Γ_K on a finite free $\mathbb{Z}_p$-module with corresponding shtukas $(\mathscr{V}, \phi_{\mathscr{V}})$ coming from Definition 3.1 and $(\mathscr{V}_{\mathrm{dR}}, \phi_{\mathscr{V}_{\mathrm{dR}}})$ coming from [Pappas and Rapoport 2024, Definition 2.6.4]. By the first part of the proof, we know both $(\mathscr{V}, \phi_{\mathscr{V}})$ and $(\mathscr{V}_{\mathrm{dR}}, \phi_{\mathscr{V}_{\mathrm{dR}}})$ correspond to the finite free $\mathbb{Z}_p$-module underlying T. Moreover, the corresponding $B^+_{\mathrm{dR}}(C)$-lattice Ξ is given by $D_{\mathrm{dR}}(T[1/p]) \otimes_K B^+_{\mathrm{dR}}(C)$ in both cases. Indeed, for $(\mathscr{V}_{\mathrm{dR}}, \phi_{\mathscr{V}_{\mathrm{dR}}})$ this follows from [loc. cit., Proposition 2.6.3]. On the other hand, let (M, φ_M) denote the BKF-module coming from the evaluation of $U(T)$ on $(W(\mathcal{O}^\flat_C), \ker(\theta))$. Then since $(\mathscr{V}, \phi_{\mathscr{V}})$ is the shtuka associated to $U(T)$ as in Definition 3.1, it follows from the proof of [Bhatt and Scholze 2023, Theorem 5.2] that the $B^+_{\mathrm{dR}}(C)$-lattice associated to $(\mathscr{V}, \phi_{\mathscr{V}})$ is $\phi^* M \otimes_{W(\mathcal{O}^\flat_C)} B^+_{\mathrm{dR}}(C)$. But by [loc. cit., Remark 5.4], this agrees with $D_{\mathrm{dR}}(T[1/p]) \otimes_K B^+_{\mathrm{dR}}(C)$. □

Suppose α is a $\mathcal{G}$-valued crystalline representation of Γ_K. For any representation (V, ρ) of the generic fiber G of $\mathcal{G}$, the representation $\alpha(V, \rho) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of Γ_K is crystalline, therefore Hodge–Tate. The functor assigning to any (V, ρ) the grading on $D_{\mathrm{HT}}(V)$ defines a graded fiber functor on $\mathrm{Rep}_{\mathbb{Q}_p}(G_{\overline{K}})$ in the sense of [Ziegler 2015, Definition 3.1], and therefore it follows from the Tannakian formalism (see, e.g., [Ziegler 2015, Construction 3.4]) that the grading on $D_{\mathrm{HT}}(V)$ as V varies is determined by a cocharacter μ of $G_{\overline{K}}$, called the *Hodge–Tate cocharacter* for α.

Lemma 3.15. *Let α be a $\mathcal{G}$-valued crystalline representation of Γ_K with Hodge–Tate cocharacter μ. Then the shtuka $(\mathscr{P}^\alpha, \phi_{\mathscr{P}^\alpha})$ over $\mathrm{Spd}(\mathcal{O}_K)$ is bounded by μ.*

Proof. The $\mathcal{G}$-shtuka $\mathscr{P}^\alpha_{\mathrm{dR}}$ is bounded by μ by [Pappas and Rapoport 2024, Definition 2.6.6], so Lemma 3.14 implies that $\mathscr{P}^\alpha_{\mathrm{Spd}(K)}$ is bounded by μ. Since $\mathrm{Spd}(\mathcal{O}_K)$ is topologically flat by [Anschütz et al. 2022, Lemma 2.17], the result follows from Lemma 2.1. □

4. Shimura varieties of toral type

4.1. ***Shimura varieties and crystalline representations.*** In this section we recall some definitions and notation from the theory of (global) Shimura varieties. Let (G, X) be a Shimura datum, meaning that G is a connected reductive group over $\mathbb{Q}$, and X is a $\mathsf{G}(\mathbb{R})$-conjugacy class of homomorphisms

$$h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m \to \mathsf{G}_{\mathbb{R}}$$

satisfying the axioms (SV1)-(SV3) in [Milne 2005, Defnition 5.5]. Associated to any $h \in X$ is a cocharacter μ_h of $\mathsf{G}_{\mathbb{C}}$, whose conjugacy class is defined over a number field E, called the reflex field of (G, X). Denote by $\{\mu\}$ the $G(\overline{\mathbb{Q}})$-conjugacy class of any μ_h for $h \in X$. We will also refer to any choice of an element of $\{\mu\}$ as μ.

Let $\mathbb{A}_f$ denote the finite adeles over $\mathbb{Q}$ and $\mathbb{A}^p_f$ denote the finite adeles away from p. For any compact open subgroup $\mathsf{K} \subset \mathsf{G}(\mathbb{A}^p_f)$, we can attach to (G, X) and K the Shimura variety $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)$, which is a quasiprojective variety over $\mathbb{C}$ whose $\mathbb{C}$-points are given by

$$\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)(\mathbb{C}) = \mathsf{G}(\mathbb{Q})\backslash X \times \mathsf{G}(\mathbb{A}_f)/\mathsf{K}.$$

There exists a canonical model for $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)$ over the reflex field E, which we will denote by $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)_{\mathsf{E}}$.

Let Z denote the center of G, and let Z° be the identity component of Z. We write Z_{ac} denote the anticuspidal part of Z° in the sense of [Kisin et al. 2021, Definition 1.5.4]. Then $\mathsf{Z}^\circ/\mathsf{Z}_{ac}$ is cuspidal in the sense of [Kisin et al. 2021], i.e., has equal $\mathbb{Q}$-split rank and $\mathbb{R}$-split rank. Denote by G^c the quotient $\mathsf{G}/\mathsf{Z}_{ac}$, and for any subgroup $\mathsf{H} \subset \mathsf{G}(\mathbb{A}_f)$, denote by H^c the image of H under $\mathsf{G} \to \mathsf{G}^c$.

Fix a prime p, and let $G = \mathsf{G} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $G^c = \mathsf{G}^c \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Assume $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p \subset \mathsf{G}(\mathbb{A}_f)$, where K_p and K^p are compact open subgroups of $\mathsf{G}(\mathbb{Q}_p)$ and $\mathsf{G}(\mathbb{A}_f^p)$, respectively. We will henceforth assume K^p is neat.

Suppose K_p is a parahoric subgroup of $G(\mathbb{Q}_p)$. Then K_p is the connected stabilizer of a point x in the extended Bruhat–Tits building $\mathscr{B}(G, \mathbb{Q}_p)$ for G over $\mathbb{Q}_p$. We will denote by $\mathcal{G}$ the corresponding parahoric group scheme defined in [Bruhat and Tits 1984], so $\mathcal{G}$ is a smooth affine group scheme over $\mathbb{Z}_p$ which satisfies $\mathcal{G}(\mathbb{Z}_p) = \mathsf{K}_p$ and $\mathcal{G}_{\mathbb{Q}_p} = G$. Since $G \to G^c$ is a central extension, by [Landvogt 2000, Theorem 2.1.8] we obtain a canonical $G(\mathbb{Q}_p)$-equivariant map of extended Bruhat–Tits buildings $\mathscr{B}(G, \mathbb{Q}_p) \to \mathscr{B}(G^c, \mathbb{Q}_p)$. Let x^c denote the image of x under this map, and denote by $\mathcal{G}^c$ the parahoric group scheme corresponding to x^c. By [Bruhat and Tits 1984, 1.7.6], the homomorphism $G \to G^c$ extends to a homomorphism of $\mathbb{Z}_p$-group schemes $\mathcal{G} \to \mathcal{G}^c$, see for comparison a related discussion in [Kisin and Pappas 2018, 1.1.3]. On $\mathbb{Z}_p$-points, $\mathcal{G}(\mathbb{Z}_p) \to \mathcal{G}^c(\mathbb{Z}_p)$ factors as a composition

$$\mathcal{G}(\mathbb{Z}_p) = \mathsf{K}_p \twoheadrightarrow \mathsf{K}_p^c \hookrightarrow \mathcal{G}^c(\mathbb{Z}_p). \tag{4-1}$$

Remark 4.1. A more concrete description of $\mathcal{G}^c$ is available in some cases. If $Z_{ac} = \ker(G \to G^c)$ is R-smooth in the sense of [Kisin and Zhou 2021, Definition 2.4.3], then $\mathcal{G}^c$ is the quotient of $\mathcal{G}$ by the Zariski closure $\mathcal{Z}_{ac}$ of Z_{ac} inside of $\mathcal{G}$ by [loc. cit., Proposition 2.4.14]. This happens in particular if G splits over a tamely ramified extension, see [loc. cit., 2.4.5]; see [Kisin and Pappas 2018, Proposition 1.1.4].

In the remainder of this section, we focus on the case where $\mathsf{G} = \mathsf{T}$ is a $\mathbb{Q}$-torus. Then the unique parahoric subgroup K_p of $T(\mathbb{Q}_p)$ corresponds to the identity component $\mathcal{T}$ of the Néron model of T, and the conjugacy class X reduces to a single homomorphism $h : \mathbb{S} \to \mathsf{T}_{\mathbb{R}}$. For any neat compact open subgroup $\mathsf{K} \subset \mathsf{T}(\mathbb{A}_f)$, we obtain a zero-dimensional Shimura variety, which is given by a finite set of points

$$\mathrm{Sh}_{\mathsf{K}}(\mathsf{T}, \{h\}) = \coprod_{i \in I} \operatorname{Spec} \mathsf{E}_i, \tag{4-2}$$

where each E_i is a finite extension of E.

We will define below certain $\mathcal{T}^c(\mathbb{Z}_p)$-valued crystalline representations of $\mathrm{Gal}(\bar{E}/E)$, which will be used to define shtukas over an integral model for $\mathrm{Sh}_{\mathsf{K}}(\mathsf{T}, \{h\})$. Since T is a torus, $\{\mu\}$ is a singleton set, and therefore μ is defined over E. Denote by $r(\mu)^{\mathrm{alg}}$ the following homomorphism of algebraic groups over $\mathbb{Q}$:

$$\mathrm{Res}_{\mathsf{E}/\mathbb{Q}} \mathbb{G}_m \xrightarrow{\mathrm{Res}_{\mathsf{E}/\mathbb{Q}} \mu} \mathrm{Res}_{\mathsf{E}/\mathbb{Q}} \mathsf{T} \xrightarrow{\mathrm{Nm}_{\mathsf{E}/\mathbb{Q}}} \mathsf{T}. \tag{4-3}$$

Applying $r(\mu)^{\mathrm{alg}}$ to $\mathbb{A}$-points, we obtain a homomorphism $\mathbb{A}_{\mathsf{E}}^\times \to \mathsf{T}(\mathbb{A})$ which, by functoriality, sends $\mathsf{E}^\times = (\mathrm{Res}_{\mathsf{E}/\mathbb{Q}} \mathbb{G}_m)(\mathbb{Q})$ into $\mathsf{T}(\mathbb{Q})$. Hence, $r(\mu)^{\mathrm{alg}}$ induces a homomorphism

$$\mathsf{E}^\times \backslash \mathbb{A}_{\mathsf{E}}^\times \to \mathsf{T}(\mathbb{Q}) \backslash \mathsf{T}(\mathbb{A}).$$

For each open compact subgroup K of $\mathsf{T}(\mathbb{A}_f)$, the quotient $\mathsf{T}(\mathbb{Q})\backslash\mathsf{T}(\mathbb{A}_f)/\mathsf{K}$ is finite, so the natural map

$$\mathsf{T}(\mathbb{Q})\backslash\mathsf{T}(\mathbb{A}) \to \mathsf{T}(\mathbb{Q})\backslash\mathsf{T}(\mathbb{A}_f)/\mathsf{K}$$

factors through

$$\pi_0(\mathsf{T}(\mathbb{Q})\backslash\mathsf{T}(\mathbb{A})),$$

and the composition

$$\mathsf{E}^\times\backslash\mathbb{A}_\mathsf{E}^\times \to \mathsf{T}(\mathbb{Q})\backslash\mathsf{T}(\mathbb{A}_f)/\mathsf{K}$$

factors through

$$\pi_0(\mathsf{E}^\times\backslash\mathbb{A}_\mathsf{E}^\times) \cong \mathrm{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E})$$

via the global Artin homomorphism, Art_E (which we normalize geometrically):

$$\mathsf{E}^\times\backslash\mathbb{A}_\mathsf{E}^\times \xrightarrow{\mathrm{Art}_\mathsf{E}} \mathrm{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E}) \to \mathsf{T}(\mathbb{Q})\backslash\mathsf{T}(\mathbb{A}_f)/\mathsf{K}.$$

We will denote the resulting factorization by $r(\mu)_\mathsf{K} : \mathrm{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E}) \to \mathsf{T}(\mathbb{Q})\backslash\mathsf{T}(\mathbb{A}_f)/\mathsf{K}$.

We can now pin down the fields E_i appearing in (4-2) more precisely. Let E_K be the finite extension of E with the property that

$$\mathrm{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E}_\mathsf{K}) = \ker(r(\mu)_\mathsf{K}). \tag{4-4}$$

Lemma 4.2. *Each of the fields* E_i *is isomorphic to* E_K.

Proof. This follows by recalling the definition of the canonical model for $\mathrm{Sh}_\mathsf{K}(\mathsf{T}, \{h\})$ over E. For $\sigma \in \mathrm{Gal}(\bar{\mathsf{E}}/\mathsf{E})$ and $x \in \mathrm{Sh}_\mathsf{K}(\mathsf{T}, \{h\})(\bar{E}) \cong \mathsf{T}(\mathbb{Q})\backslash\mathsf{T}(\mathbb{A}_f)/\mathsf{K}$, one uses $r(\mu)_\mathsf{K}$ to define the reciprocity law

$$\sigma(x) = r(\mu)_\mathsf{K}(\sigma)\cdot x, \tag{4-5}$$

which determines a continuous action of $\mathrm{Gal}(\bar{E}/E)$ on the finite set $\mathrm{Sh}_\mathsf{K}(\mathsf{T}, \{h\})$, and therefore the structure of a finite étale E-scheme on $\mathrm{Sh}_\mathsf{K}(\mathsf{T}, \{h\})$. The isomorphisms $\mathsf{E}_i \cong \mathsf{E}_\mathsf{K}$ then follow immediately from the definition of the action (4-5). ☐

Passing to the limit along open compact subsets K of $\mathsf{T}(\mathbb{A}_f)$, we obtain from the collection $\{r(\mu)_\mathsf{K}\}$ a map

$$r(\mu) : \mathrm{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E}) \to \mathsf{T}(\mathbb{Q})^-\backslash\mathsf{T}(\mathbb{A}_f),$$

where $\mathsf{T}(\mathbb{Q})^-$ is the closure of $\mathsf{T}(\mathbb{Q})$ in $\mathsf{T}(\mathbb{A}_f)$. Fix now a neat compact open subgroup $\mathsf{K} \subset \mathsf{T}(\mathbb{A}_f)$, and let pr_K be the projection $\mathsf{T}(\mathbb{Q})^-\backslash\mathsf{T}(\mathbb{A}_f) \to \mathsf{T}(\mathbb{Q})\backslash\mathsf{T}(\mathbb{A}_f)/\mathsf{K}$. The kernel of pr_K is $\mathsf{T}(\mathbb{Q})^-\backslash\mathsf{T}(\mathbb{Q})^-\mathsf{K}$, so by definition $r(\mu)$ induces a map

$$r(\mu)_\mathsf{K} : \mathrm{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E}_\mathsf{K}) \to \mathsf{T}(\mathbb{Q})^-\backslash\mathsf{T}(\mathbb{Q})^-\mathsf{K},$$

which we also denote by $r(\mu)_\mathsf{K}$.

Recall that K^c denotes the image in $\mathsf{T}^c(\mathbb{A}_f)$ of any subgroup K of $\mathsf{T}(\mathbb{A}_f)$. Note that if K is neat, then K^c is neat as well. By cuspidality of T^c, $\mathsf{T}^c(\mathbb{Q})$ is discrete in $\mathsf{T}(\mathbb{A}_f)$ by [Milne 2005, Theorem 5.26], and $\mathsf{K}^c \cap \mathsf{T}^c(\mathbb{Q}) = \{1\}$; see also Lemma 1.5.5 and the proof of Lemma 1.5.7 in [Kisin et al. 2021]. Then $\mathsf{T} \to \mathsf{T}^c$ induces

$$\mathsf{T}(\mathbb{Q})^-\backslash\mathsf{T}(\mathbb{Q})^-\mathsf{K} \to \mathsf{T}^c(\mathbb{Q})\backslash\mathsf{T}^c(\mathbb{Q})\mathsf{K}^c \cong \mathsf{K}^c. \tag{4-6}$$

Denote by $r(\mu)_{\mathsf{K},p}$ the composition

$$r(\mu)_{\mathsf{K},p} : \mathrm{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E}_{\mathsf{K}}) \xrightarrow{r(\mu)_{\mathsf{K}}} \mathsf{T}(\mathbb{Q})^-\backslash\mathsf{T}(\mathbb{Q})^-\mathsf{K} \to \mathsf{K}^c \hookrightarrow \mathsf{T}^c(\mathbb{A}_f) \to T^c(\mathbb{Q}_p),$$

where the last arrow denotes the projection. Let v denote the place above p induced by $\mathsf{E}_{\mathsf{K}} \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and let E_{K} denote the completion of E_{K} at v. Then the composition of $r(\mu)_{\mathsf{K},p}$ with the morphism of Galois groups

$$\Gamma_{E_{\mathsf{K}}} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/E_{\mathsf{K}}) \twoheadrightarrow \mathrm{Gal}(E^{\mathrm{ab}}/E_{\mathsf{K}}) \to \mathrm{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E}_{\mathsf{K}})$$

induces a $T^c(\mathbb{Q}_p)$-valued representation of $\Gamma_{E_{\mathsf{K}}}$,

$$r(\mu)_{\mathsf{K},p,\mathrm{loc}} : \Gamma_{E_{\mathsf{K}}} \to T^c(\mathbb{Q}_p). \tag{4-7}$$

Notice that if $\mathsf{K} = \mathcal{T}(\mathbb{Z}_p)U^p$ for some small enough compact open subgroup U^p of $\mathsf{T}(\mathbb{A}_f^p)$, then $r(\mu)_{\mathsf{K},p,\mathrm{loc}}$ actually lands in $\mathcal{T}^c(\mathbb{Z}_p)$, so in this case it induces a $\mathcal{T}^c$-valued representation of $\Gamma_{E_{\mathsf{K}}}$.

Suppose now $\rho : T^c \to \mathrm{GL}(W)$ is an algebraic representation of T^c on a finite dimensional $\mathbb{Q}_p$-vector space W. By composition with ρ, $r(\mu)_{\mathsf{K},p,\mathrm{loc}}$ induces a representation of $\Gamma_{E_{\mathsf{K}}}$

$$r(\mu,\rho)_{\mathsf{K},p} : \Gamma_{E_{\mathsf{K}}} \to T^c(\mathbb{Q}_p) \to \mathrm{GL}(W).$$

Lemma 4.3. *Assume that $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p$ where K^p is a neat compact open subgroup of $\mathsf{T}(\mathbb{A}_f^p)$. For any representation $\rho : T^c \to \mathrm{GL}(W)$ of T^c on a finite dimensional $\mathbb{Q}_p$-vector space W, the induced $\Gamma_{E_{\mathsf{K}}}$-representation $r(\mu,\rho)_{\mathsf{K},p}$ is crystalline.*

Proof. This is a straightforward modification of [Kisin et al. 2021, Lemma 4.4]. Let us indicate the main points. By a standard result in p-adic Hodge theory (see [Conrad 2011, Proposition B.4(i)] and the subsequent remark), to show $r(\mu,\rho)_{\mathsf{K},p}$ is crystalline it is enough to show that the composition

$$\mathcal{O}_{E_{\mathsf{K}}}^\times \hookrightarrow E_{\mathsf{K}}^\times \to \Gamma_{E_{\mathsf{K}}}^{\mathrm{ab}} \xrightarrow{r(\mu,\rho)_{\mathsf{K},p}} \mathrm{GL}(W) \tag{4-8}$$

agrees with the restriction to $\mathcal{O}_{E_{\mathsf{K}}}^\times$ of an algebraic $\mathbb{Q}_p$-group homomorphism $\mathrm{Res}_{E_{\mathsf{K}}/\mathbb{Q}_p}\mathbb{G}_m \to \mathrm{GL}(W)$.

Let f be the homomorphism of topological groups

$$f : E_{\mathsf{K}}^\times \xrightarrow{\mathrm{Art}_{E_{\mathsf{K}}}} \Gamma_{E_{\mathsf{K}}}^{\mathrm{ab}} \to \mathrm{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E}_{\mathsf{K}}) \to (\mathsf{K} \cap \mathsf{T}(\mathbb{Q})^-)\backslash\mathsf{K},$$

where $\mathrm{Art}_{E_{\mathsf{K}}}$ is the local Artin map for E_{K} (normalized geometrically), and denote by f^c the composition

$$f^c : E_{\mathsf{K}}^\times \xrightarrow{f} \mathsf{T}(\mathbb{Q})^-\backslash\mathsf{T}(\mathbb{Q})^-\mathsf{K} \to \mathsf{K}^c \hookrightarrow \mathsf{T}^c(\mathbb{A}_f).$$

By local-global compatibility, (4-8) agrees with the composition

$$\mathcal{O}_{E_{\mathsf{K}}}^\times \hookrightarrow E_{\mathsf{K}}^\times \xrightarrow{f^c} T^c(\mathbb{A}_f) \to T^c(\mathbb{Q}_p) \xrightarrow{\rho} \mathrm{GL}(W),$$

so it is enough to show that f^c restricted to $\mathcal{O}_{E_{\mathsf{K}}}^\times$ is algebraic.

Let f_1 be the homomorphism of $\mathbb{Q}_p$-groups

$$f_1 : \mathrm{Res}_{E_{\mathsf{K}}/\mathbb{Q}_p} \mathbb{G}_m \xrightarrow{\mathrm{Nm}_{E_{\mathsf{K}}/E}} \mathrm{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m \hookrightarrow (\mathrm{Res}_{\mathsf{E}/\mathbb{Q}} \mathbb{G}_m) \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{r(\mu)^{\mathrm{alg}} \otimes_{\mathbb{Q}} \mathbb{Q}_p} T,$$

and let f_1^c be the composition of f_1 with $T \to T^c$. Applied to $\mathbb{Q}_p$-points, f_1^c induces a group homomorphism $E_{\mathsf{K}}^\times \to T^c(\mathbb{Q}_p) \hookrightarrow \mathsf{T}^c(\mathbb{A}_f)$. By construction (and local-global compatibility, once more), for any $x \in E_{\mathsf{K}}^\times$, the images of $f_1(x)$ and $f(x)$ agree in $\mathsf{T}(\mathbb{Q})^- \backslash \mathsf{T}(\mathbb{A}_f)$, and therefore the images of $f_1^c(x)$ and $f^c(x)$ agree in $\mathsf{T}^c(\mathbb{Q}) \backslash \mathsf{T}^c(\mathbb{A}_f)$.

From here the arguments of [Kisin et al. 2021, Lemma 4.4] go through verbatim: If $x \in \mathcal{O}_{E_{\mathsf{K}}}^\times$, cuspidality of T^c forces the images of $f_1^c(x)$ and $f^c(x)$ to agree in $T^c(\mathbb{A}_f)$, and the result follows. □

4.2. *The shtuka on the generic fiber.* In this section we return to the general setup at the beginning of Section 4.1 in order to explain how Pappas and Rapoport construct a shtuka over the generic fiber of a given Shimura variety with parahoric level structure.

Let (G, X) be a Shimura datum with reflex field E. Let v denote a place of E corresponding to an embedding $\mathsf{E} \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and let E denote the completion of E at v. If L is an extension of $\mathbb{Q}$, denote by $\mathcal{C}(L)$ the set of $\mathsf{G}(L)$-conjugacy classes of cocharacters $\mathbb{G}_{m,L} \to \mathsf{G}_L$. By [Kottwitz 1984, Lemma 1.1.3], the embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ induces a bijection $\mathcal{C}(\overline{\mathbb{Q}}) \to \mathcal{C}(\overline{\mathbb{Q}}_p)$, so the $G(\overline{\mathbb{Q}})$-conjugacy class of characters $\{\mu_h\}$ coming from (G, X) determines a unique $G(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters $\{\mu\}$ of $G_{\overline{\mathbb{Q}}_p}$. One can check that the map $\mathcal{C}(\overline{\mathbb{Q}}) \to \mathcal{C}(\overline{\mathbb{Q}}_p)$ is equivariant for the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and it follows that E is the local reflex field for $(G, \{\mu\})$, i.e., it is the field of definition for the conjugacy class $\{\mu\}$.

For any neat compact open subgroup $\mathsf{K} \subset \mathsf{G}(\mathbb{A}_f)$, let

$$\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)_E = \mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)_{\mathsf{E}} \otimes_{\mathsf{E}} E.$$

Suppose $\mathsf{K} = \mathsf{K}_p \mathsf{K}^p$, with $\mathsf{K}_p = \mathcal{G}(\mathbb{Z}_p)$ a parahoric subgroup, and $\mathsf{K}^p \subset \mathsf{G}(\mathbb{A}_f^p)$ neat. For each $\mathsf{K}'_p \subset \mathsf{K}_p$, denote by K' the product $\mathsf{K}'_p \mathsf{K}^p$. Then the morphism

$$\mathrm{Sh}_{\mathsf{K}'}(\mathsf{G}, X)_E \to \mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)_E \tag{4-9}$$

is a finite étale Galois cover, whose Galois group we denote by $\mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}'} / \mathrm{Sh}_{\mathsf{K}})$. We also consider the infinite level Shimura variety

$$\mathrm{Sh}_{\mathsf{K}^p}(\mathsf{G}, X)_E := \varprojlim_{\mathsf{K}'_p \subset \mathsf{K}_p} \mathrm{Sh}_{\mathsf{K}'_p \mathsf{K}^p}(\mathsf{G}, X)_E.$$

Since the transition morphisms (4-9) are affine, this limit can be taken in the category of schemes, by [Stacks 2005–, Tag 01YX]. Define

$$\mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}^p} / \mathrm{Sh}_{\mathsf{K}}) = \varprojlim_{\mathsf{K}'_p \subset \mathsf{K}_p} \mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}'} / \mathrm{Sh}_{\mathsf{K}}).$$

Then $\mathrm{Sh}_{\mathsf{K}^p}(\mathsf{G}, X)_E \to \mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)_E$ is a torsor for the profinite group $\mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}^p} / \mathrm{Sh}_{\mathsf{K}})$ in the pro-étale topology on the scheme $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)_E$. We denote this pro-étale $\mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}^p} / \mathrm{Sh}_{\mathsf{K}})$-torsor by $\mathbb{P}_{0,\mathsf{K}}$. We can make $\mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}^p} / \mathrm{Sh}_{\mathsf{K}})$ explicit as follows. Denote by $\mathsf{Z}(\mathbb{Q})^-$ the closure of $\mathsf{Z}(\mathbb{Q})$ in $\mathsf{Z}(\mathbb{A}_f)$, and by $\mathsf{Z}(\mathbb{Q})^-_{\mathsf{K}}$ the intersection $\mathsf{Z}(\mathbb{Q})^- \cap \mathsf{K}$. Then

$$\mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}^p} / \mathrm{Sh}_{\mathsf{K}}) = \mathsf{K}_p/\mathsf{Z}(\mathbb{Q})^-_{\mathsf{K},p},$$

where $\mathsf{Z}(\mathbb{Q})^-_{\mathsf{K},p}$ is the closure of the image of $\mathsf{Z}(\mathbb{Q})^-_{\mathsf{K}}$ under the projection $\mathsf{G}(\mathbb{A}_f) \to \mathsf{G}(\mathbb{Q}_p)$, see [Kisin et al. 2021, Section 1.5.8].

Since K^p is neat, [Kisin et al. 2021, Lemma 1.5.7] implies that $\mathsf{Z}(\mathbb{Q})^-_{\mathsf{K},p} \subset \mathsf{Z}_{ac}(\mathbb{Q}_p)$. Hence $\mathsf{K}_p \to G^c(\mathbb{Q}_p)$ factors through $\mathsf{K}_p/\mathsf{Z}(\mathbb{Q})^-_{\mathsf{K},p}$, and therefore $\mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}^p} / \mathrm{Sh}_{\mathsf{K}})$ admits K^c_p as a quotient. In particular, the composition (4-1) can be extended to

$$\mathcal{G}(\mathbb{Z}_p) = \mathsf{K}_p \twoheadrightarrow \mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}^p} / \mathrm{Sh}_{\mathsf{K}}) \twoheadrightarrow \mathsf{K}^c_p \hookrightarrow \mathcal{G}^c(\mathbb{Z}_p). \tag{4-10}$$

However, note that the second surjection is never injective if Z_{ac} is nontrivial, contrary to the claim in [Milne 1990, Section III, Remark 6.1]. For details, see [Lan and Stroh 2018, (2)].

Let $\mathrm{Sh}(\mathsf{G}, X)^{\mathrm{ad}}_E$ be the analytic adic space associated to $\mathrm{Sh}(\mathsf{G}, X)_E$, and let $\mathbb{P}^{\mathrm{ad}}_{0,\mathsf{K}}$ be the pro-étale $\mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}^p} / \mathrm{Sh}_{\mathsf{K}})$-torsor on $\mathrm{Sh}(\mathsf{G}, X)^{\mathrm{ad}}_E$ obtained from $\mathbb{P}_{0,\mathsf{K}}$. We would like to associate to $\mathbb{P}^{\mathrm{ad}}_{0,\mathsf{K}}$ a "$\mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}^p} / \mathrm{Sh}_{\mathsf{K}})$-shtuka". However, our machinery for shtukas only applies to groups which are algebraizable, and this is not a priori the case for $\mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}^p} / \mathrm{Sh}_{\mathsf{K}})$. Instead, we will replace $\mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}^p} / \mathrm{Sh}_{\mathsf{K}})$ by the closely related group $\mathcal{G}^c(\mathbb{Z}_p)$, which *is* algebraizable. Essentially, we view $\mathcal{G}^c(\mathbb{Z}_p)$ as a sort of "algebraizable hull" of $\mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}^p} / \mathrm{Sh}_{\mathsf{K}})$. In fact, at least when $\mathcal{G}$ is reductive (and likely more generally), this can be made precise; see [Imai et al. 2023, Proposition 4.8].

Using the final two maps in (4-10) we define the contracted product

$$\mathbb{P}_{\mathsf{K}} := \mathbb{P}^{\mathrm{ad}}_{0,\mathsf{K}} \times^{\mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}^p} / \mathrm{Sh}_{\mathsf{K}})} \mathcal{G}^c(\mathbb{Z}_p), \tag{4-11}$$

which is a pro-étale $\mathcal{G}^c(\mathbb{Z}_p)$-torsor $\mathbb{P}_{\mathsf{K}}$ on $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)^{\mathrm{ad}}_E$. From every finite-dimensional $\mathbb{Q}_p$-representation of G^c, we can obtain from $\mathbb{P}_{\mathsf{K}}$ a $\mathbb{Z}_p$-local system on $\mathrm{Sh}(\mathsf{G}, X)^{\mathrm{ad}}_E$, as follows. Let $\rho : G^c \to \mathrm{GL}(W)$ of G^c be a finite dimensional $\mathbb{Q}_p$-representation of G^c, and fix a $\mathbb{Z}_p$-lattice $\Lambda \subset W$ such that $\rho(\mathcal{G}^c(\mathbb{Z}_p)) \subset \mathrm{GL}(\Lambda)$. The contracted product

$$\mathbb{L}_{\rho,\Lambda} := \mathbb{P}_{\mathsf{K}} \times^{\underline{\mathcal{G}^c(\mathbb{Z}_p)}} \underline{\Lambda}$$

is a pro-étale $\mathbb{Z}_p$-local system on $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)^{\mathrm{ad}}_E$. Denote by μ^c the cocharacter given by the composition of μ with $G \to G^c$.

Proposition 4.4. *There exists a $\mathcal{G}^c$-shtuka $\mathscr{P}_{\mathsf{K},E}$ over* $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)^{\lozenge}_E \to \mathrm{Spd}(E)$ *with one leg bounded by μ^c which is associated to $\mathbb{P}_{\mathsf{K}}$ in the sense of* [Pappas and Rapoport 2024, Section 2.5]. *Furthermore, $\mathscr{P}_{\mathsf{K},E}$ supports prime-to-p Hecke correspondences.*

Proof. When $\mathsf{G} = \mathsf{G}^c$ (i.e., Z_s is trivial), we have $\mathrm{Gal}(\mathrm{Sh}_{\mathsf{K}^p} / \mathrm{Sh}_{\mathsf{K}}) = \mathcal{G}(\mathbb{Z}_p) = \mathcal{G}^c(\mathbb{Z}_p)$, and in this case the proposition reduces to [Pappas and Rapoport 2024, Proposition 4.1.2]. In general it follows similarly once one proves that $\mathbb{P}_{\mathsf{K}}$ is a de Rham $\mathcal{G}^c(\mathbb{Z}_p)$-torsor in the sense of [loc. cit., Definition 2.5.5] and that the Hodge–Tate cocharacter for $\mathbb{P}_{\mathsf{K}}$ is given by μ^c.

Fix a representation (Λ, ρ) of G^c. To prove $\mathbb{L}_{\rho,\Lambda}$ is de Rham, one proceeds in the same way as the proof of [Liu and Zhu 2017, Theorem 1.2]. In particular, by [loc. cit., Theorem 3.9(iv)] and the fact that the set of special points in each connected component of $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)$ is nonempty [Milne 2005, Lemma 13.5], we reduce to showing that if $x = [h, a]_{\mathsf{K}} \in \mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)$ is a special point with h factoring through a maximal torus $\mathsf{T} \subset \mathsf{G}$, then the stalk $\mathbb{L}_{\rho,\Lambda,\bar{x}}$ of $\mathbb{L}_{\rho,\Lambda}$ at a geometric point $\bar{x}$ above x is de Rham.

Since $\mathsf{T}_{\mathbb{R}}$ stabilizes the special point x, it is $\mathbb{R}$-anisotropic modulo the center of G, and therefore the restriction to T of any representation of G^c factors through T^c. If ρ is a representation of G^c, let ρ' denote its restriction to T^c. As in [Liu and Zhu 2017, Lemma 4.8], we see that $\mathbb{L}_{\rho,\Lambda,\bar{x}}$ is isomorphic to $r(\mu, \rho')_{\mathsf{K},p}$. Hence the fact that $\mathbb{L}_{\rho,\Lambda,\bar{x}}$ is de Rham follows from Lemma 4.3.

Finally, for the boundedness by μ^c, we first observe that the arguments of [Kisin et al. 2021, Proposition 4.3.14] show that the Hodge–Tate cocharacter of $r(\mu, \rho')_{\mathsf{K},p}$ is μ^c. Thus the same is true of $\mathbb{L}_{\rho,\Lambda,\bar{x}}$, independent of the choice of x. Therefore by [Pappas and Rapoport 2024, Proposition 2.5.3 and Definition 2.6.6] the $\mathcal{G}^c$-shtuka $\mathscr{P}_{\mathsf{K},E}$ is bounded by μ^c. □

4.3. ***The conjecture of Pappas and Rapoport.*** In this section we state the conjecture of Pappas and Rapoport on the existence of canonical integral models for Shimura varieties. Let (G, b, μ) be a local Shimura datum in the sense of [Scholze and Weinstein 2020, Definition 24.1.1], that is, suppose G is a reductive $\mathbb{Q}_p$-group, μ is a geometric conjugacy class of minuscule cocharacters, and b is a σ-conjugacy class of elements of $G(\breve{\mathbb{Q}}_p)$, which we further assume lies in the set $B(G, \mu^{-1})$ of neutral acceptable elements in the sense of [Rapoport and Viehmann 2014, Definition 2.3]. We take also a parahoric integral model $\mathcal{G}$ for G over $\mathbb{Z}_p$. We write E for the reflex field of μ, and k for an algebraic closure of the residue field of E.

We write $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ for the *integral local Shimura variety associated with* $(\mathcal{G}, b, \mu)$ [Scholze and Weinstein 2020, Section 25.1]. This is the functor $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ on Perf_k that sends S to the set of isomorphism classes of tuples $(S^\sharp, \mathscr{P}, \phi_{\mathscr{P}}, i_r)$, where $S^\sharp$ is an untilt of S over $\mathrm{Spa}(\mathcal{O}_{\breve{E}})$, $(\mathscr{P}, \phi_{\mathscr{P}})$ is a $\mathcal{G}$-shtuka over S with one leg along $S^\sharp$ bounded by μ, and i_r is an isomorphism of G-torsors

$$i_r : G\big|_{\mathcal{Y}_{[r,\infty)}(S)} \xrightarrow{\sim} \mathscr{P}\big|_{\mathcal{Y}_{[r,\infty)}(S)}$$

for large enough r, under which $\phi_{\mathscr{P}}$ is identified with $\phi_b := b \times \mathrm{Frob}_S$. Here an isomorphism of tuples

$$(S^\sharp, \mathscr{P}, \phi_{\mathscr{P}}, i_{r_1}) \xrightarrow{\sim} ((S')^\sharp, \mathscr{P}', \phi_{\mathscr{P}'}, i'_{r_2})$$

is a pair of isomorphisms (α, β), where $\alpha : S^\sharp \xrightarrow{\sim} (S')^\sharp$ is an isomorphism whose tilt factors as $(S^\sharp)^\flat \xrightarrow{\sim} S \xleftarrow{\sim} ((S')^\sharp)^\flat$, and where $\beta : (\mathscr{P}, \phi_{\mathscr{P}}) \xrightarrow{\sim} (\mathscr{P}', \phi_{\mathscr{P}'})$ is an isomorphism of $\mathcal{G}$-shtukas which satisfies $i'_r \circ i_r^{-1} = \beta$ for $r \geq \max(r_1, r_2)$.

When $p \neq 2$ and (G, b, μ) is of abelian type, the main result of [Pappas and Rapoport 2022] states that $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ is representable by a formal scheme $\mathscr{M}_{\mathcal{G},b,\mu}$ which is normal and flat and formally locally of finite type over $\mathcal{O}_{\breve{E}}$ (in fact, this holds slightly more generally, see [loc. cit.]).

The formation of integral local Shimura varieties is functorial. Indeed, let $(G, b, \mu) \to (G', b', \mu')$ be a morphism of local Shimura data, and suppose the map $G \to G'$ extends to $\mathcal{G} \to \mathcal{G}'$. Then we obtain a morphism of integral local Shimura varieties

$$\rho : \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G}',b',\mu'} \times_{\mathrm{Spd}(\mathcal{O}_{\breve{E}'})} \mathrm{Spd}(\mathcal{O}_{\breve{E}}) \tag{4-12}$$

given on points by pushing forward shtukas along $\mathcal{G} \to \mathcal{G}'$. Moreover, the morphism ρ has the following property. Suppose $\mathscr{P}^{\mathrm{univ}}$ and $(\mathscr{P}')^{\mathrm{univ}}$ are the universal shtukas over $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ and $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$, respectively. Then

$$\rho^*(\mathscr{P}')^{\mathrm{univ}} = \mathscr{P}^{\mathrm{univ}} \times^{\mathcal{G}} \mathcal{G}'. \tag{4-13}$$

Indeed, this follows from the definition of (4-12) and those of the respective universal objects.

Let (G, b, μ) be a local Shimura datum, and let $\mathcal{G}$ be a parahoric model for G over $\mathbb{Z}_p$. We denote by $X_{\mathcal{G}}(b, \mu^{-1})$ the (b, μ^{-1})-*admissible locus* in the Witt vector affine Grassmannian. This is the functor which assigns to perfect k-algebras R the set of isomorphism classes of pairs $(\mathcal{P}, \alpha)$, where $\mathcal{P}$ is a $\mathcal{G}$-torsor over $\mathrm{Spec}(W(R))$, and α is an isomorphism of $\mathcal{G}$-torsors

$$\alpha : \mathcal{G}_{W(R)[1/p]} \xrightarrow{\sim} \mathcal{P}_{W(R)[1/p]}$$

over $W(R)[1/p]$ such that $\phi_{\mathcal{P}} = \alpha \circ \phi_b \circ \phi^*(\alpha)^{-1}$ defines the structure of a meromorphic Frobenius crystal (in the sense of [Pappas and Rapoport 2024, Definition 2.3.3]), such that the $\mathcal{G}$-shtuka $(\mathscr{P}, \phi_{\mathscr{P}})$ associated to $(\mathcal{P}, \phi_{\mathcal{P}})$ by [Pappas and Rapoport 2024, Theorem 2.3.5] has a leg along the divisor $p = 0$ and is bounded by μ. We note that $X_{\mathcal{G}}(b, \mu^{-1})$ is sometimes referred to as the affine Deligne–Lusztig variety associated to $\mathcal{G}$, b, and μ^{-1}, but we reserve that moniker for its k-points.

The functor $X_{\mathcal{G}}(b, \mu^{-1})$ is representable by a perfect scheme which is locally perfectly of finite type over k [Pappas and Rapoport 2024, Section 3.3]. Moreover, by [Gleason 2021, Proposition 2.30], there is a natural isomorphism

$$(\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu})_{\mathrm{red}} \cong X_{\mathcal{G}}(b, \mu^{-1}), \tag{4-14}$$

where $(\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu})_{\mathrm{red}}$ denotes the reduced locus of the v-sheaf $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ in the sense of [Gleason 2024]. Let $K = \mathcal{G}(\mathbb{Z}_p)$, $\breve{K} = \mathcal{G}(\breve{\mathbb{Z}}_p)$, and let $\mathrm{Adm}^K(\mu^{-1})$ denote the μ^{-1}-admissible locus in the sense of [Rapoport 2005]. Then, the isomorphism (4-14) implies, in particular, that we have the identity

$$\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}(\mathrm{Spd}(k)) = X_{\mathcal{G}}(b, \mu^{-1})(k) = \{g\breve{K} = G(\breve{\mathbb{Q}}_p)/\breve{K} \mid g^{-1}b\phi(g) \in \mathrm{Adm}^K(\mu^{-1})\}.$$

In other words, $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}(\mathrm{Spd}(k))$ is the affine Deligne–Lusztig variety associated to $\mathcal{G}$, b, and μ^{-1}. If $b \in \mathrm{Adm}^K(\mu^{-1})$, then $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ has a canonical $\mathrm{Spd}(k)$-valued *base point*,

$$x_0 \in \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}(\mathrm{Spd}(k)), \tag{4-15}$$

corresponding to the trivial coset in $G(\breve{\mathbb{Q}}_p)/\breve{K}$. From the perspective of the moduli functor, the base point associates to S in Perf_k the tuple $(S, \mathcal{G}|_{\mathcal{Y}_{[0,\infty)}}, \phi_b, \mathrm{id})$.

By [Gleason 2024, Theorem 2], $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ is a kimberlite; see [Gleason 2024] for details on this terminology. In particular, there is a continuous specialization map

$$\mathrm{sp} : |\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}| \to |(\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu})_{\mathrm{red}}| = |X_{\mathcal{G}}(b, \mu^{-1})|.$$

The *formal completion of* $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ *along a point* $x \in \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}(\mathrm{Spd}(k))$ is the sub-v-sheaf $\widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}}_{/x}$ whose points for S in Perf_k are given by

$$\widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}}_{/x}(S) = \{y : S \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu} \mid \mathrm{sp} \circ y(|S|) \subset \{x\}\}.$$

Suppose $\rho : (G, b, \mu) \to (G', b', \mu')$ is a morphism of local Shimura data which induces an isomorphism $\rho_{\mathrm{ad}} : G_{\mathrm{ad}} \xrightarrow{\sim} G'_{\mathrm{ad}}$, and suppose moreover that ρ extends to a morphism $\mathcal{G} \to \mathcal{G}'$ of parahoric models. If $\mathcal{G}$ and $\mathcal{G}'$ correspond to the same point in the common building of G_{ad} and G'_{ad}, then by [Pappas and Rapoport 2022, Proposition 5.3.1], the morphism (4-12) induces an isomorphism

$$\hat{\rho} : \widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}}_{/x} \xrightarrow{\sim} \widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{G}',b',\mu'}}_{/\rho(x)} \times_{\mathrm{Spd}(\mathcal{O}_{\breve{E}'})} \mathrm{Spd}(\mathcal{O}_{\breve{E}}) \tag{4-16}$$

for any point $x \in \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}(\mathrm{Spd}(k))$. In particular, if $G = T$ is a torus, and $\mathcal{G} = \mathcal{T}$ is the identity component of the Néron model for T, then the structure morphism to $\mathrm{Spd}(\mathcal{O}_{\breve{E}})$ defines an isomorphism

$$\widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{T},b,\mu}}_{/x} \xrightarrow{\sim} \mathrm{Spd}(\mathcal{O}_{\breve{E}}) \tag{4-17}$$

for any $x \in \mathcal{M}^{\mathrm{int}}_{\mathcal{T},b,\mu}$, by applying the functoriality (4-16) to the morphism from (T, b, μ) to the trivial local Shimura datum.

Let us now return to the general global setting of Section 4.1 in order to state the conjecture of Pappas and Rapoport [2024]. We retain the notation from the beginning of Section 4.1 as well, so (G, X) is a global Shimura datum, μ is the corresponding geometric conjugacy class of cocharacters for G, E is the reflex field, $G = \mathsf{G}_{\mathbb{Q}_p}$ is the corresponding p-adic group, E is the local reflex field, and $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p \subset \mathsf{G}(\mathbb{A}_f)$ is the level subgroup with $\mathsf{K}_p = \mathcal{G}(\mathbb{Z}_p)$ is a parahoric subgroup and $\mathsf{K}^p \subset \mathsf{G}(\mathbb{A}_f^p)$ is neat. We furthermore have the cuspidal quotient G^c, its corresponding p-adic group G^c, and the parahoric group scheme $\mathcal{G}^c$.

Suppose we are given a system of normal integral models $\mathscr{S}_{\mathsf{K}}$ of $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)_E$, each equipped with a $\mathcal{G}^c$-shtuka $\mathscr{P}_{\mathsf{K}}$ defined over $\mathscr{S}_{\mathsf{K}}^{\lozenge} \to \mathrm{Spd}(\mathcal{O}_E)$ which is bounded by μ, and which extends the $\mathcal{G}^c$-shtuka $\mathscr{P}_{\mathsf{K},E}$ over $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)_E^{\lozenge} \to \mathrm{Spd}(E)$ defined in Proposition 4.4. Then for any point $x \in \mathscr{S}_{\mathsf{K}}(k)$, we have a canonically defined σ-conjugacy class $[b_x] \in B(G^c)$ coming from $\mathscr{P}_{\mathsf{K}}$. Indeed, by [Pappas and Rapoport 2024, Example 2.4.9], the pullback $x^*\mathscr{P}_{\mathsf{K}}$ defines a $\mathcal{G}^c$-torsor $\mathscr{P}_x$ over $\mathrm{Spec}(W(k))$ along with an isomorphism

$$\phi_x : \phi^*(\mathscr{P}_x)[1/p] \xrightarrow{\sim} \mathscr{P}_x[1/p],$$

where here ϕ denotes the Frobenius for $W(k)$. Then any choice of trivialization yields an element $b_x \in G^c(\breve{\mathbb{Q}}_p)$, and changing the trivialization corresponds to applying σ-conjugation by some element of $\mathcal{G}^c(\breve{\mathbb{Z}}_p)$. Since the shtuka $\mathscr{P}_{\mathsf{K}}$ is bounded by μ^c, the resulting element of $B(G^c)$ lies in $B(G^c, (\mu^c)^{-1})$. Denote by x_0 the base point of $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c,b_x,\mu^c}(\mathrm{Spd}(k))$ as in (4-15).

The following is a slightly modified version of [Pappas and Rapoport 2024, Conjecture 4.2.2].

Conjecture 4.5 (Pappas and Rapoport). *Fix a parahoric subgroup* K_p *with corresponding* $\mathbb{Z}_p$*-group scheme* $\mathcal{G}$. *Then for every neat compact open subgroup* $\mathsf{K}^p \subset \mathsf{G}(\mathbb{A}_f^p)$, *there exists a normal flat model* $\mathscr{S}_{\mathsf{K}}$ *of* $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)_E$ *over* $\mathcal{O}_E$, *such that the system* $(\mathscr{S}_{\mathsf{K}})_{\mathsf{K}^p}$ *satisfies the following properties*:

(a) *For every discrete valuation ring R of characteristic $(0, p)$ over $\mathcal{O}_E$,*

$$(\varprojlim_{\mathsf{K}^p} \mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)_E)(R[1/p]) = (\varprojlim_{\mathsf{K}^p} \mathscr{S}_{\mathsf{K}})(R).$$

If $\mathrm{Sh}_{\mathsf{K}}(G, X)_E$ is proper over $\mathrm{Spec}(E)$ then $\mathscr{S}_{\mathsf{K}}$ is proper over $\mathrm{Spec}(\mathcal{O}_E)$. In addition, the system $\mathscr{S}_{\mathsf{K}}$ supports prime-to-p Hecke correspondences, i.e., for $g \in \mathsf{G}(\mathbb{A}_f^p)$ and K'^p with $g\mathsf{K}'^p g^{-1} \subset \mathsf{K}^p$, there are finite étale morphisms $[g] : \mathscr{S}_{\mathsf{K}'} \to \mathscr{S}_{\mathsf{K}}$ which extend the natural maps $[g] : \mathrm{Sh}_{\mathsf{K}_p\mathsf{K}'^p}(\mathsf{G}, X)_E \to \mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p}(\mathsf{G}, X)_E$.

(b) *The $\mathcal{G}^c$-shtuka $\mathscr{P}_{\mathsf{K},E}$ extends to a $\mathcal{G}^c$-shtuka $\mathscr{P}_{\mathsf{K}}$ on $(\mathscr{S}_{\mathsf{K}})^{\Diamond/}$.*

(c) *For $x \in \mathscr{S}_{\mathsf{K}}(k)$ and b_x defined as above, there is an isomorphism of v-sheaves over $\mathrm{Spd}(\mathcal{O}_E)$*

$$\Theta_x : \widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c, b_x, \mu^c}}_{/x_0} \xrightarrow{\sim} (\widehat{\mathscr{S}}_{\mathsf{K}/x})^{\Diamond},$$

under which the pullback shtuka $\Theta_x^*(\mathscr{P}_{\mathsf{K}})$ coincides with the universal shtuka on $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c, b_x, \mu^c}$.

In the case where G is itself cuspidal (i.e., $\mathsf{G} = \mathsf{G}^c$), this is exactly [Pappas and Rapoport 2024, Conjecture 4.2.2], which is proved in [loc. cit.] for Shimura varieties of Hodge type, that is, those for which there is an embedding of (G, X) into a Shimura datum for a Siegel-type Shimura datum. We note that cuspidality is automatic in the Hodge-type case by [Kisin et al. 2021, Lemma 5.1.2], so the more general formulation of the conjecture is unnecessary for the results of [Pappas and Rapoport 2024]. For Shimura varieties of toral type (the case of interest below) or of abelian type, there is no reason to expect cuspidality in general. We have formulated the more general version of the conjecture with this in mind.

Moreover, Pappas and Rapoport [2024, Theorem 4.2.4] proved that an integral model satisfying the properties of Conjecture 4.5 is uniquely determined.

Theorem 4.6 (Pappas and Rapoport). *There is at most one system of normal flat models $\mathscr{S}_{\mathsf{K}}$ of $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)_E$ over $\mathcal{O}_E$, for $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p$ with variable neat K^p, which satisfies the properties in Conjecture 4.5.*

Proof. The proof in [Pappas and Rapoport 2024, Theorem 4.2.4] addresses the case where $\mathsf{G} = \mathsf{G}^c$. The proof in general follows by replacing the integral local Shimura variety for $\mathcal{G}$ with the one for $\mathcal{G}^c$. ☐

4.4. *Proof of Theorem A.* In this section we prove Conjecture 4.5 for Shimura varieties defined by tori. As in Section 4.1, let $(\mathsf{T}, \{h\})$ be a Shimura datum defined by a torus T over $\mathbb{Q}$, let T^c be its maximal cuspidal quotient, and let E denote the reflex field. Let $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p$ with K^p neat, and $\mathsf{K}_p = \mathcal{T}(\mathbb{Z}_p)$ for $\mathcal{T}$ the identity component of the locally finite-type Néron model of $T = \mathsf{T}_{\mathbb{Q}_p}$, so K_p is the unique parahoric subgroup of $T(\mathbb{Q}_p)$. We obtain a zero-dimensional Shimura variety

$$\mathrm{Sh}_{\mathsf{K}}(\mathsf{T}, \{h\}) = \coprod_{i \in I} \mathrm{Spec}\, \mathsf{E}_i, \tag{4-18}$$

where each E_i is a finite extension of E. Moreover, by Lemma 4.2, we know that each E_i is isomorphic to the field E_{K} defined in (4-4).

As in Section 4.2, let v be the place of E above p corresponding to our chosen embedding $\mathsf{E} \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and let E denote the completion of E at v. Let $\mathcal{O}_E$ and $\mathcal{O}_{E_i}$ denote the rings of integers in E and E_i, respectively. Then

$$\mathscr{S}_{\mathsf{K}}(\mathsf{T}, \{h\}) := \coprod_{i \in I} \operatorname{Spec} \mathcal{O}_{E_i}$$

provides us with an integral model for $\mathrm{Sh}_{\mathsf{K}}(\mathsf{T}, \{h\})_E$.

We begin by using Lemma 4.3 to extend the $\mathcal{T}^c$-shtuka $\mathscr{P}_{\mathsf{K},E}$ over $\mathrm{Sh}_{\mathsf{K}}(\mathsf{T}, \{h\})^{\diamondsuit}$ of Proposition 4.4 to all of $\mathscr{S}_{\mathsf{K}}(\mathsf{T}, \{h\})^{\diamondsuit/}$. By Lemma 4.3, the $\mathcal{T}^c$-valued representation induced by the homomorphism

$$r(\mu)_{\mathsf{K},p,\mathrm{loc}} : \Gamma_{E_{\mathsf{K}}} \to \mathcal{T}^c(\mathbb{Z}_p) \tag{4-19}$$

is crystalline. Define $\mathscr{P}_{\mathsf{K},i}$ to be the $\mathcal{T}^c$-shtuka over $\mathrm{Spd}(\mathcal{O}_{E_i})$ associated by Definition 3.11 to the $\mathcal{T}^c$-valued crystalline representation of Γ_{E_i} given by (4-19). Since $\mathrm{Spec}(\mathcal{O}_{E_i})$ is proper over $\mathrm{Spec}(\mathbb{Z}_p)$, we have

$$\mathrm{Spd}(\mathcal{O}_{E_i}) \xrightarrow{\sim} (\mathrm{Spec}(\mathcal{O}_{E_i}))^{\diamondsuit/}$$

for every i, by [Huber 1994, Remark 4.6(iv)], see (2-4) and the discussion preceding it. By patching together the collection $\{\mathscr{P}_{\mathsf{K},i}\}_{i \in I}$ we can define a $\mathcal{T}^c$-shtuka $\mathscr{P}_{\mathsf{K}}$ over all of $\mathscr{S}_{\mathsf{K}}(\mathsf{T}, \{h\})^{\diamondsuit/}$.

Proposition 4.7. *The $\mathcal{T}^c$-shtuka $\mathscr{P}_{\mathsf{K}}$ over $\mathscr{S}_{\mathsf{K}}(\mathsf{T}, \{h\})^{\diamondsuit/}$ is bounded by μ^c, and it extends the $\mathcal{T}^c$-shtuka $\mathscr{P}_{\mathsf{K},E}$ over* $\mathrm{Sh}_{\mathsf{K}}(\mathsf{T}, \{h\})$.

Proof. It's enough to show the proposition for each $i \in I$. Note that the Hodge–Tate cocharacter of each $r(\mu, \rho)_{\mathsf{K},p}$ is given by μ^c by [Kisin et al. 2021, Lemma 4.4], so the first point follows from Lemma 3.15.

The second point follows from Lemma 3.14, since for each (Λ, ρ), the stalk of the $\mathbb{Z}_p$-local system $\mathbb{L}_{\rho,\Lambda}$ from Section 4.2 at a geometric point $\bar{x}_i$ over $\mathrm{Spec}(E_i)$ is given by the representation $r(\mu, \rho)_{\mathsf{K},p}$ of $\Gamma_{E_{\mathsf{K}}}$. □

This proves part (b) of Conjecture 4.5 for $\mathrm{Sh}_{\mathsf{K}}(\mathsf{T}, \{h\})$. Next we prove part (c) of the conjecture. Let us recall the statement. Let $\mathscr{S}_{\mathsf{K}} = \mathscr{S}_{\mathsf{K}}(\mathsf{T}, \{h\})$, let $x \in \mathscr{S}_{\mathsf{K}}(k)$, and let b_x be defined as in Conjecture 4.5. We want to show there is an isomorphism

$$\Theta_x : \widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{T}^c, b_x, \mu^c}}_{/x_0} \xrightarrow{\sim} (\widehat{\mathscr{S}}_{\mathsf{K}/x})^{\diamondsuit}$$

of v-sheaves over $\mathrm{Spd}(\mathcal{O}_{\breve{E}})$ such that the pullback $\Theta_x^*(\mathscr{P}_{\mathsf{K}})$ coincides with the universal shtuka on $\mathcal{M}^{\mathrm{int}}_{\mathcal{T}^c, b_x, \mu^c}$.

Lemma 4.8. *Let $x \in \mathscr{S}_{\mathsf{K}}(k)$. Then there is a natural isomorphism*

$$\mathrm{Spd}(\mathcal{O}_{\breve{E}}) \xrightarrow{\sim} (\widehat{\mathscr{S}}_{\mathsf{K}/x})^{\diamondsuit}.$$

Proof. Recall that each E_i is isomorphic to E_{K} (see Lemma 4.2), so it is enough to show E_{K} is an unramified extension of E. By definition of E_{K}, we have

$$\mathrm{Gal}(E^{\mathrm{ab}}/E_{\mathsf{K}}) = \ker(\mathrm{Gal}(E^{\mathrm{ab}}/E) \hookrightarrow \mathrm{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E}) \xrightarrow{r(\mu)_{\mathsf{K}}} \mathsf{T}(\mathbb{Q})\backslash \mathsf{T}(\mathbb{A}_f)/\mathsf{K}).$$

By local class field theory, if E^{unr} denotes the maximal unramified extension of E in $\overline{\mathbb{Q}}_p$, then the local Artin map restricts to an isomorphism $\mathcal{O}_E^\times \xrightarrow{\sim} \mathrm{Gal}(E^{\mathrm{ab}}/E^{\mathrm{unr}})$. Therefore, to show E_{K} is unramified, it is enough to show that $\mathcal{O}_E^\times$ is in the kernel of the composition

$$E^\times \hookrightarrow \mathsf{E}^\times\backslash\mathbb{A}_{\mathsf{E}}^\times \xrightarrow{\mathrm{Art}_{\mathsf{E}}} \mathrm{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E}) \xrightarrow{r(\mu)_{\mathsf{K}}} \mathsf{T}(\mathbb{Q})\backslash\mathsf{T}(\mathbb{A}_f)/\mathsf{K}. \tag{4-20}$$

The composition (4-20) can be rewritten as

$$E^\times \hookrightarrow \mathsf{E}^\times\backslash\mathbb{A}_{\mathsf{E}}^\times \xrightarrow{r(\mu)^{\mathrm{alg}}(\mathbb{A})} \mathsf{T}(\mathbb{Q})\backslash\mathsf{T}(\mathbb{A}) \xrightarrow{\mathrm{pr}_{\mathsf{K}}} \mathsf{T}(\mathbb{Q})\backslash\mathsf{T}(\mathbb{A}_f)/\mathsf{K} \tag{4-21}$$

by the definition of $r(\mu)_{\mathsf{K}}$. The composition of the first two arrows of (4-21) is given by the evaluation on $\mathbb{Q}_p$ of the homomorphism of $\mathbb{Q}_p$-group schemes

$$\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m \hookrightarrow (\mathrm{Res}_{\mathsf{E}/\mathbb{Q}}\mathbb{G}_m)_{\mathbb{Q}_p} \xrightarrow{r(\mu)^{\mathrm{alg}}} T, \tag{4-22}$$

composed with $T(\mathbb{Q}_p) \hookrightarrow \mathsf{T}(\mathbb{A}) \to \mathsf{T}(\mathbb{Q})\backslash\mathsf{T}(\mathbb{A})$. The groups $\mathcal{O}_E^\times$ and $\mathcal{T}(\mathbb{Z}_p)$ are identified with the kernels of the Kottwitz homomorphisms for $\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m$ and T, respectively, see [Rapoport 2005]. The functoriality of the Kottwitz homomorphism (see, e.g., [Kaletha and Prasad 2023, Lemma 11.5.1]) implies that (4-22) sends $\mathcal{O}_E^\times$ into $\mathcal{T}(\mathbb{Z}_p) \subset \mathsf{K}$, and the result follows. □

Combining Lemma 4.8 with the isomorphism (4-17), we obtain an isomorphism

$$\Theta_x : \widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{T}^c,b_x,\mu^c/x_0}} \xrightarrow{\sim} (\widehat{\mathscr{S}}_{\mathsf{K}/x})^\diamond. \tag{4-23}$$

We remark that the isomorphism in Lemma 4.8 is, in fact, the inverse of the structure morphism for $(\widehat{\mathscr{S}}_{\mathsf{K}/x})^\diamond$ over $\mathrm{Spd}(\mathcal{O}_{\breve{E}})$, so in particular Θ_x is a uniquely determine $\mathrm{Spd}(\mathcal{O}_{\breve{E}})$-morphism. It remains to study the pullback of $\mathscr{P}_{\mathsf{K}}$ along Θ_x.

Recall that T^c is the maximal cuspidal quotient of T. As in Section 4.1, the map $\mathsf{T} \to \mathsf{T}^c$ extends to $\mathcal{T} \to \mathcal{T}^c$, where $\mathcal{T}^c$ is the connected Néron model of $\mathsf{T}^c_{\mathbb{Q}_p}$. Let $\mathsf{K}' = \mathcal{T}^c(\mathbb{Z}_p)\mathsf{K}^{p,c}$, where $\mathsf{K}^{p,c}$ is the image of K^p under $\mathsf{T} \to \mathsf{T}^c$, and let E^c denote the reflex field of the Shimura datum $(\mathsf{T}^c, \{h^c\})$, and E^c denote the completion of E^c at the place v' induced by v. Let E_i be defined as in (4-18), and let $E_i = \mathsf{E} \otimes_{\mathsf{E}} E$. Then, by the valuative criterion for properness of $\mathscr{S}_{\mathsf{K}'}$ over $\mathcal{O}_{E^c}$, the morphism

$$\mathrm{Spec}(E_i) \hookrightarrow \mathrm{Sh}_{\mathsf{K}}(\mathsf{T}, \{h\})_E \to \mathrm{Sh}_{\mathsf{K}'}(\mathsf{T}^c, \{h^c\}) \otimes_{\mathsf{E}^c} E \to \mathscr{S}_{\mathsf{K}'} \otimes_{\mathcal{O}_{\mathsf{E}^c}} \mathcal{O}_E$$

extends to a morphism $\mathrm{Spec}(\mathcal{O}_{E_i}) \to \mathscr{S}_{\mathsf{K}'} \otimes_{\mathcal{O}_{\mathsf{E}^c}} \mathcal{O}_E$. Taken together for each i, we obtain a morphism of integral models

$$\mathscr{S}_{\mathsf{K}} \to \mathscr{S}_{\mathsf{K}'} \otimes_{\mathcal{O}_{E^c}} \mathcal{O}_E. \tag{4-24}$$

As in the case of $\mathscr{S}_{\mathsf{K}}$, we have

$$\mathscr{S}_{\mathsf{K}'} = \coprod_i \mathrm{Spec}(\mathcal{O}_{E_{\mathsf{K}'}}),$$

where $E_{\mathsf{K}'}$ is defined as in (4-4). We see that $E_{\mathsf{K}'} \subset E_{\mathsf{K}}$, hence $E_{\mathsf{K}'}$ is a finite unramified extension of E, and $\mathcal{O}_{\mathsf{K}'} \to \mathcal{O}_{\mathsf{K}}$ is finite étale. It follows that, for any point $x \in \mathscr{S}_{\mathsf{K}}(k)$, we have a commutative diagram

$$\begin{array}{ccc} & \mathrm{Spd}(\mathcal{O}_{\breve{E}}) & \\ \swarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\searrow \\ (\widehat{\mathscr{S}_{\mathsf{K}/x}})^{\lozenge} & \xrightarrow{\ \sim\ } & (\widehat{\mathscr{S}_{\mathsf{K}'/x'}})^{\lozenge} \end{array} \tag{4-25}$$

where x' denotes the image of x in $\mathscr{S}_{\mathsf{K}'}(k)$, and the bottom arrow is induced by (4-24).

We claim that the pullback of $\mathscr{P}_{\mathsf{K}'}$ along (4-24) is $\mathscr{P}_{\mathsf{K}}$. By [Pappas and Rapoport 2024, Corollary 2.7.10], it is enough to show that $\mathscr{P}_{\mathsf{K}',E'}$ pulls back to $\mathscr{P}_{\mathsf{K},E}$, and by functoriality of the construction in [loc. cit., Definition 2.6.6], we are in turn reduced to showing that the pullback of the $\mathbb{P}_{\mathsf{K}'}$ is $\mathbb{P}_{\mathsf{K}}$, where $\mathbb{P}_{\mathsf{K}'}$ and $\mathbb{P}_{\mathsf{K}}$ are defined as in (4-11). But this is straightforward to check from the definition of $\mathbb{P}_{\mathsf{K}}$; see, e.g., [Imai et al. 2023, (4.3.5)].

Now, by the diagram (4-25), to show that $\mathscr{P}_{\mathsf{K}}$ pulls back to the universal shtuka along Θ_x, it is enough to show the corresponding result for $\mathscr{P}_{\mathsf{K}'}$. Hence for the remainder of this section we assume $\mathsf{T} = \mathsf{T}^c$.

The idea for computing the pullback of $\mathscr{P}_{\mathsf{K}}$ is to use functoriality to reduce the result to Lubin–Tate theory. Let T_1 be the $\mathbb{Q}_p$-torus

$$T_1 = \mathrm{Res}_{E_{\mathsf{K}}/\mathbb{Q}_p} \mathbb{G}_m,$$

with $\mathbb{Z}_p$-model given by the Iwahori group scheme $\mathcal{T}_1 = \mathrm{Res}_{\mathcal{O}_{E_{\mathsf{K}}}/\mathbb{Z}_p} \mathbb{G}_m$. We identify $(T_1)_{\overline{\mathbb{Q}}_p}$ with $\prod_{\tau \in \mathrm{Hom}_{\mathbb{Q}_p}(E_{\mathsf{K}}, \overline{\mathbb{Q}}_p)} \mathbb{G}_{m,\overline{\mathbb{Q}}_p}$, and we define the cocharacter $\mu_1 : \mathbb{G}_m \to (T_1)_{\overline{\mathbb{Q}}_p}$ by $z \mapsto (z, 1, \dots, 1)$, with the first factor indexed by the chosen embedding $E_{\mathsf{K}} \hookrightarrow \overline{\mathbb{Q}}_p$. Denote by f_1 the homomorphism of $\mathbb{Q}_p$-group schemes given by

$$f_1 : T_1 = \mathrm{Res}_{E_{\mathsf{K}}/\mathbb{Q}_p} \mathbb{G}_m \xrightarrow{N_{E_{\mathsf{K}}/E}} \mathrm{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m \hookrightarrow (\mathrm{Res}_{\mathsf{E}/\mathbb{Q}} \mathbb{G}_m)_{\mathbb{Q}_p} \xrightarrow{r(\mu)^{\mathrm{alg}}_{\mathbb{Q}_p}} T,$$

where $r(\mu)^{\mathrm{alg}}$ is the morphism defined in (4-3). One can check that $\mu = f_1 \circ \mu_1$, so f_1 extends to a morphism of local Shimura data $(T_1, b_1, \mu_1) \to (T, b_x, \mu)$, where b_1 is the unique element of $B(T_1, \mu_1^{-1})$. Moreover, f_1 extends to a morphism $\mathcal{T}_1 \to \mathcal{T}$ by the Néron mapping property and functoriality of identity components, so by [Pappas and Rapoport 2022, Proposition 5.3.1], the morphism of $\mathrm{Spd}(\mathcal{O}_{\breve{E}})$-$v$-sheaves $\mathcal{M}^{\mathrm{int}}_{\mathcal{T}_1,b_1,\mu_1} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{T},b_x,\mu}$ induces an isomorphism

$$\hat{f}_1 : \widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{T}_1,b_1,\mu_1}}_{/x_1} \xrightarrow{\sim} \widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{T},b_x,\mu}}_{/x_0},$$

where x_1 is the base point of $\mathcal{M}^{\mathrm{int}}_{\mathcal{T}_1,b_1,\mu_1}$ (see (4-16)). Hence we obtain a commutative diagram

$$\begin{array}{ccccc} & \overset{\Psi_{x,1}}{\nearrow} & \widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{T}_1,b_1,\mu_1}}_{/x_1} & \hookrightarrow & \mathcal{M}^{\mathrm{int}}_{\mathcal{T}_1,b_1,\mu_1} \\ \mathrm{Spd}(\mathcal{O}_{\breve{E}}) & & \wr\downarrow & & \downarrow \\ & \underset{\Psi_x}{\searrow} & \widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{T},b_x,\mu}}_{/x_0} & \hookrightarrow & \mathcal{M}^{\mathrm{int}}_{\mathcal{T},b_x,\mu} \end{array} \tag{4-26}$$

where $\Psi_{x,1}$ and Ψ_x are the inverses of the isomorphisms coming from (4-17). We remark that the left triangle commutes by because $\hat{f}_1$ is a morphism over $\mathrm{Spd}(\mathcal{O}_{\breve{E}})$, and because $\Psi_{x,1}$ an Ψ_x are the inverses of the structure morphisms.

Denote by $\mathscr{P}^{\mathrm{univ}}$ and $\mathscr{P}_1^{\mathrm{univ}}$ the universal shtukas on $\mathcal{M}^{\mathrm{int}}_{\mathcal{T},b_x,\mu}$ and $\mathcal{M}^{\mathrm{int}}_{\mathcal{T}_1,b_1,\mu_1}$, respectively, and let $\mathscr{P}_{\mathsf{K},x}$ denote the restriction of $\mathscr{P}_{\mathsf{K}}$ to $(\widehat{\mathscr{S}_{\mathsf{K}/x}})^{\lozenge}$. Since Θ_x is an isomorphism, to show $\Theta_x^*(\mathscr{P}_{\mathsf{K},x}) \cong \mathscr{P}^{\mathrm{univ}}$ it is enough to show $\Psi_x^*(\mathscr{P}^{\mathrm{univ}}) \cong \mathscr{P}_{\mathsf{K},x}$. By (4-13), we have $\hat{f}_1^{\,*}(\mathscr{P}^{\mathrm{univ}}) \cong \mathscr{P}_1^{\mathrm{univ}} \times^{\mathcal{T}_1} \mathcal{T}$, so by (4-26) we see that

$$\Psi_x^*(\mathscr{P}^{\mathrm{univ}}) \cong \Psi_{x,1}^*(\mathscr{P}_1^{\mathrm{univ}}) \times^{\mathcal{T}_1} \mathcal{T}. \tag{4-27}$$

Let $\Gamma_{E_{\mathsf{K}},0}$ denote the inertia subgroup of $\Gamma_{E_{\mathsf{K}}}$, and let

$$\mathrm{LT} : \Gamma_{E_{\mathsf{K}},0} \to \mathcal{O}_{E_{\mathsf{K}}}^{\times} \tag{4-28}$$

denote the Lubin–Tate character [Serre 1979, Section 2.1] Then LT induces a $\mathcal{T}_1$-valued crystalline representation α_0 of $\Gamma_{E_{\mathsf{K}},0}$. By the arguments in [Kisin et al. 2021, Proposition 4.3.14], the restriction of $r(\mu)_{\mathsf{K},p,\mathrm{loc}}$ (see (4-7)) to $\Gamma_{E_{\mathsf{K}},0}$ is given by the composition

$$\Gamma_{E_{\mathsf{K}},0} \xrightarrow{\mathrm{LT}} \mathcal{O}_{E_{\mathsf{K}}}^{\times} = \mathcal{T}_1(\mathbb{Z}_p) \xrightarrow{f_1} \mathcal{T}(\mathbb{Z}_p).$$

Denote by $\mathscr{P}_1$ the $\mathcal{T}_1$-shtuka associated to α_0 by Definition 3.11. Then Lemma 3.12 implies that there is a natural isomorphism

$$\mathscr{P}_1 \times^{\mathcal{T}_1} \mathcal{T} \xrightarrow{\sim} \mathscr{P}_{\mathsf{K},x}. \tag{4-29}$$

By combining (4-27) and (4-29), we see that we are reduced to showing the following proposition.

Proposition 4.9. *There is a natural isomorphism of shtukas*

$$\Psi_{x,1}^*(\mathscr{P}_1^{\mathrm{univ}}) \cong \mathscr{P}_1.$$

Proof. First note that the local Shimura datum (T_1, b_1, μ_1) along with the parahoric group scheme $\mathcal{T}_1$ actually come from an RZ-datum of EL-type; see [Scholze and Weinstein 2020, Section 24.3]). Fix a uniformizer π of $\mathcal{O}_{E_{\mathsf{K}}}$. In this case, the RZ-datum is given by the tuple $\mathcal{D} = (E_{\mathsf{K}}, E_{\mathsf{K}}, \mathcal{O}_{E_{\mathsf{K}}}, \mathcal{L})$, where $\mathcal{L}$ is the lattice chain given by multiples of $\mathcal{O}_{E_{\mathsf{K}}}$, i.e., $\mathcal{L} = \{\pi^k \mathcal{O}_{E_{\mathsf{K}}}\}_{k\in\mathbb{Z}}$. Denote by $\mathcal{M}_{\mathrm{LT}}$ the Rapoport–Zink formal scheme associated to $\mathcal{D}$. By [loc. cit., Corollary 25.1.3], there is a natural isomorphism

$$(\mathcal{M}_{\mathrm{LT}})^{\lozenge} \xrightarrow{\sim} \mathcal{M}^{\mathrm{int}}_{\mathcal{T}_1,b_1,\mu_1}.$$

By [loc. cit., Proposition 18.4.1], the morphism

$$\mathrm{Spd}(\mathcal{O}_{\breve{E}}) \xrightarrow{\Psi_{x,1}} \widehat{\mathcal{M}^{\mathrm{int}}_{\mathcal{T}_1,b_1,\mu_1}}_{/x_1} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{T}_1,b_1,\mu_1}$$

is induced by a unique morphism of formal schemes over $\mathrm{Spf}(\mathcal{O}_{\breve{E}})$

$$\mathrm{Spf}(\mathcal{O}_{\breve{E}}) \to \mathcal{M}_{\mathrm{LT}}. \tag{4-30}$$

By [Rapoport and Zink 1996, 3.78], for any locally p-nilpotent $\mathcal{O}_{\breve{E}}$-scheme S, $\mathcal{M}_{\mathrm{LT}}(S)$ parametrizes pairs (X, ρ) consisting of a p-divisible group over S with an action of $\mathcal{O}_E$, such that the induced action of $\mathcal{O}_E$ on Lie X agrees with the natural one, and ρ is a quasiisogeny between the p-divisible group X_{b_1} associated to b_1 by Dieudonné theory and X modulo p. In particular, the morphism (4-30) determines a p-divisible group X over $\mathrm{Spf}(\mathcal{O}_{\breve{E}})$ (equivalently, over $\mathrm{Spec}(\mathcal{O}_{\breve{E}})$, see [de Jong 1995, Lemma 2.4.4]), and the shtuka $\Psi_{x,1}^*(\mathscr{P}_1^{\mathrm{univ}})$ is given by the shtuka associated to X by [Pappas and Rapoport 2024, Example 2.3.2].

By deformation theory [Drinfeld 1974, Proposition 4.2], up to isomorphism there is a unique p-divisible group X over $\mathrm{Spec}(\mathcal{O}_{\breve{E}})$ as in the preceding paragraph. Hence X is given by the base change to $\mathrm{Spf}(\mathcal{O}_{\breve{E}})$ of the Lubin–Tate formal group X_{LT} over $\mathrm{Spec}(\mathcal{O}_{E_{\mathsf{K}}})$. It follows that the representation of $\Gamma_{E_{\mathsf{K}},0}$ given by the Tate module of X is the restriction of the representation of $\Gamma_{E_{\mathsf{K}}}$ given by the Tate module of X_{LT}. On the other hand, by [Serre 1968, A.4 Proposition 4], the latter is given by the Lubin–Tate character, LT (see (4-28)). Therefore, to complete the proof of the proposition, it remains only to show that, for any p-divisible group X over $\mathrm{Spec}(\mathcal{O}_{\breve{E}})$, the shtuka associated to X by [Pappas and Rapoport 2024, Example 2.3.2] is the same as that associated to the crystalline representation given by the Tate module of X by Definition 3.1.

Let $\mathcal{M}_{\Delta}(X)$ be the prismatic Dieudonné crystal associated to X by [Anschütz and Le Bras 2023]. By [loc. cit., Proposition 4.3.7], the shtuka associated to X by [Pappas and Rapoport 2024, Example 2.3.2] is given by the pullback of the BKF-module over $\mathrm{Spd}(\mathcal{O}_{\breve{E}})$ given by restricting $\mathcal{M}_{\Delta}(X)$ to perfect prisms as in Section 3.1. On the other hand, by [Du et al. 2024, Proposition 3.34], the prismatic F-crystal associated to $T_p(X)$ is exactly $\mathcal{M}_{\Delta}(X)$. Thus it follows from the construction in 3.1 that the shtuka over $\mathrm{Spd}(\mathcal{O}_{\breve{E}})$ associated to $T_p(X)$ coincides with the one associated to X by [Pappas and Rapoport 2024, Example 2.3.2]. This proves the claim from the previous paragraph, and completes the proof of the proposition. □

Theorem 4.10. *Conjecture 4.5 holds when* $\mathsf{G} = \mathsf{T}$ *is a torus.*

Proof. For the first part of part (a), we can reduce to a single compact open subgroup $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p$, in which case the result follows by the valuative criterion for properness of $\mathcal{O}_{E_{\mathsf{K}}}$ over $\mathcal{O}_E$. The rest of part (a) follows from the construction of $\mathscr{S}_{\mathsf{K}}$ and Lemma 4.8. Part (b) follows from Proposition 4.7, and part (c) follows from Proposition 4.9 and the remarks above it. □

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The geometric Breuil–Mézard conjecture for two-dimensional potentially Barsotti–Tate Galois representations

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We establish a geometrization of the Breuil–Mézard conjecture for potentially Barsotti–Tate representations, as well as of the weight part of Serre's conjecture, for moduli stacks of two-dimensional mod p representations of the absolute Galois group of a p-adic local field. These results are first proved for the stacks of our earlier papers, and then transferred to the stacks of Emerton and Gee by means of a comparison of versal rings.

1. Introduction

Let $K/\mathbb{Q}_p$ be a finite extension with residue field k, let $\overline{K}$ be an algebraic closure of K, and let $d \geq 1$ be a positive integer. Emerton and Gee [2023] have constructed moduli stacks of representations of the absolute Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$, globalizing Mazur's classical deformation theory of Galois representations. These stacks are expected to be the backbone of a categorical p-adic Langlands correspondence, playing the role anticipated by the stacks of [Dat et al. 2020; Zhu 2020] in the $\ell \neq p$ setting.

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To be precise, the book [Emerton and Gee 2023] defines the category $\mathcal{X}_d$ fibered in groupoids over $\operatorname{Spf}\mathbb{Z}_p$ whose A-valued points, for any p-adically complete $\mathbb{Z}_p$-algebra A, are the groupoid of rank d projective étale (φ, Γ)-modules with A-coefficients. Then the finite type points of $\mathcal{X}_d$ correspond to representations $\bar{r} : G_K \to \mathrm{GL}_d(\overline{\mathbb{F}}_p)$, the versal rings of $\mathcal{X}_d$ at finite type points recover classical Galois deformation rings, and one has the following, which is one of the main results of [loc. cit.].

Theorem 1.1 [Emerton and Gee 2023, Theorem 1.2.1]. *Each $\mathcal{X}_d$ is a Noetherian formal algebraic stack. Its underlying reduced substack $\mathcal{X}_{d,\mathrm{red}}$ is an algebraic stack of finite type over $\mathbb{F}_p$, and is equidimensional of dimension $[K : \mathbb{Q}_p]\binom{d}{2}$. The irreducible components of $\mathcal{X}_{d,\mathrm{red}}$ have a natural bijective labeling by Serre weights.*

Recall that a *Serre weight* in this context is an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_d(k)$ (or rather an isomorphism class thereof). The description in [Emerton and Gee 2023] of the labeling of components of $\mathcal{X}_{d,\mathrm{red}}$ by Serre weights is to some extent combinatorial. Namely, it is shown that each irreducible component of $\mathcal{X}_{d,\mathrm{red}}$ has a dense set of $\overline{\mathbb{F}}_p$-points which are successive extensions of characters, with extensions as nonsplit as possible. The restrictions of these characters to the inertia group yield discrete data (their tame inertia weights) which, together with some further information about peu and très ramifiée extensions, amounts precisely to the data of (a highest weight of) a Serre weight.

It is expected, however, that there is another description of the irreducible components of $\mathcal{X}_{d,\mathrm{red}}$ that is more precise and more informative from the perspective of the p-adic Langlands program. The weight part of Serre's conjecture, as described for instance in [Gee et al. 2018, Section 3], associates to each $\bar{r} : G_K \to \mathrm{GL}_d(\overline{\mathbb{F}}_p)$ a set of Serre weights $W(\bar{r})$. One expects for each Serre weight σ that there is a set of components of $\mathcal{X}_{d,\mathrm{red}}$, including the irreducible component labeled by σ in [Emerton and Gee 2023], the union of whose $\overline{\mathbb{F}}_p$-points are *precisely* the representations $\bar{r}$ with $\sigma \in W(\bar{r})$. Equivalently, after adding additional labels to some of the components (so that components will be labeled by a set of weights, rather than a single weight), the set $W(\bar{r})$ is precisely the collection of labels of the various components of $\mathcal{X}_{d,\mathrm{red}}$ on which $\bar{r}$ lies.

One of the aims of this paper is to establish this expectation in the case $d = 2$, taking as input the weight part of Serre's conjecture for GL_2 [Gee et al. 2015] and the Breuil–Mézard conjecture for two-dimensional potentially Barsotti–Tate representations [Gee and Kisin 2014], and thus obtain a description of *all* the finite type points of each irreducible component of $\mathcal{X}_{2,\mathrm{red}}$, as opposed to just a dense set of points. Indeed we have the following theorem, which can be regarded as a geometrization of the weight part of Serre's conjecture for GL_2. If σ is a Serre weight, let $\mathcal{X}^{\sigma}_{d,\mathrm{red}}$ denote the irreducible component of $\mathcal{X}_{d,\mathrm{red}}$ labeled by σ in [Emerton and Gee 2023]. (We refer the reader to Section 2 for any unfamiliar notation or terminology in what follows.)

Theorem 1.2. *Suppose $p > 2$. For each Serre weight σ we define a cycle Z^{σ} as follows*:

- $Z^{\sigma} = \mathcal{X}^{\sigma}_{2,\mathrm{red}}$ *if the weight σ is not Steinberg.*
- $Z^{\chi\otimes\mathrm{St}} = \mathcal{X}^{\chi}_{2,\mathrm{red}} + \mathcal{X}^{\chi\otimes\mathrm{St}}_{2,\mathrm{red}}$ *if the weight $\sigma \cong \chi \otimes \mathrm{St}$ is Steinberg.*

Then $\sigma \in W(\bar{r})$ if and only if $\bar{r}$ lies in the support of Z^{σ}.

Indeed a stronger statement is true: the cycles $Z^\sigma = \mathcal{X}^\sigma_{2,\mathrm{red}}$ (for σ non-Steinberg) and $Z^{\chi\otimes\mathrm{St}} = \mathcal{X}^\chi_{2,\mathrm{red}} + \mathcal{X}^{\chi\otimes\mathrm{St}}_{2,\mathrm{red}}$ constitute the cycles in a geometric version of the Breuil–Mézard conjecture (to be explained below).

We emphasize that the existence of such a geometric interpretation of the sets $W(\bar{r})$ is far from obvious, and indeed we know of no direct proof using any of the explicit descriptions of $W(\bar{r})$ in the literature; it seems hard to understand in any explicit way which Galois representations arise as the limits of a family of extensions of given characters, and the description of the sets $W(\bar{r})$ is very complicated (for example, the description in [Buzzard et al. 2010] relies on certain Ext groups of crystalline characters). Our proof is indirect, and ultimately makes use of a description of $W(\bar{r})$ given in [Gee and Kisin 2014], which is in terms of potentially Barsotti–Tate deformation rings of $\bar{r}$ and is motivated by the Taylor–Wiles method. We interpret this description in the geometric language of [Emerton and Gee 2014], which we in turn interpret as the formal completion of a "geometric Breuil–Mézard conjecture" for our stacks.

The proof of Theorem 1.2 entwines the main results of the book [Emerton and Gee 2023] with the results of our papers [Caraiani et al. 2022; 2024]. Indeed Theorem 1.2 is (more or less) stated at [Emerton and Gee 2023, Theorem 8.6.2], but the argument given there makes reference to (an earlier version of) this paper.[1] We should therefore explain more precisely what are the contributions of this paper.

For each Hodge type λ and inertial type τ, the book [Emerton and Gee 2023] constructs a closed substack $\mathcal{X}^{\lambda,\tau}_d \subset \mathcal{X}_d$ parametrizing d-dimensional potentially crystalline representations of G_K of Hodge type λ and inertial type τ. When $d = 2$, λ is trivial, and τ is tame, these are stacks of potentially Barsotti–Tate representations of type τ, and we write $\mathcal{X}^{\tau,\mathrm{BT}}_2$ instead.

The papers [Caraiani et al. 2022; 2024] construct and study another stack $\mathcal{Z}^{\mathrm{dd}}$ which can be regarded as a stack of *tamely* potentially Barsotti–Tate representations; as well as a closed substack $\mathcal{Z}^\tau \subset \mathcal{Z}^{\mathrm{dd}}$, for each tame type τ, of potentially Barsotti–Tate representations of type τ. Our stacks $\mathcal{Z}^\tau$ are presumably isomorphic to the stacks $\mathcal{X}^{\tau,\mathrm{BT}}_2$,[2] but literally they are different stacks, constructed differently: the stacks $\mathcal{Z}^\tau$ are stacks of étale φ-modules with tame descent data, constructed by taking the scheme-theoretic image of a stack $\mathcal{C}^{\tau,\mathrm{BT}}$ of Breuil–Kisin modules with tame descent data; whereas the $\mathcal{X}^{\tau,\mathrm{BT}}_2$ are stacks of étale (φ, Γ)-modules, constructed by taking the scheme-theoretic image of a stack of Breuil–Kisin–Fargues modules satisfying a descent condition. In practice it seems to be easier to compute with the stacks $\mathcal{Z}^\tau$ than the stacks $\mathcal{X}^{\tau,\mathrm{BT}}_2$.

The properties of $\mathcal{Z}^{\mathrm{dd}}$ and $\mathcal{Z}^\tau$ that we will use in this paper are recalled in detail in Section 3, but we mention two crucial properties now:

- It is proved in [Caraiani et al. 2024] by a local model argument that the special fiber of $\mathcal{C}^{\tau,\mathrm{BT}}$ is reduced. As a consequence so is its scheme-theoretic image $\mathcal{Z}^{\tau,1}$ in $\mathcal{Z}^\tau$. The stack $\mathcal{Z}^{\mathrm{dd},1}$, the scheme-theoretic image in $\mathcal{Z}^{\mathrm{dd}}$ of the special fiber of $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$, is similarly reduced.

[1] The reference [CEGS19, Theorem 5.2.2] in [Emerton and Gee 2023] is Theorem 6.2 of this paper, while the reference [CEGS19, Lemma B.5] in [Emerton and Gee 2023] is [Caraiani et al. 2024, Lemma A.5].

[2] Added in revision: In fact this has now been proved; see [Bellovin et al. 2024, Theorem 4.5].

- It is shown in [Caraiani et al. 2022] that the irreducible components of $\mathcal{Z}^{\tau,1}$ are in bijection with the Jordan–Hölder factors of $\bar{\sigma}(\tau)$; the component corresponding to σ has a dense set of $\overline{\mathbb{F}}_p$-points $\bar{r}$ such that $W(\bar{r}) = \{\sigma\}$.

Here $\sigma(\tau)$ is the representation of $\mathrm{GL}_2(\mathcal{O}_K)$ corresponding to τ under the inertial local Langlands correspondence, and $\bar{\sigma}(\tau)$ is its reduction modulo p. Note the similarity between the second of these two properties, and the labeling by Serre weights in Theorem 1.1.

These properties are combined in Section 4 to prove that the special fiber of $\mathcal{Z}^{\tau}$ is generically reduced. (Note that in general the special fiber of $\mathcal{Z}^{\tau}$ need not be the same as $\mathcal{Z}^{\tau,1}$; and similarly for the special fiber of $\mathcal{Z}^{\mathrm{dd}}$ vis-à-vis $\mathcal{Z}^{\mathrm{dd},1}$.) From this we deduce the following theorem about the special fibers of potentially Barsotti–Tate deformation rings, which seems hard to prove purely in the setting of formal deformations. Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$, with residue field $\mathbb{F}$.

Theorem 1.3. *Let $\bar{r} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation, and τ a tame type. Let $R_{\bar{r}}^{\tau,\mathrm{BT}}$ be the universal framed deformation $\mathcal{O}$-algebra parametrizing potentially Barsotti–Tate lifts of $\bar{r}$ of type τ. Then $R_{\bar{r}}^{\tau,\mathrm{BT}} \otimes_{\mathcal{O}} \mathbb{F}$ is generically reduced.*

We anticipate that this result will be of independent interest. For example, Caraiani in joint work with James Newton [2023], has used this result in the proof of a modularity lifting theorem in the Barsotti–Tate case for GL_2 over a CM field; this modularity lifting theorem is used, in turn, to deduce the modularity of elliptic curves over $\mathbb{Q}(\sqrt{-d})$ for $d \in \{1, 2, 3, 5\}$.

We remark that très ramifiée representations do not have tamely potentially Barsotti–Tate lifts, hence do not correspond to finite type points on $\mathcal{Z}^{\mathrm{dd},1}$. Equivalently (see [Caraiani et al. 2024, Lemma A.5]), the Jordan–Hölder factors of $\bar{\sigma}(\tau)$ for tame types τ are never Steinberg, and therefore the stacks $\mathcal{Z}^{\tau,1}$ and $\mathcal{Z}^{\mathrm{dd},1}$ do not have irreducible components corresponding to Steinberg weights. So, although $\mathcal{Z}^{\mathrm{dd},1}$ and $\mathcal{X}_{2,\mathrm{red}}$ cannot be isomorphic, we anticipate (but do not prove) that there is an isomorphism between $\mathcal{Z}^{\mathrm{dd},1}$ and the union of the non-Steinberg components of $\mathcal{X}_{2,\mathrm{red}}$, along the same lines as [Bellovin et al. 2024, Theorem 4.5].

If σ is a non-Steinberg weight, then σ can be written as a virtual linear combination of representations $\bar{\sigma}(\tau)$ in the Grothendieck group of $\mathrm{GL}_2(k)$. In Section 5 this observation is translated into a special case of the classical geometric Breuil–Mézard conjecture [Emerton and Gee 2014]; we globalize this in Section 6 to prove the following theorem, which is the main result of this paper.

Theorem 1.4. *The irreducible components of $\mathcal{Z}^{\mathrm{dd},1}$ are in bijection with non-Steinberg Serre weights; write $\overline{\mathcal{Z}}(\sigma)$ for the component corresponding to σ. Then:*

(1) *The finite type points of $\overline{\mathcal{Z}}(\sigma)$ are precisely the representations $\bar{r} : G_K \to \mathrm{GL}_2(\mathbb{F}')$ having σ as a Serre weight.*

(2) *The stack $\mathcal{Z}^{\tau,1}$ is equal to $\bigcup_{\sigma \in \mathrm{JH}(\bar{\sigma}(\tau))} \overline{\mathcal{Z}}(\sigma)$.*

Part (1) of the theorem is the analogue of Theorem 1.2 for the stacks $\mathcal{Z}^{\mathrm{dd},1}$, while part (2) is a geometrization of the Breuil–Mézard conjecture for our tamely potentially Barsotti–Tate stacks. Theorem 1.3

is used crucially in the proof of Theorem 1.4, to confirm that each component $\bar{\mathcal{Z}}(\sigma)$ contributes with multiplicity at most one to the cycle of $R^{\tau,\mathrm{BT}} \otimes_{\mathcal{O}} \mathbb{F}$. We emphasize that, to this point, our results are independent from those of [Emerton and Gee 2023].

As explained above, our construction excludes the très ramifiée representations, which are twists of certain extensions of the trivial character by the mod p cyclotomic character. From the point of view of the weight part of Serre's conjecture, they are precisely the representations which admit a twist of the Steinberg representation as their only Serre weight. In accordance with the picture described above, this means that the full moduli stack of two-dimensional representations of G_K can be obtained from our stack by adding in the irreducible components consisting of the très ramifiée representations. This is carried out by extending our results to the stacks of [Emerton and Gee 2023], using the full strength of [loc. cit.].

In particular, it is proved in [loc. cit.] that the classical (numerical) Breuil–Mézard conjecture is equivalent to a geometrized Breuil–Mézard conjecture for the stacks $\mathcal{X}_d^{\lambda,\tau}$ of [loc. cit.]. Taking the Breuil–Mézard conjecture for potentially Barsotti–Tate representations [Gee and Kisin 2014] as input, they obtain the following theorem.

Theorem 1.5 [Emerton and Gee 2023]. *There exist effective cycles Z^{σ} (elements of the free group on the irreducible components of $\mathcal{X}_{2,\mathrm{red}}$, with nonnegative coefficients) such that for all inertial types τ, the cycle of the special fiber of $\mathcal{X}_2^{\tau,\mathrm{BT}}$ is equal to $\sum_\sigma m_\sigma(\tau)\cdot Z^\sigma$, where $\bar{\sigma}(\tau) = \sum_\sigma m_\sigma(\tau)\cdot\sigma$ in the Grothendieck group of* $\mathrm{GL}_2(k)$.

We stress that this theorem of [Emerton and Gee 2023] is for all inertial types, in contrast to the Breuil–Mézard result of Theorem 1.4(2) which is only for tame types; in particular the cycles for Steinberg weights σ do occur. In fact the theorem can be (and is) extended to cover potentially semistable representations of Hodge type 0 as well.

It remains to prove that the cycles Z^σ are as in Theorem 1.2, i.e., to check that $Z^\sigma = \mathcal{X}^\sigma_{2,\mathrm{red}}$ when σ is non-Steinberg, whereas $Z^{\chi\otimes\mathrm{St}} = \mathcal{X}^{\chi}_{2,\mathrm{red}} + \mathcal{X}^{\chi\otimes\mathrm{St}}_{2,\mathrm{red}}$. This is where the results of the present paper enter. We argue by transferring results from $\mathcal{Z}^\tau$ to $\mathcal{X}_2^{\tau,\mathrm{BT}}$ via a consideration of versal rings, without comparing the two stacks directly.[3] In particular the ring $R_{\bar{r}}^{\tau,\mathrm{BT}}$ is a versal ring to $\mathcal{X}^{\tau,\mathrm{BT}}$ at the point corresponding to $\bar{r}$; and so the formula $Z^\sigma = \mathcal{X}^\sigma_{2,\mathrm{red}}$ in the non-Steinberg case will follow by an application of Theorem 1.3 for a suitably chosen $\bar{r}$. The Steinberg case is handled directly using a semistable deformation ring. This completes the proof.

2. Notation and conventions

Galois theory. Let $p > 2$ be a prime number, and fix a finite extension $K/\mathbb{Q}_p$, with residue field k of cardinality p^f. In this paper we will study various stacks that are closely related to the representation theory of G_K, the absolute Galois group of K.

[3] Added in revision: It is now also possible to transfer these results using [Bellovin et al. 2024, Theorem 4.5].

Our representations of G_K will have coefficients in $\overline{\mathbb{Q}}_p$, a fixed algebraic closure of $\mathbb{Q}_p$ whose residue field we denote by $\overline{\mathbb{F}}_p$. Let E be a finite extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$. Write $\mathcal{O}$ for the ring of integers in E, with uniformizer ϖ and residue field $\mathbb{F} \subset \overline{\mathbb{F}}_p$.

As is often the case, we assume that our coefficients are "sufficiently large". Specifically, if L is the quadratic unramified extension of K, we assume that E admits an embedding of $K' = L(\pi^{1/(p^{2f}-1)})$ for some uniformizer π of K. Write l for the residue field of L.

Fix an embedding $\sigma_0 : k \hookrightarrow \mathbb{F}$, and recursively define $\sigma_i : k \hookrightarrow \mathbb{F}$ for all $i \in \mathbb{Z}$ so that $\sigma_{i+1}^p = \sigma_i$. For each i we define the fundamental character ω_{σ_i} to be the composite

$$I_K \longrightarrow \mathcal{O}_K^\times \longrightarrow k^\times \xrightarrow{\sigma_i} \overline{\mathbb{F}}_p^\times,$$

where the map $I_K \to \mathcal{O}_K^\times$ is induced by the restriction of the inverse of the Artin map, which we normalize so that uniformizers correspond to geometric Frobenius elements.

Inertial local Langlands. A two-dimensional *tame inertial type* is (the isomorphism class of) a tamely ramified representation $\tau : I_K \to \mathrm{GL}_2(\overline{\mathbb{Z}}_p)$ that extends to a representation of G_K and whose kernel is open. Such a representation is of the form $\tau \simeq \eta \oplus \eta'$, and we say that τ is a *tame principal series type* if η and η' both extend to characters of G_K. Otherwise, $\eta' = \eta^q$, and η extends to a character of G_L. In this case we say that τ is a *tame cuspidal type*. In either case $\tau|_{I_{K'}}$ is trivial, since τ is tame, and therefore a potentially crystalline representation of G_K with inertial type τ will become crystalline over K'.

Henniart [2002] associates a finite-dimensional irreducible E-representation $\sigma(\tau)$ of $\mathrm{GL}_2(\mathcal{O}_K)$ to each inertial type τ; we refer to this association as the *inertial local Langlands correspondence*. Since we are only working with tame inertial types, this correspondence can be made very explicit, as in [Caraiani et al. 2022, Section 1.2]. (Since we will not directly use the explicit description in this paper, we will not repeat it here.)

Serre weights and tame types. By definition, a *Serre weight* is an irreducible $\mathbb{F}$-representation of $\mathrm{GL}_2(k)$. Then, concretely, a Serre weight is of the form

$$\bar{\sigma}_{\vec{t},\vec{s}} := \bigotimes_{j=0}^{f-1} (\det{}^{t_j} \operatorname{Sym}^{s_j} k^2) \otimes_{k,\sigma_j} \mathbb{F},$$

where $0 \le s_j, t_j \le p-1$ and not all t_j are equal to $p-1$. We say that a Serre weight is *Steinberg* if $s_j = p-1$ for all j, and *non-Steinberg* otherwise.

Let τ be a tame inertial type. Write $\bar{\sigma}(\tau)$ for the semisimplification of the reduction modulo p of a $\mathrm{GL}_2(\mathcal{O}_K)$-stable $\mathcal{O}$-lattice in $\sigma(\tau)$. The action of $\mathrm{GL}_2(\mathcal{O}_K)$ on $\bar{\sigma}(\tau)$ factors through $\mathrm{GL}_2(k)$, so the Jordan–Hölder factors $\mathrm{JH}(\bar{\sigma}(\tau))$ of $\bar{\sigma}(\tau)$ are Serre weights. By the results of [Diamond 2007], these Jordan–Hölder factors of $\bar{\sigma}(\tau)$ are pairwise nonisomorphic, and are parametrized by a certain set $\mathcal{P}_\tau$ that we now recall.

Suppose first that $\tau = \eta \oplus \eta'$ is a tame principal series type. Set $f' = f$ in this case. We define $0 \le \gamma_i \le p-1$ (for $i \in \mathbb{Z}/f\mathbb{Z}$) to be the unique integers not all equal to $p-1$ such that $\eta(\eta')^{-1} = \prod_{i=0}^{f-1} \omega_{\sigma_i}^{\gamma_i}$.

If instead $\tau = \eta \oplus \eta'$ is a cuspidal type, set $f' = 2f$. We define $0 \le \gamma_i \le p-1$ (for $i \in \mathbb{Z}/f'\mathbb{Z}$) to be the unique integers such that $\eta(\eta')^{-1} = \prod_{i=0}^{f'-1} \omega_{\sigma_i'}^{\gamma_i}$. Here $\sigma_0' : l \to \overline{\mathbb{F}}_p^\times$ is a fixed choice of embedding extending σ_0, $(\sigma_{i+1}')^p = \sigma_i'$ for all i, and the fundamental characters $\omega_{\sigma_i'} : I_L \to \overline{\mathbb{F}}_p^\times$ for each $\sigma_i' : l \to \overline{\mathbb{F}}_p^\times$ are defined in the same way as the ω_{σ_i}.

If τ is scalar then we set $\mathcal{P}_\tau = \{\varnothing\}$. Otherwise we have $\eta \ne \eta'$, and we let $\mathcal{P}_\tau$ be the collection of subsets $J \subset \mathbb{Z}/f'\mathbb{Z}$ satisfying the conditions

- if $i-1 \in J$ and $i \notin J$ then $\gamma_i \ne p-1$, and
- if $i-1 \notin J$ and $i \in J$ then $\gamma_i \ne 0$

and, in the cuspidal case, satisfying the further condition that $i \in J$ if and only if $i+f \notin J$.

The Jordan–Hölder factors of $\bar{\sigma}(\tau)$ are by definition Serre weights, and are parametrized by $\mathcal{P}_\tau$ as follows; see [Emerton et al. 2015, Sections 3.2 and 3.3]. For any $J \subseteq \mathbb{Z}/f'\mathbb{Z}$, we let δ_J denote the characteristic function of J, and if $J \in \mathcal{P}_\tau$ we define $s_{J,i}$ by

$$s_{J,i} = \begin{cases} p-1-\gamma_i-\delta_{J^c}(i) & \text{if } i-1 \in J, \\ \gamma_i - \delta_J(i) & \text{if } i-1 \notin J, \end{cases}$$

and we set $t_{J,i} = \gamma_i + \delta_{J^c}(i)$ if $i-1 \in J$ and 0 otherwise. Write $\vec{s}$ for the tuple $(s_{J,i})_{0 \le i < f}$, suppressing the J from the notation for readability, and similarly for $\vec{t}$.

In the principal series case we let $\bar{\sigma}(\tau)_J := \bar{\sigma}_{\vec{t},\vec{s}} \otimes \eta' \circ \det$ for each $J \in \mathcal{P}_\tau$; the $\bar{\sigma}(\tau)_J$ are precisely the Jordan–Hölder factors of $\bar{\sigma}(\tau)$.

In the cuspidal case, one checks that $s_{J,i} = s_{J,i+f}$ for all i, and also that the character

$$\eta' \cdot \prod_{i=0}^{f'-1} (\sigma_i')^{t_{J,i}} : l^\times \to \mathbb{F}^\times$$

factors as $\theta \circ N_{l/k}$ where $N_{l/k}$ is the norm map. We let $\bar{\sigma}(\tau)_J := \bar{\sigma}_{0,\vec{s}} \otimes \theta \circ \det$, again for $J \in \mathcal{P}_\tau$; the $\bar{\sigma}(\tau)_J$ are precisely the Jordan–Hölder factors of $\bar{\sigma}(\tau)$.

p-adic Hodge theory. We normalize Hodge–Tate weights so that all Hodge–Tate weights of the cyclotomic character are equal to -1. We say that a potentially crystalline representation $r : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ has *Hodge type* 0, or is *potentially Barsotti–Tate*, if for each $\varsigma : K \hookrightarrow \overline{\mathbb{Q}}_p$, the Hodge–Tate weights of r with respect to ς are 0 and 1. (Note that this is a more restrictive definition of potentially Barsotti–Tate than is sometimes used; however, we will have no reason to deal with representations with nonregular Hodge–Tate weights, and so we exclude them from consideration. Note also that it is more usual in the literature to say that r is potentially Barsotti–Tate if it is potentially crystalline, and $r^\vee$ has Hodge type 0.)

We say that a potentially crystalline representation

$$r : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

has *inertial type* τ if the traces of elements of I_K acting on τ and on

$$\mathrm{D}_{\mathrm{pcris}}(r) = \varinjlim_{K'/K} (\mathrm{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_r)^{G_{K'}}$$

are equal (here V_r is the underlying vector space of V_r). A representation $\bar{r} : G_K \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ *has a potentially Barsotti–Tate lift of type* τ if and only if $\bar{r}$ admits a lift to a representation $r : G_K \to \mathrm{GL}_2(\bar{\mathbb{Z}}_p)$ of Hodge type 0 and inertial type τ.

Serre weights of mod p Galois representations. Given a continuous representation $\bar{r} : G_K \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$, there is an associated (nonempty) set of Serre weights $W(\bar{r})$, defined to be the set of Serre weights $\bar{\sigma}_{\vec{t},\vec{s}}$ such that $\bar{r}$ has a crystalline lift whose Hodge–Tate weights are as follows: for each embedding $\sigma_j : k \hookrightarrow \mathbb{F}$ there is an embedding $\widetilde{\sigma}_j : K \hookrightarrow \bar{\mathbb{Q}}_p$ lifting σ_j such that the $\widetilde{\sigma}_j$-labeled Hodge–Tate weights of r are $\{-s_j - t_j, 1 - t_j\}$, and the remaining $(e-1)f$ pairs of Hodge–Tate weights of r are all $\{0, 1\}$.

There are in fact several different definitions of $W(\bar{r})$ in the literature; as a result of the papers [Barnet-Lamb et al. 2013; Gee and Kisin 2014; Gee et al. 2015], these definitions are known to be equivalent up to normalization. The normalizations in this paper are the same as those of [Caraiani et al. 2022; 2024]; see [Caraiani et al. 2022, Section 1.2]for a detailed discussion of these normalizations. In particular we have normalized the set of Serre weights so that $\bar{r}$ has a potentially Barsotti–Tate lift of type τ if and only if $W(\bar{r}) \cap \mathrm{JH}(\bar{\sigma}(\tau)) \neq \varnothing$ [Caraiani et al. 2024, Lemma A.5].

Stacks. We follow the terminology of [Stacks 2005–]; in particular, we write "algebraic stack" rather than "Artin stack". More precisely, an algebraic stack is a stack in groupoids in the *fppf* topology, whose diagonal is representable by algebraic spaces, which admits a smooth surjection from a scheme. See [Stacks 2005–, Tag 026N] for a discussion of how this definition relates to others in the literature, and [Stacks 2005–, Tag 04XB] for key properties of morphisms representable by algebraic spaces.

For a commutative ring A, an *fppf stack over* A (or *fppf* A-stack) is a stack fibered in groupoids over the big *fppf* site of $\operatorname{Spec} A$. Following [Emerton 2019, Definitions 5.3 and 7.6], an *fppf* stack in groupoids $\mathcal{X}$ over a scheme S is called a *formal algebraic stack* if there is a morphism $U \to \mathcal{X}$, whose domain U is a formal algebraic space over S (in the sense of [Stacks 2005–, Tag 0AIL]), and which is representable by algebraic spaces, smooth, and surjective.

Let $\operatorname{Spf} \mathcal{O}$ denote the affine formal scheme (or affine formal algebraic space, in the terminology of [Stacks 2005–]) obtained by ϖ-adically completing $\operatorname{Spec} \mathcal{O}$. A formal algebraic stack $\mathcal{X}$ over $\operatorname{Spec} \mathcal{O}$ is called ϖ-adic if the canonical map $\mathcal{X} \to \operatorname{Spec} \mathcal{O}$ factors through $\operatorname{Spf} \mathcal{O}$, and if the induced map $\mathcal{X} \to \operatorname{Spf} \mathcal{O}$ is algebraic, i.e., representable by algebraic stacks (in the sense of [Stacks 2005–, Tag 06CF] and [Emerton 2019, Definition 3.1]).

3. Moduli stacks of Breuil–Kisin modules and étale φ-modules

The main object of study in this paper is the ϖ-adic formal algebraic stack $\mathcal{Z}^{\mathrm{dd}}$ that was introduced and studied in [Caraiani et al. 2022; 2024], and whose $\bar{\mathbb{F}}_p$-points are naturally in bijection with the continuous representations $\bar{r} : G_K \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ admitting a potentially Barsotti–Tate lift of some tame type. In this section we review the construction and known properties of $\mathcal{Z}^{\mathrm{dd}}$, as well as those of several other closely related stacks.

Stacks of Breuil–Kisin modules. For each tame type τ, there is a ϖ-adic formal algebraic stack $\mathcal{C}^{\tau,\mathrm{BT}}$ whose $\mathrm{Spf}(\mathcal{O}_{E'})$-points, for any finite extension E'/E, are the Breuil–Kisin modules corresponding to two-dimensional potentially Barsotti–Tate representations of type τ. This stack is constructed in several steps, which we review in brief. (We refer the reader to [Caraiani et al. 2024] as well as to the summary in [Caraiani et al. 2022, Section 2.3] for complete definitions, recalling here only what will be used in this paper.)

For each integer $a \geq 1$, we write $\mathcal{C}^{\mathrm{dd},a}$ for the *fppf* stack over $\mathcal{O}/\varpi^a$ which associates to any $\mathcal{O}/\varpi^a$-algebra A the groupoid $\mathcal{C}^{\mathrm{dd},a}(A)$ of rank 2 Breuil–Kisin modules of height at most 1 with A-coefficients and descent data from K' to K. Set $\mathcal{C}^{\mathrm{dd}} = \varinjlim_a \mathcal{C}^{\mathrm{dd},a}$. The closed substack $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$ of $\mathcal{C}^{\mathrm{dd}}$ is cut out by a Kottwitz-type determinant condition, which can be thought of (on the Galois side) as cutting out the tamely potentially Barsotti–Tate representations from among all tamely potentially crystalline representations with Hodge–Tate weights in $\{0, 1\}$. By [Caraiani et al. 2024, Corollary 4.2.13] the stack $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$ then decomposes as a disjoint union of closed substacks $\mathcal{C}^{\tau,\mathrm{BT}}$, one for each tame type τ, consisting of Breuil–Kisin modules with descent data of type τ. Finally, for each $a \geq 1$ we write $\mathcal{C}^{\tau,\mathrm{BT},a} = \mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$, and similarly for $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a}$. The following properties are established in [loc. cit., Corollaries 3.1.8 and 4.5.3, Proposition 5.2.21].

Theorem 3.1. *The stacks $\mathcal{C}^{\mathrm{dd},a}$, $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a}$, and $\mathcal{C}^{\tau,\mathrm{BT},a}$ are algebraic stacks of finite type over $\mathcal{O}$, while the stacks $\mathcal{C}^{\mathrm{dd}}$, $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$, and $\mathcal{C}^{\tau,\mathrm{BT}}$ are ϖ-adic formal algebraic stacks. Moreover*:

(1) $\mathcal{C}^{\tau,\mathrm{BT}}$ *is analytically normal*, *Cohen–Macaulay*, *and flat over $\mathcal{O}$.*

(2) *The stacks $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a}$ and $\mathcal{C}^{\tau,\mathrm{BT},a}$ are equidimensional of dimension $[K : \mathbb{Q}_p]$.*

(3) *The special fibers $\mathcal{C}^{\mathrm{dd},\mathrm{BT},1}$ and $\mathcal{C}^{\tau,\mathrm{BT},1}$ are reduced.*

Galois moduli stacks. Let $\mathcal{R}_{K'}^{\mathrm{dd},a}$ be the *fppf* $\mathbb{F}$-stack which associates to any $\mathcal{O}/\varpi^a$-algebra A the groupoid $\mathcal{R}_{K'}^{\mathrm{dd},a}(A)$ of rank 2 étale φ-modules with A-coefficients and descent data from K' to K. We will usually suppress K' from the notation. Inverting u on Breuil–Kisin modules gives a proper morphism $\mathcal{C}^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a}$, which then restricts to proper morphisms $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ as well as $\mathcal{C}^{\tau,\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ for each τ.

Emerton and Gee [2021] developed a theory of scheme-theoretic images of proper morphisms $\mathcal{X} \to \mathcal{F}$ of stacks over a locally Noetherian base-scheme S, where $\mathcal{X}$ is an algebraic stack which is locally of finite presentation over S, and the diagonal of $\mathcal{F}$ is representable by algebraic spaces and locally of finite presentation. This theory applies in particular to each of the morphisms of the previous paragraph (even though $\mathcal{R}^{\mathrm{dd},a}$ is *not* algebraic). We define $\mathcal{Z}^{\mathrm{dd},a}$ and $\mathcal{Z}^{\tau,a}$ to be the scheme-theoretic images of the morphisms $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ and $\mathcal{C}^{\tau,\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$, respectively. Set $\mathcal{Z}^{\mathrm{dd}} = \varinjlim_a \mathcal{Z}^{\mathrm{dd},a}$ and $\mathcal{Z}^{\tau} = \varinjlim_a \mathcal{Z}^{\tau,a}$. The following theorem combines [Caraiani et al. 2024, Theorem 5.1.2, Proposition 5.1.4, Lemma 5.1.8, Proposition 5.2.20].

Theorem 3.2. *The stacks $\mathcal{Z}^{\mathrm{dd},a}$ and $\mathcal{Z}^{\tau,a}$ are algebraic stacks of finite type over $\mathcal{O}$, while the stacks $\mathcal{Z}^{\mathrm{dd}}$ and $\mathcal{Z}^{\tau}$ are ϖ-adic formal algebraic stacks. Moreover*:

(1) *The stacks* $\mathcal{Z}^{\mathrm{dd},a}$ *and* $\mathcal{Z}^{\tau,a}$ *are equidimensional of dimension* $[K:\mathbb{Q}_p]$.

(2) *The stacks* $\mathcal{Z}^{\mathrm{dd},1}$ *and* $\mathcal{Z}^{\tau,1}$ *are reduced.*

(3) *The* $\overline{\mathbb{F}}_p$*-points of* $\mathcal{Z}^{\mathrm{dd},1}$ *are naturally in bijection with the continuous representations* $\bar{r}: G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *which are not a twist of a très ramifiée extension of the trivial character by the mod p cyclotomic character. Similarly, the* $\overline{\mathbb{F}}_p$*-points of* $\mathcal{Z}^{\tau,1}$ *are naturally in bijection with the continuous representations* $\bar{r}: G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *which have a potentially Barsotti–Tate lift of type* τ.

In particular the stack $\mathcal{Z}^{\mathrm{dd},1}$ *is the underlying reduced substack of* $\mathcal{Z}^{\mathrm{dd},a}$ *for each* $a \geq 1$, *as well as of* $\mathcal{Z}^{\mathrm{dd}}$; *and similarly for the stacks* $\mathcal{Z}^{\tau,1}$.

Remark 3.3. We stress that the morphism $\mathcal{Z}^{\mathrm{dd},a} \hookrightarrow \mathcal{Z}^{\mathrm{dd}} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$ need not be an isomorphism *a priori*, and we have no reason to expect that it is. However, our results in the next section will prove that it is *generically* an isomorphism for all $a \geq 1$.

Versal rings and deformation rings. Let x be an $\mathbb{F}'$-point of $\mathcal{Z}^{\tau,a}$, corresponding to the representation $\bar{r}: G_K \to \mathrm{GL}_2(\mathbb{F}')$. We will usually write $R_{\bar{r}}^{\tau,\mathrm{BT}}$ for the reduced and p-torsion free quotient of the universal framed deformation ring of $\bar{r}$ whose $\overline{\mathbb{Q}}_p$-points correspond to the potentially Barsotti–Tate lifts of $\bar{r}$ of type τ. (In Section 5 we will denote this ring instead by $R_{\bar{r},0,\tau}$, for ease of comparison with the paper [Gee and Kisin 2014].)

It is explained in [Caraiani et al. 2024, Section 5.2] that there are versal rings $R_x^{\tau,a}$ to $\mathcal{Z}^{\tau,a}$ at the point x, such that the following holds. (These rings are denoted $R^{\tau,a}$ in [Caraiani et al. 2024]; we include the subscript x here to emphasize the dependence on the point x.)

Proposition 3.4 [Caraiani et al. 2024, Proposition 5.2.19]. *We have* $\varprojlim R_x^{\tau,a} = R_{\bar{r}}^{\tau,\mathrm{BT}}$; *thus* $R_{\bar{r}}^{\tau,\mathrm{BT}}$ *is a versal ring to* $\mathcal{Z}^{\tau}$ *at* x.

Similarly there is a versal ring $R_x^{\mathrm{dd},a}$ to $\mathcal{Z}^{\mathrm{dd},a}$ at x, and each $R_x^{\tau,a}$ is a quotient of $R_x^{\mathrm{dd},a}$.

Irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$ ***and*** $\mathcal{Z}^{\tau,1}$**.** Fix a tame type τ, and recall that we set $f' = f$ if the type τ is principal series, while $f' = 2f$ if the type τ is cuspidal. We say that a subset $J \subset \mathbb{Z}/f'\mathbb{Z}$ is a *profile* if

- τ is scalar and $J = \varnothing$,
- τ is a nonscalar principal series type and J is arbitrary, or
- τ is cuspidal and J has the property that $i \in J$ if and only if $i + f \notin J$.

Thus there are exactly 2^f profiles if τ is nonscalar. The set $\mathcal{P}_\tau$ introduced in Section 2 is a subset of the set of profiles.

To each profile J, the discussion in [Caraiani et al. 2022, Section 4.2.7] associates a closed substack $\overline{\mathcal{C}}(J)$ of $\mathcal{C}^{\tau,\mathrm{BT},1}$. The stack $\overline{\mathcal{Z}}(J)$ is then defined to be the scheme-theoretic image of $\overline{\mathcal{C}}(J)$ under the map $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{Z}^{\tau,1}$.

The following description of the irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$ and $\mathcal{Z}^{\tau,1}$ is proved in [Caraiani et al. 2022, Proposition 5.1.13, Theorem 5.1.17, Corollary 5.3.3, Theorem 5.4.3]; the description of the components of $\mathcal{Z}^{\tau,1}$ in part (3) of the theorem is analogous to the description of the components of $\mathcal{X}_{2,\mathrm{red}}$ of Theorem 1.1.

Theorem 3.5. *The irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$ and $\mathcal{Z}^{\tau,1}$ are as follows*:

(1) *The irreducible components of $\mathcal{C}^{\tau,1}$ are precisely the $\bar{\mathcal{C}}(J)$ for profiles J, and if $J \neq J'$ then $\bar{\mathcal{C}}(J) \neq \bar{\mathcal{C}}(J')$.*

(2) *The irreducible components of $\mathcal{Z}^{\tau,1}$ are precisely the $\overline{\mathcal{Z}}(J)$ for profiles $J \in \mathcal{P}_\tau$, and if $J \neq J'$ then $\overline{\mathcal{Z}}(J) \neq \overline{\mathcal{Z}}(J')$.*

(3) *For each $J \in \mathcal{P}_\tau$ there is a dense open substack $\mathcal{U}$ of $\bar{\mathcal{C}}(J)$ such that the map $\bar{\mathcal{C}}(J) \to \overline{\mathcal{Z}}(J)$ restricts to an open immersion on $\mathcal{U}$.*

(4) *For each $J \in \mathcal{P}_\tau$, there is a dense set of finite type points of $\overline{\mathcal{Z}}(J)$ with the property that the corresponding Galois representations have $\bar{\sigma}(\tau)_J$ as a Serre weight, and which furthermore admit a unique Breuil–Kisin model of type τ as defined in* [Caraiani et al. 2022, Definition 5.1.16].

Remark 3.6. If $\bar{\sigma}(\tau)_J = \bar{\sigma}_{\vec{t},\vec{s}}$, then the dense set of finite type points of $\overline{\mathcal{Z}}(J)$ produced in the proof of [Caraiani et al. 2022, Theorem 5.1.17], as claimed in Theorem 3.5(4), consists of points corresponding to reducible representations $\bar{r}$ such that $\bar{r}|_{I_K}$ is an extension of $\bar{\varepsilon}^{-1}\prod_{i=0}^{f-1}\omega_{\sigma_i}^{t_i}$ by $\prod_{i=0}^{f-1}\omega_{\sigma_i}^{s_i+t_i}$ (necessarily peu ramifiée in case the ratio of the two characters is cyclotomic).

Remark 3.7. We emphasize in Theorem 3.5 that the components of $\mathcal{Z}^{\tau,1}$ are indexed by profiles $J \in \mathcal{P}_\tau$, *not* by all profiles. If $J \notin \mathcal{P}_\tau$, then by [Caraiani et al. 2022, Theorem 5.1.12] the stack $\overline{\mathcal{Z}}(J)$ has dimension strictly smaller than $[K:\mathbb{Q}_p]$, and so is properly contained in some component of $\mathcal{Z}^{\tau,1}$.

Remark 3.8. Strictly speaking, in the principal series case Theorem 3.5 is proved in [Caraiani et al. 2022] for stacks of Breuil–Kisin modules and of étale φ-modules with descent data from $K(\pi^{1/(p^f-1)})$ to K, rather than from our K' to K. But in fact we can replace $K(\pi^{1/(p^f-1)})$ with any extension of prime-to-p degree that remains Galois over K, such as K', without changing the resulting stacks. This follows from [Caraiani et al. 2024, Proposition 4.3.1(2)], together with the isomorphism $\mathcal{R}^{\mathrm{dd}}_{K(\pi^{1/(p^f-1)})} \xrightarrow{\sim} \mathcal{R}^{\mathrm{dd}}_{K'}$ of stacks of étale φ-modules (which is easier than *loc. cit.*, since multiplication by u on an étale φ-module is bijective, and can be proved by the same argument as in [Emerton and Gee 2023, Corollary 2.3.21]).

4. Generic reducedness of Spec $R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi$

Fix a Galois representation $\bar{r}: G_K \to \mathrm{GL}_2(\mathbb{F}')$, where $\mathbb{F}'/\mathbb{F}$ is a finite extension. Our goal in this section is to prove that the scheme Spec $R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi$ is generically reduced; this will be a key ingredient in the proof of our geometric Breuil–Mézard result in Section 6. Recall that a scheme is generically reduced if it contains an open reduced subscheme whose underlying topological space is dense; in the case of a Noetherian affine scheme Spec A, this is equivalent to requiring that the localization of A at each of its minimal primes is reduced.

We may of course suppose that $R_{\bar{r}}^{\tau,\mathrm{BT}} \neq 0$, so that $\bar{r}$ has a potentially Barsotti–Tate lift of type τ, and so corresponds to a finite type point $x : \operatorname{Spec} \mathbb{F}' \to \mathcal{Z}^{\tau,a}$ for any $a \geq 1$. It follows from Proposition 3.4 that $\operatorname{Spec} R_x^{\tau,a}$ is a closed subscheme of $\operatorname{Spec} R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi^a$, but we have no reason to believe that equality holds. It follows from Theorem 3.2(2), together with Lemma 4.5 below, that $\operatorname{Spec} R_x^{\tau,1}$ is the underlying reduced subscheme of $\operatorname{Spec} R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi$, so that equality holds in the case $a = 1$ if and only if $\operatorname{Spec} R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi$ is reduced; however, again, we have no reason to believe that this holds in general.

Nevertheless it *is* the case that $\operatorname{Spec} R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi$ is generically reduced; see Theorem 4.6 below. We will deduce Theorem 4.6 from the following global statement.

Proposition 4.1. *Let τ be a tame type. There is a dense open substack $\mathcal{U}$ of $\mathcal{Z}^\tau$ such that $\mathcal{U}_{/\mathbb{F}}$ is reduced.*

Proof. The proposition will follow from an application of Lemma A.6, and the key to this application will be to find a candidate open substack $\mathcal{U}^1$ of $\mathcal{Z}^{\tau,1}$, which we will do using our study of the irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$ and $\mathcal{Z}^{\tau,1}$.

Recall that, for each profile $J \in \mathcal{P}_\tau$, we let $\overline{\mathcal{Z}}(J)$ denote the scheme-theoretic image of $\bar{\mathcal{C}}(J)$ under the proper morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{Z}^{\tau,1}$. Each $\overline{\mathcal{Z}}(J)$ is a closed substack of $\mathcal{Z}^{\tau,1}$, and so, if we let $\mathcal{V}(J)$ be the complement in $\mathcal{Z}^{\tau,1}$ of the union of the $\overline{\mathcal{Z}}(J')$ for all profiles $J' \neq J$ then $\mathcal{V}(J)$ is a dense open substack of $\overline{\mathcal{Z}}(J)$, by Theorem 3.5(2) and Remark 3.7 (the former in consideration of $J' \in \mathcal{P}_\tau$, the latter for $J' \notin \mathcal{P}_\tau$). The preimage $\mathcal{W}(J)$ of $\mathcal{V}(J)$ in $\mathcal{C}^{\tau,\mathrm{BT},1}$ is therefore a dense open substack of $\bar{\mathcal{C}}(J)$. Possibly shrinking $\mathcal{W}(J)$ further, we may suppose by Theorem 3.5(3) that the morphism $\mathcal{W}(J) \to \mathcal{Z}^{\tau,1}$ is an open immersion.

Write $|\cdot|$ for the underlying topological space of a stack. The complement $|\bar{\mathcal{C}}(J)| \setminus |\mathcal{W}(J)|$ is a closed subset of $|\bar{\mathcal{C}}(J)|$, and thus of $|\mathcal{C}^{\tau,\mathrm{BT},1}|$, and its image under the proper morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{Z}^{\tau,1}$ is a closed subset of $|\mathcal{Z}^{\tau,1}|$, which is (e.g., for dimension reasons[4]) a proper closed subset of $|\overline{\mathcal{Z}}(J)|$; so if we let $\mathcal{U}(J)$ be the complement in $\mathcal{V}(J)$ of this image, then $\mathcal{U}(J)$ is open and dense in $\overline{\mathcal{Z}}(J)$, and the morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \times_{\mathcal{Z}^{\tau,1}} \mathcal{U}(J) \to \mathcal{U}(J)$ is a monomorphism. Set $\mathcal{U}^1 = \bigcup_J \mathcal{U}(J)$. Since the $\mathcal{U}(J)$ are pairwise disjoint by construction,

$$\mathcal{C}^{\tau,\mathrm{BT},1} \times_{\mathcal{Z}^{\tau,1}} \mathcal{U}^1 \to \mathcal{U}^1 \tag{4.1.1}$$

is again a monomorphism. By construction (taking into account Theorem 3.5(2)), $\mathcal{U}^1$ is dense in $\mathcal{Z}^{\tau,1}$.

Now let $\mathcal{U}$ denote the open substack of $\mathcal{Z}^\tau$ corresponding to $\mathcal{U}^1$. Since $|\mathcal{Z}^\tau| = |\mathcal{Z}^{\tau,1}|$, we see that $\mathcal{U}$ is dense in $\mathcal{Z}^\tau$. We have seen in the previous paragraph that the statement of Lemma A.6(5) holds (taking

[4]Choose, as we may, a surjective smooth morphism $U \to \overline{\mathcal{Z}}(J)$ with U a finite type $\mathbb{F}$-scheme. Let $V := \bar{\mathcal{C}}(J) \times_{\overline{\mathcal{Z}}(J)} U$. Then the projection $V \to \bar{\mathcal{C}}(J)$ is again smooth and surjective, while the projection $f : V \to U$ is representable by schemes and proper; in particular, V is also a finite type $\mathbb{F}$-scheme. Let $W := \mathcal{W}(J) \times_{\overline{\mathcal{Z}}(J)} U$. Then W is a dense open subscheme of U, equipped with a section $W \to V$ which realizes it as a dense open subscheme of V as well. Write $Y := V \setminus W$ (a closed subset of V) and $X = f(Y)$ (a closed subset of U). If T is any irreducible component of V, then $\dim T = \dim(T \cap W) = \dim(f(T) \cap W) = \dim f(T)$ (in particular, $f(T)$ is an irreducible component of U; and each irreducible component of U arises in this manner), while $\dim(T \cap Y) < \dim T$. Thus also $\dim f(T \cap Y) < \dim T = \dim f(T)$, so that $f(T \cap Y)$ is a proper closed subset of $f(T)$. Since $Y = \bigcup_T (T \cap Y)$, we see that $X = \bigcup_T f(T \cap Y)$ is a proper closed subset of U, whose complement U' is dense in U. The image of U' is then a dense (equivalently, nonempty) open substack of $\overline{\mathcal{Z}}(J)$; and this image coincides with the complement in $\overline{\mathcal{Z}}(J)$ of the image of $\bar{\mathcal{C}}(J) \setminus \mathcal{W}(J)$ (since U' is the preimage of this complement, by construction).

$a = 1$, $\mathcal{X} = \mathcal{C}^{\tau,\mathrm{BT}}$, and $\mathcal{Y} = \mathcal{Z}^{\tau}$); so Lemma A.6 implies that, for each $a \geq 1$, the closed immersion

$$\mathcal{U} \times_{\mathcal{Z}^{\tau}} \mathcal{Z}^{\tau,a} \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^{a} \tag{4.1.2}$$

is an isomorphism.

In particular, since the closed immersion $\mathcal{U}^1 = \mathcal{U} \times_{\mathcal{Z}^{\tau}} \mathcal{Z}^{\tau,1} \to \mathcal{U}_{/\mathbb{F}}$ is an isomorphism, we may regard $\mathcal{U}_{/\mathbb{F}}$ as an open substack of $\mathcal{Z}^{\tau,1}$. Since $\mathcal{Z}^{\tau,1}$ is reduced, by Theorem 3.2(2), so is its open substack $\mathcal{U}_{/\mathbb{F}}$. This completes the proof of the proposition. □

Corollary 4.2. *Let τ be a tame type. There is a dense open substack $\mathcal{U}$ of $\mathcal{Z}^{\tau}$ such that we have an isomorphism $\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{Z}^{\tau}} \mathcal{U} \xrightarrow{\sim} \mathcal{U}$, as well as isomorphisms*

$$\mathcal{U} \times_{\mathcal{Z}^{\tau}} \mathcal{C}^{\tau,\mathrm{BT},a} \xrightarrow{\sim} \mathcal{U} \times_{\mathcal{Z}^{\tau}} \mathcal{Z}^{\tau,a} \xrightarrow{\sim} \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^{a},$$

for each $a \geq 1$.

Proof. Taking $\mathcal{U}$ as constructed in the proof of Proposition 4.1, this follows from Lemma A.6 applied to the monomorphism (4.1.1) and isomorphism (4.1.2). □

Remark 4.3. More colloquially, Corollary 4.2 shows that for each tame type τ, there is an open dense substack $\mathcal{U}$ of $\mathcal{Z}^{\tau}$ consisting of Galois representations which have a unique Breuil–Kisin model of type τ.

Lemma 4.4. *If $\mathcal{U}$ is an open substack of $\mathcal{Z}^{\tau}$ satisfying the condition of Proposition 4.1, and if $T \to \mathcal{Z}^{\tau}_{/\mathbb{F}}$ is a smooth morphism whose source is a scheme, then $T \times_{\mathcal{Z}^{\tau}_{/\mathbb{F}}} \mathcal{U}_{/\mathbb{F}}$ is reduced, and is a dense open subscheme of T.*

Proof. Since $\mathcal{Z}^{\tau}_{/\mathbb{F}}$ is a Noetherian algebraic stack (being of finite presentation over $\operatorname{Spec}\mathbb{F}$), the open immersion

$$\mathcal{U}_{/\mathbb{F}} \to \mathcal{Z}^{\tau}_{/\mathbb{F}}$$

is quasicompact [Stacks 2005–, Tag 0CPM], and has dense image by assumption. Again by assumption, it factors through $\mathcal{Z}^{\tau}_{\mathrm{red}}$ $(= (\mathcal{Z}^{\tau}_{/\mathbb{F}})_{\mathrm{red}})$, and the resulting open immersion

$$\mathcal{U}_{/\mathbb{F}} \to \mathcal{Z}^{\tau}_{\mathrm{red}}$$

is again quasicompact with dense image. Since its target is reduced, it is in fact scheme-theoretically dominant.

Now the given smooth morphism $T \to \mathcal{Z}^{\tau}_{/\mathbb{F}}$ base-changes to a smooth morphism

$$T \times_{\mathcal{Z}^{\tau}_{/\mathbb{F}}} \mathcal{Z}^{\tau}_{\mathrm{red}} \to \mathcal{Z}^{\tau}_{\mathrm{red}},$$

whose source is equal to the underlying reduced scheme T_{red} of T. (Indeed the source is reduced, because property of being reduced is local for the smooth topology, [Stacks 2005–, Tag 04YH]; and it is a closed subscheme of T with underlying topological space equal to that of T, by [Stacks 2005–, Tag 04XH].) Since smooth morphisms are in particular flat, the pullback

$$T \times_{\mathcal{Z}^{\tau}_{/\mathbb{F}}} \mathcal{U}_{/\mathbb{F}} = T_{\mathrm{red}} \times_{\mathcal{Z}^{\tau}_{\mathrm{red}}} \mathcal{U}_{/\mathbb{F}} \to T_{\mathrm{red}}$$

is an open immersion with dense image; here we use the fact that for a quasicompact morphism, the property of being scheme-theoretically dominant is preserved by flat base-change. Since the source of this pullback is open in T_{red}, it is itself reduced. □

The following result is standard, but we recall the proof for the sake of completeness.

Lemma 4.5. *Let T be a Noetherian scheme, all of whose local rings at finite type points are G-rings. If T is reduced (resp. generically reduced), then so are all of its complete local rings at finite type points.*

Proof. Let t be a finite type point of T, and write $A := \mathcal{O}_{T,t}$. Then A is a (generically) reduced local G-ring, and we need to show that its completion $\widehat{A}$ is also (generically) reduced. Let $\widehat{\mathfrak{p}}$ be a (minimal) prime of $\widehat{A}$; since $A \to \widehat{A}$ is (faithfully) flat, $\widehat{\mathfrak{p}}$ lies over a (minimal) prime $\mathfrak{p}$ of A by the going-down theorem.

Then $A_{\mathfrak{p}}$ is reduced by assumption, and we need to show that $\widehat{A}_{\widehat{\mathfrak{p}}}$ is reduced. By [Stacks 2005–, Tag 07QK], it is enough to show that the morphism $A \to \widehat{A}_{\widehat{\mathfrak{p}}}$ is regular. Both A and $\widehat{A}$ are G-rings (the latter by [Stacks 2005–, Tag 07PS]), so the composite

$$A \to \widehat{A} \to (\widehat{A}_{\widehat{\mathfrak{p}}})\widehat{\ }$$

is a composite of regular morphisms, and is thus a regular morphism by [Stacks 2005–, Tag 07QI].

This composite factors through the natural morphism $A_{\mathfrak{p}} \to (\widehat{A}_{\widehat{\mathfrak{p}}})\widehat{\ }$, so this morphism is also regular. Factoring it as the composite

$$A_{\mathfrak{p}} \to \widehat{A}_{\widehat{\mathfrak{p}}} \to (\widehat{A}_{\widehat{\mathfrak{p}}})\widehat{\ },$$

it follows from [Stacks 2005–, Tag 07NT] that $A_{\mathfrak{p}} \to \widehat{A}_{\widehat{\mathfrak{p}}}$ is regular, as required. □

Finally, we are ready to prove the main result of this section.

Theorem 4.6. *For any tame type τ, the scheme* $\operatorname{Spec} R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi$ *is generically reduced, with underlying reduced subscheme* $\operatorname{Spec} R_x^{\tau,1}$.

Proof. By Proposition 3.4, we have a versal morphism

$$\operatorname{Spf} R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi \to \mathcal{Z}^{\tau}_{/\mathbb{F}}$$

at the point of $\mathcal{Z}^{\tau}_{/\mathbb{F}}$ corresponding to $\bar{r} : G_K \to \mathrm{GL}_2(\mathbb{F}')$. Since $\mathcal{Z}^{\tau}_{/\mathbb{F}}$ is an algebraic stack of finite presentation over $\mathbb{F}$ (as $\mathcal{Z}^{\tau}$ is a ϖ-adic formal algebraic stack of finite presentation over $\operatorname{Spf} \mathcal{O}$), we may apply [Stacks 2005–, Tag 0DR0] to this morphism so as to find a smooth morphism $V \to \mathcal{Z}^{\tau}_{/\mathbb{F}}$ with source a finite type $\mathcal{O}/\varpi$-scheme, and a point $v \in V$ with residue field $\mathbb{F}'$, such that there is an isomorphism $\widehat{\mathcal{O}}_{V,v} \cong R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi$, compatible with the given morphism to $\mathcal{Z}^{\tau}_{/\mathbb{F}}$. Proposition 4.1 and Lemma 4.4 taken together show that V is generically reduced, and so the result follows from Lemma 4.5. □

5. A case of the classical geometric Breuil–Mézard conjecture

In this section, by combining the methods of [Emerton and Gee 2014] and [Gee and Kisin 2014] we prove a special case of the classical geometric Breuil–Mézard conjecture [Emerton and Gee 2014, Conjecture 4.2.1]. This result is "globalized" in Section 6.

Let $\bar{r}: G_K \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation, and let $R^{\square}_{\bar{r}}$ be the universal framed deformation $\mathcal{O}$-algebra for $\bar{r}$. In this section we write $R^{\square}_{\bar{r},0,\tau}$ for the quotient of $R^{\square}_{\bar{r}}$ that elsewhere we have denoted $R^{\tau,\mathrm{BT}}_{\bar{r}}$. We use the more cumbersome notation $R^{\square}_{\bar{r},0,\tau}$ here to make it easier for the reader to refer to [Emerton and Gee 2014] and [Gee and Kisin 2014].

By [Emerton and Gee 2014, Proposition 4.1.2], $R^{\square}_{\bar{r},0,\tau}/\varpi$ is zero if $\bar{r}$ has no potentially Barsotti–Tate lifts of type τ, and otherwise it is equidimensional of dimension $4+[K:\mathbb{Q}_p]$. Each $\operatorname{Spec} R^{\square}_{\bar{r},0,\tau}/\varpi$ is a closed subscheme of $\operatorname{Spec} R^{\square}_{\bar{r}}/\varpi$, and we write $Z(R^{\square}_{\bar{r},0,\tau}/\varpi)$ for the corresponding cycle, as in [loc. cit., Definition 2.2.5]. This is a formal sum of the irreducible components of $\operatorname{Spec} R^{\square}_{\bar{r},0,\tau}/\varpi$, weighted by the multiplicities with which they occur.

Lemma 5.1. *If σ is a non-Steinberg Serre weight of $\mathrm{GL}_2(k)$, then there are integers $n_\tau(\sigma)$ such that $\sigma = \sum_\tau n_\tau(\sigma)\bar{\sigma}(\tau)$ in the Grothendieck group of mod p representations of $\mathrm{GL}_2(k)$, where the τ run over the tame types.*

Proof. This is an immediate consequence of the surjectivity of the natural map from the Grothendieck group of $\overline{\mathbb{Q}}_p$-representations of $\mathrm{GL}_2(k)$ to the Grothendieck group of $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k)$ [Serre 1977, Section III, Theorem 33], together with the observation that the reduction of the Steinberg representation of $\mathrm{GL}_2(k)$ is precisely $\bar{\sigma}_{\vec{0},\vec{p-1}}$. $\square$

Let σ be a non-Steinberg Serre weight of $\mathrm{GL}_2(k)$, so that by Lemma 5.1 we can write

$$\sigma = \sum_\tau n_\tau(\sigma)\bar{\sigma}(\tau) \tag{5.1.1}$$

in the Grothendieck group of mod p representations of $\mathrm{GL}_2(k)$. Note that the integers $n_\tau(\sigma)$ are not uniquely determined; however, all our constructions elsewhere in this paper will be (nonobviously!) independent of the choice of the $n_\tau(\sigma)$. We also write

$$\bar{\sigma}(\tau) = \sum_\sigma m_\sigma(\tau)\sigma;$$

since $\bar{\sigma}(\tau)$ is multiplicity-free, each $m_\sigma(\tau)$ is equal to 0 or 1. Then

$$\sigma = \sum_{\sigma'}\Bigl(\sum_\tau n_\tau(\sigma)m_{\sigma'}(\tau)\Bigr)\sigma',$$

and therefore

$$\sum_\tau n_\tau(\sigma)m_{\sigma'}(\tau) = \delta_{\sigma,\sigma'}. \tag{5.1.2}$$

For each non-Steinberg Serre weight σ, we set

$$\mathcal{C}_\sigma := \sum_\tau n_\tau(\sigma) Z(R^{\square}_{\bar{r},0,\tau}/\varpi),$$

where the sum ranges over the tame types τ, and the integers $n_\tau(\sigma)$ are as in (5.1.1). By definition this is a formal sum with (possibly negative) multiplicities of irreducible subschemes of $\operatorname{Spec} R^{\square}_{\bar{r}}/\varpi$; recall that we say that it is *effective* if all of the multiplicities are nonnegative.

Theorem 5.2. *Let σ be a non-Steinberg Serre weight. Then the cycle $\mathcal{C}_\sigma$ is effective, and is nonzero precisely when $\sigma \in W(\bar{r})$. It is independent of the choice of integers $n_\tau(\sigma)$ satisfying* (5.1.1). *For each tame type τ, we have*

$$Z(R^{\square}_{\bar{r},0,\tau}/\varpi) = \sum_{\sigma\in \mathrm{JH}(\bar{\sigma}(\tau))} \mathcal{C}_\sigma.$$

Proof. We will argue exactly as in the proof of [Emerton and Gee 2014, Theorem 5.5.2] (taking $n = 2$), and we freely use the notation and definitions of [loc. cit.]. Since $p > 2$, we have $p \nmid n$ and thus a suitable globalization $\bar{\rho}$ exists provided that [loc. cit., Conjecture A.3] holds for $\bar{r}$. The latter follows from the proof of Theorem A.2 of [Gee and Kisin 2014] (which shows that $\bar{r}$ has a potentially Barsotti–Tate lift) and Lemma 4.4.1 of [loc. cit.]. (which shows that any potentially Barsotti–Tate representation is potentially diagonalizable). These same results also show that the equivalent conditions of [Emerton and Gee 2014, Lemma 5.5.1] hold in the case that $\lambda_v = 0$ for all v, and in particular in the case that $\lambda_v = 0$ and τ_v is tame for all v, which is all that we will require.

By [Emerton and Gee 2014, Lemma 5.5.1(5)], we see that for each choice of tame types τ_v, we have

$$Z(\bar{R}_\infty/\varpi) = \sum_{\bigotimes_{v\mid p}\sigma_v} \prod_{v\mid p} m_{\sigma_v}(\tau_v) Z'_{\bigotimes_{v\mid p}\sigma_v}(\bar{\rho}). \tag{5.2.1}$$

Now, by definition we have

$$Z(\bar{R}_\infty/\varpi) = \prod_{v\mid p} Z(R^{\square}_{\bar{r},0,\tau_v}/\varpi) \times Z(\mathbb{F}[\![x_1,\dots,x_{q-[F^+:\mathbb{Q}]n(n-1)/2}, t_1,\dots,t_{n^2}]\!]). \tag{5.2.2}$$

Fix a non-Steinberg Serre weight $\sigma = \bigotimes_v \sigma_v$, and sum over all choices of types τ_v, weighted by $\prod_{v\mid p} n_{\tau_v}(\sigma_v)$. We obtain

$$\sum_\tau \prod_{v\mid p} n_{\tau_v}(\sigma_v) \prod_{v\mid p} Z(R^{\square}_{\bar{r},0,\tau_v}/\varpi) \times Z(\mathbb{F}[\![x_1,\dots,x_{q-[F^+:\mathbb{Q}]n(n-1)/2}, t_1,\dots,t_{n^2}]\!])$$
$$= \sum_\tau \prod_{v\mid p} n_{\tau_v}(\sigma_v) \sum_{\bigotimes_{v\mid p}\sigma'_v} \prod_{v\mid p} m_{\sigma'_v}(\tau_v) Z'_{\bigotimes_{v\mid p}\sigma'_v}(\bar{\rho}),$$

which by (5.1.2) simplifies to

$$\prod_{v\mid p} \mathcal{C}_{\sigma_v} \times Z(\mathbb{F}[\![x_1,\dots,x_{q-[F^+:\mathbb{Q}]n(n-1)/2}, t_1,\dots,t_{n^2}]\!]) = Z'_{\bigotimes_{v\mid p}\sigma_v}(\bar{\rho}). \tag{5.2.3}$$

Since $Z'_{\bigotimes_{v\mid p}\sigma_v}(\bar{\rho})$ is effective by definition (as it is defined as a positive multiple of the support cycle of a patched module), this shows that every $\prod_{v\mid p} \mathcal{C}_{\sigma_v}$ is effective. We conclude that either every $\mathcal{C}_\sigma$ is effective, or that every $-\mathcal{C}_\sigma$ is effective.

Substituting (5.2.3) and (5.2.2) into (5.2.1), we see that

$$\prod_{v\mid p} Z(R^{\square}_{\bar{r},0,\tau_v}/\varpi) \times Z(\mathbb{F}[\![x_1,\dots,x_{q-[F^+:\mathbb{Q}]n(n-1)/2}, t_1,\dots,t_{n^2}]\!])$$
$$= \prod_{v\mid p} \Bigl(\sum_{\sigma\in \mathrm{JH}(\bar{\sigma}(\tau))} \mathcal{C}_{\sigma_v} \Bigr) \times Z(\mathbb{F}[\![x_1,\dots,x_{q-[F^+:\mathbb{Q}]n(n-1)/2}, t_1,\dots,t_{n^2}]\!]), \tag{5.2.4}$$

and we deduce that either

$$Z(R^{\square}_{\bar{r},0,\tau}/\varpi) = \sum_\sigma m_\sigma(\tau)\mathcal{C}_\sigma$$

for all τ, or

$$Z(R^{\square}_{\bar{r},0,\tau}/\varpi) = -\sum_{\sigma} m_\sigma(\tau)\mathcal{C}_\sigma$$

for all τ.

Since each $Z(R^{\square}_{\bar{r},0,\tau}/\varpi)$ is effective, the second possibility holds if and only if every $-\mathcal{C}_\sigma$ is effective (since either all the $-\mathcal{C}_\sigma$ are effective, or all the $\mathcal{C}_\sigma$ are effective). It remains to show that this possibility leads to a contradiction. Now, if $Z(R^{\square}_{\bar{r},0,\tau}/\varpi) = -\sum_\sigma m_\sigma(\tau)\mathcal{C}_\sigma$ for all τ, then substituting into the definition $\mathcal{C}_\sigma = \sum_\tau n_\tau(\sigma)Z(R^{\square}_{\bar{r},0,\tau}/\varpi)$, we obtain

$$\mathcal{C}_\sigma = \sum_{\sigma'}\biggl(\sum_{\tau} n_\tau(\sigma)m_{\sigma'}(\tau)\biggr)(-\mathcal{C}_{\sigma'}),$$

and applying (5.1.2), we obtain $\mathcal{C}_\sigma = -\mathcal{C}_\sigma$, so that $\mathcal{C}_\sigma = 0$ for all σ. Thus all the $\mathcal{C}_\sigma$ are effective, as claimed.

Since $Z'_{\bigotimes_{v\mid p}\sigma_v}(\bar{\rho})$ by definition depends only on (the global choices in the Taylor–Wiles method, and) $\bigotimes_{v\mid p}\sigma_v$, and *not* on the particular choice of the $n_\tau(\sigma)$, it follows from (5.2.3) that $\mathcal{C}_\sigma$ is also independent of this choice.

Finally, note that by definition $Z'_{\bigotimes_{v\mid p}\sigma_v}(\bar{\rho})$ is nonzero precisely when σ_v is in the set $W^{\mathrm{BT}}(\bar{r})$ defined in [Gee and Kisin 2014, Section 3]; but by the main result of [Gee et al. 2015], this is precisely the set $W(\bar{r})$. □

Remark 5.3. As we do not use wildly ramified types elsewhere in the paper, we have restricted the statement of Theorem 5.2 to the case of tame types; but the statement admits a natural extension to the case of wildly ramified inertial types (with some components now occurring with multiplicity greater than one), and the proof goes through unchanged in this more general setting.

6. The geometric Breuil–Mézard conjecture for the stacks $\mathcal{Z}^{\mathrm{dd},1}$

We now prove our main results on the irreducible components of $\mathcal{Z}^{\mathrm{dd},1}$. We do this by a slightly indirect method, defining certain formal sums of these irreducible components which we then compute via the geometric Breuil–Mézard conjecture, and in particular the results of Section 5.

By Theorem 3.2, $\mathcal{Z}^{\mathrm{dd},1}$ is reduced and equidimensional, and each $\mathcal{Z}^{\tau,1}$ is a union of some of its irreducible components. Let $K(\mathcal{Z}^{\mathrm{dd},1})$ be the free abelian group generated by the irreducible components of $\mathcal{Z}^{\mathrm{dd},1}$. We say that an element of $K(\mathcal{Z}^{\mathrm{dd},1})$ is *effective* if the multiplicity of each irreducible component is nonnegative. We say that an element of $K(\mathcal{Z}^{\mathrm{dd},1})$ is *reduced and effective* if the multiplicity of each irreducible component is 0 or 1; we will sometimes abuse language by identifying a reduced and effective cycle with the reduced union of the irreducible components appearing in it.

Let x be a finite type point of $\mathcal{Z}^{\mathrm{dd},1}$, corresponding to a representation $\bar{r} : G_K \to \mathrm{GL}_2(\mathbb{F}')$, and recall that $R_x^{\mathrm{dd},1}$ is a versal ring to $\mathcal{Z}^{\mathrm{dd},1}$ having each $R_x^{\tau,1}$ as a quotient. Since $\mathcal{Z}^{\tau,1}$ is a union of irreducible components of $\mathcal{Z}^{\mathrm{dd},1}$, Spec $R_x^{\tau,1}$ is a union of irreducible components of Spec $R_x^{\mathrm{dd},1}$.

Let $K(R_x^{\mathrm{dd},1})$ be the free abelian group generated by the irreducible components of Spec $R_x^{\mathrm{dd},1}$. By [Stacks 2005–, Tag 0DRB, Tag 0DRD], there is a natural multiplicity-preserving surjection from the set of irreducible components of Spec $R_x^{\mathrm{dd},1}$ to the set of irreducible components of $\mathcal{Z}^{\mathrm{dd},1}$ which contain x. Using this surjection, we can define a group homomorphism

$$K(\mathcal{Z}^{\mathrm{dd},1}) \to K(R_x^{\mathrm{dd},1})$$

in the following way: we send any irreducible component $\mathcal{Z}$ of $\mathcal{Z}^{\mathrm{dd},1}$ which contains x to the formal sum of the irreducible components of Spec $R_x^{\mathrm{dd},1}$ in the preimage of $\mathcal{Z}$ under this surjection, and we send every other irreducible component to 0.

Lemma 6.1. *An element $\overline{\mathcal{T}}$ of $K(\mathcal{Z}^{\mathrm{dd},1})$ is effective if and only if for every finite type point x of $\mathcal{Z}^{\mathrm{dd},1}$, the image of $\overline{\mathcal{T}}$ in $K(R_x^{\mathrm{dd},1})$ is effective. We have $\overline{\mathcal{T}} = 0$ if and only if its image is 0 in every $K(R_x^{\mathrm{dd},1})$.*

Proof. The "only if" direction is trivial, so we need only consider the "if" implication. Write $\overline{\mathcal{T}} = \sum_{\overline{\mathcal{Z}}} a_{\overline{\mathcal{Z}}} \overline{\mathcal{Z}}$, where the sum runs over the irreducible components $\overline{\mathcal{Z}}$ of $\mathcal{Z}^{\mathrm{dd},1}$, and the $a_{\overline{\mathcal{Z}}}$ are integers.

Suppose first that the image of $\overline{\mathcal{T}}$ in $K(R_x^{\mathrm{dd},1})$ is effective for every x; we then have to show that each $a_{\overline{\mathcal{Z}}}$ is nonnegative. To see this, fix an irreducible component $\overline{\mathcal{Z}}$, and choose x to be a finite type point of $\mathcal{Z}^{\mathrm{dd},1}$ which is contained in $\overline{\mathcal{Z}}$ and in no other irreducible component of $\mathcal{Z}^{\mathrm{dd},1}$. Then the image of $\overline{\mathcal{T}}$ in $K(R_x^{\mathrm{dd},1})$ is equal to $a_{\overline{\mathcal{Z}}}$ times a nonempty sum of irreducible components of Spec $R_x^{\mathrm{dd},1}$. By hypothesis, this must be effective, which implies that $a_{\overline{\mathcal{Z}}}$ is nonnegative, as required.

Finally, if the image of $\overline{\mathcal{T}}$ in $K(R_x^{\mathrm{dd},1})$ is 0, then $a_{\overline{\mathcal{Z}}} = 0$; so if this holds for all x, then $\overline{\mathcal{T}} = 0$. □

For each tame type τ, we let $\mathcal{Z}(\tau)$ denote the formal sum of the irreducible components of $\mathcal{Z}^{\tau,1}$, considered as an element of $K(\mathcal{Z}^{\mathrm{dd},1})$. By Lemma 5.1, for each non-Steinberg Serre weight σ of $\mathrm{GL}_2(k)$ there are integers $n_\tau(\sigma)$ such that $\sigma = \sum_\tau n_\tau(\sigma)\bar{\sigma}(\tau)$ in the Grothendieck group of mod p representations of $\mathrm{GL}_2(k)$, where the τ run over the tame types. We set

$$\mathcal{Z}(\sigma) := \sum_\tau n_\tau(\sigma)\mathcal{Z}(\tau) \in K(\mathcal{Z}^{\mathrm{dd},1}).$$

The integers $n_\tau(\sigma)$ are not necessarily unique, but it follows from the following result that $\mathcal{Z}(\sigma)$ is independent of the choice of $n_\tau(\sigma)$, and is reduced and effective.

Theorem 6.2. (1) *Each $\mathcal{Z}(\sigma)$ is an irreducible component of $\mathcal{Z}^{\mathrm{dd},1}$.*

(2) *The finite type points of $\mathcal{Z}(\sigma)$ are precisely the representations $\bar{r} : G_K \to \mathrm{GL}_2(\mathbb{F}')$ having σ as a Serre weight.*

(3) *For each tame type τ, we have $\mathcal{Z}(\tau) = \sum_{\sigma \in \mathrm{JH}(\bar{\sigma}(\tau))} \mathcal{Z}(\sigma)$.*

(4) *Every irreducible component of $\mathcal{Z}^{\mathrm{dd},1}$ is of the form $\mathcal{Z}(\sigma)$ for some unique non-Steinberg Serre weight σ.*

(5) *For each tame type τ, and each $J \in \mathcal{P}_\tau$, we have $\mathcal{Z}(\bar{\sigma}(\tau)_J) = \overline{\mathcal{Z}}(J)$.*

Proof. Let x be a finite type point of $\mathcal{Z}^{\mathrm{dd},1}$ corresponding to $\bar{r} : G_K \to \mathrm{GL}_2(\mathbb{F}')$, and write $\mathcal{Z}(\sigma)_x$, $\mathcal{Z}(\tau)_x$ for the images in $K(R_x^{\mathrm{dd},1})$ of $\mathcal{Z}(\sigma)$ and $\mathcal{Z}(\tau)$ respectively. Each Spec $R_x^{\tau,1}$ is a closed subscheme of

Spec $R^{\square}$, the universal framed deformation $\mathcal{O}_{E'}$-algebra for $\bar{r}$, so we may regard the $\mathcal{Z}(\tau)_x$ as formal sums (with multiplicities) of irreducible subschemes of Spec $R^{\square}/\pi$.

By definition, $\mathcal{Z}(\tau)_x$ is just the underlying cycle of Spec $R_x^{\tau,1}$. By Theorem 4.6, this is equal to the underlying cycle of Spec $R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi$. Consequently, $\mathcal{Z}(\sigma)_x$ is the cycle denoted by $\mathcal{C}_\sigma$ in Section 5. It follows from Theorem 5.2 that:

- $\mathcal{Z}(\sigma)_x$ is effective, and is nonzero precisely when σ is a Serre weight for $\bar{r}$.
- For each tame type τ, we have $\mathcal{Z}(\tau)_x = \sum_{\sigma \in \mathrm{JH}(\bar{\sigma}(\tau))} \mathcal{Z}(\sigma)_x$.

Applying Lemma 6.1, we see that each $\mathcal{Z}(\sigma)$ is effective, and that (3) holds. Since $\mathcal{Z}^{\tau,1}$ is reduced, $\mathcal{Z}(\tau)$ is reduced and effective, so it follows from (3) that each $\mathcal{Z}(\sigma)$ is reduced and effective. Since x is a finite type point of $\mathcal{Z}(\sigma)$ if and only if $\mathcal{Z}(\sigma)_x \neq 0$, we have also proved (2).

Since every irreducible component of $\mathcal{Z}^{\mathrm{dd},1}$ is an irreducible component of some $\mathcal{Z}^{\tau,1}$, in order to prove (1) and (4) it suffices to show that for each τ, every irreducible component of $\mathcal{Z}^{\tau,1}$ is of the form $\mathcal{Z}(\bar{\sigma}(\tau)_J)$ for some J, and that each $\mathcal{Z}(\bar{\sigma}(\tau)_J)$ is irreducible. Since τ is fixed for the rest of the argument, let us simplify notation by writing $\bar{\sigma}_J$ for $\bar{\sigma}(\tau)_J$. Now, by Theorem 3.5(2), we know that $\mathcal{Z}^{\tau,1}$ has exactly $\#\mathcal{P}_\tau$ irreducible components, namely the $\bar{\mathcal{Z}}(J')$ for $J' \in \mathcal{P}_\tau$. On the other hand, the $\mathcal{Z}(\bar{\sigma}_J)$ are reduced and effective, and since by the results of [Gee et al. 2015] there exist representations admitting $\bar{\sigma}_J$ as their unique non-Steinberg Serre weight,[5] it follows from (2) that for each J, there must be a $J' \in \mathcal{P}_\tau$ such that $\bar{\mathcal{Z}}(J')$ contributes to $\mathcal{Z}(\bar{\sigma}_J)$, but not to any $\mathcal{Z}(\bar{\sigma}_{J''})$ for $J'' \neq J$.

Since $\mathcal{Z}(\tau)$ is reduced and effective, and the sum in (3) is over $\#\mathcal{P}_\tau$ weights σ, it follows that we in fact have $\mathcal{Z}(\bar{\sigma}_J) = \bar{\mathcal{Z}}(J')$. This proves (1) and (4), and to prove (5), it only remains to show that $J' = J$. To see this, note that by (2), $\mathcal{Z}(\bar{\sigma}_J) = \bar{\mathcal{Z}}(J')$ has a dense open substack whose finite type points have $\bar{\sigma}_J$ as their unique non-Steinberg Serre weight (namely the complement of the union of the $\mathcal{Z}(\sigma')$ for all $\sigma' \neq \bar{\sigma}_J$). By Theorem 3.5(4), it also has a dense open substack whose finite type points have $\bar{\sigma}_{J'}$ as a Serre weight. Considering any finite type point in the intersection of these dense open substacks, we see that $\bar{\sigma}_J = \bar{\sigma}_{J'}$, so that $J = J'$, as required. □

7. The geometric Breuil–Mézard conjecture for the stacks $\mathcal{X}_{2,\mathrm{red}}$

We now explain how to transfer our results from the stacks $\mathcal{Z}^{\mathrm{dd},1}$ to the stacks $\mathcal{X}_{2,\mathrm{red}}$ of [Emerton and Gee 2023]. The book [loc. cit.] establishes an equivalence between the classical "numerical" Breuil–Mézard conjecture and the geometric Breuil–Mézard conjecture for the stacks $\mathcal{X}_{2,\mathrm{red}}$. (Indeed the implication from the former to the latter only requires the classical Breuil–Mézard conjecture at a single sufficiently generic point of each component of $\mathcal{X}_{2,\mathrm{red}}$.)

[5]More precisely, it follows from the results of [Gee et al. 2015] that there exist $[K : \mathbb{Q}]$-dimensional extension spaces V of reducible representations admitting $\bar{\sigma}_J$ as a Serre weight, such that the representations in V admitting at least one other non-Steinberg weight lie in a finite union of proper subspaces; moreover the number of subspaces in this finite union is independent of $\mathbb{F}'$.

The Breuil–Mézard conjecture for two-dimensional potentially Barsotti–Tate representations is established in [Gee and Kisin 2014], and this is extended to two-dimensional potentially semistable representations of Hodge type 0 in [Emerton and Gee 2023, Theorem 8.6.1]. The arguments of [loc. cit., Section 8.3] translate this into the following theorem. Here $\mathcal{X}_2^{\tau,\mathrm{BT},\mathrm{ss}}$ denotes the stack of two-dimensional potentially semistable representations of Hodge type 0 constructed in [loc. cit.], while $\sigma^{\mathrm{ss}}(\tau)$ is as in [loc. cit., Theorem 8.2.1].

Theorem 7.1 [Emerton and Gee 2023]. *There exist effective cycles Z^σ (elements of the free group on the irreducible components of $\mathcal{X}_{2,\mathrm{red}}$, with nonnegative coefficients) such that for all inertial types τ,*

- *the cycle of the special fiber of $\mathcal{X}_2^{\tau,\mathrm{BT}}$ is equal to $\sum_\sigma m_\sigma(\tau)\cdot Z^\sigma$, while*
- *the cycle of the special fiber of $\mathcal{X}_2^{\tau,\mathrm{BT},\mathrm{ss}}$ is equal to $\sum_\sigma m_\sigma^{\mathrm{ss}}(\tau)\cdot Z^\sigma$.*

Here $\bar\sigma(\tau)=\sum_\sigma m_\sigma(\tau)\cdot\sigma$ and $\bar\sigma^{\mathrm{ss}}(\tau)=\sum_\sigma m_\sigma^{\mathrm{ss}}(\tau)\cdot\sigma$ in the Grothendieck group of $\mathrm{GL}_2(k)$.

Corollary 7.2 [Emerton and Gee 2023]. *Let $\bar r: G_K\to\mathrm{GL}_2(\mathbb{F}')$ be a continuous Galois representation, corresponding to a finite type point of $\mathcal{X}_{2,\mathrm{red}}$. For each Serre weight σ we have $\sigma\in W(\bar r)$ if and only if $\bar r$ lies in the support of Z^σ.*

Proof. This follows by the argument in [Emerton and Gee 2023, Section 8.4]: the Breuil–Mézard multiplicity $\mu_\sigma(\bar r)$ is nonzero if and only if Z^σ is supported at $\bar r$. More precisely, invoking Theorem 8.6.1 of [loc. cit.] in place of Conjecture 8.2.2 of [loc. cit.], the next-to-last paragraph of [loc. cit., Section 8.4] shows that $\bar r$ lies in the support of Z^σ if and only if σ is an element of the weight set $W_{\mathrm{BT}}(\bar r)=\{\sigma:\mu_\sigma(\bar r)>0\}$. But $W_{\mathrm{BT}}(\bar r)=W(\bar r)$ by the main results of [Gee et al. 2015]. ☐

Remark 7.3. The cycles Z^σ of Theorem 7.1 are constructed in [Emerton and Gee 2023, Section 8.3] as follows. For each Serre weight σ', the smooth points of $\mathcal{X}_{2,\mathrm{red}}^{\sigma'}$ that furthermore do not lie on any other component of $\mathcal{X}_{2,\mathrm{red}}$ are dense. Choose such a point $\bar r_{\sigma'}$ and let $\{\mu_\sigma(\bar r_{\sigma'})\}$ be the multiplicities in the Breuil–Mézard conjecture for $\bar r_{\sigma'}$. Then $Z^\sigma:=\sum_{\sigma'}\mu_\sigma(\bar r_{\sigma'})\cdot\mathcal{X}_{2,\mathrm{red}}^{\sigma'}$.

It remains to compute the cycles Z^σ. We begin with the following observation, which could be proved with modest effort by calculating dimensions of families of extensions and using the results of [Gee et al. 2015], but is also easily deduced from the results of Section 6.

Lemma 7.4. *Let σ,σ' be Serre weights and suppose that σ' is non-Steinberg. Then $\mathcal{X}_{2,\mathrm{red}}^{\sigma}$ contains at least one finite type point corresponding to a representation $\bar r$ with $\sigma'\notin W(\bar r)$.*

Proof. Suppose first that σ is non-Steinberg. By Theorem 6.2 the component $\mathcal{Z}(\sigma)$ of $\mathcal{Z}^{\mathrm{dd},1}$ has a dense open set $U(\sigma)$ whose finite type points $\bar r$ have no non-Steinberg Serre weights other than σ: take $\mathcal{Z}(\sigma)\setminus\bigcup_{\sigma'\neq\sigma}\mathcal{Z}(\sigma')$. The finite type points of $\mathcal{Z}(\sigma)$ described in Remark 3.6 are also dense; therefore at least one of them (indeed a dense set of them) lies in $U(\sigma)$. Let $\bar r$ be such a representation. The finite type points of $\mathcal{Z}(\sigma)$ described in Remark 3.6 are precisely the family of niveau 1 representations in the description of $\mathcal{X}_{2,\mathrm{red}}^{\sigma}$ of Theorem 1.1 (and whose construction can be found in the proof of [Emerton and Gee 2023, Theorem 5.5.12] and its correction in the errata to [Emerton and Gee 2023]). So $\bar{r}$ lies on $\mathcal{X}^{\sigma}_{2,\mathrm{red}}$ as well, and by construction the only non-Steinberg weight in $W(\bar{r})$ is σ. This completes the non-Steinberg case.

If instead σ is Steinberg, then by construction, and by the second bullet point in [Emerton and Gee 2023, Definition 5.5.1], $\mathcal{X}^{\sigma}_{2,\mathrm{red}}$ contains representations $\bar{r}$ that are très ramifiée. But très ramifiée representations have no non-Steinberg Serre weights by [Caraiani et al. 2024, Lemma A.5]. □

Remark 7.5. Once we have proved Theorem 7.6 below, "at least one finite type point" in the statement of Lemma 7.4 can be promoted to "a dense set of finite type points" by taking $\mathcal{X}^{\sigma}_{2,\mathrm{red}} \setminus \bigcup_{\sigma' \neq \sigma} \mathcal{X}^{\sigma'}_{2,\mathrm{red}}$.

We now reach our main theorem.

Theorem 7.6. *Suppose $p > 2$. We have*

- $Z^{\sigma} = \mathcal{X}^{\sigma}_{2,\mathrm{red}}$, *if the weight σ is not Steinberg, while*
- $Z^{\chi\otimes\mathrm{St}} = \mathcal{X}^{\chi}_{2,\mathrm{red}} + \mathcal{X}^{\chi\otimes\mathrm{St}}_{2,\mathrm{red}}$ *if the weight $\sigma \cong \chi \otimes \mathrm{St}$ is Steinberg.*

In particular $\sigma \in W(\bar{r})$ if and only if $\bar{r}$ lies on $\mathcal{X}^{\sigma}_{2,\mathrm{red}}$ if σ is not Steinberg, or on $\mathcal{X}^{\chi}_{2,\mathrm{red}} \cup \mathcal{X}^{\chi\otimes\mathrm{St}}_{2,\mathrm{red}}$ if $\sigma \cong \chi \otimes \mathrm{St}$ is Steinberg.

Essentially the same statement appears at [Emerton and Gee 2023, Theorem 8.6.2], and the argument given there invokes an earlier version of this paper.[6] The proof we give below is independent of the proof of [loc. cit., Theorem 8.6.2], but has the same major beats, and is rearranged to delay until as late as possible making any references to the earlier parts of this paper. Indeed the proof now only invokes the generic reducedness of Theorem 4.6 (but certainly invokes it in a crucial way); we hope that this clarifies the various dependencies involved.

However, we also take the opportunity to repair a small gap in the argument given at [Emerton and Gee 2023, Theorem 8.6.2]. It is claimed there that [CEGS19, Theorem 5.2.2(2)] (i.e., our Theorem 6.2(2)) implies that $\mathcal{X}^{\sigma}_{2,\mathrm{red}}$ has a dense set of finite type points corresponding to representations $\bar{r}$ whose only non-Steinberg Serre weight is σ. Although the conclusion is certainly true, the deduction is incorrect, or at least seems to presume that $\mathcal{X}^{\sigma}_{2,\mathrm{red}}$ can be identified with our $\mathcal{Z}(\sigma)$. We replace this claim with an argument using Lemma 7.4, which does follow from Theorem 6.2 (or from the results of [Gee et al. 2015], as previously noted). The claim about points of $\mathcal{X}^{\sigma}_{2,\mathrm{red}}$ will follow once Theorem 7.6 is proved, as explained in Remark 7.5.

Proof of Theorem 7.6. Consider first the non-Steinberg case. The finite type points in the support of the effective cycle Z^{σ} are precisely the representations having σ as a Serre weight. By Lemma 7.4 each component $\mathcal{X}^{\sigma'}_{2,\mathrm{red}}$ with $\sigma' \neq \sigma$ has a finite type point for which σ is not a Serre weight; therefore $\mathcal{X}^{\sigma'}_{2,\mathrm{red}}$ cannot occur in the cycle Z^{σ}. It follows that $Z^{\sigma} = \mu_{\sigma}(\bar{r}_{\sigma}) \cdot \mathcal{X}^{\sigma}_{2,\mathrm{red}}$ with $\bar{r}_{\sigma}$ as in Remark 7.3, and that $\mu_{\sigma}(\bar{r}_{\sigma'}) = 0$ for all $\sigma' \neq \sigma$. Note that we can already deduce that $\sigma \in W(\bar{r})$ if and only if $\bar{r}$ lies on $\mathcal{X}^{\sigma}_{2,\mathrm{red}}$.

Now consider the Steinberg case; by twisting it will suffice to consider the weight $\sigma = \mathrm{St}$. The type $\sigma^{\mathrm{ss}}(\mathrm{triv})$ is the Steinberg type, and so by Theorem 7.1 the cycle of the special fiber of $\mathcal{X}_2^{\mathrm{triv},\mathrm{BT},\mathrm{ss}}$

[6]As mentioned in the Introduction, the reference [CEGS19, Theorem 5.2.2] in [Emerton and Gee 2023] is Theorem 6.2 of this paper, while the reference [CEGS19, Lemma B.5] in [Emerton and Gee 2023] is [Caraiani et al. 2024, Lemma A.5].

is equal to Z^{St}. In particular the finite type points of Z^{St} are precisely the representations $\bar{r}$ having a semistable lift of Hodge type 0. Such a lift is either crystalline, in which case the trivial Serre weight is a Serre weight for $\bar{r}$, and by the previous paragraph $\bar{r}$ is a finite type point of $\mathcal{X}_{2,\mathrm{red}}^{\mathrm{triv}}$; or else the lift is semistable noncrystalline, in which case $\bar{r}$ is an unramified twist of an extension of the inverse of the cyclotomic character by the trivial character. Such an extension is either peu ramifiée, in which case again $\mathrm{triv} \in W(\bar{r})$, and $\bar{r}$ lies on $\mathcal{X}_{2,\mathrm{red}}^{\mathrm{triv}}$ by the first paragraph of the proof; or else it is tres̀ ramifiée, and is a finite type point of $\mathcal{X}_{2,\mathrm{red}}^{\mathrm{St}}$ (as a member of the family of niveau 1 representations defining $\mathcal{X}_{2,\mathrm{red}}^{\mathrm{St}}$). Since all the finite type points of the support of Z^{St} are contained in $\mathcal{X}_{2,\mathrm{red}}^{\mathrm{triv}} \cup \mathcal{X}_{2,\mathrm{red}}^{\mathrm{St}}$ it follows that $Z^{\mathrm{St}} = \mu_{\mathrm{St}}(\bar{r}_{\mathrm{triv}})\mathcal{X}_{2,\mathrm{red}}^{\mathrm{triv}} + \mu_{\mathrm{St}}(\bar{r}_{\mathrm{St}})\mathcal{X}_{2,\mathrm{red}}^{\mathrm{St}}$, and that $\mu_{\mathrm{St}}(\bar{r}_{\sigma'}) = 0$ for all $\sigma' \neq \mathrm{triv}, \mathrm{St}$.

In the remainder of the argument we can and do assume that $\bar{r}_{\mathrm{triv}}$ is an extension of an unramified twist of the inverse cyclotomic character by a *different* unramified character: these are dense in $\mathcal{X}_{2,\mathrm{red}}^{\mathrm{triv}}$ by the dominance of the eigenvalue morphism of [Emerton and Gee 2023, Theorem 5.5.12(2)].

It remains to determine the multiplicities $\mu_\sigma(\bar{r}_\sigma)$, $\mu_{\mathrm{St}}(\bar{r}_{\mathrm{triv}})$, and $\mu_{\mathrm{St}}(\bar{r}_{\mathrm{St}})$. Each of these is positive because $\bar{r}_\sigma$ does have σ as a Serre weight, while $\bar{r}_{\mathrm{triv}}$ has St as a Serre weight because each extension as in the previous paragraph has a crystalline lift with labeled Hodge–Tate weights $(p, 0)$ at one embedding lifting each embedding $k \hookrightarrow \overline{\mathbb{F}}_p$, and labeled Hodge–Tate weights $(1, 0)$ at all other embeddings.

Suppose that σ is non-Steinberg. Choose any tame type τ such that $\bar{\sigma}(\tau)$ has σ as a Jordan–Hölder factor. The ring $R_{\bar{r}_\sigma}^{\tau,\mathrm{BT}}$ is versal to $\mathcal{X}_2^{\tau,\mathrm{BT}}$ at $\bar{r}_\sigma$, and since $\bar{r}_\sigma$ is a smooth point of $\mathcal{X}_{2,\mathrm{red}}$ the underlying reduced of $\operatorname{Spec} R_{\bar{r}_\sigma}^{\tau,\mathrm{BT}}/\varpi$ is smooth. But $\operatorname{Spec} R_{\bar{r}_\sigma}^{\tau,\mathrm{BT}}/\varpi$ is generically reduced by Theorem 4.6. We deduce that the Hilbert–Samuel multiplicity of $R_{\bar{r}_\sigma}^{\tau,\mathrm{BT}}/\varpi$ is 1, and therefore $\mu_\sigma(\bar{r}_\sigma) \leq 1$. Since $\mu_\sigma(\bar{r}_\sigma)$ is positive it must be equal to 1. Alternately, it follows from Theorem 6.2 and the isomorphism $\mathcal{Z}^\tau \cong \mathcal{X}_2^{\tau,\mathrm{BT}}$ of [Bellovin et al. 2024, Theorem 4.5] that the cycle of the special fiber of $\mathcal{X}_2^{\tau,\mathrm{BT}}$ is reduced and effective; since by Theorem 7.1 this cycle contains $\mathcal{Z}^\sigma$ as a summand, we see again that $\mu_\sigma(\bar{r}_\sigma) \leq 1$.

Our chosen $\bar{r}_{\mathrm{triv}}$ does not have any semistable noncrystalline lifts, and the semistable Hodge type 0 deformation ring of $\bar{r}_{\mathrm{triv}}$ is simply a crystalline deformation ring, indeed one of the flat deformation rings studied by Kisin [2009]. The argument in the previous paragraph showed that the Hilbert–Samuel multiplicity of $R_{\bar{r}_{\mathrm{cris}}}^{\mathrm{triv},\mathrm{BT}}/\varpi$ is 1. It follows that $\mu_{\mathrm{St}}(\bar{r}_{\mathrm{triv}}) \leq 1$, and since it is positive it must be precisely 1. On the other hand the semistable Hodge type 0 deformation ring of the très ramifiée representation $\bar{r}_{\mathrm{St}}$ is an ordinary deformation ring, hence formally smooth, and we obtain $\mu_{\mathrm{St}}(\bar{r}_{\mathrm{St}}) = 1$. ☐

Remark 7.7. The finite type points of $X_{2,\mathrm{red}}^{\mathrm{St}}$ are precisely those $\bar{r}$ having a semistable noncrystalline lift, i.e., the unramified twists of an extension of the inverse of the cyclotomic character by the trivial character; for the details see [Emerton and Gee 2023, Lemma 8.6.4].

Appendix: A lemma on formal algebraic stacks

We suppose given a commutative diagram of morphisms of formal algebraic stacks:

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ & \searrow & \downarrow \\ & & \operatorname{Spf} \mathcal{O} \end{array}$$

We suppose that each of $\mathcal{X}$ and $\mathcal{Y}$ is quasicompact and quasiseparated, and that the horizontal arrow is scheme-theoretically dominant, in the sense of [Emerton 2019, Definition 6.13]. We furthermore suppose that the morphism $\mathcal{X} \to \operatorname{Spf} \mathcal{O}$ realizes $\mathcal{X}$ as a finite type ϖ-adic formal algebraic stack.

Concretely, if we write $\mathcal{X}^a := \mathcal{X} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$, then each $\mathcal{X}^a$ is an algebraic stack, locally of finite type over $\operatorname{Spec} \mathcal{O}/\varpi^a$, and there is an isomorphism $\varinjlim_a \mathcal{X}^a \xrightarrow{\sim} \mathcal{X}$. Furthermore, the assumption that the horizontal arrow is scheme-theoretically dominant means that we may find an isomorphism $\mathcal{Y} \cong \varinjlim_a \mathcal{Y}^a$, with each $\mathcal{Y}^a$ being a quasicompact and quasiseparated algebraic stack, and with the transition morphisms being thickenings, such that the morphism $\mathcal{X} \to \mathcal{Y}$ is induced by a compatible family of morphisms $\mathcal{X}^a \to \mathcal{Y}^a$, each of which is scheme-theoretically dominant. (The $\mathcal{Y}^a$ are uniquely determined by the requirement that for all $b \geq a$ large enough so that the morphism $\mathcal{X}^a \to \mathcal{Y}$ factors through $\mathcal{Y} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^b$, $\mathcal{Y}^a$ is the scheme-theoretic image of the morphism $\mathcal{X}^a \to \mathcal{Y} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^b$. In particular, $\mathcal{Y}^a$ is a closed substack of $\mathcal{Y} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$.)

It is often the case, in the preceding situation, that $\mathcal{Y}$ is also a ϖ-adic formal algebraic stack. For example, we have the following result. (Note that the usual graph argument shows that the morphism $\mathcal{X} \to \mathcal{Y}$ is necessarily algebraic, i.e., representable by algebraic stacks, in the sense of [Stacks 2005–, Tag 06CF] and [Emerton 2019, Definition 3.1]. Thus it makes sense to speak of it being proper, following [loc. cit., Definition 3.11].)

Proposition A.1. *Suppose that the morphism $\mathcal{X} \to \mathcal{Y}$ is proper, and that $\mathcal{Y}$ is locally Ind-finite type over* $\operatorname{Spec} \mathcal{O}$ (*in the sense of* [Emerton 2019, Remark 8.30]). *Then $\mathcal{Y}$ is a ϖ-adic formal algebraic stack.*

Proof. This is an application of [loc. cit., Proposition 10.5]. □

A key point is that, because the formation of scheme-theoretic images is not generally compatible with nonflat base-change, the closed immersion

$$\mathcal{Y}^a \hookrightarrow \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}/\varpi^a \tag{A.2}$$

is typically *not* an isomorphism, even if $\mathcal{Y}$ is a ϖ-adic formal algebraic stack. Our goal in the remainder of this discussion is to give a criterion (involving the morphism $\mathcal{X} \to \mathcal{Y}$) on an open substack $\mathcal{U} \hookrightarrow \mathcal{Y}$ which guarantees that the closed immersion $\mathcal{U} \times_{\mathcal{Y}} \mathcal{Y}^a \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$ induced by (A.2) *is* an isomorphism.

We begin by establishing a simple lemma. For any $a \geq 1$, we have the 2-commutative diagram:

$$\begin{array}{ccc} \mathcal{X}^a & \longrightarrow & \mathcal{Y}^a \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array} \tag{A.3}$$

Similarly, if $b \geq a \geq 1$, then we have the 2-commutative diagram:

$$\begin{array}{ccc} \mathcal{X}^a & \longrightarrow & \mathcal{Y}^a \\ \downarrow & & \downarrow \\ \mathcal{X}^b & \longrightarrow & \mathcal{Y}^b \end{array} \tag{A.4}$$

Lemma A.5. *Each of the diagrams* (A.3) *and* (A.4) *is* 2-*Cartesian.*

Proof. We may embed the diagram (A.3) in the larger 2-commutative diagram:

$$\begin{array}{ccccc} \mathcal{X}^a & \longrightarrow & \mathcal{Y}^a & \xrightarrow{\text{(A.2)}} & \mathcal{Y}\otimes_{\mathcal{O}}\mathcal{O}/\varpi^a \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} & = & \mathcal{Y} \end{array}$$

Since the outer rectangle is manifestly 2-Cartesian, and since (A.2) is a closed immersion (and thus a monomorphism), we conclude that (A.3) is indeed 2-Cartesian.

A similar argument shows that (A.4) is 2-Cartesian. □

We next note that, since each of the closed immersions $\mathcal{Y}^a \hookrightarrow \mathcal{Y}$ is a thickening, giving an open substack $\mathcal{U} \hookrightarrow \mathcal{Y}$ is equivalent to giving an open substack $\mathcal{U}^a \hookrightarrow \mathcal{Y}^a$ for some, or equivalently, every, choice of $a \geq 1$; the two pieces of data are related by the formulas $\mathcal{U}^a := \mathcal{U} \times_{\mathcal{Y}} \mathcal{Y}^a$ and $\varinjlim_a \mathcal{U}^a \xrightarrow{\sim} \mathcal{U}$.

Lemma A.6. *Suppose that $\mathcal{X} \to \mathcal{Y}$ is proper. If $\mathcal{U}$ is an open substack of $\mathcal{Y}$, then the following conditions are equivalent*:

(1) *The morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{U} \to \mathcal{U}$ is a monomorphism.*

(2) *The morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{U} \to \mathcal{U}$ is an isomorphism.*

(3) *For every $a \geq 1$, the morphism $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is a monomorphism.*

(4) *For every $a \geq 1$, the morphism $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is an isomorphism.*

(5) *For some $a \geq 1$, the morphism $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is a monomorphism.*

(6) *For some $a \geq 1$, the morphism $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is an isomorphism.*

Furthermore, if these equivalent conditions hold, then the closed immersion $\mathcal{U}^a \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$ is an isomorphism, for each $a \geq 1$.

Proof. The key point is that Lemma A.5 implies that the diagram

$$\begin{array}{ccc} \mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a & \longrightarrow & \mathcal{U}^a \\ \downarrow & & \downarrow \\ \mathcal{X} \times_{\mathcal{Y}} \mathcal{U} & \longrightarrow & \mathcal{U} \end{array}$$

is 2-Cartesian, for any $a \geq 1$, and similarly, that if $b \geq a \geq 1$, then the diagram

$$\begin{array}{ccc} \mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a & \longrightarrow & \mathcal{U}^a \\ \downarrow & & \downarrow \\ \mathcal{X}^b \times_{\mathcal{Y}^b} \mathcal{U}^b & \longrightarrow & \mathcal{U}^b \end{array}$$

is 2-Cartesian. Since the vertical arrows of this latter diagram are finite order thickenings, we find (by applying the analogue of [Stacks 2005–, Tag 09ZZ] for algebraic stacks, whose straightforward deduction from that result we leave to the reader) that the top horizontal arrow is a monomorphism if and only if the bottom horizontal arrow is. This shows the equivalence of (3) and (5). Since the morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{U} \to \mathcal{U}$ is obtained as the inductive limit of the various morphisms $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$, we find that (3) implies (1) (by applying e.g., [Emerton 2019, Lemma 4.11(1)], which shows that the inductive limit of monomorphisms is a monomorphism), and also that (4) implies (2) (the inductive limit of isomorphisms being again an isomorphism).

Conversely, if (1) holds, then the base-changed morphism

$$\mathcal{X} \times_{\mathcal{Y}} (\mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a) \to \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$$

is a monomorphism. The source of this morphism admits an alternative description as $\mathcal{X}^a \times_{\mathcal{Y}} \mathcal{U}$, which the 2-Cartesian diagram at the beginning of the proof allows us to identify with $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a$. Thus we obtain a monomorphism

$$\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}_{/}\varpi^a.$$

Since this monomorphism factors through the closed immersion $\mathcal{U}^a \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$, we find that each of the morphisms of (3) is a monomorphism; thus (1) implies (3). Similarly, (2) implies (4), and also implies that the closed immersion $\mathcal{U}^a \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$ is an isomorphism, for each $a \geq 1$.

Since clearly (4) implies (6), while (6) implies (5), to complete the proof of the proposition, it suffices to show that (5) implies (6). Suppose then that $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is a monomorphism. Since $\mathcal{U}^a \hookrightarrow \mathcal{Y}^a$ is an open immersion, it is in particular flat. Since $\mathcal{X}^a \to \mathcal{Y}^a$ is scheme-theoretically dominant and quasicompact (being proper), any flat base-change of this morphism is again scheme-theoretically dominant, as well as being proper. Thus we see that $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is a scheme-theoretically dominant proper monomorphism, i.e., a scheme-theoretically dominant closed immersion, i.e., an isomorphism, as required. □

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On reduced arc spaces of toric varieties

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An arc space of an affine cone over a projective toric variety is known to be nonreduced in general. It was demonstrated recently that the reduced scheme structure of arc spaces is very meaningful from algebro-geometric, representation-theoretic and combinatorial points of view. In this paper we develop a general machinery for the description of the reduced arc spaces of affine cones over toric varieties. We apply our techniques to a number of classical cases and explore some connections with representation theory of current algebras.

Introduction

Let V be the affine cone over a normal projective toric variety. In this paper we study the arc space over V. Roughly speaking, the arc space consists of $\Bbbk[[t]]$-points of V [Nash 1995; Ein and Mustaţă 2009; Frenkel 2007; Mustaţă 2007], where $\Bbbk$ is the base field. If V is embedded into an affine space with coordinates $X_1, \dots, X_n$, then each variable X_i produces a sequences of variables $X_i^{(j)}$ with generating series $X_i(s) = \sum_{j\ge 0} X_i^{(j)} s^j$. The relations cutting out V inside the affine space are replaced with an infinite number of relations appearing as coefficients of powers of s. For example, the relation $X_1X_3 - X_2^2$ defining the degree two Veronese curve gives $X_1(s)X_3(s) - X_2^2(s)$ and one is interested in the ideal, generated by the s-coefficients of this relation. We denote the arc space by $J^\infty(V)$. Arc spaces have attracted a lot of attention (see, e.g., [Anderson and Stapledon 2013; Arakawa and Moreau 2021; Bourqui and Sebag 2019b; de Fernex and Docampo 2020; Gorsky 2013; Ishii 2007]) due to the beauty of the theory and various connections with singularity theory, theory of motivic integration, differential algebra, representation theory, vertex algebras and other mathematical subjects. The particular case of arc spaces

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of toric varieties was also studied by several authors. For instance, in [Ishii and Kollár 2003] the Nash problem is positively solved for toric varieties; in [Ishii 2004] the orbits of the $J^\infty((\Bbbk^\times)^n)$-action are studied; in [Bourqui and Sebag 2019a] the finite formal model for toric singularities is constructed; for other results see also [Reguera 2023; Mourtada 2011].

One interesting feature of arc spaces is that $J^\infty(V)$ may be nonreduced even if V is. The simplest example relevant to our studies is the cone over the degree three Veronese curve. It is defined inside $\mathbb{A}^4$ by three quadratic equations on variables X_0, X_1, X_2, X_3, so the arc space $J^\infty(V)$ is defined by three series of relations. However, in order to get the reduced scheme structure of $J^\infty(V)$ one has to add another series of quadratic relations.

Studying the reduced scheme $J^\infty_{\mathrm{red}}(V)$ is an important problem, related to singularity theory [Mustaţă 2001], differential algebra [Boulier et al. 1995], representation theory [Feigin and Makedonskyi 2020; Dumanski and Feigin 2023], invariant theory [Linshaw and Song 2024], vertex algebras [Feigin and Makedonskyi 2019], and other areas. Our main motivation for the study of the reduced schemes comes from geometric representation theory due to the connection with the theory of semi-infinite flag varieties (see [Dumanski and Feigin 2023; Dumanski et al. 2021; Feigin and Makedonskyi 2020; Makedonskyi 2022]). In particular, in [Feigin and Makedonskyi 2020; Makedonskyi 2022; Dumanski and Feigin 2023] the complete list of relations is found for the arc space of certain varieties (proving, in particular, that the radical is differentially finitely generated). Let us also mention several other results. In [Boulier et al. 1995] an algorithmic approach to the problem of finding the radical of a differential ideal is developed; in [Sebag 2011; Kpognon and Sebag 2017] the question of reducedness of the arc space of a plane curve is investigated; in [Feigin and Makedonskyi 2019] a vertex algebra structure on a reduced arc-ring for flag varieties is introduced; see also [Bourqui and Haiech 2021; Morán Cañón and Sebag 2020; 2022] for other results.

Let $V = \operatorname{Spec}(R)$. We denote by $J^\infty(R)$ the ring of functions on $J^\infty(V)$ and by $J^\infty_{\mathrm{red}}(R)$ the ring of functions on $J^\infty_{\mathrm{red}}(V)$. The goal of this paper is two-fold. First, we develop a technique for the study of reduced arc rings for toric varieties in terms of certain spaces of symmetric polynomials. Second, we apply our construction to the description of $J^\infty_{\mathrm{red}}(R)$ for several classical examples. Let us describe our results in more detail.

We note that since V is an affine cone over a projective variety, there are two natural gradings on $J^\infty_{\mathrm{red}}(R)$: the z-grading defined by attaching degree 1 to each variable $X_i^{(j)}$ and the q-grading attaching degree j to $X_i^{(j)}$. These gradings define a graded character of the arc rings, which are series in z and q. Note that this graded character of the rings $J^\infty(R)$ and $J^\infty_{\mathrm{red}}(R)$ is of great importance in singularity theory and combinatorics; see [Ait El Manssour and Pogudin 2024; Bai et al. 2020; Mourtada 2014; 2023; Bruschek et al. 2013]. The two main problems we address in this paper are as follows:

- Describe $J^\infty_{\mathrm{red}}(R)$ in terms of generators and relations.
- Find the graded character of $J^\infty_{\mathrm{red}}(R)$.

These problems are resolved for affine cones over certain (not necessarily toric) projective varieties in [Feigin and Makedonskyi 2020; Makedonskyi 2022; Sottile and Sturmfels 2001]. For instance, for

Veronese embeddings of the projective line the z-homogeneous components of the reduced arc rings are isomorphic to the dual global Demazure modules (see [Dumanski and Feigin 2023]). More generally, in favorable situations the following nice properties hold:

- The ideal of relations is generated in degree two by explicitly given relations.
- For each $L \geq 0$ the q-character (the Hilbert series with respect to the grading by q-degree) of the z-degree L homogeneous component of the arc ring admits the factorization $\chi_L(q)/(q)_L$ for certain polynomial $\chi_L(q)$.

(Here $(q)_L = (1-q)\cdots(1-q^L)$.) We note that the factor $1/(q)_L$ is the q-character of space of symmetric functions in L variables. In a favorable situation the character has the above form because

- the L-th z-homogeneous component of the arc ring admits a free action of the ring of symmetric polynomials in L variables.

For a normal lattice polytope $P \subset \mathbb{R}^n$ we denote by $X(P)$ the corresponding projective toric variety equipped with a projective embedding (see Section 1.1 for definitions and constructions). The variety $X(P)$ is the projective spectrum of the toric ring $R(P)$. The ring $R(P)$ is known to be generated inside a polynomial ring in $z_1, \dots, z_n$ and w by the elements $Y_{\bar{\alpha}^1}, \dots, Y_{\bar{\alpha}^m}$, where $Y_{\bar{\alpha}^i}$ are the exponents in z variables of the integer points $\bar{\alpha}^i$ of P multiplied by an extra variable w. One also has a presentation of $R(P)$ as the quotient of the polynomial ring in variables $X_{\bar{\alpha}^1}, \dots, X_{\bar{\alpha}^m}$ corresponding to the elements $Y_{\bar{\alpha}^i}$.

Similarly to the finite picture, the reduced arc ring $J^\infty_{\text{red}}(R(P))$ we are interested in is generated by the coefficients of the currents $Y_{\bar{\alpha}^i}(s)$. Thus, the reduced arc ring has two realizations: as a subring of the polynomial ring in variables $z_i^{(j)}$, $w^{(j)}$ and as a quotient ring of the polynomial ring in variables $X^{(j)}_{\bar{\alpha}^i}$.

We introduce a family of subspaces $A_{\bar{r}} \subset J^\infty_{\text{red}}(R(P))$ labeled by tuples $\bar{r} = (r_1, \dots, r_m)$ of nonnegative integers. The space $A_{\bar{r}}$ is realized inside the polynomial ring in variables $z_i^{(j)}$; it is spanned as a vector space by elements of the form

$$Y^{(j_1^1)}_{\bar{\alpha}^1} \cdots Y^{(j^1_{r_1})}_{\bar{\alpha}^1} \cdots Y^{(j_1^m)}_{\bar{\alpha}^m} \cdots Y^{(j^m_{r_m})}_{\bar{\alpha}^m}$$

for all $j_u^v \in \mathbb{N}$. Since the ring $J^\infty_{\text{red}}(R(P))$ is spanned by such elements, it can be presented as the sum of subspaces $A_{\bar{r}}$. This sum is clearly not direct. In order to control the ring $J^\infty_{\text{red}}(R(P))$ we introduce an order $\preceq$ on the set of m-tuples $\bar{r}$ and describe the subquotients

$$\sum_{\bar{r}' \preceq \bar{r}} A_{\bar{r}'} \Big/ \sum_{\bar{r}' \prec \bar{r}} A_{\bar{r}'} \tag{0-1}$$

in terms of certain symmetric polynomials. More precisely, let $\Lambda_{\bar{r}}(\boldsymbol{t}) = \Bbbk[t_i^{(j)}]^{\times \mathfrak{S}_{r_i}}_{i=1,\dots,m;\, j=1,\dots,r_i}$ be the ring of partially symmetric polynomials; see (2-2) and discussions after it for the precise definition. We construct embeddings

$$\Big(\sum_{\bar{r}' \preceq \bar{r}} A_{\bar{r}'} \Big/ \sum_{\bar{r}' \prec \bar{r}} A_{\bar{r}'}\Big)^* \subset \Lambda_{\bar{r}}(\boldsymbol{t})$$

(here we mean the restricted dual with respect to the natural grading; see Section 2). In Section 2 (see Proposition 2.8) we prove:

Theorem 0.1. *There is a homogeneous element* $K \in \Lambda_{\bar{r}}(\boldsymbol{t})$ *such that*

$$\Big(\sum_{\bar{r}' \preceq \bar{r}} A_{\bar{r}'} \Big/ \sum_{\bar{r}' \prec \bar{r}} A_{\bar{r}'}\Big)^* \subset K\Lambda_{\bar{r}}(\boldsymbol{t}).$$

In Section 3 (see Corollary 3.7 and Lemma 3.17) we prove:

Theorem 0.2. *There is a homogeneous element* $\Gamma \in \Lambda_{\bar{r}}(\boldsymbol{t})$ *such that*

$$\Big(\sum_{\bar{r}' \preceq \bar{r}} A_{\bar{r}'} \Big/ \sum_{\bar{r}' \prec \bar{r}} A_{\bar{r}'}\Big)^* \supset \Gamma\Lambda_{\bar{r}}(\boldsymbol{t}).$$

We also prove the following corollary:

Corollary 0.3. *If* $K = \Gamma$, *then for all* $L \geq 0$ *the degree* L z*-homogeneous component of the reduced arc ring admits a free action of the algebra of symmetric polynomials in* L *variables.*

The proofs are constructive, i.e., we compute elements K and Γ. In the rest of the paper we provide several examples when $K = \Gamma$, and we call such polytopes and corresponding varieties favorable. In particular, starting from a one-dimensional case (with P being a segment and $X(P)$ isomorphic to a projective line) we develop an inductive procedure providing a family of favorable polytopes. Namely, for a convex function $\zeta : P \to \mathbb{R}_{\geq 0}$ with $\zeta(\bar{\alpha}^i) = \zeta_i \in \mathbb{Z}$, set $\zeta_{\max} = \max\{\zeta_i\}$. Denote by P^ζ the polytope in $\mathbb{R}^{n+1}$ obtained as the convex hull of $P \times \zeta_{\max}$ and all $\bar{\alpha}^i \times (\zeta_{\max} - \zeta_i)$.

We prove the following theorem (see Theorem 4.20 for a precise statement).

Theorem 0.4. *Let* P *be a* d*-dimensional lattice polytope such that* $X(P)$ *is favorable. Then* $X(P^\zeta)$ *is also favorable provided certain conditions are satisfied.*

As a corollary we derive the following:

Corollary 0.5. *The toric varieties* $X(P)$ *are favorable for* P *being a simplex, a parallelepiped, or a Hirzebruch trapezoid. In all these cases the radical of the ideal of the corresponding arc scheme is finitely generated as a differential ideal.*

In the Appendix we also develop a completely different technique to study the case when P is the product of simplices. This technique uses the representation theory of current algebras and the geometry of the semi-infinite flag variety.

The paper is organized as follows. In Section 1 we collect the main definitions and basic properties of the main objects of study: toric varieties, arc schemes, arc rings, supersymmetric polynomials. In Section 2 we develop a general machinery for the description of arc rings in terms of certain subspaces of symmetric polynomials. The approach is applied to the case of toric varieties in Section 3. In Section 4 we present an inductive procedure for constructing a large family of favorable toric arc spaces. The construction is applied in Section 5 where we work out explicitly several classical examples. Finally, in the Appendix we provide a representation theoretic description of the arc rings of the Veronese–Segre embeddings.

1. Preliminaries and generalities

We work over an algebraically closed field $\Bbbk$ of characteristic zero. For variables $z_1, \dots, z_m$ and vector $\bar{a} = (a_1, \dots, a_m) \in \mathbb{Z}^m$ we use the multi-index notation

$$z^{\bar{a}} = z_1^{a_1} \dots z_m^{a_m}.$$

1.1. ***Toric varieties.*** The basic properties of toric varieties recalled here can be found in [Cox et al. 2011].

Definition 1.1. A toric variety V is an algebraic variety V containing the algebraic torus $(\Bbbk^\times)^k$ as an open dense subset such that the natural action of the torus on itself can be extended to an action on the variety.

Definition 1.2. An ideal in a polynomial ring is called *toric* if it is prime and generated by binomials. A ring is toric if it is the quotient of a polynomial ring by a toric ideal.

Proposition 1.3. *Affine toric varieties are precisely those varieties which have the form* Spec R *for a toric ring* R.

For a rational cone $\sigma \subset \mathbb{R}^m$ its set of integer points $\sigma \cap \mathbb{Z}^m$ forms a semigroup.

Proposition 1.4. *Normal affine toric varieties are precisely those varieties which have the form*

$$\operatorname{Spec}(\Bbbk[\sigma \cap \mathbb{Z}^m])$$

for some m and rational cone $\sigma \subset \mathbb{R}^m$ of dimension m.

A convex polytope $P \subset \mathbb{R}^m$ is said to be normal if for any integer $k > 0$ every integer point in its dilation kP can be expressed as the sum of k (not necessarily distinct) integer points in P. In other words, this means that the set $kP \cap \mathbb{Z}^m$ is the k-fold Minkowski sum of $P \cap \mathbb{Z}^m$ with itself.

Consider a normal convex polytope $P \subset \mathbb{R}^m$. Define a polyhedral cone $C(P) \subset \mathbb{R}^{m+1}$ to be generated by vectors $(a, 1) \in \mathbb{R}^{m+1}$, $a \in P$. The semigroup of this cone $C(P) \cap \mathbb{Z}^{m+1}$ consists of pairs (b, k), where b is an integer point in kP. By normality this semigroup is generated by the points $(a, 1)$, $a \in P \cap \mathbb{Z}^m$. The ring $R(P) := \Bbbk[C(P) \cap \mathbb{Z}^{m+1}]$ is naturally graded, the k-th graded component is equal to the linear span of $z^{\bar{a}} w^k$, $\bar{a} \in kP \cap \mathbb{Z}^m$.

Definition 1.5. The projective toric variety of P is $\operatorname{Proj}(R(P))$.

We denote the generators $z^{\bar{a}} w$ of the ring $R(P)$ by $Y_{\bar{a}}$, $\bar{a} \in P \cap \mathbb{Z}^m$.

Example 1.6. Let $P \subset \mathbb{R}^2$ be the convex hull of points $(0, 0)$, $(1, 0)$, $(1, 1)$, i.e., a unimodular simplex. Then $R(P) \subset \Bbbk[z_1, z_2, w]$ is generated by monomials w; $z_1 w$; $z_1 z_2 w$. Clearly this algebra is freely generated by these monomials, so it is isomorphic to the polynomial ring in 3 variables. All of them have degree 1, so the corresponding variety is isomorphic to a projective plane.

1.2. ***Arc spaces.*** In this subsection we recall the basic properties of arc spaces; see [Chambert-Loir et al. 2018; Ein and Mustaţă 2009; Nash 1995].

Definition 1.7. Let V be a variety over $\Bbbk$. The arc (or jet-∞) space $J^\infty(V)$ is a scheme for which there exists a natural isomorphism of functors from $\mathbf{CAlg}_\Bbbk$ (commutative algebras) to **Set**

$$\mathrm{Hom}(\mathrm{Spec}(-), J^\infty(V)) \simeq \mathrm{Hom}(\mathrm{Spec}(-[\![s]\!]), V).$$

Note that if $V = \mathrm{Spec}(R)$, this definition is equivalent to the existence of a natural isomorphism

$$\mathrm{Hom}(\Bbbk[J^\infty(V)], -) \simeq \mathrm{Hom}(R, -[\![s]\!]). \tag{1-1}$$

We immediately observe that $J^\infty(V)$ is unique if it exists, since $\Bbbk[J^\infty(V)]$ represents the functor $\mathrm{Hom}(R, -[\![s]\!])$. Accordingly, the natural isomorphisms in Definition 1.7 and in (1-1) must also be unique. Throughout the paper we mostly deal with affine varieties (actually, with affine cones of projective varieties). The existence of $J^\infty(V)$ is deduced from the following explicit construction.

Definition 1.8. For a finitely generated $\Bbbk$-algebra R choose generators and relations:

$$R \simeq \Bbbk[X_1, \dots, X_m]/\langle f_1(X_1, \dots, X_m), \dots, f_\ell(X_1, \dots, X_m)\rangle.$$

Consider the polynomial ring $\Bbbk[\{X_i^{(j)}\}]$, where $i = 1, \dots, m$ and $j = 0, 1, \dots$. The *arc ring* $J^\infty(R)$ is the quotient of $\Bbbk[\{X_i^{(j)}\}]$ by the ideal generated by all s-coefficients in series

$$f_k(X_1(s), \dots, X_m(s)), \tag{1-2}$$

where $k = 1, \dots, \ell$ and $X_i(s) := \sum_{j=0}^\infty X_i^{(j)} s^j$.

Proposition 1.9. *For an affine variety* $V = \mathrm{Spec}(R)$ *the scheme*

$$J^\infty(V) = \mathrm{Spec}\, J^\infty(R)$$

satisfies Definition 1.7. In particular, $J^\infty(R)$ *does not depend on the chosen presentation of* R.

In the notation of Definition 1.8 let $Y_i \in R$ be the image of X_i and denote by $Y_i^{(j)} \in J^\infty(R)$ the image of $X_i^{(j)}$. Consider a $\Bbbk$-algebra A and a homomorphism $\iota : R \to A[\![s]\!]$ with

$$\iota(Y_i) = \sum_{j=0}^\infty a_i^{(j)} s^j. \tag{1-3}$$

Then we have a homomorphism $\iota' : J^\infty(R) \to A$ with $\iota'(Y_i^{(j)}) = a_i^{(j)}$ We obtain a morphism from $\mathrm{Hom}(J^\infty(R), A)$ to $\mathrm{Hom}(R, A[\![s]\!])$ given by $\iota \mapsto \iota'$, together these morphisms provide the functorial isomorphism from Definition 1.7. We see that J^∞ is a functor from the category of finitely generated commutative algebras to the category $\mathbf{CAlg}_\Bbbk$. In geometric terms we can view J^∞ as a functor from the category of varieties to the category of schemes which satisfies $\mathrm{Spec} \circ J^\infty \simeq J^\infty \circ \mathrm{Spec}$.

Example 1.10. Consider $R = \Bbbk[X_{00}, X_{01}, X_{10}, X_{11}]/\langle X_{00}X_{11} - X_{10}X_{01}\rangle$. Then the ring $J^\infty(R)$ is defined by relations which are s-coefficients of

$$X_{00}(s)X_{11}(s) - X_{10}(s)X_{01}(s).$$

In other words, the ideal of relations is generated by the expressions

$$\sum_{i=0}^{k} X_{00}^{(i)} X_{11}^{(k-i)} - \sum_{i=0}^{k} X_{01}^{(i)} X_{10}^{(k-i)}.$$

Proposition 1.9 provides a canonical bijection from $\mathrm{Hom}(J^\infty(R), J^\infty(R))$ to $\mathrm{Hom}(R, J^\infty(R)[\![s]\!])$. Let ι be the image of the identity under this bijection. For every $r \in R$ we obtain elements $r^{(j)} \in J^\infty(R)$ by setting

$$\iota(r) = \sum_{j=0}^{\infty} r^{(j)} s^j = r(s).$$

Explicitly, if R is generated by $Y_1, \dots, Y_m$ and polynomial p is such that $r = p(Y_1, \dots, Y_m)$, then $r(s) = p(Y_1(s), \dots, Y_m(s))$. In particular, $r \mapsto r(s)$ is injective.

Let $\varphi : R \to S$ be a homomorphism of finitely generated $\mathbb{k}$-algebras. The homomorphism

$$J^\infty(\varphi) : J^\infty(R) \to J^\infty(S)$$

is given by

$$J^\infty(\varphi)(r^{(j)}) = \varphi(r)^{(j)}. \tag{1-4}$$

It is clear that the functor J^∞ preserves surjections. However, this functor need not preserve injections, such examples will be discussed in this paper. More precisely, we will consider situations where R is embedded into a polynomial ring S but $J^\infty(R)$ has a nontrivial nilradical which, in particular, implies that $J^\infty(R)$ cannot be a subring of $J^\infty(S)$ (which is itself a polynomial ring). See, for example, Lemma 1.31.

The fact that the arc space of a reduced ring can be nonreduced is, in a sense, the only obstacle to preserving injections (at least if the rings are nice enough). We have a functor J^∞_{red} from $\mathbf{CAlg}_{\mathbb{k}}$ to itself which takes R to its *reduced arc ring*, i.e., $J^\infty(R)$ modulo its nilradical. We will now show that J^∞_{red} preserves injections.

Theorem 1.11. *Suppose $\phi : X \to Y$ is a dominant morphism of varieties. Then the corresponding morphism of arc spaces $J^\infty(\phi) : J^\infty(X) \to J^\infty(Y)$ is dominant.*

Proof. We note that the restriction of ϕ to the smooth locus X^{reg} gives a dominant map $X^{\mathrm{reg}} \to Y$. To prove the theorem it suffices to show that $J^\infty(X^{\mathrm{reg}}) \to J^\infty(Y)$ is dominant. Hence, we may (and will) assume that X is smooth.

By the Chevalley constructibility theorem, the dominance is equivalent to the fact that $\phi(X)$ contains a dense open subset of Y, denote it by U. Then the restriction of ϕ to $\phi^{-1}(U)$ can be decomposed as $\phi^{-1}(U) \xrightarrow{\phi_1} U \xrightarrow{\phi_2} Y$ with surjective ϕ_1 and open embedding ϕ_2. Using that $\phi^{-1}(U) \subset X$ is smooth, we can apply the "generic smoothness on the target" theorem [Vakil 2017, Theorem 25.3.3] to ϕ_1, and obtain that there is an open dense $V \subset U$ such that ϕ_1 is smooth on $\phi_1^{-1}(V)$. Restricting to $\phi_1^{-1}(V)$, we have

$$\phi_1^{-1}(V) \xrightarrow{\phi_1} V \xrightarrow{\psi} U \xrightarrow{\phi_2} Y,$$

where ϕ_1 is surjective smooth and ψ, ϕ_2 are open embeddings.

Passing to arcs, we get that $J^\infty(\phi_1)$ is surjective [Ein and Mustaţă 2009, Remark 2.10] and $J^\infty(\psi)$, $J^\infty(\phi_2)$ are open embeddings [Ein and Mustaţă 2009, Lemma 2.3]. Note also that by Kolchin's theorem [Ein and Mustaţă 2009, Theorem 3.3] $J^\infty(U)$, $J^\infty(Y)$ are irreducible and hence the open embeddings $J^\infty(\psi)$, $J^\infty(\phi_2)$ are dominant. Therefore, the composition is dominant and the theorem is proven. □

In this paper we study the ring $J^\infty_{\mathrm{red}}(R)$ for a toric ring R embedded into a polynomial algebra. The following corollary implies that the reduced arc ring $J^\infty_{\mathrm{red}}(R)$ can be described using arc map of the embedding.

Corollary 1.12. *Suppose $f : R \hookrightarrow L$ is an injective map of integral finitely generated $\Bbbk$-algebras. Then the corresponding map $J^\infty_{\mathrm{red}}(R) \to J^\infty_{\mathrm{red}}(L)$ is also injective.*

Proof. The corresponding map $f^* : \operatorname{Spec} L \to \operatorname{Spec} R$ is dominant. Applying Theorem 1.11 we obtain that $J^\infty(f)^* : \operatorname{Spec} J^\infty(L) \to \operatorname{Spec} J^\infty(R)$ is dominant. But that is equivalent to the fact that $\ker(J^\infty(f))$ is contained in the nilradical of $J^\infty(R)$ and we are done. □

The next proposition shows how the arc space of a closure is related to the closure of the arc space.

Proposition 1.13. *Let $\phi : X \to Y$ be a morphism of varieties. Then*

$$J^\infty(\overline{\operatorname{im}\phi}) = \overline{\operatorname{im}(J^\infty(\phi))},$$

where both sides are endowed with the reduced scheme structure.

Proof. Note that for any morphism of schemes $A \to B$ there exists unique decomposition $A \to C \to B$ such that the first arrow is dominant and the second is a closed embedding. We prove that both decompositions

$$J^\infty(X) \xrightarrow{J^\infty(\phi)} \overline{\operatorname{im}(J^\infty(\phi))} \hookrightarrow J^\infty(Y) \tag{1-5}$$

and

$$J^\infty(X) \xrightarrow{J^\infty(\phi)} J^\infty(\overline{\operatorname{im}\phi}) \hookrightarrow J^\infty(Y) \tag{1-6}$$

satisfy this property. This will imply the proposition.

Decomposition (1-5) satisfies the desired properties just by definition (the first map is dominant and the second is a closed embedding).

In order to obtain the same for (1-6) we first consider the decomposition $X \to \overline{\operatorname{im}\phi} \to Y$ of the map ϕ, this decomposition has the desired properties. Applying the functor J^∞ we obtain that in decomposition (1-6) the first arrow is dominant due to Theorem 1.11 and the second is a closed embedding. □

Corollary 1.14. *Let X be a variety with an action of an algebraic group G. This induces the action of $J^\infty(G) = G[[t]]$ on $J^\infty(X)$. Then for any point $x \in X \hookrightarrow J^\infty(X)$ and orbits $G.x \subset X$, $G[[t]].x \subset J^\infty(X)$ one has $J^\infty(\overline{G.x}) = \overline{G[[t]].x}$, where both sides are endowed with the reduced scheme structure.*

Proof. Use Proposition 1.13 for the map $G \to X$ defined by $g \mapsto g.x$. □

The corollary above implies that the arc ring $J^\infty_{\mathrm{red}}(R)$ of an affine toric variety, which we mainly study in this paper, is isomorphic to the coordinate ring of the cone over the closure of the orbit of the toric arc group.

To end this section we provide an example of how one may find relations in a ring $J^\infty_{\mathrm{red}}(R)$. We first prove two general (and well-known) lemmas concerning an arbitrary commutative $\Bbbk$-algebra A. The lemmas can be seen to be equivalent but it is convenient for us to use them both.

Lemma 1.15. *Consider elements $a_i \in A, i = 0, 1, \ldots$. Assume that the series $a(s) = \sum_{i=0}^\infty a_i s^i$ is nilpotent. Then all the coefficients a_i are nilpotent.*

Proof. Clearly a_0 is nilpotent. Assume that $a_i,\ i < j$ are nilpotent. Then

$$a(s) - \sum_{i=0}^{j-1} a_i s^i$$

is nilpotent and thus its leading term $a_j s^j$ is nilpotent. This completes the proof by induction. □

Lemma 1.16. *Let d be a $\Bbbk$-linear derivation of the algebra A. If $a \in A$ is nilpotent, then $d(a)$ is nilpotent.*

Proof. Suppose $a^n = 0$, then $d^n(a^n) = 0$. Expanding $d^n(a^n)$ via the Leibniz rule we obtain $n!\, d(a)^n + ab$ for some $b \in A$. Hence $d(a)^n = -ab/n!$ is nilpotent and so is $d(a)$. □

Example 1.17 [Bourqui and Sebag 2019b]. Here and further we write $a \equiv b$ for elements a, b of a ring to denote that $a - b$ is nilpotent. Consider the ring R generated by Y_1 and Y_2 satisfying the single relation $Y_1 Y_2 = 0$. Then the ring $J^\infty(R)$ is generated by $Y_1^{(j)}$, $Y_2^{(j)}$ satisfying the relations

$$Y_1(s) Y_2(s) = 0.$$

We show by induction that for $n \geq 0$ the series

$$Y_1(s) \frac{\partial^n Y_2(s)}{\partial s^n} \tag{1-7}$$

is nilpotent in $J^\infty(R)[[s]]$. Indeed, assume that

$$Y_1(s) \frac{\partial^{n-1} Y_2(s)}{\partial s^{n-1}}$$

is nilpotent. Using Lemma 1.16 we have

$$Y_1(s) \frac{\partial^n Y_2(s)}{\partial s^n} \equiv -\frac{\partial Y_1(s)}{\partial s} \frac{\partial^{n-1} Y_2(s)}{\partial s^{n-1}}.$$

Multiplying both sides by the left-hand side part we obtain

$$\left(Y_1(s) \frac{\partial^n Y_2(s)}{\partial s^n} \right)^2 \equiv -Y_1(s) \frac{\partial^n Y_2(s)}{\partial s^n} \frac{\partial Y_1(s)}{\partial s} \frac{\partial^{n-1} Y_2(s)}{\partial s^{n-1}}.$$

By the induction hypothesis the right-hand side is nilpotent; hence so is (1-7). In fact, one may check that the coefficients of the series (1-7) span the same linear space as the elements $Y_1^{(i)} Y_2^{(j)}$ for all $i, j \geq 0$. Thus the quotient of $J^\infty(R)$ modulo its nilradical is spanned by the images of all $Y_1^{(i_1)} \cdots Y_1^{(i_M)}$ and $Y_2^{(j_1)} \cdots Y_2^{(j_N)}$. Using this and the results of Section 2.3 one can show that the nilradical is generated by coefficients of series (1-7).

1.3. ***Derivations of arc rings.*** In this subsection we define two Lie algebra actions on an arbitrary arc ring, these actions will be instrumental to our construction.

Consider any commutative $\Bbbk$-algebra Θ and an odd formal variable τ (i.e., $\tau^2 = 0$). Let $\varkappa : \Theta[\tau] \twoheadrightarrow \Theta$ be the natural projection, i.e., $\varkappa(\Theta\tau) = 0$. In the following lemma and throughout the text we only consider $\Bbbk$-derivations.

Lemma 1.18. *The derivations of Θ are in one-to-one correspondence with the homomorphisms $\delta : \Theta \to \Theta[\tau]$ for which $\varkappa \circ \delta = 1$. A derivation d corresponds to the homomorphism $1 + \tau d$.*

Proof. For any derivation $d \in \operatorname{Der}\Theta$ one evidently has

$$\varkappa \circ (1 + \tau d) = 1.$$

Suppose that we have a homomorphism $\delta : \Theta \to \Theta[\tau]$ such that $\varkappa \circ \delta = 1$. Then we have

$$\delta = 1 + \tau d$$

for a linear map d. For any $x, y \in \Theta$,

$$xy + \tau d(xy) = (1 + \tau d)(xy) = (1 + \tau d)(x)(1 + \tau d)(y) = xy + \tau(xd(y) + d(x)y).$$

Hence d is a derivation. □

Fix a finitely generated $\Bbbk$-algebra R for the rest of this section. Consider a continuous derivation $d \in \operatorname{Der}\Bbbk[\![s]\!]$, we can extend d to a derivation of $J^\infty(R)[\![s]\!] = J^\infty(R) \mathbin{\widehat{\otimes}} \Bbbk[\![s]\!]$. Here continuity is with respect to the standard topology on $\Bbbk[\![s]\!]$ and $\widehat{\otimes}$ denotes the completed tensor product with respect to this topology; see, for instance, [Stacks]. Lemma 1.18 applied to $\Theta = J^\infty(R)[\![s]\!]$ and the obtained derivation provide maps

$$J^\infty(R)[\![s]\!] \to J^\infty(R)[\![s]\!][\tau] \to J^\infty(R)[\![s]\!],$$

and hence the maps

$$\operatorname{Hom}(R, J^\infty(R)[\![s]\!]) \to \operatorname{Hom}(R, J^\infty(R)[\![s]\!][\tau]) \to \operatorname{Hom}(R, J^\infty(R)[\![s]\!]),$$

with composition equal to identity. Applying isomorphism (1-1) we have maps φ and π

$$\operatorname{Hom}(J^\infty(R), J^\infty(R)) \xrightarrow{\varphi} \operatorname{Hom}(J^\infty(R), J^\infty(R)[\tau]) \xrightarrow{\pi} \operatorname{Hom}(J^\infty(R), J^\infty(R)) \tag{1-8}$$

with composition equal to identity.

Proposition 1.19. *The image of the identity under φ has the form $1 + \tau\rho(d)$ for some $\rho(d) \in \operatorname{Der} J^\infty(R)$. The map $-\rho$ defines a Lie algebra action of* $\operatorname{Der}^c \Bbbk[\![s]\!]$ (*continuous derivations*) *on* $J^\infty(R)$.

Proof. By construction, the map π is obtained by applying the functor $\operatorname{Hom}(J^\infty(R), -)$ to the projection $\varkappa : J^\infty(R)[\tau] \twoheadrightarrow J^\infty(R)$, i.e., it is the composition with $\varkappa$ on the left. In particular, if $\delta = \varphi(\mathrm{id}_{J^\infty(R)})$, then $\mathrm{id}_{J^\infty(R)} = \varkappa\delta$. By Lemma 1.18, we indeed have $\delta = 1 + \tau\rho(d)$ for some $\rho(d) \in \operatorname{Der} J^\infty(R)$.

Let us show that

$$-\rho : \operatorname{Der}^c \Bbbk[\![s]\!] \to \operatorname{Der} J^\infty(R)$$

is a Lie algebra homomorphism. The derivation d is given by a $(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0})$-matrix B such that

$$d(s^j) = \sum_{i \geq 0} B_{i,j} s^i$$

for any $j \geq 0$, i.e., d maps a series with coefficients $(c_0, c_1, \dots)$ to the series with coefficients $B(c_0, c_1, \dots)^T$. From the definitions we see that for any $r \in R$ and $i \geq 0$ we then have

$$\varphi(\mathrm{id}_{J^\infty(R)})(r^{(i)}) = r^{(i)} + \tau \sum_{j \geq 0} B_{i,j} r^{(j)}.$$

In other words, $\rho(d)$ maps a linear combination of the $r^{(j)}$ with coefficients $(c_0, c_1, \dots)$ to their linear combination with coefficients $(c_0, c_1, \dots)B$. This shows that ρ is an antihomomorphism, i.e, $-\rho$ is indeed a homomorphism. □

We will use the shorthand notation $dX = -\rho(d)(X)$ for $d \in \operatorname{Der}^c \Bbbk[\![s]\!]$ and $X \in J^\infty(R)$. Recall that the Lie algebra $\operatorname{Der}^c \Bbbk[\![s]\!]$ is spanned by the derivations $d_k = s^{k+1} \frac{\partial}{\partial s}$ with $k \geq -1$. These derivations act on $J^\infty(R)$ as follows (here and further $r^{(i)} = 0$ for any $r \in R$ and $i < 0$).

Proposition 1.20. *For any $r \in R$, $i \geq 0$ and $k \geq -1$ we have*

$$d_k(r^{(i)}) = (i-k) r^{(i-k)}. \tag{1-9}$$

Proof. Setting $d = d_k$ within the proof of the previous proposition we see that the matrix B is given by $B_{i,j} = j$ when $i - j = k$ and $B_{i,j} = 0$ otherwise. In particular, the proof then implies that

$$d_k(r^{(i)}) = \sum_{j > 0} B_{i,j} r^{(j)} = (i-k) r^{(i-k)}.$$ □

From (1-4) and Proposition 1.20 we see that J^∞ respects the action of $\operatorname{Der}^c \Bbbk[\![s]\!]$. In other words, J^∞ as a functor from $\mathbf{CAlg}_{\Bbbk}^{\mathrm{fin}}$ to the category of commutative $\operatorname{Der}^c \Bbbk[\![s]\!]$-differential algebras.

We next describe another Lie algebra action on $J^\infty(R)$. Fix a presentation

$$R \simeq \Bbbk[X_1, \dots, X_m] / \langle f_1(X_1, \dots, X_m), \dots, f_\ell(X_1, \dots, X_m) \rangle$$

and let $Y_i \in R$ be the image of X_i.

Definition 1.21. For a derivation $\mathfrak{d} \in \operatorname{Der} R$ and $k \geq 0$ we consider $\mathfrak{d}^{(k)} \in \operatorname{Der} J^\infty(R)$ defined on generators by

$$\mathfrak{d}^{(k)}(Y_i^{(j)}) := \mathfrak{d}(Y_i)^{(j-k)}.$$

Lemma 1.22. *The derivations $\mathfrak{d}^{(k)}$ are well-defined and independent of the presentation.*

Proof. Choose polynomials $p_1, \dots, p_m$ in $\Bbbk[X_1, \dots, X_m]$ such that $p_i(Y_1, \dots, Y_m) = \mathfrak{d} Y_i$ for all Y_i. We have a unique derivation $\tilde{\mathfrak{d}}$ of $\Bbbk[X_1, \dots, X_m]$ given by $\tilde{\mathfrak{d}} X_i = p_i$. We also consider the derivation $\tilde{\mathfrak{d}}^{(k)}$ of $\Bbbk[\{X_i^{(j)}\}]$ such that $\tilde{\mathfrak{d}}^{(k)}(X_i^{(j)})$ is the coefficient of s^{j-k} in $p_i(X_1(s), \dots, X_m(s))$ or 0 if $j < k$. Furthermore, we extend $\tilde{\mathfrak{d}}^{(k)}$ to a derivation of $\Bbbk[\{X_i^{(j)}\}][\![s]\!]$ by applying it coefficientwise. Note that

$$\tilde{\mathfrak{d}}^{(k)}(X_i(s)) = s^k p_i(X_1(s), \dots, X_m(s)). \tag{1-10}$$

Now, for any $p \in \Bbbk[X_1, \dots, X_m]$ its image $\tilde{\mathfrak{d}}(p)$ projects to $\mathfrak{d}(p(Y_1, \dots, Y_m)) \in R$. Hence $\tilde{\mathfrak{d}}$ preserves the ideal of relations in R, i.e., for all f_o we have polynomials a_q such that

$$\tilde{\mathfrak{d}}(f_o) = \sum_{q=1}^{\ell} a_q f_q.$$

From (1-10) we then obtain

$$\tilde{\mathfrak{d}}^{(k)}(f_o(X_1(s), \dots, X_m(s))) = s^k \sum_{q=1}^{\ell} a_q(X_1(s), \dots, X_m(s)) f_q(X_1(s), \dots, X_m(s)).$$

This means that $\tilde{\mathfrak{d}}^{(k)} \in \operatorname{Der} \Bbbk[\{X_i^{(j)}\}]$ preserves the defining ideal of $J^\infty(R)$ generated by all coefficients of the series $f_q(X_1(s), \dots, X_m(s))$. By projecting onto $J^\infty(R)$ we obtain a derivation which maps $Y_i^{(j)}$ to the coefficient of s^{j-k} in $p_i(Y_1(s), \dots, Y_m(s))$. The latter coefficient is precisely $\mathfrak{d}(Y_i)^{(j-k)}$ and the obtained derivation satisfies the definition of $\mathfrak{d}^{(k)}$.

To show that $\mathfrak{d}^{(k)}$ is independent of the presentation extend $\mathfrak{d}^{(k)}$ to a derivation of $J^\infty(R)[\![s]\!]$ coefficientwise. We have the embedding ι of R into $J^\infty(R)[\![s]\!]$ with $\iota(r) = r(s)$. Now we note that

$$s^k \iota \mathfrak{d}(Y_i) = \mathfrak{d}^{(k)} \iota(Y_i),$$

i.e., the maps $s^k \iota \mathfrak{d} \iota^{-1}$ and $\mathfrak{d}^{(k)}$ agree on the $\iota(Y_i)$, and hence on all of $\iota(R)$. In other words,

$$\mathfrak{d}^{(k)}(r^{(j)}) = \mathfrak{d}(r)^{(j-k)} \tag{1-11}$$

for any $r \in R$ and $j \geq 0$ which is independent of the presentation. □

Lemma 1.23. *The current Lie algebra* $(\operatorname{Der} R)[s] = \operatorname{Der} R \otimes_{\Bbbk} \Bbbk[s]$ *acts by derivations on* $J^\infty(R)$ *by* $\mathfrak{d}s^k$ *acting as* $\mathfrak{d}^{(k)}$.

Proof. For any two derivations $\mathfrak{d}_1, \mathfrak{d}_2 \in \operatorname{Der}(D)$ and $k_1, k_2 \geq 0$ we need to prove

$$[\mathfrak{d}_1^{(k_1)}, \mathfrak{d}_2^{(k_2)}] = [\mathfrak{d}_1, \mathfrak{d}_2]^{(k_1+k_2)}.$$

However, from (1-11) we indeed have, for any $r \in R$,

$$[\mathfrak{d}_1^{(k_1)}, \mathfrak{d}_2^{(k_2)}](r^{(j)}) = ([\mathfrak{d}_1, \mathfrak{d}_2](r))^{(j-k_1-k_2)} = [\mathfrak{d}_1, \mathfrak{d}_2]^{(k_1+k_2)}(r^{(j)}).$$

Since the $r^{(j)}$ generate $J^\infty(R)$, the two derivations agree on all of $J^\infty(R)$. □

Let $\tilde{\tau} : \mathfrak{a} \to \mathfrak{b}$ be a morphism of Lie algebras. Then there exists the morphism

$$\tilde{\tau}[s] : \mathfrak{a}[s] \to \mathfrak{b}[s], \quad as^k \mapsto \tilde{\tau}(a)s^k.$$

Lemma 1.24. *Let R, L be associative algebras, $\mathfrak{g}_R, \mathfrak{g}_L$ be Lie algebras acting on them by derivations. Let τ, $\tilde{\tau}$ be two morphisms $\tau : R \to L$, $\tilde{\tau} : \mathfrak{g}_R \to \mathfrak{g}_L$ such that τ is equivariant with respect to $\tilde{\tau}$, i.e., for any $d \in \mathfrak{g}_R$, $r \in R$,*

$$\tau(d(r)) = \tilde{\tau}(d)(\tau(r)).$$

Then $J^\infty(\tau)$ is equivariant with respect to $\tilde{\tau}[s]$.

Proof. This follows from (1-11) and the definition of equivariance. □

1.4. ***Supersymmetric polynomials.*** In this subsection we discuss the properties of supersymmetric polynomials. Take two groups of variables $p^{(1)}, \dots, p^{(b)}, q^{(1)}, \dots, q^{(c)}$.

Definition 1.25. A polynomial $f \in \Bbbk[p^{(1)}, \dots, p^{(b)}, q^{(1)}, \dots, q^{(c)}]^{\mathfrak{S}_b \times \mathfrak{S}_c}$ symmetric in the $p^{(i)}$ and the $q^{(j)}$ is supersymmetric if $f|_{p^{(b)} \mapsto Z, q^{(c)} \mapsto Z}$ is independent of Z. We denote the ring of such polynomials by $\Omega_{b,c}$.

Proposition 1.26. *There exists the map $\psi_{b,c} : \Omega_{b,c} \to \Omega_{b-1,c-1}$:*

$$\psi_{b,c} : f \mapsto f|_{p^{(b)} \mapsto Z, q^{(c)} \mapsto Z}.$$

The kernel of $\psi_{b,c}$ is the ideal

$$\prod_{1 \le i \le b, 1 \le j \le c} (p^{(i)} - q^{(j)}) \Bbbk[p^{(1)}, \dots, p^{(b)}, q^{(1)}, \dots, q^{(c)}]^{\mathfrak{S}_b \times \mathfrak{S}_c}.$$

Proof. For a polynomial $f \in \Omega_{b,c}$ the image $\psi_{b,c}(f)$ is symmetric in $p^{(1)}, \dots, p^{(b-1)}$ and $q^{(1)}, \dots, q^{(c-1)}$. We have

$$\psi_{b,c}(f)|_{p^{(b-1)} \mapsto Z', q^{(c-1)} \mapsto Z'} = f|_{p^{(b)} \mapsto Z, q^{(c)} \mapsto Z, p^{(b-1)} \mapsto Z', q^{(c-1)} \mapsto Z'}.$$

Therefore it is independent of Z'. Hence $\psi_{b,c}(f) \in \Omega_{b-1,c-1}$.

Assume now $f \in \ker(\psi_{b,c})$. Since f is divisible by $(p^{(b)} - q^{(c)})$ and symmetric in $p^{(1)}, \dots p^{(b)}$ and $q^{(1)}, \dots q^{(c)}$, we see that f is divisible by $\prod_{1 \le i \le b, 1 \le j \le c} (p^{(i)} - q^{(j)})$. Conversely

$$\prod_{1 \le i \le b, 1 \le j \le c} (p^{(i)} - q^{(j)})_{p^{(b)} \mapsto Z, q^{(c)} \mapsto Z} = 0.$$

Thus the principal ideal $\prod_{1 \le i \le b, 1 \le j \le c} (p^{(i)} - q^{(j)}) \Omega_{b,c}$ is annihilated by $\psi_{b,c}$. □

The proof of the following proposition can be found in [Macdonald 1995, Section 1.3.23].

Proposition 1.27. *The ring $\Omega_{b,c}$ has two sets of generators,*

$$p_k(p^{(1)}, \dots, p^{(b)}; q^{(1)}, \dots, q^{(c)}) = \sum_{i=1}^{b} (p^{(i)})^k - \sum_{i=1}^{c} (q^{(i)})^k \quad \textit{and} \quad h_k(p^{(1)}, \dots, p^{(b)}; q^{(1)}, \dots, q^{(c)}),$$

where

$$\frac{\prod_{i=1}^{b} (1 - T p^{(i)})}{\prod_{i=1}^{c} (1 - T q^{(i)})} = \sum_{k=0}^{\infty} h_k(p^{(1)}, \dots, p^{(b)}; q^{(1)}, \dots, q^{(c)}) T^k$$

for a formal variable T. In particular $\psi_{b,c}$ is surjective.

1.5. ***Associated graded algebras and initial ideals.*** Consider the polynomial ring $\Bbbk[X_1, \dots, X_n]$. Let $\prec$ be a monomial order on monomials in $X_1, \dots, X_n$, i.e., a total order on monomials such that if $a \prec b$ then for any monomial c: $ac \prec bc$. In other words, $\prec$ is a total semigroup order on $\mathbb{N}^n$ (see [Miller and Sturmfels 2005]).

The ring $\Bbbk[X_1, \dots, X_n]$ is naturally $\mathbb{N}^n$-graded. All graded components are one-dimensional and are spanned by monomials:

$$\Bbbk[X_1, \dots, X_n][\bar{r}] = \Bbbk \prod X_i^{r_i}.$$

The ring $\mathbb{J}_n = J^\infty(\Bbbk[X_1, \dots, X_n])$ is the polynomial ring in variables $X_i^{(j)}$ with $1 \le i \le n$ and $j \ge 0$. This ring has an induced $\mathbb{N}^n$-grading where graded component $\mathbb{J}_n[\bar{r}]$ has the basis

$$\left\{ \prod_{1 \le i \le n, j=0,1,\dots} (X_i^{(j)})^{b_i^j} \,\middle|\, \sum_{j=0}^{\infty} b_i^j = r_i,\ i = 1, \dots, n \right\}.$$

$\mathbb{J}_n$ has two filtrations by the ordered semigroup $(\mathbb{N}^n, \prec)$. The first has components

$$\mathbb{J}_n[\prec \bar{r}] = \bigoplus_{\bar{r}' \prec \bar{r}} J^\infty(\Bbbk[X_1, \dots, X_n])[\bar{r}'],$$

while the second has components

$$\mathbb{J}_n[\preceq \bar{r}] = \bigoplus_{\bar{r}' \preceq \bar{r}} J^\infty(\Bbbk[X_1, \dots, X_n])[\bar{r}'].$$

Both filtrations are multiplicative: for any $\bar{r}_1, \bar{r}_2 \in \mathbb{N}^n$ we have

$$\mathbb{J}_n[\prec \bar{r}_1]\mathbb{J}_n[\prec \bar{r}_2] \subset \mathbb{J}_n[\prec \bar{r}_1 + \bar{r}_2] \quad \text{and} \quad \mathbb{J}_n[\preceq \bar{r}_1]\mathbb{J}_n[\preceq \bar{r}_2] \subset \mathbb{J}_n[\preceq \bar{r}_1 + \bar{r}_2].$$

Let $\mathcal{I} \subset \mathbb{J}_n$ be an ideal. Denote $R = \mathbb{J}_n/\mathcal{I}$ and let π be the natural projection $\pi : \mathbb{J}_n \twoheadrightarrow R$. The filtrations on $\mathbb{J}_n$ induce filtrations on R by $R[\prec \bar{r}] = \pi(\mathbb{J}_n[\prec \bar{r}])$ and $R[\preceq \bar{r}] = \pi(\mathbb{J}_n[\preceq \bar{r}])$. These filtrations are again multiplicative and we have the associated $\mathbb{N}^n$-graded algebra

$$\operatorname{gr}_\prec R = \bigoplus_{\bar{r} \in \mathbb{N}^n} R[\preceq \bar{r}]/R[\prec \bar{r}].$$

Next, for any nonzero element $g \in \mathbb{J}_n$ consider its decomposition

$$g = \sum_{\bar{r} \in \mathbb{N}^n} g[\bar{r}], \quad g[\bar{r}] \in \mathbb{J}_n[\bar{r}].$$

Let $\bar{r}'$ be the largest $\bar{r}$ with respect to $\prec$ such that $g[\bar{r}]$ is nonzero. Then $g[\bar{r}']$ is denoted by $\operatorname{in}_\prec g$ and is called the *initial part* of g. The *initial ideal* of $\mathcal{I}$ with respect to $\prec$ is the linear span of $\{\operatorname{in} g \mid g \in \mathcal{I}\}$. This space is denoted by $\operatorname{in}_\prec \mathcal{I}$, one easily checks that it is an ideal in $\mathbb{J}_n$. By construction, the ideal $\operatorname{in}_\prec \mathcal{I}$ is homogeneous with respect to the $\mathbb{N}^n$-grading. Therefore, the ring $\mathbb{J}_n/(\operatorname{in}_\prec \mathcal{I})$ is naturally $\mathbb{N}^n$-graded.

Next, let us observe that $\mathbb{J}_n$ has a further $\mathbb{N}$-grading grad given by $\operatorname{grad} X_i^{(j)} = j$. Together with the grading considered above this turns $\mathbb{J}_n$ into a $\mathbb{N}^{n+1}$-graded ring. We now assume that $\mathcal{I}$ is homogeneous with respect to grad. This gives us an induced $\mathbb{N}$-grading grad on R. Note that every subspace $\mathbb{J}_n[\prec \bar{r}]$ and $\mathbb{J}_n[\preceq \bar{r}]$ is also grad-homogeneous, and hence so is every $R[\prec \bar{r}]$ and $R[\preceq \bar{r}]$. As a result we obtain an induced grading grad on $\operatorname{gr}_\prec R$ which makes the latter $\mathbb{N}^{n+1}$-graded. Furthermore, since $\mathcal{I}$ is grad-homogeneous, so is any initial ideal. Consequently, $\operatorname{in}_\prec \mathcal{I}$ and $\mathbb{J}_n/\operatorname{in}_\prec \mathcal{I}$ are $\mathbb{N}^{n+1}$-graded.

Proposition 1.28. *The rings* $\mathrm{gr}_\prec R$ *and* $\mathbb{J}_n/\operatorname{in}_\prec \mathcal{I}$ *are isomorphic as* $\mathbb{N}^{n+1}$*-graded algebras.*

Proof. We may consider the associated graded algebra $\mathrm{gr}_\prec \mathbb{J}_n$, note that it is naturally identified with $\mathbb{J}_n$. Since the filtrations on R are defined as projections of the filtrations on $\mathbb{J}_n$, we obtain a surjective homomorphism $\pi' : \mathrm{gr}_\prec \mathbb{J}_n \to \mathrm{gr}_\prec R$ preserving the $\mathbb{N}^{n+1}$-grading. In terms of this identification, $p \in \mathbb{J}_n[\bar{r}]$ lies in $\ker \pi'$ if and only if $p - q \in \mathbb{J}_n[\prec \bar{r}]$ for some $q \in \mathcal{I}$. In other words: $p \in \ker \pi'$ if and only if $p = \operatorname{in}_\prec q$ for some $q \in \mathcal{I}$. This proves the proposition. □

Remark 1.29. This proposition is a special case of the well-known and rather general phenomenon of isomorphisms between associated graded rings and quotients by initial ideals. See, for instance, [Kaveh and Manon 2019, Lemma 3.4] or [Makhlin 2022, Propositions 1.1 and 8.1] for statements of this form for finitely generated commutative and associative algebras.

Choose a grad-homogeneous generating set $\{f_1, \dots, f_\ell\}$ of $\mathcal{I}$. We denote by $\mathcal{J}$ the ideal generated by the polynomials $\operatorname{in}_\prec f_i$. This ideal is $\mathbb{N}^{n+1}$-homogeneous and is contained in $\operatorname{in}_\prec \mathcal{I}$. This provides a surjection $\mathbb{J}_n/\mathcal{J} \twoheadrightarrow \mathbb{J}_n/\operatorname{in}_\prec \mathcal{I}$ of $\mathbb{N}^{n+1}$-graded algebras. In view of Proposition 1.28, we have the following.

Corollary 1.30. *There exists a surjective homomorphism of* $\mathbb{N}^{n+1}$*-graded algebras from* $\mathbb{J}_n/\mathcal{J}$ *to* $\mathrm{gr}_\prec R$.

1.6. ***Unit cubes.*** For an integer $\ell \geq 2$ let $\mathbb{R}^{[\ell]}$ be the space of real-valued functions on the set of subsets of $[\ell]$. Consider the unit cube $\mathcal{C}_\ell \subset \mathbb{R}^{[\ell]}$, its vertices are the indicator functions $\mathbf{1}_I$ with $I \subset [\ell]$. $R(\mathcal{C}_\ell)$ is generated by elements Y_I which satisfy

$$Y_I Y_J - Y_{I\cap J} Y_{I\cup J}. \tag{1-12}$$

Consider the arc ring $J^\infty(R(\mathcal{C}_\ell))$. It is generated by variables $Y_I^{(j)}$, $j \geq 0$, satisfying the relations

$$Y_I(s) Y_J(s) - Y_{I\cap J}(s) Y_{I\cup J}(s) = 0. \tag{1-13}$$

Lemma 1.31. *The coefficients of the following series are nilpotent in* $J^\infty(R(\mathcal{C}_\ell))$ *for any* $0 \leq k \leq \ell - 2$:

$$W_{\ell,k} = \sum_{I \subset [2,\ell]} (-1)^{|I|} Y_{I\cup\{1\}}(s) \frac{\partial^k Y_{[2,\ell]\setminus I}(s)}{\partial s^k}. \tag{1-14}$$

Proof. We prove this lemma by induction on ℓ by showing that the series $W_{\ell,k}$ is nilpotent in $J^\infty(R(\mathcal{C}_\ell))[\![s]\!]$, the lemma then follows by Lemma 1.15. In the case $\ell = 2$ we only have $k = 0$ and $W_{\ell,k}$ vanishes by (1-13).

For the induction step we first consider the case $k < \ell - 2$. We break up $W_{\ell,k}$ into two sums and show that each summand can be interpreted as a similar relation for $\ell - 1$. First consider the sum

$$\sum_{J \subset [2,\ell-1]} (-1)^{|J|+1} Y_{J\cup\{1,\ell\}}(s) \frac{\partial^k Y_{[2,\ell-1]\setminus J}(s)}{\partial s^k}. \tag{1-15}$$

Every $I \subset [\ell]$ corresponds to a vertex v^I of $\mathcal{C}_\ell$ such that v^I_i is 1 if $i \in I$ and 0 otherwise. The subscripts appearing in (1-15) are those subsets which either contain both 1 and ℓ or neither of them. The corresponding vertices are the vertices of an $(\ell-1)$-dimensional parallelepiped D. Note that $R(D)$ is embedded into $R(\mathcal{C}_\ell)$ as the subalgebra generated by the Y_I with $v^I \in D$. Consequently, we can view $J^\infty(R(D))$ as the subalgebra in $J^\infty(R(\mathcal{C}_\ell))$ generated by the corresponding $Y_I^{(j)}$.

Now, D can be identified with $\mathcal{C}_{\ell-1}$ by projecting along the ℓ-th coordinate. This projection is unimodular (it identifies the lattice $\operatorname{span}(D) \cap \mathbb{Z}^\ell$ with $\mathbb{Z}^{\ell-1}$); therefore it induces isomorphisms $R(D) \simeq R(\mathcal{C}_{\ell-1})$ and $J^\infty(R(D)) \simeq J^\infty(R(\mathcal{C}_{\ell-1}))$. The image of (1-15) under the latter isomorphism is $-W_{\ell-1,k}$ which is nilpotent by the induction hypothesis.

The summands in $W_{\ell,k}$ not found in (1-15) are those featuring subscripts which contain exactly one of 1 and ℓ. The corresponding vertices again form the vertex set of an $(\ell-1)$-dimensional parallelepiped and the same argument shows that the remaining sum is also nilpotent.

Now we consider $k = \ell - 2$. First, for integers $n \in [2, \ell]$, $m \geq 0$ and $p \in [0, m]$ consider the expression

$$\widetilde{W}_{n,m,p} = \sum_{I \subset [\ell-n+2,\ell]} (-1)^{|I|} \frac{\partial^p Y_{I \cup \{\ell-n+1\}}(s)}{\partial s^p} \frac{\partial^{m-p} Y_{[\ell-n+2,\ell]\setminus I}(s)}{\partial s^{m-p}} \in J^\infty(R(\mathcal{C}_\ell))[\![s]\!].$$

In particular, $\widetilde{W}_{\ell,m,0} = W_{\ell,m}$ for $m \leq \ell - 2$.

Let n be either $\ell - 1$ or ℓ. By, respectively, either the induction hypothesis applied to the Boolean lattice of subsets in $[2, \ell]$ or the above case $k < \ell - 2$, we have $\widetilde{W}_{n,m,0} \equiv 0$ for $m \leq \ell - 3$ (recall that $a \equiv b$ means that $a - b$ is nilpotent). Then, by Lemma 1.16,

$$\frac{\partial}{\partial s} \widetilde{W}_{n,m,0} = \widetilde{W}_{n,m+1,0} + \widetilde{W}_{n,m+1,1} \equiv 0$$

for $m \leq \ell - 3$. We deduce that $\widetilde{W}_{n,m,1} \equiv 0$ for $m \leq \ell - 3$. Iterating this process we obtain that all $\widetilde{W}_{n,m,p} \equiv 0$ when $m \leq \ell - 3$. Hence, by writing the derivative of $\widetilde{W}_{n,\ell-3,p} \equiv 0$ for any $p \in [0, \ell - 3]$ we obtain $\widetilde{W}_{n,\ell-2,p} = -\widetilde{W}_{n,\ell-2,p+1}$. Together these $\ell - 2$ equivalences provide

$$\widetilde{W}_{n,\ell-2,0} \equiv (-1)^{\ell-2} \widetilde{W}_{n,\ell-2,\ell-2} \quad \text{for } n = \ell - 1, \ell. \tag{1-16}$$

Now choose $J \subset [2, \ell]$. Multiplying $W_{\ell,\ell-2}$ by $Y_J(s)$ and using the relations (1-13) we have

$$Y_J(s) \sum_{I \subset [2,\ell]} (-1)^{|I|} Y_{I \cup \{1\}}(s) \frac{\partial^{\ell-2} Y_{[2,\ell]\setminus I}(s)}{\partial s^{\ell-2}} = Y_{J \cup \{1\}}(s) \sum_{I \subset [2,\ell]} (-1)^{|I|} Y_I(s) \frac{\partial^{\ell-2} Y_{[2,\ell]\setminus I}(s)}{\partial s^{\ell-2}}.$$

In view of (1-16) for $n = \ell - 1$, we have

$$\sum_{I \subset [2,\ell]} (-1)^{|I|} Y_I(s) \frac{\partial^{\ell-2} Y_{[2,\ell]\setminus I}(s)}{\partial s^{\ell-2}} = -\widetilde{W}_{\ell-1,\ell-2,0} + (-1)^{\ell-2} \widetilde{W}_{\ell-1,\ell-2,\ell-2} \equiv 0. \tag{1-17}$$

We see that $Y_J(s) W_{\ell,\ell-2} \equiv 0$ for any $Y_J(s)$ with $J \subset [2, \ell]$. By applying (1-16) for $n = \ell$ we also have $Y_J(s) \widetilde{W}_{\ell,\ell-2,\ell-2} \equiv 0$. However, it is evident from the relations (1-13) that $J^\infty(R(\mathcal{C}_\ell))[\![s]\!]$ admits an involution exchanging $Y_I^{(j)}$ and $Y_{[\ell]\setminus I}^{(j)}$. Applying this involution to $Y_J(s) \widetilde{W}_{\ell,\ell-2,\ell-2}$ we obtain $Y_{J'}(s) W_{\ell,\ell-2}$, where $1 \in J'$, and we see that this product is also nilpotent. As a result, all $Y_J(s) W_{\ell,\ell-2} \equiv 0$ and, since $W_{\ell,\ell-2}^2$ is a sum of multiples of various $Y_J(s) W_{\ell,\ell-2}$, we have $W_{\ell,\ell-2} \equiv 0$. □

Remark 1.32. Consider $r \leq \ell$ and an affine embedding $\mathcal{C}_r \hookrightarrow \mathcal{C}_\ell$ mapping vertices to vertices. This induces a map $f : J^\infty(R(\mathcal{C}_r)) \to J^\infty(R(\mathcal{C}_\ell))$ and any $f(W_{r,k})$ will lie in the nilradical of $J^\infty(R(\mathcal{C}_\ell))$. It can be shown that all elements of this form generate the nilradical, more specifically, a minimal set of such relations is obtained in Section 5.1.

2. Symmetric polynomials and dual spaces of graded components

2.1. ***Arcs over the polynomial ring.*** In this subsection we use the notation from Section 1.5 and discuss the arc ring $\mathbb{J}_n$ of a polynomial ring. Note that, similarly to $\mathbb{J}_n$, any arc ring $J^\infty(R)$ has a grading grad given by $\operatorname{grad} r^{(d)} = d$ for any $r \in R$. If B is any grad-homogeneous subspace, quotient or subquotient of an arc ring, we will denote by $B^{(d)}$ its component spanned by b with $\operatorname{grad} b = d$. In particular, if all $B^{(d)}$ are finite-dimensional (for example, when $B = \mathbb{J}_n[\bar{a}]$, $\bar{a} \in \mathbb{N}^n$) we will understand the dual space B^* to be the graded dual with respect to grad.

Choose $\bar{a} \in \mathbb{N}^n$ and consider $a_1 + \cdots + a_n$ formal variables

$$s_1^{(1)}, \dots, s_1^{(a_1)}, \dots, s_n^{(1)}, \dots, s_n^{(a_n)}.$$

Let $\epsilon_i \in \mathbb{N}^n$ be the vector with its i-th coordinate equal to 1 and all other coordinates 0. We have

$$X_i(s_i^{(j)}) \in \mathbb{J}_n[\epsilon_i] \mathbin{\widehat{\otimes}} \Bbbk[s_i^{(j)}],$$

where we view any $\mathbb{J}_n[\bar{a}]$ as a topological vector space with a neighborhood base at 0 consisting of the spaces $\bigoplus_{d \geq N} \mathbb{J}_n[\bar{a}]^{(d)}$ for $N \geq 0$. Consider the expression

$$\begin{aligned} U_{\bar{a}}(X_1, \dots, X_n) &:= X_1(s_1^{(1)}) \cdots X_1(s_{a_1}^{(1)}) \cdots X_n(s_1^{(n)}) \cdots X_n(s_{a_n}^{(n)}) \\ &\in \mathbb{J}_n[\bar{a}] \mathbin{\widehat{\otimes}} \Bbbk[s_1^{(1)}, \dots, s_1^{(a_1)}, \dots, s_n^{(1)}, \dots, s_n^{(a_n)}]. \end{aligned} \tag{2-1}$$

We have

$$U_{\bar{a}}(X_1, \dots, X_n) = \sum_{(k_i^j \mid 1 \leq i \leq n, 1 \leq j \leq a_i) \in \mathbb{N}^{a_1 + \cdots + a_n}} \prod_{1 \leq i \leq n, 1 \leq j \leq a_i} (s_i^{(j)})^{k_i^j} \prod_{1 \leq i \leq n, 1 \leq j \leq a_i} X_i^{(k_i^j)}.$$

Collecting all terms with the same $\prod_{1 \leq i \leq n, 1 \leq j \leq a_i} X_i^{(k_i^j)}$ we obtain

$$\begin{aligned} &U_{\bar{a}}(X_1, \dots, X_n) \\ &= \sum_{\substack{(k_i^j \mid 1 \leq i \leq n, 1 \leq j \leq a_i) \in \mathbb{N}^{a_1 + \cdots + a_n} \text{ with} \\ k_i^1 \leq k_i^2 \leq \cdots \leq k_i^{a_i} \text{ for every } i}} \Biggl(\prod_{1 \leq i \leq n, 1 \leq j \leq a_i} X_i^{(k_i^j)} \Biggl(\sum_{(\sigma_1, \dots, \sigma_n) \in \mathfrak{S}_{a_1} \times \cdots \times \mathfrak{S}_{a_n}} \prod_{1 \leq i \leq n, 1 \leq j \leq a_i} (s_i^{(\sigma_i(j))})^{k_i^j} \Biggr) \Biggr). \end{aligned}$$

Therefore the coefficients at $\prod_{1 \leq i \leq n, 1 \leq j \leq a_i} X_i^{(k_i^j)}$ form a linear basis of the ring

$$\Lambda_{\bar{a}}(\{s_i^{(j)}\}_{1 \leq i \leq n, 1 \leq j \leq a_i}) := \Bbbk[s_i^{(j)}]_{1 \leq i \leq n, 1 \leq j \leq a_i}^{\mathfrak{S}_{a_1} \times \cdots \times \mathfrak{S}_{a_n}} \tag{2-2}$$

of partially symmetric functions. Here the factor $\mathfrak{S}_{a_1}$ permutes the variables $s_1^{(1)}, \dots, s_1^{(a_1)}$ and so on. We will shorten the above notation to $\Lambda_{\bar{a}}(\boldsymbol{s})$, where we denote $\boldsymbol{s} = \{s_i^{(j)}\}_{1 \leq i \leq n, 1 \leq j \leq a_i}$. In particular, we have

$$U_{\bar{a}}(X_1, \dots, X_n) \in \mathbb{J}_n[\bar{a}] \mathbin{\widehat{\otimes}} \Lambda_{\bar{a}}(\boldsymbol{s}). \tag{2-3}$$

Consider a linear functional $\xi \in \mathbb{J}_n[\bar{a}]^*$ and let 1 be the identity isomorphism on $\Lambda_{\bar{a}}(\boldsymbol{s})$. We have the map $\xi \mathbin{\widehat{\otimes}} 1 : \mathbb{J}_n[\bar{a}] \mathbin{\widehat{\otimes}} \Lambda_{\bar{a}}(\boldsymbol{s}) \to \Lambda_{\bar{a}}(\boldsymbol{s})$ and denote

$$f_\xi := (\xi \mathbin{\widehat{\otimes}} 1)(U_{\bar{a}}(X_1, \dots, X_n)) \in \Lambda_{\bar{a}}(\boldsymbol{s}). \tag{2-4}$$

Set $\mathbb{J}_n^* = \bigoplus_{\bar{a} \in \mathbb{N}^n} \mathbb{J}_n[\bar{a}]^*$. We immediately obtain:

Lemma 2.1. *The map $\gamma_{\bar{a}} : \xi \mapsto f_\xi$ is a linear isomorphism from $\mathbb{J}_n[\bar{a}]^*$ to $\Lambda_{\bar{a}}(\boldsymbol{s})$. The map $\gamma = \bigoplus_{\bar{a} \in \mathbb{N}^n} \gamma_{\bar{a}}$ is an isomorphism from $\mathbb{J}_n^*$ to $\bigoplus_{\bar{a} \in \mathbb{N}^n} \Lambda_{\bar{a}}(\boldsymbol{s})$.*

For vectors $\bar{a}, \bar{b} \in \mathbb{N}^n$ we have a natural map

$$\Delta_{a,b} : \Lambda_{\bar{a}+\bar{b}}(\boldsymbol{s}) \hookrightarrow \Lambda_{\bar{a}}(\boldsymbol{s}) \otimes \Lambda_{\bar{b}}(\{s_i^{(j)}\}_{1 \le i \le n, 1+a_i \le j \le b_i + a_i}) \simeq \Lambda_{\bar{a}}(\boldsymbol{s}) \otimes \Lambda_{\bar{b}}(\boldsymbol{s}) \tag{2-5}$$

because any polynomial symmetric in $s_i^{(1)}, \dots, s_i^{(a_i)}, s_i^{(a_i+1)}, \dots, s_i^{(a_i+b_i)}$ is symmetric separately in $s_i^{(1)}, \dots, s_i^{(a_i)}$ and $s_i^{(a_i+1)}, \dots, s_i^{(a_i+b_i)}$. We choose a concrete isomorphism on the right in (2-5) by identifying the variable $s_i^{(a_i+j)}$ in $\Lambda_{\bar{b}}(\{s_i^{(j)}\}_{1 \le i \le n, 1+a_i \le j \le b_i + a_i})$ with the variable $s_i^{(j)}$ in $\Lambda_{\bar{b}}(\boldsymbol{s})$. Then (2-5) provides a comultiplication map Δ on the space $\bigoplus_{\bar{a} \in \mathbb{N}^n} \Lambda_{\bar{a}}(\boldsymbol{s})$. This comultiplication is obviously coassociative.

Proposition 2.2. *The comultiplication map Δ is dual to the multiplication in the ring $\mathbb{J}_n$ with respect to γ.*

Proof. Consider the comultiplication $\widetilde{\Delta} = (\gamma^{-1} \otimes \gamma^{-1}) \Delta \gamma$ on $\mathbb{J}_n^*$. We need to show that for any $X, Y \in \mathbb{J}_n$ and $\xi \in \mathbb{J}_n^*$ we have $\xi(XY) = \widetilde{\Delta}(\xi)(X \otimes Y)$. By linearity it suffices to consider the case when X and Y are monomials and $\xi(Z) = 1$ for some monomial Z while ξ vanishes on all other monomials. We are to check that $\widetilde{\Delta}(\xi)(X \otimes Y)$ equals 1 if $XY = Z$ and 0 otherwise. This follows directly from the definitions of γ and Δ. ☐

In the following corollary and below we often identify $\mathbb{J}_n[a]^*$ with $\Lambda_{\bar{a}}(\boldsymbol{s})$ via $\gamma_{\bar{a}}$.

Corollary 2.3. (a) *Consider an ideal $I \subset \mathbb{J}_n$. Then the subspace in $\bigoplus_{\bar{a} \in \mathbb{N}^n} \Lambda_{\bar{a}}(\boldsymbol{s})$ which annihilates I is a subcoalgebra.*

(b) *Consider a subalgebra $S \subset \mathbb{J}_n$. Then the subspace in $\bigoplus_{\bar{a} \in \mathbb{N}^n} \Lambda_{\bar{a}}(\boldsymbol{s})$ which annihilates S is a coideal.*

2.2. ***Action of the current Lie algebra on dual components.*** Since the polynomial algebra $R = \Bbbk[X_1, \dots, X_n]$ is $\mathbb{N}^n$-graded, it is acted upon by an n-dimensional abelian Lie algebra $\mathfrak{h} = \langle h_1, \dots, h_n \rangle$. For $r \in R[\bar{a}]$ we have $h_i(p) = a_i p$. The Lie algebra $\mathfrak{h}$ acts by derivations and, applying Definition 1.21, we obtain derivations $h_i^{(k)}$ on $\mathbb{J}_n$ given by

$$h_{i_1}^{(k)}(X_{i_2}^{(j)}) = \delta_{i_1,i_2} X_{i_2}^{(j-k)}.$$

By Lemma 1.23 we have an action of $\mathfrak{h}[s]$ on $\mathbb{J}_n$ with $h_i s^k$ acting as $h_i^{(k)}$. This $\mathfrak{h}[s]$-action is faithful; hence $h_i s^k \mapsto h_i^{(k)}$ provides an embedding $\mathfrak{h}[s] \hookrightarrow \operatorname{Der} \mathbb{J}_n$ and allows us to identify $h_i s^k$ with $h_i^{(k)}$. This action also preserves the $\mathbb{N}^n$-grading, i.e.,

$$\mathfrak{h}[s](\mathbb{J}_n[\bar{a}]) \subset \mathbb{J}_n[\bar{a}].$$

We may extend the action of $\mathfrak{h}[s]$ to $\mathbb{J}_n[\![s]\!]$ (coefficientwise) and $\mathbb{J}_n \mathbin{\widehat{\otimes}} \Lambda_{\bar{a}}(\boldsymbol{s})$ (by acting on the first factor). In particular, we have

$$h_{i_1}^{(k)} X_{i_2}(s) = \delta_{i_1,i_2} s^k X_{i_2}(s). \tag{2-6}$$

Lemma 2.4.
$$h_i^{(k)}(U_{\bar{a}}(X_1, \dots, X_n)) = U_{\bar{a}}(X_1, \dots, X_n) \sum_{j=1}^{a_i} (s_i^{(j)})^k.$$

Proof. This follows from (2-6) by the Leibniz rule. □

Proposition 2.5. *The action of* $\mathfrak{h}[s]$ *on* $\Lambda_{\bar{a}}(\boldsymbol{s})$ *dual* (*with respect to* $\gamma_{\bar{a}}$) *to the* $\mathfrak{h}[s]$*-action on* $\mathbb{J}_n[\bar{a}]$ *is given by*

$$h_i^{(k)}(f) = -\sum_{j=1}^{a_i} (s_i^{(j)})^k f$$

for any $f \in \Lambda_{\bar{a}}(\boldsymbol{s})$.

Proof. Choose $f \in \Lambda_{\bar{a}}(\boldsymbol{s})$ and let $\xi = \gamma_{\bar{a}}^{-1}(f)$. Then for the dual $\mathfrak{h}[s]$-action we have

$$\begin{aligned} h_i^{(k)}(f) &= (h_i^{(k)}(\xi) \mathbin{\widehat{\otimes}} 1)(U_{\bar{a}}(X_1, \dots, X_n)) = -(\xi \mathbin{\widehat{\otimes}} 1)(h_i^{(k)}(U_{\bar{a}}(X_1, \dots, X_n))) \\ &= -(\xi \mathbin{\widehat{\otimes}} 1)(U_{\bar{a}}(X_1, \dots, X_n)) \sum_{j=1}^{a_i} (s_i^{(j)})^k = -\sum_{j=1}^{a_i} (s_i^{(j)})^k f. \end{aligned} \tag{2-7}$$

This completes the proof. □

The action (1-9) of the Lie algebra $\mathrm{Der}^c\, \mathbb{k}[\![s]\!]$ also preserves graded components of the arc ring $\mathbb{J}_n$.

Proposition 2.6. *The action of* $\mathrm{Der}^c\, \mathbb{k}[\![s]\!]$ *on* $\Lambda_{\bar{a}}(\boldsymbol{s})$ *dual to the action on* $\mathbb{J}_n[\bar{a}]$ *is given by*

$$d_k(f) = -\Bigg(\sum_{1 \le i \le n, 1 \le j \le a_i} (s_i^{(j)})^{k+1} \frac{\partial}{\partial s_i^{(j)}} \Bigg) f.$$

Proof. We have

$$d_k(X_i(s_i^{(j)})) = (s_i^{(j)})^{k+1} \frac{\partial}{\partial s_i^{(j)}} X_i(s_i^{(j)}).$$

Therefore, by the Leibniz rule,

$$d_k(U_{\bar{a}}(X_1, \dots, X_n)) = \Bigg(\sum_{1 \le i \le n, 1 \le j \le a_i} (s_i^{(j)})^{k+1} \frac{\partial}{\partial s_i^{(j)}} \Bigg) U_{\bar{a}}(X_1, \dots, X_n).$$

The proof is now completed similarly to the previous proposition. □

2.3. ***Arcs and quadratic monomial ideals.*** Consider a set of pairs

$$\pi = \{(\alpha_1, \beta_1), \dots, (\alpha_p, \beta_p)\}, \quad \alpha_i, \beta_i \in [n],\ \alpha_i < \beta_i$$

and the monomial ideal $\mathcal{M}_\pi = \langle X_{\alpha_i} X_{\beta_i} \rangle_{i=1,\dots,p}$ in $\mathbb{k}[X_1, \dots, X_n]$. The goal of this subsection is to describe the space dual to the ring $J^\infty(\mathbb{k}[X_1, \dots, X_n]/\mathcal{M}_\pi)$ in terms of symmetric functions.

We have the surjection $\rho : \mathbb{J}_n \twoheadrightarrow J^\infty(\Bbbk[X_1, \dots, X_n]/\mathcal{M}_\pi)$ with kernel generated by coefficients of the series

$$X_{\alpha_i}(s)X_{\beta_i}(s), \quad (\alpha_i, \beta_i) \in \pi. \tag{2-8}$$

This kernel is $\mathbb{N}^n$-homogeneous; hence, the ring $J^\infty(\Bbbk[X_1, \dots, X_n]/\mathcal{M}_\pi)$ is $\mathbb{N}^n$-graded and ρ restricts to $\mathbb{N}^n$-homogeneous surjections $\rho[\bar{a}]$ from $\mathbb{J}_n[\bar{a}]$ to $J^\infty(\Bbbk[X_1, \dots, X_n]/\mathcal{M}_\pi)[\bar{a}]$. Therefore, the (graded) dual of the homogeneous component $J^\infty(\Bbbk[X_1, \dots, X_n]/\mathcal{M}_\pi)[\bar{a}]$ can be naturally viewed as a subspace in $\mathbb{J}_n[\bar{a}]^*$.

The kernel of ρ is $\mathfrak{h}[s]$ invariant. Therefore, the map ρ is a homomorphism of $\mathfrak{h}[s]$-modules and thus the dual of the homogeneous component $J^\infty(\Bbbk[X_1, \dots, X_n]/\mathcal{M}_\pi)[\bar{a}]$ has a structure of a $\mathfrak{h}[s]$-submodule.

Proposition 2.7. *The subspace*

$$\gamma_{\bar{a}}(J^\infty(\Bbbk[X_1, \dots, X_n]/\mathcal{M}_\pi)[\bar{a}]^*) \subset \Lambda_{\bar{a}}(\boldsymbol{s})$$

is the principal ideal generated by

$$\prod_{\substack{(\alpha,\beta)\in\pi \\ 1\le i\le a_\alpha, 1\le j\le a_\beta}} (s_\alpha^{(i)} - s_\beta^{(j)}). \tag{2-9}$$

This principal ideal is a cyclic $\mathfrak{h}[s]$-module generated by (2-9) *that is isomorphic to $\Lambda_{\bar{a}}(\boldsymbol{s})$.*

Proof. In view of (2-8), the kernel of $\rho[\bar{a}]$ is the linear span of coefficients of the series

$$U_{\bar{a}-e_\alpha-e_\beta}(X_1, \dots, X_n)X_\alpha(t)X_\beta(t) = U_{\bar{a}}(X_1, \dots, X_n)|_{s_\alpha^{(a_\alpha)}\mapsto t,\, s_\beta^{(a_\beta)}\mapsto t} \tag{2-10}$$

for $(\alpha, \beta) \in \pi$, where ϵ_α is the α-th basis vector in $\mathbb{N}^n$. For $\xi \in \mathbb{J}_n[\bar{a}]^*$ we have

$$\begin{aligned}(\xi \mathbin{\widehat{\otimes}} 1)(U_{\bar{a}}(X_1, \dots, X_n)|_{s_\alpha^{(a_\alpha)}\mapsto t, s_\beta^{(a_\beta)}\mapsto t}) &= (\xi \mathbin{\widehat{\otimes}} 1)(U(X_1, \dots, X_n)_{\bar{a}})|_{s_\alpha^{(a_\alpha)}\mapsto t, s_\beta^{(a_\beta)}\mapsto t} \\ &= \gamma_{\bar{a}}(\xi)|_{s_\alpha^{(a_\alpha)}\mapsto t, s_\beta^{(a_\beta)}\mapsto t}.\end{aligned}$$

Thus, $\xi(\ker \rho[\bar{a}]) = 0$ if and only if $\gamma_{\bar{a}}(\xi)|_{s_\alpha^{(a_\alpha)}\mapsto t, s_\beta^{(a_\beta)}\mapsto t} = 0$. By symmetry, $\gamma_{\bar{a}}(\xi)$ must then be divisible by (2-9). The last claim follows from (2-7). This completes the proof. □

We also need the following generalization of this proposition. For each pair of indices (α, β), $1 \le \alpha < \beta \le n$ choose a number $k(\alpha, \beta) \in \mathbb{N}$. Denote the collection of these numbers by $\bar{k}$. Consider the ideal $\mathcal{I}_{\bar{k}}$ generated by the coefficients of

$$\left\langle X_{\alpha_i}(s)\frac{\partial^{k'} X_{\beta_i}(s)}{\partial s^{k'}}\right\rangle, \quad 1 \le \alpha < \beta \le n, \quad 0 \le k' < k(\alpha, \beta). \tag{2-11}$$

We denote by ρ the natural surjection $\mathbb{J}_n \twoheadrightarrow \mathbb{J}_n/\mathcal{I}_{\bar{k}}$ and let $\rho[\bar{a}] : \mathbb{J}_n[\bar{a}] \twoheadrightarrow (\mathbb{J}_n/\mathcal{I}_{\bar{k}})[\bar{a}]$ be its $\mathbb{N}^n$-homogeneous components. As before $\mathcal{I}_{\bar{k}}$ is $\mathfrak{h}[s]$-invariant. Therefore, $\rho[\bar{a}]$ is a map of $\mathfrak{h}[s]$-modules. As in the previous case we identify $(\mathbb{J}_n/\mathcal{I}_{\bar{k}})[\bar{a}]^*$ with a principal ideal in $\Lambda_{\bar{a}}(\boldsymbol{s})$.

Proposition 2.8. *The subspace $\gamma_{\bar a}((\mathbb{J}_n/\mathcal{I}_{\bar k})[\bar a]^*) \subset \Lambda_{\bar a}(\boldsymbol{s})$ is the principal ideal generated by*

$$\prod_{\substack{1\le\alpha<\beta\le n\\ 1\le i\le a_\alpha,1\le j\le a_\beta}} (s_\alpha^{(i)} - s_\beta^{(j)})^{k(\alpha,\beta)}. \tag{2-12}$$

This principal ideal is a cyclic $\mathfrak{h}[s]$-module generated by (2-12) *that is isomorphic to $\Lambda_{\bar a}(\boldsymbol{s})$.*

Proof. By definition, $\ker\rho[\bar a]$ is the linear span of coefficients of the series

$$U_{\bar a-e_\alpha-e_\beta}(X_1,\dots,X_n)X_\alpha(t)\frac{\partial^{k'}X_\beta(t)}{\partial t^{k'}} = \frac{\partial^{k'}U_{\bar a}(X_1,\dots,X_n)}{(\partial s_\beta^{(a_\beta)})^{k'}}\bigg|_{s_\alpha^{(a_\alpha)}\mapsto t,\, s_\beta^{(a_\beta)}\mapsto t}, \tag{2-13}$$

where $0\le k'<k$. We have, for any $\xi\in\mathbb{J}_n[\bar a]^*$,

$$(\xi\mathbin{\widehat{\otimes}}1)\bigg(\frac{\partial^{k'}U_{\bar a}(X_1,\dots,X_n)}{(\partial s_\beta^{(a_\beta)})^{k'}}\bigg|_{s_\alpha^{(a_\alpha)}\mapsto t,s_\beta^{(a_\beta)}\mapsto t}\bigg) = \frac{\partial^{k'}\xi(U_{\bar a}(X_1,\dots,X_n))}{(\partial s_\beta^{(a_\beta)})^{k'}}\bigg|_{s_\alpha^{(a_\alpha)}\mapsto t,s_\beta^{(a_\beta)}\mapsto t}$$
$$= \frac{\partial^{k'}\gamma_{\bar a}(\xi)}{(\partial s_\beta^{(a_\beta)})^{k'}}\bigg|_{s_\alpha^{(a_\alpha)}\mapsto t,s_\beta^{(a_\beta)}\mapsto t}.$$

Thus, $\xi(\ker\rho[\bar a])=0$ if and only if for any $\alpha<\beta$ and $0\le k'<k(\alpha,\beta)$ we have

$$\frac{\partial^{k'}\gamma_{\bar a}(\xi)}{(\partial s_\beta^{(a_\beta)})^{k'}}\bigg|_{s_\alpha^{(a_\alpha)}\mapsto t,s_\beta^{(a_\beta)}\mapsto t} = 0.$$

By symmetry, such $\gamma_{\bar a}(\xi)$ are precisely those divisible by $(s_\alpha^{(i)}-s_\beta^{(j)})^{k(\alpha,\beta)}$ for any $\alpha<\beta$, $1\le i\le a_\alpha$ and $1\le j\le a_\beta$. The last claim is straightforward. □

Remark 2.9. Consider a finite set $S\subset\mathbb{N}^n$ and the ideal $I_S\subset\Bbbk[X_1,\dots,X_n]$ generated by monomials $X^{\bar b}$, $\bar b\in S$. Then the subspace $\gamma_{\bar a}(J^\infty(\Bbbk[X_1,\dots,X_n]/I_S)[\bar a]^*)\subset\Lambda_{\bar a}(\boldsymbol{s})$ is the ideal

$$\{f \text{ such that } f|_{s_1^{(1)},\dots,s_1^{(b_1)},\dots,s_n^{(1)},\dots,s_n^{(b_n)}\mapsto t} = 0 \text{ for all } \bar b\in S\}.$$

This generalizes Propositions 2.7 and 2.8, the proof being similar. We note that in this general case the dual space is still an ideal in $\Lambda_{\bar a}(\boldsymbol{s})$ which, however, need not be principal.

3. Arc rings of toric rings

3.1. ***Graded components of toric rings.*** Consider a normal lattice polytope $P\subset\mathbb{R}^n_{\ge0}$ and the corresponding toric ring $R(P)$. Note that $R(P)$ is invariant under shifts of P so we do not lose generality by assuming that P lies in the positive orthant.

We have an embedding

$$\eta: R(P)\hookrightarrow\Bbbk[z_1,\dots,z_n,w]$$

with $\eta(Y_{\bar\alpha})=z^{\bar\alpha}w$ for $\bar\alpha\in P\cap\mathbb{Z}^n$. We study the corresponding map of arc rings

$$J^\infty(\eta): J^\infty R(P)\to J^\infty(\Bbbk[z_1,\dots,z_n,w]).$$

By definition, applying $J^\infty(\eta)$ coefficientwise we have $J^\infty(\eta)(Y_{\bar\alpha}(s)) = z(s)^{\bar\alpha} w(s)$. Let $\{\bar\alpha^1, \dots, \bar\alpha^m\}$ be the set of integer points in P. We have a surjection

$$\tau : \Bbbk[X_{\bar\alpha^1}, \dots, X_{\bar\alpha^m}] \twoheadrightarrow R(P),$$

where $\tau(X_{\bar\alpha^i}) = Y_{\bar\alpha^i}$. We also have the map

$$\eta \circ \tau : \Bbbk[X_{\bar\alpha^1}, \dots, X_{\bar\alpha^m}] \to \Bbbk[z_1, \dots, z_n, w]$$

and, for a vector $\bar r = (r_i) \in \mathbb{N}^m$,

$$\eta \circ \tau(X^{\bar r}) = z^{\sum_{i=1}^m (r_i \bar\alpha^i)} w^{\sum_{i=1}^m r_i}.$$

Thus, $\eta \circ \tau$ maps monomials in the $X_{\bar\alpha^i}$ to monomials in the z_i and w. For $\bar a \in \mathbb{N}^n$, $L \in \mathbb{N}$ we define

$$\mathcal{R}(\bar a, L) = \left\{ \bar r \in \mathbb{N}^m \;\middle|\; \sum_{i=1}^m r_i \bar\alpha^i = \bar a \text{ and } \sum_{i=1}^m r_i = L \right\}.$$

Then $\eta \circ \tau(X^{\bar r}) = z^{\bar a} w^L$ if and only if $\bar r \in \mathcal{R}(\bar a, L)$.

The map $J^\infty(\tau) : \mathbb{J}_m \to J^\infty R(P)$ is surjective, since $J^\infty(\tau)(X^{(j)}_{\bar\alpha^i}) = Y^{(j)}_{\bar\alpha^i}$ (more generally, J^∞ is easily seen to preserve epimorphisms). Therefore, we have

$$J^\infty(\eta)(J^\infty R(P)) = J^\infty(\eta \circ \tau)(\mathbb{J}_m) \subset J^\infty(\Bbbk[z_1, \dots, z_n, w])$$

and we have a surjective map

$$\nu : J^\infty_{\text{red}}(R(P)) \twoheadrightarrow J^\infty(\eta)(J^\infty R(P)) = J^\infty(\eta \circ \tau)(\mathbb{J}_m). \tag{3-1}$$

By Corollary 1.12, ν is an isomorphism.

For the rest of this section we fix a monomial order $\prec$ on $\Bbbk[X_{\bar\alpha^1}, \dots, X_{\bar\alpha^m}]$ as well as the corresponding order on $\mathbb{N}^m$. As discussed in Section 1.5, we have two $(\mathbb{N}^m, \prec)$-filtrations with components $\mathbb{J}_m[\prec \bar r]$ and $\mathbb{J}_m[\preceq \bar r]$. Since both $J^\infty_{\text{red}}(R(P))$ and $J^\infty(\eta \circ \tau)(\mathbb{J}_m)$ are surjective images of $\mathbb{J}_m$ they admit induced $(\mathbb{N}^m, \prec)$-filtrations which are seen to be identified by the isomorphism ν. Since ν respects the grading grad, we obtain:

Lemma 3.1. *For any $d \in \mathbb{N}$ and $\bar r \in \mathbb{N}^m$ the map ν induces an isomorphism of vector spaces*

$$\mathrm{gr}_\prec J^\infty_{\text{red}} R(P)[\bar r]^{(d)} \simeq \mathrm{gr}_\prec J^\infty(\eta \circ \tau)(\mathbb{J}_m)[\bar r]^{(d)}.$$

3.2. ***Dual of the inclusion map.*** Retaining notation from the previous subsection consider $\bar r \in \mathcal{R}(\bar a, L) \subset \mathbb{N}^m$ for some $\bar a \in \mathbb{N}^n$ and $L \in \mathbb{N}$. Consider the subspace $A_{\bar r} \subset J^\infty(\Bbbk[z_1, \dots, z_n, w])$ spanned by elements of the form

$$\eta(Y_{\bar\alpha^1})^{(j_1^1)} \cdots \eta(Y_{\bar\alpha^1})^{(j_{r_1}^1)} \cdots \eta(Y_{\bar\alpha^m})^{(j_1^m)} \cdots \eta(Y_{\bar\alpha^m})^{(j_{r_m}^m)} = J^\infty(\eta)\left(\prod_{i=1}^m \prod_{k=1}^{r_m} Y_{\bar\alpha^i}^{(j_k^i)} \right)$$

for all $j_u^v \in \mathbb{N}$. This subspace coincides with $J^\infty(\eta \circ \tau)(\mathbb{J}_m[\bar r])$.

In this subsection we study the dual map to the inclusion

$$\varphi_{\bar r} : A_{\bar r} \hookrightarrow J^\infty(\Bbbk[z_1, \dots, z_n, w])[\bar a, L], \tag{3-2}$$

where on the right we have a component of the natural $\mathbb{N}^{n+1}$-grading.

We consider the following series in variables $t_i^{(j)}$ with $1 \le i \le m$ and $1 \le j \le r_i$:

$$\begin{aligned} V_{\bar r} &:= \eta(Y_{\bar\alpha^1})(t_1^{(1)}) \cdots \eta(Y_{\bar\alpha^1})(t_1^{(r_1)}) \cdots \eta(Y_{\bar\alpha^m})(t_m^{(1)}) \cdots \eta(Y_{\bar\alpha^m})(t_m^{(r_m)}) \\ &= (z^{\bar\alpha^1} w)(t_1^{(1)}) \cdots (z^{\bar\alpha^1} w)(t_1^{(r_1)}) \cdots (z^{\bar\alpha^m} w)(t_m^{(1)}) \cdots (z^{\bar\alpha^m} w)(t_m^{(r_m)}). \end{aligned} \tag{3-3}$$

From the definitions we have:

Proposition 3.2. *The space $A_{\bar r}$ is spanned by the coefficients of $V_{\bar r}$ as of a series in the $t_i^{(j)}$.*

By Lemma 2.1 we obtain a linear isomorphism

$$\gamma_{\bar a, L} : J^\infty(\Bbbk[z_1, \dots, z_n, w])[\bar a, L]^* \to \Lambda_{\bar a}(\boldsymbol{s}) \otimes \Lambda_L(\{s_w^{(j)}\}_{j=1,\dots,L}).$$

We will denote the right-hand side above by $\Lambda_{\bar a, L}(\boldsymbol{s})$.

Recall that $\bar a = \sum_{j=1}^m r_j \bar\alpha^j$ and $L = \sum_{i=1}^m r_i$. Consider the homomorphism of polynomial algebras

$$\varphi_{\bar r}^\vee : \Bbbk[s_i^{(j)}, s_w^{(j')}]_{\substack{1\le i\le n \\ 1\le j\le a_i \\ 1\le j'\le L}} \to \Bbbk[t_i^{(j)}]_{\substack{1\le i\le m \\ 1\le j\le r_i}}$$

given by

$$\varphi_{\bar r}^\vee(s_i^{(j)}) = \varphi_{\bar r}^\vee(s_w^{(j')}) = \begin{cases} t_1^{(1)} & \text{if } j \in [1, \alpha_i^1] \text{ and } j' = 1, \\ \vdots & \\ t_1^{(r_1)} & \text{if } j \in [(r_1 - 1)\alpha_i^1 + 1, r_1\alpha_i^1] \text{ and } j' = r_1, \\ t_2^{(1)} & \text{if } j \in [r_1\alpha_i^1 + 1, r_1\alpha_i^1 + \alpha_i^2] \text{ and } j' = r_1 + 1, \\ \vdots & \\ t_2^{(r_2)} & \text{if } j \in [r_1\alpha_i^1 + (r_2 - 1)\alpha_i^2 + 1, r_1\alpha_i^1 + r_2\alpha_i^2] \text{ and } j' = r_1 + r_2, \\ \vdots & \end{cases} \tag{3-4}$$

In other words, for a given $1 \le i \le n$ we divide the tuple $s_i^{(1)}, \dots, s_i^{(a_i)}$ into L groups of consecutive elements so that the first r_1 groups contain α_i^1 elements, the next r_2 groups contain α_i^2 elements, etc. Then $\varphi_{\bar r}^\vee$ maps the elements in the first r_1 groups to, respectively, $t_1^{(1)}, \dots, t_1^{(r_1)}$, the elements in the next r_2 groups to, respectively, $t_2^{(1)}, \dots, t_2^{(r_2)}$ and so on. As for the variables $s_w^{(1)}, \dots, s_w^{(L)}$, the first r_1 of them are mapped to, respectively, $t_1^{(1)}, \dots, t_1^{(r_1)}$, the next r_2 are mapped to, respectively, $t_2^{(1)}, \dots, t_2^{(r_2)}$, etc.

Remark 3.3. The map $\varphi_{\bar r}^\vee$ can be described in the following way. The ring $\Lambda_{\bar a, L}(\boldsymbol{s})$ is generated by power sums

$$p_k(s_i^{(1)}, \dots, s_i^{(a_i)}) = \sum_{j=1}^{a_i} (s_i^{(j)})^k \quad \text{and} \quad p_k(s_w^{(1)}, \dots, s_w^{(L)}) = \sum_{j=1}^{L} (s_w^{(j)})^k.$$

We have

$$\varphi_{\bar r}^\vee(p_k(s_i^{(1)},\dots,s_i^{(a_i)}))=\sum_{j=1}^m \alpha_i^j\, p_k(t_j^{(1)},\dots,t_j^{(r_j)}) \quad\text{and}\quad \varphi_{\bar r}^\vee(p_k(s_w^{(1)},\dots,s_w^{(L)}))=\sum_{j=1}^m p_k(t_j^{(1)},\dots,t_j^{(r_j)}).$$

In particular, $\varphi_{\bar r}^\vee(\Lambda_{\bar a,L}(\boldsymbol{s}))$ is generated as a ring by the elements $\sum_{j=1}^m \alpha_i^j\, p_k(t_j^{(1)},\dots,t_j^{(r_j)})$, $i=1,\dots,n$ and $\sum_{j=1}^m p_k(t_j^{(1)},\dots,t_j^{(r_j)})$.

Example 3.4. Let P be the convex hull of $\{(0,0),(1,2),(2,1)\}$ and let $\alpha_1=(0,0)$, $\alpha_2=(1,1)$, $\alpha_3=(2,1)$, $\alpha_4=(1,2)$. Consider $\bar r=(1,1,1,1)\in\mathcal{R}((4,4),4)$. Then $\varphi_{\bar r}^\vee$ is given by

$$s_w^{(1)}\mapsto t_1^{(1)},\quad s_w^{(2)},s_1^{(1)},s_2^{(1)}\mapsto t_2^{(1)},\quad s_w^{(3)},s_1^{(2)},s_1^{(3)},s_2^{(2)}\mapsto t_3^{(1)},\quad s_w^{(4)},s_1^{(4)},s_2^{(3)},s_2^{(4)}\mapsto t_4^{(1)}$$

and we have

$$\begin{aligned}
\varphi_{\bar r}^\vee(p_k(s_w^{(1)},s_w^{(2)},s_w^{(3)},s_w^{(4)}))&=p_k(t_1^{(1)})+p_k(t_2^{(1)})+p_k(t_3^{(1)})+p_k(t_4^{(1)}),\\
\varphi_{\bar r}^\vee(p_k(s_1^{(1)},s_1^{(2)},s_1^{(3)},s_1^{(4)}))&=p_k(t_2^{(1)})+2p_k(t_3^{(1)})+p_k(t_4^{(1)}),\\
\varphi_{\bar r}^\vee(p_k(s_2^{(1)},s_2^{(2)},s_2^{(3)},s_2^{(4)}))&=p_k(t_2^{(1)})+p_k(t_3^{(1)})+2p_k(t_4^{(1)}).
\end{aligned}$$

Lemma 3.5. *The image $\varphi_{\bar r}^\vee(\Lambda_{\bar a,L}(\boldsymbol{s}))$ is contained in the subring $\Lambda_{\bar r}(\boldsymbol{t})$.*

Proof. The ring $\Lambda_{\bar a,L}(\boldsymbol{s})$ decomposes into a product of its subspaces as

$$\Lambda_{\bar a,L}(\boldsymbol{s})=\Lambda_{a_1}(\{s_1^{(j)}\}_{j=1,\dots,a_1})\cdots\Lambda_{a_n}(\{s_n^{(j)}\}_{j=1,\dots,a_n})\Lambda_L(\{s_w^{(j)}\}_{j=1,\dots,L}).$$

The image under $\varphi_{\bar r}^\vee$ of a polynomial in $\Lambda_{a_i}(\{s_i^{(j)}\}_{j=1,\dots,a_i})$ is evidently symmetric in $t_\ell^{(1)},\dots,t_\ell^{(r_\ell)}$ for each ℓ. The same holds for any polynomial in the image of $\Lambda_L(\{s_w^{(j)}\}_{j=1,\dots,L})$ and the lemma follows. □

Recall the inclusion map $\varphi_{\bar r}$ defined in (3-2). We denote by

$$\varphi_{\bar r}^*: J^\infty(\Bbbk[z_1,\dots,z_n,w])[\bar a,L]^*\to A_{\bar r}^*$$

the (graded) dual map for this inclusion.

Theorem 3.6. *The annihilator of $A_{\bar r}$ in $J^\infty(\Bbbk[z_1,\dots,z_n,w])[\bar a,L]^*$ is the subspace $\ker(\varphi_{\bar r}^\vee\gamma_{\bar a,L})$. In other words, there exists a linear isomorphism*

$$\varepsilon_{\bar r}: A_{\bar r}^*\xrightarrow{\sim}\varphi_{\bar r}^\vee(\Lambda_{\bar a,L}(\boldsymbol{s}))$$

such that $\varepsilon_{\bar r}\varphi_{\bar r}^*=\varphi_{\bar r}^\vee\gamma_{\bar a,L}$.

Proof. The two claims are equivalent because the annihilator of $A_{\bar r}$ is $\ker\varphi_{\bar r}^*$, i.e., both state that $\varphi_{\bar r}^*$ and $\varphi_{\bar r}^\vee\gamma_{\bar a,L}$ have the same kernel.

Recall the expression $U_{\bar a}(z_1,\dots,z_n)$ defined in (2-1) and consider

$$U=U_{\bar a}(z_1,\dots,z_n)w(s_w^{(1)})\dots w(s_w^{(L)})\in J^\infty(\Bbbk[z_1,\dots,z_n,w])[\bar a,L]\,\widehat{\otimes}\,\Lambda_{\bar a,L}(\boldsymbol{s}).$$

By definition we have $(1\,\widehat{\otimes}\,\varphi_{\bar r}^\vee)(U)=V_{\bar r}$.

Consider a functional $\xi \in J^\infty(\Bbbk[z_1,\dots,z_n,w])[\bar a, L]^*$. We are to show that $\xi(A_{\bar r}) = 0$ if and only if $\varphi^\vee_{\bar r}\gamma_{\bar a,L}(\xi) = 0$. Note that we may view $V_{\bar r}$ as an element of

$$J^\infty(\Bbbk[z_1,\dots,z_n,w])[\bar a, L]\,\widehat{\otimes}\,\Lambda_{\bar r}(\boldsymbol{t})$$

and that, in view of Proposition 3.2, $\xi(A_{\bar r}) = 0$ if and only if $(\xi\,\widehat{\otimes}\,1)(V_{\bar r}) = 0$. However,

$$(\xi\,\widehat{\otimes}\,1)(V_{\bar r}) = (\xi\,\widehat{\otimes}\,1)(1\,\widehat{\otimes}\,\varphi^\vee_{\bar r})(U) = (1\,\widehat{\otimes}\,\varphi^\vee_{\bar r})(\xi\,\widehat{\otimes}\,1)(U) = \varphi^\vee_{\bar r}(\gamma_{\bar a,L}(\xi)). \qquad \square$$

Recall that the monomial order $\prec$ induces a filtration on and an associated graded for $J^\infty(\eta\circ\tau)(\mathbb{J}_m)$. We can write the filtration components as

$$J^\infty(\eta\circ\tau)(\mathbb{J}_m)[\preceq \bar r] = \sum_{\bar r'\preceq \bar r} A_{\bar r}$$

and similarly for $J^\infty(\eta\circ\tau)(\mathbb{J}_m)[\prec \bar r]$. This lets us write the components of the associated graded as

$$\mathrm{gr}_\prec J^\infty(\eta\circ\tau)(\mathbb{J}_m)[\bar r] = A_{\bar r}\Big/\Bigg(A_{\bar r}\cap \sum_{\bar r'\prec\bar r,\,\bar r'\in\mathcal{R}(\bar a,L)} A_{\bar r'}\Bigg).$$

We can now apply Theorem 3.6 to identify the (graded) dual space $\mathrm{gr}_\prec J^\infty(\eta\circ\tau)(\mathbb{J}_m)[\bar r]^*$ with an image of $\varphi^\vee_{\bar r}$.

Corollary 3.7. *We have an isomorphism of* grad-*graded spaces*

$$\mathrm{gr}_\prec J^\infty(\eta\circ\tau)(\mathbb{J}_m)[\bar r]^* \simeq \varphi^\vee_{\bar r}\Bigg(\bigcap_{\bar r'\prec\bar r,\,\bar r'\in\mathcal{R}(\bar a,L)} \ker(\varphi^\vee_{\bar r'})\Bigg),$$

where ker *denotes the kernel in* $\Lambda_{\bar a,L}(\boldsymbol{s})$.

3.3. ***Current algebra action and $\mathcal{A}$-freeness.*** In this subsection we denote $R = R(P)$. Recall that each ring $S \in \{R, J^\infty(R), J^\infty_{\mathrm{red}}(R)\}$ possesses an $\mathbb{N}^{n+1}$-grading with components $S[\bar a, L]$. The ring R admits a natural action of a one-dimensional Lie algebra $\mathfrak{h} = \Bbbk h$ given by $hr = Lr$ for $r \in R[\bar a, L]$. In view of Lemma 1.23 we obtain an $\mathfrak{h}[s]$-action on $J^\infty(R)$ given by $hs^k(r^{(j)}) = (hr)^{(j-k)}$. By Lemma 1.16 we have an induced $\mathfrak{h}[s]$-action on $J^\infty_{\mathrm{red}}(R)$. Clearly, the components $J^\infty_{\mathrm{red}}(R)[\bar a, L]$ are preserved by the action of $\mathfrak{h}[s]$.

Next, consider the algebra $\Bbbk[z_1,\dots,z_n,w]$, it also admits an $\mathfrak{h}$-action given by $h(M) = LM$ for $M = z_{i_1}\dots z_{i_N}w^L$. By Lemma 1.23 we obtain an action of $\mathfrak{h}[s]$ on $J^\infty\Bbbk[z_1,\dots,z_n,w]$. Now recall the isomorphism (3-1),

$$\nu : J^\infty_{\mathrm{red}}(R) \to J^\infty(\eta\circ\tau)(\mathbb{J}_m).$$

Since the right-hand side above is a subspace in $J^\infty\Bbbk[z_1,\dots,z_n,w]$, we will view ν as an embedding into the latter. It is then seen from the definitions that ν is a homomorphism of $\mathfrak{h}[s]$-algebras with respect to the $\mathfrak{h}[s]$-action on $J^\infty_{\mathrm{red}}(R)$ defined in the previous paragraph.

Lemma 3.8. *Every subspace $A_{\bar r} \subset \nu(J^\infty_{\mathrm{red}}(R))$ is preserved by the $\mathfrak{h}[s]$-action.*

Proof. For any $\bar{\alpha}^i$ we have $hs^k((z^{\bar{\alpha}^i} w)^{(j)}) = (z^{\bar{\alpha}^i} w)^{(j-k)}$. Hence, applying hs^k coefficientwise we have $hs^k((z^{\bar{\alpha}^i} w)(t)) = t^k (z^{\bar{\alpha}^i} w)(t)$. Therefore,

$$hs^k(V_{\bar{r}}) = \left(\sum_{i=1}^{m} \sum_{\ell=1}^{r_i} (t_i^{(\ell)})^k\right) V_{\bar{r}}. \tag{3-5}$$

Proposition 3.2 now provides the claim. □

Let us fix $L \in \mathbb{N}$ and $\bar{a} \in \mathbb{N}^n$.

Corollary 3.9. *For any $\bar{r} \in \mathcal{R}(\bar{a}, L)$ we have an $\mathfrak{h}[s]$-action on the subquotient*

$$\sum_{\bar{r}' \preceq \bar{r}, \bar{r}' \in \mathcal{R}(\bar{a}, L)} A_{\bar{r}'} \Big/ \sum_{\bar{r}' \prec \bar{r}, \bar{r}' \in \mathcal{R}(\bar{a}, L)} A_{\bar{r}'}.$$

Recall that for every L the space $J^{\infty}_{\mathrm{red}}(R)[\bar{a}, L]$ is graded by grad and the homogeneous components are finite-dimensional. We denote by $J^{\infty}_{\mathrm{red}}(R)[\bar{a}, L]^*$ the graded dual space. We denote by $\mathcal{A}_L \subset U(\mathfrak{h}[s])$ the subalgebra generated by $hs, \ldots, hs^L$. We will be interested in situations when the action of $\mathcal{A}_L$ on $J^{\infty}_{\mathrm{red}}(R)[\bar{a}, L]^*$ is free for all $\bar{a}$ and L.

Remark 3.10. An analogous freeness property holds for the homogeneous coordinate ring of the Plücker embedding and it has important representation theoretic meaning (see [Fourier and Littelmann 2007; Feigin and Makedonskyi 2020; Kato 2018]). This property also holds for the Veronese curve (see [Dumanski and Feigin 2023]). It is interesting to investigate the graded rank of these free modules. For example, in the case of the Plücker embedding they are equal to Macdonald polynomials for $t = 0$; see [Ion 2003; Naoi 2012].

For $\bar{r} \in \mathcal{R}(\bar{a}, L)$ we define the following $\mathfrak{h}[s]$-action on the algebra $\Lambda_{\bar{r}}(\boldsymbol{t})$:

$$hs^{\ell}(f) = \left(\sum_{i=1}^{m} \sum_{k=1}^{r_i} (t_i^{(k)})^{\ell}\right) f. \tag{3-6}$$

Proposition 3.11. *The space $\varphi^{\vee}_{\bar{r}}(\Lambda_{\bar{a},L}(\boldsymbol{s}))$ is invariant under the action* (3-6)*. The action on this subspace is dual to the action of $\mathfrak{h}[s]$ on $A_{\bar{r}}$ with respect to the isomorphism $\varepsilon_{\bar{r}}$* (*Theorem 3.6*).

Proof. To prove the first claim note that the first factor in the right-hand side of (3-6) lies in $\varphi^{\vee}_{\bar{r}}(\Lambda_{\bar{a},L}(\boldsymbol{s}))$ by Remark 3.3. The second claim follows from (3-5). □

Lemma 3.12. *Assume that the natural action of $\mathcal{A}_L$ is free on $\varphi^{\vee}_{\bar{r}}\big(\bigcap_{\bar{r}' \prec \bar{r}, \bar{r}' \in \mathcal{R}(\bar{a}, L)} \ker(\varphi^{\vee}_{\bar{r}'})\big)$ for every $\bar{r} \in \mathcal{R}(\bar{a}, L)$. Then it is free on the space $J^{\infty}(\eta \circ \tau)(\mathbb{J}_m)[\bar{a}, L]^*$.*

Proof. One sees that the isomorphism in Corollary 3.7 is one of $\mathfrak{h}[s]$-modules; thus $J^{\infty}(\eta \circ \tau)(\mathbb{J}_m)[\bar{a}, L]^*$ has a filtration with free subquotients. However, free modules are projective; hence $J^{\infty}(\eta \circ \tau)(\mathbb{J}_m)[\bar{a}, L]^*$ is isomorphic to the direct sum of its subquotients. □

We will need the following general fact about symmetric polynomials.

Lemma 3.13. *Let $n = n_1 + \cdots + n_a$. The ring $\Bbbk[x_1, \dots, x_n]^{\mathfrak{S}_{n_1}\times\cdots\times\mathfrak{S}_{n_a}}$ is free of q-rank $\binom{n}{n_1\cdots n_a}_q$ over its subring $\Bbbk[x_1, \dots, x_n]^{\mathfrak{S}_n}$.*

Proof. We are going to prove that it is a direct summand of a free module. Consider the tower of ring extensions,

$$\Bbbk[x_1, \dots, x_n]^{\mathfrak{S}_n} \subset \Bbbk[x_1, \dots, x_n]^{\mathfrak{S}_{n_1}\times\cdots\times\mathfrak{S}_{n_a}} \subset \Bbbk[x_1, \dots, x_n].$$

Note that the rightmost ring in this tower is isomorphic to

$$\bigotimes_{i=1}^{a} \Bbbk[x_{n_1+\cdots+n_{i-1}+1}, \dots, x_{n_1+\cdots+n_i}]$$

and hence is free of q-rank $[n_1]_q! \cdots [n_a]_q!$ over the second ring in this tower, which is isomorphic to

$$\bigotimes_{i=1}^{a} \Bbbk[x_{n_1+\cdots+n_{i-1}+1}, \dots, x_{n_1+\cdots+n_i}]^{\mathfrak{S}_{n_i}}$$

(here we use the classical fact that a polynomial ring is free over its subring of symmetric polynomials). Therefore, $\Bbbk[x_1, \dots, x_n]^{\mathfrak{S}_{n_1}\times\cdots\times\mathfrak{S}_{n_a}}$ is a direct summand in $\Bbbk[x_1, \dots, x_n]$ as a $\Bbbk[x_1, \dots, x_n]^{\mathfrak{S}_{n_1}\times\cdots\times\mathfrak{S}_{n_a}}$-module, and, in particular, as a $\Bbbk[x_1, \dots, x_n]^{\mathfrak{S}_n}$-module. Hence, $\Bbbk[x_1, \dots, x_n]^{\mathfrak{S}_{n_1}\times\cdots\times\mathfrak{S}_{n_a}}$ is a projective module over $\Bbbk[x_1, \dots, x_n]^{\mathfrak{S}_n}$ which by the Quillen–Suslin theorem implies freeness.

Since the ranks of free modules in a tower of extensions multiply, the q-rank of our extension can be expressed as a quotient of ranks

$$\frac{[n]_q!}{[n_1]_q! \cdots [n_a]_q!} = \binom{n}{n_1 \cdots n_a}_q. \qquad \square$$

Denote by I the kernel of the projection $\mathbb{J}_m \to J^\infty_{\mathrm{red}}(R(P))$. By Proposition 1.28 the quotient $\mathbb{J}_m / \mathrm{in}_\prec I$ is isomorphic to $\mathrm{gr}_\prec J^\infty_{\mathrm{red}}(R(P))$.

Theorem 3.14. *Assume that* $\mathrm{in}_\prec I$ *is generated by relations of the form* (2-11). *Then*

$$\mathrm{gr}_\prec(J^\infty_{\mathrm{red}}(R(P)))[\bar{a}, L] \simeq J^\infty_{\mathrm{red}}(R(P))[\bar{a}, L]$$

as a $\mathcal{A}_L$- module and the dual of this module is free.

Proof. Consider $\bar{r} \in \mathcal{R}(\bar{a}, L)$. By Proposition 2.8 the space $\mathrm{gr}_\prec(J^\infty_{\mathrm{red}}(R(P)))[\bar{r}]^*$ is a free module over $\Lambda_{\bar{r}}(\boldsymbol{t})$. By (3-6) the algebra $\mathcal{A}_L$ can be embedded into $\Lambda_{\bar{r}}(\boldsymbol{t})$ as the subalgebra of polynomials symmetric in all $t_i^{(k)}$. Therefore, applying Lemma 3.13, we deduce that $\mathrm{gr}_\prec(J^\infty_{\mathrm{red}}(R(P)))[\bar{r}]^*$ is free over $\mathcal{A}_L$. Lemma 3.12 now shows that $\mathrm{gr}_\prec(J^\infty_{\mathrm{red}}(R(P)))[\bar{a}, L]^*$ is free. $\square$

3.4. *Technical lemma.* Consider integers $m \geq 1$ and $\zeta_1, \dots, \zeta_m \geq 0$ together with a vector $\bar{a} = (a_1, \dots, a_m, a) \in \mathbb{N}^{m+1}$. We also consider variables $u_i^{(j)}$ with $1 \leq i \leq m$, $1 \leq j \leq a_i$ and $s^{(j)}$ with $1 \leq j \leq a$. We work with the ring

$$\Lambda_{\bar{a}}(\boldsymbol{u}, \boldsymbol{s}) := \Bbbk[u_1^{(1)}, \dots, u_1^{(a_1)}, \dots, u_m^{(1)}, \dots, u_m^{(a_m)}, s^{(1)}, \dots, s^{(a)}]^{\mathfrak{S}_{a_1}\times\cdots\times\mathfrak{S}_{a_m}\times\mathfrak{S}_a}.$$

Within the set of collections of positive integers $\bar{r} = (r_{i,j})_{1\le i\le m, 0\le j\le \zeta_i}$ we distinguish the subset

$$\mathcal{R}(\bar{a}) := \left\{ \bar{r} \;\middle|\; \sum_{j=0}^{\zeta_i} r_{i,j} = a_i,\; \sum_{1\le i\le m,\, 0\le j\le \zeta_i} r_{i,j}\, j = a \right\}.$$

For each $\bar{r} \in \mathcal{R}(\bar{a})$ we have the ring

$$\Lambda_{\bar{r}}(\boldsymbol{t}) := \Bbbk[t_{i,j}^{(1)}, \dots, t_{i,j}^{(r_{i,j})}]_{1\le i\le m, 0\le j\le \zeta_i}^{\times_{i,j} \mathfrak{S}_{r_{i,j}}}.$$

We now define a map $\psi_{\bar{r}} : \Lambda_{\bar{a}}(\boldsymbol{u}, \boldsymbol{s}) \to \Lambda_{\bar{r}}(\boldsymbol{t})$. As for $\varphi_{\bar{r}}^{\vee}$ above we define $\psi_{\bar{r}}$ as a map between rings of all polynomials and then restrict to subrings of symmetric polynomials, the fact that the image lies in $\Lambda_{\bar{r}}(\boldsymbol{t})$ is checked similarly to Lemma 3.5. Consider a variable $u_i^{(k)}$ and note that we have a unique $j_0 \in [0, \zeta_i]$ for which $\sum_{j=1}^{j_0-1} r_{i,j} < k \le \sum_{j=1}^{j_0} r_{i,j}$. Denote $\ell = k - \sum_{j=1}^{j_0} r_{i,j}$, then we set $\psi_{\bar{r}}(u_i^{(k)}) = t_{i,j_0}^{(\ell)}$. Now consider a variable $s^{(k)}$. We have a unique $i_0 \in [1, m]$ for which

$$\sum_{1\le i\le i_0-1,\, 0\le j\le \zeta_i} j r_{i,j} < k \le \sum_{1\le i\le i_0,\, 0\le j\le \zeta_i} j r_{i,j}.$$

Denote the first sum above by S. We also have a unique $j_0 \in [0, \zeta_{i_0}]$ for which

$$S + \sum_{j=1}^{j_0-1} j r_{i_0,j} < k \le S + \sum_{j=1}^{j_0} j r_{i_0,j}.$$

Finally, denote

$$\ell = \left\lceil \frac{k - S - \sum_{j=1}^{j_0-1} j r_{i_0,j}}{j_0} \right\rceil,$$

note that $\ell \in [1, r_{i_0,j_0}]$. We set $\psi_{\bar{r}}(s^{(k)}) = t_{i_0,j_0}^{(\ell)}$.

Note that $\psi_{\bar{r}}$ establishes a bijection between all variables of the form $u_i^{(k)}$ and all variables of the form $t_{i,j}^{(\ell)}$ for every i. Meanwhile, the number of variables of the form $s^{(k)}$ which are mapped into a given $t_{i,j}^{(\ell)}$ is equal to j.

Remark 3.15. The map $\psi_{\bar{r}}$ appears naturally in the following context. Consider lattice polytopes $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^{n+1}$ such that the following holds. Let $\bar{\alpha}^1, \dots, \bar{\alpha}^m$ be the integer points of P. Then the integer points of Q are precisely the points $\bar{\beta}_{i,j} = \bar{\alpha}^i \times j$ with $1 \le i \le m$, $0 \le j \le \zeta_i$. In particular, the projection of Q along the $(n+1)$-st coordinate is P. The resulting embedding $Q \hookrightarrow P \times \mathbb{R}$ gives rise to the map

$$\Bbbk[X_{i,j}]_{1\le i\le m, 0\le j\le \zeta_i} \to \Bbbk[X_i]_{1\le i\le m} \otimes \Bbbk[z] \tag{3-7}$$

taking $X_{i,j}$ to $X_i z^j$. We may now successively apply J^∞ to this map, restrict to the component of grading $\bar{r}$ on the left and the component of grading $\bar{a}$ on the right and then dualize both sides. We obtain a map $\psi_{\bar{r}} : \Lambda_{\bar{a}}(\boldsymbol{u}, \boldsymbol{s}) \to \Lambda_{\bar{r}}(\boldsymbol{t})$, this is precisely the map defined above.

Consider a total order $\prec$ on pairs (i, j), $1 \le i \le m$, $0 \le j \le \zeta_i$ with the property $(i, j) \prec (i, j')$ if $j < j'$. One has an induced lexicographic order on vectors $\bar{r}_{i,j}$ such that $\bar{r} \prec \bar{r}'$ if and only if the $\prec$-minimal (i, j) for which $r_{i,j} \ne r'_{i,j}$ satisfies $r_{i,j} < r'_{i,j}$.

Definition 3.16. For $(i,j) \prec (i',j')$ let $\varkappa((i,j),(i',j'))$ be the number of ℓ such that $(i,j) \prec (i',\ell)$ and $\ell < j'$.

In other words, let $e_{\min}$ be the minimal number e such that $(i,j) \prec (i',e)$. Then $\varkappa((i,j),(i',j')) = j' - e_{\min}$. The key claim of this subsection is as follows.

Lemma 3.17.
$$\psi_{\bar r}\Bigl(\bigcap_{\bar r \succ \bar r'} \ker(\psi_{\bar r'})\Bigr) \supset \prod_{\substack{(i,j)\prec(i',j'),\\ 1\le \ell\le r_{i,j},\\ 1\le \ell'\le r_{i',j'}}} (t_{i,j}^{(\ell)} - t_{i',j'}^{(\ell')})^{\varkappa((i,j),(i',j'))} \Lambda_{\bar r}(\boldsymbol{t}).$$

This lemma will be proved by induction reducing the case of vectors $\bar a$ and $\bar r$ to smaller vectors defined as follows. Choose $1 \le \tilde i \le m$ with $\zeta_{\tilde i} > 0$. For $\max(a_{\tilde i} - a, 0) \le \ell \le a_{\tilde i}$ we define the truncation operator

$$\mathrm{tc}_\ell^{\tilde i}(\bar a) = (a_1, \dots, a_{\tilde i - 1}, a_{\tilde i} - \ell, a_{\tilde i + 1}, \dots, a_m, a - a_{\tilde i} + \ell) \in \mathbb{N}^{m+1}.$$

The vector $\bar r$ is transformed by the map $\mathrm{tc}^{\tilde i}$ with

$$\mathrm{tc}^{\tilde i}(\bar r)_{i,j} = \begin{cases} r_{i,j} & \text{if } i \ne \tilde i, \\ r_{i,j+1} & \text{if } i = \tilde i, \end{cases}$$

where $1 \le i \le m$ and $1 \le j \le \zeta_i$ for $i \ne \tilde i$ while $1 \le j \le \zeta_{\tilde i} - 1$ otherwise. Note that we have

$$\mathrm{tc}^{\tilde i}(\bar r) \in \mathcal{R}(\mathrm{tc}^{\tilde i}_{r_{\tilde i,0}}(\bar a)).$$

We now show that $\psi_{\bar r}$ factors through $\psi_{\mathrm{tc}^{\tilde i}(\bar r)}$.

We define the map

$$\chi_\ell^{\tilde i} : \Lambda_{\bar a}(\boldsymbol{u}, \boldsymbol{s}) \to \Lambda_{\mathrm{tc}_\ell^{\tilde i}(\bar a)}(\boldsymbol{u}, \boldsymbol{s}) \otimes \Bbbk[t_{\tilde i,0}^{(1)}, \dots, t_{\tilde i,0}^{(\ell)}]^{\mathfrak{S}_\ell}$$

by

$$u_i^{(j)} \mapsto u_i^{(j)} \quad \text{if } i \ne \tilde i, \qquad u_{\tilde i}^{(j)} \mapsto \begin{cases} u_{\tilde i}^{(j-\ell)} & \text{if } j > \ell, \\ t_{\tilde i,0}^{(j)} & \text{if } j \le \ell, \end{cases} \qquad s^{(j)} \mapsto \begin{cases} u_{\tilde i}^{(j)} & \text{if } j \le a_{\tilde i} - \ell, \\ s^{(j - a_{\tilde i} + \ell)} & \text{if } j > a_{\tilde i} - \ell. \end{cases}$$

Let $\mathrm{sh}_{\tilde i}$ be the index shift operator,

$$\mathrm{sh}_{\tilde i}(t_{i,j}) = t_{i,j} \quad \text{if } i \ne \tilde i, \quad \mathrm{sh}_{\tilde i}(t_{\tilde i,j}) = t_{\tilde i,j+1}.$$

Then we have the identity

$$\psi_{\bar r} = ((\mathrm{sh}_{\tilde i} \circ \psi_{\mathrm{tc}^{\tilde i}(\bar r)}) \otimes \mathrm{id}) \circ \chi^{\tilde i}_{r_{\tilde i,0}}. \tag{3-8}$$

For clarity let us point out the following. As mentioned, the maps $\psi_\bullet$, $\chi_\bullet^\bullet$ and $\mathrm{sh}_\bullet$ are defined as maps between polynomial rings and then restricted to symmetric polynomials. It should be noted that identity (3-8) does not hold for these maps between polynomial rings (only for the restrictions) because the left-hand side and the right-hand side may not coincide on some $s^{(i)}$ (but always coincide up to the action of the corresponding symmetric group).

Remark 3.18. Identity (3-8) can be interpreted in terms of Remark 3.15 as follows. The map (3-7) decomposes as

$$\Bbbk[X_{i,j}]_{1\le i\le m, 0\le j\le \zeta_i} \to \Bbbk[X_i]_{1\le i\le m}\otimes \Bbbk[z]\otimes \Bbbk[X_{\tilde{i},0}] \to \Bbbk[X_i]_{1\le i\le m}\otimes \Bbbk[z].$$

Here the left arrow maps $X_{i,j}$ with $i\ne\tilde{i}$ to $X_i z^j$, maps $X_{\tilde{i},j}$ with $j\ge 1$ to $X_{\tilde{i}}z^{j-1}$ and preserves $X_{\tilde{i},0}$. The right arrow preserves z and X_i with $i\ne\tilde{i}$ while mapping $X_{\tilde{i}}$ to $X_{\tilde{i}}z$ and $X_{\tilde{i},0}$ to $X_{\tilde{i}}$. We again apply J^∞, restrict to components of gradings $\bar{r}$, $\mathrm{tc}^{\tilde{i}}_{r_{\tilde{i},0}}(\bar{a})\oplus r_{\tilde{i},0}$ and $\bar{a}$ and then dualize. One may check that on the right we will obtain the map $\chi^{\tilde{i}}_{r_{\tilde{i},0}}$ and on the left we will obtain $(\mathrm{sh}_{\tilde{i}}\circ\psi_{\mathrm{tc}^{\tilde{i}}(\bar{r})})\otimes\mathrm{id}$ (the latter can be seen by further decomposing the left arrow).

For $\ell < a_{\tilde{i}}$ we have

$$\ker\chi^{\tilde{i}}_\ell \supset \ker\chi^{\tilde{i}}_{\ell+1}.$$

Moreover, we have $\chi^{\tilde{i}}_\ell = \tau\circ\chi^{\tilde{i}}_{\ell+1}$, where

$$\tau : \Lambda_{\mathrm{tc}^{\tilde{i}}_{\ell+1}(\bar{a})}(\boldsymbol{u},\boldsymbol{s})\otimes\Bbbk[t^{(1)}_{\tilde{i},0},\dots,t^{(\ell+1)}_{\tilde{i},0}]^{\mathfrak{S}_{\ell+1}} \to \Lambda_{\mathrm{tc}^{\tilde{i}}_{\ell+1}(\bar{a})}(\boldsymbol{u},\boldsymbol{s})\otimes\Bbbk[u^{(a_{\tilde{i}}-\ell)}_{\tilde{i}}]\otimes\Bbbk[t^{(1)}_{\tilde{i},0},\dots,t^{(\ell)}_{\tilde{i},0}]^{\mathfrak{S}_\ell},$$
$$\tau(t^{(\ell+1)}_{\tilde{i},0}) = \tau(s^{(a-a_{\tilde{i}}+\ell+1)}) = u^{(a_{\tilde{i}}-\ell)}_{\tilde{i}}$$

and τ sends any variable (on the left) different from $t^{(\ell+1)}_{\tilde{i},0}$ and $s^{(a-a_{\tilde{i}}+\ell+1)}$ to the same variable on the right.

Lemma 3.19. $$\chi^{\tilde{i}}_{\ell+1}(\ker\chi^{\tilde{i}}_\ell)\supset \prod_{\substack{j=1,\dots,a-a_{\tilde{i}}+\ell+1,\\ j'=1,\dots,\ell+1}}(s^{(j)}-t^{(j')}_{\tilde{i},0})(\Lambda_{\mathrm{tc}^{\tilde{i}}_{\ell+1}(\bar{a})}(\boldsymbol{u},\boldsymbol{s})\otimes\Bbbk[t^{(1)}_{\tilde{i},0},\dots,t^{(\ell+1)}_{\tilde{i},0}]^{\mathfrak{S}_{\ell+1}}).$$

Proof. One has

$$\chi^{\tilde{i}}_{\ell+1}(p_k(u^{(1)}_{\tilde{i},0},\dots,u^{(a_{\tilde{i}})}_{\tilde{i},0})) = p_k(u^{(1)}_{\tilde{i},0},\dots,u^{(a_{\tilde{i}}-\ell-1)}_{\tilde{i},0}) + p_k(t^{(1)}_{\tilde{i},0},\dots,t^{(\ell+1)}_{\tilde{i},0}),$$
$$\chi^{\tilde{i}}_{\ell+1}(p_k(s^{(1)},\dots,s^{(a)})) = p_k(u^{(1)}_{\tilde{i},0},\dots,u^{(a_{\tilde{i}}-\ell-1)}_{\tilde{i},0}) + p_k(s^{(1)},\dots,s^{(a-a_{\tilde{i}}+\ell+1)}).$$

Therefore, the image of $\chi^{\tilde{i}}_{\ell+1}$ contains the differences

$$p_k(t^{(1)}_{\tilde{i},0},\dots,t^{(\ell+1)}_{\tilde{i},0}) - p_k(s^{(1)},\dots,s^{(a-a_{\tilde{i}}+\ell+1)})$$

which generate the ring $\Omega_{\ell+1,a-a_{\tilde{i}}+\ell+1}$ of supersymmetric polynomials in variables $t^{(1)}_{\tilde{i},0},\dots,t^{(\ell+1)}_{\tilde{i},0}$ and $s^{(1)},\dots,s^{(a-a_{\tilde{i}}+\ell+1)}$ (Proposition 1.27). We deduce that the image of $\chi^{\tilde{i}}_{\ell+1}$ contains the ring

$$S = \Omega_{\ell+1,a-a_{\tilde{i}}+\ell+1}\otimes\Lambda_{a_{\tilde{i}}-\ell-1}(\boldsymbol{u}_{\tilde{i}})\otimes\bigotimes_{i\ne\tilde{i}}\Lambda_{a_i}(\boldsymbol{s}_i).$$

The right-hand side in the statement of our lemma is equal to $\ker\tau\cap S$ due to Proposition 1.26. However, $\chi^{\tilde{i}}_{\ell+1}(\ker\chi^{\tilde{i}}_\ell) = \mathrm{im}\,\chi^{\tilde{i}}_{\ell+1}\cap\ker\tau\supset S\cap\ker\tau$. □

Proof of Lemma 3.17. Let $(\tilde{i},0)$ be the smallest pair (i,j) in the order $\prec$. By definition we have

$$\bar{r}\succ\bar{r}' \iff r_{\tilde{i},0} > r'_{\tilde{i},0} \text{ or } (r_{\tilde{i},0} = r'_{\tilde{i},0} \text{ and } \mathrm{tc}^{\tilde{i}}(\bar{r})\succ\mathrm{tc}^{\tilde{i}}(\bar{r}')),$$

where $\mathrm{tc}^{\tilde{i}}(\bar{r})$ and $\mathrm{tc}^{\tilde{i}}(\bar{r}')$ are also compared lexicographically.

Using Lemma 3.19 and identity (3-8) we have

$$\chi^{\tilde{i}}_{r_{\tilde{i},0}}\Bigg(\bigcap_{\bar{r}':r_{\tilde{i},0}>r'_{\tilde{i},0}}\ker(\psi_{\bar{r}'})\Bigg)\supset\prod_{\substack{\ell=1,\dots,a-a_{\tilde{i}}+r_{\tilde{i},0},\\ \ell'=1,\dots,r_{\tilde{i},0}}}(s^{(\ell)}-t^{(\ell')}_{\tilde{i},0})\Lambda_{\mathrm{tc}^{\tilde{i}}_{r_{\tilde{i},0}}(\bar{a})}(\boldsymbol{u},\boldsymbol{s})\otimes\Bbbk[t^{(1)}_{\tilde{i},0},\dots,t^{(r_{\tilde{i},0})}_{\tilde{i},0}]^{\mathfrak{S}_{r_{\tilde{i},0}}}. \tag{3-9}$$

We proceed by induction on $\sum_{i=1}^{m}\zeta_i$. By the induction hypothesis we have

$$\psi_{\mathrm{tc}^{\tilde{i}}(\bar{r})}\Bigg(\bigcap_{\mathrm{tc}^{\tilde{i}}(\bar{r})\succ\mathrm{tc}^{\tilde{i}}(\bar{r}')}\ker(\psi_{\mathrm{tc}^{\tilde{i}}(\bar{r}')})\Bigg)\supset\prod_{\substack{(i,j)\prec(i',j'),\\ 1\le\ell\le\mathrm{tc}^{\tilde{i}}(r)_{i,j},\\ 1\le\ell'\le\mathrm{tc}^{\tilde{i}}(r)_{i',j'}}}(t^{(\ell)}_{i,j}-t^{(\ell')}_{i',j'})^{\varkappa((i,j),(i',j'))}\Lambda_{\mathrm{tc}^{\tilde{i}}(\bar{r})}(\boldsymbol{t}).$$

Let $\bar{r}'\prec\bar{r}$ be such that $r'_{\tilde{i},0}=r_{\tilde{i},0}$. Consider the set $F_{\bar{r}'}$ consisting of all

$$f\in\Lambda_{\mathrm{tc}^{\tilde{i}}_{r_{\tilde{i},0}}(\bar{a})}(\boldsymbol{u},\boldsymbol{s})\otimes\Bbbk[t^{(1)}_{\tilde{i},0},\dots,t^{(r_{\tilde{i},0})}_{\tilde{i},0}]^{\mathfrak{S}_{r_{\tilde{i},0}}}$$

such that $(\psi_{\mathrm{tc}^{\tilde{i}}(\bar{r}')}\otimes\mathrm{id})f=0$. Then, by (3-8),

$$\psi_{\bar{r}}(\ker(\psi_{\bar{r}'}))=((\mathrm{sh}_{\tilde{i}}\circ\psi_{\mathrm{tc}^{\tilde{i}}(\bar{r})})\otimes\mathrm{id})\circ\chi^{\tilde{i}}_{r_{\tilde{i},0}}(\ker(\psi_{\bar{r}'})),$$

and hence we get (note that $\mathrm{sh}_{\tilde{i}}$ is bijective)

$$\begin{aligned}\psi_{\bar{r}}(\ker(\psi_{\bar{r}'}))&=((\mathrm{sh}_{\tilde{i}}\circ\psi_{\mathrm{tc}^{\tilde{i}}(\bar{r})})\otimes\mathrm{id})(\ker(\psi_{\mathrm{tc}^{\tilde{i}}(\bar{r}')}\otimes\mathrm{id})\cap(\mathrm{im}\,\chi^{\tilde{i}}_{r_{\tilde{i},0}}))\\&\supset((\mathrm{sh}_{\tilde{i}}\circ\psi_{\mathrm{tc}^{\tilde{i}}(\bar{r})})\otimes\mathrm{id})\prod_{\substack{\ell=1,\dots,a-a_{\tilde{i}}+r_{\tilde{i},0},\\ \ell'=1,\dots,r_{\tilde{i},0}}}(s^{(\ell)}-t^{(\ell')}_{\tilde{i},0})F_{\bar{r}'}\end{aligned}$$

due to Lemma 3.19. Taking the intersection one gets

$$\psi_{\bar{r}}\Bigg(\bigcap_{\bar{r}'\prec\bar{r},r'_{\tilde{i},0}=r_{\tilde{i},0}}\ker(\psi_{\bar{r}'})\Bigg)\supset((\mathrm{sh}_{\tilde{i}}\circ\psi_{\mathrm{tc}^{\tilde{i}}(\bar{r})})\otimes\mathrm{id})\prod_{\substack{\ell=1,\dots,a-a_{\tilde{i}}+r_{\tilde{i}0},\\ \ell'=1,\dots,r_{\tilde{i},0}}}(s^{(\ell)}-t^{(\ell')}_{\tilde{i},0})\bigcap_{\bar{r}'\prec\bar{r},r'_{\tilde{i},0}=r'_{\tilde{i},0}}F_{\bar{r}'}.$$

Therefore, using (3-9) we obtain

$$\psi_{\bar{r}}\Bigg(\bigcap_{\bar{r}'\prec\bar{r}}\ker(\psi_{\bar{r}'})\Bigg)\supset((\mathrm{sh}_{\tilde{i}}\circ\psi_{\mathrm{tc}^{\tilde{i}}(\bar{r})})\otimes\mathrm{id})\prod_{\substack{\ell=1,\dots,a-a_{\tilde{i}}+r_{\tilde{i},0},\\ \ell'=1,\dots,r_{\tilde{i},0}}}(s^{(\ell)}-t^{(\ell')}_{\tilde{i},0})\bigcap_{\bar{r}'\prec\bar{r},r'_{\tilde{i},0}=r'_{\tilde{i},0}}F_{\bar{r}'}.$$

Definition 3.16 implies

$$(\mathrm{sh}_{\tilde{i}}\circ\psi_{\mathrm{tc}^{\tilde{i}}(\bar{r})})\otimes\mathrm{id}\Bigg(\prod_{\substack{\ell=1,\dots,a_s-a_{\tilde{i}}+r_{\tilde{i},0},\\ \ell'=1,\dots,r_{\tilde{i},0}}}(s^{(\ell)}-t^{(\ell')}_{\tilde{i},0})\Bigg)=\prod_{(i,j)\succ(\tilde{i},0)}\prod_{\substack{\ell=1,\dots,r_{i,j},\\ \ell'=1,\dots,r_{\tilde{i},0}}}(t^{(\ell)}_{i,j}-t^{(\ell')}_{\tilde{i},0})^{\varkappa((i,j),(\tilde{i},0))}.$$

By induction hypothesis we have

$$\begin{aligned}&((\mathrm{sh}_{\tilde{i}}\circ\psi_{\mathrm{tc}^{\tilde{i}}(\bar{r})})\otimes\mathrm{id})\Bigg(\bigcap_{\bar{r}'\prec\bar{r},r'_{\tilde{i},0}=r'_{\tilde{i},0}}F_{\bar{r}'}\Bigg)\\&\quad\supset\prod_{(i,j)\succ(i',j')\succ(\tilde{i},0)}\prod_{\substack{\ell=1,\dots,r_{i,j},\\ \ell'=1,\dots,r_{i',j'}}}(t^{(\ell)}_{i,j}-t^{(\ell')}_{i',j'})^{\varkappa((i,j),(i',j'))}\Lambda_{\mathrm{tc}^{\tilde{i}}_{r_{\tilde{i},0}}(\bar{a})}(\boldsymbol{u},\boldsymbol{s})\otimes\Bbbk[t^{(1)}_{\tilde{i},0},\dots,t^{(r_{\tilde{i},0})}_{\tilde{i},0}]^{\mathfrak{S}_{r_{\tilde{i},0}}}.\end{aligned}$$

□

4. Inductive construction

4.1. ***Veronese curve.*** In this subsection we study the arc schemes corresponding to the segments $[0,\zeta]$ (recall that a one-dimensional polytope is affine equivalent to a segment). By definition, the Veronese ring $R([0,\zeta])$ is generated by $Y_\alpha = z^\alpha w$, $\alpha = 0,\dots,\zeta$. The ring $J^\infty_{\mathrm{red}}(R([0,\zeta]))$ of reduced arcs over $R([0,\zeta])$ was studied in [Dumanski and Feigin 2023]. Here we study this ring by methods developed above. Namely, we use the results from Section 1.6 on the structure of the reduced arc schemes of the cubes and Lemma 3.17.

Recall the notation Y_I for the generators of the arc rings of cubes.

For any $0 \le \alpha < \beta \le \zeta$ we define the linear map of polytopes

$$[0,1]^{\beta-\alpha} \to [0,\zeta], \quad (d_1,\dots,d_{\beta-\alpha}) \mapsto \alpha + \sum_{i=1}^{\beta-\alpha} d_i.$$

This map defines a homomorphism of corresponding toric rings $R([0,1])^{\beta-\alpha} \to R([0,\zeta])$:

$$\iota_{\alpha,\beta} : Y_I \mapsto Y_{\alpha+|I|}.$$

Lemma 4.1. *For integers $0 \le \alpha < \beta \le \zeta$ and $0 \le k' \le \beta-\alpha-2$ the coefficients of the following series are zero in the ring $J^\infty_{\mathrm{red}}(R([0,\zeta]))$:*

$$W'_{\alpha,\beta,k'} = \sum_{i=0}^{\beta-\alpha-1} (-1)^i \binom{\beta-\alpha-1}{i} \frac{\partial^{k'} Y_{\alpha+i}(s)}{\partial s^{k'}} Y_{\beta-i}(s).$$

Proof. Consider the map $J^\infty(\iota_{\alpha,\beta}) : J^\infty(R([0,1]^{\beta-\alpha})) \to J^\infty(R([0,\zeta]))$. Recall the elements $W_{\beta-\alpha,k'} \in J^\infty(R([0,1]^{\beta-\alpha}))$. By Lemma 1.31 the coefficients of $W_{\beta-\alpha,k'}$ are nilpotent. However,

$$J^\infty(\iota_{\alpha,\beta})(W_{\beta-\alpha,k'}) = W'_{\alpha,\beta,k'}.$$

Thus the coefficients of $W'_{\beta-\alpha,k'}$ are nilpotent. Hence they are equal to 0 in $J^\infty_{\mathrm{red}}(R([0,\zeta]))$. □

Definition 4.2. The ring $\overline{J^\infty_{\mathrm{red}}(R([0,\zeta]))}$ is the quotient of $\Bbbk[X_i^{(j)}]_{0\le i\le\zeta,\, j\ge 0}$ modulo the ideal generated by coefficients of the series

$$\sum_{i=0}^{\beta-\alpha-1} (-1)^i \binom{\beta-\alpha-1}{i} \frac{\partial^{k'} X_{\alpha+i}(s)}{\partial s^{k'}} X_{\beta-i}(s) \tag{4-1}$$

for all $0 \le \alpha < \beta \le \zeta$, $0 \le k' \le \beta-\alpha-2$.

Clearly, there is a surjective homomorphism $\overline{J^\infty_{\mathrm{red}}(R([0,\zeta]))} \to J^\infty_{\mathrm{red}}(R([0,\zeta]))$.

Consider the following (weight-restricted lexicographic) order on the semigroup $\mathbb{N}^{\zeta+1}$: we say that $(u_0,\dots,u_\zeta)$ is less than $(v_0,\dots,v_\zeta)$ if $u_0+\dots+u_\zeta < v_0+\dots+v_\zeta$ or $u_0+\dots+u_\zeta = v_0+\dots+v_\zeta$ and there exists an i such that $u_i < v_i$ and $u_j = v_j$ for $j < i$. We have an induced order $\prec$ on the graded components of $J^\infty(\Bbbk[X_0,\dots,X_\zeta])$. The next lemma is obvious.

Lemma 4.3. *The initial part of* (4-1) *is equal to*

$$\frac{\partial^{k'} X_\alpha(s)}{\partial s^{k'}} X_\beta(s). \tag{4-2}$$

We define the ring $\widehat{J^\infty_{\mathrm{red}}(R([0,\zeta]))}$ to be a quotient of $\Bbbk[X_i^{(j)}]_{i=0,\dots,\zeta,\, j\geq 0}$ by the ideal generated by coefficients of (4-2). These relations are homogeneous with respect to the grading $\prec$. Thus the ring $\widehat{J^\infty_{\mathrm{red}}(R([0,\zeta]))}$ has a natural $\mathbb{N}^{\zeta+1}$-grading and we denote the $\bar{r}$-th graded component by $\widehat{J^\infty_{\mathrm{red}}(R([0,\zeta]))}[\bar{r}]$. Proposition 2.8 applied to this graded component gives us the following.

Proposition 4.4. *We have the following isomorphism of* grad*-graded spaces*:

$$(\widehat{J^\infty_{\mathrm{red}}(R([0,\zeta]))}[\bar{r}])^* \simeq \prod_{0\leq\alpha<\beta\leq\zeta,1\leq i\leq r_\alpha,1\leq j\leq r_\beta} (t_\alpha^{(i)} - t_\beta^{(j)})^{\beta-\alpha-1} \Bbbk[t_\alpha^{(j)}]_{0\leq\alpha\leq\zeta,1\leq j\leq r_\alpha}^{\mathfrak{S}_{r_0}\times\cdots\times\mathfrak{S}_{r_\zeta}}.$$

Taking into account the grading grad we extend the $\mathbb{N}^{\zeta+1}$-grading above to the $\mathbb{N}^{\zeta+2}$-grading. Corollary 1.30 implies:

Lemma 4.5. *There is the following surjective map of* $\mathbb{N}^{\zeta+2}$*-graded rings*:

$$\widehat{J^\infty_{\mathrm{red}}(R([0,\zeta]))} \twoheadrightarrow \mathrm{gr}_\prec J^\infty_{\mathrm{red}}(R([0,\zeta])).$$

We take a vector $\bar{r} = (r_0,\dots,r_\zeta)$ and denote

$$\sum_{i=0}^{\zeta} i r_i = a, \qquad \sum_{i=0}^{\zeta} r_i = L.$$

Recall the notation $\bar{r} \in \mathcal{R}(a, L)$. We consider the rings of symmetric functions $\Lambda_{a,L}(\boldsymbol{s})$ and $\Lambda_{\bar{r}}(\boldsymbol{t})$. We have the map $\varphi_{\bar{r}} : A_{\bar{r}} \to J^\infty(\Bbbk[z_1, w])[a, L]$ for $\bar{r} \in \mathcal{R}(a, L)$ and the map $\varphi_{\bar{r}}^\vee : \Lambda_{a,L}(\boldsymbol{s}) \to \Lambda_{\bar{r}}(\boldsymbol{t})$ defined by the following formulas (see (3-4)):

$$\begin{gathered} s_w^{(1)} \mapsto t_0^{(1)}; \quad \dots; \quad s_w^{(r_0)} \mapsto t_0^{(r_0)}; \quad s_w^{(r_0+1)}, s_1^{(1)} \mapsto t_1^{(1)}; \quad \dots; \quad s_w^{(r_0+r_1)}, s_1^{(r_1)} \mapsto t_1^{(r_1)}; \\ s_w^{(r_0+r_1+1)}, s_1^{(r_1+1)}, s_1^{(r_1+2)} \mapsto t_2^{(1)}; \quad \dots, s_w^{(r_0+r_1+r_2)}, s_1^{(r_1+2r_2-1)}, s_1^{(r_1+2r_2)} \mapsto t_2^{(r_2)}; \\ s_w^{(r_0+r_1+r_2+1)}, s_1^{(r_1+2r_2+1)}, s_1^{(r_1+2r_2+2)}, s_1^{(r_1+2r_2+3)} \mapsto t_3^{(1)} \dots. \end{gathered} \tag{4-3}$$

One has

$$\varphi_{\bar{r}}^\vee(p_k(s_w^{(1)},\dots,s_w^{(L)})) = \sum_{i=0}^{\zeta} p_k(t_i^{(1)},\dots,t_i^{(r_i)}),$$

$$\varphi_{\bar{r}}^\vee(p_k(s_1^{(1)},\dots,s_1^{(a)})) = \sum_{i=0}^{\zeta} i\, p_k(t_i^{(1)},\dots,t_i^{(r_i)}).$$

Lemma 3.17 with $m = 1$ implies:

Proposition 4.6. *For any* $\bar{r} \in \mathcal{R}(a, L)$,

$$\varphi_{\bar{r}}^\vee\Bigl(\bigcap_{\bar{r}'\in\mathcal{R}(a,L),\bar{r}'\prec\bar{r}} \ker(\varphi_{\bar{r}'}^\vee)\Bigr) \supset \prod_{0\leq\alpha<\beta\leq\zeta,1\leq i\leq r_\alpha,1\leq j\leq r_\beta} (t_\alpha^{(i)} - t_\beta^{(j)})^{\beta-\alpha-1} \Lambda_{\bar{r}}(\boldsymbol{t}). \tag{4-4}$$

Thus we reobtain the following theorem from [Dumanski and Feigin 2023].

Theorem 4.7. *For any $\bar{r}$ grad-graded vector spaces $\mathrm{gr}_{\prec} J^{\infty}_{\mathrm{red}}(R([0,\zeta]))[\bar{r}]$ and $(\widehat{J^{\infty}_{\mathrm{red}}(R([0,\zeta]))})[\bar{r}]$ are naturally isomorphic. One has the isomorphism of graded rings*

$$J^{\infty}_{\mathrm{red}}(R([0,\zeta])) \simeq \overline{J^{\infty}_{\mathrm{red}}(R([0,\zeta]))}.$$

Corollary 4.8. *The action of $\mathcal{A}_L$ is free on $J^{\infty}_{\mathrm{red}}(R([0,\zeta]))[L]^*$.*

Proof. This follows from Theorem 3.14. □

4.2. ***Two-dimensional case.*** Let $f:[0,\eta]\to\mathbb{R}_{\geq 0}$ be a convex function such that for any integer $i\in[0,\eta]$: $\zeta_i := f(i)\in\mathbb{N}$. In this subsection we consider the case of the two-dimensional polygon obtained as the convex hull of the segment $[0,\eta]\times\{0\}$ and all points $(i, f(i))$, $i\in[0,\eta]$. In other words, we consider a positive integer η and let $\zeta_0,\dots,\zeta_\eta$ be a tuple of nonnegative integers such that $\zeta_0-\zeta_1\leq\zeta_1-\zeta_2\leq\cdots\leq\zeta_{\eta-1}-\zeta_\eta$. Then the integer points of the corresponding convex lattice polygon are (i,j), $0\leq i\leq\eta$, $0\leq j\leq\zeta_i$.

Example 4.9. The case

$$\zeta=\begin{cases}\zeta(\alpha)=\alpha/b & \text{if } \alpha\in[0,b],\\ \zeta(\alpha)=1 & \text{if } \alpha\in[b,\eta],\end{cases}\qquad 0\leq b\leq\eta,$$

corresponds to Hirzebruch surface.

Example 4.10. The case $\zeta(\alpha)=\alpha$, $\alpha\in[0,\eta]$ corresponds to Veronese embedding of $\mathbb{P}^2$.

It will be convenient for us to reflect this polygon across the horizontal line $j=\zeta_{\max}/2$, where $\zeta_{\max}=\max\{\zeta_i\mid i=0,\dots,\eta\}$. The integer points of the reflected polytope P are (i,j) with $0\leq i\leq\eta$ and $\zeta_{\max}-\zeta_i\leq j\leq\zeta_{\max}$. An example of such a reflected polygon P is seen in the Figure 1.

Consider two distinct integer points $\bar{\alpha}=(\alpha_1,\alpha_2)$, $\bar{\beta}=(\beta_1,\beta_2)\in P$. We assume that $\alpha_1\leq\beta_1$ and if $\alpha_1=\beta_1$, then $\alpha_2\leq\beta_2$. We will now define a certain polygonal curve with integer vertices connecting these two points. Denote the vertices of this curve by $(i_0,j_0),\dots,(i_{k+1},j_{k+1})$. We require the following to hold:

- $(i_0,j_0)=\bar{\alpha}$ and $(i_{k+1},j_{k+1})=\bar{\beta}$.
- The sequence $i_0,\dots,i_{k+1}$ is weakly increasing.
- The sequence $j_0,\dots,j_{k+1}$ is weakly monotonic, it increases (weakly) if $\alpha_2\leq\beta_2$ and decreases (weakly) if $\alpha_2\geq\beta_2$.
- For every $1\leq\ell\leq k+1$ we either have $i_\ell-i_{\ell-1}=1$ or $i_\ell=i_{\ell-1}$ and $|j_\ell-j_{\ell-1}|=1$.
- The curve is concave in the sense that a segment connecting two points of the curve cannot contain points lying below the curve.
- All $(i_\ell,j_\ell)\in P$.

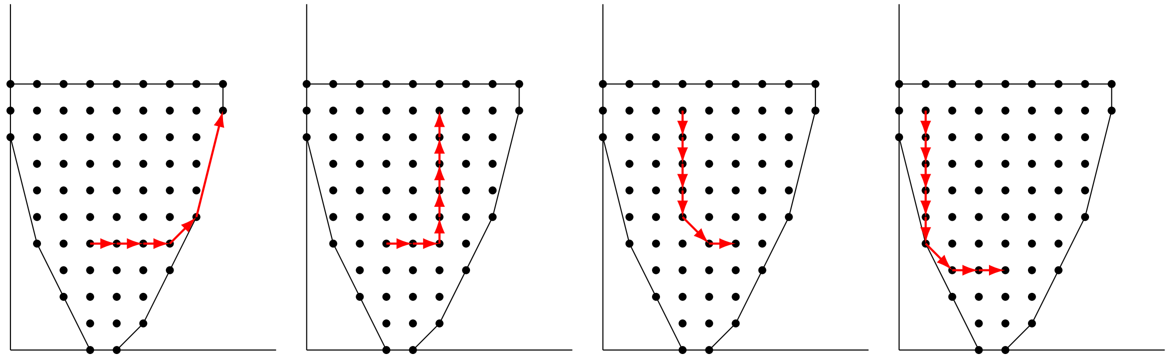

Figure 1. Concave polygonal curves with integer vertices in P.

This provides a concave polygonal curve with integer vertices in P which is directed either towards the top-right or the bottom-right. However, these conditions still leave plenty of degrees of freedom, the curve we choose can be described as follows (modulo one exceptional case which will be discussed below). The curve contains as many vertices as possible on the horizontal line $i = \min(\alpha_2, \beta_2)$ (i.e., that which contains the lower endpoint) and as many vertices as possible on the vertical line which contains the higher endpoint (i.e., either $j = \alpha_1$ or $j = \beta_1$ depending on $\max(\alpha_2, \beta_2)$). All intermediate vertices (if any) are chosen as vertices of P. In other words, for $1 \le \ell \le k+1$ let $e_\ell = (i_\ell, j_\ell) - (i_{\ell-1}, j_{\ell-1})$ be the vector corresponding to the ℓ-th segment, explicit formulas for these vectors can be given as follows.

Suppose $\alpha_2 \le \beta_2$, i.e., the curve is directed towards the top-right (Figure 1, first and second image). Choose the maximal $u \le \beta_1$ for which $(\alpha_1, u) \in P$.

$$\begin{aligned} e_1 &= \cdots = e_{u-\alpha_1} = (1, 0),\\ e_{u-\alpha_1+1} &= (1, \zeta_{\max} - \zeta_{u+1} - \alpha_2),\\ e_{u-\alpha_1+2} &= (1, \zeta_{u+2} - \zeta_{u+1}), \quad \ldots, \quad e_{\beta_1-\alpha_1} = (1, \zeta_{\beta_1-1} - \zeta_{\beta_1}),\\ e_{\beta_1-\alpha_1+1} &= \cdots = e_{k+1} = (0, 1). \end{aligned}$$

Now suppose $\alpha_2 > \beta_2$ and choose the minimal $u \ge \alpha_1$ for which $(u, \beta_2) \in P$. First suppose that $u = \alpha_1$ (so that $(\alpha_1, \beta_2) \in P$, see Figure 1, third image), this is the exceptional case mentioned above, note vector $e_{\alpha_2-\beta_2}$. Here we set

$$e_1 = \cdots = e_{\alpha_2-\beta_2-1} = (0, -1), \quad e_{\alpha_2-\beta_2} = (1, -1), \quad e_{\alpha_2-\beta_2+1} = \cdots = e_{k+1} = (1, 0).$$

Finally, suppose $(\alpha_1, \beta_2) \notin P$ (Figure 1, fourth image). We proceed similarly to the case of $\alpha_2 < \beta_2$:

$$\begin{aligned} e_1 &= \ldots = e_{\alpha_2-(\zeta_{\max}-\zeta_{\alpha_1})} = (0, -1),\\ e_{\alpha_2-(\zeta_{\max}-\zeta_{\alpha_1})+1} &= (1, \zeta_{\alpha_1} - \zeta_{\alpha_1+1}), \quad \ldots, \quad e_{\alpha_2-(\zeta_{\max}-\zeta_{\alpha_1})+u-1-\alpha_1} = (1, \zeta_{u-2} - \zeta_{u-1}),\\ e_{\alpha_2-(\zeta_{\max}-\zeta_{\alpha_1})+u-\alpha_1} &= (1, \zeta_{\max} - \zeta_{u-1} - \beta_2),\\ e_{\alpha_2-(\zeta_{\max}-\zeta_{\alpha_1})+u-\alpha_1+1} &= \cdots = e_{k+1} = (1, 0). \end{aligned}$$

We will use the notation $k(\bar\alpha,\bar\beta)=k(\bar\beta,\bar\alpha)=k$ and $e_\ell(\bar\alpha,\bar\beta)=e_\ell$. Note the following two properties of these vectors, both immediate from concavity and monotonicity:

- For any subset $I\subset\{1,\dots,k(\bar\alpha,\bar\beta)+1\}$ one has $\bar\alpha+\sum_{j\in I}e_j(\bar\alpha,\bar\beta)\in P$.
- $\bar\alpha+\sum_{j=1}^{k(\bar\alpha,\bar\beta)+1}e_j(\bar\alpha,\bar\beta)=\bar\beta$ and there is no proper nonempty $I\subset\{1,\dots,k(\bar\alpha,\bar\beta)+1\}$ such that $\bar\alpha+\sum_{j\in I}e_j(\bar\alpha,\bar\beta)\in\{\bar\alpha,\bar\beta\}$.

We will also write $\varkappa(\alpha,\beta)$ for the number of vertical segments, i.e., we set

$$\varkappa(\alpha,\beta)=k(\alpha,\beta)+1-(\beta_1-\alpha_1)=\begin{cases}\beta_2-\alpha_2 & \text{if } \alpha_2\le\beta_2 \text{ and } (\beta_1,\alpha_2)\in P,\\ \beta_2-(\zeta_{\max}-\zeta_{\beta_1}) & \text{if } \alpha_2\le\beta_2 \text{ and } (\beta_1,\alpha_2)\notin P,\\ \alpha_2-\beta_2-1 & \text{if } \alpha_2>\beta_2 \text{ and } (\alpha_1,\beta_2)\in P,\\ \alpha_2-(\zeta_{\max}-\zeta_{\alpha_1}) & \text{if } \alpha_2>\beta_2 \text{ and } (\alpha_1,\beta_2)\notin P.\end{cases}\tag{4-5}$$

Proposition 4.11. *The coefficients of the following series are zero in the ring $J^\infty_{\mathrm{red}}(R(P))$:*

$$W'_{\bar\alpha,\bar\beta,k'}=\sum_{I\subset\{1,\dots,k(\bar\alpha,\bar\beta)\}}(-1)^{|I|}\frac{\partial^{k'}Y_{\bar\alpha+\sum_{j\in I}e_j(\bar\alpha,\bar\beta)}(s)}{\partial s^{k'}}Y_{\bar\beta-\sum_{j\in I}e_j(\bar\alpha,\bar\beta)}(s),\quad k'=0,\dots,k(\bar\alpha,\bar\beta)-1.\tag{4-6}$$

Proof. We define the map $[0,1]^{k(\bar\alpha,\bar\beta)+1}\to P$:

$$(x_1,\dots,x_{k(\bar\alpha,\bar\beta)+1})\mapsto\bar\alpha+x_1e_1(\bar\alpha,\bar\beta)+\dots+x_{k(\bar\alpha,\bar\beta)+1}e_{k(\bar\alpha,\bar\beta)+1}(\bar\alpha,\bar\beta).$$

This map of polytopes defines the map of rings $\eta:R(\mathcal{B}_{k(\bar\alpha,\bar\beta)+1})\to R(P)$. Applying the corresponding arc map to $W_{k(\bar\alpha,\bar\beta)+1,k'}$ (1-14) we have

$$J^\infty(\eta)(W_{k(\bar\alpha,\bar\beta)+1,k'})=W'_{\bar\alpha,\bar\beta,k'}.$$

Thus the element $W'_{\bar\alpha,\bar\beta,k'}$ is nilpotent. This completes the proof. □

We define a lexicographic order on points $\bar\alpha\in P$:

$$\bar\alpha<\bar\beta\quad\text{if } \alpha_2<\beta_2 \text{ or } \alpha_2=\beta_2 \text{ and } \alpha_1<\beta_1.$$

We define the following monomial order on the monomials in the variables $X_{\bar\alpha}$, $\bar\alpha\in P$. Let $X^{\bar r}:=\prod_{\bar\alpha\in P}X_{\bar\alpha}^{r_{\bar\alpha}}$. We define $r_i:=\sum_j r_{(i,j)}$, $L(r):=\sum_{\bar\alpha\in P}r_{\bar\alpha}$. If

$$(L(\bar r),r_0,r_1,\dots,r_\eta)>(L(\bar r'),r'_0,r'_1,\dots,r'_\eta)$$

in the standard lexicographic order, then $X^{\bar r}\succ X^{\bar r'}$. For $\bar r=(r_{\bar\alpha})_{\bar\alpha\in P}$, $\bar r'=(r'_{\bar\alpha})_{\bar\alpha\in P}$ such that

$$(L(\bar r),r_0,r_1,\dots,r_\eta)=(L(\bar r'),r'_0,r'_1,\dots,r'_\eta)$$

the order $\prec$ is lexicographic: we consider the minimal α with respect $<$ for which $r_{\bar\alpha}\ne r'_{\bar\alpha}$ and set $X^{\bar r}\prec X^{\bar r'}$ if $r_{\bar\alpha}<r'_{\bar\alpha}$.

As before, we have an induced order on the graded parts of the ring $J^\infty(\Bbbk[X_\alpha])$ and a filtration on the ring $J^\infty(R(P))$. Let $W'_{\bar\alpha,\bar\beta,k'}(X)$ be defined as the right-hand side of (4-6) with $Y_\bullet$ replaced by $X_\bullet$.

Lemma 4.12. *The initial part* $\mathrm{in}_{\prec} W'_{\bar\alpha,\bar\beta,k'}(X)$ *is equal to*

$$\frac{\partial^{k'} X_{\bar\alpha}(s)}{\partial s^{k'}} X_{\bar\beta}(s).$$

Proof. We need to prove that $X_{\bar\alpha+\sum_{j\in I} e_j(\bar\alpha,\bar\beta)} X_{\bar\beta-\sum_{j\in I} e_j(\bar\alpha,\bar\beta)} \prec X_{\bar\alpha} X_{\bar\beta}$. This is clear from the definition of the order. □

Corollary 4.13. *The dual of the graded component* $\mathrm{gr}_{\prec} J^{\infty}_{\mathrm{red}} R(P)[\bar r]$ *has a natural homogeneous* $\mathfrak{h}[s]$*-equivariant embedding into the space*

$$\prod_{\bar\alpha<\bar\beta\in P} \prod_{1\le i\le r_{\bar\alpha}, 1\le j\le r_{\bar\beta}} (t^{(i)}_{\bar\alpha} - t^{(j)}_{\bar\beta})^{k(\bar\alpha,\bar\beta)} \Lambda_{\bar r}(\boldsymbol{t}).$$

The rest of this section is dedicated to the proof of the following theorem.

Theorem 4.14. *The coefficients of the series* $W'_{\bar\alpha,\bar\beta,k'}(X)$ *with distinct* $\bar\alpha, \bar\beta \in P$ *and* $k' \in [0, k(\bar\alpha, \bar\beta) - 1]$ *generate the ideal of relations in* $J^{\infty}_{\mathrm{red}}(R(P))$. *Initial parts* $\mathrm{in}_{\prec} W'_{\bar\alpha,\bar\beta,k'}(X)$ *generate the corresponding initial ideal.*

Proof. As before we study the dual map for the inclusion $\varphi_{\bar r} : A_{\bar r} \hookrightarrow J^{\infty}(\Bbbk[z_1, z_2, w])[\bar a, L]$, where $\bar a = \sum_{\bar\alpha\in P} r_{\bar\alpha}\bar\alpha$, $L = L(\bar r)$. Recall the notation $\bar r \in \mathcal{R}(\bar a, L)$. The image of the map $\varphi^{\vee}_{\bar r} : \Lambda_{a_1,a_2,L}(\boldsymbol{s}) \to \Lambda_{\bar r}(\boldsymbol{t})$,

$$s_1^{(\alpha_1^1(j-1)+1)}, \dots, s_1^{(\alpha_1^1(j-1)+\alpha_1^1)}, s_2^{(\alpha_2^1(j-1)+1)}, \dots, s_2^{(\alpha_2^1(j-1)+\alpha_2^1)}, s_w^{(j)} \mapsto t^{(j)}_{\bar\alpha^1}, \quad j = 1, \dots, r_{\bar\alpha^1};$$

$$s_1^{(r_{\bar\alpha^1}\alpha_1^1+\alpha_1^2(j-1)+1)}, \dots, s_1^{(r_{\bar\alpha^1}\alpha_1^1+\alpha_1^2(j-1)+\alpha_1^2)}, s_2^{(r_{\bar\alpha^1}\alpha_2^1+\alpha_2^2(j-1)+1)}, \dots,$$
$$s_2^{(r_{\bar\alpha^1}\alpha_2^1+\alpha_2^2(j-1)+\alpha_2^2)}, s_w^{(r_{\bar\alpha^1}+j)} \mapsto t^{(j)}_{\bar\alpha^2}, \quad j = 1, \dots, r_{\bar\alpha^2};$$

$$\dots$$

is isomorphic to the dual space of $A_{\bar r}$. The dual space to $\sum_{\bar r'\preceq\bar r} A_{\bar r} / \sum_{\bar r'\prec\bar r} A_{\bar r}$ is isomorphic to the space $\varphi^{\vee}_{\bar r}\big(\bigcap_{\bar r'\in\mathcal{R}(\bar a,L),\bar r'\prec\bar r} \ker(\varphi^{\vee}_{\bar r'})\big)$. Our goal is to compute this space.

We consider the set of variables $u_i^{(j)}$, $i = 0, \dots, \eta$, $j = 1, \dots, r_i$. Consider the map

$$\varphi^{\vee}_{(r_0,\dots,r_\eta)} \otimes \mathrm{id} : \Lambda_{a_1,L}(\boldsymbol{s}) \otimes \Lambda_{a_2}(\boldsymbol{s}_2) = \Lambda_{a_1,a_2,L}(\boldsymbol{s}) \to \Lambda_{(r_0,\dots,r_\eta,a_2)}(\boldsymbol{u}, \boldsymbol{s}_2)$$

(see Section 4.1):

$$s_w^{(j)} \mapsto u_0^{(j)}, \qquad j = 1, \dots, r_0;$$
$$s_1^{(j)}, s_w^{(r_0+j)} \mapsto u_1^{(r_0+j)}, \qquad j = 1, \dots, r_1;$$
$$s_1^{(r_1+2j-1)}, s_1^{(r_1+2j)}, s_w^{(r_0+r_1+j)} \mapsto u_1^{(r_0+r_1+j)}, \quad j = 1, \dots, r_2; \dots; \quad s_2^{(i)} \mapsto s_2^{(i)}.$$

We have an order on vectors $\mathrm{pr}_2(\bar r) := (r_0, \dots, r_\eta)$ as in the previous section. By definition,

- if $\bar r' \prec \bar r$, then $\mathrm{pr}_2(\bar r') \le \mathrm{pr}_2(\bar r)$;
- the map $\varphi^{\vee}_{\bar r}$ is right divisible by $\varphi^{\vee}_{\mathrm{pr}_2(\bar r)}$.

More precisely we define the map $\tilde{\psi}_{\bar{r}}:\Lambda_{(r_0,\dots,r_\eta,a_2)}(\boldsymbol{u},\boldsymbol{s}_2)\to\Lambda_{\bar{r}}(\boldsymbol{t})$,

$$u_i^{(j)},\, s_2^{((\zeta_{\max}-\zeta_i)(j-1)+1)},\dots,s_2^{((\zeta_{\max}-\zeta_i)j)}\mapsto t^{(j)}_{i,(\zeta_{\max}-\zeta_i)},\qquad j=1,\dots,r_{i,\zeta_{\max}-\zeta_i};$$

$$u_i^{(r_{i,\zeta_{\max}-\zeta_i}+j)},\, s_2^{(r_{i,\zeta_0-\zeta_i}(\zeta_{\max}-\zeta_i)+(\zeta_{\max}-\zeta_i+1)(j-1)+1)},\dots,\\ s_2^{(r_{i,\zeta_{\max}-\zeta_i}(\zeta_{\max}-\zeta_i)+(\zeta_{\max}-\zeta_i+1)j)}\mapsto t^{(j)}_{i,(\zeta_{\max}-\zeta_i+1)},\qquad j=1,\dots,r_{i,\zeta_{\max}-\zeta_i+1};\dots$$

Then

$$\varphi^\vee_{\bar{r}}=\tilde{\psi}_{\bar{r}}\circ(\varphi^\vee_{\mathrm{pr}_2(\bar{r})}\otimes\mathrm{id}).$$

Thus we have

$$\begin{aligned}\varphi^\vee_{\mathrm{pr}_2(\bar{r})}\otimes\mathrm{id}\Big(\bigcap_{\bar{r}'\prec\bar{r}}\ker\varphi^\vee_{\bar{r}'}\Big)\\ &\supset\prod_{1\le i<j\le\eta}\ \prod_{1\le\ell\le r_i,1\le\ell'\le r_j}(u_i^{(\ell)}-u_j^{(\ell')})^{j-i-1}\Lambda_{(r_0,\dots,r_\eta,a_2)}(\boldsymbol{u},\boldsymbol{s}_2)\cap\bigcap_{\bar{r}'\prec\bar{r},\mathrm{pr}_2(\bar{r}')=(r_0,\dots,r_\eta)}\ker(\tilde{\psi}_{\bar{r}'})\\ &\supset\prod_{1\le i<j\le\eta}\ \prod_{1\le\ell\le r_i,1\le\ell'\le r_j}(u_i^{(\ell)}-u_j^{(\ell')})^{j-i-1}\Big(\bigcap_{\bar{r}'\prec\bar{r},\mathrm{pr}_2(\bar{r}')=(r_0,\dots,r_\eta)}\ker(\tilde{\psi}_{\bar{r}'})\Big).\end{aligned}$$

Therefore

$$\varphi^\vee_{\bar{r}}\Big(\bigcap_{r'\prec r}\ker\varphi^\vee_{\bar{\rho}'}\Big)\supset\tilde{\psi}_{\bar{r}}\Big(\prod_{\substack{0\le a<b\le\eta,\\1\le\ell\le r_i,1\le\ell'\le r_j}}(u_i^{(\ell)}-u_j^{(\ell')})^{b-a-1}\Big)\tilde{\psi}_{\bar{r}}(\Lambda_{(r_0,\dots,r_\eta,a_2)}(\boldsymbol{u},\boldsymbol{s}_2)).$$

Consider now the set of $\bar{r}'\prec\bar{r}$ such that $\mathrm{pr}_2(\bar{r}')=\bar{\rho}$, where $\bar{\rho}=\mathrm{pr}_2(\bar{r})$. One has

$$\varphi^\vee_{\bar{r}}\Big(\bigcap_{\bar{r}'\prec\bar{r},\mathrm{pr}_2(\bar{r}')=\bar{\rho}}\ker\varphi^\vee_{\bar{\rho}'}\Big)\supset\tilde{\psi}_{\bar{r}}\Big(\prod_{\substack{0\le a<b\le\eta,\\1\le i\le r_a,1\le j\le r_b}}(u_a^{(i)}-u_b^{(j)})^{b-a-1}\Big)\tilde{\psi}_{\bar{r}}\Big(\bigcap_{\bar{r}'\prec\bar{r},\mathrm{pr}_2(\bar{r}')=\bar{\rho}}\ker(\tilde{\psi}_{\bar{r}'})\Big).\tag{4-7}$$

Let us study the two factors in (4-7) separately. For the first factor one has

$$\tilde{\psi}_{\bar{r}}\Big(\prod_{\substack{0\le a<b\le\eta,\\1\le i\le r_a,1\le j\le r_b}}(u_a^{(i)}-u_b^{(j)})^{b-a-1}\Big)=\prod_{\bar{\alpha},\bar{\beta}\in P,\alpha_1<\beta_1}\ \prod_{1\le\ell\le r_{\bar{\alpha}},1\le\ell'\le r_{\bar{\beta}}}(t_{\bar{\alpha}}^{(\ell)}-t_{\bar{\beta}}^{(\ell')})^{\beta_1-\alpha_1-1}.$$

Now let us compute the second factor. We define the map

$$g_{\bar{\rho}}:\Lambda_{(\bar{\rho},a_2)}(\boldsymbol{u},\boldsymbol{s}_2)\to\Lambda_{\left(\bar{\rho},a_2-\sum_{i=0}^{\eta}r_i(\zeta_{\max}-\zeta_i)\right)}(\boldsymbol{u},\boldsymbol{s}_2),$$

$$s_2^{\left(\sum_{i'=0}^{i-1}r_{i'}(\zeta_{\max}-\zeta_{i'})+(\ell-1)(\zeta_{\max}-\zeta_i)+1\right)},\dots,s_2^{\left(\sum_{i'=0}^{i-1}r_{i'}(\zeta_{\max}-\zeta_{i'})+\ell(\zeta_{\max}-\zeta_i)\right)},\qquad u_i^{(\ell)}\mapsto u_i^{(\ell)},$$

$$s_2^{(\ell)}\mapsto s_2^{\left(\ell-\sum_{i=0}^{\eta}r_i(\zeta_{\max}-\zeta_i)\right)},\qquad \ell\ge\sum_{i=0}^{\eta}r_i(\zeta_{\max}-\zeta_i).$$

Then

$$\tilde{\psi}_{\bar{r}}=\psi_{d(\bar{r})}\circ g_{\bar{\rho}},$$

where $d(\bar r) = d(\bar r)_{i,j},\ i = 0, \dots, \eta,\ j = 0, \dots, \zeta_i,\ d(\bar r)_{i,j} = r_{i,j+\zeta_{\max}-\zeta_i}$ and $\psi_{d(\bar r)}$ is defined in the following way (it differs from the $\psi_{d(\bar r)}$ defined in Section 3.4 only by a change of the subscripts):

$$u_i^{(j)} \mapsto t^{(j)}_{i,(\zeta_{\max}-\zeta_i)}, \quad j = 1, \dots, r_{i,\zeta_{\max}-\zeta_i};$$
$$u_i^{(r_{i,\zeta_{\max}-\zeta_i}+j)}, s_2^{(j)} \mapsto t^{(j)}_{i,(\zeta_{\max}-\zeta_i+1)}, \quad j = 1, \dots, r_{i,\zeta_{\max}-\zeta_i+1};$$
$$u_i^{(r_{i,\zeta_{\max}-\zeta_i}+r_{i,\zeta_{\max}-\zeta_i+1}+j)}, s_2^{(r_{i,\zeta_{\max}-\zeta_i+1}+2j-1)}, s_2^{(r_{i,\zeta_{\max}-\zeta_i+1}+2j)} \mapsto t^{(j)}_{i,(\zeta_{\max}-\zeta_i+2)}, \quad j = 1, \dots, r_{i,\zeta_{\max}-\zeta_i+2}; \dots$$

The map $g_{\bar\rho}$ is surjective. Thus

$$\tilde\psi_{\bar r}\Bigg(\bigcap_{\bar r'\prec \bar r,\, \mathrm{pr}_2(\bar r')=\rho} \ker(\tilde\psi_{\bar r'})\Bigg) = \psi_{d(\bar r)}\Bigg(\bigcap_{\bar r'\succ \bar r,\, \mathrm{pr}_2(\bar r')=\rho} \ker(\psi_{d(\bar r')})\Bigg).$$

Note that the order on vectors $\bar r$ gives the lexicographic order on vectors $d(\bar r)$ with respect to the following order on coordinates: $(i, j) \prec (i', j')$ if and only if $j + (\zeta_{\max} - \zeta_i) < j' + (\zeta_{\max} - \zeta_{i'})$ or $j + (\zeta_{\max} - \zeta_i) = j' + (\zeta_{\max} - \zeta_{i'})$ and $i < i'$. So using Lemma 3.17 we have

$$\psi_{d(\bar r)}\Bigg(\bigcap_{\bar r'\prec \bar r,\, \mathrm{pr}_2(\bar r')=\bar\rho} \ker(\psi_{d(\bar r')})\Bigg) \supset \prod_{\bar\alpha<\bar\beta}\ \prod_{1\le \ell\le r_{\bar\alpha},\, 1\le \ell'\le r_{\bar\beta}} (t_{\bar\alpha}^{(\ell)} - t_{\bar\beta}^{(\ell)})^{\varkappa(\bar\alpha,\bar\beta)} \Lambda_{\bar r}(\boldsymbol t)$$

(see (4-5)). Multiplying the two factors of (4-7) we obtain

$$\varphi^\vee_{\bar r}\Bigg(\bigcap_{r'\prec r} \ker(\varphi^\vee_{\bar r'})\Bigg) \supset \prod_{\bar\alpha<\bar\beta}\ \prod_{1\le \ell\le r_{\bar\alpha},\, 1\le \ell'\le r_{\bar\beta}} (t_{\bar\alpha}^{(\ell)} - t_{\bar\beta}^{(\ell')})^{k(\bar\alpha,\bar\beta)} \Lambda_{\bar r}(\boldsymbol t).$$

Therefore, due to Corollary 4.13 these spaces are equal. Thus $\mathrm{in}_\prec W'_{\bar\alpha,\bar\beta,k'}(X)$ generate the defining ideal of $\mathrm{gr}_\prec J^\infty_{\mathrm{red}}(R(P))$. This completes the proof of the theorem. □

Corollary 4.15. *The action of $\mathcal A_L$ is free on $J^\infty_{\mathrm{red}}(R(P))[L]^*$.*

4.3. *Higher-dimensional case.* In this subsection we study a family of polytopes such that the (dual of the) reduced arc rings of the corresponding toric varieties admit a free action of the polynomial algebras.

Consider a convex lattice polytope $P \subset \mathbb R^n$. Let $\bar\alpha^1, \dots, \bar\alpha^m$ be the tuple of integer points of P, $\prec$ be a monomial order on $\Bbbk[X_{\bar\alpha^1}, \dots, X_{\bar\alpha^m}]$, $\gamma(\bar\alpha^i, \bar\alpha^j) = \gamma(\bar\alpha^i, \bar\alpha^j)$ with $1 \le i, j \le m$ be a collection of nonnegative integers. Assume that for any $i \ne j$ there exists a set of vectors $e_1(\bar\alpha^i, \bar\alpha^j), \dots, e_{\gamma(\bar\alpha^i,\bar\alpha^j)+1}(\bar\alpha^i, \bar\alpha^j)$ with the following properties:

- $\bar\alpha^i + \sum_{\ell=1}^{\gamma(\bar\alpha^i,\bar\alpha^j)+1} e_\ell(\bar\alpha^i, \bar\alpha^j) = \bar\alpha^j$.
- For any $J \subset \{1, \dots, \gamma(\bar\alpha^i, \bar\alpha^j) + 1\}$: $\bar\alpha^i + \sum_{\ell\in J} e_\ell(\bar\alpha^i, \bar\alpha^j) \in P$.
- For any nonempty proper subset $J \subset \{1, \dots, \gamma(\bar\alpha^i, \bar\alpha^j) + 1\}$,
$$X_{\bar\alpha^i} X_{\bar\alpha^j} \prec X_{\bar\alpha^i + \sum_{\ell\in J} e_\ell(\bar\alpha^i,\bar\alpha^j)} X_{\bar\alpha^j - \sum_{\ell\in J} e_\ell(\bar\alpha^i,\bar\alpha^j)}.$$
- $e_\ell(\bar\alpha^j, \bar\alpha^i) = -e_\ell(\bar\alpha^i, \bar\alpha^j)$.

We call $(P, \prec, \gamma, \{e_\ell\})$ a *cube generating data.*

We define the linear maps of polytopes $\eta_{\bar\alpha^i,\bar\alpha^j} : [0,1]^{\gamma(\bar\alpha^i,\bar\alpha^j)+1} \to P$,

$$\eta_{\bar\alpha^i,\bar\alpha^j}\biggl(\sum_{\ell=1}^{\gamma(\bar\alpha^i,\bar\alpha^j)+1} x_\ell e_\ell\biggr) = \bar\alpha^i + \sum_{\ell=1}^{\gamma(\bar\alpha^i,\bar\alpha^j)+1} x_\ell e_\ell(\bar\alpha^i,\bar\alpha^j), \tag{4-8}$$

where $\{e_\ell\}$ is the basis vectors of $\mathbb{R}^{\gamma(\bar\alpha^i,\bar\alpha^j)+1}$, $0 \le x_\ell \le 1$. The corresponding map of rings

$$R([0,1]^{\gamma(\bar\alpha^i,\bar\alpha^j)+1}) \to R(P)$$

will also be denoted by $\eta_{\bar\alpha^i,\bar\alpha^j}$.

For any $k \le \gamma(\bar\alpha^i,\bar\alpha^j) - 1$,

$$J^\infty(\eta_{\bar\alpha^i,\bar\alpha^j})(W_{\gamma(\bar\alpha^i,\bar\alpha^j)+1,k})$$

is nilpotent. Therefore the coefficients of the series

$$W_{\bar\alpha^i,\bar\alpha^j,k}(X) := \sum_{I\subset\{2,\dots,\gamma(\bar\alpha^i,\bar\alpha^j)+1\}} (-1)^{|I|} X_{\bar\alpha^i+\sum_{\ell\in I\cup\{1\}} e_\ell(\bar\alpha^i,\bar\alpha^j)}(s)\,\frac{\partial^k X_{\bar\alpha^j-\sum_{\ell\in I\cup\{1\}} e_\ell(\bar\alpha^i,\bar\alpha^j)}(s)}{\partial s^k}$$

belong to the defining ideal of $J^\infty_{\mathrm{red}}(R(P))$ for $k \le \gamma(\bar\alpha^i,\bar\alpha^j) - 1$. By construction we have

$$\mathrm{in}_\prec W_{\bar\alpha^i,\bar\alpha^j,k}(X) = X_{\bar\alpha^i}(s)\,\frac{\partial^k X_{\bar\alpha^j}(s)}{\partial s^k}. \tag{4-9}$$

Recall that in view of Proposition 1.28 the coefficients of (4-9) belong to the defining ideal of the associated graded ring $\mathrm{gr}_\prec J^\infty_{\mathrm{red}}(R(P))$. Suppose that, moreover, they generate this ideal. In this case we call the data $(P,\prec,\gamma,\{e_\ell\})$ a *strict cube generating data*.

Proposition 4.16. *Consider a strict cube generating data* $(P,\prec,\gamma(\bar\alpha^i,\bar\alpha^j),\{e_\ell(\bar\alpha^i,\bar\alpha^j)\})$. *Then for any* $\bar\rho\in\mathbb{N}^m$ *we have the following isomorphism of* grad*-graded spaces*:

$$\mathrm{gr}_\prec J^\infty_{\mathrm{red}}(R(P))[\bar\rho]^* \simeq \prod_{i<j}\ \prod_{\substack{1\le\ell\le\rho_i\\ 1\le\ell'\le\rho_j}} (t_i^{(\ell)} - t_j^{(\ell')})^{\gamma(\bar\alpha^i,\bar\alpha^j)}\Lambda_{\bar\rho}(\boldsymbol{t}).$$

Proof. This follows from Proposition 2.8. □

Corollary 4.17. *For $L\in\mathbb{N}$ the action of $\mathcal{A}_L$ is free on $J^\infty_{\mathrm{red}}(R(P))[L]^*$.*

Proof. This follows from Theorem 3.14. □

Consider a convex function $\zeta : P \to \mathbb{R}_{\ge 0}$ with $\zeta(\bar\alpha^i) = \zeta_i \in \mathbb{Z}$, set $\zeta_{\max} = \max\{\zeta_i\}$. Denote by P^ζ the polytope in $\mathbb{R}^{n+1}$ obtained as the convex hull of $P\times\zeta_{\max}$ and all $\bar\alpha^i \times (\zeta_{\max} - \zeta_i)$. In what follows we construct a cube generating data for P^ζ starting from a cube generating data for P satisfying the following condition. For any $\bar\alpha^i,\bar\alpha^j \in P$ such that $\zeta_i \le \zeta_j$, there exist nonnegative integers $f_1(\bar\alpha^i,\bar\alpha^j),\dots,f_{\gamma(\bar\alpha^i,\bar\alpha^j)+1}(\bar\alpha^i,\bar\alpha^j)$ such that

- $\zeta_i + \sum_{\ell=1}^{\gamma(\bar\alpha^i,\bar\alpha^j)+1} f_\ell(\bar\alpha^i,\bar\alpha^j) = \zeta_j$;
- for any $J\subset\{1,\dots,\gamma(\bar\alpha^i,\bar\alpha^j)+1\}$: $\zeta_i + \sum_{\ell\in J} f_\ell(\bar\alpha^i,\bar\alpha^j) \le \zeta\bigl(\bar\alpha^i + \sum_{\ell\in J} e_\ell(\bar\alpha^i,\bar\alpha^j)\bigr)$.

Example 4.18. Consider the cube generating data $(P, \prec, \gamma, \{e_\ell\})$. Assume that for any $\bar\alpha^i, \bar\alpha^j$ the ℓ-th coordinate of every $e_p(\bar\alpha^i, \bar\alpha^j)$ is either nonnegative or nonpositive. Then any convex function of the form $\zeta(\bar\alpha) = \zeta(\alpha_\ell)$, i.e., depending only on the ℓ-th coordinate admits such a set of integers $f_p(\bar\alpha^i, \bar\alpha^j)$. Indeed, without loss of generality we may assume that $\zeta(\bar\alpha^i) \geq \zeta(\bar\alpha^j)$. We may consider an enumeration of vectors $e_p(\bar\alpha^i, \bar\alpha^j)$ such that the absolute value of the ℓ-th coordinate is weakly increasing. Let x be the smallest number such that $\zeta\big(\bar\alpha^i + \sum_{p=1}^{x} e_p(\bar\alpha^i, \bar\alpha^j)\big) < \zeta(\bar\alpha^i)$. We put

$$f_q(\bar\alpha^i, \bar\alpha^j) = \begin{cases} 0 & \text{if } 1 \leq q < x, \\ \zeta\big(\bar\alpha^i + \sum_{p=1}^{x} e_p(\bar\alpha^i, \bar\alpha^j)\big) - \zeta(\bar\alpha^i) & \text{if } q = x, \\ \zeta\big(\bar\alpha^i + \sum_{p=1}^{q} e_p(\bar\alpha^i, \bar\alpha^j)\big) - \zeta\big(\bar\alpha^i + \sum_{p=1}^{q-1} e_p(\bar\alpha^i, \bar\alpha^j)\big) & \text{if } x < q \leq \gamma(\bar\alpha^i, \bar\alpha^j) + 1. \end{cases}$$

Note that the numbers $f_q(\bar\alpha^i, \bar\alpha^j)$ are all either nonnegative or nonpositive.

For $\bar r = (r_{(\bar\alpha^i,a)})_{(\bar\alpha^i,a)\in P^\zeta}$ we define $\mathrm{pr}_{n+1}(\bar r) = (\mathrm{pr}_{n+1}(\bar r))_{\bar\alpha^i}$, $i = 1, \dots, m$ by the formula

$$(\mathrm{pr}_{n+1}(\bar r))_{\bar\alpha^i} = \sum_{a=\zeta_{\max}-\zeta_i}^{\zeta_{\max}} r_{(\bar\alpha^i,a)}.$$

For $(\bar\alpha^i, a), (\bar\alpha^j, b) \in P^\zeta \cap \mathbb{Z}^{n+1}$ we let

$$(\bar\alpha^i, a) < (\bar\alpha^j, b) \quad \text{if } a < b \text{ or } a = b \text{ and } i < j. \tag{4-10}$$

We define an order $\prec^\zeta$ on monomials in $\Bbbk[X_{(\bar\alpha^i,a)}]$, $(\bar\alpha^i, a) \in P^\zeta \cap \mathbb{Z}^{n+1}$ in the following way. For r, $r' \in \mathbb{N}^{P^\zeta \cap \mathbb{Z}^{n+1}}$ with $\mathrm{pr}_{n+1}(\bar r) \prec \mathrm{pr}_{n+1}(\bar r')$ we set $\bar r \prec^\zeta \bar r'$ and $X^{\bar r} \prec^\zeta X^{\bar r'}$. If $\mathrm{pr}_{n+1}(\bar r) = \mathrm{pr}_{n+1}(\bar r')$, then $\prec^\zeta$ compares r to r' and $X^{\bar r}$ to $X^{\bar r'}$ lexicographically with respect to the order (4-10).

Suppose that $(\bar\alpha^i, a) \leq (\bar\alpha^j, b)$. We set

$$\gamma((\bar\alpha^i, a), (\bar\alpha^j, b)) = \gamma(\bar\alpha^i, \bar\alpha^j) + \varkappa((\bar\alpha^i, a), (\bar\alpha^j, b)),$$

where

$$\varkappa((\bar\alpha^i, a), (\bar\alpha^j, b)) = \begin{cases} b - (\zeta_{\max} - \zeta_j) & \text{if } \zeta_{\max} - \zeta_j > a, \\ b - a & \text{if } \zeta_{\max} - \zeta_j \leq a \text{ and } i \leq j, \\ b - a - 1 & \text{if } \zeta_{\max} - \zeta_j \leq a \text{ and } i > j. \end{cases} \tag{4-11}$$

We construct the vectors $e_\ell((\bar\alpha^i, a), (\bar\alpha^j, b))$ in the following way.

Suppose that $\zeta_{\max} - \zeta_j > a$. Let y be the number such that

$$\sum_{\ell=1}^{y-1} f_\ell(\bar\alpha^i, \bar\alpha^j) < a - (\zeta_{\max} - \zeta_i) \leq \sum_{\ell=1}^{y} f_\ell(\bar\alpha^i, \bar\alpha^j).$$

Then

$$e_\ell((\bar\alpha^i, a), (\bar\alpha^j, b)) = \begin{cases} (e_\ell(\bar\alpha^i, \bar\alpha^j), 0) & \text{if } \ell = 1, \dots, y-1, \\ (e_y(\bar\alpha^i, \bar\alpha^j), f_y(\bar\alpha^i, \bar\alpha^j) - (\zeta_{\max} - \zeta_i)) & \text{if } \ell = y, \\ (e_\ell(\bar\alpha^i, \bar\alpha^j), f_\ell(\bar\alpha^i, \bar\alpha^j)), & \text{if } \ell = y+1, \dots, \gamma(\bar\alpha^i, \bar\alpha^j) + 1, \\ (0, 1), & \text{if } \ell = \gamma(\bar\alpha^i, \bar\alpha^j) + 2, \dots, \gamma((\bar\alpha^i, a), (\bar\alpha^j, b)) + 1. \end{cases}$$

If $a \ge \zeta_{\max} - \zeta_j$ and $i \le j$, then

$$e_\ell((\bar\alpha^i, a), (\bar\alpha^j, b)) = \begin{cases} (e_\ell(\bar\alpha^i, \bar\alpha^j), 0) & \text{if } \ell = 1, \dots, \gamma(\bar\alpha^i, \bar\alpha^j) + 1, \\ (0, 1) & \text{if } \ell = \gamma(\bar\alpha^i, \bar\alpha^j) + 2, \dots, \gamma(\bar\alpha^i, \bar\alpha^j) + b - a + 1. \end{cases}$$

If $a \ge \zeta_{\max} - \zeta_j$ and $i > j$, then

$$e_\ell((\bar\alpha^i, a), (\bar\alpha^j, b)) = \begin{cases} (e_\ell(\bar\alpha^i, \bar\alpha^j), 0) & \text{if } \ell = 1, \dots, \gamma(\bar\alpha^i, \bar\alpha^j), \\ (e_{\gamma(\bar\alpha^i, \bar\alpha^j)+1}, 1) & \text{if } \ell = \gamma(\bar\alpha^i, \bar\alpha^j) + 1, \\ (0, 1) & \text{if } \ell = \gamma(\bar\alpha^i, \bar\alpha^j) + 2, \dots, \gamma(\bar\alpha^i, \bar\alpha^j) + b - a. \end{cases}$$

For $a > b$ we put $e_\ell((\bar\alpha^i, a), (\bar\alpha^j, b)) = -e_\ell((\bar\alpha^j, b), (\bar\alpha^i, a))$. The following lemma follows directly from the definitions.

Lemma 4.19. *$(P^\zeta, \prec^\zeta, \gamma, \{e_\ell\})$ satisfy the conditions for cube generating data.*

We consider the maps

$$\eta_{(\bar\alpha^i, a), (\bar\alpha^j, b)} : [0, 1]^{\gamma((\bar\alpha^i, a), (\bar\alpha^j, b))+1} \to P^\zeta$$

corresponding to this data. By construction and Proposition 2.7 we have

$$(\mathrm{gr}_\prec J^\infty_{\mathrm{red}}(R(P^\zeta))[\bar r])^* \hookrightarrow \prod_{(\bar\alpha^i, a) \prec (\bar\alpha^j, b)} \prod_{\substack{1 \le \ell \le r_{(\bar\alpha^i, a)}, \\ 1 \le \ell' \le r_{(\bar\alpha^j, b)}}} (t^{(\ell)}_{(\bar\alpha^i, a)} - t^{(\ell')}_{(\bar\alpha^j, b)})^{\gamma((\bar\alpha^i, a), (\bar\alpha^j, b))} \Lambda_{\bar r}(\boldsymbol{t}). \tag{4-12}$$

We now show that this inclusion is, in fact, an equality.

Theorem 4.20. *Assume that $(P, \prec, \gamma, \{e_\ell\})$ is a strict cube generating data. Then $(P^\zeta, \prec^\zeta, \gamma, \{e_\ell\})$ is a strict cube generating data.*

Proof. In this proof we compute the space $(\mathrm{gr}_\prec J^\infty_{\mathrm{red}}(R(P^\zeta))[\bar r])^*$ in the same way as in the proof of Theorem 4.14. As before we study the dual map for the inclusion

$$\varphi_{\bar r} : A_{\bar r} \hookrightarrow J^\infty(\Bbbk[z_1, z_2, \dots, z_{n+1}, w])[(\bar a, a_{n+1}), L],$$

where

$$(\bar a, a_{n+1}) = \sum_{(\bar\alpha, \alpha_{n+1}) \in P^\zeta} r_{(\bar\alpha, \alpha_{n+1})} (\bar\alpha, \alpha_{n+1}), \quad L = \sum_{(\bar\alpha, \alpha_{n+1}) \in P^\zeta} r_{(\bar\alpha, \alpha_{n+1})}.$$

The image of the map $\varphi^\vee_{\bar r} : \Lambda_{(\bar a, a_{n+1}), L}(\boldsymbol{s}) \to \Lambda_{\bar r}(\boldsymbol{t})$,

$$s_1^{(\alpha_1^1(j-1)+1)}, \dots, s_1^{(\alpha_1^1(j-1)+\alpha_1^1)}, \dots, s_{n+1}^{((\zeta_{\max}-\zeta_1)(j-1)+1)}, \dots,$$
$$s_{n+1}^{((\zeta_{\max}-\zeta_1)j)}, s_w^{(j)} \mapsto t^{(j)}_{(\bar\alpha^1, \zeta_{\max}-\zeta_i)}, \quad j = 1, \dots, r_{(\bar\alpha^1, \zeta_{\max}-\zeta_1)};$$

$$s_1^{(r_{(\bar\alpha^1, \zeta_{\max}-\zeta_1)}\alpha_1^1 + \alpha_1^1(j-1)+1)}, \dots, s_{n+1}^{(r_{(\bar\alpha^1, \zeta_{\max}-\zeta_1)}(\zeta_{\max}-\zeta_i) + (\zeta_{\max}-\zeta_1+1)j)},$$
$$s_w^{(j)} \mapsto t^{(j)}_{(\bar\alpha^1, \zeta_{\max}-\zeta_i)}, \quad j = r_{(\bar\alpha^1, \zeta_{\max}-\zeta_1)} + 1, \dots, r_{(\bar\alpha^1, \zeta_{\max}-\zeta_1)} + r_{(\bar\alpha^1, \zeta_{\max}-\zeta_1-1)};$$

$$\dots$$

is isomorphic to the dual space of $A_{\bar r}$. Here for every integer point $(\bar\alpha,\alpha_{n+1})\in P^\zeta$ and $1\le i\le n+1$ the map $\varphi^\vee_{\bar r}$ takes $r_{(\bar\alpha,\alpha_{n+1})}\alpha_i$ variables of the form $s_i^\bullet$ to variables of the form $t^\bullet_{(\bar\alpha,\alpha_{n+1})}$, it also takes $r_{(\bar\alpha,\alpha_{n+1})}$ of the $s_w^\bullet$ to the $t^\bullet_{(\bar\alpha,\alpha_{n+1})}$. The dual space to $A_{\bar r}\big/\big(A_{\bar r}\cap\big(\bigoplus_{r'\prec r}A_{\bar r}\big)\big)$ is isomorphic to the space $\varphi^\vee_{\bar r}\big(\bigcap_{r'\prec r}\ker(\varphi^\vee_{\bar r'})\big)$. Our goal is to compute this space.

Denote $\bar\rho:=\mathrm{pr}_{n+1}(\bar r)$. We consider the set of variables $u_i^{(j)}=u_{\alpha^i}^{(j)}$, $i\in P$, $j=1,\dots,\rho_i$. Consider the map $\varphi^\vee_{\bar\rho}\otimes\mathrm{id}:\Lambda_{(\bar a,a_{n+1}),L}(\boldsymbol{s})\to\Lambda_{(\bar\rho,a_{n+1})}(\boldsymbol{u},\boldsymbol{s}_{n+1})$, i.e., $s^{(i)}_{n+1}\mapsto s^{(i)}_{n+1}$ and it is equal to $\varphi^\vee_{\bar\rho}$ on $s_1^{(i)},\dots,s_n^{(i)},s_w^{(i)}$ (but with t replaced by u).

By definition,

- if $\bar r'\prec\bar r$, then $\mathrm{pr}_{n+1}(\bar r')\preceq\mathrm{pr}_{n+1}(\bar r)$;
- the map $\varphi^\vee_{\bar r}$ is right divisible by $\varphi^\vee_\rho$.

More precisely, we define the map $\tilde\psi_{\bar r}:\Lambda_{(\bar\rho,a_{n+1})}(\boldsymbol{u},\boldsymbol{s}_{n+1})\to\Lambda_{\bar r}(\boldsymbol{t})$,

$$u_i^{(j)},s_{n+1}^{((\zeta_{\max}-\zeta_i)(j-1)+1)},\dots,s_{n+1}^{((\zeta_{\max}-\zeta_i)j)}\mapsto t^{(j)}_{(\bar\alpha^i,(\zeta_{\max}-\zeta_i))},\qquad j=1,\dots,r_{i,\zeta_{\max}-\zeta_i};$$

$$u_i^{(r_{i,\zeta_{\max}-\zeta_i}+j)},s_{n+1}^{(r_{i,\zeta_{\max}-\zeta_i}(\zeta_{\max}-\zeta_i)+(\zeta_{\max}-\zeta_i+1)(j-1)+1)},\dots,$$
$$s_{n+1}^{(r_{i,\zeta_{\max}-\zeta_i}(\zeta_{\max}-\zeta_i)+(\zeta_{\max}-\zeta_i+1)j)}\mapsto t^{(j)}_{(\bar\alpha^i(\zeta_{\max}-\zeta_i+1))},\qquad j=1,\dots,r_{i,\zeta_{\max}-\zeta_i+1};$$
$$\dots$$

Here for every $(\bar\alpha_i,\alpha_{n+1})\in P^\zeta$ we have $r_{(\bar\alpha_i,a_{n+1})}$ variables of the form $u_i^\bullet$ and α_{n+1} variables of the form $s^\bullet_{n+1}$ being mapped to variables of the form $t^\bullet_{(\bar\alpha_i,\alpha_{n+1})}$. Then $\varphi^\vee_{\bar r}=\tilde\psi_{\bar r}\circ(\varphi^\vee_\rho\otimes\mathrm{id})$.

By the definition of strict cube generating data and Proposition 1.28, the right-hand sides of (4-9) generate the ring $\mathrm{gr}_\prec J^\infty_{\mathrm{red}}(R(P))$. A component of the latter ring is considered in Corollary 3.7 and Proposition 2.8 provides

$$\varphi^\vee_{\bar\rho}\otimes\mathrm{id}\Big(\bigcap_{\bar\rho'\prec\rho}\ker(\varphi^\vee_{\bar\rho'}\otimes\mathrm{id})\Big)=\prod_{1\le i<j\le m}\ \prod_{1\le\ell\le\rho_i,1\le\ell'\le\rho_j}(u^{(\ell)}_{\bar\alpha^i}-u^{(\ell')}_{\bar\alpha^j})^{\gamma(\bar\alpha^i,\bar\alpha^j)}\Lambda_{(\rho,\alpha_{n+1})}(\boldsymbol{u},\boldsymbol{s}_{n+1}).$$

Consequently,

$$\varphi^\vee_\rho\otimes\mathrm{id}\Big(\bigcap_{\mathrm{pr}_{n+1}(\bar r')\prec\bar\rho}\ker\varphi^\vee_{\bar r'}\Big)\supset\prod_{1\le i<j\le m}\ \prod_{1\le\ell\le\rho_i,1\le\ell'\le\rho_j}(u^{(\ell)}_{\bar\alpha_i}-u^{(\ell')}_{\bar\alpha_j})^{\gamma(\bar\alpha^i,\bar\alpha^j)}\Lambda_{(\rho,\alpha_{n+1})}(\boldsymbol{u},\boldsymbol{s}_{n+1}).$$

Therefore,

$$\begin{aligned}\varphi^\vee_\rho\otimes\mathrm{id}\Big(\bigcap_{\bar r'\prec\bar r}\ker\varphi^\vee_{\bar r'}\Big)&\\ \supset\prod_{1\le i<j\le m}\ \prod_{1\le\ell\le\rho_i,1\le\ell'\le\rho_j}&(u^{(\ell)}_{\bar\alpha_i}-u^{(\ell')}_{\bar\alpha_j})^{\gamma(\bar\alpha^i,\bar\alpha^j)}\Lambda_{(\rho,\alpha_{n+1})}(\boldsymbol{u},\boldsymbol{s}_{n+1})\cap\bigcap_{\bar r'\prec^\zeta\bar r,\mathrm{pr}_{n+1}(\bar r')=\bar\rho}\ker(\tilde\psi_{\bar r'})\\ \supset\prod_{1\le i<j\le m}\ \prod_{1\le\ell\le\rho_i,1\le\ell'\le\rho_j}&(u^{(\ell)}_{\bar\alpha_i}-u^{(\ell')}_{\bar\alpha_j})^{\gamma(\bar\alpha^i,\bar\alpha^j)}\Big(\bigcap_{\bar r'\prec^\zeta\bar r,\mathrm{pr}_{n+1}(\bar r')=\bar\rho}\ker(\tilde\psi_{\bar r'})\Big).\end{aligned}$$

Hence

$$\varphi_{\bar r}^{\vee}\Big(\bigcap_{r'\prec^{\zeta} r}\ker\varphi_{\bar r'}^{\vee}\Big)\supset\tilde\psi_{\bar r}\Big(\prod_{1\le i<j\le m}\prod_{\substack{1\le\ell\le\rho_i,\\1\le\ell'\le\rho_j}}(u_{\bar\alpha_i}^{(\ell)}-u_{\bar\alpha_j}^{(\ell')})^{\gamma(\bar\alpha^i,\bar\alpha^j)}\Big)\tilde\psi_{\bar r}\Big(\bigcap_{\bar r'\prec^{\zeta}\bar r,\mathrm{pr}_{n+1}(\bar r')=\bar\rho}\ker(\tilde\psi_{\bar r'})\Big).\tag{4-13}$$

Let us study the two factors above separately. By definition of $\tilde\psi_{\bar r}$ we have

$$\tilde\psi_{\bar r}\Big(\prod_{1\le i<j\le m} i\prod_{\substack{1\le\ell\le\rho_i,\\1\le\ell'\le\rho_j}}(u_{\bar\alpha_i}^{(\ell)}-u_{\bar\alpha_j}^{(\ell')})^{\gamma(\bar\alpha^i,\bar\alpha^j)}\Big)=\prod_{\substack{(\bar\alpha^i,\alpha^i_{n+1}),(\bar\alpha^j,\alpha^j_{n+1})\in P^{\zeta}\\ \text{where } i<j}}\ \prod_{\substack{1\le\ell\le r_{(\bar\alpha^i,\alpha^i_{n+1})},\\1\le\ell'\le r_{(\bar\alpha^j,\alpha^j_{n+1})}}}(t^{(\ell)}_{(\bar\alpha^i,\alpha^i_{n+1})}-t^{(\ell')}_{(\bar\alpha^j,\alpha^j_{n+1})})^{\gamma(\bar\alpha^i,\bar\alpha^j)}.$$

Define the map

$$\begin{gathered}g_{\bar\rho}:\Lambda_{(\bar\rho,a_{n+1})}(\boldsymbol u,\boldsymbol s_{n+1})\to\Lambda_{\left(\bar\rho,a_{n+1}-\sum_{\alpha_i\in P}\rho_i(\zeta_{\max}-\zeta_i)\right)}(\boldsymbol u,\boldsymbol s_{n+1}),\\
s_{n+1}^{\left(\sum_{i'=1}^{i-1}\rho_{i'}(\zeta_{\max}-\zeta_{i'})+(\ell-1)(\zeta_{\max}-\zeta_i)+1\right)},\dots,s_{n+1}^{\left(\sum_{i'=1}^{i-1}\rho_{i'}(\zeta_{\max}-\zeta_{i'})+\ell(\zeta_{\max}-\zeta_i)\right)},\quad u_i^{(\ell)}\mapsto u_i^{(\ell)},\\
s_{n+1}^{(\ell)}\mapsto s_{n+1}^{\left(\ell-\sum_{i=0}^{m}\rho_i(\zeta_{\max}-\zeta_i)\right)},\quad \ell\ge\sum_{\alpha_i\in P}\rho_i(\zeta_{\max}-\zeta_i).\end{gathered}$$

Here $\zeta_{\max}-\zeta_i$ variables of the form $s^{\bullet}_{n+1}$ are mapped to every variable of the form $u_i^{\bullet}$ while the remaining $s^{\bullet}_{n+1}$ are mapped to (distinct) $s^{\bullet}_{n+1}$. We have

$$\tilde\psi_{\bar r}=\psi_{d(\bar r)}\circ g_{\bar\rho},$$

where $d(\bar r)=(d(\bar r)_{(\bar\alpha^i,a)}),\ i=0,\dots,m,\ a=0,\dots,\zeta_i,\ d(\bar r)_{(\bar\alpha^i,a)}=\bar r_{(\bar\alpha^i,a+\zeta_{\max}-\zeta_i)}$ and $\psi_{d(\bar r)}$ is defined in the following way (it differs from the $\psi_{d(\bar r)}$ defined in Section 3.4 only by a change of the subscripts):

$$\begin{aligned}
u_{\bar\alpha^i}^{(j)}&\mapsto t^{(j)}_{(\bar\alpha^i,\zeta_{\max}-\zeta_i)}, && j=1,\dots,r_{\bar\alpha^i,(\zeta_{\max}-\zeta_i)};\\
u_{\bar\alpha^i}^{(r_{(\bar\alpha^i,\zeta_{\max}-\zeta_i)}+j)},\ s_{n+1}^{(j)}&\mapsto t^{(j)}_{(\bar\alpha^i,\zeta_{\max}-\zeta_i+1)}, && j=1,\dots,r_{(\bar\alpha^i,\zeta_{\max}-\zeta_i+1)};\\
u_i^{(r_{(\bar\alpha^i,\zeta_{\max}-\zeta_i)}+r_{(\bar\alpha^i,\zeta_{\max}-\zeta_i+1)}+j)},&\\
s_{n+1}^{(r_{(\bar\alpha^i,\zeta_{\max}-\zeta_i+1)}+2j-1)},\ s_{n+1}^{(r_{(\bar\alpha^i,\zeta_{\max}-\zeta_i+1)}+2j)}&\mapsto t^{(j)}_{(\bar\alpha^i,\zeta_{\max}-\zeta_i+2)}, && j=1,\dots,r_{(\bar\alpha^i,\zeta_{\max}-\zeta_i+2)};\\
&\dots
\end{aligned}$$

Here distinct variables of the form $u^{\bullet}_{\bar\alpha_i}$ are mapped to distinct variables of the form $t^{\bullet}_{(\bar\alpha_i,a)}$ and $a-(\zeta_{\max}-\zeta_i)$ variables of the form $s^{\bullet}_{n+1}$ are mapped to each variable $t^{\bullet}_{(\bar\alpha_i,a)}$. The map $g_{\bar\rho}$ is surjective. Thus

$$\tilde\psi_{\bar r}\Big(\bigcap_{\bar r'\prec\bar r,\mathrm{pr}_{n+1}(\bar r')=\bar\rho}\ker(\tilde\psi_{\bar r'})\Big)=\psi_{d(\bar r)}\Big(\bigcap_{\bar r'\prec\bar r,\mathrm{pr}_{n+1}(\bar r')=\bar\rho}\ker(\psi_{d(\bar r')})\Big).$$

Note that the order $\prec^{\zeta}$ on vectors $\bar r$ defines an order on the vectors $d(\bar r)$. This order is lexicographic with respect to the following order on coordinates: $(\bar\alpha^i,a)<_d(\bar\alpha^j,b)$ if and only if $a+(\zeta_{\max}-\zeta_i)<b+(\zeta_{\max}-\zeta_j)$

or $a+(\zeta_{\max}-\zeta_i)=b+(\zeta_{\max}-\zeta_j)$ and $i<j$. Thus using Lemma 3.17 we have

$$\psi_{d(\bar r)}\Bigg(\bigcap_{\bar r'\prec\bar r,\mathrm{pr}_{n+1}(\bar r')=\bar\rho}\ker(\psi_{d(\bar r')})\Bigg) \supset \prod_{\substack{(\bar\alpha^i,\alpha^i_{n+1})<(\bar\alpha^j,\alpha^j_{n+1})\\ \in P^\zeta}}\ \prod_{\substack{1\le\ell\le r_{(\bar\alpha^i,\alpha^i_{n+1})},\\ 1\le\ell'\le r_{(\bar\alpha^j,\alpha^j_{n+1})}}}\big(t^{(\ell)}_{(\bar\alpha^i,\alpha^i_{n+1})}-t^{(\ell')}_{(\bar\alpha^j,\alpha^j_{n+1})}\big)^{\varkappa((\bar\alpha^i,\alpha^i_{n+1}),(\bar\alpha^j,\alpha^j_{n+1}))}\Lambda_{\bar r}(\boldsymbol{t})$$

(see (4-11)). We obtain

$$\varphi^\vee_{\bar r}\Bigg(\bigcap_{\bar r'\prec\bar r}\ker(\varphi^\vee_{\bar r'})\Bigg)\supset \prod_{\substack{(\bar\alpha^i,\alpha^i_{n+1})<(\bar\alpha^j,\alpha^j_{n+1})\\ \in P^\zeta}}\ \prod_{\substack{1\le\ell\le r_{(\bar\alpha^i,\alpha^i_{n+1})},\\ 1\le\ell'\le r_{(\bar\alpha^j,\alpha^j_{n+1})}}}\big(t^{(\ell)}_{(\bar\alpha^i,\alpha^i_{n+1})}-t^{(\ell')}_{(\bar\alpha^j,\alpha^j_{n+1})}\big)^{\gamma((\bar\alpha^i,\alpha^i_{n+1}),(\bar\alpha^j,\alpha^j_{n+1}))}\Lambda_{\bar r}(\boldsymbol{t}).$$

Therefore the map (4-12) is bijective. □

5. Examples

5.1. *Parallelepiped.* Consider the polytope $P_{d_1,\dots,d_n}:=(x_1,\dots,x_n)\in\mathbb{R}^n,\ 0\le x_i\le d_i,\ d_1,\dots,d_n\in\mathbb{N}$. Choose $d_{n+1}\in\mathbb{N}$, and define $\zeta(\bar\alpha)=d_{n+1}$, $\bar\alpha\in P_{d_1,\dots,d_n}$. Then we have

$$P_{d_1,\dots,d_n,d_{n+1}}=P^\zeta_{d_1,\dots,d_n}.$$

By induction we can construct a strict cube generating data for this polytope (in this example $f_i(\bar\alpha,\bar\beta)=0$ for all $\bar\alpha,\bar\beta,i$). In particular, the action of $\mathcal{A}_L$ on the L-th graded component of $J^\infty_{\mathrm{red}}(R(P_{d_1,\dots,d_n}))^*$ is free. Let us compute the graded dimension of this component.

We define an order on $P_{d_1,\dots,d_n}\cap\mathbb{Z}^n$ by setting $\bar\alpha<_n\bar\beta$ if for the largest k such that $\alpha_k\ne\beta_k$ we have $\alpha_k<\beta_k$. Note that this order corresponds to the order (4-10), i.e., $<_{n+1}=<^\zeta_n$. Let us compute $\gamma(\bar\alpha,\bar\beta)$ for $\bar\alpha,\bar\beta\in P_{d_1,\dots,d_n}$. First, by (4-11) for $0\le i\le n-1$ and $(\bar\alpha,\alpha_{i+1})\le_{i+1}(\bar\beta,\beta_{i+1})\in P^{\zeta_i}_{d_1,\dots,d_i}$, where $\zeta_i\equiv d_{i+1}$, we have

$$\varkappa((\bar\alpha,\alpha_{i+1}),(\bar\beta,\beta_{i+1}))=\begin{cases}\beta_{i+1}-\alpha_{i+1} & \text{if } \bar\alpha\le_i\bar\beta,\\ \bar\beta_{i+1}-\bar\alpha_{i+1}-1 & \text{if } \bar\beta<_i\bar\alpha.\end{cases}\tag{5-1}$$

Therefore for $\bar\alpha,\bar\beta\in P_{d_1,\dots,d_n}$ we have

$$\gamma(\bar\alpha,\bar\beta)=\sum_{i=1}^n|\beta_i-\alpha_i|-S,$$

where S is the number of i for which the largest $j<i$ with $\alpha_j\ne\beta_j$ satisfies $(\alpha_i-\beta_i)(\alpha_j-\beta_j)<0$ (or no such j exists). We note that if all $d_i=1$, then we recover Remark 1.32.

Note that the graded dimension of $\Lambda_{\bar r}(\boldsymbol{s})$ is equal to

$$\prod_{\bar\alpha\in P_{d_1,\dots,d_n}}\prod_{\ell=1}^{r_{\bar\alpha}}\frac{1}{1-q^\ell}.$$

Using Proposition 2.8 we can now write the graded dimension of $J^{\infty}_{\mathrm{red}}R(P_{d_1,\dots,d_n})[L]$ as

$$\sum_{\bar r \mid \sum_{\bar\alpha\in P_{d_1,\dots,d_n}} r_{\bar\alpha}=L} v^{\sum_{\bar\alpha\in P_{d_1,\dots,d_n}} r_{\bar\alpha}\bar\alpha} \prod_{\bar\alpha\in P_{d_1,\dots,d_n}} \prod_{\ell=1}^{r_{\bar\alpha}} \frac{1}{1-q^{\ell}} \prod_{\bar\alpha<\bar\beta\in P_{d_1,\dots,d_n}} q^{r_{\bar\alpha}r_{\bar\beta}\gamma(\bar\alpha,\bar\beta)}.$$

Here variables $v_1,\dots,v_n$ correspond to the n-dimensional torus action on $R(P_{d_1,\dots,d_n})$ (and, subsequently, on $J^{\infty}_{\mathrm{red}}R(P_{d_1,\dots,d_n})$) while q corresponds to the grading grad.

5.2. ***Simplex.*** Consider the polytope

$$P_{n,d}=\{(x_1,\dots,x_n)\in\mathbb{R}^n \mid d\ge x_n\ge\cdots\ge x_1\ge 0\}$$

for $d\in\mathbb{N}$. Consider $\zeta((x_1,\dots,x_n))=d-x_n$. Then $P^{\zeta}_{n,d}$ is equal to the polytope $P_{n+1,d}$. Thus by induction we can construct a strict cube generating data for all the polytopes $P_{n,d}$ using the construction from Example 4.18. In particular, the action of $\mathcal{A}_L$ on the L-th graded component of $J^{\infty}_{\mathrm{red}}(R(P_{n,d}))^*$ is free.

We define an order on $P_{n,d}\cap\mathbb{Z}^n$ by setting $\bar\alpha<_n\bar\beta$ if for the largest k such that $\alpha_k\ne\beta_k$ we have $\alpha_k<\beta_k$. Note that this order corresponds to the order (4-10), i.e., $<_{n+1}=<^{\zeta}$. Let us compute $\gamma(\bar\alpha,\bar\beta)$ for $\bar\alpha,\bar\beta\in P_{n,d}$. First, by (4-11) for $1\le i\le n-1$ and $(\bar\alpha,\alpha_{i+1})\le_{i+1}(\bar\beta,\beta_{i+1})\in P_{i,d}$ we have

$$\varkappa((\bar\alpha,\alpha_{i+1}),(\bar\beta,\beta_{i+1}))=\begin{cases}\beta_{i+1}-\beta_i & \text{if } \beta_i>\alpha_{i+1},\\ \beta_{i+1}-\alpha_{i+1} & \text{if } \beta_i\le\alpha_{i+1} \text{ and } \bar\alpha\le\bar\beta,\\ \beta_{i+1}-\alpha_{i+1}-1 & \text{if } \beta_i\le\alpha_{i+1} \text{ and } \bar\alpha>\bar\beta.\end{cases} \tag{5-2}$$

Now, for $\bar\alpha,\bar\beta\in P_{n,d}$ and $i=1,\dots,n-1$ we define

$$\varkappa_i(\bar\alpha,\bar\beta)=\begin{cases}\beta_{i+1}-\beta_i & \text{if } \beta_i>\alpha_{i+1},\\ \beta_{i+1}-\alpha_{i+1} & \text{if } \beta_i\le\alpha_{i+1} \text{ and } (\alpha_1,\dots,\alpha_i)\le(\beta_1,\dots,\beta_i),\\ \beta_{i+1}-\alpha_{i+1}-1 & \text{if } \beta_i\le\alpha_{i+1} \text{ and } (\alpha_1,\dots,\alpha_i)>(\beta_1,\dots,\beta_i),\end{cases} \tag{5-3}$$

and

$$\varkappa_0(\bar\alpha,\bar\beta)=\begin{cases}|\alpha_1-\beta_1|-1 & \text{if } \alpha_1\ne\beta_1\\ 0 & \text{otherwise.}\end{cases}$$

Then by induction we have

$$\gamma(\bar\alpha,\bar\beta)=\sum_{i=0}^{n-1}\varkappa_i(\bar\alpha,\bar\beta).$$

Note that $R(P_{n,d})$ is naturally embedded into $R(P_{n,1})$ as a sum of homogeneous components and $R(P_{n,1})$ is a polynomial algebra in $n+1$ variables which is acted upon by $\mathfrak{sl}_{n+1}(\Bbbk)$. Consider the restriction of this action to $R(P_{n,d})$. By Lemmas 1.23 and 1.16 we have an action of $\mathfrak{sl}_{n+1}(\Bbbk)[s]$ on $J^{\infty}_{\mathrm{red}}R(P_{n,d})$ and for any $L\in\mathbb{N}$ the graded component $J^{\infty}_{\mathrm{red}}R(Q)[L]$ is preserved by this action. Due to [Dumanski and Feigin 2023] we have that $J^{\infty}_{\mathrm{red}}R(Q)[L]$ is isomorphic to the global Demazure module $\mathbb{D}_{d,L\omega_1}$. The graded dimension of $J^{\infty}_{\mathrm{red}}R(Q)[L]$ is equal to

$$\sum_{\bar r \mid \sum_{\bar\alpha\in P_{n,d}} r_{\bar\alpha}=L} v^{\sum_{\bar\alpha\in P_{d_1,\dots,d_n}} r_{\bar\alpha}} \prod_{\bar\alpha\in P_{n,d}} \prod_{\ell=1}^{r_{\bar\alpha}} \frac{1}{1-q^{\ell}} \prod_{\bar\alpha<\bar\beta\in P_{n,d}} q^{r_{\bar\alpha}r_{\bar\beta}\gamma(\bar\alpha,\bar\beta)}, \tag{5-4}$$

where the v_i correspond to coordinates in a maximal torus of $\mathfrak{sl}_{n+1}(\Bbbk)$.

Appendix: Veronese–Segre embeddings

In this appendix we deal with a special kind of toric projective embeddings. We use an approach, completely different to the one in the main body of the paper, to study their (reduced) coordinate rings and compute their characters. This approach is based on the representation theory of the current algebras. It would be interesting to investigate the combinatorial identities, obtained by comparing the formulae for the graded characters of the form (5-4) with the ones of the form (A-3), (A-4).

For a semisimple Lie algebra $\mathfrak{g}$ we denote by $V_\lambda^{\mathfrak{g}}$ and $D_{d,\lambda}^{\mathfrak{g}}$ the irreducible $\mathfrak{g}$-module of highest weight λ and the $\mathfrak{g}[t]$ affine Demazure module of level d and highest weight $d\lambda$ (see [Fourier and Littelmann 2007; Chari and Venkatesh 2015] for details on affine Demazure modules).

Let $\mathfrak{g}_1, \dots, \mathfrak{g}_m$ be simple Lie algebras. To the algebra $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ one can associate the affine Kac–Moody Lie algebra $\widehat{\mathfrak{g}_1} \oplus \cdots \oplus \widehat{\mathfrak{g}_m}$. Consider the m-tuple $(\Lambda_1, \dots \Lambda_m)$, where Λ_i is an affine integrable $\widehat{\mathfrak{g}_i}$-weight. Then $L(\Lambda_1) \otimes \cdots \otimes L(\Lambda_m)$ naturally is an irreducible $\widehat{\mathfrak{g}_1} \oplus \cdots \oplus \widehat{\mathfrak{g}_m}$-module, where $L(\Lambda_i)$ is an irreducible integrable $\widehat{\mathfrak{g}_i}$-module.

Let $D^{\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m}_{(d_1,\dots,d_m),(\lambda_1,\dots,\lambda_m)} \subset L(\Lambda_1) \otimes \cdots \otimes L(\Lambda_m)$ be an affine Demazure module, where $d_1, \dots, d_k$ are positive integers and λ_i is a dominant weight of a simple Lie algebra $\mathfrak{g}_i$. It is clear that

$$D^{\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m}_{(d_1,\dots,d_m),(\lambda_1,\dots,\lambda_m)} \simeq D^{\mathfrak{g}_1}_{d_1,\lambda_1} \otimes \cdots \otimes D^{\mathfrak{g}_m}_{d_m,\lambda_m}.$$

In this appendix we deal with the arc space of the following composition of Veronese and Segre embeddings:

$$\begin{aligned} \mathbb{P}(\Bbbk^{n_1}) \times \cdots \times \mathbb{P}(\Bbbk^{n_m}) &\hookrightarrow \mathbb{P}(S^{d_1}\Bbbk^{n_1}) \times \cdots \times \mathbb{P}(S^{d_m}\Bbbk^{n_m}) \\ &\hookrightarrow \mathbb{P}((S^{d_1}\Bbbk^{n_1}) \otimes \cdots \otimes (S^{d_m}\Bbbk^{n_m})). \end{aligned} \tag{A-1}$$

The image of this map is a projective toric variety, corresponding to the product of simplices.

Consider the semisimple group $\mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_m}$ and its flag variety $\mathcal{B} = (\mathrm{SL}_{n_1}/B_{n_1}) \times \cdots \times (\mathrm{SL}_{n_m}/B_{n_m})$. We consider each space $\Bbbk^{n_i}$, appearing in (A-1) as the first fundamental $\mathfrak{sl}_{n_i}$-module, so $\Bbbk^{n_i} \simeq V_{\omega_1}^{\mathfrak{sl}_{n_i}}$ and $S^{d_i}\Bbbk^{n_i} \simeq V_{d_i\omega_1}^{\mathfrak{sl}_{n_i}}$. Therefore the image of the map (A-1) is isomorphic to the image of the map

$$\mathcal{B} \to \mathbb{P}(V_{d_1\omega_1}^{\mathfrak{sl}_{n_1}} \otimes \cdots \otimes V_{d_m\omega_1}^{\mathfrak{sl}_{n_m}}).$$

In order to study the arc space of this image, one should replace the flag variety $\mathcal{B}$ by the semi-infinite flag variety $\mathfrak{Q}$ of the semisimple group $\mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_m}$ (see [Finkelberg and Mirković 1999; Braverman and Finkelberg 2014; Kato 2018]) and study the homogeneous coordinate ring of the map

$$\mathfrak{Q} \to \mathbb{P}((V_{d_1\omega_1}^{\mathfrak{sl}_{n_1}} \otimes \cdots \otimes V_{d_m\omega_1}^{\mathfrak{sl}_{n_m}})[[t]]).$$

Using Corollary 1.14 as well as [Dumanski and Feigin 2023, Proposition 4.1] applied to the algebra $\mathfrak{sl}_{n_1} \oplus \cdots \oplus \mathfrak{sl}_{n_m}$ and its weight $(d_1\omega_1, \dots, d_m\omega_1)$ we can describe the coordinate ring as

$$\Gamma = \bigoplus_{\ell \geq 0} (D^{\mathfrak{sl}_{n_1} \oplus \cdots \oplus \mathfrak{sl}_{n_m}}_{(d_1,\dots,d_m),(\omega_1,\dots,\omega_1)}[t]^{\odot \ell})^*$$

(we use that $V^{\mathfrak{sl}_{n_i}}_{d_i\omega_1} \simeq D^{\mathfrak{sl}_{n_i}}_{d_i,\omega_i}$). Here for a Lie algebra $\mathfrak{a}$ and its cyclic module W with a cyclic vector w we denote by $W^{\odot \ell}$ the Cartan component $\mathrm{U}(\mathfrak{a}).w^{\otimes \ell} \subset W^{\otimes \ell}$. We are going to compute the character of the ℓ-th homogeneous component $\Gamma_\ell \subset \Gamma$.

The modules of the form $\Gamma_\ell = D^{\mathfrak{sl}_{n_1}\oplus\cdots\oplus\mathfrak{sl}_{n_m}}_{(d_1,\dots,d_m),(\omega_1,\dots,\omega_1)}[t]^{\odot \ell}$ were studied in [Dumanski and Feigin 2023; Dumanski et al. 2021]. In particular, it was shown that Γ_ℓ admits an action of the algebra $\mathcal{A}_\ell$, which in this case is isomorphic to the algebra of symmetric polynomials in ℓ variables. It was also shown that the fiber of Γ_ℓ with respect to this algebra at a generic point $\boldsymbol{c} = (c_1, \dots, c_\ell)$ is isomorphic to

$$\Gamma_\ell \otimes_{\mathcal{A}_\ell} \Bbbk_{\boldsymbol{c}} \simeq \bigotimes_{i=1}^{\ell} D^{\mathfrak{sl}_{n_1}\oplus\cdots\oplus\mathfrak{sl}_{n_m}}_{(d_1,\dots,d_m),(\omega_1,\dots,\omega_1)}(c_i), \tag{A-2}$$

where $\Bbbk_{\boldsymbol{c}}$ is the quotient of $\mathcal{A}_\ell$ by the maximal ideal, corresponding to $\boldsymbol{c}$. It was also proved that to show that Γ_ℓ is free over $\mathcal{A}_\ell$ it is sufficient to obtain the surjection

$$\Gamma_\ell \otimes_{\mathcal{A}_\ell} \Bbbk_0 \twoheadleftarrow D^{\mathfrak{sl}_{n_1}\oplus\cdots\oplus\mathfrak{sl}_{n_m}}_{(d_1,\dots,d_m),(\ell\omega_1,\dots,\ell\omega_1)},$$

where the latter has the same dimension as the right-hand side of (A-2). But this was shown for restriction to any $\mathfrak{sl}_{n_i}$ in [Dumanski and Feigin 2023, Proposition 3.2]. The surjectivity of the whole map follows. Therefore we proved the following:

Lemma A.1. *Module Γ_ℓ is free over the algebra of symmetric polynomials in ℓ variables $\mathcal{A}_\ell$. The fiber at* 0 *with respect to this algebra is isomorphic to*

$$D^{\mathfrak{sl}_{n_1}\oplus\cdots\oplus\mathfrak{sl}_{n_m}}_{(d_1,\dots,d_m),(\ell\omega_1,\dots,\ell\omega_1)}.$$

Thus, we obtain the desired:

Corollary A.2. *The character of the ℓ-th homogeneous component of the arc space of the projective embedding* (A-1) *is equal to*

$$\mathrm{ch}\mathcal{A}_\ell \mathrm{ch} D^{\mathfrak{sl}_{n_1}\oplus\cdots\oplus\mathfrak{sl}_{n_m}}_{(d_1,\dots,d_m),(\ell\omega_1,\dots,\ell\omega_1)} = \frac{1}{(q)_\ell}\prod_{i=1}^{m} \mathrm{ch} D^{\mathfrak{sl}_{n_i}}_{d_i,\ell\omega_1}. \tag{A-3}$$

The last factor in this expression was found in [Feigin et al. 2004, Theorem 2.11]. Namely, in the notation of [Feigin et al. 2004],

$$D^{\mathfrak{sl}_{n_i}}_{d_i,\ell\omega_1} = \underbrace{V^{\mathfrak{sl}_{n_i}}_{d_i\omega_1} * \cdots * V^{\mathfrak{sl}_{n_i}}_{d_i\omega_1}}_{\ell} = V(\boldsymbol{n}_i, \boldsymbol{d}_i),$$

where

$$\boldsymbol{n}_i = (\underbrace{n_i, \dots, n_i}_{\ell}), \quad \boldsymbol{d}_i = (\underbrace{d_i, \dots, d_i}_{\ell}),$$

and $*$ denotes the fusion product. Now [Feigin et al. 2004, Formula (2.52)] gives that

$$\mathrm{ch} D^{\mathfrak{sl}_{n_i}}_{d_i,\ell\omega_1} = \sum_{\substack{\lambda\in\mathbb{Z}^{n_i}\\ |\lambda| = d_i\ell}} e^{\lambda_2\omega_1} \cdots e^{\lambda_{n_i}\omega_{n_i-1}} \tilde{S}_{\lambda,(\ell^{d_i})}(q), \tag{A-4}$$

where $\lambda = (\lambda_1, \dots, \lambda_{n_i})$, e^{ω_i} stay for the $\mathfrak{sl}_{n_i}$-weights and $\tilde{S}_{\lambda,\mu}$ is the q-supernomial coefficient, defined in [Schilling 2002, Formula (2.1)].

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Divisibility of character values of the symmetric group by prime powers

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In memory of Chandra Sekhar Raju

Let k be a positive integer. We show that, as n goes to infinity, almost every entry of the character table of S_n is divisible by k. This proves a conjecture of Miller.

1. Introduction

It is a standard fact that the irreducible characters of S_n take only integer values for every natural number n. In 2017, Miller [12] computed the character tables of S_n for all $n \leq 38$ and looked at various statistical properties of these integers as n grew. His computations suggested that

(1) the density of even entries tended to 1 as n tended to infinity,

(2) the density of entries divisible by 3, the density of entries divisible by 5, and the density of entries divisible by 7 increase with n,

(3) about half of the nonzero entries were positive,

(4) and the density of zeros in the character table decreased as n grew, but not very quickly.

Based on this first observation, Miller [12; 14] conjectured that as n goes to infinity, almost every entry of the character table of the symmetric group S_n is even. Following partial progress due to McKay [11], Gluck [6], Ganguly, Prasad and Spallone [5], and Morotti [16], Peluse proved this conjecture in [18]. Based on the second observation, Miller [12; 14] also conjectured, more generally, that for any fixed prime p, almost every entry of the character table of S_n is a multiple of p as n goes to infinity. We proved this conjecture in [19], with a uniform upper bound for the number of entries not divisible by a fixed prime. Recently, Miller [13] conjectured, even more generally, that for any fixed prime power q, almost every entry of the character table of S_n is a multiple of q as n goes to infinity. In this paper, we prove this most general of Miller's conjectures.

Theorem 1.1. *Let n be large and $q \leq 10^{-3} \log n/(\log\log n)^2$ be a prime power. The number of entries in the character table of S_n that are not divisible by q is at most*

$$O(p(n)^2 \exp(-(\log\log n)^2)).$$

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It follows immediately from Theorem 1.1 and the union bound that almost every entry of the character table of S_n is divisible by any fixed positive integer as n goes to infinity.

Corollary 1.2. *Let k be any positive integer. Then, as n goes to infinity, the proportion of entries in the character table of S_n that are not divisible by k tends to* 0.

Our methods do not seem to shed any light on Miller's third and fourth observations. Most interesting to us is the question of what proportion of character table entries are zero, and recent large scale simulations of Miller and Scheinerman [15] suggest that the proportion of zeros tends to 0 as n tends to infinity. Combining the Murnaghan–Nakayama rule and an old result of Erdős and Lehner [2] on the distribution of the largest part of a uniformly random partition of n produces a proportion of about $2/\log n$ zeros in the character table of S_n, and no lower bound of a larger order of magnitude seems to be known. In the related setting of finite simple groups of Lie type, Larsen and Miller [8] have shown that almost every character table entry is zero as the rank goes to infinity.

2. Proof outline

For any partitions λ and μ of n, let χ_μ^λ denote the value of the irreducible character of S_n corresponding to λ on the conjugacy class of elements with cycle type corresponding to μ. In [19], our argument proceeded by combining two key facts: (i) if μ contains a part substantially larger than the typical largest part of a random partition, then $\chi_\mu^\lambda = 0$ for almost every λ, and (ii) if ν is another partition of n that is obtained from μ by combining p parts of the same size m into one part of size pm, then $\chi_\mu^\lambda \equiv \chi_\nu^\lambda \pmod p$ for every λ. We showed that, for almost every μ, repeatedly combining p parts of the same size in this manner produces a partition $\tilde{\mu}$ containing a very large part, large enough so that $\chi_{\tilde{\mu}}^\lambda$ must be zero for almost every λ. Our main result on the divisibility of character values by primes then followed from the fact that $\chi_\mu^\lambda \equiv \chi_{\tilde{\mu}}^\lambda \pmod p$ for every λ.

The second key fact generalizes to a congruence of character value modulo prime powers in a straightforward manner.

Lemma 2.1. *Let p^r be a power of the prime p. Suppose that μ is a partition of n, and that ν is another partition of n obtained from μ by replacing p^r parts of the same size m by p^{r-1} parts of size pm. Then for all partitions λ of n, we have*

$$\chi_\mu^\lambda \equiv \chi_\nu^\lambda \pmod{p^r}.$$

However, when $r > 1$, it is no longer the case that starting from a typical partition μ of n and repeatedly combining p^r parts of the same size m into p^{r-1} parts of size pm produces a partition $\tilde{\mu}$ containing a part substantially larger than the largest part of a typical partition of n. The argument from [19] that worked for primes thus breaks down for all other prime powers.

The key idea used to overcome this barrier is a new condition for character values of the symmetric group to be divisible by a fixed prime power, which we prove by exploiting certain symmetries that appear after applying the Murnaghan–Nakayama rule multiple times.

Theorem 2.2. *Let $n, m_1, \dots, m_r$ be distinct positive integers. Let μ be a partition of n containing parts of size $m_1, \dots, m_r$, each appearing at least p^{r-1} times. If λ is a $\left(\sum_{i=1}^r k_i m_i\right)$-core partition of n for all r-tuples $(k_1, \dots, k_r)$ of integers $0 \le k_1, \dots, k_r \le p^{r-1}$ for which some $k_i = p^{r-1}$, then*

$$p^r \mid \chi^\lambda_\mu.$$

Starting with a partition μ of n, repeatedly combine p^r parts of the same size m into p^{r-1} parts of size pm, until the process terminates in a partition $\tilde{\mu}$ where no part appears more than $p^r - 1$ times. As a preliminary to applying Theorem 2.2 we show that for a typical partition μ, the resulting partition $\tilde{\mu}$ will have r parts that are suitably large, and with each appearing at least p^{r-1} times.

Proposition 2.3. *Starting with a partition μ of n, repeatedly replace every occurrence of p^r parts of the same size m by p^{r-1} parts of size pm until we arrive at a partition $\tilde{\mu}$ where no part appears more than $p^r - 1$ times. Then, except for*

$$O(p(n)\exp(-n^{1/20p^r}))$$

initial partitions μ, the partition $\tilde{\mu}$ contains at least r distinct parts $m_1, \dots, m_r$, each appearing at least p^{r-1} times and satisfying

$$p^{r-1}m_j > \left(1 + \frac{1}{6p^r}\right)\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n.$$

This holds uniformly for $p^r \le 10^{-3}\log n/(\log\log n)^2$.

The significance of the lower bound on $p^{r-1}m_j$ in Proposition 2.3 is that it lies beyond the threshold of values t such that almost every partition of n is a t-core.

Lemma 2.4. *Let $1 \le L \le \log n/\log\log n$ be a real number. Then, for any given integer t with*

$$t \ge \left(1 + \frac{1}{L}\right)\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n,$$

all but

$$O\left(p(n)\frac{\log n}{n^{1/2L}}\right)$$

partitions of n are t-cores.

We can swiftly deduce our main result, Theorem 1.1, from the results stated above.

Deducing Theorem 1.1. Let μ be a partition of n, and suppose that $\tilde{\mu}$ is as in Proposition 2.3. Then, for all but at most

$$O(p(n)\exp(-n^{1/20p^r}))$$

choices of μ, the partition $\tilde{\mu}$ contains at least r distinct parts $m_1, \dots, m_r$, each appearing at least p^{r-1} times and satisfying

$$p^{r-1}m_j > \left(1 + \frac{1}{6p^r}\right)\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n. \tag{2-1}$$

Consider any r-tuple $(k_1, \dots, k_r)$ with $0 \le k_1, \dots, k_r \le p^{r-1}$ and $k_i = p^{r-1}$ for some i. Then $k_1 m_1 + \cdots + k_r m_r$ also exceeds the bound in (2-1), so that by Lemma 2.4 all but $O(p(n)(\log n)/n^{1/2L})$ partitions λ of n are $(k_1 m_1 + \cdots + k_r m_r)$-cores. Since there are at most $r(p^{r-1}+1)^{r-1}$ such r-tuples $(k_1, \dots, k_r)$, by the union bound we see that all but at most

$$O\biggl(p(n)\frac{\log n}{n^{1/12p^r}} r(p^{r-1}+1)^{r-1}\biggr)$$

partitions λ of n are $(k_1 m_1 + \cdots + k_r m_r)$-cores for all choices of the r-tuple $(k_1, \dots, k_r)$.

Theorem 2.2 now shows that p^r divides $\chi^\lambda_{\tilde{\mu}}$, and since $\chi^\lambda_\mu \equiv \chi^\lambda_{\tilde{\mu}} \pmod{p^r}$ by Lemma 2.1, it also follows that p^r divides χ^λ_μ. Putting everything together, we conclude that the number of partitions λ and μ with $p^r \nmid \chi^\lambda_\mu$ is at most

$$O\biggl(p(n)^2\biggl(\exp(-n^{1/(20p^r)}) + \frac{1}{n^{1/13p^r}} r(p^{r-1}+1)^{r-1}\biggr)\biggr) = O(p(n)^2 \exp(-(\log\log n)^2)),$$

in the range $p^r \le 10^{-3} \log n/(\log\log n)^2$. □

The rest of the paper is organized as follows. We will prove Lemmas 2.1 and 2.4 in Section 3, Theorem 2.2 in Sections 4, 5, 6, and 7, and Proposition 2.3 in Sections 8 and 9.

3. Proofs of Lemmas 2.1 and 2.4

We begin by proving the two lemmas stated in the previous section.

Proof of Lemma 2.1. We claim that if $Q \in \mathbb{Z}[x_1, \dots, x_k]$ is a polynomial with integer coefficients, then

$$Q(x_1, \dots, x_k)^{p^r} \equiv Q(x_1^p, \dots, x_k^p)^{p^{r-1}} \pmod{p^r}.$$

As is well known, we may write

$$Q(x_1, \dots, x_k)^p = Q(x_1^p, \dots, x_k^p) + p \cdot R(x_1, \dots, x_k) \tag{3-1}$$

for some $R \in \mathbb{Z}[x_1, \dots, x_k]$, which establishes the claim when $r = 1$. For $r > 1$, raise both sides of (3-1) to the power p^{r-1}, and expand using the binomial theorem:

$$\begin{aligned} Q(x_1, \dots, x_k)^{p^r} &= (Q(x_1^p, \dots, x_k^p) + p \cdot R(x_1, \dots, x_k))^{p^{r-1}} \\ &= Q(x_1^p, \dots, x_k^p)^{p^{r-1}} + \sum_{\ell=1}^{p^{r-1}} \binom{p^{r-1}}{\ell} Q(x_1^p, \dots, x_k^p)^{p^{r-1}-\ell} (pR(x_1, \dots, x_k))^\ell. \end{aligned}$$

Note that for $1 \le \ell \le p^{r-1}$

$$p^\ell \binom{p^{r-1}}{\ell} = p^\ell \frac{p^{r-1}}{\ell} \binom{p^{r-1}-1}{\ell-1} \equiv 0 \pmod{p^r},$$

since the power of p dividing ℓ is certainly at most $\ell - 1$. This establishes our claim.

The lemma now follows by applying this observation to the polynomials appearing in Frobenius's formula for the character values χ^λ_μ and χ^λ_ν; see Chapter 4 of [4]. □

Proof of Lemma 2.4. The proof is essentially identical to that of Proposition 1 of [19], but we include the short argument for completeness. Since every partition of n is a t-core for $t > n$, we may naturally assume that $t \le n$. From Lemma 5 of [16], we know that at most $(t+1)p(n-t)$ partitions of n are not t-cores. By the asymptotic formula

$$p(m) \sim \frac{1}{4\sqrt{3}m} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{m}\right)$$

for the partition function, we have

$$(t+1)p(n-t) \ll \frac{t+1}{n-t+1} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n-t}\right) \le \frac{t+1}{n-t+1} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n} - \frac{\pi t}{\sqrt{6n}}\right).$$

In the range $n \ge t \ge (1+1/L)(\sqrt{6}/2\pi)\sqrt{n}\log n$, the right-hand side above is maximized at the lower endpoint $t = (1+1/L)(\sqrt{6}/2\pi)\sqrt{n}\log n$. It follows that the number of partitions of n that are not t-cores is

$$\ll \frac{\log n}{\sqrt{n}} n^{-(1+1/L)/2} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right) \ll p(n)\frac{\log n}{n^{1/2L}},$$

where the last step uses again the asymptotic for the partition function. □

4. Partitions and Abaci

The proof of Theorem 2.2 requires the machinery of the *abacus* associated to a partition. A good reference for this theory is Section 2.7 of [7], and we recall some salient facts below.

4.1. ***The notion of an abacus.*** An abacus is a bi-infinite sequence of 0's and 1's beginning with an infinite sequence of 1's and ending with an infinite sequence of 0's.

More formally, let

$$\mathcal{S} := \{s : \mathbb{Z} \to \{0, 1\} : \text{there exists a } k \ge 0 \text{ such that } s(-i) = 1 \text{ and } s(i) = 0 \text{ for all } i \ge k\}$$

denote the set of all sequences of 0's and 1's indexed using the integers, that begin with an infinite sequence of 1's and end with an infinite sequence of 0's. For example,

$$\ldots, 1, \ldots, 1, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 0, \ldots, 0, \ldots$$

is in $\mathcal{S}$. We consider two sequences s and s' in $\mathcal{S}$ to be equivalent if there is some integer j such that $s(i) = s'(i-j)$ for all i, that is, if s' can be produced by shifting the terms in s by j. This is an equivalence relation, and an abacus refers to an equivalence class in $\mathcal{S}$ under this relation. We denote by $\mathcal{A}$ the set of such abaci, so that by an element a of $\mathcal{A}$ we mean the equivalence class consisting of some sequence $s \in \mathcal{S}$ together with all its shifts.

4.2. ***The abacus associated to a partition.*** We now show how abaci are in one-to-one correspondence with partitions of integers. Starting with an integer partition λ, we construct an abacus $a_\lambda \in \mathcal{A}$ as follows. Draw the Young diagram of λ, and trace out the boundary of the diagram, moving from the lower left-hand

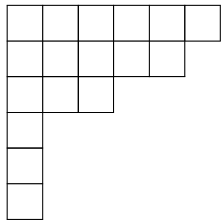

Figure 1. The Young diagram of (6, 5, 3, 1, 1, 1).

corner to the upper right-hand corner, writing a 0 for each horizontal move and a 1 for each vertical move. Then prepend an infinite string of 1's and append an infinite string of 0's to find a representative of the corresponding element a_λ of $\mathcal{A}$.

This procedure is easily reversed, and starting with an abacus a in $\mathcal{A}$ we obtain a Young diagram, which corresponds to a partition λ. If $s \in \mathcal{S}$ is a representative of a, then the partition λ is a partition of the integer $n(a)$ which counts the number of pairs of indices (i, j) with $i < j$ such that $s(i) = 0$ and $s(j) = 1$.

To illustrate, consider the partition (6, 5, 3, 1, 1, 1), whose Young diagram is pictured in Figure 1. If we start in the lower left-hand corner of this diagram and move along the boundary to the upper right-hand corner, we move right, up three times, right twice, up, right twice, up, right, and up. The correspondence described above produces the string

$$0, 1, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, \tag{4-1}$$

which we can turn into a bi-infinite sequence by adding an infinite sequence of 1's to the beginning and an infinite sequence of 0's to the end:

$$\ldots, 1, \ldots, 1, 1, 1, 0, 1, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, \ldots, 0, \ldots. \tag{4-2}$$

The equivalence class of this sequence is the abacus associated to (6, 5, 3, 1, 1, 1).

4.3. ***Hooks and border strips.*** Let λ be a partition. The *hook* h associated to a box b in the Young diagram of λ consists of the box b together with all the boxes directly to its right and directly below it. The hook-length of h, denoted by $\ell(h)$, is the number of boxes contained in the hook. The *height* of the hook h, denoted by $\mathrm{ht}(h)$, is one less than the number of rows in the Young diagram of λ that contain a box of h. Associated to each hook is a *border strip* (also known as a skew hook), denoted $\mathrm{bs}(h)$, which is the connected region of boundary boxes of the Young diagram running from the rightmost to the bottommost box of h. Removing such a border strip leaves behind a smaller Young diagram. These notions play a prominent role in the representation theory of the symmetric group, and in particular feature in the Murnaghan–Nakayama rule for computing character values, which we next recall; see Theorem 2.4.7 of [7], and also Chapter 4 of [4].

Theorem 4.1 (the Murnaghan–Nakayama rule). *Let n and t be positive integers, with $t \leq n$. Let $\sigma \in S_n$ be of the form $\sigma = \tau \cdot \rho$, where ρ is a t-cycle, and τ is a permutation of S_n with support disjoint from ρ. Let λ be a partition of n. Then*

$$\chi^{\lambda}(\sigma) = \sum_{\substack{h \in \lambda \\ \ell(h)=t}} (-1)^{\mathrm{ht}(h)} \chi^{\lambda \setminus \mathrm{bs}(h)}(\tau). \tag{4-3}$$

Above, $\chi^{\lambda}(\sigma)$ denotes the value of the character of the irreducible representation of S_n corresponding to the partition λ, evaluated on the conjugacy class of σ, $\lambda \setminus \mathrm{bs}(h)$ denotes the partition of $n - t$ obtained by removing the border strip $\mathrm{bs}(h)$ from the Young diagram of λ, and $\chi^{\lambda \setminus \mathrm{bs}(h)}(\tau)$ denotes the character value of the irreducible representation of S_{n-t} corresponding to the partition $\lambda \setminus \mathrm{bs}(h)$ evaluated on the conjugacy class of τ.

The abacus notation helps with thinking about hook lengths and border strips. Let λ be a partition, let a_λ denote the corresponding abacus, and let s be a representative in $\mathcal{S}$ for the abacus a_λ. Each hook h in the Young diagram of λ is in natural one-to-one correspondence with a pair of indices (i, j), $i < j$, with $s(i) = 0$ and $s(j) = 1$. The length of the hook h is $j - i$. In particular, the partition λ contains no hooks of length t (that is, λ is a t-core) if and only if there is no pair of indices $(i, i+t)$ with $s(i) = 0$ and $s(i+t) = 1$. The height of the hook h equals the number of 1's in the sequence s lying strictly between the 0 at index i and the 1 at index j:

$$\mathrm{ht}(h) = \#\{i < k < j : s(k) = 1\}.$$

Further, the abacus notation gives an easy description of the result of removing a border strip from a partition. Define, for any pair of distinct integers (i, j) the operator $T_{ij} : \mathcal{S} \to \mathcal{S}$ that swaps the terms indexed by i and j in a bi-infinite sequence $s \in \mathcal{S}$ and leaves all other entries fixed. Thus for $s \in \mathcal{S}$

$$(T_{ij}s)(k) = \begin{cases} s(k), & k \neq i, j, \\ s(j), & k = i, \\ s(i), & k = j. \end{cases}$$

With this notation in place, suppose λ is a partition, and $s \in a_\lambda$ is a representative of the abacus of λ. Let h be a hook of λ, corresponding to the pair of indices (i, j) (with $i < j$) in s. Then $T_{ij}s$ is a representative of the abacus associated to $\lambda \setminus \mathrm{bs}(h)$.

Returning to our example of the partition $(6, 5, 3, 1, 1, 1)$, Figure 2 contains its Young diagram again, but now with each box filled in with the corresponding hook-length. The unique hook h of length 5 in the diagram corresponds to the pair of indices $(5, 10)$ of the sequence (4-1). If we remove the corresponding border strip, we obtain the diagram pictured in Figure 3, which corresponds to the partition $(6, 5, 3, 1, 1, 1) \setminus \mathrm{bs}(h) = (6, 2, 1, 1, 1, 1)$ and the bi-infinite sequence

$$\ldots, 1, \ldots, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0 \ldots, 0, \ldots$$

of 0's and 1's. Note that if we swap the 0 and 1 corresponding to the hook h in the representative (4-2) of $a_{(6,5,3,1,1,1)}$, then we get an equivalent bi-infinite sequence.

11	7	6	4	3	1
9	5	4	2	1	
6	2	1			
3					
2					
1					

Figure 2. Hook-lengths for (6, 5, 3, 1, 1, 1).

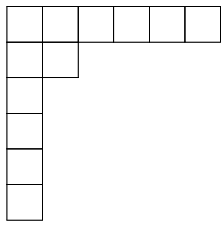

Figure 3. The Young diagram of (6, 2, 1, 1, 1, 1).

4.4. ***Removing several hooks in succession.*** In our work below, we will need to remove several hooks (more precisely, the border strips corresponding to those hooks) in succession from a partition. By removing a sequence of hooks $h_1, \ldots, h_R$ from a partition λ, we mean the following: h_1 is a hook of λ, h_2 is a hook of $\lambda \setminus \mathrm{bs}(h_1)$, h_3 is a hook of $\lambda \setminus \mathrm{bs}(h_1) \setminus \mathrm{bs}(h_2)$, and so on, until we arrive at h_R which is a hook of $\lambda \setminus \mathrm{bs}(h_1) \cdots \setminus \mathrm{bs}(h_{R-1})$, and when this is removed we obtain the final partition $\lambda' = \lambda \setminus \mathrm{bs}(h_1) \cdots \setminus \mathrm{bs}(h_R)$.

Let s be a representative of the abacus a_λ associated to λ. Let (i_1, j_1) denote the pair of indices in s corresponding to the hook h_1, (i_2, j_2) the corresponding pair to h_2 (which, recall, is a hook of $\lambda \setminus \mathrm{bs}(h_1)$ corresponding to the bi-infinite sequence $T_{i_1,j_1}s$), and so on. Thus, the sequence of hooks $h_1, \ldots, h_R$ may be encoded by the R-tuple of pairs $((i_1, j_1), (i_2, j_2), \ldots, (i_R, j_R))$, and the process of removing these hooks results in the sequence

$$s' = T_{i_R,j_R} T_{i_{R-1},j_{R-1}} \cdots T_{i_1,j_1} s.$$

The sequence s' is a representative of the abacus $a_{\lambda'}$ associated to the partition λ'.

Of particular interest for us will be the situation where all the hooks have the same length, m say. Here $j_k = i_k + m$ for all $1 \le k \le R$, and we may encode the sequence of hooks by simply the R-tuple $(i_1, \ldots, i_R)$. Note that the indices $i_1, \ldots, i_R$ may contain repeats, but there are also constraints, such as $i_2 \neq i_1$ (since $(i_1, i_1 + m)$ is a hook in s and so it cannot be a hook in $T_{i_1,i_1+m}s$).

5. Plan of the proof of Theorem 2.2

We begin by restating Theorem 2.2 in terms of values of irreducible characters at elements of S_n, which will make the notation involved in its proof cleaner.

Theorem 5.1 (an equivalent formulation of Theorem 2.2). *Let $n, m_1, \dots, m_r$ be distinct positive integers. Let $\sigma \in S_n$ be a permutation of the form*

$$\sigma = \tau \cdot \prod_{i=1}^{r} \prod_{j=1}^{p^{r-1}} \rho_i^{(j)},$$

where each $\rho_i^{(j)}$ is a cycle of length m_i, the supports of all the cycles $\rho_i^{(j)}$ are disjoint, and $\tau \in S_n$ is a permutation with support disjoint from those of the $\rho_i^{(j)}$. Suppose that λ is a $\left(\sum_{i=1}^{r} k_i m_i\right)$-core partition of n for all r-tuples $(k_1, \dots, k_r)$ of integers $0 \le k_1, \dots, k_r \le p^{r-1}$ for which some $k_i = p^{r-1}$. Then

$$p^r \mid \chi^\lambda(\sigma).$$

The proof of Theorem 5.1 rests on the following crucial proposition, which is based on applying the Murnaghan–Nakayama rule p^{r-1} times.

Proposition 5.2. *Let r, m and n be positive integers. Let $\sigma \in S_n$ be of the form*

$$\sigma = \tau \cdot \prod_{j=1}^{p^{r-1}} \rho^{(j)},$$

where each $\rho^{(j)}$ is an m-cycle, with all the cycles $\rho^{(j)}$ being disjoint, and with $\tau \in S_n$ being a permutation whose support is disjoint from all the cycles $\rho^{(j)}$. Denote by L the set of partitions of $n - p^{r-1}m$ that can be obtained from λ by removing, in succession, p^{r-1} border strips of length m. If λ is a $p^{r-1}m$-core partition of n, then

$$\chi^\lambda(\sigma) = p \sum_{\lambda' \in L} \epsilon_{\lambda'} \chi^{\lambda'}(\tau),$$

where each $\epsilon_{\lambda'}$ is an integer.

We will quickly deduce Theorem 5.1 (and hence Theorem 2.2) from Proposition 5.2 and the following simple observation.

Lemma 5.3. *Let n, t and m be positive integers. Let λ be a partition of n which is both a t-core and a $(t+m)$-core. Let λ' be a partition of $n - m$ that can be obtained by removing a border strip of length m from λ. Then λ' is a t-core.*

Proof. If λ has no hook (and thus no border strip) of length m then the lemma holds vacuously. Suppose that λ' arises from removing the border strip corresponding to the hook h of length m in λ. Let a_λ be the abacus of λ, and s be a representative bi-infinite sequence in a_λ. Suppose the hook h corresponds to the pair of indices $(i, i+m)$ with $s(i) = 0$ and $s(i+m) = 1$, so that the partition λ' corresponds to the abacus containing $s' = T_{i,i+m}s$.

If λ' is not a t-core, then there must exist a pair of indices $(j, j+t)$ with $s'(j) = 0$ and $s'(j+t) = 1$. Since the entries of s and s' differ only at the indices i and $i+m$, and since λ is a t-core, we must have either $j = i+m$, or $j+t = i$. If $j = i+m$, then $s(i) = 0$ and $s(i+t+m) = s'(j+t) = 1$ which contradicts the assumption that λ is a $(t+m)$-core. If $j = i-t$, then $s(i-t) = s'(j) = 0$ and $s(i+m) = 1$, which again contradicts the assumption that λ is a $(t+m)$-core. □

Deducing Theorem 5.1 from Proposition 5.2. Apply Proposition 5.2 first with $m = m_r$ to obtain

$$\chi^\lambda(\sigma) = p \sum_{\lambda' \in L} \epsilon_{\lambda'} \chi^{\lambda'}\left(\tau \prod_{i=1}^{r-1} \prod_{j=1}^{p^{r-1}} \rho_i^{(j)}\right).$$

If t is any number of the form $t = \sum_{i=1}^{r-1} k_i m_i$ where the k_i lie in $[0, p^{r-1}]$ with at least one of them being p^{r-1}, then λ is a $(t + k_r m_r)$-core for all $0 \le k_r \le p^{r-1}$. Since any $\lambda' \in L$ arises from λ by removing p^{r-1} border strips of length m_r, it follows by p^{r-1} applications of Lemma 5.3 that λ' is a t-core.

We may now repeat this argument, applying Proposition 5.2 to each $\lambda' \in L$ and now removing p^{r-1} border strips of length m_{r-1}. Applications of Lemma 5.3 show that the new partitions λ'' that arise are $\left(\sum_{i=1}^{r-2} k_i m_i\right)$-cores for all choices of $0 \le k_i \le p^{r-1}$ with some $k_i = p^{r-1}$.

Carrying this argument out r times, we obtain the desired result. □

The proof of Proposition 5.2 depends on the following two lemmas, which we shall prove in the next two sections.

Lemma 5.4. *Let λ be a partition, and let λ' be obtained from λ by removing a sequence of R border strips of the same length m. Let $h_1, \ldots, h_R$ be a sequence of R hooks of length m which may be removed from the initial partition λ to result in the final partition λ'. Then*

$$(-1)^{\mathrm{ht}(h_1)+\cdots+\mathrm{ht}(h_R)} = \epsilon(\lambda, \lambda')$$

where the sign $\epsilon(\lambda, \lambda') = \pm 1$ depends only on the initial and final partitions λ and λ' and is the same for all such possible sequences of hooks.

We are grateful to the referee for pointing out that Lemma 5.4 may be found in the literature as Proposition 2.2 of [17]. In the interest of keeping our exposition self-contained, we include the short proof of Lemma 5.4 in Section 6.

Lemma 5.5. *Let λ be a $p^{r-1}m$-core partition, and let λ' be a partition that can be obtained from λ by removing $R = p^{r-1}$ border strips of length m. The number of tuples $(i_1, \ldots, i_R)$ such that*

$$s' = T_{i_R, i_R+m} T_{i_{R-1}, i_{R-1}+m} \cdots T_{i_1, i_1+m} s$$

is a multiple of p. Here s is a representative of the abacus of λ, and the partition λ' corresponds to the abacus containing s'.

Once Lemmas 5.4 and 5.5 are in place, it is a simple matter to deduce Proposition 5.2.

Deducing Proposition 5.2. We apply the Murnaghan–Nakayama rule repeatedly while removing in succession $R = p^{r-1}$ hooks of length m from λ. This will result in an expression for $\chi^\lambda(\sigma)$ of the form $\sum_{\lambda' \in L} c_{\lambda'} \chi^{\lambda'}(\tau)$, for suitable integers $c_{\lambda'}$ which we must show are multiples of p. Now

$$c_{\lambda'} = \sum_{(i_1, \ldots, i_R)} (-1)^{\mathrm{ht}(h_1)+\cdots+\mathrm{ht}(h_R)}$$

where the sum is over all R-tuples $(i_1, \ldots, i_R)$ corresponding to hooks $h_1, \ldots, h_R$, which when removed from λ in order result in the partition λ'. Lemma 5.4 tells us that the sign $(-1)^{\mathrm{ht}(h_1)+\cdots+\mathrm{ht}(h_R)}$ is the same for all suitable tuples $(i_1, \ldots, i_R)$, and Lemma 5.5 tells us that the number of such R-tuples is a multiple of p. □

6. Parity of heights of hooks: Proof of Lemma 5.4

Let λ be a partition, and s a representative of the abacus a_λ associated to λ. Augment s by coloring a finite number N of 1's in s with distinct colors, taking care to color all the 1's appearing to the right of the first zero in s. The 1's appearing to the left of the first 0 are unimportant, but we allow the flexibility of coloring some of them since this situation may arise at an intermediate step when we remove hooks from λ. Note that the number of 1's appearing to the right of the first zero equals the number of rows in the partition λ. Thus N is at least the number of rows in λ. Color these 1's in the order of their appearance in s using the colors $c_1, \ldots, c_N$. Call the augmented sequence $\hat{s}$.

We begin with a general observation on removing hooks. Suppose (i, j) is a pair of indices corresponding to a hook h in s (at the moment the hook can have any length $j - i$). Removing this hook produces the sequence $T_{i,j}s$. Considering the augmented sequence $\hat{s}$, we have the corresponding augmented sequence $T_{i,j}\hat{s}$ after removing this hook. If we consider the sequence of colors among the 1's in this sequence, we obtain a permutation π_{ij}, say, of the original sequence of colors $(c_1, \ldots, c_N)$ — the 1 appearing in $(T_{i,j}\hat{s})(i)$ has the color of the 1 in $\hat{s}(j)$, and all other 1's in $T_{ij}(\hat{s})$ retain their color in $\hat{s}$. If the height of the hook removed is k, then note that $\hat{s}$ had k colored 1's between $s(i) = 0$ and $s(j) = 1$ and the permutation π_{ij} can be obtained by making k-transpositions, each time swapping the color of the 1 at position j by the color immediately preceding it. Thus $(-1)^k = (-1)^{\mathrm{ht}(h)}$ equals the sign of the permutation π_{ij}.

If we remove hooks $h_1, \ldots, h_\ell$ in succession (again, their lengths could be arbitrary), then the associated permutations of colors multiply, and therefore so do the signs of these permutations. Thus, after removing these hooks in succession we would arrive at a permutation π of the sequence of colors $(c_1, \ldots, c_N)$ and

$$(-1)^{\mathrm{ht}(h_1)+\mathrm{ht}(h_2)+\cdots+\mathrm{ht}(h_\ell)} = \mathrm{sgn}(\pi).$$

We now turn to the situation of the lemma, where a sequence $h_1, \ldots, h_R$ of R hooks is removed all of length m. Our observation above shows that removing these hooks in order leads to the sequence $\hat{s}'$ where the color of the 1's is given by a permutation π of the original sequence of colors $c_1, \ldots, c_N$. Further the sign of this permutation $\mathrm{sgn}(\pi)$ equals $(-1)^{\mathrm{ht}(h_1)+\cdots+\mathrm{ht}(h_R)}$.

To complete the proof, we will show that every way of removing R hooks of length m that leads to the partition λ' results in the same permutation of colors π. Consider the subsequence of $\hat{s}$ obtained by restricting to a progression $\pmod{m}$: namely, $(\hat{s}(a + \ell m))_{\ell\in\mathbb{Z}}$. There are m such subsequences corresponding to $a = 1, \ldots, m$. Since the hooks removed all have length m, each removal of a hook affects only the terms within one of these subsequences, leaving all the other subsequences unaltered. Further within any particular subsequence $(\hat{s}(a + \ell m))_{\ell\in\mathbb{Z}}$, it is impossible to alter the original sequence of colors by removing any sequence of hooks of length m. Therefore we can determine uniquely the color of any element in $\hat{s}'$: the 1's appearing in this sequence in the progression $a \pmod m$ have colors determined by their order of appearance in the original sequence s.

7. Proof of Lemma 5.5

Let λ be a $p^{r-1}m$-core partition, and let s be a representative of its abacus. Let s' be the sequence obtained by removing a sequence of $R = p^{r-1}$ border strips of length m from λ, and let λ' be the partition associated to s'. Our goal is to show that the number of ways of reaching λ' starting from λ is a multiple of p.

Let us first note that when $r = 1$, it is impossible to remove a border strip of length m from λ, since λ is an m-core partition by assumption. Thus the number of ways here is 0, and the lemma holds (vacuously). Henceforth, assume that $r \geq 2$.

For each $a = 1, \ldots, m$, consider the subsequences of s and s' obtained by restricting to the progression $a \pmod m$: thus, set

$$s(a; m) = (s(a + \ell m))_{\ell \in \mathbb{Z}}, \quad s'(a; m) = (s'(a + \ell m))_{\ell \in \mathbb{Z}}.$$

We may think of $s(a; m)$ and $s'(a; m)$ as corresponding to partitions $\lambda(a; m)$ and $\lambda'(a; m)$, and note that a hook of length m in the partition λ corresponds to a hook of length 1 (or simply a border square) in the partition $\lambda(a; m)$ (for some choice of a). Since $\lambda'(a; m)$ arises from $\lambda(a; m)$ by removing some number of hooks of length 1, the Young diagram of the partition $\lambda'(a; m)$ is contained in the Young diagram of the partition $\lambda(a; m)$ (that is, $\lambda_i(a; m) \geq \lambda'_i(a; m)$ for all i). The difference between the Young diagram of $\lambda(a; m)$ and $\lambda'(a; m)$ (in other words, the boxes in $\lambda(a; m)$ that are not in $\lambda'(a; m)$) is a skew diagram, which we denote by $\lambda(a; m)/\lambda'(a; m)$. Let ℓ_a denote the size of this skew diagram $|\lambda(a; m)/\lambda'(a; m)|$, so that ℓ_a hooks of length 1 must be removed from $\lambda(a; m)$ to reach $\lambda'(a; m)$. Since a total of $R = p^{r-1}$ hooks of length m are removed to go from λ to λ', note that

$$R = p^{r-1} = \sum_{a=1}^{m} \ell_a.$$

The number of ways to go from $\lambda(a; m)$ to $\lambda'(a; m)$ by removing successively ℓ_a hooks of length 1 equals the number of standard Young tableaux of skew shape $\lambda(a; m)/\lambda'(a; m)$, which we denote (in the usual notation) by $f_{\lambda(a;m)/\lambda'(a;m)}$. Recall that a standard Young tableau of this skew shape is a numbering of the boxes in the skew diagram using the numbers 1 to ℓ_a such that the entries are increasing from left to right in each row, and increasing down each column. Each such tableau corresponds to a way of removing hooks, by removing boxes in descending order of their entries.

We can now count the number of ways of going from λ to λ' by removing R hooks of length m. Note that removing a hook from one subsequence $s(a; m)$ has no impact on the hooks in any of the other subsequences. Therefore the desired number of ways to proceed from λ to λ' equals

$$\binom{p^{r-1}}{\ell_1, \ell_2, \ldots, \ell_m} \prod_{a=1}^{m} f_{\lambda(a;m)/\lambda'(a;m)}.$$

The multinomial coefficient

$$\binom{p^{r-1}}{\ell_1, \ell_2, \ldots, \ell_m}$$

is a multiple of p, except in the situation where $\ell_a = p^{r-1}$ for some a (and all other ℓ_j are 0). Thus we are left with the case when all the hooks of length m in going from λ to λ' are confined to one subsequence $s(a; m)$. So far, we have not made use of the condition that λ is a $p^{r-1}m$-core, and it is only in this case that we need this assumption. The assumption implies that $\lambda(a; m)$ is p^{r-1}-core, and so the skew diagram $\lambda(a; m)/\lambda'(a; m)$ (which has size $\ell_a = p^{r-1}$) cannot be a border strip of $\lambda(a; m)$. In this situation, it turns out that $f_{\lambda(a;m)/\lambda'(a;m)}$ is a multiple of p. This is implied by our next lemma, which is perhaps of independent interest.

Lemma 7.1. *Let π and τ be two partitions, with the Young diagram of π containing the Young diagram of τ (thus $\pi_i \geq \tau_i$ for all i). Suppose the skew diagram π/τ is not a border strip of the partition π (equivalently, either π/τ is disconnected, or it contains a 2×2 square), and that $|\pi/\tau| = p^t$ is a prime power (with $t > 0$). Then the number of standard Young tableaux of skew shape π/τ, denoted $f_{\pi/\tau}$, is a multiple of p.*

Proof. First suppose that π/τ is disconnected, and is composed of $k \geq 2$ maximally connected skew shapes $S_1, \ldots, S_k$, with $|S_j| = s_j \geq 1$. Then

$$f_{\pi/\tau} = \binom{p^t}{s_1, \ldots, s_k} f_{S_1} \cdots f_{S_k},$$

is clearly a multiple of p.

Now suppose that π/τ is a connected skew shape, but contains a 2×2 square so that it is not a border strip of π. Since $f_{\pi/\tau}$ depends only on the shape π/τ, we may assume that π is minimal, having only as many rows and columns as needed for the skew shape π/τ. Then the maximal hook length of π equals the number of border squares of π, which is strictly smaller than $|\pi/\tau| = p^t$ (since π/τ is not a border strip by assumption).

It is a basic fact (see Section I.9 of [9], for example — the identity below follows from (9.1) of [9] by taking the Hall inner product of both sides with the symmetric function $e_1^{p^t}$) that

$$f_{\pi/\tau} = \sum_{\nu \vdash p^t} f_\nu c^{\pi}_{\tau\nu},$$

where the sum is over partitions ν of $|\pi/\tau| = p^t$, $f_\nu = \chi^{\nu}_{(1,\ldots,1)}$ is the degree of the irreducible character corresponding to ν and the $c^{\pi}_{\tau\nu}$ are the Littlewood–Richardson coefficients (which are integers). By Lemma 2.1, $f_\nu \equiv \chi^{\nu}_{(p^t)} \pmod{p}$, so that $p \mid f_\nu$ unless ν is a hook of length p^t. Suppose now that ν is a hook of length p^t. Here we use that the Littlewood–Richardson coefficient $c^{\pi}_{\tau\nu}$ is zero unless the Young diagram of the partition ν is contained in that of π (see Section I.9 of [9] once again). But all the hooks of π have length $< p^t$, and therefore π cannot contain a hook ν of length p^t. Thus either $c^{\pi}_{\tau\nu} = 0$ or $p \mid f_\nu$, and therefore the lemma follows. □

8. Preliminaries for the proof of Proposition 2.3

As in [19], let $\tilde{p}(k)$ denote the number of partitions of a nonnegative integer k into powers of p, with the convention that $\tilde{p}(0) = 1$. Denote by $F_p(t)$ the associated generating function

$$F_p(t) := \sum_{k=0}^{\infty} \tilde{p}(k) e^{-k/t} = \prod_{j=0}^{\infty} (1 - e^{-p^j/t})^{-1},$$

where $t > 0$ is a real number. We begin by recalling some estimates from our prior work [19].

Lemma 8.1 [19, Lemma 2]. *When $0 < t \le 1$, we have $F_p(t) = O(1)$, and when $t \ge 1$, we have*

$$\frac{(\log t)^2}{2 \log p} + \frac{1}{2} \log t + O(1) \le \log F_p(t) \le \frac{(\log t)^2}{2 \log p} + \frac{1}{2} \log t + \frac{1}{8} \log p + O(1).$$

More precise results are known for fixed primes p, as partitions into prime powers have been studied extensively since the work of Mahler [10] and de Bruijn [1]. We will only require the estimates of Lemma 8.1, which are cruder but uniform in p.

Given a partition μ of k into powers of p, let $\tilde{\mu}$ denote the partition obtained by repeatedly replacing every occurrence of p^r parts of the same size p^j by p^{r-1} parts of size p^{j+1} until no part appears more than $p^r - 1$ times. For every nonnegative integer s, define $\tilde{p}(k; s)$ to be the number of partitions μ of k into powers of p such that $\tilde{\mu}$ does not contain (at least) p^{r-1} parts of the same size p^j for any $j \ge s$. The second lemma of this section gives a useful lower bound for the difference between $\tilde{p}(k)$ and $\tilde{p}(k; s)$.

Lemma 8.2. *For all $s \ge 2$ and $k \ge p^{r+s-1}(1 + 4/s)$, we have*

$$\tilde{p}(k) - \tilde{p}(k; s) \ge \frac{p^{s(s-1)/2}}{(s-1)^{s-1}}.$$

Proof. We will construct at least $p^{s(s-1)/2}/(s-1)^{s-1}$ partitions of k counted in $\tilde{p}(k)$ but not in $\tilde{p}(k; s)$. For each $1 \le i \le s - 1$, pick an integer a_i in the range

$$0 \le a_i \le \frac{p^{s-i}}{s-1}.$$

Each choice of $a_1, \dots, a_{s-1}$ gives a partition μ counted in $\tilde{p}(k)$ by using a_i copies of p^i for $1 \le i \le s - 1$ and $k - \sum_{i=1}^{s-1} a_i p^i$ copies of 1. The number of such partitions is

$$\prod_{i=1}^{s-1} \left\lceil \frac{p^{s-i}}{s-1} \right\rceil \ge \prod_{i=1}^{s-1} \frac{p^{s-i}}{s-1} = \frac{p^{s(s-1)/2}}{(s-1)^{s-1}}.$$

Note that if $i > s - \log(s-1)/\log p$, then a_i must be zero, so that all of these partitions have largest part at most $p^s/(s-1)$.

We must check that each such μ is not counted in $\tilde{p}(k; s)$; that is, that the corresponding $\tilde{\mu}$ contains at least p^{r-1} copies of some part p^j with $j \ge s$. Suppose that this is not the case. Notice that, by construction, the number of times any part appears in μ is congruent modulo p^{r-1} to the number of times it appears in $\tilde{\mu}$. Since no part can appear more than $p^r - 1$ times in $\tilde{\mu}$, it follows that any part that appears fewer than p^{r-1} times or more than $p^r - p^{r-1}$ times in $\tilde{\mu}$ must have appeared in the original partition μ. Since all the parts of μ are below $p^s/(s-1)$, we conclude that $\tilde{\mu}$ can contain (i) at most $p^r - 1$ copies of any part p^j with $p^j \le p^s/(s-1)$, (ii) at most $p^r - p^{r-1}$ copies of any part p^j with $p^s/(s-1) < p^j \le p^{s-1}$, and (iii) no parts of size p^j with $j \ge s$. But these constraints imply that

$$k = |\tilde{\mu}| \le (p^r - 1) \sum_{p^j \le p^s/(s-1)} p^j + (p^r - p^{r-1}) \sum_{p^s/(s-1) < p^j \le p^{s-1}} p^j$$
$$< (p^{r-1} - 1)\frac{p^s}{(s-1)}\left(1 - \frac{1}{p}\right)^{-1} + (p^r - p^{r-1})p^{s-1}\left(1 - \frac{1}{p}\right)^{-1} < p^{r+s-1}\left(1 + \frac{4}{s}\right),$$

which contradicts our assumption on the size of k. □

9. Proof of Proposition 2.3

Let $\mathcal{L}$ be a set of positive integers coprime to p, and define $p(n; \mathcal{L}, s)$ to be the number of partitions μ of n for which $\tilde{\mu}$ contains fewer than p^{r-1} parts of the same size ℓp^j for every $\ell \in \mathcal{L}$ and $j \ge s$. We will prove Proposition 2.3 by obtaining an upper bound for $p(n; \mathcal{L}, s)$ for well-chosen $\mathcal{L}$ and s.

Lemma 9.1. *Suppose that n is large and $p^r \le 10^{-3} \log n / \log\log n$. Put*

$$x = \frac{\sqrt{6n}}{\pi}, \quad s = \left\lfloor \frac{\log \sqrt{n}}{ep^r} \right\rfloor, \tag{9-1}$$

and let $\mathcal{L}$ be the set of integers in the interval $[L, L + x/p^{r+s-1}]$ that are coprime to p, where L is a parameter lying in the range

$$\frac{\sqrt{6n}}{2\pi p^{r+s-1}} \le L \le \left(1 + \frac{1}{5p^r}\right)\frac{\sqrt{6n}}{2\pi p^{r+s-1}} \log n. \tag{9-2}$$

Then

$$p(n; \mathcal{L}, s) \ll p(n) n^{3/4} \exp(-n^{1/(16p^r)}).$$

Before proving the lemma, let us see how Proposition 2.3 would follow. Choose r distinct values L_j (with $1 \le j \le r$) all in the range

$$\left(1 + \frac{1}{6p^r}\right)\frac{\sqrt{6n}}{2\pi p^{r+s-1}} \log n \le L_j \le \left(1 + \frac{1}{5p^r}\right)\frac{\sqrt{6n}}{2\pi p^{r+s-1}} \log n,$$

such that the corresponding sets $\mathcal{L}_j$ are all disjoint. A partition μ for which $\tilde{\mu}$ does not contain r distinct parts $m_1, \dots, m_r$ each appearing at least p^{r-1} times and suitably large as desired in the proposition, must be counted among some $p(n; \mathcal{L}_j, s)$ with $1 \le j \le r$. Thus by Lemma 9.1 the number of such bad partitions μ is

$$\le \sum_{j=1}^{r} p(n; \mathcal{L}_j, s) \ll r p(n) n^{3/4} \exp(-n^{1/(16p^r)}) \ll p(n) n \exp(-n^{1/(16p^r)}) \ll p(n) \exp(-n^{1/(20p^r)}),$$

as claimed.

Proof of Lemma 9.1. Consider the process of going from a partition μ to $\tilde{\mu}$ by combining p^r parts of the same size m into p^{r-1} parts of size pm. Suppose that ℓ is coprime to p, and that the sum of all parts of the form ℓp^j appearing in μ equals ℓk. Restricting our attention to these parts, we may think of μ as giving rise to a partition of k into powers of p, and then $\tilde{\mu}$ correspondingly gives a partition of k into powers of p obtained by repeatedly combining p^r parts of size p^j into p^{r-1} parts of size p^{j+1}. It follows that $p(n; \mathcal{L}, s)$ is the coefficient of z^n in the generating function

$$\prod_{\substack{\ell\notin\mathcal{L}\\(\ell,p)=1}} \prod_{j=0}^{\infty}(1-z^{\ell p^j})^{-1} \prod_{\ell\in\mathcal{L}}\biggl(\sum_{k=0}^{\infty} \tilde{p}(k; s)z^{\ell k}\biggr),$$

which equals

$$\prod_{i=1}^{\infty}(1-z^i)^{-1}\prod_{\ell\in\mathcal{L}}\biggl(\frac{\sum_{k=0}^{\infty}\tilde{p}(k;s)z^{\ell k}}{\sum_{k=0}^{\infty}\tilde{p}(k)z^{\ell k}}\biggr).$$

Since all of the coefficients in the generating function for $p(n; \mathcal{L}, s)$ are nonnegative, we must have, for any $0 < z < 1$,

$$p(n;\mathcal{L},s) \le \frac{1}{z^n}\prod_{i=1}^{\infty}(1-z^i)^{-1}\prod_{\ell\in\mathcal{L}}\biggl(\frac{\sum_{k=0}^{\infty}\tilde{p}(k;s)z^{\ell k}}{\sum_{k=0}^{\infty}\tilde{p}(k)z^{\ell k}}\biggr). \tag{9-3}$$

Recall that $x = \sqrt{6n}/\pi$, and take $z = e^{-1/x}$ in the bound (9-3). Then, by the asymptotic formula for the partition function and basic estimates for the generating function of the number of partitions (see Section VIII.6 of [3]), we obtain

$$p(n;\mathcal{L},s) \ll n^{3/4}p(n)\prod_{\ell\in\mathcal{L}}\biggl(\frac{\sum_{k=0}^{\infty}\tilde{p}(k;s)z^{\ell k}}{\sum_{k=0}^{\infty}\tilde{p}(k)z^{\ell k}}\biggr) \ll n^{3/4}p(n)\exp(-\Delta), \tag{9-4}$$

where

$$\Delta := \sum_{\ell\in\mathcal{L}}\frac{1}{F_p(x/\ell)}\sum_{k=0}^{\infty}(\tilde{p}(k)-\tilde{p}(k;s))e^{-\ell k/x}.$$

Our work so far applies to any set $\mathcal{L}$ of integers that are coprime to p, and we now proceed to the situation at hand. The lower bound on L and our choice of x give, for all $\ell\in\mathcal{L}$, the bound

$$F_p\Bigl(\frac{x}{\ell}\Bigr) \le F_p\Bigl(\frac{x}{L}\Bigr) \le F_p\Bigl(\frac{2p^{r+s-1}}{\log n}\Bigr).$$

From this estimate, our choice of $\mathcal{L}$, and Lemma 8.2 it follows that

$$\Delta \ge \frac{1}{F_p(2p^{r+s-1}/\log n)}\sum_{\substack{L\le\ell\le L+x/p^{r+s-1}\\(\ell,p)=1}}\ \sum_{k\ge p^{r+s-1}(1+4/s)}\frac{p^{s(s-1)/2}}{(s-1)^{s-1}}e^{-\ell k/x}. \tag{9-5}$$

For ℓ in the range $L \le \ell \le L + x/p^{r+s-1}$, we have

$$\sum_{k\ge p^{r+s-1}(1+4/s)} e^{-\ell k/x} \ge \exp\biggl(-\frac{\ell p^{r+s-1}}{x}\Bigl(1+\frac{4}{s}\Bigr)\biggr)\frac{e^{-\ell/x}}{1-e^{-\ell/x}} \ge \frac{x}{2L}\exp\biggl(-\Bigl(\frac{Lp^{r+s-1}}{x}+1\Bigr)\Bigl(1+\frac{4}{s}\Bigr)\biggr).$$

Inserting this into the right-hand side of (9-5) and noting that (since p^{r+s-1} is small in comparison to x)

$$|\mathcal{L}| \geq \left(1 - \frac{1}{p}\right)\frac{x}{p^{r+s-1}} - 2 \geq \frac{x}{3p^{r+s-1}},$$

we obtain (using our choice of x and the range for L)

$$\begin{aligned}\Delta &\geq \frac{1}{F_p(2p^{r+s-1}/\log n)} \cdot \frac{p^{s(s-1)/2}}{(s-1)^{s-1}} \cdot \frac{x}{3p^{r+s-1}} \cdot \frac{x}{2L} \exp\left(-\left(\frac{Lp^{r+s-1}}{x} + 1\right)\left(1 + \frac{4}{s}\right)\right) \\ &\geq \frac{1}{6}\frac{1}{F_p(2p^{r+s-1}/\log n)} \cdot \frac{p^{s(s-1)/2}}{(s-1)^{s-1}} \cdot \frac{x}{\log n} \cdot \exp\left(-\left(\frac{Lp^{r+s-1}}{x} + 1\right)\left(1 + \frac{4}{s}\right)\right).\end{aligned}$$

Using Lemma 8.1 and the bound $p^r \leq \log\sqrt{n}$, it follows that

$$\begin{aligned}\log F_p\left(\frac{2p^{r+s-1}}{\log n}\right) &\leq \frac{1}{2\log p}\left(\log\frac{p^{r+s-1}}{\log\sqrt{n}}\right)^2 + \frac{1}{2}\log\frac{p^{r+s-1}}{\log\sqrt{n}} + \frac{1}{8}\log p + O(1) \\ &\leq \frac{1}{2\log p}\left(\log\frac{p^{r+s-1}}{\log\sqrt{n}}\right)^2 + \frac{s}{2}\log p + O(1).\end{aligned}$$

Therefore

$$\begin{aligned}\log\frac{p^{s(s-1)/2}}{F_p(2p^{r+s-1}/\log n)(s-1)^{s-1}} &\geq \frac{s^2}{2}\log p - \frac{1}{2\log p}\left(\log\frac{p^{r+s-1}}{\log\sqrt{n}}\right)^2 - s\log ps + O(1) \\ &\geq s\log\frac{\log\sqrt{n}}{p^r s} - \frac{(\log\log\sqrt{n})^2}{2\log p} + O(1).\end{aligned}$$

Recalling our choice of s, we conclude that

$$\begin{aligned}\log\Delta &\geq s\log\frac{\log\sqrt{n}}{p^r s} + \log\sqrt{n} - \frac{(\log\log\sqrt{n})^2}{2\log p} - \log\log n - \frac{Lp^{r+s-1}}{x}\left(1 + \frac{4}{s}\right) + O(1) \\ &\geq \left(1 + \frac{1}{ep^r}\right)\log\sqrt{n} - \frac{Lp^{r+s-1}}{x} - \log\log n - \frac{(\log\log\sqrt{n})^2}{2\log p} + O(1) \\ &\geq \left(\frac{1}{ep^r} - \frac{1}{5p^r}\right)\log\sqrt{n} - (\log\log n)^2 + O(1) \geq \frac{\log n}{15p^r} - (\log\log n)^2 + O(1),\end{aligned}$$

upon using the upper bound on L in (9-2). In the range $p^r \leq 10^{-3}\log n/(\log\log n)^2$ we find

$$\log\Delta \geq \frac{\log n}{16p^r} + O(1),$$

which when used in (9-4) yields the lemma. □

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Index of coregularity zero log Calabi–Yau pairs

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We study the index of log Calabi–Yau pairs (X, B) of coregularity 0. We show that $2\lambda(K_X + B) \sim 0$, where λ is the Weil index of (X, B). This is in contrast to the case of klt Calabi–Yau varieties, where the index can grow doubly exponentially with the dimension. Our sharp bound on the index extends to the context of generalized log Calabi–Yau pairs, semi-log canonical pairs, and isolated log canonical singularities of coregularity 0. As a consequence, we show that the index of a variety appearing in the Gross–Siebert program or in the Kontsevich–Soibelman program is at most 2. Finally, we discuss applications to Calabi–Yau varieties endowed with a finite group action, including holomorphic symplectic varieties endowed with a purely nonsymplectic automorphism.

1. Introduction

A *log Calabi–Yau pair of coregularity* 0 is a pair (X, B) consisting of a proper normal variety X and an effective $\mathbb{Q}$-divisor $B \sim_{\mathbb{Q}} -K_X$ such that (X, B) is log canonical and whose dual complex $\mathcal{DMR}(X, B)$ (see Definition 2.2) has maximal dimension $\dim \mathcal{DMR}(X, B) = \dim X - 1$. For instance, a complete normal toric variety with its toric boundary is an example of a log Calabi–Yau pair of coregularity 0.

In this article, we study the index of log Calabi–Yau pairs of coregularity 0.

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Keywords: Calabi–Yau, index conjecture, dual complex, minimal model program, mirror symmetry.

1.1. ***Index conjecture.*** From the point of view of the minimal model program (MMP), Calabi–Yau varieties are fundamental building blocks of algebraic varieties. Their canonical line bundle is numerically trivial, i.e., its restriction to any curve has degree 0. By the abundance conjecture, which is known in this special case [Gongyo 2013], the canonical divisor of a Calabi–Yau variety is $\mathbb{Q}$-linearly trivial, i.e., it is a torsion element of the Picard group. The smallest positive integer m for which $mK_X \sim 0$ is known as the (Cartier) *index* of the Calabi–Yau variety. For instance, $m = 1$ if $\dim X = 1$, and m divides 12 if $\dim X = 2$ and X is smooth; see [Beauville 1996, Corollary VIII.7].

Inspired by the MMP, it is natural to study the same question for pairs (X, B) with mild singularities. Here, mild singularities mean either klt or log canonical. When we consider mildly singular varieties or pairs, the situation gets more interesting. In this case, the index will also depend on the coefficient set of B. For instance, if we consider 2-dimensional klt Calabi–Yau pairs (X, B) with standard coefficients, the largest possible index of $K_X + B$ is 66; see [Ishii 2000, Proposition 4.9] for the bound and [Kondō 1992, Main Theorem and Section 3] for its sharpness. It is conjectured that, once we fix the dimension and coefficient set, there are only finitely many possible indices for the divisor $K_X + B$. This conjecture is known as the *index conjecture* in birational geometry. It is known up to dimension 3 due to the work of Jiang [2021] and Xu [2020]. In [Esser et al. 2022], the authors construct a sequence of klt Calabi–Yau varieties X_d for which the index grows doubly exponentially with the dimension d. These Calabi–Yau varieties, regarded as log Calabi–Yau pairs with empty boundary, have coregularity equal to their dimension. In contrast, our results show that, if the coregularity is minimal (i.e., equal to 0), the index does not depend on the dimension but only on the coefficients set.

1.2. ***Main theorem.*** We focus on log Calabi–Yau pairs (X, B) of coregularity 0; see Definition 2.4. These pairs arise naturally in mirror symmetry, and they are often referred to as pairs of maximal intersection; see [Kollár and Xu 2016]. Indeed, the condition that the dual complex has maximal dimension means that, up to a dlt modification, the maximal number of irreducible components of $B^{=1}$ intersecting at some point is the maximal possible, namely $\dim X$. These pairs also appear in relation to maximally unipotent degenerations of Calabi–Yau varieties; see Section 7 and also [Kollár and Xu 2016; Nicaise and Xu 2016; Nicaise et al. 2019]. In this setting, we prove that the index of $K_X + B$ is independent of the dimension and only depends on the coefficient set of B, and it is actually 1 or 2 if B is reduced.

Theorem 1. *Let $(X, B, \boldsymbol{M})$ be a projective generalized log Calabi–Yau pair of coregularity 0 and Weil index λ. Then, we have that $\lambda'(K_X + B + \boldsymbol{M}_X) \sim 0$, where $\lambda' = \mathrm{lcm}(\lambda, 2)$.*

We recall the notion of generalized log Calabi–Yau pair in Section 2; see Definition 3.4 for the Weil index of a generalized pair. Note that in order to have the linear equivalence $D \sim 0$, the divisor D must be Weil. Hence, multiplying $K_X + B + \boldsymbol{M}_X$ by λ is necessary to compute its Cartier index. In Example 9.1, we show that taking the least common multiple of λ and 2 is needed in general. The following special case of our main result is a key step in the proof of Theorem 1.

Proposition 2. *Let (X, B) be a proper log Calabi–Yau pair of coregularity 0 with $B = B^{=1}$. Then, we have that $2(K_X + B) \sim 0$.*

In many circumstances, it is important to deal with pairs with coefficients greater than $\frac{1}{2}$. For instance, in moduli theory, this implies the flatness of the boundary divisors; see [Kollár 2023, Theorem 2.82]. In the context of log Calabi–Yau pairs of coregularity 0 and large boundary, we obtain the following corollary.

Corollary 3. *Let (X, B) be a projective log Calabi–Yau pair of coregularity 0. Assume that all the coefficients of B are greater than or equal to $\frac{1}{2}$. Then, the divisor $2B$ is reduced and $2(K_X + B) \sim 0$. In particular, if the coefficients of B are strictly greater than $\frac{1}{2}$, then B is integral and $2(K_X + B) \sim 0$.*

1.3. ***Orientability of dual complexes.*** The dual complex $\mathcal{D}(B)$ of a dlt pair (X, B) is a simplicial object encoding the combinatorial information about multiple intersections of irreducible components of B; see Definition 2.1. It has been clear since [Kollár and Xu 2016, Section 17] that the index of (X, B) is a measure of the orientability of $\mathcal{D}(B)$. What Corollary 4 says is that the index of (X, B) does not contain any information but the orientability of $\mathcal{D}(B)$.

Corollary 4. *Let (X, B) be a proper dlt log Calabi–Yau pair of coregularity 0 and dimension $n + 1$ with $B = B^{=1}$. Then, there exists an isomorphism of singular and coherent cohomology groups*

$$H^n(\mathcal{D}(B), \mathbb{K}) \simeq H^0(X, \mathcal{O}_X(K_X + B)),$$

and the following facts hold:

(i) *$K_X + B \sim 0$ if and only if $\mathcal{D}(B)$ is an orientable closed pseudomanifold.*

(ii) *If $\mathcal{D}(B)$ is not an orientable closed pseudomanifold, there exists a quasiétale double cover $(\widetilde{X}, \widetilde{B}) \to (X, B)$ such that $\mathcal{D}(\widetilde{B})$ is an orientable closed pseudomanifold and $\mathcal{D}(B) \simeq \mathcal{D}(\widetilde{B})/(\mathbb{Z}/2\mathbb{Z})$.*

We refer the reader to Definitions 5.1 and 5.3 for the orientability of pseudomanifolds. In the previous corollary, quasiétale means étale in codimension 1. Note that the quotient map $\mathcal{D}(\tilde{B}) \to \mathcal{D}(B)$ in Corollary 4 is not necessarily a topological covering space. In fact, closed pseudomanifolds admit no orientable topological covering spaces in general, and branched orientation covers are the best we can expect; see Example 9.7 and [Matthews 2016, Section 5.2].

1.4. ***Mirror symmetry.*** Mirror symmetry predicts a relation between pairs of Calabi–Yau varieties. The goal is to develop a general mechanism to produce mirror pairs and describe how this duality exchanges geometric structures through the two sides of the mirror. Various approaches have been developed to achieve this purpose, e.g., the SYZ conjecture [Strominger et al. 2001], the Gross–Siebert program [Gross 2013], and the Kontsevich–Soibelman program [Kontsevich and Soibelman 2006]. In all these programs, one tries to degenerate the Calabi–Yau variety to a simple normal crossing Calabi–Yau variety of coregularity 0, also known as *large complex limit* or *maximal degeneration*. Here, we show that the Calabi–Yau varieties X that appear in a maximal degeneration have index at most 2, namely $2K_X \sim 0$.[1]

[1]Here, a Calabi–Yau variety that appears in a maximal degeneration satisfies $K_X \sim_{\mathbb{Q}} 0$, while some authors require $K_X \sim 0$. Our theorems show that $K_X \sim 0$ or $2K_X \sim 0$ are automatic.

Theorem 5. *Let $\pi : \mathcal{X} \to C$ be a minimal family of Calabi–Yau varieties of coregularity 0 over $c_0 \in C$. Let U be the maximal open subset over which the family is locally stable. Then, for every $c \in U$, we have that $2K_{\mathcal{X}_c} \sim 0$.*

See Definition 7.1 for the notion of a minimal family of Calabi–Yau varieties of coregularity 0 over a point, and Definition 7.3 for the notion of local stability of a family. Theorem 5 is a special case of a more general statement regarding Calabi–Yau pairs; see Theorem 7.6.

Corollary 6. *Let $\pi : \mathcal{X} \to C$ be a minimal family of Calabi–Yau varieties of coregularity 0 over $c_0 \in C$ of relative dimension n. Suppose that the pair $(\mathcal{X}, \mathcal{X}_{c_0})$ is dlt. Then we have*

$$H^n(\mathcal{D}(\mathcal{X}_{c_0}), \mathbb{K}) \simeq H^0(\mathcal{X}_c, \mathcal{O}_{\mathcal{X}_c}(K_{\mathcal{X}_c})). \tag{1-1}$$

In particular, if $K_{\mathcal{X}_c} \not\sim 0$, then $\mathcal{D}(\mathcal{X}_{c_0})$ is not orientable.

We can improve the estimate on the index in Theorem 5 by imposing additional hypotheses on the special fiber, which are natural, especially from the point of view of the Gross–Siebert program.

Theorem 7. *Let $\pi : \mathcal{X} \to C$ be a minimal family of Calabi–Yau varieties of coregularity 0 over $c_0 \in C$. Assume that*

- *the fiber $\mathcal{X}_{c_0}$ is reduced and toric simple normal crossing, i.e., a union of smooth toric varieties intersecting along toric strata; and*
- *the dual complex $\mathcal{D}(\mathcal{X}_{c_0})$ is simply connected.*

Let U be the maximal open subset over which the family is locally stable. Then, for every $c \in U$, we have that $K_{\mathcal{X}_c} \sim 0$.

1.5. *Index of semi-log canonical pairs.* In order to establish the applications to mirror symmetry in Section 1.4, we will need to prove a version of Theorem 1 for semi-log canonical pairs; see [Kollár 2013, Section 5] for a definition of these pairs. The key idea is to produce a section on a suitable normal modification of the pair which is invariant under certain birational transformations. These sections are called *admissible*; see Definition 6.1. We obtain the following Theorem 8 regarding the existence of admissible sections on log Calabi–Yau pairs of coregularity 0. To this end, given a pair (X, B), we denote by $\mathrm{Bir}(X, B)$ the group of crepant birational automorphisms of (X, B); see also [Hacon and Xu 2016, Definition 1.1].

Theorem 8. *Let (X, B) be a projective log Calabi–Yau pair of coregularity 0 and Weil index λ. Set $\lambda' = \mathrm{lcm}(\lambda, 2)$. Then, there is a nontrivial section $s \in H^0(X, \mathcal{O}_X(\lambda'(K_X + B)))$ such that $g^*s = s$ for every $g \in \mathrm{Bir}(X, B)$.*

Theorem 8 may be regarded as an effective version of the boundedness of B-representations; see [Hacon and Xu 2016, Theorem 1.2] or [Fujino and Gongyo 2014, Theorem 3.15]. As a consequence, we show the following corollary.

Corollary 9. *Let (X, B) be a projective semi-log canonical Calabi–Yau pair of coregularity 0 and Weil index λ. Then, we have that $\lambda'(K_X + B) \sim 0$, where $\lambda' = \mathrm{lcm}(\lambda, 2)$.*

Corollary 9 is an immediate consequence of Theorems 1 and 8. Our techniques do not apply to the case of generalized semi-log canonical pairs; see Remark 1.2.

1.6. ***Index of log canonical singularities.*** By adjunction, the exceptional divisor of a dlt modification of an isolated log canonical singularity is a projective semi-log canonical log Calabi–Yau pair. Therefore, Corollary 9 provides a bound on the index of log canonical singularities.

Corollary 10. *Let $(X, B; x)$ be the germ of a log canonical pair of coregularity 0 and Weil index λ. Suppose that the pair (X, B) is dlt away from x. Then, we have that $\lambda'(K_X + B) \sim 0$, where $\lambda' = \mathrm{lcm}(\lambda, 2)$.*

If the pair $(X, B; x)$ has only standard coefficients, one can drop the assumption that the log canonical germ $(X, B; x)$ is dlt away from x, as shown by Fujino in [2001, Theorem 4.18]. In particular, Corollary 10 and [Fujino 2001, Theorem 4.18] specialize to Corollary 11, which was first proved by Ishii [2000, Theorem 4.5]. Note that the Hodge theoretic assumption in [loc. cit.] coincides with our condition of coregularity 0; see [Kollár and Xu 2016, Claim 32.3].

Corollary 11. *The index of an isolated log canonical singularity $(X; x)$ of coregularity 0 is either 1 or 2.*

The only possible log canonical surface singularities of coregularity 0 and index 2 are double quotients of cusp singularities; see, e.g., [Ishii 2014, Section 7.8]. Examples in dimension 3 appear in [Kollár 2013, Propositions 3.54 and 3.59] provided that the 2-manifold F in [loc. cit.] is chosen to be nonorientable.

1.7. ***Calabi–Yau and holomorphic symplectic varieties with group actions.*** We apply Theorem 5 to study degenerations of quotients of Calabi–Yau varieties, and in particular of holomorphic symplectic varieties.

Theorem 12. *Let X be a projective log canonical variety with $K_X \sim 0$ and let G be a finite group acting on X freely in codimension 1. Assume that the quotient X/G is a fiber of a minimal family of Calabi–Yau pairs of coregularity 0. Then, the order of the character $\rho : G \to \mathrm{GL}(H^0(X, \mathcal{O}_X(K_X)))$ is at most 2.*

One may drop the assumption that G acts freely in codimension 1 and replace X with a log Calabi–Yau pair with standard coefficients. For expository reasons, we postpone it to Theorem 8.1. In particular, the result imposes strong constraints on degenerations of holomorphic symplectic varieties with a nonsymplectic automorphism. See Section 8 for definitions.

Corollary 13. *Let $\pi^* : \mathcal{X}^* \to C^*$ be projective family of type III of holomorphic symplectic varieties of dimension $2n$ with a finite order automorphism $g : \mathcal{X}^* \to \mathcal{X}^*$ acting purely nonsymplectically on the fibers $\mathcal{X}^*_c$ for any $c \in C^*$. Then the order of g divides $2n$.*

Remark 1.1. The case of K3 surfaces is proved in [Alexeev et al. 2024, Corollary 3.16] or [Matsumoto 2023] via Hodge theory. The arguments of [Alexeev et al. 2024, Section 3] extend verbatim to the case of higher dimensional primitive symplectic varieties, see Definition 8.2. This means that if the fibers $\mathcal{X}^*_c$ in Corollary 13 are primitive symplectic, the order of g is exactly 2.

1.8. ***Strategy of the proof of Theorem 1.*** We distinguish two cases: either $B^{<1}+\boldsymbol{M}_X$ is numerically trivial, or not. In the first case, we can actually assume that $B^{=1}=B$ and $\boldsymbol{M}=0$, as explained in Section 3.3, and so Theorem 1 reduces to Proposition 2. In Sections 5.1 and 5.2 we provide a (self-contained) topological proof of Proposition 2. We explain the relation between the index of a log Calabi–Yau pair and the orientability of its dual complex. In brief, we show that the cohomology group $H^0(X, \mathcal{O}_X(K_X+B))$ can be endowed with an integral structure given by the top cohomology of the dual complex $\mathcal{D}(B)$. The factor 2 in the index then serves to control the orientation of $\mathcal{D}(B)$.

Following a suggestion of Kollár, we also present a second proof of Proposition 2 in Section 5.3. In this case, the presence of the factor 2 reflects the fact that Poincaré residue maps are defined only up to a sign and the fact that the rational canonical form dx/x on $\mathbb{P}^1$ has residues 1 and -1 at its poles. See Remark 5.8 for a comparison between the two approaches.

If instead $B^{<1}+\boldsymbol{M}_X \not\equiv 0$, we adopt an inductive strategy from birational geometry. First, by the work of Filipazzi and Svaldi, we may assume that $B^{=1}$ fully supports a big divisor; see [Filipazzi and Svaldi 2023]. Then, we can run a $(K_X+B^{=1})$-MMP which terminates with a Mori fiber space $Y \to Z$. Let B_Y be the push-forward of B on Y. Since $(Y, B_Y, \boldsymbol{M})$ and $(X, B, \boldsymbol{M})$ are crepant equivalent, they have the same index; see Corollary 3.3. Thus, it suffices to control the index of $(Y, B_Y, \boldsymbol{M})$. The divisor $B^{=1}$ fully supports a big divisor, so some component S of ${B_Y}^{=1}$ is ample over the base. Since (Y, B_Y) is dlt, such component S is normal. We can perform adjunction to S and obtain a generalized log Calabi–Yau pair structure $(S, B_S, \boldsymbol{N})$. A careful analysis of the coefficients of B_S and $\boldsymbol{N}$ shows that $\lambda'(K_S+B_S+\boldsymbol{N}_S)$ is Weil, where $\lambda'=\operatorname{lcm}(\lambda, 2)$; see Section 3.2. By adjunction, $(S, B_S, \boldsymbol{N})$ is a generalized log Calabi–Yau pair. By induction on the dimension, we conclude that $\lambda'(K_S+B_S+\boldsymbol{N}_S)\sim 0$. Then, using the positivity of S and a relative version of Kawamata–Viehweg vanishing over Z, we conclude that $\lambda'(K_X+B+\boldsymbol{M}_X)\sim 0$; see Section 4.

Remark 1.2. We avoid the use of nonnormal generalized pairs. In order to control the index of nonnormal pairs, it is often necessary to use gluing theory. For recent advances in this direction, see for instance [Hu 2021; Liu and Xie 2023].

2. Notation

We work over a field $\mathbb{K}$ of characteristic 0. We refer to [Kollár and Mori 1998] for the standard terminology in birational geometry. For the language of generalized pairs and b-divisors, we refer to [Filipazzi and Svaldi 2023]. Recall that a *generalized log Calabi–Yau pair* is a projective generalized log canonical pair $(X, B, \boldsymbol{M})$ such that $K_X+B+\boldsymbol{M}_X\sim_{\mathbb{Q}} 0$.

Our varieties are connected and quasiprojective unless otherwise stated. Given a normal variety X and an open subset $U\subset X$, we say that U is *big* if $\operatorname{codim}_X(X\setminus U)\geq 2$.

In this work, all divisors have coefficients in $\mathbb{Q}$. Unless otherwise stated, a *divisor* means a Weil $\mathbb{Q}$-divisor. Given two proper birational normal varieties X and X' and a $\mathbb{Q}$-Cartier divisor D on X with $D\sim_{\mathbb{Q}} 0$, the *crepant trace* D' of D is defined as follows: we take a common resolution X'' with morphisms

$p : X'' \to X$ and $q : X'' \to X'$. Then, we set $D' := q_* p^* D$. If $(X, B, \boldsymbol{M})$ is a generalized log Calabi–Yau pair, its trace $(X', B', \boldsymbol{M})$ on X' is defined in the same fashion, where we put $D = K_X + B + \boldsymbol{M}_X$.

Definition 2.1 (dual complex). Let (X, B) be a dlt pair with $B^{=1} = \sum_{i \in I} B_i$. The *dual complex* of (X, B), denoted by $\mathcal{D}(B)$, is the regular Δ-complex whose vertices are in correspondence with the irreducible components of $B^{=1}$ and whose k-faces correspond to connected components of $B_{i_0} \cap \cdots \cap B_{i_k}$.

Definition 2.2. Let $(X, B, \boldsymbol{M})$ be a generalized log Calabi–Yau pair (not necessarily dlt). Then $\mathcal{DMR}(X, B, \boldsymbol{M})$ is the PL-homeomorphism type of the dual complex of a generalized dlt modification $(X', B', \boldsymbol{M})$ of $(X, B, \boldsymbol{M})$, i.e.,

$$\mathcal{DMR}(X, B, \boldsymbol{M}) \simeq_{\mathrm{PL}} \mathcal{D}(B').$$

Remark 2.3. The notation $\mathcal{DMR}$ stands for "Dual complex of a Minimal divisorial log terminal partial Resolution". The definition is independent of the choice of the dlt modification by [de Fernex et al. 2017, Proposition 11], up to PL-homeomorphism. In Berkovich geometry, the cone over $\mathcal{DMR}(X, B)$ is also called the (essential) skeleton of (X, B); see, e.g., [Mauri et al. 2022, Section 3].

Definition 2.4 (coregularity). Let c be a natural number. A generalized log canonical pair $(X, B, \boldsymbol{M})$ has *coregularity* c if some (any) generalized dlt modification $(X', B', \boldsymbol{M})$ of $(X, B, \boldsymbol{M})$ has a generalized log canonical center of dimension c, and c is minimal among the dimensions of the generalized log canonical centers of $(X', B', \boldsymbol{M})$. Equivalently, $(X, B, \boldsymbol{M})$ has coregularity c if the dual complex $\mathcal{DMR}(X, B, \boldsymbol{M})$ has dimension $\dim X - c - 1$.

We refer to [Figueroa et al. 2022; Moraga 2024] for more details about the notion of coregularity.

3. Preliminaries

In this section, we collect some preliminaries about Cartier indices, Weil indices of generalized pairs, and numerically trivial moduli parts. We prove some related technical statements that will be used in the proof of the main theorem.

3.1. ***Cartier index of a numerically trivial divisor.*** In this section, we prove some statements regarding the Cartier index of a numerically trivial divisor.

Lemma 3.1. *Let $D \sim_{\mathbb{Q}} 0$ be a numerically trivial integral $\mathbb{Q}$-Cartier divisor on a normal proper variety X. If $H^0(X, \mathcal{O}_X(D)) \neq 0$, then D is Cartier and $D \sim 0$.*

Proof. Let $s \in \Gamma(X, \mathcal{O}_X(D))$ be a nonzero section. Let U be any smooth big open subset of X. In particular, D is Cartier along U. Since $D \sim_{\mathbb{Q}} 0$ and X is integral, s is a nowhere vanishing section on U, inducing a trivialization of $\mathcal{O}_X(D)$ along U. Since $\mathcal{O}_X(D)$ is a reflexive sheaf and U is a big open subset, s is a nowhere vanishing section trivializing $\mathcal{O}_X(D)$. Thus, D is a linearly trivial Cartier divisor. □

The following lemma shows that the Cartier index of a $\mathbb{Q}$-trivial $\mathbb{Q}$-Cartier divisor is independent of the birational model. In particular, in order to control the index of a log Calabi–Yau pair, we are allowed to take dlt modifications and run minimal model programs.

Lemma 3.2. *Let X be a proper normal variety, and let D be a $\mathbb{Q}$-Cartier divisor with $D \sim_{\mathbb{Q}} 0$. Also, let X' be a proper normal variety that is birational to X, and let D' be the crepant trace of D on X'. Then, for any integer m, $mD \sim 0$ if and only if $mD' \sim 0$.*

Proof. Let X'' be a common resolution of X and X', and let $p : X'' \to X$ and $q : X'' \to X'$ be the corresponding morphisms. Let D'' be the crepant trace of D on X''. By the symmetry of the problem, it suffices to show the following: if $mD \sim 0$ holds, then so does $mD' \sim 0$. Fix such m. By the projection formula, the linear equivalence is preserved by pull-back, and we have $mD'' \sim 0$. Then, the linear equivalence is preserved by the push-forward q_*, as q is a birational morphism of normal varieties. Indeed, we may find a big open subset $U' \subset X'$ over which q is an isomorphism. Then, as $mD'' \sim 0$, we have $mD' \sim 0$ along U'. Finally, by the S_2 property, $mD' \sim 0$ holds true on the whole X'. □

We have the following immediate corollary.

Corollary 3.3. *Let $(X', B', \boldsymbol{M})$ be a proper generalized log Calabi–Yau pair that is crepant birational to the proper generalized log Calabi–Yau pair $(X, B, \boldsymbol{M})$. Then, for any integer m, $m(K_{X'} + B' + \boldsymbol{M}_{X'}) \sim 0$ if and only if $m(K_X + B + \boldsymbol{M}_X) \sim 0$.*

3.2. ***Weil index of a generalized pair.*** In this subsection, we study the Weil index of coregularity 0 log Calabi–Yau pairs under adjunction.

Definition 3.4 (Weil index of a generalized pair). Let $(X, B, \boldsymbol{M})$ be a generalized log canonical pair. Let $\Lambda \subset \mathbb{Q}$ be the smallest set of positive rational numbers such that

- the coefficients of B are contained in Λ; and
- we can write $\boldsymbol{M}_{X'} = \sum_j \lambda_j M_j$, where $X' \to X$ is a model where $\boldsymbol{M}$ descends, $\lambda_j \in \Lambda$, and each M_j is nef Cartier.

Then, the *Weil index of the generalized pair* $(X, B, \boldsymbol{M})$ is the smallest positive integer for which $\lambda\Lambda \subset \mathbb{N}$.

Example 3.5. The Weil index of the log canonical pair $(X, B = \sum_i (p_i/q_i)B_i)$, whose coefficients p_i/q_i are irreducible fractions is the least common multiple of the denominators q_i.

We show that the Weil index of a log pair of coregularity 0 is stable under adjunction. To this end, we will introduce some notation and recall a statement from [Figueroa et al. 2022].

Notation 3.6. Let Λ be a finite subset of $[0, 1]$, and r be a nonnegative rational number. We define the set

$$D_\Lambda(r) := \left\{ 1 - \frac{1}{m} + \frac{m_0 r + \sum_{i=1}^{k} m_i \lambda_i}{m} \;\middle|\; \lambda_i \in \Lambda \cup \{0, 1\}, m, m_i \in \mathbb{Z}_{>0}, \text{ and } k \in \mathbb{Z}_{\geq 0} \right\} \cap [0, 1],$$

where we mean the empty sum when $k = 0$. If $r = 0$, then we set $D_\Lambda := D_\Lambda(0)$. If $\Lambda = \mathbb{Z}[1/\lambda] \cap [0, 1]$ for some positive integer λ, then we denote the set $D_\Lambda(r)$ simply by $D_\lambda(r)$.

Lemma 3.7. *Let $\Lambda \subset [0, 1]$ be a finite set and r be a nonnegative rational number. Then, we have*

(i) $D_{D_\Lambda} = D_\Lambda \cup \{1\}$; *and*

(ii) $D_{D_\Lambda}(r) = D_\Lambda(r) \cup \{1\}$.

Proof. The first identity is proved in [McKernan and Prokhorov 2004, Lemma 4.4], and the second follows verbatim from the proof of [loc. cit., Lemma 4.4]. □

Lemma 3.8 follows from [Figueroa et al. 2022, Lemma 3.2 and Remark 3.3].

Lemma 3.8. *Let $(X, B, \boldsymbol{M})$ be a projective generalized log Calabi–Yau pair of coregularity 0. Let λ be a positive integer, $\Lambda \subset \mathbb{Z}\left[\frac{1}{\lambda}\right] \cap [0, 1]$ be a finite set, and $r \in \Lambda$. Assume that $\lambda\boldsymbol{M}$ is b-Cartier, the coefficients of B belong to Λ, and $r < 1$ is a coefficient of B. Then, there exists a generalized log Calabi–Yau pair $(\mathbb{P}^1, B_{\mathbb{P}^1}, \boldsymbol{L})$ satisfying the following conditions:*

- *The Weil index of $\boldsymbol{L}_{\mathbb{P}^1}$ divides λ.*
- *The coefficients of $B_{\mathbb{P}^1}$ belong to D_Λ.*
- *There is $q \in \mathbb{P}^1$ such that $\operatorname{coeff}_q(B_{\mathbb{P}^1}) = 1$ holds.*
- *There is $p \in \mathbb{P}^1$, with $p \neq q$, for which $\operatorname{coeff}_p(B_{\mathbb{P}^1}) \in D_\Lambda(r) \cap [0, 1)$ holds.*

In particular, if $\boldsymbol{M} = 0$, then we can choose $\boldsymbol{L} = 0$ as well.

Now, we turn to prove the statement about the Weil index under adjunction.

Lemma 3.9. *Let $(X, B, \boldsymbol{M})$ be a generalized log canonical pair of Weil index λ. Let $\lambda' := \operatorname{lcm}(\lambda, 2)$. Let S be a prime component of $B^{=1}$. Assume that S is projective. Let $S^\nu \to S$ be its normalization. Let $(S^\nu, B_{S^\nu}, \boldsymbol{N})$ be the generalized pair obtained by generalized adjunction. Assume that $(S^\nu, B_{S^\nu}, \boldsymbol{N})$ is generalized log Calabi–Yau of coregularity 0. Then, the Weil index of the generalized pair $(S^\nu, B_{S^\nu}, \boldsymbol{N})$ divides λ'.*

In particular, if $(X, B, \boldsymbol{M})$ is a generalized log Calabi–Yau pair of Weil index λ and coregularity 0, then the Weil index of $(S^\nu, B_{S^\nu}, \boldsymbol{N})$ divides λ'.

Proof. Up to passing to a dlt modification, we can assume that $(X, B, \boldsymbol{M})$ is generalized dlt and that $S = S^\nu$. Indeed, the Weil index of a generalized log canonical pair coincides with the Weil index of any crepant dlt modification, and any dlt modification of $(X, B, \boldsymbol{M})$ restricts to a dlt modification of $(S, B_S, \boldsymbol{N})$. Notice that we may assume that the dlt modification is projective over the original model, thus the projectivity of S is preserved. Also, the statement is clear for what concerns $\boldsymbol{N}$, since it is the restriction of the b-divisor $\boldsymbol{M}$. Thus, in the rest of the proof, we will focus on the boundary divisor B_S. By adjunction, the coefficients of B_S belong to $D_\lambda \cup \{1\}$; see [Birkar 2019, proof of Lemma 3.3] where p and $\mathfrak{R}$ in [loc. cit.] stand for λ and $\mathbb{Z}[1/\lambda] \cap [0, 1]$, respectively. Now, let P be a prime divisor on S. We can write

$$\operatorname{coeff}_P(B_S) = \frac{p_0}{\lambda_0},$$

where the fraction is irreducible. For the purpose of the proof, we may assume that $p_0 < \lambda_0$. We show that λ_0 divides λ'. If X is smooth at P, then the statement is clear. Thus, we have that $\operatorname{coeff}_P(\operatorname{Diff}_S(0)) = 1 - 1/m$ for some integer $m \geq 2$, see [Kollár 2013, Section 4.1]. Then, we have that

$$\operatorname{coeff}_P(B_S) \geq \operatorname{coeff}_P(\operatorname{Diff}_S(0)) = 1 - \frac{1}{m} \geq \frac{1}{2}.$$

If $p_0/\lambda_0 = \frac{1}{2}$, then $\lambda_0 = 2$ divides λ'. From now on, we assume that $p_0/\lambda_0 > \frac{1}{2}$. We apply Lemma 3.8 to the generalized pair $(S, B_S, \boldsymbol{N})$ with $r = p_0/\lambda_0$. By generalized divisorial adjunction, the coefficients of B_S form a finite subset of D_λ and $\lambda \boldsymbol{N}$ is b-Cartier. Then, we can find a generalized pair structure $(\mathbb{P}^1, B_{\mathbb{P}^1}, \boldsymbol{L})$ and two distinguished points $p, q \in \mathbb{P}^1$ satisfying the following conditions:

- The Weil index of $\boldsymbol{L}_{\mathbb{P}^1}$ divides λ.
- The divisor $B_{\mathbb{P}^1}$ has coefficient 1 at q.
- We have $\operatorname{coeff}_p(B_{\mathbb{P}^1}) \in D_{D_\lambda}(p_0/\lambda_0)$. By Lemma 3.7(ii), $\operatorname{coeff}_p(B_{\mathbb{P}^1}) \in D_\lambda(p_0/\lambda_0)$ and we can write
$$\operatorname{coeff}_p(B_{\mathbb{P}^1}) = 1 - \frac{1}{m_p} + \frac{m_0 p_0/\lambda_0 + \sum_{i=1}^k m_i p_i/\lambda}{m_p};$$
where $m_i, m_p \in \mathbb{Z}_{>0}$, $k \in \mathbb{Z}_{\geq 0}$, and p_i are nonnegative integers with $0 \leq p_i \leq \lambda$.
- For every $s \in \mathbb{P}^1 \setminus \{p, q\}$, we have $\operatorname{coeff}_s(B_{\mathbb{P}^1}) \in D_{D_\lambda}$. By Lemma 3.7(i), $\operatorname{coeff}_s(B_{\mathbb{P}^1}) \in D_\lambda$ and we can write
$$\operatorname{coeff}_s(B_{\mathbb{P}^1}) = 1 - \frac{1}{m_s} + \frac{\sum_{i=1}^k m_i p_i/\lambda}{m_s},$$
where $m_i, m_p, k \in \mathbb{Z}_{>0}$, and p_i are nonnegative integers with $0 \leq p_i \leq \lambda$.

Note that $m_0 = 1$ holds. Otherwise $\operatorname{coeff}_p(B_{\mathbb{P}^1}) > 1$ would follow, which contradicts the fact that $(\mathbb{P}^1, B_{\mathbb{P}^1}, \boldsymbol{L})$ is generalized log canonical. If $m_{s_0} \geq 2$ for some $s_0 \in \mathbb{P}^1 \setminus \{p, q\}$, then we would have
$$\deg(B_{\mathbb{P}^1}) \geq \operatorname{coeff}_q(B_{\mathbb{P}^1}) + \operatorname{coeff}_p(B_{\mathbb{P}^1}) + \operatorname{coeff}_{s_0}(\mathbb{P}^1) > 1 + \tfrac{1}{2} + \tfrac{1}{2} = 2,$$
leading to a contradiction. Thus, we have that $m_s = 1$ for every $s \in \mathbb{P}^1 \setminus \{p, q\}$. Taking the degree of $K_{\mathbb{P}^1} + B_{\mathbb{P}^1} + M_{\mathbb{P}^1}$, we obtain a relation of the form
$$1 - \frac{1}{m} + \frac{p_0/\lambda_0 + \sum_{i=1}^k m_i p_i/\lambda}{m} + \sum_{j=1}^{\ell} \frac{n_j p'_j}{\lambda} = 1,$$
where each p_i and each p'_j are positive integers. Hence, we may write
$$p_0 \lambda_0^{-1} = \left(\lambda - \sum_{i=1}^k m_i p_i - \sum_{j=1}^{\ell} m n_j p'_j\right)\lambda^{-1}.$$
We conclude that λ_0 divides λ, and so λ_0 divides λ' as well. This finishes the proof. □

3.3. *Numerically trivial moduli part.* In this subsection, we show two lemmas regarding generalized pairs with numerically trivial moduli part.

Lemma 3.10. *Let $(X, B, \boldsymbol{M})$ be a projective generalized log Calabi–Yau pair. Assume one of the following two conditions*:

- *$(X, B, \boldsymbol{M})$ has coregularity 0.*
- *X is rationally connected and of klt-type.*

Let c be the b-Cartier index of $\boldsymbol{M}$. If $\boldsymbol{M}_X$ is $\mathbb{Q}$-Cartier with $\boldsymbol{M}_X \equiv 0$, then $c\boldsymbol{M} \sim 0$ as b-divisor.

Proof. Let $\pi : X' \to X$ be a projective resolution of singularities of X where $\boldsymbol{M}$ descends. By the negativity lemma [Kollár and Mori 1998, Lemma 3.39], we have $\boldsymbol{M}_{X'} = \pi^* \boldsymbol{M}_X \equiv 0$. Let c be the Weil index of $\boldsymbol{M}_{X'}$. Since X' is smooth, then $c\boldsymbol{M}_{X'}$ is Cartier. By [Filipazzi and Svaldi 2023, Theorem 4.2] and [Hacon and McKernan 2007] (note that just [Hacon and McKernan 2007] suffices for case (ii)), X' is rationally connected, so $H^1(X', \mathcal{O}_{X'}) = 0$ and $\mathrm{Pic}(X') = \mathrm{NS}(X')$. This implies that $c\boldsymbol{M}_{X'}$ is a torsion Cartier divisor, i.e., $c\boldsymbol{M}_{X'} \sim_{\mathbb{Q}} 0$. Since a rationally connected variety has no nontrivial étale covers (see, e.g., [Debarre 2001, Corollary 4.18(b)]), the étale cover associated with $c\boldsymbol{M}_{X'} \sim_{\mathbb{Q}} 0$ must be trivial, which means that $c\boldsymbol{M}_{X'} \sim 0$. □

We have the following immediate corollary.

Corollary 3.11. *Let $(X, B, \boldsymbol{M})$ be a projective generalized log Calabi–Yau pair of coregularity* 0. *Assume that $\boldsymbol{M}_X$ is $\mathbb{Q}$-Cartier with $\boldsymbol{M}_X \equiv 0$. Let λ be a positive integer such that $\lambda\boldsymbol{M}$ is b-Cartier. Then, Theorem 1 holds for $(X, B, \boldsymbol{M})$ and λ if and only if so does for the pair (X, B) and λ.*

Proof. By Lemma 3.10, we have $\lambda\boldsymbol{M} \sim 0$ as b-divisor. Then, the claim follows, as the statement of Theorem 1 concerns linear equivalence up to multiplication by $\lambda' = \mathrm{lcm}(\lambda, 2)$. □

3.4. ***Coregularity and finite quotients.*** In this subsection, we show that the coregularity is preserved by finite maps.

Proposition 3.12. *Let (X, Δ_X) and (Y, Δ_Y) be two log canonical pairs with a generically finite surjective morphism $f : (X, \Delta_X) \to (Y, \Delta_Y)$. Assume that $K_X + \Delta_X = f^*(K_Y + \Delta_Y)$. Then, we have* $\mathrm{coreg}(X, \Delta_X) = \mathrm{coreg}(Y, \Delta_Y)$.

Proof. Throughout the proof, given a log canonical pair (W, Γ), we denote by $\mathrm{dmLCC}(W, \Gamma)$ the minimum among the dimensions of the log canonical centers of (W, Γ). Note that

$$\mathrm{dmLCC}(W, \Gamma) \leq \mathrm{coreg}(W, \Gamma), \tag{3-1}$$

and the equality holds if (W, Γ) is dlt. In particular, if (W, Γ) and (W', Γ') are crepant birational dlt pairs, then

$$\mathrm{coreg}(W, \Gamma) = \mathrm{dmLCC}(W, \Gamma) = \mathrm{dmLCC}(W', \Gamma') = \mathrm{coreg}(W', \Gamma') \tag{3-2}$$

by [de Fernex et al. 2017, Proposition 11].

Since the coregularity is preserved by crepant birational morphisms, we may replace (X, Δ_X) with the pair induced on the Stein factorization of $X \to Y$. In particular, we may assume that f is finite.

Let $(Y', \Delta_{Y'})$ be a dlt modification of (Y, Δ_Y), and let X' denote the normalization of the main component of $X \times_Y Y'$, and let $\pi : X' \to X$ be the induced morphism. Define $K_{X'} + \Delta_{X'} = \pi^*(K_X + \Delta_X)$. Then, by [Kollár and Mori 1998, Proposition 5.20], $(X', \Delta_{X'})$ is a pair. Since the coregularity is preserved by crepant birational morphisms, we may replace (X, Δ_X) and (Y, Δ_Y) with $(X', \Delta_{X'})$ and $(Y', \Delta_{Y'})$, respectively. So, in the following, we may assume that (Y, Δ_Y) is dlt.

Now, let $(\tilde{X}, \Delta_{\tilde{X}})$ denote a dlt modification of (X, Δ_X), and let $(\tilde{Y}, \Delta_{\tilde{Y}})$ denote the pair induced on the Stein factorization of $\tilde{X} \to Y$.

Since $(\tilde{X}, \Delta_{\tilde{X}}) \to (X, \Delta_X)$ and $(\tilde{Y}, \Delta_{\tilde{Y}}) \to (Y, \Delta_Y)$ are crepant birational and by [Kollár and Mori 1998, Proposition 5.20], we have the inequalities

$$\mathrm{dmLCC}(Y, \Delta_Y) = \mathrm{dmLCC}(X, \Delta_X) \leq \mathrm{dmLCC}(\tilde{X}, \Delta_{\tilde{X}}) = \mathrm{dmLCC}(\tilde{Y}, \Delta_{\tilde{Y}}).$$

By (3-1) and (3-2), since (Y, Δ) is dlt and crepant to $(\tilde{Y}, \Delta_{\tilde{Y}})$, we have

$$\mathrm{dmLCC}(Y, \Delta_Y) \geq \mathrm{dmLCC}(\tilde{Y}, \Delta_{\tilde{Y}}).$$

Putting together the two chains of inequalities, we conclude that

$$\mathrm{coreg}(Y, \Delta_Y) = \mathrm{dmLCC}(Y, \Delta_Y) = \mathrm{dmLCC}(X, \Delta_X) = \mathrm{dmLCC}(\tilde{X}, \Delta_{\tilde{X}}) = \mathrm{coreg}(X, \Delta_X). \qquad \square$$

If $f : (X, \Delta_X) \to (Y, \Delta_Y)$ is a finite Galois morphism, the following stronger version of Proposition 3.12 holds. See Definition 2.2 for the definition of $\mathcal{DMR}(X, \Delta_X)$.

Proposition 3.13. *Let $f : (X, \Delta_X) \to (Y, \Delta_Y)$ be a finite Galois morphism of log canonical pairs (X, Δ_X) and (Y, Δ_Y) with Galois group G. Assume that $K_X + \Delta_X = f^*(K_Y + \Delta_Y)$. Then, the group G acts by* PL*-homeomorphisms on the dual complex $\mathcal{D}(X, \Delta_X)$, and there exists a* PL*-homeomorphism*

$$\mathcal{DMR}(Y, \Delta_Y) \simeq_{\mathrm{PL}} \mathcal{DMR}(X, \Delta_X)/G.$$

In particular, we have $\mathrm{coreg}(X, \Delta_X) = \mathrm{coreg}(Y, \Delta_Y)$.

Proof. The same argument of [Mauri et al. 2022, Theorem F] continues to work if we replace the assumption in [loc. cit.], namely that f is not ramified in codimension 1, with the current crepant assumption $K_X + \Delta_X = f^*(K_Y + \Delta_Y)$. $\square$

4. Fractional and moduli divisors

In this section, we prove an inductive statement for the main theorem when the fractional or moduli parts are nontrivial.

Proposition 4.1. *Let $(X, B, \boldsymbol{M})$ be a projective generalized log Calabi–Yau pair of coregularity* 0 *and Weil index λ. Suppose that $B^{<1} + \boldsymbol{M}_X \not\equiv 0$ and that Theorem 1 holds in dimension less than* $\dim X$. *Then $\lambda'(K_X + B + \boldsymbol{M}_X) \sim 0$, where $\lambda' = \mathrm{lcm}(\lambda, 2)$.*

Proof. We proceed by induction on the dimension. The base case is $\dim X = 1$; by the assumption $B^{<1} + \boldsymbol{M} \not\equiv 0$, it follows that $X = \mathbb{P}^1$. Then, the claim follows immediately. Thus, in the rest of the proof, we will address the inductive step, and we will assume that Theorem 1 is known in lower dimension.

By Corollary 3.3, we may replace $(X, B, \boldsymbol{M})$ in its crepant birational class, as long as we control the Weil index of the new pair. We can apply [Filipazzi and Svaldi 2023, Corollary 4.4] and assume the following conditions:

(1) The generalized pair $(X, B, \boldsymbol{M})$ is $\mathbb{Q}$-factorial and generalized dlt.

(2) There is a fibration $X \to W$ such that every generalized log canonical center of $(X, B, \boldsymbol{M})$ dominates W.

(3) The divisor $B^{=1}$ fully supports a divisor that is big and semiample over W.

As $(X, B, \boldsymbol{M})$ has coregularity 0 and is generalized dlt, it has a generalized log canonical center of dimension 0. By (2) above, we conclude that W is a point. Hence $B^{=1}$ fully supports a big divisor. Notice that, by the properties of the birational transformations in [Filipazzi and Svaldi 2023, Theorem 4.2] and dlt modifications needed to apply [loc. cit., Corollary 4.4], all the new divisors extracted in the process appear in B with coefficient 1. Thus, the Weil index λ remains unchanged by this reduction. Similarly, the property $B^{<1} + \boldsymbol{M}_X \not\equiv 0$ is preserved.

Now, we run a $(K_X + B^{=1})$-MMP with scaling, which terminates with a Mori fiber space $g : Y \to Z$, as $B^{<1} + \boldsymbol{M}_X \not\equiv 0$. Let $(Y, B_Y, \boldsymbol{M})$ be the induced generalized log Calabi–Yau pair, which is not necessarily generalized dlt. We observe that the pair $(Y, B_Y^{=1})$ is dlt, and $B_Y^{=1} \neq 0$ dominates Z since $B_Y^{=1}$ supports a big divisor. Furthermore, the Weil index $\tilde{\lambda}$ of $(Y, B_Y, \boldsymbol{M})$ divides the Weil index λ of $(X, B, \boldsymbol{M})$, since we may have contracted some components of $B^{<1}$.

Let S be an irreducible component of $B_Y^{=1}$ dominating Z. Notice that S is normal, since $(Y, B_Y^{=1})$ is dlt. The generalized pair $(S, B_S, \boldsymbol{N})$ obtained by generalized adjunction

$$K_S + B_S + \boldsymbol{N}_S \sim_{\mathbb{Q}} (K_Y + B_Y + \boldsymbol{M}_Y)|_S \tag{4-1}$$

is log Calabi–Yau, and its Weil index divides $\lambda' = \operatorname{lcm}(\lambda, 2)$ by Lemma 3.9. By induction on the dimension, we obtain

$$\lambda'(K_S + B_S + \boldsymbol{N}_S) \sim 0.$$

The coefficients of B_S and $\boldsymbol{N}_S$ control the coefficients of $\operatorname{Diff}_S(0)$, as we explain in what follows. By [Kollár 2013, Section 3.35], at the codimension 2 points of X contained in S, X has cyclic singularities. Then, given a prime divisor P in S, an étale local neighborhood of a general point $p \in P$ is isomorphic to

$$(p \in (X, B, \boldsymbol{M})) \simeq (0 \in (\mathbb{A}^2 = (x, y), (x = 0) + c(y = 0)))/(\mathbb{Z}/m\mathbb{Z}) \times \mathbb{A}^{\dim X - 2},$$

where $S = (x = 0)$ and $S' = (y = 0)$. Since the class group of $Z := \mathbb{A}^2/(\mathbb{Z}/m\mathbb{Z})$ is generated by S', there exists an integer μ such that locally

$$\lambda'(K_X + B + \boldsymbol{M}_X) \sim \mu S'. \tag{4-2}$$

By adjunction, $S'|_S \sim (1/m)\{0\}$. Since the denominators of the coefficients of B_S and $\boldsymbol{N}_S$ divide λ', $\lambda'(K_S + B_S + \boldsymbol{N}_S)$ is a Weil divisor on the smooth germ S of p, and it is $\mathbb{Q}$-linearly trivial by (4-1). Therefore, $\lambda'(K_S + B_S + \boldsymbol{N}_S)$ is actually a trivial Cartier divisor at p, and $\lambda'(K_X + B + \boldsymbol{M}_X)|_S \sim 0$. Since we write

$$0 \sim \lambda'(K_X + B + M)|_S \sim \mu S'|_S \sim \frac{\mu}{m}\{0\},$$

we conclude that m divides μ. In particular, we have that $\mu S'$ is a Cartier divisor, as m is the order of the class group of X at $\{0\}$. By the linear equivalence (4-2), we conclude that the divisor $\lambda'(K_X+B+M)$ is Cartier in codimension 2.

Then, by [Birkar 2019, Section 2.41 and Lemma 2.42], we have the following short exact sequence

$$0\to \mathcal{O}_Y(\lambda'(K_Y+B_Y+\boldsymbol{M}_Y)-S)\to \mathcal{O}_Y(\lambda'(K_Y+B_Y+\boldsymbol{M}_Y))\to \mathcal{O}_S(\lambda'(K_S+B_S+\boldsymbol{N}_S))\to 0. \quad (4\text{-}3)$$

Since $\lambda'(K_Y+B_Y+\boldsymbol{M}_Y)-S\sim_{\mathbb{Q},g}-S$, the divisor $-S$ is g-ample, and $\dim Z<\dim Y$, we have

$$g_*\mathcal{O}_Y(\lambda'(K_Y+B+\boldsymbol{M}_Y)-S)=0.$$

Similarly, we write

$$\lambda'(K_Y+B_Y+\boldsymbol{M}_Y)-S\sim_{\mathbb{Q},g}-S\sim_{\mathbb{Q},g}K_Y+(B_Y-S+\boldsymbol{M}_Y).$$

Note that Y is klt and $B_Y-S+\boldsymbol{M}_Y=B_Y^{<1}+\boldsymbol{M}_Y+(B_Y^{=1}-S)$ is g-ample, since g is a Mori fiber space and the divisor $B_Y^{<1}+\boldsymbol{M}_Y$ is g-ample. Thus, by the relative version of Kawamata–Viehweg vanishing, we have

$$R^1g_*\mathcal{O}_Y(\lambda'(K_Y+B_Y+\boldsymbol{M}_Y)-S)=0.$$

Therefore, by pushing forward (4-3) via g, we obtain

$$g_*\mathcal{O}_Y(\lambda'(K_Y+B_Y+\boldsymbol{M}_Y))\simeq g_*\mathcal{O}_S(\lambda'(K_S+B_S+\boldsymbol{N}_S)).$$

Now, taking global sections, we have

$$H^0(Y,\mathcal{O}_Y(\lambda'(K_Y+B+\boldsymbol{M}_Y)))=H^0(S,\mathcal{O}_S(\lambda'(K_S+B_S+\boldsymbol{N}_S)))=H^0(S,\mathcal{O}_S)\neq 0. \quad (4\text{-}4)$$

By Lemma 3.1, (4-4) implies that $\lambda'(K_Y+B_Y+\boldsymbol{M}_Y)\sim 0$. Therefore, by Corollary 3.3, we conclude that $\lambda'(K_X+B_X+\boldsymbol{M}_X)\sim 0$. ☐

5. Index of log Calabi–Yau pairs and orientability of dual complexes

In this section, we prove the main theorem of this article.

5.1. ***Orientability of dual complexes.*** In this subsection, we recall the concept of pseudomanifold and dual complex, and we prove some facts about their orientability for log Calabi–Yau pairs.

Definition 5.1 (pseudomanifold). A topological space T is called a n-dimensional *pseudomanifold with boundary* if it admits a triangulation $\mathcal{T}$ satisfying the following conditions:

(1) (pure dimension) $T=|\mathcal{T}|$ is the union of all n-simplices.

(2) (nonbranching) Every $(n-1)$-simplex is a face of precisely one or two n-simplices.

(3) (strong connectedness) For every pair of n-simplices σ and σ' in T, there is a sequence of n-simplices

$$\sigma=\sigma_0,\sigma_1,\dots,\sigma_l=\sigma'$$

such that the intersection $\sigma_i\cap\sigma_{i+1}$ is a $(n-1)$-simplex for all i.

The boundary of T, denoted by ∂T, is the union of all the $(n-1)$-simplices that are faces of only one n-simplex. We say that T is a *closed pseudomanifold* if $\partial T = \varnothing$.

Lemma 5.2. *Let T be a n-dimensional pseudomanifold with boundary, then $H^n(T, \mathbb{Z}) = \mathbb{Z}$ or 0. If $\partial T \neq \varnothing$, then $H^n(T, \mathbb{Z}) = 0$.*

Proof. See [Hatcher 2002, Section 3.3, page 238]. □

Definition 5.3 (orientable pseudomanifold). A n-dimensional closed pseudomanifold T is *orientable* if $H^n(T, \mathbb{Z}) = \mathbb{Z}$.

Lemma 5.4. *The dual complex $\mathcal{D}(B)$ of a dlt log Calabi–Yau pair (X, B) is a pseudomanifold with boundary.*

Proof. See [Nicaise and Xu 2016, Theorem 4.1.4].[2] □

Proposition 5.5 (orientability of dual complexes). *Let (X, B) be a projective dlt log Calabi–Yau pair of coregularity 0 with $B = B^{=1}$. Then, the dual complex $\mathcal{D}(B)$ is an orientable closed pseudomanifold if and only if $K_X + B \sim 0$.*

Proof. Set $n + 1 = \dim X$. The short exact sequence

$$0 \to \mathcal{O}_X(-B) \to \mathcal{O}_X \to \mathcal{O}_B \to 0$$

induces the long exact sequence in cohomology

$$H^n(X, \mathcal{O}_X) \to H^n(B, \mathcal{O}_B) \to H^{n+1}(X, \mathcal{O}_X(-B)) \simeq H^0(X, \mathcal{O}_X(K_X + B)) \to H^{n+1}(X, \mathcal{O}_X).$$

Since (X, B) has coregularity 0, X is rationally connected by [Kollár and Xu 2016, Section 18], so we have

$$H^n(X, \mathcal{O}_X) = H^{n+1}(X, \mathcal{O}_X) = 0.$$

Thus, $K_X + B \sim 0$ is equivalent to $H^n(B, \mathcal{O}_B) \neq 0$; see also Lemma 3.1. Observe that $H^n(\mathcal{D}(B), \mathbb{K}) \simeq H^n(B, \mathcal{O}_B)$ by [Kollár and Xu 2016, Lemma 25] and its generalization to the dlt setting in [Kollár et al. 2018, Proposition A.7]. We conclude that the orientability of $\mathcal{D}(B)$ (i.e., $H^n(\mathcal{D}(B), \mathbb{K}) \neq 0$) implies $H^n(B, \mathcal{O}_B) \neq 0$, or equivalently $K_X + B \sim 0$. The converse follows from [Kollár and Xu 2016, Claim 32.3]. □

5.2. ***Dual complexes of log Calabi–Yau pairs in coregularity zero.*** In this subsection, we prove the statements in Section 1.2 and Section 1.3 including the main theorem of the paper.

Proof of Proposition 2. Let m be the smallest integer such that $m(K_X + B) \sim 0$. Correspondingly, there exists a degree m quasiétale cyclic cover

$$q : (\widetilde{X}, \widetilde{B}) \to (X, B), \tag{5-1}$$

[2] The authors of [Nicaise and Xu 2016] work in the context of degenerations of Calabi–Yau varieties, but their proof works verbatim for log Calabi–Yau pairs.

with $K_{\widetilde{X}}+\widetilde{B}\sim 0$, alias the index-1 cover of the pair (X, B). The cover induces a quotient map of regular Δ-complexes

$$\mathcal{D}(\widetilde{B})\to\mathcal{D}(\widetilde{B})/(\mathbb{Z}/m\mathbb{Z})\simeq\mathcal{D}(B);$$

see [Kollár and Xu 2016, Section 17]. Set $\dim X=n+1$ and suppose $m\neq 1$. By Proposition 5.5, we have

$$H^n(\mathcal{D}(\widetilde{B}),\mathbb{Z})=\mathbb{Z},\quad \text{but}\quad H^n(\mathcal{D}(B),\mathbb{Z})=0.$$

Note that the Galois group $\mathbb{Z}/m\mathbb{Z}$ of q acts on $H^n(\mathcal{D}(\widetilde{B}),\mathbb{Z})=\mathbb{Z}$ and exchanges the generators 1 and -1. Otherwise, we would have

$$\mathbb{Q}=H^n(\mathcal{D}(\widetilde{B}),\mathbb{Q})^{\mathbb{Z}/m\mathbb{Z}}=H^n(\mathcal{D}(B),\mathbb{Z})\otimes\mathbb{Q}=0,$$

where the second equality follows from [Bredon 1972, Theorem III.2.4]. Therefore, there exists a group homomorphism $\mathbb{Z}/m\mathbb{Z}\to\mathbb{Z}/2\mathbb{Z}\simeq\{1,-1\}$ that is nontrivial. In particular, m is even.

Now, let (Y, B_Y) be the dlt pair corresponding to the quotient $\mathbb{Z}/m\mathbb{Z}\to\mathbb{Z}/2\mathbb{Z}$ as in [Kollár and Xu 2016, Theorem 2(5)]. We obtain the sequence of quotient maps

$$(\widetilde{X},\widetilde{B})\xrightarrow{\frac{m}{2}:1}(Y, B_Y)\xrightarrow{2:1}(X, B)$$
$$\mathcal{D}(\widetilde{B})\xrightarrow{\frac{m}{2}:1}\mathcal{D}(B_Y)\xrightarrow{2:1}\mathcal{D}(B).$$

Taking cohomology and using [Bredon 1972, Theorem III.2.4], we obtain

$$H^n(\mathcal{D}(\widetilde{B}),\mathbb{Q})=\mathbb{Q}\longleftarrow H^n(\mathcal{D}(B_Y),\mathbb{Q})=H^n(\mathcal{D}(\widetilde{B}),\mathbb{Q})^{\mathbb{Z}/\frac{m}{2}\mathbb{Z}}=\mathbb{Q}\longleftarrow H^n(\mathcal{D}(B),\mathbb{Q})=0,$$

i.e., $K_Y+B_Y\sim 0$ by Proposition 5.5 and so $m=2$. The norm of the section trivializing K_Y+B_Y descends to a section of $2(K_X+B)$. □

Proof of Corollary 4. This is immediate from Proposition 5.5 and the proof of Proposition 2. □

Proof of Theorem 1. We proceed by induction on the dimension of X. The base case is $\dim X=1$, in which case $X=\mathbb{P}^1$ and the statement is clear. Thus, in the rest of the proof, we will address the inductive step, and we will assume that Theorem 1 holds in lower dimension.

By Corollary 3.3, we may assume that $(X, B, \boldsymbol{M})$ is generalized dlt. If $B^{<1}+\boldsymbol{M}_X\not\equiv 0$, then the statement follows from Proposition 4.1. Thus, we may assume that $B^{<1}+\boldsymbol{M}_X\equiv 0$. Then, by Corollary 3.11, we may assume that $\boldsymbol{M}=0$ as a b-divisor. But then, we have that $(X, B, \boldsymbol{M})=(X, B)$ is a dlt pair with $B=B^{=1}$, and we conclude by Proposition 2. □

Proof of Corollary 3. Let Λ be the set of coefficients of B. By assumption, we have that $\Lambda\subset\left[\frac{1}{2},1\right]$. We first show that $2B$ is integral. Then, by Theorem 1, we obtain that $2(K_X+B)\sim 0$.

Since $\Lambda\subset\left[\frac{1}{2},1\right]$, the set D_Λ (see Notation 3.6) is contained in $\{0\}\cup\left[\frac{1}{2},1\right]$. The proof is elementary: an element of D_Λ is of the form

$$1-\frac{1}{m}+\frac{\sum_{i=1}^k m_i\lambda_i}{m}=\begin{cases}\sum_{i=1}^k m_i\lambda_i\geq\frac{1}{2}\text{ or }=0 & \text{if } m=1,\\ \geq 1-\frac{1}{m}\geq\frac{1}{2} & \text{if } m>1,\end{cases}\tag{5-2}$$

where $m, m_i, k\in\mathbb{Z}_{>0}$ and $\lambda_i\in\Lambda\cup\{0,1\}$.

Let $r \in \left[\frac{1}{2}, 1\right)$ be a coefficient of B, different from 1. By Lemmas 3.7 and 3.8, there exists a log Calabi–Yau pair $(\mathbb{P}^1, B_{\mathbb{P}^1})$ satisfying the following conditions:

- The coefficients of $B_{\mathbb{P}^1}$ belong to D_Λ.
- There is $q \in \mathbb{P}^1$ such that $\operatorname{coeff}_q(B_{\mathbb{P}^1}) = 1$.
- There is $p \in \mathbb{P}^1$, with $p \neq q$, for which $\operatorname{coeff}_p(B_{\mathbb{P}^1}) \in D_\Lambda(r)$.

By (5-2), we know that $\operatorname{coeff}_x(B_{\mathbb{P}^1}) \geq \frac{1}{2}$ for any $x \in \operatorname{Supp}(B_{\mathbb{P}^1})$. Then, we deduce that

$$2 = \deg(B_{\mathbb{P}^1}) = \operatorname{coeff}_p(B_{\mathbb{P}^1}) + \operatorname{coeff}_q(B_{\mathbb{P}^1}) + \sum_{s \notin \{p,q\}} \operatorname{coeff}_s(B_{\mathbb{P}^1}) \geq \frac{3}{2} + \sum_{s \notin \{p,q\}} \operatorname{coeff}_s(B_{\mathbb{P}^1}). \tag{5-3}$$

First, assume that $\operatorname{coeff}_s(B_{\mathbb{P}^1}) > 0$ for some $s \notin \{p, q\}$. Due to (5-2) and (5-3), there must be a single such s, and it appears in $B_{\mathbb{P}^1}$ with coefficient $\frac{1}{2}$, so that

$$\operatorname{coeff}_p(B_{\mathbb{P}^1}) = 1 - \frac{1}{m_p} + \frac{m_0 r + \sum_{i=1}^k m_i \lambda_i}{m_p} = \frac{1}{2}.$$

The equality implies $m_p = k = m_1 = m_0 = 1$, $\lambda_1 = 0$ and $r = \frac{1}{2}$.

Now, assume that $B_{\mathbb{P}^1}$ is supported only on p and q. By (5-3), we write

$$\operatorname{coeff}_p(B_{\mathbb{P}^1}) = 1 - \frac{1}{m_p} + \frac{m_0 r + \sum_{i=1}^k m_i \lambda_i}{m_p} = 1.$$

So we have that $m_p = k = m_1 = m_0 = 1$, and $\lambda_1 = \frac{1}{2}$ and $r = \frac{1}{2}$. We conclude that the coefficients of B are either $\frac{1}{2}$ or 1. This means that $2B$ is an integral divisor.

Finally, if $\Lambda \subset \left(\frac{1}{2}, 1\right]$, i.e., $r \neq \frac{1}{2}$, the previous argument forces $r = 1$, i.e., B is reduced. □

5.3. *Index and residues.* In this subsection, we present an alternative proof of Proposition 2 that does not use the language of dual complexes. This alternative approach was kindly suggested to us by János Kollár.

Let (X, B) be a proper dlt Calabi–Yau pair of coregularity 0 and $\{Z_i\}_i$ be the set of its 0-dimensional log canonical centers. Let r be a positive integer such that $r(K_X + B)$ is Cartier. In particular, we have $\mathcal{O}_X(r(K_X + B)) \simeq \omega_X^{[r]}(rB) \simeq (\omega_X(B))^{\otimes r}$. Then, the Poincaré residue maps (see [Kollár 2013, 4.18])

$$\mathcal{R}^r_{X \to Z_i} : (\omega_X(B))^{\otimes r} \to \omega_{Z_i}^{\otimes r}$$

induce a $\mathbb{K}$-linear residue map in cohomology

$$\mathcal{R}^r := (\dots, \mathcal{R}^r_{X \to Z_i,*}, \dots) : H^0(X, \mathcal{O}_X(r(K_X + B))) \to \bigoplus_i H^0(Z_i, \omega_{Z_i}^{\otimes r}) \simeq \bigoplus_i \mathbb{K}. \tag{5-4}$$

The maps are defined up to sign for r odd, and they are unique for r even; see again [Kollár 2013, 4.18].

Lemma 5.6. *Let (X, B) be a proper dlt pair of coregularity 0, and G be a subgroup of $\operatorname{Aut}(X, B)$, i.e., a subgroup of the group of automorphisms $g : X \to X$ such that $g^*(B) = B$. For any even integer r, the map $\mathcal{R}^r$ is G-equivariant with respect to the natural action induced on its domain and codomain.*

Proof. Locally around a 0-dimensional log canonical center Z_i, a section s of $H^0(X, \mathcal{O}_X(r(K_X+B)))$ is given by

$$a\prod_j \left(\frac{dx_j}{x_j}\right)^{\otimes r},$$

where $a \in \mathcal{O}_{X,Z_i}$ and $\prod_j x_j$ is a local equation for B at Z_i. For any $g \in G$, the function $y_j := x_i \circ g$ is a local equation for an irreducible component of B passing through $g^{-1}(Z_i)$. We obtain

$$\begin{aligned} g^*(\mathcal{R}^r_{X\to Z_i,*}(s)) &= a(Z_i) = a\circ g(g^{-1}(Z_i)) \\ &= \mathcal{R}^r_{X\to g^{-1}(Z_i),*}\left(a\circ g\prod_j\left(\frac{dy_j}{y_j}\right)^{\otimes r}\right) = \mathcal{R}^r_{X\to g^{-1}(Z_i),*}(g^*s), \end{aligned} \tag{5-5}$$

i.e., $\mathcal{R}^r$ is G-equivariant.[3] □

Lemma 5.7. *For any* 0*-dimensional log canonical centers* Z_1 *and* Z_2 *of the proper dlt Calabi–Yau pair* (X, B), *we have* $\mathcal{R}^r_{X\to Z_1,*} = \pm\mathcal{R}^r_{X\to Z_2,*}$. *In particular, if r is even, we have* $\mathcal{R}^r_{X\to Z_1,*} = \mathcal{R}^r_{X\to Z_2,*}$.

Proof. Without loss of generality, we can assume that Z_1 and Z_2 belong to the same 1-dimensional log canonical center $W \simeq \mathbb{P}^1$. Indeed, any two minimal log canonical centers of a log Calabi–Yau pair are $\mathbb{P}^1$-linked; see [Kollár 2013, Theorem 4.40]. In our setup, this means that there exists a sequence of 0-dimensional log canonical centers $Z'_1, \dots, Z'_s$ with $Z'_1 = Z_1$ and $Z'_s = Z_2$ such that Z'_k and Z'_{k+1} are contained in the same 1-dimensional log canonical center, which is necessarily isomorphic to $\mathbb{P}^1$. Now, the Poincaré residue map $\mathcal{R}^r_{X\to Z_i}$ factors as

$$(\omega_X(B))^{\otimes r} \xrightarrow{\mathcal{R}^r_{X\to W}} (\omega_W(Z_1+Z_2))^{\otimes r} \xrightarrow{\mathcal{R}^r_{W\to Z_i}} \omega_{Z_i}^{\otimes r},$$

where $i = 1, 2$. A generator of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r(K_{\mathbb{P}^1}+Z_1+Z_2)))$ is $(dx/x)^{\otimes r}$, which has residue 1 at Z_1 and $(-1)^r$ at Z_2. □

Alternative proof of Proposition 2. Let $q : (\widetilde{X}, \widetilde{B}) \to (X, B)$ be the index-1 cover defined in (5-1). Up to a dlt modification, we can suppose that (X, B) is dlt, and so $(\widetilde{X}, \widetilde{B})$ is too. Let $\{Z_i\}_i$ be the 0-dimensional log canonical centers of $(\widetilde{X}, \widetilde{B})$. Since the Galois group $\mathbb{Z}/m\mathbb{Z}$ of the cover q is a subgroup of $\mathrm{Aut}(\widetilde{X}, \widetilde{B})$, the map

$$\mathcal{R}^2 = (\dots, \mathcal{R}^2_{\widetilde{X}\to Z_i,*}, \dots) : H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(2(K_{\widetilde{X}}+\widetilde{B}))) \to \bigoplus_i H^0(Z_i, \omega_{Z_i}^{\otimes r}) \simeq \bigoplus_i \mathbb{K}$$

is $\mathbb{Z}/m\mathbb{Z}$-equivariant by Lemma 5.6. The image of $\mathcal{R}^2$ is the line spanned by the vector $(1, \dots, 1) \in \bigoplus_i \mathbb{K}$ by Lemma 5.7, and it is fixed by $\mathbb{Z}/m\mathbb{Z}$, since the Galois group acts on $\bigoplus_i \mathbb{K}$ by permuting the copies of $\mathbb{K}$.

Since $(\widetilde{X}, \widetilde{B})$ is log Calabi–Yau, $\mathcal{R}^2$ is injective. Therefore, we obtain that $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(r(K_{\widetilde{X}}+\widetilde{B})))$ is invariant under the action of $\mathbb{Z}/m\mathbb{Z}$. Then, a trivializing section of $2(K_{\widetilde{X}}+\widetilde{B})$ descends to a section of $2(K_X+B)$. In particular, this implies that $m = 2$. □

[3] Note that if r is odd, the equality (5-5) holds in general only up to sign.

Remark 5.8. The two proposed proofs of Proposition 2 are essentially equivalent. One should regard the residue map (5-4) as the restriction of a global orientation of the dual complex $\mathcal{D}(\widetilde{B})$ to a local orientation at a point of a maximal cell σ in $\mathcal{D}(\widetilde{B})$

$$H^n(\mathcal{D}(\widetilde{B}), \mathbb{Z}) \to H^n(\mathcal{D}(\widetilde{B}), \mathcal{D}(\widetilde{B}) \setminus \sigma, \mathbb{Z}) \simeq H^{n-1}(\partial\sigma, \mathbb{Z}) \simeq \mathbb{Z};$$

see [Hatcher 2002, Section 3.3, page 234].

6. Index of semi-log canonical Calabi–Yau pairs

In this section, we prove a version of our main theorem for semi-log canonical pairs (Corollary 9). This is a key ingredient for the proof of Theorem 5; see Section 7. For the concept of semi-dlt and semi-log canonical pair, we refer the readers to [Kollár 2013, Section 5]. We will use the language of admissible and preadmissible sections.

Definition 6.1. Let (X, B) be a projective semi-dlt pair, possibly disconnected, of dimension n. Assume that $m(K_X+B)$ is Cartier. Let (X^ν, B^ν) be the normalization of (X, B). Let $D^\nu \subset X^\nu$ be the normalization of the double locus $D \subset X$. We write (D^ν, B_{D^ν}) for the dlt pair obtained by the adjunction of (X^ν, B^ν) to D^ν. We write X_i^ν for the connected components of X^ν and (X_i^ν, B_i^ν) for the pairs obtained by restriction of (X^ν, B^ν) to X_i^ν. We define *preadmissible* and *admissible* sections in $H^0(X, \mathcal{O}_X(m(K_X+B)))$ inductively using the following rules:

(1) A section
$$s \in H^0(X, \mathcal{O}_X(m(K_X + B))$$
is *preadmissible* if its restriction $s|_{D^\nu} \in H^0(D^\nu, \mathcal{O}_{D^\nu}(m(K_{D^\nu} + B_{D^\nu})))$ is admissible. The set of preadmissible sections is denoted by $PA(X, m(K_X + B))$.

(2) A section
$$s \in H^0(X, \mathcal{O}_X(m(K_X + B))$$
is *admissible* if it is preadmissible and for every crepant birational map $g : (X_i^\nu, B_i^\nu) \dashrightarrow (X_j^\nu, B_j^\nu)$, we have that $g^*(s|_{X_i^\nu}) = s|_{X_j^\nu}$. The set of admissible sections is denoted by $A(X, m(K_X + B))$.

In what follows, we recall several lemmas regarding preadmissible and admissible sections. The following lemma is clear by definition; see [Gongyo 2013, Remark 5.2].

Lemma 6.2. *Let (X, B) be a projective semi-dlt pair. Let $\pi : (X^\nu, B^\nu) \to (X, B)$ be its normalization. Assume $m(K_X + B) \sim 0$. A section $s \in H^0(X, \mathcal{O}_X(m(K_X + B)))$ is preadmissible (resp. admissible) if and only if $\pi^*s \in H^0(X^\nu, \mathcal{O}_{X^\nu}(m(K_{X^\nu} + B^\nu)))$ is preadmissible (resp. admissible).*

Preadmissible sections of normalizations are designed to descend to nonnormal varieties, as explained in the following statement; see, e.g., [Fujino 2000, Lemma 4.2].

Lemma 6.3. *Let (X, B) be a projective semi-log canonical pair for which $m(K_X + B)$ is Weil. Let $(X^\nu, B^\nu) \to (X, B)$ be its normalization and (Y, B_Y) a dlt modification of (X^ν, B^ν). Assume $m(K_Y + B_Y)$ is Cartier. Then, $s \in PA(Y, m(K_Y + B_Y))$ descends to $H^0(X, \mathcal{O}_X(m(K_X + B)))$.*

In the context of connected dlt pairs, which are not klt, the set of admissible sections is the same as the set of preadmissible sections; see, e.g., [Xu 2020, Proposition 3.2.15].

Lemma 6.4. *Consider (X, B) a connected projective dlt pair. Assume that (X, B) is not klt. Assume that $m(K_X + B) \sim 0$ where m is even. Then, we have that*

$$PA(X, m(K_X + B)) = A(X, m(K_X + B)).$$

On the other hand, in the dlt setting, we can lift admissible sections from the boundary to preadmissible sections on the whole pair; see, e.g., [Xu 2020, Lemma 3.2.14].

Lemma 6.5. *Assume that (X, B) is a possibly disconnected, projective dlt pair. Assume that $m(K_X+B)\sim 0$ for some even integer m. Assume that*

$$s \in A(B^{=1}, m(K_X + B)|_{B^{=1}}).$$

Then, there exists

$$t \in PA(X, m(K_X + B))$$

such that $t|_{B^{=1}} = s$.

Finally, the following lemma allows us to produce admissible sections on possibly disconnected dlt pairs, once we know the existence of admissible sections on connected dlt pairs. For the proof, see [Xu 2020, Proposition 3.2.8].

Lemma 6.6. *Let (X, B) be a, possibly disconnected, projective dlt log Calabi–Yau pair. Assume that for every component (X_i, B_i) of (X, B), we have a nontrivial section in $A(X_i, m(K_{X_i} + B_i))$. Then, we have that $A(X, m(K_X + B))$ contains a nontrivial section.*

Now, we turn to prove the main theorem of this section.

Theorem 6.7. *Let (X, B) be a, possibly disconnected, projective dlt log Calabi–Yau pair whose connected components all have coregularity 0 and Weil index λ. Then, we have that $A(X, \lambda'(K_X + B))$ admits a nontrivial section, where $\lambda' = \mathrm{lcm}(\lambda, 2)$.*

Proof. We proceed by induction on the dimension. The statement is trivial if $\dim X = 0$. Assume that the statement holds in dimension $n - 1$. By Theorem 1, we have that $\lambda'(K_X + B) \sim 0$. Denote by B_i the (normal) irreducible components of $B^{=1}$. The pairs $(B_i, \mathrm{Diff}_{B_i}(B - B_i))$, obtained by adjunction, are projective dlt log Calabi–Yau of dimension $n - 1$, coregularity 0 and Weil index that divides λ'; see [Kollár 2013, Theorem 4.40] and Lemma 3.9. Their disjoint union

$$\Delta := \bigsqcup_i B_i \xrightarrow{\nu} B^{=1} = \bigcup_i B_i$$

is the normalization of $B^{=1}$. Consider the pair

$$(\Delta, B_\Delta) := \bigsqcup_i (B_i, \mathrm{Diff}_{B_i}(B - B_i)).$$

By induction on the dimension, there exist nontrivial sections

$$0 \neq s_{B_i} \in A(B_i, \lambda'(K_{B_i} + \mathrm{Diff}_{B_i}(B - B_i))).$$

Then, by Lemma 6.6, there exists a section

$$0 \neq s_\Delta \in A(\Delta, \lambda'(K_\Delta + B_\Delta)),$$

and by Lemma 6.3 it descends to

$$s_{B^{=1}} \in H^0(B^{=1}, \lambda'(K_X + B)|_{B^{=1}}).$$

Since $\nu^* s_{B^{=1}} = s_\Delta$, and by Lemma 6.2, we actually have

$$s_{B^{=1}} \in A(B^{=1}, \lambda'(K_X + B)|_{B^{=1}}).$$

Finally, by Lemmas 6.5 and 6.4, there exists an admissible section

$$0 \neq s_X \in PA(X, \lambda'(K_X + B_X)) = A(X, \lambda'(K_X + B)).$$

This finishes the induction step. ☐

Remark 6.8. The coregularity of a connected semi-log canonical log Calabi–Yau pair is the coregularity of a (or any) connected component of its normalization with the induced boundary. Note that the independence of the connected component follows easily from [Kollár 2013, Theorem 4.40 and Proposition 5.12].

Proof of Theorem 8. By Theorem 6.7, we have a nontrivial admissible section $s \in A(X, \lambda'(K_X + B))$, where $\lambda' = \mathrm{lcm}(\lambda, 2)$. By definition of admissible section, it follows that $g^* s = s$ for every $g \in \mathrm{Bir}(X, B)$. ☐

Proof of Corollary 9. Let $X^\nu \to X$ be the normalization of X and $Y \to X^\nu$ be a dlt modification. We write $\pi : Y \to X$ and $K_Y + B_Y = \pi^*(K_X + B)$. Hence, the pair (Y, B_Y) is dlt log Calabi–Yau of coregularity 0 and its Weil index is λ. By Theorem 6.7, we conclude that $A(Y, \lambda'(K_Y + B_Y))$ admits a nontrivial section. Then, the statement follows from Lemma 6.3. ☐

Proof of Corollary 10. Let $(X, B; x)$ be a germ as in the statement. Since $(X, B; x)$ has coregularity 0, the point x is a log canonical center of $(X, B; x)$. If $(X, B; x)$ is dlt at x, then the claim is immediate, as (X, B) has simple normal crossings at x and so $\lambda(K_X + B) \sim 0$. Thus, in the rest of the proof, we may assume that $(X, B; x)$ is not dlt at x.

Now, let $g : (Y, B_Y := g_*^{-1}(B) + E) \to (X, B)$ be a $\mathbb{Q}$-factorial dlt modification of $(X, B; x)$ such that g is an isomorphism over the simple normal crossing locus of (X, B), where E is the sum of the g-exceptional divisors; see [Kollár 2013, Corollary 1.36]. Since $(X, B; x)$ is not dlt, it follows that E is nonempty. Furthermore, since the pair $(X, B; x)$ is dlt away from x and g is an isomorphism at a general point of the log canonical centers of (X, B) contained in $X \setminus \{x\}$, it follows that $g(E) = x$, and we can also assume that $E = \mathrm{Supp}(g^{-1}(x))$ by [Nakamura 2021, Theorem 2.5]. In particular, we have

$$(K_Y + B_Y)|_E \sim_{\mathbb{Q}} g^*(K_X + B)|_E \sim_{\mathbb{Q}} 0.$$

We denote by (E, B_E) the pair induced on E by (Y, B_Y) via adjunction. Then, (E, B_E) is a projective semi-dlt log Calabi–Yau pair whose Weil index divides λ' by Lemma 3.9. Hence, by Corollary 9, we have $\lambda'(K_Y + B_Y)|_E \sim 0$.

Now, consider the short exact sequence

$$0 \to \mathcal{O}_Y(\lambda'(K_Y + B_Y) - E) \to \mathcal{O}_Y(\lambda'(K_Y + B_Y)) \to \mathcal{O}_E(\lambda'(K_Y + B_Y)) \to 0. \tag{6-1}$$

Note that the pair $(Y, g_*^{-1}B)$ is dlt and $g|_Z : Z \to f(Z)$ is birational for every log canonical center of $(Y, g_*^{-1}B)$. This can be showed as follows. Assume that Z is a log canonical center of $(Y, g_*^{-1}B)$ such that $g|_Z : Z \to f(Z)$ is not birational, and denote by d its codimension in Y. Since g is an isomorphism at a general point of the log canonical centers of (X, B) contained in $X \setminus \{x\}$, we must have $g(Z) = \{x\}$. Since $(Y, g_*^{-1}B)$ is dlt and Z has codimension d, Z is a connected component of the intersection of d prime divisors in $(g_*^{-1}B)^{=1}$. Since it is also contained in some irreducible components of E, we get the sought contradiction, as the pair $(Y, g_*^{-1}(B) + E)$ is dlt.

Since we have

$$-E \sim_{\mathbb{Q}} \lambda'(K_Y + B_Y) - E \sim_{\mathbb{Q}} K_Y + g_*^{-1}B,$$

the vanishing result [Kollár 2013, Corollary 10.38] gives $R^1 g_* \mathcal{O}_Y(\lambda'(K_Y + B_Y) - E) = 0$. Therefore, by pushing forward (6-1) via g, we obtain

$$\mathcal{O}_X(\lambda'(K_X + B)) \simeq g_* \mathcal{O}_Y(\lambda'(K_Y + B_Y)) \to \mathcal{O}_{\{x\}} \to 0.$$

This gives a section of $\mathcal{O}_Y(\lambda'(K_X + B))$ that does not vanish at x, i.e., $\mathcal{O}_X(\lambda'(K_X + B)) \sim 0$. □

7. Applications to mirror symmetry

We now study the index of Calabi–Yau varieties that appear in mirror symmetry. First, we introduce the notion of a minimal family of Calabi–Yau varieties of coregularity 0 over a point. This notion is modeled on the *large complex limit* or *maximal degeneration* condition; see also Remark 7.2.

Definition 7.1. Let (C, c_0) be a pointed smooth curve. Let $\pi : \mathcal{X} \to C$ be a flat projective morphism with $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_C$, and $\mathcal{B}$ an effective divisor such that every irreducible component of $\operatorname{Supp}(\mathcal{B})$ dominates C. Then $\pi : (\mathcal{X}, \mathcal{B}) \to C$ is a *minimal family of log Calabi–Yau pairs* if $K_{\mathcal{X}} + \mathcal{B} + \mathcal{X}_{c_0,\mathrm{red}} \sim_{\mathbb{Q},C} 0$, where $\mathcal{X}_{c_0,\mathrm{red}}$ is the reduced fiber over c_0. If $\mathcal{B} = 0$, we say it is a *minimal family of Calabi–Yau varieties*.

Further, we say that the family has *coregularity* 0 *over* $c_0 \in C$ if

- the pair $(\mathcal{X}, \mathcal{B} + \mathcal{X}_{c_0,\mathrm{red}})$ is log canonical; and
- the restriction $(\mathcal{X}|_V, \mathcal{B}|_V + \mathcal{X}_{c_0,\mathrm{red}})$ has coregularity 0 for any neighborhood $V \subseteq C$ of c_0.

Remark 7.2 (maximal degeneration). Let $\pi : \mathcal{X} \to C$ be a minimal family of Calabi–Yau varieties of coregularity over c_0. Let K denote the fraction field of the formal ring $\widehat{\mathcal{O}}_{C,c_0}$. If the K-variety $X := \mathcal{X} \times_C \operatorname{Spec} K$ is smooth with trivial canonical bundle, Definition 7.1 recovers the notion of maximally unipotent degeneration in [Nicaise and Xu 2016, 4.2.4(4)]; see also [Kollár et al. 2018, Sections 6.1

and 6.2]. Our weaker definition allows mild singularities on the general fiber, and we do not prescribe the index of the general fiber.

Definition 7.3 (locally stable family). Let C be a smooth curve. A family $\pi : (\mathcal{X}, \mathcal{B}) \to C$ is *locally stable* if the pair $(\mathcal{X}, \mathcal{B} + \mathcal{X}_c)$ is semi-log canonical for any closed point $c \in C$.

In the following lemma we show that, up to base change, we can suppose that the fibers of a minimal family of log Calabi–Yau pairs are all reduced and semi-log canonical.

Lemma 7.4. *Let $\pi : (\mathcal{X}, \mathcal{B}) \to C$ be a minimal family of log Calabi–Yau pairs of coregularity 0 over $c_0 \in C$. There exists a finite morphism $(C', c_0') \to (C, c_0)$ and a birational transformation $\mathcal{Y} \to \mathcal{X} \times_C C'$ such that $\pi' : (\mathcal{Y}, \mathcal{D}) \to C'$ is a minimal family of log Calabi–Yau pairs of coregularity 0 over $c_0' \in C'$ with the additional property that $\pi' : (\mathcal{Y}, \mathcal{D}) \to C'$ is locally stable at c_0'.*

Let U be the maximal open subset over which π is locally stable. Then, for every $c \in U$, there exists a $c' \in C'$ such that $(\mathcal{Y}_{c'}, \mathcal{D}_{c'}) \simeq (\mathcal{X}_c, \mathcal{B}_c)$.

Remark 7.5. The divisor $\mathcal{D}$ in the statement of Lemma 7.4 is defined as follows. Over $C' \setminus \{c_0'\}$, $\mathcal{D} \times_{C'} C' \setminus \{c_0'\}$ is the boundary obtained by crepant pull-back to $\mathcal{Y} \times_{C'} C' \setminus \{c_0'\}$ of the pair $(\mathcal{X}, \mathcal{B}) \times_C C \setminus \{c_0\}$. Then, $\mathcal{D}$ is the divisor obtained by taking the closure of $\mathcal{D} \times_{C'} C' \setminus \{c_0'\}$ in $\mathcal{Y}$.

Proof. Let U be the maximal open subset of C where π is locally stable. Without loss of generality, we may assume that $U \cup \{c_0\} = C$. Then, by weak semistable reduction [Abramovich and Karu 2000], there is a finite morphism $(C', c_0') \to (C, c_0)$ such that the fiber $\mathcal{Y}_{c_0'}$ over c_0' of $\mathcal{Y}$ is reduced. Here, $\mathcal{Y}$ denotes the main component of the normalization of $\mathcal{X} \times_C C'$. By assumption, we have $K_{\mathcal{X}} + \mathcal{B} + \mathcal{X}_{c_0,\mathrm{red}} \sim_{\mathbb{Q},C} 0$, and so by base change and the ramification formula, we obtain $K_{\mathcal{Y}} + \mathcal{D} + \mathcal{Y}_{c_0'} \sim_{\mathbb{Q},C'} 0$. Furthermore, by construction, the family $\pi' : (\mathcal{Y}, \mathcal{D}) \to C'$ is locally stable.

Now, we show that $(\mathcal{Y}, \mathcal{D} + \mathcal{Y}_{c_0'})$ has coregularity 0 over c_0'. Up to a dlt modification, we can suppose that $(\mathcal{X}, \mathcal{D} + \mathcal{X}_{c_0,\mathrm{red}})$ is dlt. In a neighborhood of a log canonical center of dimension 0, the pair $(\mathcal{Y}, \mathcal{D}' + \mathcal{Y}_{c_0'})$ is not necessarily dlt, but it is *qdlt* (see [de Fernex et al. 2017, Section 5]) and still of coregularity 0 over c_0' as we wanted; see also [Nicaise and Xu 2016, Lemma 4.1.9]. □

Theorem 7.6. *Let $\pi : (\mathcal{X}, \mathcal{B}) \to C$ be a minimal family of Calabi–Yau varieties of coregularity 0 over $c_0 \in C$. Assume that $\lambda\mathcal{B}$ is integral, where λ is a positive integer. Let U be the maximal open subset over which the family is locally stable. Then, for every $c \in U$, we have that $\lambda'(K_{\mathcal{X}_c} + \mathcal{B}_c) \sim 0$, where $\lambda' = \mathrm{lcm}(\lambda, 2)$.*

Proof. By shrinking C, we may assume that the open subset in the statement satisfies $C \setminus \{c_0\} \subset U$. Then, by Lemma 7.4, we can suppose that π is locally stable, i.e., $U = C$. In particular, $K_{\mathcal{X}} + \mathcal{B}$ is $\mathbb{Q}$-Cartier. By Corollary 9, we have $\lambda'(K_{\mathcal{X}_{c_0}} + \mathcal{B}_{c_0}) \sim 0$. In particular, $\lambda'(K_{\mathcal{X}_{c_0}} + \mathcal{B}_{c_0})$ is Cartier. By [Kollár 2023, Proposition 2.79], we have $\mathcal{O}_{\mathcal{X}}(\lambda'(K_{\mathcal{X}/C} + \mathcal{B}))|_{\mathcal{X}_{c_0}} \simeq \mathcal{O}_{\mathcal{X}_{c_0}}(\lambda'(K_{\mathcal{X}_{c_0}} + \mathcal{B}_{c_0}))$. Since $\mathcal{X}_{c_0}$ is a Cartier divisor in $\mathcal{X}$, by Nakayama's lemma, it follows that $\lambda'(K_{\mathcal{X}} + \mathcal{B})$ is Cartier in a neighborhood of $\mathcal{X}_{c_0}$. By the Calabi–Yau condition in Definition 7.1, $\mathcal{O}_{\mathcal{X}}(\pm\lambda'(K_{\mathcal{X}} + \mathcal{B}))$ is a relatively semiample line bundle over a neighborhood of c_0. Then, by [Kollár 2023, Corollary 2.65], $h^0(\mathcal{X}_c, \mathcal{O}_{\mathcal{X}_c}(\lambda'(K_{\mathcal{X}_c} + \mathcal{B}_c))) = 1$ for c

in a neighborhood of c_0. Henceforth, by Lemma 3.1, we conclude that $\lambda'(K_{\mathcal{X}_c}+\mathcal{B}_c)\sim 0$ for c in a neighborhood of c_0.

Now, since π is a locally stable family and $\lambda\mathcal{B}$ is integral, $\omega^{[\lambda']}_{\mathcal{X}/C}(\lambda'\mathcal{B})$ is a flat family of divisorial sheaves, see [Kollár 2023, Proposition 2.79.3 and Section 3.3]. Then, by [Kollár 2023, Theorem 3.32], $h^0(\mathcal{X}_c,\mathcal{O}_{\mathcal{X}_c}(\lambda'(K_{\mathcal{X}_c}+\mathcal{B}_c)))\geq 1$ for all $c\in C$. Since $K_{\mathcal{X}_c}+\mathcal{B}_c$ is torsion for all $c\in C$, we have that $h^0(\mathcal{X}_c,\mathcal{O}_{\mathcal{X}_c}(\lambda'(K_{\mathcal{X}_c}+\mathcal{B}_c)))=1$ for all $c\in C$. By Lemma 3.1, this concludes the proof. □

Corollary 7.7. *Let $\pi:\mathcal{X}\to C$ be a locally stable minimal family of Calabi–Yau varieties of coregularity 0 over $c_0\in C$. Suppose that the general fiber $\mathcal{X}_c$ is klt. If $K_{\mathcal{X}_c}\not\sim 0$, then $\mathcal{DMR}(\mathcal{X},\mathcal{X}_{c_0})$ is not orientable.*

Proof. Up to a small $\mathbb{Q}$-factorial modification of $\mathcal{X}$, we can suppose that $\mathcal{X}$ has only klt $\mathbb{Q}$-factorial singularities. Take a dlt modification $(\mathcal{Y},\mathcal{Y}_{c_0,\mathrm{red}})$ of $(\mathcal{X},\mathcal{X}_{c_0})$ as constructed in [Kollár 2013, Section 1.35]. In particular, observe that $\mathcal{Y}\to\mathcal{X}$ is an isomorphism over the open set $\mathcal{X}\setminus\mathcal{X}_{c_0}$ as $\mathcal{X}$ has only klt $\mathbb{Q}$-factorial singularities.

By definition of dual complex of a log Calabi–Yau pair (compare Definition 2.2), we write

$$\mathcal{DMR}(\mathcal{X},\mathcal{X}_{c_0})\simeq_{\mathrm{PL}}\mathcal{D}(\mathcal{Y}_{c_0,\mathrm{red}}).$$

Since $\pi:\mathcal{X}\to C$ has coregularity 0 over c_0, there exists an isomorphism

$$H^n(\mathcal{D}(\mathcal{Y}_{c_0,\mathrm{red}}),\mathbb{K})\simeq H^n(\mathcal{Y}_{c_0,\mathrm{red}},\mathcal{O}_{\mathcal{Y}_{c_0,\mathrm{red}}}),$$

with $n=\dim\mathcal{Y}_{c_0,\mathrm{red}}$, see [Kollár and Xu 2016, Lemma 25] and [Kollár et al. 2018, Proposition A.7]. Recall that the fibers of π have du Bois singularities, and $\mathcal{Y}\to\mathcal{X}$ is an isomorphism over the open set $\mathcal{X}\setminus\mathcal{X}_{c_0}$. Then, [Schwede 2007, Theorem 4.3] gives the following isomorphism:

$$H^n(\mathcal{Y}_{c_0,\mathrm{red}},\mathcal{O}_{\mathcal{Y}_{c_0,\mathrm{red}}})\simeq H^n(\mathcal{X}_{c_0},\mathcal{O}_{\mathcal{X}_{c_0}}).$$

By [Kollár 2023, Theorem 2.62], we also have

$$H^n(\mathcal{X}_{c_0},\mathcal{O}_{\mathcal{X}_{c_0}})\simeq H^n(\mathcal{X}_c,\mathcal{O}_{\mathcal{X}_c}).$$

Now, since the general fiber $\mathcal{X}_c$ is Cohen–Macaulay, by duality we obtain

$$H^n(\mathcal{X}_c,\mathcal{O}_{\mathcal{X}_c})\simeq H^0(\mathcal{X}_c,\mathcal{O}_{\mathcal{X}_c}(K_{\mathcal{X}_c})).$$

Finally, we conclude

$$H^n(\mathcal{DMR}(\mathcal{X},\mathcal{X}_{c_0}))\simeq H^0(\mathcal{X}_c,\mathcal{O}_{\mathcal{X}_c}(K_{\mathcal{X}_c})).$$

If $K_{\mathcal{X}_c}\not\sim 0$, then $H^n(\mathcal{DMR}(\mathcal{X},\mathcal{X}_{c_0}),\mathbb{K})=0$, i.e., $\mathcal{DMR}(\mathcal{X},\mathcal{X}_{c_0})$ is not orientable. □

Proof of Corollary 6. Corollary 6 is a special case of Corollary 7.7. □

See Example 9.4 for an instance of Corollary 6.

Proof of Theorem 7. Up to shrinking C, we can assume that π is locally stable. By the same proof of Theorem 5, it suffices to show that $K_{\mathcal{X}_{c_0}}\sim 0$, or $K_{\mathcal{X}}+\mathcal{X}_{c_0}\sim 0$ in a neighborhood of $\mathcal{X}_{c_0}$.

First, we show that the restriction of $K_{\mathcal{X}}+\mathcal{X}_{c_0}$ to each component Δ_i of the central fiber is linearly trivial. To this end, note that the $\mathbb{Q}$-divisor

$$K_{\Delta_i}+\mathrm{Diff}_{\Delta_i}(\mathcal{X}_{c_0}-\Delta_i)\sim_{\mathbb{Q}}(K_{\mathcal{X}}+\mathcal{X}_{c_0})|_{\Delta_i}\sim_{\mathbb{Q}}0$$

is actually a linearly trivial Cartier divisor, since the inequality

$$0\sim_{\mathbb{Q}}K_{\Delta_i}+\mathrm{Diff}_{\Delta_i}(\mathcal{X}_{c_0}-\Delta_i)\geq K_{\Delta_i}+\sum_{j\neq i}\Delta_j|_{\Delta_i}\sim 0$$

implies that $K_{\Delta_i}+\mathrm{Diff}_{\Delta_i}(\mathcal{X}_{c_0}-\Delta_i)=K_{\Delta_i}+\sum_{j\neq i}\Delta_j$, and $K_{\Delta_i}+\sum_{j\neq i}\Delta_j\sim 0$ as for any toric pair.

Let m be the smallest positive integer for which $m(K_{\mathcal{X}}+\mathcal{X}_{c_0})\sim 0$ holds, up to eventually shrinking C. Take the degree m quasiétale cover $q:\mathcal{X}'\to\mathcal{X}$ associated to $m(K_{\mathcal{X}}+\mathcal{X}_{c_0})\sim 0$. By [Kollár 2013, (4.2.7)], there exists an open subset $\Delta_i^\circ\subseteq\Delta_i$, with $\mathrm{codim}_{\Delta_i}(\Delta_i\setminus\Delta_i^\circ)\geq 2$, such that for any $k>0$

$$\omega_{\mathcal{X}}^{[k]}(k\mathcal{X}_{c_0})|_{\Delta_i^\circ}\simeq\omega_{\mathcal{X}_{c_0}}^{[k]}(k\,\mathrm{Diff}_{\Delta_i}(\mathcal{X}_{c_0}-\Delta_i))|_{\Delta_i^\circ}\sim 0.$$

This implies that $q^{-1}(\Delta_i^\circ)$ consists of m disjoint copies of Δ_i°. By the smoothness of Δ_i and purity of the branch locus, or by [Kollár 2023, Lemma 2.93], the same holds for $q^{-1}(\Delta_i)$. Therefore, the induced morphism $\mathcal{X}'_{c_0}\to\mathcal{X}_{c_0}$ is a finite étale cover of degree m. Since every irreducible component Δ_i of $\mathcal{X}_{c_0}$ is simply connected, it follows that

$$\pi_1^{\text{ét}}(\mathcal{X}_{c_0})\simeq\pi_1(\mathcal{D}(\mathcal{X}_{c_0}))\simeq\{1\}$$

by [Kollár and Xu 2016, Lemma 26] and by the hypothesis on the simple connectedness of $\mathcal{D}(\mathcal{X}_{c_0})$. This implies that $m=1$ and finishes the proof. □

Remark 7.8. In Theorem 7, we require that the special fiber $\mathcal{X}_{c_0}=\sum\Delta_i$ is toric, but we could weaken it by requiring that all the log Calabi–Yau pairs $(\Delta_i,\mathrm{Diff}_{\Delta_i}(\mathcal{X}_{c_0}-\Delta_i))$ have index 1. By Theorem 5, the cover $q:\mathcal{X}'\to\mathcal{X}$ in the proof of Theorem 7 has actually degree at most 2. Therefore, one could also replace the assumption on the simple connectedness of the dual complex with the weaker condition that $\pi_1(\mathcal{D}(\mathcal{X}_{c_0}))$ has no index-2 subgroups.

8. Quotients of Calabi–Yau and holomorphic symplectic varieties

As a consequence of Theorem 5, we obtain the following bound on the index of quotients of log Calabi–Yau varieties.

Theorem 8.1. *Let*

- (X,Δ_X) *be a projective log Calabi–Yau pair with* $K_X+\Delta_X\sim_{\mathbb{Q}}0$, *where* $\Delta_X=\sum_i(1-1/m_i)\Delta_i$ *has standard coefficients*;
- G *be a finite subgroup of* $\mathrm{Aut}(X,\Delta_X)$, *i.e., the group of automorphisms* $g:X\to X$ *such that* $g^*\Delta_X=\Delta_X$; *and*
- (Y,Δ_Y) *be the pair given by the quotient* $Y:=X/G$ *and the boundary* Δ_Y *induced from* (X,Δ_X) *by the Riemann–Hurwitz formula as in* [Kollár 2013, (2.41.6)].

Assume that (Y, Δ_Y) *is a fiber of a minimal family* $\pi : (\mathcal{Y}, \mathcal{B}) \to C$ *of Calabi–Yau pairs of coregularity* 0 *over* $c_0 \in C$ *such that* $\mathrm{coeff}(\mathcal{B}) = \mathrm{coeff}(\Delta_Y)$. *Then the index of* (Y, Δ_Y) *is at most* 2, *or equivalently the order of the character* $\rho : G \to \mathrm{GL}(H^0(X, \mathcal{O}_X(K_X + \Delta_X)))$ *is at most* 2.

Note that the condition $\mathrm{coeff}(\mathcal{B}) = \mathrm{coeff}(\Delta_Y)$ is satisfied by a general fiber of the family π.

Proof. Assume that (Y, Δ_Y) admits a degeneration $\pi : (\mathcal{Y}, \mathcal{B}) \to C$ as in the statement. Up to a finite base change ramified at c_0, by Proposition 3.12, we may further assume that $\mathcal{Y}_{c_0}$ is reduced and hence $\pi : (\mathcal{Y}, \mathcal{B}) \to C$ is a locally stable family near c_0. Then, by adjunction, $(\mathcal{Y}_{c_0}, \mathcal{B}_{c_0})$ is a semi-log canonical Calabi–Yau pair of coregularity 0.

By the Riemann–Hurwitz formula, the coefficients of Δ_Y are standard, see [Kollár 2013, (2.41.6)]. Then, by hypothesis, the coefficients of $\mathcal{B}$ are standard as well. Since standard coefficients are greater than or equal to $\frac{1}{2}$, components of $\mathcal{B}$ that intersect along a divisor D_{c_0} of $\mathcal{Y}_{c_0}$ must have coefficient $\frac{1}{2}$, so that D_{c_0} appears with coefficient 1 in $\mathcal{B}_{c_0}$. In particular, also $\mathcal{B}_{c_0}$ has standard coefficients.

Then, by applying Lemma 3.8 to the normalization of $(\mathcal{Y}_{c_0}, \mathcal{B}_{c_0})$, it follows that the coefficients of $\mathcal{B}_{c_0}$, and hence of $\mathcal{B}$, are in $\{\frac{1}{2}, 1\}$. Therefore, by Theorem 7.6, we have that $2(K_Y + \Delta_Y) \sim 0$. Since the sections of $\mathcal{O}_Y(2(K_Y + \Delta_Y))$ correspond to G-invariant sections of $\mathcal{O}_X(2(K_X + \Delta_X))$, it follows that ρ^2 is the trivial character. □

We now apply Theorem 8.1 to degeneration of holomorphic symplectic varieties.

Definition 8.2. A normal proper variety X with canonical (Gorenstein) singularities is:

(1) *Holomorphic symplectic* if it admits a nondegenerate closed holomorphic 2-form $\omega_X \in H^0(X^{\mathrm{reg}}, \Omega^2_{X^{\mathrm{reg}}})$ on its regular locus.

(2) *Primitive symplectic* if it is holomorphic symplectic, $H^1(X, \mathcal{O}_X) = 0$ and $H^0(X, \Omega_X^{[2]})$ is generated by ω_X.

Definition 8.3. An automorphism $g : X \to X$ of a holomorphic symplectic variety X is *nonsymplectic* if $g^*\omega_X \neq \omega_X$, and it is *purely nonsymplectic* if $(g^k)^*\omega_X \neq \omega_X$ for all powers $g^k \neq 1$, e.g., a nonsymplectic automorphism of prime order.

If g is finite of order m,[4] any nonsymplectic automorphism descends to a purely nonsymplectic automorphism on the quotient X/g^{k_0}, where g^{k_0} is the minimal symplectic power of g.

Let (C, c_0) denote the germ of a smooth curve at a point $c_0 \in C$ and let $C^* := C \setminus \{c_0\}$. Let $\pi^* : X^* \to C^*$ be a projective family of holomorphic symplectic varieties.

Definition 8.4. The degeneration π^* is *of type III* if, up to finite base change, π^* extends to a minimal family $\pi : \mathcal{X} \to C$ of Calabi–Yau varieties of coregularity 0 over c_0.

See [Kollár et al. 2018, Theorem 0.11] for a discussion of the other types of degenerations and the relation with the coregularity of the central fiber.

[4]Note that if X is a projective primitive symplectic, a purely nonsymplectic automorphism is automatically finite by the same proof of [Huybrechts 2016, Corollary 3.4].

Proof of Corollary 13. Let $\pi : \mathcal{X} \to C$ be an extension of $\pi^* : \mathcal{X}^* \to C^*$, projective over C. Up to a birational modification, we can suppose that $g : \mathcal{X}^* \to \mathcal{X}^*$ extends to a regular morphism on $g : \mathcal{X} \to \mathcal{X}$, e.g., take the normalization of the main component of the graph of the rational map $\mathcal{X} \dashrightarrow \prod_{i=1}^{m-1} \mathcal{X}$ sending $x \mapsto (g(x), \dots, g^{m-1}(x))$, where m is the order of g. The semistable reduction theorem for Calabi–Yau varieties in [Fujino 2011] works in the equivariant setting too: it suffices to take g-equivariant log resolutions [Kollár 2007, Proposition 3.9.1] and to replace the ordinary MMP with a g-equivariant analog. Therefore, we can suppose that $\pi : \mathcal{X} \to C$ is a minimal family of Calabi–Yau varieties endowed with an automorphism $g : \mathcal{X} \to \mathcal{X}$ acting fiberwise and such that the pair $(\mathcal{X}, \mathcal{X}_{c_0} = \mathcal{X}_{c_0,\mathrm{red}})$ is dlt. In particular, $g^*(K_{\mathcal{X}} + \mathcal{X}_{c_0}) \sim_{\mathbb{Q}} K_{\mathcal{X}} + \mathcal{X}_{c_0} \sim_{\mathbb{Q},C} 0$.

Consider the pair $(\mathcal{Y}, \Delta_{\mathcal{Y}})$ given by the quotient $\mathcal{Y} := \mathcal{X}/g$ and the boundary $\Delta_{\mathcal{Y}}$ induced from $(\mathcal{X}, \mathcal{X}_{c_0})$ by the Riemann–Hurwitz formula as in [Kollár 2013, (2.41.6)]. Let $\mathcal{B}$ be the sum of the components of $\Delta_{\mathcal{Y}}$ dominating C. By construction, we have

$$0 \sim_{\mathbb{Q},C} K_{\mathcal{Y}} + \Delta_{\mathcal{Y}} = K_{\mathcal{Y}} + \mathcal{B} + \mathcal{Y}_{c_0} = K_{\mathcal{Y}} + \mathcal{B} + \mathcal{Y}_{c_0,\mathrm{red}}.$$

Hence, the quotient family $\pi_Y : (\mathcal{Y}, \Delta_{\mathcal{Y}}) \to C$ is a minimal family of Calabi–Yau varieties. If $\mathcal{X}^*$ is of type III, then the pair $(\mathcal{X}, \mathcal{X}_{c_0})$ has coregularity 0, and so the pair $(\mathcal{Y}, \Delta_{\mathcal{Y}})$ has coregularity 0 too by Proposition 3.12. Let $\omega \in H^0(\mathcal{X}_c, \Omega^{[2]}_{\mathcal{X}_c})$. Since g acts purely nonsymplectically on the general fiber $\mathcal{X}_c$, then $g^*\omega = \zeta_m \omega$, where ζ_m is a primitive m-th root of unity, and so $g^*\omega^n = \zeta_m^n \omega^n \in H^0(\mathcal{X}_c, \mathcal{O}_{\mathcal{X}_c}(K_{\mathcal{X}_c}))$. The index of the general fiber $\mathcal{Y}_c$ is equal to the order of ζ_m^n. By Theorem 8.1, we conclude that m divides $\dim \mathcal{X}_c = 2n$. □

Remark 8.5. The dual complex $\mathcal{DMR}(\mathcal{Y}, \Delta_{\mathcal{Y}})$ is PL-homeomorphic to the quotient $\mathcal{D}(\mathcal{X}_{c_0})/g$ by Proposition 3.13. In particular, if g is a nonsymplectic involution and $\dim \mathcal{X}_c \equiv 2 \mod 4$, then $\mathcal{DMR}(\mathcal{Y}, \Delta_{\mathcal{Y}})$ is not orientable, since the identity (1-1) is g-equivariant, and g acts nontrivially on $H^0(\mathcal{X}_c, \mathcal{O}_{\mathcal{X}_c}(K_{\mathcal{X}_c}))$.

Remark 8.6. Let $\mathcal{F}$ be the moduli space of (markable) primitive symplectic varieties of dimension n with a purely nonsymplectic automorphism of order 2. A consequence of Remark 1.1 is that the Baily–Borel compactification of the period domain of $\mathcal{F}$ does not contain 0-cusps. We refer to [Alexeev et al. 2024, Section 2] for an explanation of the terminology. In fact, we do not know any place in the literature where the details of the construction of these moduli spaces are carried out for arbitrary primitive symplectic varieties. For that, one can adapt the arguments in [loc. cit., Section 2] for K3 surfaces and [Boissière et al. 2016, Section 4]. This goes beyond the scope of this paper, so we omit the details here.

9. Examples

In this section, we collect some examples showing that the statement of the theorem is optimal.

Example 9.1 [Kollár and Xu 2016, Example 60]. Set $X_1 := \mathbb{P}^1$ and $\Delta_1 := (0 : 1) + (1 : 0)$. Let τ_1 denote the involution $\tau_1 : (x : y) \mapsto (y : x)$. Then, for every integer $n \geq 2$, we set $(X_n, \Delta_n) := (X_1, \Delta_1)^n$, while τ_n denotes the involution $\tau_n := (\tau_1, \dots, \tau_1)$ that acts diagonally as τ_1 on each factor. We define $(Y_n, B_n) := (X_n, \Delta_n)/(\tau_n)$.[5]

Lemma 9.2. *If $n \geq 2$, (Y_n, B_n) is a dlt log Calabi–Yau pair, and B_n is a reduced Cartier divisor.*

Proof. If $n \geq 2$, the action of τ_n is free in codimension 1. The morphism $X_n \to Y_n$ is étale in codimension 1, so the pair (Y_n, B_n) is log Calabi–Yau. The quotient $X_n \to Y_n$ is étale along Δ_n, and B_n is a reduced Cartier divisor with only simple normal crossings, since Δ_n is so. Hence, (Y_n, B_n) is a dlt pair. □

Lemma 9.3. *If $n \geq 2$ is odd, $K_{Y_n} + B_n \not\sim 0$ but $2(K_{Y_n} + B_n) \sim 0$.*

Proof. We observe that the log canonical divisor $K_{Y_n} + B_n$ is not Cartier. Indeed, the involution τ_n does not preserve the volume form $dz_1/z_1 \wedge dz_2/z_2 \wedge \cdots \wedge dz_n/z_n$ on $\mathbb{G}_m^n = X_n \setminus \Delta_n$, where $z_i := x/y$ is a coordinate on the torus of the i-th copy of $\mathbb{P}^1$ in $X_n = (\mathbb{P}^1)^n$. However, $2(K_{Y_n} + B_n) \sim 0$, since the class group of Y_n has only 2-torsion.

Alternatively, note that the dual complex $\mathcal{D}(\Delta_n)$ is an n-dimensional hyperoctahedron, thus it is PL-homeomorphic to $\mathbb{S}^{n-1}$. The involution τ_n induces the antipodal involution on $\mathbb{S}^{n-1}$. Hence, $\mathcal{D}(B_n) \simeq \mathbb{RP}^{n-1}$, which is not orientable for $n \geq 2$ odd. By Proposition 5.5, it follows that $K_{Y_n} + B_n \not\sim 0$, while $2(K_{Y_n} + B_n) \sim 0$ by descending $K_{X_n} + \Delta_n \sim 0$. □

Examples of the behavior in Lemma 9.3 in even dimension $n > 2$ can be achieved by replacing (Y_n, B_n) by $(Y_{n-1}, B_{n-1}) \times (X_1, \Delta_1)$; see also [Mauri 2020, Example 7.4]. This shows that the least common multiple in Theorem 1 and Proposition 2 is indeed necessary. In view of Theorem 7, it is also interesting to note that B_n is a toric simple normal crossing variety but $\pi_1(B_n) \simeq \mathbb{Z}/2\mathbb{Z}$ and $K_{B_n} \not\sim 0$.

Example 9.4. With the notation of Example 9.1, let $[x_i : y_i]$ be homogeneous coordinates on the i-th copy of $\mathbb{P}^1$ in $X_n = (\mathbb{P}^1)^n$. The pencil

$$\mathcal{X} := \left\{ ([x_1 : y_1], \dots, [x_n : y_n], t) \in X_n \times \mathbb{A}^1 \,\middle|\, \prod_i x_i^2 + \prod_i y_i^2 + t \prod_i x_i y_i = 0 \right\} \to \mathbb{A}^1$$

is an example of a minimal family of Calabi–Yau varieties of coregularity 0 at $t = 0$, locally stable in a neighborhood of $t = 0$. The fiber $\mathcal{X}_0$ is the toric boundary of X_n, so the assumptions of Theorem 7 hold for $\mathcal{X}$. Therefore, the general fiber $\mathcal{X}_t$ satisfies $K_{\mathcal{X}_t} \sim 0$, which is also clear since $\mathcal{X}_t$ is a smooth anticanonical hypersurface of X_n.

Note that $\mathcal{X}$ is τ_n-invariant. The quotient $\mathcal{Y} := \mathcal{X}/\tau_n \to \mathbb{A}^1$ is still a minimal family of Calabi–Yau varieties of coregularity 0 at $t = 0$, locally stable in a neighborhood of $t = 0$. The fiber $\mathcal{Y}_0$ is the boundary B_n. We are no longer in the assumptions of Theorem 7 as $\pi_1(B_n) \simeq \mathbb{Z}/2\mathbb{Z}$. In fact we have $K_{\mathcal{Y}_t} \not\sim 0$, but $2K_{\mathcal{Y}_t} \sim 0$ according to Theorem 5. For instance, if $n = 3$ and t is general, $\mathcal{Y}_t$ is an Enriques surface.

[5] For $n = 2$, the variety Y_2 is a so-called Fano–Enriques threefold, and in characteristic 2, it is a Fano compactification of a terminal non-Cohen–Macaulay singularity studied in [Totaro 2019, Theorem 5.1].

Example 9.5 [Kollár and Xu 2016, Example 61]. The cyclic group $\mathbb{Z}/m\mathbb{Z}$ acts on $\mathbb{P}^{m-1}$, with $m > 2$, by permuting the homogeneous coordinates $[x_1 : \cdots : x_m]$. Set $\Delta := \{x_1 \cdot \cdots \cdot x_m = 0\}$. The quotient $(Y, B) := (\mathbb{P}^{m-1}, \Delta)/(\mathbb{Z}/m\mathbb{Z})$ is a log Calabi–Yau pair with reduced boundary and with complicated self-intersection, and it has index 2 if m is even, and index 1 if m is odd. Indeed, a generator of $H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(K_{\mathbb{P}^m} + \Delta))$ is given by

$$\omega := \sum_i (-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

A permutation σ of the coordinates sends ω to $\mathrm{sgn}(\sigma) \cdot \omega$, so σ preserves $\omega^{\otimes 2}$, i.e., $2(K_Y + B) \sim 0$. This shows that although the order of the group we quotient by (i.e., m) can be arbitrarily large, the index of the quotient log Calabi–Yau pair (Y, B) is always at most 2.[6]

Example 9.6. Theorem 1 does not impose any constraints on the index of X, i.e., the smallest positive integer n such that nK_X is Cartier. Take for instance the weighted projective space $\mathbb{P}(1, 1, n)$ with its toric boundary. For $n > 2$, the Cartier index of $\mathbb{P}(1, 1, n)$ is n, while the index of any toric pair is 1.

Example 9.7. In general, a pseudomanifold need not have an orientable double cover which is a topological covering space. For instance, we can consider the suspensions $S\mathbb{S}^2$ and $S\mathbb{RP}^2$, with vertices $\{a, b\}$ and $\{p, q\}$, respectively. Notice that the pseudomanifold $S\mathbb{S}^2$ is orientable while $S\mathbb{RP}^2$ is not. The space $S\mathbb{RP}^2$ is simply connected, so it does not admit any topological covering space. On the other hand, the morphism $S\mathbb{S}^2 \to S\mathbb{RP}^2$ provides an orientable double cover ramified at $\{p, q\}$. This is an incarnation of a more general fact, as every pseudomanifold admits a *branched* orientation double cover; see [Matthews 2016, Section 5.2].

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