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In local relative p-adic Hodge theory, we show that the Galois cohomology of a finite-height crystalline representation (up to a twist) is essentially computed via the (Fontaine–Messing) syntomic complex with coefficients in the associated F-isocrystal. In global applications, for smooth (p-adic formal) schemes, we establish a comparison between the syntomic complex with coefficients in a locally free Fontaine–Laffaille module and the p-adic nearby cycles of the associated étale local system on the (rigid) generic fibre.

1. Introduction

Let p denote a fixed prime and κ a perfect field of characteristic p. Let K be a mixed characteristic complete discrete valuation field with ring of integers O_K and residue field κ, and let $F := W(\kappa)[1/p]$ be the fraction field of the ring of p-typical Witt vectors with coefficients in κ. Fontaine's *crystalline conjecture* for a proper and smooth O_K-scheme relates the p-adic étale cohomology of its generic fibre to the crystalline cohomology of its special fibre. Fontaine and Messing [1987] initiated a program for proving the crystalline conjecture via *syntomic* methods. By subsequent works of Kato and Messing [1992] and Kato [1994] and with the remarkable work of Tsuji [1999], the crystalline conjecture was shown to be true. There have been several proofs and generalisations of the crystalline comparison theorem: [Andreatta and Iovita 2013; Beilinson 2012; Bhatt et al. 2018; Colmez and Nizioł 2017; Diao et al. 2023; Faltings 1989; 2002; Guo and Reinecke 2024; Nizioł 1998; Scholze 2013; Tsuji 1999; Yamashita and Yasuda 2014].

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1.1. ***p-adic nearby cycles.*** Let $\mathfrak{X}$ be a smooth (p-adic formal) O_K-scheme with (rigid) generic fibre X and special fibre $\mathfrak{X}_\kappa$. Let $j : X_{\text{ét}} \to \mathfrak{X}_{\text{ét}}$ and $i : \mathfrak{X}_{\kappa,\text{ét}} \to \mathfrak{X}_{\text{ét}}$ denote natural morphisms of sites. For $r \geqslant 0$, let $\mathscr{S}_n(r)_{\mathfrak{X}}$ denote the syntomic sheaf modulo p^n on $\mathfrak{X}_{\kappa,\text{ét}}$ (see Sections 7 and 8 for the definition of the syntomic complex). Fontaine and Messing [1987] constructed a period morphism from the syntomic complex to the complex of p-adic nearby cycles,

$$\alpha_{r,n}^{\text{FM}} : \mathscr{S}_n(r)_{\mathfrak{X}} \to i^*\mathrm{R}j_*\mathbb{Z}/p^n(r)'_X, \tag{1-1}$$

where

$$\mathbb{Z}_p(r)' := \frac{1}{a(r)!\, p^{a(r)}}\mathbb{Z}_p(r)$$

for $r = (p-1)a(r)+b(r)$ with $0 \leqslant b(r) < p-1$. For $\mathfrak{X}$ a smooth and proper O_K-scheme and $0 \leqslant r \leqslant p-1$, by truncating (1-1) in degree $\leqslant r$, the map $\alpha_{r,n}^{\text{FM}}$ is known to be a quasi-isomorphism by [Kato 1987; 1994; Kurihara 1987; Tsuji 1999]. Tsuji [1996] generalised this result to proper and semistable schemes and nontrivial étale local systems arising from (the pullback of) Fontaine–Laffaille modules over O_F; see [Fontaine and Laffaille 1982]. Moreover, Colmez and Nizioł [2017] proved a similar result for semistable (p-adic formal) schemes and constant coefficients, without any restrictions on r. In particular, for a smooth (p-adic formal) scheme, we have the following.

Theorem 1.1 [Colmez and Nizioł 2017, Theorem 1.1]. *For $0 \leqslant k \leqslant r$, the natural map*

$$\alpha_{r,n}^{\text{FM}} : \mathcal{H}^k(\mathscr{S}_n(r)_{\mathfrak{X}}) \to i^*\mathrm{R}^k j_*\mathbb{Z}/p^n(r)'_X$$

is a p^N-isomorphism; i.e., its kernel and cokernel are killed by p^N, where $N = N(e, p, r) \in \mathbb{N}$ depends on the absolute ramification index e of K, prime p and twist r, but not on X or n.

The proof of Theorem 1.1 in [Colmez and Nizioł 2017] works by reducing the problem to the local setting; i.e., one works over the p-adic completion of an étale algebra over $O_K[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ for some indeterminates $X_1, \dots, X_d$. Locally, Colmez and Nizioł also show that it is enough to work with p-adic formal schemes and deduce the result for schemes by invoking Elkik's approximation theorem and a form of rigid GAGA; see [Colmez and Nizioł 2017, §5.1].

For simplicity in the introduction, we will let R be the p-adic completion of $O_F[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ and $S := O_K \otimes_{O_F} R$; see Assumption 2.1 for a more general setup. Let $G_S := \pi_1^{\text{ét}}(S[1/p], \bar{\eta})$ for a fixed geometric generic point of $\mathrm{Sp}(S[1/p])$. Denote by $\mathrm{Syn}(S, r)$ the r-th Tate twist of the (log-) syntomic complex; see [Colmez and Nizioł 2017, §3.3] for details.

Theorem 1.2 [Colmez and Nizioł 2017, Theorem 1.6]. *If K contains enough roots of unity, then the maps*

$$\alpha_r^{\mathcal{L}\text{az}} : \tau_{\leqslant r}\,\mathrm{Syn}(S, r) \to \tau_{\leqslant r}\mathrm{R}\Gamma_{\text{cont}}(G_S, \mathbb{Z}_p(r)),$$
$$\alpha_{r,n}^{\mathcal{L}\text{az}} : \tau_{\leqslant r}\,\mathrm{Syn}(S, r)_n \to \tau_{\leqslant r}\mathrm{R}\Gamma_{\text{cont}}(G_S, \mathbb{Z}/p^n(r)) \to \tau_{\leqslant r}\mathrm{R}\Gamma((\mathrm{Sp}\,S[1/p])_{\text{ét}}, \mathbb{Z}/p^n(r))$$

are p^{Nr}-quasi-isomorphisms for a universal constant N; i.e., N does not depend on p, X, K, n or r.

One of our main goals in this article is to generalise Theorem 1.2 by studying syntomic complexes with coefficients. Subsequently, by "glueing" the local results for relative Fontaine–Laffaille modules, we will obtain a global generalisation of Theorem 1.1. Note that, in the local setting, on the étale side, by using a $K(\pi, 1)$-lemma (see [Scholze 2013, Theorem 4.9]), we can reduce to the setting of $\mathbb{Z}_p$-representations of G_R. Then, due to the "crystalline" nature of our goal, we will consider G_R-stable $\mathbb{Z}_p$-lattices inside "finite-height" crystalline representations of G_R and certain natural invariants attached to such representations as in [Abhinandan 2025, §4].

1.2. ***Finite-height representations.*** Fix $p \geqslant 3$, $m \in \mathbb{N}_{\geqslant 2}$, $K = F(\zeta_{p^m})$ and $\varpi = \zeta_{p^m} - 1$ (see Remark 1.8 on the rationale behind our assumptions). Fix an algebraically closed field $\overline{\mathrm{Fr}(R)}$ containing $\bar{F}$ an algebraic closure of F, and set $F_\infty := F(\mu_{p^\infty}) \subset \bar{F}$. Let $\bar{R}$ denote the union of finite R-subalgebras $R' \subset \overline{\mathrm{Fr}(R)}$ such that $R'[1/p]$ is étale over $R[1/p]$. Set

$$\begin{aligned} R_\infty &:= \bigcup_{n \in \mathbb{N}} R[\mu_{p^n}, X_1^{1/p^n}, \dots, X_d^{1/p^n}], \\ G_R &:= \mathrm{Gal}(\bar{R}[1/p]/R[1/p]), \\ \Gamma_R &:= \mathrm{Gal}(R_\infty[1/p]/R[1/p]), \\ H_R &:= \mathrm{Ker}(G_R \twoheadrightarrow \Gamma_R), \end{aligned}$$

and note that $\Gamma_R = \Gamma'_R \rtimes \Gamma_F$, where we have the isomorphisms

$$\begin{aligned} \Gamma'_R &:= \mathrm{Gal}(R_\infty[1/p]/F_\infty R[1/p]) \xrightarrow{\sim} \mathbb{Z}_p(1)^d, \\ \Gamma_F &:= \mathrm{Gal}(F_\infty/F) \xrightarrow{\sim} \mathbb{Z}_p^\times. \end{aligned}$$

Recall that [Fontaine 1990] showed a categorical equivalence between $\mathbb{Z}_p$-representations of G_F and étale (φ, Γ_F)-modules over a certain period ring $\boldsymbol{A}_F$. These results were generalised to the relative setting in [Andreatta 2006] to establish a categorical equivalence between $\mathbb{Z}_p$-representations of G_R and étale (φ, Γ_R)-modules over a certain period ring $\boldsymbol{A}_R$; see Section 2.4. Moreover, the work of Fontaine [1982; 1994a; 1994b] on crystalline representations of G_F was generalised to the relative case in [Brinon 2008] via the construction of a fully faithful functor $\mathcal{O}\boldsymbol{D}_{\mathrm{cris}}$ from the category of crystalline representations of G_R to the category of filtered (φ, ∂)-modules over $R[1/p]$; see Section 2.3.

Let $q = \varphi(\pi)/\pi$ belong to $\boldsymbol{A}_R$, where π is the usual element of Fontaine; see Section 2.2. In [Abhinandan 2025], we studied finite q-height representations of G_R, a notion parallel to the arithmetic case, i.e., $R = O_F$ in [Berger 2004; Colmez 1999; Wach 1996; 1997]; see [Abhinandan 2025, Remark 1.4]. A representation T in $\mathrm{Rep}_{\mathbb{Z}_p,\mathrm{free}}(G_R)$ is of finite q-height if it admits a unique (φ, Γ_R)-module over a certain subring $\boldsymbol{A}_R^+ \subset \boldsymbol{A}_R$ satisfying certain conditions on the (φ, Γ_R)-action (see Definition 3.1); the aforementioned $\boldsymbol{A}_R^+$-module is called the *Wach module* associated to T and denoted by $\boldsymbol{N}(T)$. Moreover, we showed that finite q-height representations are closely related to crystalline representations via a certain period ring $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\bar{R})$, where the former is equipped with structures induced from the latter; see [Abhinandan 2025, §4.3].

Theorem 1.3 [Abhinandan 2025, Theorem 4.24 and Proposition 4.27]. *Let T be a $\mathbb{Z}_p$-representation of G_R and assume that T is of positive finite q-height. Then $V := T[1/p]$ is a positive crystalline representation and we have an isomorphism of $R[1/p]$-modules*

$$\mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V) \xleftarrow{\sim} (\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T))^{\Gamma_R}[1/p]$$

compatible with the respective Frobenii, filtrations and connections.

1.3. *Syntomic coefficients and* (φ, Γ)*-modules.* In this subsection, we will assume the following: Let T be a $\mathbb{Z}_p$-representation of G_R of positive finite q-height $s \in \mathbb{N}$ and set $V := T[1/p]$; see Definition 3.1. Assume that $\boldsymbol{N}(T)$ is free of rank $\mathrm{rk}_{\mathbb{Z}_p} T$ over $\boldsymbol{A}_R^+$ and $M \subset \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$ is a finite free R-submodule of rank $\mathrm{rk}_{\mathbb{Z}_p} T$ such that $M[1/p] \xrightarrow{\sim} \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$ is an isomorphism and satisfies Assumption 5.1; see Example 5.2 for obtaining M from $\boldsymbol{N}(T)$.

Our objective is to compute the continuous G_R-cohomology of $T(r)$ using the syntomic complex for R with coefficients in $M \subset \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$. Set $S = R[\varpi]$ and note that we have a divided power thickening $R_\varpi^{\mathrm{PD}} \twoheadrightarrow S$ (using an "arithmetic" variable X_0, see Section 2.5), and the ring R_ϖ^{PD} is equipped with a Frobenius endomorphism φ; let $\Omega^1_{R_\varpi^{\mathrm{PD}}}$ denote the p-adic completion of the module of differentials of R_ϖ^{PD} with respect to $\mathbb{Z}$. Set

$$M_\varpi^{\mathrm{PD}} := R_\varpi^{\mathrm{PD}} \otimes_R M$$

and equip it with the induced supplementary structures to obtain a filtered de Rham complex (see Section 5.1)

$$\mathrm{Fil}^r \mathscr{D}^\bullet_{S,M} := \mathrm{Fil}^r M_\varpi^{\mathrm{PD}} \to \mathrm{Fil}^{r-1} M_\varpi^{\mathrm{PD}} \otimes_{R_\varpi^{\mathrm{PD}}} \Omega^1_{R_\varpi^{\mathrm{PD}}} \to \mathrm{Fil}^{r-2} M_\varpi^{\mathrm{PD}} \otimes_{R_\varpi^{\mathrm{PD}}} \Omega^2_{R_\varpi^{\mathrm{PD}}} \to \cdots.$$

Definition 1.4. Define the *syntomic complex* of S with coefficients in M and its modulo p^n-version as

$$\mathrm{Syn}(S, M, r) := \big[\mathrm{Fil}^r \mathscr{D}^\bullet_{S,M} \xrightarrow{p^r - p^\bullet\varphi} \mathscr{D}^\bullet_{S,M}\big],$$
$$\mathrm{Syn}(S, M, r)_n := \mathrm{Syn}(S, M, r) \otimes \mathbb{Z}/p^n$$

for $n \geqslant 1$.

Theorem 1.5 (Theorem 5.5). *Let T be a positive finite q-height $\mathbb{Z}_p$-representation of G_R of height s as above, and take $r \in \mathbb{N}$ such that $r \geqslant s+1$. Then, there exist p^N-quasi-isomorphisms*

$$\alpha_r^{\mathcal{L}\mathrm{az}} : \tau_{\leqslant r-s-1} \mathrm{Syn}(S, M, r) \simeq \tau_{\leqslant r-s-1} \mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T(r)),$$
$$\alpha_{r,n}^{\mathcal{L}\mathrm{az}} : \tau_{\leqslant r-s-1} \mathrm{Syn}(S, M, r)_n \simeq \tau_{\leqslant r-s-1} \mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T/p^n(r)),$$

where $N = N(T, e, r) \in \mathbb{N}$ depends on the representation T, $e = [K : F]$ and the twist r.

Similarly, we have a filtered de Rham complex with coefficients in M, and one can also define the *syntomic complex* of R with coefficients in M. Using Theorem 1.5 for $\varpi = \zeta_{p^2} - 1$ and Galois descent (see Lemma 6.21), we obtain the following.

Corollary 1.6 (Corollary 5.9). *Let T be a positive finite q-height $\mathbb{Z}_p$-representation of G_R of height s as above, and take $r \in \mathbb{N}$ such that $r \geqslant s+1$. Then, there exist p^N-quasi-isomorphisms*

$$\alpha_r^{\mathcal{L}\mathrm{az}} : \tau_{\leqslant r-s-1} \operatorname{Syn}(R, M, r) \simeq \tau_{\leqslant r-s-1} \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, T(r)),$$
$$\alpha_{r,n}^{\mathcal{L}\mathrm{az}} : \tau_{\leqslant r-s-1} \operatorname{Syn}(R, M, r)_n \simeq \tau_{\leqslant r-s-1} \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, T/p^n(r)),$$

where $N = N(p, r, s) \in \mathbb{N}$ depends on the prime p, twist r and height s of T.

The proof of Theorem 5.5 is broadly divided into two main steps. First, we modify the syntomic complex with coefficients in M and relate it to a "differential" Koszul complex with coefficients in $\boldsymbol{N}(T)$; see Proposition 5.28. Next, we modify the Koszul complex from the first step to obtain a Koszul complex computing the continuous G_S-cohomology of $T(r)$; see Theorem 5.5 and Proposition 6.1. The key idea behind relating these two steps is the comparison isomorphism in [Abhinandan 2025, Theorem 4.24] and a Poincaré lemma; see Section 5.6. Our proof of Theorem 5.5 is inspired by [Colmez and Nizioł 2017], however our setting demands several nontrivial generalisations of the ideas in [loc. cit.].

Remark 1.7. Setting $T = \mathbb{Z}_p$ in Theorem 1.5, we obtain a statement similar to Theorem 1.1 (note that we truncate in degree $\leqslant r-1$ as we are working with the syntomic complex instead of the log-syntomic complex as in [Colmez and Nizioł 2017]).

Remark 1.8. In Theorem 1.5 we restrict to a finite cyclotomic K/F because we are using the cyclotomic Frobenius ($X_0 \mapsto (1+X_0)^p - 1$) in Definition 1.4 instead of the Kummer Frobenius ($X_0 \mapsto X_0^p$) as in [Colmez and Nizioł 2017]. For K/F finite, one should use Kummer Frobenius to define a log-syntomic complex (log-structure with respect to X_0). Then it should be possible to obtain Theorem 1.5 for all finite extensions K/F (with truncation in degree $\leqslant r-s$ as in [Colmez and Nizioł 2017]). Furthermore, to obtain the statement over $\overline{F}$, one could pass to the limit over all finite extensions K/F. Alternatively, one could directly work over $\mathbb{C}_p = \hat{\overline{F}}$ as in [Gilles 2023] to avoid complications arising from Frobenius on X_0. In the latter case, our proofs can be adapted to obtain Theorem 1.5 for $S = R\widehat{\otimes}_{O_F} O_{\mathbb{C}_p}$ (with truncation in degrees $\leqslant r-s-1$).

Remark 1.9. The case $p=2$ is different from $p \geqslant 3$ as, for $p=2$, the constant N in Theorem 1.5 also depends on the relative dimension of R/O_F; see [Colmez and Nizioł 2017, Lemma 3.11].

Next, using the fundamental exact sequence in p-adic Hodge theory (2-2), one can define a local Fontaine–Messing period map for T as in Theorem 1.5; see Section 6.7. Then, we show the following.

Theorem 1.10 (Theorem 6.19). *The period map $\tilde{\alpha}_{r,n,S}^{\mathrm{FM}}$ is $p^{N(T,e,r)}$-equal to $\alpha_{r,n}^{\mathcal{L}\mathrm{az}}$ from Theorem 1.5.*

1.4. ***Fontaine–Laffaille modules and p-adic nearby cycles.*** In this subsection, we will specialise Theorem 1.5 to the case of global relative Fontaine–Laffaille modules introduced by Faltings [1989, §II]. Let R denote the p-adic completion of an étale algebra over $O_F[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ with nonempty geometrically integral special fibre; see Section 2.1 for details. Note that Theorem 1.5 and Corollary 1.6 are true in this setting as well. In [Abhinandan 2025, §5], we considered the category $\mathrm{MF}_{[0,s],\mathrm{free}}(R, \Phi, \partial)$

of free relative Fontaine–Laffaille modules of level $[0, s]$ (see Remark 3.26 (i)) as a full subcategory of $\mathfrak{MF}^{\nabla}_{[0,s]}(R)$ in [Faltings 1989, §II]. To any M in $\mathrm{MF}_{[0,s],\mathrm{free}}(R, \Phi, \partial)$, one can functorially attach a representation $T_{\mathrm{cris}}(M)$ in $\mathrm{Rep}_{\mathbb{Z}_p,\mathrm{free}}(G_R)$, which admits a Wach module $\boldsymbol{N}(T)$ (see [Abhinandan 2025, Theorem 5.4]) and satisfies Assumption 5.1 (see Example 5.2 (iii)). Next, let $\mathfrak{X}$ be a smooth (p-adic formal) scheme defined over O_F and cover it by affine (p-adic formal) schemes $\{\mathfrak{U}_i\}_{i \in I}$, where, for all $i \in I$, we have $\mathfrak{U}_i = \mathrm{Spec}\, A_i$ ($\mathfrak{U}_i = \mathrm{Spf}\, A_i$) such that its p-adic completion $\hat{A}_i$ is an étale algebra as above; we also fix compatible Frobenius lifts $\varphi_i : \hat{A}_i \to \hat{A}_i$. Take $\mathrm{MF}_{[0,s],\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$ to be the category of finite locally free filtered $\mathcal{O}_{\mathfrak{X}}$-modules $\mathcal{M}$ equipped with a quasinilpotent integrable connection satisfying Griffiths transversality such that there exists a covering $\{\mathfrak{U}_i\}_{i \in I}$ of $\mathfrak{X}$ as above with $\mathcal{M}_{\mathfrak{U}_i} \in \mathrm{MF}_{[0,s],\mathrm{free}}(\hat{A}_i, \Phi, \partial)$ for all $i \in I$; see Section 8.1.

To state the main global result, let $\mathfrak{X}$ be a smooth (p-adic formal) scheme defined over O_F (for $\mathfrak{X}$ a scheme, assume that it is proper or an open subscheme of a proper semistable scheme defined over O_F). Let $\mathcal{M}$ be an object of $\mathrm{MF}_{[0,s],\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$ with $0 \leqslant s \leqslant p-2$ (for $\mathfrak{X}$ an open scheme, further assume that $\mathcal{M}$ extends to the compactification of $\mathfrak{X}$, see Remark 8.4). Let $\mathbb{L}$ denote the associated $\mathbb{Z}_p$-local system on the (rigid) generic fibre X of $\mathfrak{X}$. Then, we show the following:

Theorem 1.11 (Theorem 8.8). *For $r \geqslant s+1$ and $0 \leqslant k \leqslant r-s-1$, the Fontaine–Messing period map*

$$\alpha^{\mathrm{FM}}_{r,n,\mathfrak{X}} : \mathcal{H}^k(\mathcal{S}_n(\mathcal{M}, r)_{\mathfrak{X}}) \to i^* \mathrm{R}^k j_* \mathbb{L}/p^n(r)'_X$$

is a p^N-isomorphism, where $N = N(p, r, s) \in \mathbb{N}$ depends on p, r and s but not on $\mathfrak{X}$ or n.

The proof of Theorem 1.11 proceeds by reducing to the local setting, whence we may directly apply Theorem 1.5.

Remark 1.12. In personal communications with Takeshi Tsuji, I learned that, in some unpublished work, he obtained similar results over $\overline{F}$ and large enough p. However, our respective approaches are different and this article includes more general local results and the arithmetic case as well.

Remark 1.13. Note that, from [Bhatt et al. 2019, §10], we have a prismatic syntomic complex and it is known to compute p-adic nearby cycles in the case of constant coefficients. Using the results of [Morrow and Tsuji 2020] on coefficients in integral p-adic Hodge theory and prismatic cohomology, it should be possible to obtain an integral version of our results (in the geometric case, i.e., over $\overline{F}$). Moreover, using the theory of analytic prismatic F-crystals on the absolute prismatic site from [Du et al. 2024; Guo and Reinecke 2024], we should be able to generalise those results to the arithmetic case as well. We will report on these ideas in the future.

1.5. ***Outline of the paper.*** Sections 2–6 comprise the local part of the paper, while Sections 7 and 8 consist of global applications. In Section 2.1 we describe our local setup, notation and some conventions. In Sections 2.2–2.4 we quickly recall basics of period rings, crystalline representations and relative étale (φ, Γ)-modules. Section 2.5 introduces "good" crystalline coordinates, and we define certain rings of analytic functions convergent on some annulus following [Colmez and Nizioł 2017, §2]; these rings are denoted as $R_{\varpi}^{\star}$ for $\star \in \{+, \mathrm{PD}, [u], [u, v], (0, v]+\}$, where we can take $u = p/(p-1)$ and $v = p-1$. In Section 2.6, we equip these rings with a Frobenius endomorphism and, in Section 2.7, we consider their Frobenius-equivariant "cyclotomic" embedding ι_{cycl} into period rings and define $\boldsymbol{A}_{R,\varpi}^{\star}$ as the image of $R_{\varpi}^{\star}$ under ι_{cycl}. The latter enables us to relate differential operators on the ring $R_{\varpi}^{[u,v]}$ to the infinitesimal action of $\Gamma_S := \mathrm{Gal}(R_\infty[1/p]/S[1/p])$ on its "cyclotomic" image, i.e., $\boldsymbol{A}_{R,\varpi}^{[u,v]}$. Finally, in Section 2.8, we introduce certain big period rings, in particular $E_{R,\varpi}^{\star}$ and $E_{\bar{R}}^{\star}$, we study a natural filtration on the scalar extension of M to these rings and we prove a version of the filtered Poincaré lemma. The latter, together with the results of Section 3.3, are key ingredients in relating syntomic complexes with coefficients in M to Koszul complexes with coefficients in $\boldsymbol{N}(T)$. The motivation for our approach comes from the computations of [Colmez and Nizioł 2017, §2.6].

In Sections 3.1 and 3.2, we recall the notion of finite-height representations and their relationship to crystalline representations from [Abhinandan 2025], as well as prove some useful technical lemmas. In Section 3.3, we study a filtration on scalar extensions of Wach modules and prove another filtered Poincaré lemma. The local theory of relative Fontaine–Laffaille modules is recalled in Section 3.4. Section 4 recalls the definition of Koszul complexes computing continuous Γ_S-cohomology (see Section 4.2) and Lie Γ_S-cohomology (see Section 4.3).

In Section 5, we formulate our main local result, Theorem 1.5, and carry out the local syntomic computations for its proof. The aim of Section 6 is to carry out the (φ, Γ)-module side computations for the proof of Theorem 1.5. To explain the content of these two sections to the reader, we introduce the following commutative diagram of complexes (see the discussion after Theorem 6.19 for a more complete picture and explanations), where all isomorphisms are p-power quasi-isomorphisms; i.e., the kernel and the cokernel of the induced map on cohomology are killed by a fixed bounded power of p.

$$
\begin{array}{ccccccc}
\mathrm{K}_{\partial,\varphi}(\mathrm{F}^r M_{\varpi}^{\mathrm{PD}}) & \longrightarrow & C_G(\mathrm{K}_{\partial,\varphi}(\mathrm{F}^r \Delta^{\mathrm{PD}})) & \xleftarrow[\sim]{\mathrm{PL}} & C_G(\mathrm{K}_{\varphi}(\mathrm{F}^r \Delta^{\mathrm{PD},\partial})) & \longrightarrow & C_G(\mathrm{K}_{\varphi}(\mathrm{F}^r T A_{\mathrm{cris}})) \\
\wr\downarrow \tau_{\leqslant r} & & & & & & \wr\uparrow \mathrm{FES} \\
\mathrm{K}_{\partial,\varphi}(\mathrm{F}^r M_{\varpi}^{[u,v]}) & & & & & & C_G(T(r)) \\
\wr\downarrow \mathrm{PL} & & & & & & \wr\uparrow \mathrm{AS} \\
\mathrm{K}_{\partial,\varphi,\partial_A}(\mathrm{F}^r \Delta_{\varpi}^{[u,v]}) & & & & & & C_G(\mathrm{K}_{\varphi}(T A_{\bar{S}}(r))) \\
\wr\uparrow \mathrm{PL} & & & & & & \wr\uparrow \\
\mathrm{K}_{\varphi,\partial_A}(\mathrm{F}^r N_{\varpi}^{[u,v]}) & & & & & & C_{\Gamma}(\mathrm{K}_{\varphi}(D_{R_\infty}(r))) \\
\tau_{\leqslant r}\,\wr\downarrow t^{\bullet} & & & & & & \wr\uparrow \\
\mathcal{K}_{\varphi,\mathrm{Lie}\,\Gamma}(\mathrm{F}^r N_{\varpi}^{[u,v]}) & & & & & & C_{\Gamma}(\mathrm{K}_{\varphi}(D_{\varpi}(r))) \\
\wr\uparrow t^r & & & & & & \wr\uparrow \\
\mathcal{K}_{\varphi,\mathrm{Lie}\,\Gamma}(N_{\varpi}^{[u,v]}(r)) & \xleftarrow[\mathcal{L}\mathrm{az}]{\sim} & \mathcal{K}_{\varphi,\Gamma}(N_{\varpi}^{[u,v]}(r)) & \xleftarrow[\mathrm{can}]{\sim} & \mathcal{K}_{\varphi,\Gamma}(N_{\varpi}^{(0,v]+}(r)) & \xrightarrow{\sim} & \mathrm{K}_{\varphi,\Gamma}(D_{\varpi}(r)).
\end{array}
$$

In the diagram, we set

$$M_{\varpi}^{\star} = R_{\varpi}^{\star} \otimes_R M, \quad N_{\varpi}^{\star} = \boldsymbol{A}_{R,\varpi}^{\star} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T), \quad N_{\varpi}^{\star}(r) = \boldsymbol{A}_{R,\varpi}^{\star} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T(r)),$$
$$\Delta^{\mathrm{PD}} = E_{\bar{R}}^{\mathrm{PD}} \otimes_R M, \quad \Delta^{\mathrm{PD},\partial} = (\Delta^{\mathrm{PD}})^{\partial=0}, \quad \Delta_{\varpi}^{[u,v]} = E_{R,\varpi}^{[u,v]} \otimes_R M, \quad T\boldsymbol{A}_{\mathrm{cris}} = \boldsymbol{A}_{\mathrm{cris}}(\bar{R}) \otimes_{\mathbb{Z}_p} T.$$

Moreover, using the rings from the theory of (φ, Γ)-modules (see Section 2.4), we set

$$T\boldsymbol{A}^{[u,v]} = \boldsymbol{A}_{\bar{R}}^{[u,v]} \otimes_{\mathbb{Z}_p} T, \quad T\boldsymbol{A}_{\bar{R}}(r) = \boldsymbol{A}_{\bar{R}} \otimes_{\mathbb{Z}_p} T(r), \quad D_{\varpi}(r) = \boldsymbol{A}_{R,\varpi} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T(r))$$

(see Section 2.7 for $\boldsymbol{A}_{R,\varpi}$), and $D_{R_\infty}(r) = \boldsymbol{A}_{R_\infty} \otimes_{\boldsymbol{A}_{R,\varpi}} D_{\varpi}(r)$. Furthermore, we have $G = G_S$ and $\Gamma = \Gamma_S$, with C_G and C_Γ denoting the complex of continuous cochains for G and Γ, respectively. The letter "K" denotes the Koszul complex with subscripts; ∂ denotes the operators $((1+X_0)\partial/\partial X_0, \dots, X_d\partial/\partial X_d)$; the subscript Γ denotes the operators $(\gamma_0 - 1, \dots, \gamma_d - 1)$ for our choice of topological generators of Γ; the subscript $\mathrm{Lie}\,\Gamma$ denotes the operators $(\nabla_0, \dots, \nabla_d)$, with $\nabla_i = \log \gamma_i$; and the subscript ∂_A denotes $((1+X_0)\partial/\partial X_0, X_1\partial/\partial X_1, \dots, X_d\partial/\partial X_d)$ as operators on $\boldsymbol{A}_R^{[u,v]}$ and $E_R^{[u,v]}$ via the isomorphism $\iota_{\mathrm{cycl}} : R_{\varpi}^{[u,v]} \xrightarrow{\sim} \boldsymbol{A}_{R,\varpi}^{[u,v]}$. The letter "$\mathcal{K}$" denotes a certain subcomplex of the Koszul complex; see Sections 6.2–6.5.

Let us now describe the maps in the diagram. FES denotes a map coming from the fundamental exact sequences in (2-2) and (2-5). AS denotes a map originating from the Artin–Schreier theory in (2-4). PL denotes the maps coming from the filtered Poincaré lemma of Section 2.8. In the first column, the map from the first to the second row is induced by the inclusion $R_{\varpi}^{\mathrm{PD}} \subset R_{\varpi}^{[u,v]}$ (the p-power quasi-isomorphism is shown by using the operator ψ — the left inverse of φ — and p-power acyclicity of the $\psi = 0$ eigencomplexes similar to [Colmez and Nizioł 2017, §3], see Sections 5.3 and 5.4); the maps from the second to the third row and from the fourth to the third row are applications of the filtered Poincaré lemma (see Sections 5.5 and 5.6, in particular Proposition 5.28); the map from the fourth to the fifth row is given by multiplication by suitable powers of t, exploiting the relation $\partial_i = (\log \gamma_i)/t$, and the map from the sixth to the fifth row is multiplication by t^r (see Section 6.2). In the fourth column, the map from the fourth to the third row is the inflation map from Γ_S to G_S using the inclusion $\boldsymbol{A}_{R_\infty} \subset \boldsymbol{A}_{\bar{R}}$ (one could use almost étale descent to obtain the quasi-isomorphism); the map from the fifth to the fourth row uses the inclusion $\boldsymbol{A}_{R,\varpi} \subset \boldsymbol{A}_{R_\infty}$ (the quasi-isomorphism is obtained by decompletion techniques); the map from the sixth to the fifth row is the comparison between the complex computing the continuous cohomology of Γ_S and the Koszul complex as in Section 4.2. The top two maps from the first to the second column are induced by the respective inclusions $R_{\varpi}^{\mathrm{PD}} \subset E_{\bar{S}}^{\mathrm{PD}}$ and $R_{\varpi}^{[u,v]} \subset E_{\bar{S}}^{[u,v]}$. The bottom map $\mathcal{L}az$ between the first and the second column is the Lazard isomorphism discussed in Section 6.3. The bottom map from the third to the second column is induced canonically from the inclusion $\boldsymbol{A}_{R,\varpi}^{(0,v]+} \subset \boldsymbol{A}_{R,\varpi}^{[u,v]}$ (see Section 6.4). From the third to the fourth column, the top horizontal map is induced similar to (6-11) and the bottom horizontal map is induced by the inclusion $\boldsymbol{A}_{R,\varpi}^{(0,v]+} \subset \boldsymbol{A}_{R,\varpi}$ (the p-power quasi-isomorphism is proven by using the operator ψ — the left inverse of φ — and p-power acyclicity of the $\psi = 0$ eigencomplexes, a standard technique in the theory of (φ, Γ)-modules, see Sections 6.5 and 6.6).

Composition of the left vertical, bottom horizontal and right vertical arrows produces the p-power quasi-isomorphism $\alpha_r^{\mathcal{L}\mathrm{az}}$ of Theorem 1.5; composition of the top horizontal arrows gives the p-adic version of the map $\tilde{\alpha}^{\mathrm{FM}}_{r,n,S}$ of Theorem 1.10. The proof of Theorem 1.5 follows from the discussion above, and the proof of Theorem 1.10 is the content of Section 6.7.

In Section 7 we describe our global setup and define the syntomic complex with coefficients globally. In Sections 8.1 and 8.2, we describe global relative Fontaine–Laffaille modules and the global Fontaine–Messing period map as in [Tsuji 1996, §5; 1999, §3.1]. Finally, in Section 8.3, we state and prove Theorem 1.11 by first reducing the problem to the local setting via cohomological descent [Tsuji 1996; 1999] and then to the computation of Galois cohomology by a $K(\pi, 1)$-lemma [Scholze 2013], whence the claim follows from Corollary 1.6.

Notation. Let $f : C_1 \to C_2$ be a morphism of complexes. The *mapping cone* of f is the complex $\mathrm{Cone}(f)$ whose degree n part is given as $C_1^{n+1} \oplus C_2^n$ and the differential is given by

$$d(c_1, c_2) = (-d(c_1), d(c_2) - f(c_1)).$$

Furthermore, we denote the *mapping fibre* of f by

$$[C_1 \xrightarrow{f} C_2] := \mathrm{Cone}(f)[-1].$$

We also set

$$\begin{bmatrix} C_1 & \xrightarrow{f} & C_2 \\ \downarrow & & \downarrow \\ C_3 & \xrightarrow{g} & C_4 \end{bmatrix} := [[C_1 \xrightarrow{f} C_2] \to [C_3 \xrightarrow{g} C_4]].$$

In other words, this amounts to taking the total complex of the associated double complex.

2. Relative p-adic Hodge theory

In this section, we will recall some constructions and results in local relative p-adic Hodge theory from [Andreatta 2006; Andreatta and Brinon 2008; Brinon 2008] and describe some properties of the objects to be considered in Sections 3–6.

2.1. ***Setup and notation.*** Let $p \geqslant 3$ be a fixed prime and κ a perfect field of characteristic p, and set $O_F := W(\kappa)$ the ring of p-typical Witt vectors with coefficients in κ and $F := O_F[1/p]$. Let $\overline{F}$ be a fixed algebraic closure of F, so that its residue field $\bar{\kappa}$ is an algebraic closure of κ, and set $G_F := \mathrm{Gal}(\overline{F}/F)$.

Convention. We will work under the convention that $0 \in \mathbb{N}$, the set of natural numbers.

Let $Z = (Z_1, \ldots, Z_s)$ denote a set of indeterminates and, for $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}^s$ a multi-index, we will write $Z^{\boldsymbol{k}} := Z_1^{k_1} \cdots Z_s^{k_s}$. For a topological algebra Λ, we set

$$\Lambda\{Z\} := \left\{ \sum_{\boldsymbol{k} \in \mathbb{N}^s} a_{\boldsymbol{k}} Z^{\boldsymbol{k}} \;\middle|\; a_{\boldsymbol{k}} \in \Lambda \text{ and } p\text{-adically } a_{\boldsymbol{k}} \to 0 \text{ as } |\boldsymbol{k}| = \sum k_i \to +\infty \right\}.$$

Assumption 2.1. Fix $d \in \mathbb{N}$ and $X = (X_1, X_2, \ldots, X_d)$ a set of indeterminates. Let R be the p-adic completion of an étale algebra over $O_F\{X, X^{-1}\}$ with nonempty geometrically integral special fibre. In particular, we assume that $R = O_F\{X, X^{-1}\}\{Z_1, \ldots, Z_s\}/(Q_1, \ldots, Q_s)$, where $Q_i(Z_1, \ldots, Z_s)$ is in $O_F\{X, X^{-1}\}[Z_1, \ldots, Z_s]$ for $1 \leqslant i \leqslant s$ are multivariate polynomials such that $\det(\partial Q_i/\partial Z_j)_{1\leqslant i,j\leqslant s}$ is invertible in R.

Fix an algebraic closure $\overline{\mathrm{Fr}(R)}$ of $\mathrm{Fr}(R)$ containing $\bar{F}$. Let $\bar{R}$ denote the union of finite R-subalgebras $S \subset \overline{\mathrm{Fr}(R)}$ such that $S[1/p]$ is étale over $R[1/p]$. Let $\bar{\eta}$ denote a geometric point of the generic fibre $\mathrm{Sp}(R[1/p])$ (corresponding to $\overline{\mathrm{Fr}(R)}$), and let

$$G_R := \pi_1^{\text{ét}}(\mathrm{Sp}(R[1/p]), \bar{\eta}) = \mathrm{Gal}(\bar{R}[1/p]/R[1/p])$$

denote the étale fundamental group.

For $n \in \mathbb{N}$, let $F_n := F(\mu_{p^n})$. Fix some $m \in \mathbb{N}_{\geqslant 1}$ and set $K := F_m$, with ring of integers O_K. The element $\varpi = \zeta_{p^m} - 1$ in O_K is a uniformiser of K and its minimal polynomial

$$P_\varpi(X) := \frac{(1+X)^{p^m} - 1}{(1+X)^{p^{m-1}} - 1}$$

is an Eisenstein polynomial in $O_F[X]$ of degree $e := [K : F] = p^{m-1}(p-1)$. Moreover,

$$S = R[\varpi] = O_K \otimes_{O_F} R$$

is totally ramified over the prime $(p) \subset R$. Observe that we have Galois groups $G_K \lhd G_F$ and $G_S \lhd G_R$ such that $G_R/G_S = G_F/G_K = \mathrm{Gal}(K/F)$. Moreover, R and $R[\varpi]$ are *small* algebras in the sense of [Faltings 1988, §II 1(a)].

For $k \in \mathbb{N}$, let Ω_R^k denote the p-adic completion of the module of k-differentials of R relative to $\mathbb{Z}$. Then, we have $\Omega_R^1 = \bigoplus_{i=1}^d R\, d\log X_i$ and $\Omega_R^k = \bigwedge_R^k \Omega_R^1$. More explicitly, for $1 \leqslant i \leqslant d$, let us set $\partial_i := X_i d/dX_i$ as an operator on R. Then, for any f in R, its differential can be written as $df = \sum_{i=1}^d \partial_i(f)\, d\log X_i$ in Ω_R^1. Furthermore, note that $R/pR \xrightarrow{\sim} S/\varpi S$ is an isomorphism and, for any $n \in \mathbb{N}$, $R/p^n R$ is a smooth $\mathbb{Z}/p^n\mathbb{Z}$-algebra. Finally, we fix a lift $\varphi : R \to R$ of the absolute Frobenius $x \mapsto x^p$ over R/pR such that $\varphi(X_i) = X_i^p$ for $1 \leqslant i \leqslant d$.

Note that, to carry out some computations in later sections, we will need to extend our base field (hence the base ring) by adjoining a p-power root of unity (see K and $S = R[\varpi]$ above). As a consequence, we will also require period rings defined for such rings. However, we will only recall the results by fixing our base as R, because the period rings that we consider will only depend on $\bar{R}$ and we have $\bar{S} = \bar{R} \subset \overline{\mathrm{Fr}(R)} = \overline{\mathrm{Fr}(S)}$; see [Andreatta 2006; Andreatta and Brinon 2008; Brinon 2008] for general constructions.

Convention. Let A be a ring and $I \subsetneq A$ an ideal. An A-module M is said to be I-adically complete if $M \xrightarrow{\sim} \lim_n M/I^n M$ is an isomorphism.

Notation. Let A be a $\mathbb{Z}_p$-algebra. A morphism $f : M \to N$ of two A-modules is called a p^n*-isomorphism* for some $n \in \mathbb{N}$ if the kernel and cokernel of f are killed by p^n.

2.2. ***Period rings.*** Let $\mathbb{C}_p$ denote the p-adic completion of $\bar{F}$. We recall that $\bar{R}$ is the union of finite R-subalgebras $S \subset \overline{\mathrm{Fr}(R)} = \overline{\mathrm{Fr}(R[\varpi])}$ such that $S[1/p]$ is étale over $R[1/p]$. Let $\mathbb{C}^+(\bar{R})$ denote the p-adic completion of $\bar{R}$, and let $\mathbb{C}(\bar{R}) = \mathbb{C}^+(\bar{R})[1/p]$. We define the tilt of $\mathbb{C}^+(\bar{R})$ as

$$\mathbb{C}^+(\bar{R})^\flat := \lim_{x \mapsto x^p} \mathbb{C}^+(\bar{R})/p = \lim_{x \mapsto x^p} \bar{R}/p$$

and equip it with the inverse limit topology (where we equip $\bar{R}/p$ with the discrete topology), and let $\mathbb{C}(\bar{R})^\flat := \mathbb{C}^+(\bar{R})^\flat[1/p^\flat]$ for $p^\flat := (p, p^{1/p}, p^{1/p^2}, \dots) \in \mathbb{C}^+(\bar{R})^\flat$ equipped with the coarsest ring topology such that $\mathbb{C}^+(\bar{R})^\flat$ is an open subring. These rings admit a continuous action of G_R.

Let us fix $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \dots)$ in $\mathbb{C}_p^\flat$ and $X_i^\flat := (X_i, X_i^{1/p}, X_i^{1/p^2}, \dots)$ in $\mathbb{C}^+(\bar{R})^\flat$ for $1 \leqslant i \leqslant d$. Set $\boldsymbol{A}_{\mathrm{inf}}(\bar{R}) := W(\mathbb{C}^+(\bar{R})^\flat)$, the ring of p-typical Witt vectors with coefficients in $\mathbb{C}^+(\bar{R})^\flat$. The absolute Frobenius on $\mathbb{C}^+(\bar{R})^\flat$ lifts to an endomorphism $\varphi : \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \to \boldsymbol{A}_{\mathrm{inf}}(\bar{R})$, and the G_R-action extends to a continuous (for the weak topology, see [Andreatta and Iovita 2008, §2.10]) action on $\boldsymbol{A}_{\mathrm{inf}}(\bar{R})$. For $x \in \mathbb{C}^+(\bar{R})^\flat$, let $[x] = (x, 0, 0, \dots)$ in $\boldsymbol{A}_{\mathrm{inf}}(\bar{R})$ denote its Teichmüller representative. So we have $[\varepsilon]$ in $\boldsymbol{A}_{\mathrm{inf}}(\bar{R})$ with $\varphi([\varepsilon]) = [\varepsilon]^p$ and $g[\varepsilon] = [\varepsilon]^{\chi(g)}$ for g in G_R and $\chi : G_R \to \mathbb{Z}_p^\times$ the p-adic cyclotomic character. Furthermore, let $\pi := [\varepsilon] - 1$, $\pi_1 := \varphi^{-1}(\pi) = [\varepsilon^{1/p}] - 1$ and $\xi := \pi/\pi_1$. Clearly, we have that $g(\pi) = (1+\pi)^{\chi(g)} - 1$ for $g \in G_R$ and $\varphi(\pi) = (1+\pi)^p - 1$.

We will use the de Rham period rings $\boldsymbol{B}_{\mathrm{dR}}^+(\bar{R})$ and $\boldsymbol{B}_{\mathrm{dR}}(\bar{R})$ defined in [Brinon 2008, Chapitre 5] and [Abhinandan 2025, §2.1]. These are F-algebras equipped with a natural action of G_R and a G_R-stable filtration. We have that

$$t := \log[\varepsilon] = \log(1+\pi) = \sum_{k \in \mathbb{N}} (-1)^k \frac{\pi^{k+1}}{k+1}$$

converges in $\boldsymbol{B}_{\mathrm{dR}}^+(\bar{R})$ and the action on t of any g in G_R can be described by the formula $g(t) = \chi(g)t$. Moreover, we will use fat period rings $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}^+(\bar{R})$ and $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R})$ defined in [Brinon 2008, Chapitre 5] and [Abhinandan 2025, §2.1]. These are $R[1/p]$-algebras equipped with a natural action of G_R, a G_R-stable filtration and a G_R-equivariant connection satisfying Griffiths transversality with respect to the filtration. Furthermore, we have

$$(\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}^+(\bar{R}))^{\partial=0} = \boldsymbol{B}_{\mathrm{dR}}^+(\bar{R}), \quad (\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R}))^{\partial=0} = \boldsymbol{B}_{\mathrm{dR}}(\bar{R}) \quad \text{and} \quad (\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R}))^{G_R} = R[1/p].$$

We will also use the crystalline period rings $\boldsymbol{A}_{\mathrm{cris}}(\bar{R})$, $\boldsymbol{B}_{\mathrm{cris}}^+(\bar{R})$ and $\boldsymbol{B}_{\mathrm{cris}}(\bar{R})$, from [Brinon 2008, Chapitre 6] and [Abhinandan 2025, §2.2], as subrings of $\boldsymbol{B}_{\mathrm{dR}}(\bar{R})$. The ring $\boldsymbol{A}_{\mathrm{cris}}(\bar{R})$ is an O_F-algebra and $\boldsymbol{B}_{\mathrm{cris}}^+(\bar{R})$ and $\boldsymbol{B}_{\mathrm{cris}}(\bar{R})$ are F-algebras. These rings are equipped with a natural action of G_R, a G_R-stable filtration (induced from the filtration on $\boldsymbol{B}_{\mathrm{dR}}(\bar{R})$) and a G_R-equivariant Frobenius endomorphism φ. Note that t converges in $\boldsymbol{A}_{\mathrm{cris}}(\bar{R})$ and $\varphi(t) = pt$. Moreover, we will use fat period rings $\mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\bar{R})$, $\mathcal{O}\boldsymbol{B}_{\mathrm{cris}}^+(\bar{R})$ and $\mathcal{O}\boldsymbol{B}_{\mathrm{cris}}(\bar{R})$, defined in [Brinon 2008, Chapitre 6] and [Abhinandan 2025, §2.2], as subrings of $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R})$. The ring $\mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\bar{R})$ is an R-algebra and $\mathcal{O}\boldsymbol{B}_{\mathrm{cris}}^+(\bar{R})$ and $\mathcal{O}\boldsymbol{B}_{\mathrm{cris}}(\bar{R})$ are $R[1/p]$-algebras. These rings are equipped with a natural action of G_R, a G_R-stable induced filtration (from $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R})$), a G_R-equivariant Frobenius endomorphism φ and a G_R-equivariant induced connection (from $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R})$)

satisfying Griffiths transversality with respect to the filtration and commuting with φ. Taking the horizontal sections for the connection, we get

$$(\mathcal{O}\boldsymbol{A}_{\text{cris}}(\bar{R}))^{\partial=0} = \boldsymbol{A}_{\text{cris}}(\bar{R}), \quad (\mathcal{O}\boldsymbol{B}^{+}_{\text{cris}}(\bar{R}))^{\partial=0} = \boldsymbol{B}^{+}_{\text{cris}}(\bar{R}) \quad \text{and} \quad (\mathcal{O}\boldsymbol{B}_{\text{cris}}(\bar{R}))^{\partial=0} = \boldsymbol{B}_{\text{cris}}(\bar{R}),$$

and by taking G_R-invariants we get

$$(\mathcal{O}\boldsymbol{A}_{\text{cris}}(\bar{R}))^{G_R} = R \quad \text{and} \quad (\mathcal{O}\boldsymbol{B}^{+}_{\text{cris}}(\bar{R}))^{G_R} = (\mathcal{O}\boldsymbol{B}_{\text{cris}}(\bar{R}))^{G_R} = R[1/p].$$

2.2.1. *Fundamental exact sequence.* From the Artin–Schrier theory in [Andreatta and Iovita 2008, §8.1.1], we have an exact sequence

$$0 \to \mathbb{Z}_p \to \boldsymbol{A}_{\text{inf}}(\bar{R}) \xrightarrow{1-\varphi} \boldsymbol{A}_{\text{inf}}(\bar{R}) \to 0. \tag{2-1}$$

Let $r \in \mathbb{N}$, write $r = (p-1)a(r) + b(r)$, with $0 \leqslant b(r) < p-1$, and set $\mathbb{Z}_p(r)' = \mathbb{Z}_p(r)/p^{a(r)}$. From [Tsuji 1999, Theorem A3.26] and [Colmez and Nizioł 2017, Lemma 2.23], we have a p^r-exact sequence called the fundamental exact sequence in p-adic Hodge theory:

$$0 \to \mathbb{Z}_p(r)' \to \operatorname{Fil}^r \boldsymbol{A}_{\text{cris}}(\bar{R}) \xrightarrow{p^r-\varphi} \boldsymbol{A}_{\text{cris}}(\bar{R}) \to 0. \tag{2-2}$$

2.3. ***p*-adic Galois representations.** For the ring $B = \mathcal{O}\boldsymbol{B}_{\text{dR}}(\bar{R})$ and $\mathcal{O}\boldsymbol{B}_{\text{cris}}(\bar{R})$, we consider *$B$-admissible* p-adic representations in the sense of [Brinon 2008, Chapitre 8] and [Abhinandan 2025, §2.3]. We note that $\mathcal{O}\boldsymbol{B}_{\text{dR}}(\bar{R})$ is a G_R-regular $R[1/p]$-algebra. Let V be a p-adic representation of G_R, and set $\mathcal{O}\boldsymbol{D}_{\text{dR}}(V) := (\mathcal{O}\boldsymbol{B}_{\text{dR}}(\bar{R}) \otimes_{\mathbb{Q}_p} V)^{G_R}$. We say that V is de Rham if it is $\mathcal{O}\boldsymbol{B}_{\text{dR}}(\bar{R})$-admissible. The $R[1/p]$-module $\mathcal{O}\boldsymbol{D}_{\text{dR}}(V)$ is equipped with a decreasing, separated and exhaustive filtration and an integrable connection satisfying Griffiths transversality with respect to the filtration (all induced from the corresponding structures on $\mathcal{O}\boldsymbol{B}_{\text{dR}}(\bar{R}) \otimes_{\mathbb{Q}_p} V$). Furthermore, $\mathcal{O}\boldsymbol{D}_{\text{dR}}(V)$ is projective over $R[1/p]$ and of rank $\leqslant \dim(V)$. If V is de Rham, then, for all $r \in \mathbb{Z}$, the $R[1/p]$-modules $\operatorname{Fil}^r \mathcal{O}\boldsymbol{D}_{\text{dR}}(V)$ and $\operatorname{gr}^r \mathcal{O}\boldsymbol{D}_{\text{dR}}(V)$ are projective of finite type, and the collection of integers r_i for $1 \leqslant i \leqslant \dim_{\mathbb{Q}_p}(V)$ such that $\operatorname{gr}^{-r_i} \mathcal{O}\boldsymbol{D}_{\text{dR}}(V) \neq 0$ are called the *Hodge–Tate weights* of V; see [Brinon 2008, §8.3]. Moreover, we say that V is *positive* if and only if $r_i \leqslant 0$ for all $1 \leqslant i \leqslant \dim_{\mathbb{Q}_p}(V)$.

Next, we note that $\mathcal{O}\boldsymbol{B}_{\text{cris}}(\bar{R})$ is also a G_R-regular $R[1/p]$-algebra. Let V be a p-adic representation of G_R, and set $\mathcal{O}\boldsymbol{D}_{\text{cris}}(V) := (\mathcal{O}\boldsymbol{B}_{\text{cris}}(\bar{R}) \otimes_{\mathbb{Q}_p} V)^{G_R}$. We will say that V is crystalline if it is $\mathcal{O}\boldsymbol{B}_{\text{cris}}(\bar{R})$-admissible. The $R[1/p]$-module $\mathcal{O}\boldsymbol{D}_{\text{cris}}(V)$ is equipped with a Frobenius-semilinear operator φ induced from the Frobenius on $\mathcal{O}\boldsymbol{B}_{\text{cris}}(\bar{R}) \otimes_{\mathbb{Q}_p} V$, where we consider the G_R-equivariant Frobenius on $\mathcal{O}\boldsymbol{B}_{\text{cris}}(\bar{R})$. Moreover, the inclusion $\mathcal{O}\boldsymbol{B}_{\text{cris}}(\bar{R}) \subset \mathcal{O}\boldsymbol{B}_{\text{dR}}(\bar{R})$ induces an $R[1/p]$-linear inclusion $\mathcal{O}\boldsymbol{D}_{\text{cris}}(V) \subset \mathcal{O}\boldsymbol{D}_{\text{dR}}(V)$ (see [Brinon 2008, §8.2 and §8.3]), and we equip $\mathcal{O}\boldsymbol{D}_{\text{cris}}(V)$ with (induced from $\mathcal{O}\boldsymbol{D}_{\text{dR}}(V)$) filtration and connection satisfying Griffiths transversality with respect to the filtration. Additionally, we have $\partial\varphi = \varphi\partial$ over $\mathcal{O}\boldsymbol{D}_{\text{cris}}(V)$. The module $\mathcal{O}\boldsymbol{D}_{\text{cris}}(V)$ is finite projective over $R[1/p]$ of rank $\leqslant \dim(V)$. If V is crystalline, then the $R[1/p]$-linear homomorphism $1 \otimes \varphi : R[1/p] \otimes_{\varphi, R[1/p]} \mathcal{O}\boldsymbol{D}_{\text{cris}}(V) \to \mathcal{O}\boldsymbol{D}_{\text{cris}}(V)$ is an isomorphism and $\mathcal{O}\boldsymbol{D}_{\text{cris}}(V)$ is called a filtered (φ, ∂)-module.

2.4. (φ, Γ)-*modules.* In this subsection, we will briefly recall some results from the theory of relative étale (φ, Γ)-modules; see [Andreatta 2006; Andreatta and Brinon 2008; Andreatta and Iovita 2008] for details.

2.4.1. *The Galois group* Γ_R. Let $F_n = F(\mu_{p^n})$ for $n \in \mathbb{N}$, and let $F_\infty = \bigcup_n F_n$. We take R_n to be the integral closure of $R \otimes_{O_F[X^{\pm 1}]} O_{F_n}[X_1^{p^{-n}}, \ldots, X_d^{p^{-n}}]$ inside $\bar{R}[1/p]$ and set $R_\infty := \bigcup_{n \geqslant m} R_n$, noting that $F_\infty \subset R_\infty[1/p]$. From Section 2.2 recall that $\mathbb{C}(\bar{R}) = \mathbb{C}^+(\bar{R})[1/p]$ and $\mathbb{C}(\bar{R})^\flat$ denotes its tilt. The ring $\mathbb{C}(\bar{R})^\flat$ is perfect of characteristic p, and we set $\boldsymbol{A}_{\bar{R}} := W(\mathbb{C}(\bar{R})^\flat)$, the ring of p-typical Witt vectors with coefficients in $\mathbb{C}(\bar{R})^\flat$, and endow it with the weak topology; see [Andreatta and Iovita 2008, §2.10]. The absolute Frobenius over $\mathbb{C}(\bar{R})^\flat$ lifts to an endomorphism $\varphi : \boldsymbol{A}_{\bar{R}} \to \boldsymbol{A}_{\bar{R}}$, which we again call the Frobenius. The continuous action of G_R on $\mathbb{C}(\bar{R})^\flat$ extends to a continuous action on $\boldsymbol{A}_{\bar{R}}$ commuting with Frobenius. The inclusion $\bar{F} \subset \bar{R}[1/p]$ induces inclusions $\mathbb{C}_p^\flat \subset \mathbb{C}(\bar{R})^\flat$ and $\boldsymbol{A}_{\bar{F}} \subset \boldsymbol{A}_{\bar{R}}$, and the inclusion $O_{\bar{F}} \subset \bar{R}$ induces inclusions $O_{\mathbb{C}_p}^\flat \subset \mathbb{C}^+(\bar{R})^\flat$ and $\boldsymbol{A}_{\mathrm{inf}}(O_{\bar{F}}) \subset \boldsymbol{A}_{\mathrm{inf}}(\bar{R})$.

The ring $R_\infty[1/p]$ is Galois over $R[1/p]$ with Galois group $\Gamma_R := \mathrm{Gal}(R_\infty[1/p]/R[1/p])$. Consider the Galois groups

$$\Gamma_F := \mathrm{Gal}(F_\infty/F) \xrightarrow{\sim} \mathbb{Z}_p^\times, \quad \Gamma'_R := \mathrm{Gal}(R_\infty[1/p]/F_\infty R[1/p]) \xrightarrow{\sim} \mathbb{Z}_p(1)^d$$

(see [Andreatta 2006, §2.4; Brinon 2008, p. 9]), and note that we have an exact sequence

$$1 \to \Gamma'_R \to \Gamma_R \to \Gamma_F \to 1. \tag{2-3}$$

The group Γ_F can be viewed as a subgroup of Γ_R; i.e., we can take a section of the projection map in (2-3) such that, for $\gamma \in \Gamma_F$ and $g \in \Gamma'_R$, we have $\gamma g \gamma^{-1} = g^{\chi(\gamma)}$. So we can choose topological generators $\{\gamma, \gamma_1, \ldots, \gamma_d\}$ of Γ_R such that $\gamma_0 = \gamma^e$, with $\chi(\gamma_0) = \exp(p^m)$, is a topological generator of $\Gamma_K = \mathrm{Gal}(K_\infty/K)$, where $K_\infty = F_\infty$ and $e = [K : F]$. It follows that $\{\gamma_1, \ldots, \gamma_d\}$ are topological generators of Γ'_R and γ is a topological generator of Γ_F. In particular, we have the isomorphism

$$\chi : \Gamma_K = \mathrm{Gal}(F_\infty/K) \xrightarrow{\sim} 1 + p^m\mathbb{Z}_p.$$

The action of these generators on the elements of $\mathbb{C}(\bar{R})^\flat$, fixed in Section 2.2, is given as $\gamma(\varepsilon) = \varepsilon^{\chi(\gamma)}$ and $\gamma_i(\varepsilon) = \varepsilon$ for $1 \leqslant i \leqslant d$, and $\gamma_i(X_i^\flat) = \varepsilon X_i^\flat$ and $\gamma_i(X_j^\flat) = X_j^\flat$ for $i \neq j$ and $1 \leqslant j \leqslant d$.

2.4.2. *Étale* (φ, Γ_R)*-modules.* Andreatta [2006] introduced the theory of étale (φ, Γ_R)-modules for p-adic representations of G_R; see [Abhinandan 2025, §3.1] for a quick recollection. From [loc. cit.], let us recall that we have characteristic p period rings $\boldsymbol{E}^+ \subset \boldsymbol{E} \subset \mathbb{C}(\bar{R})^\flat$. Let $\bar{\pi}$ denote the reduction modulo p of $\pi \in \boldsymbol{A}_{\mathrm{inf}}(O_{F_\infty})$. Then, the characteristic p period rings above are $\bar{\pi}$-adically complete and equipped with a continuous G_R-action. Furthermore, we have rings $\boldsymbol{E}_R^+ \subset \boldsymbol{E}_R \subset \hat{R}_\infty^\flat[1/p^\flat]$, complete for the $\bar{\pi}$-adic topology and equipped with a continuous G_R-action. Moreover, we have

$$(\mathbb{C}^+(\bar{R}))^{H_R} = \hat{R}_\infty, \ \ (\mathbb{C}^+(\bar{R})^\flat)^{H_R} = \hat{R}_\infty^\flat, \ \ (\mathbb{C}(\bar{R})^\flat)^{H_R} = \hat{R}_\infty^\flat[1/p^\flat], \ \ (\boldsymbol{E}^+)^{H_R} = \boldsymbol{E}_R^+ \ \text{and} \ \boldsymbol{E}^{H_R} = \boldsymbol{E}_R.$$

In mixed characteristic, we have period rings $\boldsymbol{A}^+ \subset \boldsymbol{A} \subset W(\mathbb{C}(\bar{R})^\flat)$ equipped with an induced weak topology, an induced Frobenius endomorphism φ and a continuous G_R-action. Furthermore, we have

$\boldsymbol{A}_R^+ = \boldsymbol{A}_R \subset W(\hat{R}_\infty^\flat[1/p^\flat])$, complete for the induced weak topology and equipped with an induced Frobenius and a continuous Γ_R-action. Additionally, from [Andreatta and Iovita 2008], we have that $\boldsymbol{A}^{H_R} = \boldsymbol{A}_R$, $(\boldsymbol{A}^+)^{H_R} = \boldsymbol{A}_R^+$ and $\boldsymbol{A}/p\boldsymbol{A} = \boldsymbol{E}$, and, from [Abhinandan 2025, Remark 3.7], we have that $\boldsymbol{A}^+/p\boldsymbol{A}^+ = \boldsymbol{E}^+$.

Let D be a finitely generated $\boldsymbol{A}_R$-module equipped with a continuous (for the weak topology) and semilinear action of Γ_R and a Frobenius-semilinear and Γ_R-equivariant endomorphism φ.

Definition 2.2. The $\boldsymbol{A}_R$-module D is said to be *étale* if the linearisation of Frobenius, i.e., the natural map $1 \otimes \varphi : \boldsymbol{A}_R \otimes_{\varphi, \boldsymbol{A}_R} D \to D$, is an isomorphism.

Denote by $(\varphi, \Gamma_R)\text{-Mod}_{\boldsymbol{A}_R}^{\text{ét}}$ the category of étale (φ, Γ_R)-modules over $\boldsymbol{A}_R$ with morphisms between objects being continuous and (φ, Γ_R)-equivariant morphisms of $\boldsymbol{A}_R$-modules. Furthermore, denote by $\text{Rep}_{\mathbb{Z}_p}(G_R)$ the category of finitely generated $\mathbb{Z}_p$-modules equipped with a linear and continuous G_R-action and morphisms between objects being continuous and G_R-equivariant morphisms of $\mathbb{Z}_p$-modules. Let T denote a $\mathbb{Z}_p$-representation of G_R; then $\boldsymbol{D}(T) := (\boldsymbol{A} \otimes_{\mathbb{Z}_p} T)^{H_R}$ is an étale (φ, Γ_R)-module over $\boldsymbol{A}_R$. Furthermore, if T is finite free over $\mathbb{Z}_p$, then $\boldsymbol{D}(T)$ is finite projective over $\boldsymbol{A}_R$ of rank $\text{rk}_{\mathbb{Z}_p} T$; see [Andreatta 2006, Theorem 7.11]. Finally, the functor

$$\boldsymbol{D} : \text{Rep}_{\mathbb{Z}_p}(G_R) \to (\varphi, \Gamma_R)\text{-}\,\text{Mod}_{\boldsymbol{A}_R}^{\text{ét}}$$

induces an equivalence of categories; see [Andreatta 2006, Theorem 7.11].

2.4.3. *Overconvergent étale* (φ, Γ_R)*-modules.* In this subsection, we will quickly recall the theory of overconvergent relative étale (φ, Γ)-modules from [Andreatta and Brinon 2008], which generalises the classical results of [Cherbonnier and Colmez 1998]. Denote the natural valuation on $O_{\mathbb{C}_p}^\flat$ by $\upsilon^\flat$ and extend it to a map $\upsilon^\flat : \mathbb{C}^+(\bar{R})^\flat \to \mathbb{R} \cup \{+\infty\}$ by setting

$$\upsilon^\flat(x) = \frac{p}{p-1} \max\{n \in \mathbb{Q} \mid x \in \bar{\pi}^{-n}\mathbb{C}^+(\bar{R})^\flat\}.$$

Let $v > 0$, and let $\alpha \in O_{\mathbb{C}_p}^\flat$ such that $\upsilon^\flat(\alpha) = 1/v$. Set

$$\boldsymbol{A}_{\bar{R}}^{(0,v]} := \left\{ \sum_{k \in \mathbb{N}} p^k[x_k] \in \boldsymbol{A}_{\bar{R}} \;\middle|\; v\upsilon^\flat(x_k) + k \to +\infty \text{ when } k \to +\infty \right\},$$

$$\boldsymbol{A}_{\bar{R}}^{(0,v]+} := \left\{ \sum_{k \in \mathbb{N}} p^k[x_k] \in \boldsymbol{A}_{\bar{R}}^{(0,v]} \;\middle|\; v\upsilon^\flat(x_k) + k \geqslant 0 \right\} = p\text{-adic completion of } \boldsymbol{A}_{\text{inf}}(\bar{R})[p/[\alpha]].$$

Note that we have $\boldsymbol{A}_{\bar{R}}^{(0,v]} = \boldsymbol{A}_{\bar{R}}^{(0,v]+}[1/[p^\flat]]$. The G_R-action on $\boldsymbol{A}_{\text{inf}}(\bar{R})$ extends to these rings and it commutes with the induced Frobenius φ, where

$$\varphi(\boldsymbol{A}_{\bar{R}}^{(0,v]+}) = \boldsymbol{A}_{\bar{R}}^{(0,v/p]+} \quad \text{and} \quad \varphi(\boldsymbol{A}_{\bar{R}}^{(0,v]}) = \boldsymbol{A}_{\bar{R}}^{(0,v/p]}.$$

Moreover, we have that $\boldsymbol{A}_{\bar{R}}^{(0,v]+} \subset \boldsymbol{B}_{\text{dR}}^+(\bar{R})$ and $\boldsymbol{A}_{\bar{R}}^{(0,v]} \subset \boldsymbol{B}_{\text{dR}}(\bar{R})$ for $v \geqslant 1$; see [Colmez and Nizioł 2017, §2.4.2]. We use these embeddings to induce filtrations on $\boldsymbol{A}_{\bar{R}}^{(0,v]+}$ and $\boldsymbol{A}_{\bar{R}}^{(0,v]}$.

Definition 2.3. Define the ring of *overconvergent coefficients* as $A_{\bar{R}}^{\dagger} := \bigcup_{v \in \mathbb{Q}_{>0}} A_{\bar{R}}^{(0,v]}$. Moreover, inside $A_{\bar{R}}$, we set $A_R^{(0,v]} := A_R \cap A_{\bar{R}}^{(0,v]}$ and $A^{(0,v]} := A \cap A_{\bar{R}}^{(0,v]}$. Define

$$A_R^{\dagger} := A_R \cap A_{\bar{R}}^{\dagger} = \bigcup_{v \in \mathbb{Q}_{>0}} A_R^{(0,v]} \quad \text{and} \quad A^{\dagger} := A \cap A_{\bar{R}}^{\dagger} = \bigcup_{v \in \mathbb{Q}_{>0}} A^{(0,v]}.$$

The rings defined above are equipped with a topology described in [Andreatta and Brinon 2008, §4]. We have an embedding $A_{\bar{R}}^{\dagger} \subset A_{\bar{R}}$ compatible with the weak topology on $A_{\bar{R}}$. Furthermore, $A_{\bar{R}}^{\dagger}$ is stable under the induced Frobenius φ and the G_R-action which commutes with φ; see [Andreatta 2006, Proposition 7.2]. Finally, all rings appearing above are equipped with a (φ, G_R)-action (induced from $A_{\bar{R}}$) and from [Andreatta and Iovita 2008, Lemma 2.11] we have

$$(A^{(0,v]})^{H_R} = A_R^{(0,v]}, \quad (A^{\dagger})^{H_R} = A_R^{\dagger} \quad \text{and} \quad A_R^{\dagger}/pA_R^{\dagger} = E_R.$$

Define $(\varphi, \Gamma_R)\text{-Mod}^{\text{ét}}_{A_R^{\dagger}}$ to be the category of étale (φ, Γ_R)-modules over $A_R^{\dagger}$, similar to Definition 2.2. Let $T \in \text{Rep}_{\mathbb{Z}_p}(G_R)$; then $D^{\dagger}(T) := (A^{\dagger} \otimes_{\mathbb{Z}_p} T)^{H_R}$ is an étale (φ, Γ_R)-module over $A_R^{\dagger}$. Moreover, if T is finite free over $\mathbb{Z}_p$, then $D^{\dagger}(T)$ is finite projective over $A_R^{\dagger}$ of rank $\text{rk}_{\mathbb{Z}_p} T$. The functor

$$D^{\dagger} : \text{Rep}_{\mathbb{Z}_p}(G_R) \to (\varphi, \Gamma_R)\text{-Mod}^{\text{ét}}_{A_R^{\dagger}}$$

induces an equivalence of categories; see [Andreatta and Brinon 2008, Théorème 4.35]. Moreover, extension of scalars along $A_R^{\dagger} \to A_R$ gives the following isomorphism of étale (φ, Γ_R)-modules over A_R

$$A_R \otimes_{A_R^{\dagger}} D^{\dagger}(T) \xrightarrow{\sim} D(T).$$

Finally, we introduce the analytic rings to be used in Section 5. Let $0 < u \leqslant v$ and $\alpha, \beta \in O_{\mathbb{C}_p}^{\flat}$ such that $v^{\flat}(\alpha) = 1/v$ and $v^{\flat}(\beta) = 1/u$. We set $A_{\bar{R}}^{[u]}$ to be the p-adic completion of $A_{\text{inf}}(\bar{R})[[\beta]/p]$ and $A_{\bar{R}}^{[u,v]}$ to be the p-adic completion of $A_{\text{inf}}(\bar{R})[p/[\alpha], [\beta]/p]$. The G_R-action on $A_{\text{inf}}(\bar{R})$ extends to these rings and commutes with the extension of Frobenius to these rings, denoted again by φ. For the homomorphism φ, we have

$$\varphi(A_{\bar{R}}^{[u]}) = A_{\bar{R}}^{[u/p]} \quad \text{and} \quad \varphi(A_{\bar{R}}^{[u,v]}) = A_{\bar{R}}^{[u/p,v/p]}.$$

Moreover, we have inclusions $A_{\bar{R}}^{[u]} \subset B_{\text{dR}}^{+}(\bar{R})$ for $u \leqslant 1$ and $A_{\bar{R}}^{[u,v]} \subset B_{\text{dR}}^{+}(\bar{R})$ for $u \leqslant 1 \leqslant v$; see [Colmez and Nizioł 2017, §2.4.2]. We use these embeddings to induce filtrations on $A_{\bar{R}}^{[u]}$ and $A_{\bar{R}}^{[u,v]}$.

2.4.4. *Fundamental exact sequences.* The Artin–Schreier exact sequence in (2-1) can be upgraded to the following exact sequences (see [Andreatta and Iovita 2008, §8.1; Colmez and Nizioł 2017, Lemma 2.23]):

$$\begin{aligned} &0 \to \mathbb{Z}_p \to A_{\bar{R}} \xrightarrow{1-\varphi} A_{\bar{R}} \to 0, \\ &0 \to \mathbb{Z}_p \to A_{\bar{R}}^{(0,v]+} \xrightarrow{1-\varphi} A_{\bar{R}}^{(0,v/p]+} \to 0 \quad \text{for } v > 0. \end{aligned} \tag{2-4}$$

Furthermore, for $0 < u \leqslant 1 \leqslant v$, the p^r-exact sequence in (2-2) can be upgraded to a p^{4r}-exact sequence (see [Colmez and Nizioł 2017, Lemma 2.23]):

$$0 \to \mathbb{Z}_p(r) \to \text{Fil}^r A_{\bar{R}}^{[u,v]} \xrightarrow{p^r - \varphi} A_{\bar{R}}^{[u,v/p]} \to 0. \tag{2-5}$$

2.4.5. *The operator ψ.* Let us define a left inverse ψ of the Frobenius operator φ on the ring $\boldsymbol{A}$. From [Andreatta and Brinon 2008, Corollaire 4.10], note that the $\boldsymbol{A}$-module $\varphi^{-1}(\boldsymbol{A})$ is free with a basis given as $u_{\alpha/p} = (1+\pi)^{\alpha_0/p}[X_1^\flat]^{\alpha_1/p}\cdots[X_d^\flat]^{\alpha_d/p}$, where $\alpha = (\alpha_0, \ldots, \alpha_d)$ is a $(d+1)$-tuple with $\alpha_i \in \{0, 1, \ldots, p-1\}$ for each $0 \leqslant i \leqslant d$ (to get this statement from [loc. cit.], one should replace $\varphi^{-1}(\boldsymbol{A})$ by $\boldsymbol{A}$ there and take the p-th root of the basis elements). Define an operator (a left inverse of φ) denoted by $\psi : \boldsymbol{A} \to \boldsymbol{A}$ and given by the formula

$$x \mapsto \frac{1}{p^{d+1}} \circ \operatorname{Tr}_{\varphi^{-1}(\boldsymbol{A})/\boldsymbol{A}} \circ \varphi^{-1}(x).$$

Proposition 2.4 [Andreatta and Brinon 2008, §4.8]. *Let $x \in \boldsymbol{A}$, and write $\varphi^{-1}(x) = \sum_\alpha x_\alpha u_{\alpha/p}$. Then we have $\psi(x) = x_0$. Moreover, for the operator ψ, we have $\psi \circ \varphi = \mathrm{id}$. Furthermore, ψ commutes with the action of G_R, $\psi(\boldsymbol{A}^+) \subset \boldsymbol{A}^+$ and $\psi(\boldsymbol{A}^\dagger) \subset \boldsymbol{A}^\dagger$.*

2.5. ***Crystalline coordinates.*** Here we introduce good "crystalline" coordinates; see [Abhinandan 2025, §3.2]. Let $r_\varpi^+ = O_F[\![X_0]\!]$ and $r_\varpi = O_F[\![X_0]\!]\{X_0^{-1}\}$. Sending X_0 to $\varpi = \zeta_{p^m} - 1$ induces a surjective ring homomorphism $r_\varpi^+ \twoheadrightarrow O_K$, whose kernel is generated by a degree $e = [K : F] = p^{m-1}(p-1)$ Eisenstein polynomial $P_\varpi = P_\varpi(X_0)$. Let $R_{\varpi,\square}^+$ denote the completion of $O_F[X_0, X, X^{-1}]$ for the (p, X_0)-adic topology. Sending X_0 to ϖ induces a surjective ring homomorphism $R_{\varpi,\square}^+ \twoheadrightarrow O_K\{X, X^{-1}\}$, whose kernel is again generated by P_ϖ. Recall that R is étale over $O_F\{X, X^{-1}\}$ and we have multivariate polynomials $Q_i(Z_1, \ldots, Z_s) \in O_F\{X, X^{-1}\}[Z_1, \ldots, Z_s]$ for $1 \leqslant i \leqslant s$ such that $\det(\partial Q_i/\partial Z_j)$ is invertible in R. Set R_ϖ^+ to be the quotient of the (p, X_0)-adic completion of $R_{\varpi,\square}^+[Z_1, \ldots, Z_s]$ by the ideal $(Q_1, \ldots, Q_s)$. Again, we have that $\det(\partial Q_i/\partial Z_j)$ is invertible in R_ϖ^+ (since $R \hookrightarrow R_\varpi^+$). Hence R_ϖ^+ is étale over $R_{\varpi,\square}^+$ and smooth over O_F. Sending X_0 to ϖ induces a surjective ring homomorphism $R_\varpi^+ \twoheadrightarrow R[\varpi]$, whose kernel is again generated by P_ϖ. Since $P_\varpi \equiv X_0^e \bmod p$, we have

$$R_\varpi^+[P_\varpi^k/k!]_{k\in\mathbb{N}} = R_\varpi^+[X_0^k/[k/e]!]_{k\in\mathbb{N}}.$$

Set R_ϖ^{PD} to be the p-adic completion of $R_\varpi^+[P_\varpi^k/k!]_{k\in\mathbb{N}}$.

Recall that Ω_R^1 denotes the p-adic completion of the module of differentials of R relative to $\mathbb{Z}$, and we have

$$\Omega_R^1 = \bigoplus_{i=1}^d R\, d\log X_i \quad \text{and} \quad \Omega_R^k = \bigwedge_R^k \Omega_R^1.$$

Moreover, since R_ϖ^+ is étale over $R_{\varpi,\square}^+$, for $S = R_{\varpi,\square}^+$ or R_ϖ^+, we have

$$\Omega_S^1 = S\frac{dX_0}{1+X_0} \oplus \left(\bigoplus_{i=1}^d S\, d\log X_i\right).$$

Definition 2.5. For $0 < u \leqslant v$, define $R_\varpi^{(0,v]+}$ to be the p-adic completion of $R_\varpi^+[p^{\lceil vk/e\rceil}/X_0^k]_{k\in\mathbb{N}}$ and set $R_\varpi^{(0,v]} := R_\varpi^{(0,v]+}[1/X_0]$. Furthermore, define $R_\varpi^{[u]}$ to be the p-adic completion of $R_\varpi^+[X_0^k/p^{\lfloor uk/e\rfloor}]_{k\in\mathbb{N}}$, define $R_\varpi^{[u,v]}$ to be the p-adic completion of $R_\varpi^+[X_0^k/p^{\lfloor uk/e\rfloor}, p^{\lceil vk/e\rceil}/X_0^k]_{k\in\mathbb{N}}$ and set R_ϖ as the p-adic completion of $R_\varpi^+[1/X_0]$. We will write $R_\varpi^\star$ for $\star \in \{\ \ , +, \mathrm{PD}, [u], (0, v]+, [u, v]\}$ and, for the arithmetic case, i.e., $R = O_F$, we will write $r_\varpi^\star$ instead. Going from R_ϖ^+ to $R_\varpi^\star$ only involves the arithmetic variable X_0, so we have $R_\varpi^\star = r_\varpi^\star \widehat{\otimes}_{r_\varpi^+} R_\varpi^+$, where $\widehat{\otimes}$ denotes the p-adic completion of the usual tensor product.

Remark 2.6. Unless otherwise stated, we will assume $(p-1)/p \leqslant u \leqslant v/p < 1 < v < p$: for example, we can take $u = (p-1)/p$ and $v = p-1$.

Definition 2.7. Define a filtration on the rings in Definition 2.5 as follows:

(i) Let $S = R_\varpi^{(0,v]+}$ $(v < 1)$, $R_\varpi^{(0,v]}$ $(v < 1)$, $R_\varpi^{[u,v]}$ $(1 \notin [u, v])$ or R_ϖ. As P_ϖ is invertible in $S[1/p]$, we put the trivial filtration on S.

(ii) Let S be the placeholder for all the remaining rings in Definition 2.5; in particular, we have that P_ϖ is not invertible in $S[1/p]$. Then, there is a natural embedding

$$S \to R[\varpi, 1/p][\![P_\varpi]\!] = R[\varpi, 1/p][\![X_0 - \varpi]\!],$$

obtained by completing $S[1/p]$ for the P_ϖ-adic topology and where we note that P_ϖ and $X_0 - \varpi$ generate the same ideal in $R[\varpi, 1/p][\![P_\varpi]\!]$. We use this embedding to endow S with a natural filtration $\mathrm{Fil}^k S := S \cap P_\varpi^k R[\varpi, 1/p][\![P_\varpi]\!]$ for all $k \in \mathbb{Z}$.

Remark 2.8. Let us describe the filtration on the rings of Definition 2.7 (ii) more concretely. Note that $\mathrm{Fil}^k S = S$ for $k \leqslant 0$. For any $k \in \mathbb{N}$, the ideal $\mathrm{Fil}^k R_\varpi^{\mathrm{PD}} \subset R_\varpi^{\mathrm{PD}}$ is topologically generated by the elements $P_\varpi^n/n!$ for $n \geqslant k$; i.e., $\mathrm{Fil}^k R_\varpi^{\mathrm{PD}}$ is the closure of the ideal generated by such elements. Similarly, the ideal $\mathrm{Fil}^k R_\varpi^{[u]} \subset R_\varpi^{[u]}$ is topologically generated by the elements $P_\varpi^n/p^{\lfloor nu \rfloor}$ for $n \geqslant k$. Using this description, an easy computation shows that $\mathrm{Fil}^k R_\varpi^{[u]} \subset (P_\varpi/p)^k R_\varpi^{[u]}$. On the other hand, we have that $\mathrm{Fil}^k R_\varpi^{(0,v]+} = P_\varpi^k R_\varpi^{(0,v]+}$. By definition, note that $R_\varpi^{[u,v]} = R_\varpi^{[u]} + R_\varpi^{(0,v]+}$, so we get that the ideal $\mathrm{Fil}^k R_\varpi^{[u,v]} \subset R_\varpi^{[u,v]}$ is topologically generated by $(\mathrm{Fil}^k R_\varpi^{[u]} + \mathrm{Fil}^k R_\varpi^{(0,v]+}) R_\varpi^{[u,v]}$.

The following claim easily follows from Remark 2.8.

Lemma 2.9 [Colmez and Nizioł 2017, Lemma 2.6]. *For any $k \in \mathbb{N}$ and $f \in R_\varpi^{[u]}$, we can write $f = f_1 + f_2$ with $f_1 \in \mathrm{Fil}^k R_\varpi^{[u]}$ and $f_2 \in (1/p^{\lfloor ku \rfloor}) R_\varpi^+$.*

2.6. ***Cyclotomic Frobenius.*** In this subsection, we will define a (cyclotomic) Frobenius endomorphism and its left inverse on the rings studied in the previous section; see [Abhinandan 2025, §3.3].

Definition 2.10. Over $R_{\varpi,\square}^+$, define the (cyclotomic) Frobenius as a lift of the absolute Frobenius modulo p, denoted by $\varphi : R_{\varpi,\square}^+ \to R_{\varpi,\square}^+$ and sending $X_0 \mapsto (1 + X_0)^p - 1$ and $X_i \mapsto X_i^p$ for $1 \leqslant i \leqslant d$. Clearly, we have that $\varphi(x) - x^p$ is in $pR_{\varpi,\square}^+$ for any x in $R_{\varpi,\square}^+$. Using [Colmez and Nizioł 2017, Proposition 2.1], the Frobenius extends to an endomorphism $\varphi : R_\varpi^+ \to R_\varpi^+$. Finally, by continuity, the Frobenius admits unique extensions

$$R_\varpi^{\mathrm{PD}} \to R_\varpi^{\mathrm{PD}}, \quad R_\varpi^{[u]} \to R_\varpi^{[u]}, \quad R_\varpi^{(0,v]+} \to R_\varpi^{(0,v/p]+}, \quad R_\varpi^{[u,v]} \to R_\varpi^{[u,v/p]} \quad \text{and} \quad R_\varpi \to R_\varpi.$$

Recall that

$$r_{\varpi}^{[u]} = \left\{ \sum_{k\in\mathbb{N}} a_k p^{-\lfloor ku/e \rfloor} X_0^k \;\middle|\; a_k \in O_F \text{ goes to } 0 \text{ as } i \to +\infty \right\}.$$

Denote by $\upsilon_{X_0} : r_{\varpi}^{[u]} \to \mathbb{N} \cup \{+\infty\}$ the valuation relative to X_0; i.e., if $f = \sum b_k X_0^k$, then

$$\upsilon_{X_0}(f) = \inf\{k \in \mathbb{N} \mid b_k \neq 0\}.$$

For $N \in \mathbb{N}$, we set

$$r_{\varpi,N}^{[u]} := \{f \in r_{\varpi}^{[u]} \mid \upsilon_{X_0}(f) \geqslant N\}$$

and define $R_{\varpi,N}^{[u]}$ to be the topological closure of $r_{\varpi,N}^{[u]} \otimes_{r_{\varpi}^+} R_{\varpi}^+ \subset R_{\varpi}^{[u]}$.

Lemma 2.11. *Let $s \in \mathbb{Z}$ and $N \in \mathbb{N}_{\geqslant 1}$ such that $N \geqslant se/u(p-1)$; then $1 - p^{-s}\varphi$ is bijective on $R_{\varpi,N}^{[u]}$.*

Proof. The claim follows from [Colmez and Nizioł 2017, Lemma 3.1], where, by explicit computations, one shows that $p^{-ks}\varphi^k(R_{\varpi,N}^{[u]}) \subset p^{n(k)} R_{\varpi,N}^{[u]}$, where $n(k)$ depends on k and goes to $+\infty$ as $k \to +\infty$. So it follows that the series of operators $\sum_{k\in\mathbb{N}} p^{-ks}\varphi^k$ converge as an inverse to $1 - p^{-s}\varphi$ on $R_{\varpi,N}^{[u]}$. □

2.6.1. *The operator ψ.* Set $u_\alpha := (1+X_0)^{\alpha_0} X_1^{\alpha_1} \cdots X_d^{\alpha_d}$, where $\alpha = (\alpha_0, \ldots, \alpha_d)$ is a $(d+1)$-tuple with $\alpha_i \in \{0, \ldots, p-1\}$ for each $0 \leqslant i \leqslant d$. Over the ring R_ϖ, we have O_F-linear differential operators $\partial_0 = (1+X_0)d/dX_0$ and $\partial_i = X_i d/dX_i$ for $1 \leqslant i \leqslant d$. Therefore, for $0 \leqslant i \leqslant d$, we have that $\partial_i u_\alpha = \alpha_i u_\alpha$ and $\varphi(u_\alpha) = u_\alpha^p$.

Lemma 2.12 [Colmez and Nizioł 2017, Proposition 2.15]. *Any x in R_ϖ/p can be uniquely written as $x = \sum_\alpha c_\alpha(x)$, with $\partial_i \circ c_\alpha(x) = \alpha_i c_\alpha(x)$ for $0 \leqslant i \leqslant d$. Moreover, there exists a unique x_α in R_ϖ/p such that $c_\alpha(x) = x_\alpha^p u_\alpha$. Furthermore, if x is in R_ϖ^+/p, then $c_\alpha(x)$ belongs to R_ϖ^+/p.*

Proposition 2.13. *Any x in R_ϖ can be uniquely written as $x = \sum_\alpha c_\alpha(x)$, with $c_\alpha(x)$ in $\varphi(R_\varpi)u_\alpha$. Moreover, if x is in R_ϖ^+ with $c_\alpha(x) = \varphi(x_\alpha)u_\alpha$, then $c_\alpha(x)$ belongs to R_ϖ^+ for all α, and $\partial_i c_\alpha(x) - \alpha_i c_\alpha(x)$ belongs to pR_ϖ^+ for $0 \leqslant i \leqslant d$. Finally, if x is in $R_\varpi^{(0,v]+}$, then $c_\alpha(x)$ is in $R_\varpi^{(0,v]+}$ for all α.*

Proof. The first two claims follow from Lemma 2.12 and the last from [Colmez and Nizioł 2017, Proposition 2.15]. □

Definition 2.14. Define the left inverse ψ of the Frobenius φ on $S = R_\varpi^+$ or $S = R_\varpi$ by the formula $\psi(x) = \varphi^{-1}(c_0(x))$. Since R_ϖ is an extension of degree p^{d+1} of $\varphi(R_\varpi)$, with basis the u_α, and since $\varphi(u_\alpha) = u_\alpha^p$ for all α, we have that $\mathrm{Tr}_{R_\varpi/\varphi(R_\varpi)}(u_\alpha) = 0$ if $\alpha \neq 0$, and we can define ψ intrinsically as

$$\psi(x) := \frac{1}{p^{d+1}} \varphi^{-1} \circ \mathrm{Tr}_{R_\varpi/\varphi(R_\varpi)}(x).$$

The operator ψ defined above is closely related to the operator defined in Proposition 2.4 (also denoted by ψ; the relation will become clear in Section 2.7). Note that ψ is not a ring morphism; it is a left inverse to φ and, more generally, we have $\psi(\varphi(x)y) = x\psi(y)$. Also, we have $\partial_i \circ \varphi = p\varphi \circ \partial_i$ and $\partial_i \circ \psi = p^{-1}\psi \circ \partial_i$ for $i = 0, 1, \ldots, d$. Indeed, the first equality can be obtained by checking on the basis elements u_α and the second equality is obtained by an easy computation using Proposition 2.13.

For any $k \in \mathbb{N}$, we can write $X_0^k = \sum_{j=0}^{p-1} \varphi(a_{j,k})(1+X_0)^j$ for some $a_{j,k}$ in R_ϖ^+. Therefore, by continuity, we obtain the following.

Lemma 2.15. (i) *The definition of ψ extends to surjective maps*

$$R_\varpi^{(0,v]+} \to R_\varpi^{(0,pv]+}, \quad R_\varpi^{[u]} \to R_\varpi^{[pu]} \quad \text{and} \quad R_\varpi^{[u,v]} \to R_\varpi^{[pu,pv]}.$$

(ii) *For the same reasons, the maps $x \mapsto c_\alpha(x)$ also extend and lead to decompositions $S = \bigoplus_\alpha S_\alpha$, where $S_\alpha = S \cap \varphi(R_\varpi)u_\alpha$ for $S = R_\varpi^\star$, with $\star \in \{\ \ , +, [u], (0, v]+, [u, v]\}$. Since $\psi(x) = \varphi^{-1}(c_0(x))$, we have that $S^{\psi=0} = \bigoplus_{\alpha\neq 0} S_\alpha$.*

Lemma 2.16. *Let $S = R_\varpi^\star$ for $\star \in \{\ \ , +, [u], (0, v]+, [u, v]\}$. Then, for $0 \leqslant i \leqslant d$, the operator ∂_i on $S_\alpha^\star / pS_\alpha^\star$ is given by multiplication by α_i, where α_i is the i-th entry in $\alpha = (\alpha_0, \dots, \alpha_d)$.*

Proof. If $\star \in \{\ \ , +\}$, then the claim was already shown in Proposition 2.13. For $\star \in \{[u], (0, v]+, [u, v]\}$, the elements of $S_\alpha^\star$ are those of the form $\sum_{k\in\mathbb{Z}} p^{r_k} X_0^k x_k$, where $x_k \in S^+$ goes to 0 when $k \to +\infty$ and r_k is determined by "$\star$". Let

$$x = \sum_{k\in\mathbb{Z}} p^{r_k} X_0^k x_k.$$

Then, note that, for $1 \leqslant i \leqslant d$, we have that $\partial_i(X_0^k a_k) - \alpha_i X_0^k a_k = X_0^k(\partial_i(a_k) - \alpha_i a_k)$ belongs to pS^+ by Proposition 2.13. Therefore, the claim follows for all $1 \leqslant i \leqslant d$ and $\star \in \{\ \ , +, [u], (0, v]+, [u, v]\}$. Next, we will look at the case of $i = 0$. We first assume that x is in $S^{[u]}$ and write

$$x = \sum_{k\in\mathbb{N}} p^{r_k} x_k \sum_{j=0}^{p-1} \varphi(a_{j,k})(1+X_0)^j \quad \text{for some } a_{j,k} \in S^+.$$

Then,

$$c_\alpha(x) = \sum_{j=0}^{p-1} \sum_{k\in\mathbb{N}} p^{r_k} \varphi(a_{j,k}) c_{(\alpha_0-j,\alpha_1,\dots,\alpha_d)}(x_k)(1+X_0)^j,$$

where $\alpha_0 - j$ denotes its value modulo p. Since $\partial_0(c_{(\alpha_0-j,\alpha_1,\dots,\alpha_d)}(x_k)) - (\alpha_0 - j)c_{(\alpha_0-j,\alpha_1,\dots,\alpha_d)}(x_k)$ belongs to pS^+ and $\partial_0 \circ \varphi = p\varphi \circ \partial_0$, we get the desired conclusion for $i = 0$ and x in $S^{[u]}$. Next, assume that x is in $S^{(0,v]+}$ and, using the result for S, we get that $\partial_0(x) - \alpha_0 x$ belongs to $pS \cap S^{(0,v]+} = pS^{(0,v]+}$. Finally, by combining the results for $S^{[u]}$ and $S^{(0,v]+}$, we get the conclusion for any x in $S^{[u,v]}$. This allows us to conclude. □

Proposition 2.17. *Assume that $v < p$.*

(i) *Let $x \in R_\varpi^{\psi=0}$, then $X_0^k \psi(x) = \psi(\varphi(X_0)^k x)$ for all $k \in \mathbb{Z}$.*

(ii) $\psi(X_0^{-pN} R_\varpi^{(0,v/p]+}) \subset X_0^{-N} R_\varpi^{(0,v]+}$ *for all $N \in \mathbb{N}$.*

(iii) *The natural map $\bigoplus_{\alpha\neq 0} \varphi(R_\varpi^{(0,v]+})u_\alpha \to (R_\varpi^{(0,v/p]+})^{\psi=0}$ is an isomorphism.*

Proof. The claim in (i) follows from an elementary computation. Claims (ii) and (iii) follow from [Colmez and Nizioł 2017, Proposition 2.16]. □

2.7. ***Cyclotomic embedding.*** In this subsection, we will describe the relationships between period rings discussed in Sections 2.2 and 2.4 as well as the ring $R_\varpi^\star$, where $\star \in \{\ \ , +, \mathrm{PD}\}$. Define a morphism of rings $\iota_{\mathrm{cycl}} : R_{\varpi,\square}^+ \to \boldsymbol{A}_{\mathrm{inf}}(\bar{R})$ by sending $X_0 \mapsto \pi_m = \varphi^{-m}(\pi)$ and $X_i \mapsto [X_i^\flat]$ for $1 \leqslant i \leqslant d$. The map ι_{cycl} admits a unique extension to an embedding $R_\varpi^+ \to \boldsymbol{A}_{\mathrm{inf}}(\bar{R})$ such that $\theta \circ \iota_{\mathrm{cycl}}$ is the projection $R_\varpi^+ \twoheadrightarrow R[\varpi]$; see [Abhinandan 2025, Lemma 3.12]. This embedding commutes with the respective Frobenii; i.e., $\iota_{\mathrm{cycl}} \circ \varphi = \varphi \circ \iota_{\mathrm{cycl}}$. By continuity, the morphism ι_{cycl} extends to embeddings

$$R_\varpi^{\mathrm{PD}} \subset \boldsymbol{A}_{\mathrm{cris}}(\bar{R}), \quad R_\varpi^{[u]} \subset \boldsymbol{A}_{\bar{R}}^{[u]}, \quad R_\varpi^{(0,v]+} \subset \boldsymbol{A}_{\bar{R}}^{(0,v]+}, \quad R_\varpi^{[u,v]} \subset \boldsymbol{A}_{\bar{R}}^{[u,v]} \quad \text{and} \quad R_\varpi \subset \boldsymbol{A}_{\bar{R}}.$$

Denote by $\boldsymbol{A}_{R,\varpi}^\star$ the image of $R_\varpi^\star$ under ι_{cycl}. These rings are stable under the action of G_R and the action factors through Γ_R; we equip these rings with the induced action of Γ_R. Moreover, for $\star \in \{+, \mathrm{PD}, [u], [u, v], (0, v]+\}$, we equip $\boldsymbol{A}_{R,\varpi}^\star$ with a filtration using Definition 2.7 and ι_{cycl}. It is easy to see that, for $u \leqslant 1 \leqslant v$, the filtration on $\boldsymbol{A}_{R,\varpi}^\star$ coincides with the filtration induced via the embedding $\boldsymbol{A}_{R,\varpi}^\star \subset \boldsymbol{B}_{\mathrm{dR}}^+(\bar{R})$, where we consider the natural filtration on $\boldsymbol{B}_{\mathrm{dR}}^+(\bar{R})$; see Section 2.2. From [Colmez and Nizioł 2017, §2.4.2], note that we have (φ, Γ_R)-equivariant inclusions $\boldsymbol{A}_{R,\varpi}^{[u']} \subset \boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \subset \boldsymbol{A}_{R,\varpi}^{[u]}$ for $u \geqslant 1/(p-1)$ and $u' \leqslant 1/p$.

Note that the preceding discussion works well for $R[\varpi]$, where $\varpi = \zeta_{p^m} - 1$ with $m \geqslant 1$. For R, one can repeat the constructions above to obtain the period ring $\boldsymbol{A}_R^+ \subset \boldsymbol{A}_{R,\varpi}^+$ (see [Abhinandan 2025, §3.3.2]) equipped with an induced filtration $\mathrm{Fil}^k \boldsymbol{A}_R^+ = \boldsymbol{A}_R^+ \cap \mathrm{Fil}^k \boldsymbol{A}_{R,\varpi}^+ = \pi^k \boldsymbol{A}_R^+$ (see [Abhinandan 2025, Lemma 3.17]). We recall the following.

Lemma 2.18 [Abhinandan 2025, Lemma 3.14]. *The element t/π is a unit in*

$$\boldsymbol{A}_{F,\varpi}^{\mathrm{PD}} \subset \boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \subset \boldsymbol{A}_{R,\varpi}^{[u]} \subset \boldsymbol{A}_{R,\varpi}^{[u,v]}.$$

Lemma 2.19. *For $k \in \mathbb{Z}$ and $\star \in \{+, \mathrm{PD}, [u], [u, v]\}$, we have*

$$\mathrm{Fil}^k \boldsymbol{A}_{R,\varpi}^\star \cap \pi \boldsymbol{A}_{R,\varpi}^\star = \pi \,\mathrm{Fil}^{k-1} \boldsymbol{A}_{R,\varpi}^\star$$

as submodules of $\boldsymbol{A}_{R,\varpi}^\star$.

Proof. Let

$$A = \boldsymbol{A}_{R,\varpi}^\star \quad \text{and} \quad B = R[\varpi, 1/p][\![P_\varpi]\!] = R[\varpi, 1/p][\![X_0 - \varpi]\!]$$

(see Definition 2.7 for the latter ring), where $\varpi = \zeta_{p^m} - 1$. Using the inverse of the isomorphism $\iota_{\mathrm{cycl}} : R_\varpi^\star \xrightarrow{\sim} \boldsymbol{A}_{R,\varpi}^\star = A$, we may regard A as a subring of B. Now, we will prove the claim by induction on k. Note that the claim is trivial for $k \leqslant 0$ and, for $k = 1$, we have that $\mathrm{Fil}^k A \cap \pi A = \pi A$. So, let $k \in \mathbb{N}_{\geqslant 2}$, and assume that the claim is true for $k-1$; i.e., $\mathrm{Fil}^{k-1} A \cap \pi A = \pi \,\mathrm{Fil}^{k-2} A$. Now, note that

$$\mathrm{Fil}^k A \cap \pi A = \mathrm{Fil}^k A \cap \mathrm{Fil}^{k-1} A \cap \pi A = \mathrm{Fil}^k A \cap \pi \,\mathrm{Fil}^{k-2} A.$$

In particular, to get the claim, it is enough to show that $\mathrm{Fil}^k A \cap \pi \,\mathrm{Fil}^{k-2} A = \pi \,\mathrm{Fil}^{k-1} A$. Let x be an element of $\mathrm{Fil}^k A \cap \pi \,\mathrm{Fil}^{k-2} A$, and write $x = \pi y$ for some y in $\mathrm{Fil}^{k-2} A$. From the description of the filtration on A in Definition 2.7, it follows that we can write $x = \xi^k x'$ and $y = \xi^{k-2} y'$ for some x' and y'

in B (note that $\iota_{\mathrm{cycl}}(P_\varpi) = \xi$). Since B is ξ-torsion free and $\pi = \xi\pi_1$, we get that $\xi x' = \pi_1 y'$ in B. But we have

$$\pi_1 = (1+\pi_m)^{p^{m-1}} - 1 = (\pi_m - \varpi + \zeta_{p^m})^{p^{m-1}} - 1 = (\pi_m - \varpi)z + \zeta_p - 1$$

for some z in B and $\zeta_p = \zeta_{p^m}^{p^{m-1}}$ (note that $\pi_m = \iota_{\mathrm{cycl}}(X_0)$). Moreover, from Definition 2.7, recall that ξ and $\pi_m - \varpi$ generate the same ideal in B. Therefore, we obtain that $(\zeta_p - 1)y' = \xi x' - (\pi_m - \varpi)zy'$ is an element of ξB. As $(\zeta_p - 1)$ is a unit in B, it follows that we have $y' = \xi y''$ for some y'' in B. So, we can write $y = \xi^{k-2}y' = \xi^{k-1}y''$ and see that it belongs to $\xi^{k-1}B \cap A = \mathrm{Fil}^{k-1} A$. Hence $x = \pi y$ is an element of $\pi \,\mathrm{Fil}^{k-1} A$; in particular, $\mathrm{Fil}^k A \cap \pi \,\mathrm{Fil}^{k-2} A \subset \pi \,\mathrm{Fil}^{k-1} A$. The other inclusion, i.e., $\pi \,\mathrm{Fil}^{k-1} A \subset \mathrm{Fil}^k A \cap \pi \,\mathrm{Fil}^{k-2} A$, is obvious. □

Lemma 2.20 [Colmez and Nizioł 2017, Lemma 2.35]. *If $v < p$, then the following hold*:

(i) *The element $\pi_m^{-p^{m-1}}\pi_1$ is a unit in $\boldsymbol{A}_{R,\varpi}^{(0,v]+}$.*

(ii) *In $\boldsymbol{A}_{R,\varpi}^{(0,v]+}$, the element p is divisible by $\pi_m^{\lfloor (p-1)p^{m-1}/v\rfloor}$ and hence also by $\pi_m^{(p-1)p^{m-2}}$.*

(iii) *Let $v = p-1$. Then $\pi_m^{-p^m}\pi$ is a unit in $\boldsymbol{A}_{R,\varpi}^{(0,v/p]+}$ and $p/\pi \in \boldsymbol{A}_{R,\varpi}^{(0,v/p]+}$.*

Next, we prove some claims for the action of Γ_R.

Lemma 2.21. *Let $k \in \mathbb{N}$, and note that, for $\star \in \{+, \mathrm{PD}, [u]\}$ and $i \in \{0, 1, \dots, d\}$, we have*

$$(\gamma_i - 1)(p^m, \pi_m^{p^m})^k \boldsymbol{A}_{R,\varpi}^\star \subset (p^m, \pi_m^{p^m})^{k+1}\boldsymbol{A}_{R,\varpi}^\star.$$

Proof. Let $i = 0$, and note that we have $(\gamma_0 - 1)\pi_m = \pi x$ for some $x \in \boldsymbol{A}_{R,\varpi}^+$. Since $\pi = (1+\pi_m)^{p^m} - 1$, we get that $(\gamma_0 - 1)\pi_m$ belongs to $(p^m\pi_m, \pi_m^{p^m})\boldsymbol{A}_{R,\varpi}^+$. Moreover, $(\gamma_0 - 1)\pi_m^{p^m} = (\pi x + \pi_m)^{p^m} - \pi_m^{p^m}$ belongs to $(p^m\pi_m, \pi_m^{p^m})^2\boldsymbol{A}_{R,\varpi}^+$. Proceeding by induction on $k \geqslant 1$ and using the fact that $\gamma_0 - 1$ acts as a twisted derivation (i.e., for all x, y in $\boldsymbol{A}_{R,\varpi}^+$, we have $(\gamma_0 - 1)xy = (\gamma_0 - 1)x \cdot y + \gamma_0(x)(\gamma_0 - 1)y)$, we conclude that

$$(\gamma_0 - 1)(p^m\pi_m, \pi_m^{p^m})^k \boldsymbol{A}_{R,\varpi}^+ \subset (p^m\pi_m, \pi_m^{p^m})^{k+1}\boldsymbol{A}_{R,\varpi}^+.$$

Furthermore, any f in $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$ can be written as $f = \sum_{n\in\mathbb{N}} f_n\pi_m^n/(\lfloor n/e\rfloor!)$ such that f_n is in $\boldsymbol{A}_{R,\varpi}^+$ and goes to 0 p-adically as $n \to +\infty$. For notational convenience, we take $n = je$ for some j in $\mathbb{N}$, and see that $(\gamma_0 - 1)\pi_m^{je}/j!$ is in $(p^m, \pi_m^{p^m})\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$. Proceeding by induction on $k \geqslant 1$ and using that $\gamma_0 - 1$ acts as a twisted derivation, we conclude that

$$(\gamma_0 - 1)(p^m, \pi_m^{p^m})^k \boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \subset (p^m, \pi_m^{p^m})^{k+1}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}.$$

Next, for $1 \leqslant i \leqslant d$, note that we have that $(\gamma_i - 1)[X_i^\flat] = \pi[X_i^\flat]$ belongs to $(p^m, \pi_m^{p^m})\boldsymbol{A}_{R,\varpi}^+$ and $(\gamma_i - 1)([X_i^\flat]^{-1}) = -\pi(1+\pi)^{-1}[X_i^\flat]^{-1}$ belongs to $(p^m\pi_m, \pi_m^{p^m})\boldsymbol{A}_{R,\varpi}^+$. Proceeding by induction on $k \geqslant 0$ and using the fact that $\gamma_i - 1$ also acts as a twisted derivation, we conclude that

$$(\gamma_i - 1)(p^m\pi_m, \pi_m^{p^m})^k \boldsymbol{A}_{R,\varpi}^+ \subset (p^m\pi_m, \pi_m^{p^m})^{k+1}\boldsymbol{A}_{R,\varpi}^+.$$

Again, by the description of elements of $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$, using the discussion for $\boldsymbol{A}_{R,\varpi}^{+}$ and the fact that $\gamma_i - 1$ acts as a twisted derivation, we conclude that

$$(\gamma_i - 1)(p^m, \pi_m^{p^m})^k \boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \subset (p^m, \pi_m^{p^m})^{k+1} \boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}.$$

Finally, the claim for $\boldsymbol{A}_{R,\varpi}^{[u]}$ follows in a similar manner. □

Lemma 2.22. *We have*

$$(\gamma_0 - 1)\boldsymbol{A}_{R,\varpi}^{(0,v]+} \subset (p^m \pi_m, \pi_m^{p^m})\boldsymbol{A}_{R,\varpi}^{(0,v]+} \quad \textit{and} \quad (\gamma_i - 1)\boldsymbol{A}_{R,\varpi}^{(0,v]+} \subset \pi \boldsymbol{A}_{R,\varpi}^{(0,v]+}$$

for $i \in \{1, \dots, d\}$. *Moreover, for* $i \in \{0, 1, \dots, d\}$ *and* $k \in \mathbb{N}$, *we have*

$$(\gamma_i - 1)(p^m, \pi_m^{p^m})^k \boldsymbol{A}_{R,\varpi}^{[u,v]} \subset (p^m, \pi_m^{p^m})^{k+1} \boldsymbol{A}_{R,\varpi}^{[u,v]}.$$

Proof. Let $i = 0$. From the proof of Lemma 2.21, we have that $(\gamma_0 - 1)\pi_m$ is in $(p^m \pi_m, \pi_m^{p^m})\boldsymbol{A}_{R,\varpi}^{+}$. So we conclude that $(\gamma_0 - 1)\boldsymbol{A}_{R,\varpi}^{+}$ belongs to $(p^m \pi_m, \pi_m^{p^m})\boldsymbol{A}_{R,\varpi}^{+}$. Observe that $\gamma_0(\pi_m) = \chi(\gamma_0)\pi_m a$, where $\chi(\gamma_0) = \exp(p^m)$ is in $\mathbb{Z}_p^{\times}$ and a is a unit in $\boldsymbol{A}_{R,\varpi}^{+}$. So, we can write

$$(\gamma_0 - 1)\pi_m^{-1} = p^m z / (\chi(\gamma_0) a \pi_m),$$

and, therefore, $(\gamma_0 - 1)(p/\pi_m)$ belongs to $(p^m \pi_m, \pi_m^{p^m})\boldsymbol{A}_{R,\varpi}^{(0,v]+}$. Proceeding by induction on $k \geqslant 1$ and using the fact that $\gamma_0 - 1$ acts as a twisted derivation, we conclude that

$$(\gamma_0 - 1)(p^m \pi_m, \pi_m^{p^m})^k \boldsymbol{A}_{R,\varpi}^{(0,v]+} \subset (p^m \pi_m, \pi_m^{p^m})^{k+1} \boldsymbol{A}_{R,\varpi}^{(0,v]+}.$$

Next, for $1 \leqslant i \leqslant d$, from the analysis for $\boldsymbol{A}_{R,\varpi}^{+}$ in Lemma 2.21, we already have $(\gamma_i - 1)\boldsymbol{A}_{R,\varpi}^{+} \subset \pi \boldsymbol{A}_{R,\varpi}^{+}$. Since passing from $\boldsymbol{A}_{R,\varpi}^{+}$ to $\boldsymbol{A}_{R}^{(0,v]+}$ involves only the arithmetic variable π_m, on which γ_i acts trivially, we therefore conclude that $(\gamma_i - 1)\boldsymbol{A}_{R,\varpi}^{(0,v]+} \subset \pi \boldsymbol{A}_{R,\varpi}^{(0,v]+}$. Proceeding by induction on $k \geqslant 1$ and using that $\gamma_i - 1$ acts as a twisted derivation, we get that

$$(\gamma_i - 1)(p^m \pi_m, \pi_m^{p^m})^k \boldsymbol{A}_{R,\varpi}^{(0,v]+} \subset (p^m \pi_m, \pi_m^{p^m})^{k+1} \boldsymbol{A}_{R,\varpi}^{(0,v]+}.$$

This shows the first claim. Finally, the claim for $\boldsymbol{A}_{R,\varpi}^{[u,v]}$ follows by combining the discussion above with Lemma 2.21 for $\boldsymbol{A}_{R,\varpi}^{[u]}$. □

2.8. ***Filtered Poincaré lemma.*** Here we state and prove a filtered version of the PD-Poincaré lemma which will be useful for Section 5.

2.8.1. *Fat period rings.* We recall the definition from [Colmez and Nizioł 2017, §2.6] and [Abhinandan 2025, §3.4]. Let A and B be two p-adically complete filtered O_F-algebras. Let $\iota : B \to A$ be a continuous injective homomorphism of filtered O_F-algebras, and let $f : B \otimes_{O_F} A \to A$ denote the ring homomorphism sending $x \otimes y \mapsto \iota(x)y$.

Definition 2.23. Define E to be the p-adic completion of the divided power envelope of $B \otimes_{O_F} A$ with respect to $\operatorname{Ker} f$.

For consistency in notation, in the following definition, we write $\boldsymbol{A}_{\mathrm{cris}}(\overline{R})$ as $\boldsymbol{A}_{\overline{R}}^{\mathrm{PD}}$.

Definition 2.24. In the notation of Definition 2.23, we record the following:

(i) Let $\star \in \{\mathrm{PD}, [u], [u, v]\}$ and define $E^{\star}_{R,\varpi} = E$ for $B = R^{\star}_{\varpi}$, $A = \boldsymbol{A}^{\star}_{R,\varpi}$ and $\iota = \iota_{\mathrm{cycl}}$ (see Section 2.7).

(ii) Let $\star \in \{\mathrm{PD}, [u], [u, v]\}$ and define $E^{\star}_{\overline{R}} = E$ for $B = R^{\star}_{\varpi}$, $A = \boldsymbol{A}^{\star}_{\overline{R}}$ and $\iota = \iota_{\mathrm{cycl}}$ (see Section 2.7).

Remark 2.25. Let us note some properties of the ring E in Definition 2.24:

(i) The ring E is the p-adic completion of $B \otimes_{O_F} A$ adjoin $(x \otimes 1 - 1 \otimes \iota(x))^{[k]}$ for all x in B and $n \in \mathbb{N}$, and $(V_i - 1)^{[k]}$ for $0 \leqslant i \leqslant d$ and $k \in \mathbb{N}$, where

$$V_i = \frac{X_i \otimes 1}{1 \otimes \iota(X_i)} \quad \text{for } 1 \leqslant i \leqslant d \quad \text{and} \quad V_0 = \frac{1 + (X_0 \otimes 1)}{1 + (1 \otimes \iota(X_0))}.$$

The morphism $f : B \otimes_{O_F} A \to A$ extends uniquely to a continuous morphism $f : E \to A$.

(ii) The ring E is equipped with a $\mathbb{Z}$-indexed decreasing filtration, which we define to be $\mathrm{Fil}^r E := E$ for $r \leqslant 0$, and, for $r \geqslant 0$, define $\mathrm{Fil}^r E$ to be the topological closure of the ideal generated by elements of the form $x_1 x_2 \prod_{i=0}^{d} (V_i - 1)^{[k_i]}$, with x_1 in $\mathrm{Fil}^{r_1} B$, x_2 in $\mathrm{Fil}^{r_2} A$ and $r_1 + r_2 + \sum_{i=0}^{d} k_i \geqslant r$.

(iii) From [Colmez and Nizioł 2017, Lemma 2.36], we have that any element x in E can be uniquely written as

$$x = \sum_{\boldsymbol{k} \in \mathbb{N}^{d+1}} x_{\boldsymbol{k}} (1 - V_0)^{[k_0]} \cdots (1 - V_d)^{[k_d]},$$

with $x_{\boldsymbol{k}}$ in A for all $\boldsymbol{k} = (k_0, k_1, \ldots, k_d) \in \mathbb{N}^{d+1}$ and $x_{\boldsymbol{k}} \to 0$ as $|\boldsymbol{k}| = \sum_{i=0}^{d} k_i \to +\infty$. Moreover, x is in $\mathrm{Fil}^r E$ if and only if $x_{\boldsymbol{k}}$ is in $\mathrm{Fil}^{r - |\boldsymbol{k}|} A$ for all $\boldsymbol{k} \in \mathbb{N}^{d+1}$.

(iv) The ring E is equipped with a natural A-linear continuous de Rham differential operator $d : E \to \Omega^1_{E/A}$. Moreover, by the description of the filtration on E in (iii), it is easy to see that the differential operator satisfies Griffiths transversality with respect to the filtration; i.e., we have

$$d : \mathrm{Fil}^r E \to \mathrm{Fil}^{r-1} E \otimes_E \Omega^1_{E/A}.$$

In the special case that $\iota : B \xrightarrow{\sim} A$ is an isomorphism, we see that E is further equipped with a natural B-linear continuous de Rham differential operator $d : E \to \Omega^1_{E/B}$ satisfying Griffiths transversality with respect to the filtration.

Lemma 2.26. *Rings in Definition 2.24 have desirable properties*:

(i) *In Definition 2.24* (i), *the tensor product Frobenii $\varphi \otimes \varphi$ on $R^{\star}_{\varpi} \otimes_{O_F} \boldsymbol{A}^{\star}_{R,\varpi}$ for $\star \in \{\mathrm{PD}, [u], [u, v]\}$ extend uniquely to the respective continuous morphisms*

$$E^{\mathrm{PD}}_{R,\varpi} \to E^{\mathrm{PD}}_{R,\varpi}, \quad E^{[u]}_{R,\varpi} \to E^{[u]}_{R,\varpi} \quad \textit{and} \quad E^{[u,v]}_{R,\varpi} \to E^{[u,v/p]}_{R,\varpi}.$$

Moreover, the actions of G_R on $\boldsymbol{A}^{\star}_{R,\varpi}$ extend uniquely to the respective continuous actions of G_R on $E^{\mathrm{PD}}_{R,\varpi}$, $E^{[u]}_{R,\varpi}$ and $E^{[u,v]}_{R,\varpi}$, which commute with the respective Frobenii. Furthermore, we have (φ, G_R)-equivariant inclusions $E^{\mathrm{PD}}_{R,\varpi} \subset E^{[u]}_{R,\varpi} \subset E^{[u,v]}_{R,\varpi}$.

(ii) *In Definition 2.24 (ii), the tensor product Frobenii* $\varphi \otimes \varphi$ *on* $R_{\varpi}^{\star} \otimes_{O_F} \boldsymbol{A}_{\overline{R}}^{\star}$ *for* $\star \in \{\mathrm{PD}, [u], [u, v]\}$ *extend uniquely to the respective continuous morphisms*

$$E_{\overline{R}}^{\mathrm{PD}} \to E_{\overline{R}}^{\mathrm{PD}}, \quad E_{\overline{R}}^{[u]} \to E_{\overline{R}}^{[u]} \quad \textit{and} \quad E_{\overline{R}}^{[u,v]} \to E_{\overline{R}}^{[u,v/p]}.$$

Moreover, the actions of G_R *on* $\boldsymbol{A}_{\overline{R}}^{\star}$ *extend uniquely to the respective continuous actions of* G_R *on* $E_{\overline{R}}^{\mathrm{PD}}$, $E_{\overline{R}}^{[u]}$ *and* $E_{\overline{R}}^{[u,v]}$, *which commute with the respective Frobenii. Furthermore, we have the* (φ, G_R)*-equivariant inclusions* $E_{\overline{R}}^{\mathrm{PD}} \subset E_{\overline{R}}^{[u]} \subset E_{\overline{R}}^{[u,v]}$.

(iii) *The natural* (φ, Γ_R)*-equivariant inclusion of rings* $\boldsymbol{A}_{R,\varpi}^{\star} \subset \boldsymbol{A}_{\overline{R}}^{\star}$ *induces a natural* (φ, Γ_R)*-equivariant injective homomorphism of rings* $E_{R,\varpi}^{\star} \subset E_{\overline{R}}^{\star}$. *Moreover, the filtration and the* $\boldsymbol{A}_{R,\varpi}^{\star}$*-linear connection on* $E_{R,\varpi}^{\star}$ *are induced from the filtration and* $\boldsymbol{A}_{\overline{R}}^{\star}$*-linear connection on* $E_{\overline{R}}^{\star}$, *respectively; in particular,*

$$\mathrm{Fil}^r E_{R,\varpi}^{\star} = E_{R,\varpi}^{\star} \cap \mathrm{Fil}^r E_{\overline{R}}^{\star} \subset E_{\overline{R}}^{\star}$$

for all $r \in \mathbb{Z}$.

Proof. The first two claims follow from [Colmez and Nizioł 2017, Lemma 2.38]. The last claim follows from the description of $\boldsymbol{E}_{R,\varpi}^{\star}$ and $E_{\overline{R}}^{\star}$ in Remark 2.25 and the fact that

$$\boldsymbol{A}_{R,\varpi}^{\star} \cap \mathrm{Fil}^r \boldsymbol{A}_{\overline{R}}^{\star} = \boldsymbol{A}_{R,\varpi}^{\star} \cap \mathrm{Fil}^r \boldsymbol{B}_{\mathrm{dR}}(\overline{R}) = \mathrm{Fil}^r \boldsymbol{A}_{R,\varpi}^{\star}. \qquad \square$$

Remark 2.27. From Definition 2.24 and Lemma 2.26, we have a natural embedding $\mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\overline{R}) \subset E_{\overline{R}}^{\mathrm{PD}}$ compatible with the respective Frobenii, $\boldsymbol{A}_{\mathrm{cris}}(\overline{R})$-linear connections and actions of G_R, and the natural filtration on the former is induced from the filtration on the latter. Furthermore, from Section 3.2, recall that we have the ring $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\overline{R})$ and from [Abhinandan 2025, Remark 4.20] we have an alternative construction of $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$ using the embedding $R \subset R_{\varpi}^{\mathrm{PD}} \xrightarrow{\sim} \boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$ (the last morphism is ι_{cycl} in Section 2.7). This induces an embedding $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \subset E_{R,\varpi}^{\mathrm{PD}}$ compatible with the respective Frobenii and actions of Γ_R, and the natural filtration on the former is induced from the filtration on the latter. Denote the O_F-linear differential operator over $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$ by ∂_A and the O_F-linear differential operator over R_{ϖ}^{PD} (as well as over R) by ∂_R. Then, the induced differential operators $\partial_R \otimes 1 + 1 \otimes \partial_A$ over $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$ and $E_{R,\varpi}^{\mathrm{PD}}$ are compatible.

Lemma 2.28. *For* $r \in \mathbb{Z}$ *and* $\star \in \{+, \mathrm{PD}, [u], [u, v]\}$, *we have*

$$\mathrm{Fil}^r E_{R,\varpi}^{\star} \cap \pi E_{R,\varpi}^{\star} = \pi \, \mathrm{Fil}^{r-1} E_{R,\varpi}^{\star}$$

as submodules of $E_{R,\varpi}^{\star}$.

Proof. Let $E := E_{R,\varpi}^{\star}$ and $A := \boldsymbol{A}_{R,\varpi}^{\star}$ for $\star \in \{+, \mathrm{PD}, [u], [u, v]\}$. The claim is trivial for $r \leqslant 0$, so assume that $r \geqslant 1$. Note that we have $\pi \, \mathrm{Fil}^{r-1} E \subset \mathrm{Fil}^r E \cap \pi E$, so we need to show the reverse inclusion. Let x be any element of $\mathrm{Fil}^r E \cap \pi E$, and write $x = \pi y$ for some y in E. From the description of the filtration on E in Remark 2.25 (iii), we have a unique presentation of x as

$$\sum_{\boldsymbol{k} \in \mathbb{N}^{d+1}} x_{\boldsymbol{k}} (1 - V_0)^{[k_0]} \cdots (1 - V_d)^{[k_d]},$$

with $x_{\boldsymbol{k}}$ in $\mathrm{Fil}^{r-|\boldsymbol{k}|} A$ for all $\boldsymbol{k} \in \mathbb{N}^{d+1}$. Moreover, we have a unique presentation of y as

$$\sum_{\boldsymbol{k} \in \mathbb{N}^{d+1}} y_{\boldsymbol{k}} (1-V_0)^{[k_0]} \cdots (1-V_d)^{[k_d]},$$

with $y_{\boldsymbol{k}}$ in A for all $\boldsymbol{k} \in \mathbb{N}^{d+1}$. Then, using the equality $x = \pi y$, we get that $x_{\boldsymbol{k}} = \pi y_{\boldsymbol{k}}$ for all $\boldsymbol{k} \in \mathbb{N}^{d+1}$. Now, from Lemma 2.19 and the fact that A is π-torsion free, it follows that $x_{\boldsymbol{k}}$ is an element of $\pi\,\mathrm{Fil}^{r-|\boldsymbol{k}|-1} A$, and hence x is an element of $\pi\,\mathrm{Fil}^{r-1} E$. □

Finally, to work with various filtered modules later, we define a filtered ring (analogous to $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R})$) containing all the rings described so far and inducing the same filtrations as described above. From [Brinon 2008, Proposition 5.2.2], recall that the natural inclusion $\boldsymbol{B}_{\mathrm{dR}}^+ \subset \mathcal{O}\boldsymbol{B}_{\mathrm{dR}}^+(\bar{R})$ extends to a $\boldsymbol{B}_{\mathrm{dR}}^+$-linear isomorphism of rings $\boldsymbol{B}_{\mathrm{dR}}^+[\![T_1, \ldots, T_d]\!] \xrightarrow{\sim} \mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R})$ by sending the indeterminate T_i to $X_i - [X_i^\flat]$ for each $1 \leqslant i \leqslant d$. We enlarge $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R})$ by setting

$$\mathcal{B}^+ := \boldsymbol{B}_{\mathrm{dR}}^+[\![T_0, T_1, \ldots, T_d]\!] \quad \text{and} \quad \mathcal{B} := \mathcal{B}^+[1/t];$$

in particular, we have natural inclusions of rings $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}^+(\bar{R}) \subset \mathcal{B}^+$ and $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R}) \subset \mathcal{B}$. We equip the latter rings with filtrations similar to [Brinon 2008, p. 52]. Set $\mathrm{Fil}^r \mathcal{B}^+ := (t, T_0, \ldots, T_d)^r \mathcal{B}^+$ for all $r \in \mathbb{N}$, and $\mathrm{Fil}^r \mathcal{B}^+ = \mathcal{B}^+$ for $r < 0$. Moreover, set $\mathrm{Fil}^0 \mathcal{B} := \sum_{n \in \mathbb{N}} t^{-n} \mathrm{Fil}^n \mathcal{B}^+$ and $\mathrm{Fil}^r \mathcal{B} := t^r \mathrm{Fil}^0 \mathcal{B}$ for all $r \in \mathbb{Z}$. Similar rings were studied in [Andreatta and Iovita 2012, §3.2.1] in the more general setting of semistable schemes. Now, employing arguments similar to [Brinon 2008, Propositions 5.2.5, 5.2.6 & 5.2.8], the following is clear.

Lemma 2.29. *Let x_i denote the image of T_i in $\mathrm{gr}^1 \mathcal{B}^+$ and y_i denote the image of T_i/t in $\mathrm{gr}^0 \mathcal{B}$ for $0 \leqslant i \leqslant d$. Then, we have the isomorphisms $\mathrm{gr}^\bullet \mathcal{B}^+ \xrightarrow{\sim} \mathbb{C}(\bar{R})[t, x_0, \ldots, x_d]$, where the grading is given by the degree of the polynomial in $t, x_0, \ldots, x_d$, and $\mathrm{gr}^\bullet \mathcal{B} \xrightarrow{\sim} \mathbb{C}(\bar{R})[t, t^{-1}, y_0, \ldots, y_d]$, where the grading is given by the degree of t; in particular, we have the isomorphism $\mathrm{gr}^0 \mathcal{B}^+ \xrightarrow{\sim} \mathbb{C}(\bar{R})[y_0, \ldots, y_d]$. Moreover, the filtration on $\mathcal{B}^+$ is the same as the induced filtration from $\mathcal{B}$, i.e., $\mathrm{Fil}^r \mathcal{B}^+ = \mathrm{Fil}^r \mathcal{B} \cap \mathcal{B}^+ \subset \mathcal{B}$ for all $r \in \mathbb{Z}$.*

Remark 2.30. From Lemma 2.29 and the description of the filtration on $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}^+(\bar{R})$ in [Brinon 2008, p. 52], we see that the filtration on $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}^+(\bar{R})$ is induced from the filtration on $\mathcal{B}^+$; i.e.,

$$\mathrm{Fil}^r \mathcal{O}\boldsymbol{B}_{\mathrm{dR}}^+(\bar{R}) = \mathcal{O}\boldsymbol{B}_{\mathrm{dR}}^+(\bar{R}) \cap \mathrm{Fil}^r \mathcal{B}^+ \subset \mathcal{B}^+ \quad \text{for } r \in \mathbb{Z}.$$

Then it also follows that

$$\mathrm{Fil}^r \mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R}) = \mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R}) \cap \mathrm{Fil}^r \mathcal{B} \subset \mathcal{B} \quad \text{for } r \in \mathbb{Z}.$$

Now, recall that we have an inclusion of rings $\boldsymbol{A}_{\bar{R}}^{[u,v]} \subset \boldsymbol{B}_{\mathrm{dR}}^+(\bar{R})$ (since $u \leq 1 \leq v$) and the former is equipped with a filtration induced from the latter; see Section 2.4.3. Then, upon using the description of $E_{R,\varpi}^{[u,v]}$ from Remark 2.25 (i), we see that the preceding embedding naturally extends to an injective ring homomorphism $E_{\bar{R}}^{[u,v]} \to \mathcal{B}^+$ via $V_i - 1 \mapsto T_i/[X_i^\flat]$ for $1 \leqslant i \leqslant d$, and $V_0 - 1 \mapsto T_0/(1+\pi_m)$. Using the description of the filtration on $E_{R,\varpi}^{[u,v]}$ from Remark 2.25 and the filtration on $\mathcal{B}^+$ from above, we have the following.

Lemma 2.31. *The filtration on $E_{\bar{R}}^{[u,v]}$ is induced from the filtration on $\mathcal{B}^+$; i.e.,*

$$\mathrm{Fil}^r\, E_{\bar{R}}^{[u,v]} = E_{\bar{R}}^{[u,v]} \cap \mathrm{Fil}^r\, \mathcal{B}^+ \subset \mathcal{B}^+ \quad \textit{for all } r \in \mathbb{Z}.$$

Remark 2.32. Let S be any ring out of

$$\boldsymbol{A}_{\mathrm{cris}}(\bar{R}), \quad \mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\bar{R}), \quad \mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}, \quad R_{\varpi}^{\star}, \quad E_{R,\varpi}^{\star}, \quad E_{\bar{R}}^{\star}$$

for $\star \in \{\mathrm{PD}, [u], [u, v]\}$. Then, by Remarks 2.27 and 2.30 and Lemma 2.31, it is easy to see that

$$\mathrm{Fil}^r\, S = S[1/p] \cap \mathrm{Fil}^r\, \mathcal{B} \subset \mathcal{B} \quad \text{and} \quad \mathrm{Fil}^r(S[1/p]) := S[1/p] \cap \mathrm{Fil}^r\, \mathcal{B} = (\mathrm{Fil}^r\, S)[1/p] \subset \mathcal{B}$$

for all $r \in \mathbb{Z}$.

2.8.2. *Filtered modules.* Let V be a de Rham representation of G_R, and set $D := \mathcal{O}\boldsymbol{D}_{\mathrm{dR}}(V)$ as a finite projective $R[1/p]$-module. From Section 2.3, recall that D is equipped with a decreasing, separated and exhaustive filtration by $R[1/p]$-submodules $\{\mathrm{Fil}^r\, D\}_{r\in\mathbb{Z}}$ such that $\mathrm{Fil}^a\, D = D$ and $\mathrm{Fil}^b\, D = 0$ for some $a, b \in \mathbb{Z}$ and, for each $r \in \mathbb{Z}$, the $R[1/p]$-modules $\mathrm{Fil}^r\, D$ and $\mathrm{gr}^r\, D$ are finite projective. Recall that, by the definition of de Rham representations, we have a natural $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R})$-linear isomorphism

$$\alpha_{\mathrm{dR}} : \mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R}) \otimes_{R[1/p]} D \xrightarrow{\sim} \mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R}) \otimes_{\mathbb{Q}_p} V.$$

Extending scalars of the isomorphism α_{dR} along the natural map $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R}) \to \mathcal{B}$ from Section 2.8.1, we obtain the $\mathcal{B}$-linear isomorphism

$$\alpha_{\mathcal{B}} : \mathcal{B} \otimes_{R[1/p]} D \xrightarrow{\sim} \mathcal{B} \otimes_{\mathbb{Q}_p} V. \tag{2-6}$$

Next, let $S \subset \mathcal{B}$ be an R-subalgebra equipped with a filtration induced from the filtration on $\mathcal{B}$ (see after Lemma 2.29), such that the natural map $R \to S$ is injective and p is not invertible in S. Now, consider the $S[1/p]$-module $D_S := S \otimes_R D$. We equip D_S with the induced filtration, for each $r \in \mathbb{Z}$,

$$\mathrm{Fil}^r\, D_S := D_S \cap \alpha_{\mathcal{B}}^{-1}(\mathrm{Fil}^r\, \mathcal{B} \otimes_{\mathbb{Q}_p} V). \tag{2-7}$$

Proposition 2.33. *The filtration on D_S in (2-7) coincides with the tensor product filtration; i.e., for each $r \in \mathbb{Z}$, we have the isomorphism*

$$F^r D_S := \sum_{i+j=r}^{r} \mathrm{Fil}^i\, S \otimes_R \mathrm{Fil}^j\, D \xrightarrow{\sim} \mathrm{Fil}^r\, D_S. \tag{2-8}$$

Proof. From Lemma 2.38, we have the isomorphism

$$F^r(\mathcal{B} \otimes_{R[1/p]} D) \xrightarrow{\sim} \mathrm{Fil}^r(\mathcal{B} \otimes_{R[1/p]} D) \quad \text{for each } r \in \mathbb{Z}.$$

Then, by using Lemmas 2.35 and 2.37 below with $S' = \mathcal{B}$, we obtain the isomorphism in (2-8). ☐

Remark 2.34. If p is invertible in S, then, by using $R[1/p]$ in place of R in Proposition 2.33, we get the isomorphism

$$\sum_{i+j=r}^{r} \mathrm{Fil}^i\, S \otimes_{R[1/p]} \mathrm{Fil}^j\, D \xrightarrow{\sim} \mathrm{Fil}^r\, D_S$$

for each $r \in \mathbb{Z}$.

The following observations were used above.

Lemma 2.35. *For each $i, j, r \in \mathbb{Z}$ such that $i + j = r$, the natural map* $\mathrm{Fil}^i S \otimes_R \mathrm{Fil}^j D \to D_S$ *is injective. In particular, the filtration $\{F^r D_S\}_{r\in\mathbb{Z}}$ is a well-defined $\mathbb{Z}$-indexed decreasing filtration on D_S by $S[1/p]$-submodules. Moreover, we have that*

$$\mathrm{gr}^r_F D_S = \bigoplus_{i+j=r} \mathrm{gr}^i S \otimes_R \mathrm{gr}^j D.$$

Proof. For each $j \in \mathbb{Z}$, let us consider the following exact sequence of finite projective $R[1/p]$-modules, in particular flat R-modules:

$$0 \to \mathrm{Fil}^{j+1} D \to \mathrm{Fil}^j D \to \mathrm{gr}^j D \to 0. \tag{2-9}$$

Extending scalars in (2-9) along the natural map $R \to S$ and by decreasing induction on $j \geqslant a$, it is easy to see that the natural map

$$S \otimes_R \mathrm{Fil}^j D \to S \otimes_R \mathrm{Fil}^a D = S \otimes_R D$$

is injective. Therefore, for any $i + j = r$, it follows that the natural map

$$\mathrm{Fil}^i S \otimes_R \mathrm{Fil}^j D \hookrightarrow S \otimes_R \mathrm{Fil}^j D \to D_S$$

is injective, where the first arrow is obtained by tensoring the R-linear inclusion $\mathrm{Fil}^i S \subset S$ with the flat R-module $\mathrm{Fil}^j D$ and the second arrow is as above. Hence, for each $r \in \mathbb{Z}$, we get that

$$F^r D_S := \sum_{i+j=r} \mathrm{Fil}^i S \otimes_R \mathrm{Fil}^j D$$

is an $S[1/p]$-submodule of D_S. It is clear that the filtration is decreasing. Next, let us note that, upon tensoring (2-9) with $\mathrm{Fil}^i S$ and $\mathrm{gr}^i S$, we obtain the following R-linear commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Fil}^{i+1} S \otimes_R \mathrm{Fil}^{j+1} D & \longrightarrow & \mathrm{Fil}^{i+1} S \otimes_R \mathrm{Fil}^j D & \longrightarrow & \mathrm{Fil}^{i+1} S \otimes_R \mathrm{gr}^j D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Fil}^{i} S \otimes_R \mathrm{Fil}^{j+1} D & \longrightarrow & \mathrm{Fil}^{i} S \otimes_R \mathrm{Fil}^j D & \longrightarrow & \mathrm{Fil}^{i} S \otimes_R \mathrm{gr}^j D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{gr}^{i} S \otimes_R \mathrm{Fil}^{j+1} D & \longrightarrow & \mathrm{gr}^{i} S \otimes_R \mathrm{Fil}^j D & \longrightarrow & \mathrm{gr}^{i} S \otimes_R \mathrm{gr}^j D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2-10}$$

Since $\mathrm{Fil}^j D$ and $\mathrm{gr}^j D$ are finite projective modules over $R[1/p]$, in particular flat modules over R, we get that all rows and columns of (2-10) are exact. From the diagram, it easily follows that

$$\mathrm{gr}^r_F D_S = \bigoplus_{i+j=r} \mathrm{gr}^i S \otimes_R \mathrm{gr}^j D \quad \text{for each } r \in \mathbb{Z}.$$

□

Remark 2.36. If p is invertible in S, then, by employing arguments similar to the proof of Lemma 2.35 (using $R[1/p]$ in place of R), we see that the S-module $D_S := S \otimes_{R[1/p]} D$ is equipped with a well-defined $\mathbb{Z}$-indexed decreasing tensor product filtration by S-submodules given as

$$F^r D_S := \sum_{i+j=r} \mathrm{Fil}^i S \otimes_{R[1/p]} \mathrm{Fil}^j D.$$

Moreover, for each $r \in \mathbb{Z}$, we have that

$$\mathrm{gr}^r_F D_S = \bigoplus_{i+j=r} \mathrm{gr}^i S \otimes_{R[1/p]} \mathrm{gr}^j D[1/p].$$

Next, let $S \subset S' \subset \mathcal{B}$ be two R-subalgebras equipped with the respective induced filtrations such that the natural map $R \to S$ is injective. Set $D_S := S \otimes_R D$ and $D_{S'} := S' \otimes_R D$, equipped with the tensor product filtration as in Lemma 2.35. Then, we claim the following.

Lemma 2.37. *For each $r \in \mathbb{Z}$, we have that $F^r D_{S'} \cap D_S = F^r D_S$ as submodules of $D_{S'}$.*

Proof. We will prove the claim by assuming that p is not invertible in S'; in the case that p is invertible in either S or S', the same argument works by using Remark 2.36 and replacing R with $R[1/p]$. Now, let us first note that an easy induction on r shows that proving the equality $F^{r+1} D_{S'} \cap D_S = F^{r+1} D_S$ is equivalent to proving the equality $F^{r+1} D_{S'} \cap F^r D_S = F^{r+1} D_S$. Next, consider the following diagram with R-linear exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{r+1} D_S & \longrightarrow & F^r D_S & \longrightarrow & \mathrm{gr}^r_F D_S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F^{r+1} D_{S'} & \longrightarrow & F^r D_{S'} & \longrightarrow & \mathrm{gr}^r_F D_{S'} & \longrightarrow & 0 \end{array} \tag{2-11}$$

Note that proving the equality $F^{r+1} D_{S'} \cap F^r D = F^{r+1} D_S$ is equivalent to showing that the right vertical arrow in diagram (2-11) is injective. Now, by Lemma 2.35, we have that

$$\mathrm{gr}^r_F D_S = \bigoplus_{i+j=r} \mathrm{gr}^i S \otimes_R \mathrm{gr}^j D$$

for each $r \in \mathbb{Z}$. Similarly, we also have that $\mathrm{gr}^r_F D_{S'} = \bigoplus_{i+j=r} \mathrm{gr}^i S' \otimes_R \mathrm{gr}^j D$ for each $r \in \mathbb{Z}$. Since $\mathrm{Fil}^i S' \cap S = \mathrm{Fil}^i S$, by using a diagram similar to (2-11), it follows that the natural R-linear map $\mathrm{gr}^i S \to \mathrm{gr}^i S'$ is injective for all $i \in \mathbb{Z}$. Furthermore, as $\mathrm{gr}^i D$ is flat over R, it follows that the natural map $\mathrm{gr}^i S \otimes_R \mathrm{gr}^j D \to \mathrm{gr}^i S' \otimes_R \mathrm{gr}^j D$ is also injective. Hence we get that the right vertical arrow in (2-11) is injective, allowing us to conclude. □

Now, by using [Brinon 2008, Proposition 8.4.3], note that the isomorphism α_{dR} is compatible with the tensor product filtration of Remark 2.36 on the source and the filtration on the target is induced by the natural filtration on $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R})$. As the natural filtration on $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R})$ coincides with the induced filtration via the inclusion $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R}) \subset \mathcal{B}$ (see Remark 2.30), it follows that we have the isomorphism

$$F^r(\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R}) \otimes_{R[1/p]} D) \xrightarrow{\sim} \mathrm{Fil}^r(\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R}) \otimes_{R[1/p]} D) \quad \text{for all } r \in \mathbb{Z}.$$

Using Lemma 2.29 and an argument similar to the proof of [Brinon 2008, Proposition 8.3.2], we obtain the following.

Lemma 2.38. *The isomorphism in (2-6) induces an isomorphism*

$$\alpha_{\mathcal{B}}(F^r(\mathcal{B} \otimes_{R[1/p]} D)) \xrightarrow{\sim} \mathrm{Fil}^r \mathcal{B} \otimes_{\mathbb{Q}_p} V \quad \text{for all } r \in \mathbb{Z}.$$

In particular, we get the isomorphism $F^r(\mathcal{B} \otimes_{R[1/p]} D) \xrightarrow{\sim} \mathrm{Fil}^r(\mathcal{B} \otimes_{R[1/p]} D)$.

Proof. Note that (2-6) is an isomorphism and the filtration on D is exhaustive, so it is enough to show that the maps on the associated graded pieces, induced by (2-6), are bijective. For each $r \in \mathbb{Z}$, consider the diagram

$$\begin{array}{ccc} \bigoplus_{i+j=r} \mathrm{gr}^i \, \mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R}) \otimes_{R[1/p]} \mathrm{gr}^j M[1/p] & \xrightarrow{\sim} & \mathrm{gr}^r \, \mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R}) \otimes_{\mathbb{Q}_p} V \\ \downarrow & & \downarrow \\ \bigoplus_{i+j=r} \mathrm{gr}^i \, \mathcal{B} \otimes_{R[1/p]} \mathrm{gr}^j M[1/p] & \longrightarrow & \mathrm{gr}^r \, \mathcal{B} \otimes_{\mathbb{Q}_p} V \end{array}$$

where the top horizontal arrow is the isomorphism induced by the filtration-compatible $\mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R})$-linear isomorphism α_{dR}, the left vertical arrow is induced by the compatibility of filtrations on the source of α_{dR} and $\alpha_{\mathcal{B}}$ (see Lemma 2.37) and the right vertical arrow is induced by the compatibility of filtrations on the target of α_{dR} and $\alpha_{\mathcal{B}}$ (see Lemma 2.29 and Remark 2.30). Now, recall that from Lemma 2.29 we have the isomorphism $\mathrm{gr}^i \mathcal{B} \xrightarrow{\sim} t^i \mathbb{C}(\bar{R})[y_0, \dots, y_d]$ and from [Brinon 2008, Proposition 5.2.6] we have the isomorphism $\mathrm{gr}^i \mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R}) \xrightarrow{\sim} t^i \mathbb{C}(\bar{R})[y_1, \dots, y_d]$. In particular, we see that $\mathrm{gr}^i \mathcal{B} \xrightarrow{\sim} \mathbb{Z}[y_0] \otimes_{\mathbb{Z}} \mathrm{gr}^i \mathcal{O}\boldsymbol{B}_{\mathrm{dR}}(\bar{R})$ is an isomorphism. Therefore, it follows that the bottom horizontal arrow of the diagram above is given as the extension of scalars along $\mathbb{Z} \to \mathbb{Z}[y_0]$ of the top horizontal arrow, and hence it is also an isomorphism. ☐

Next, let us note some applications of Proposition 2.33, which will be used in Section 5.

Lemma 2.39. *Let* $S = E^{[u,v]}_{R,\varpi}$, *and set* $D_S := E^{[u,v]}_{R,\varpi} \otimes_R D$, *equipped with the tensor product filtration as in Lemma 2.35. Assume that* $\mathrm{Fil}^0 D = D$. *Then, for any* $r \in \mathbb{N}$, *we have that* $\mathrm{Fil}^r D_S \cap \pi D_S = \pi \, \mathrm{Fil}^{r-1} D_S$ *as submodules of* D_S.

Proof. The claim is trivial for $r = 0$, so assume that $r \geqslant 1$. We will prove the claim by induction on r. Note that, for $r = 1$, we have $\mathrm{Fil}^r D_S \cap \pi D_S = \pi D_S$. So, let $r \in \mathbb{N}_{\geqslant 2}$, and assume the claim is true for $r - 1$; i.e., $\mathrm{Fil}^{r-1} D_S \cap \pi D_S = \pi \, \mathrm{Fil}^{r-2} D_S$. Then, we see that

$$\mathrm{Fil}^r D_S \cap \pi D_S = \mathrm{Fil}^r D_S \cap \mathrm{Fil}^{r-1} D_S \cap \pi D_S = \mathrm{Fil}^r D_S \cap \pi \, \mathrm{Fil}^{r-2} D_S.$$

In particular, to get the claim, it is enough to show that $\mathrm{Fil}^r D_S \cap \pi \, \mathrm{Fil}^{r-2} D_S = \pi \, \mathrm{Fil}^{r-1} D_S$. Now, consider the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Fil}^{r-1} D_S & \longrightarrow & \mathrm{Fil}^{r-2} D_S & \longrightarrow & \mathrm{gr}^{r-2} D_S & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & \mathrm{Fil}^r D_S & \longrightarrow & \mathrm{Fil}^{r-1} D_S & \longrightarrow & \mathrm{gr}^{r-1} D_S & \longrightarrow & 0 \end{array} \tag{2-12}$$

where the left and middle vertical arrows are multiplication-by-π and the right vertical arrow is the induced map, which we again denote as multiplication-by-π. Note that all the vertical arrows in (2-12) are R-linear.

Moreover, from diagram (2-12), we see that showing the equality $\mathrm{Fil}^r D_S \cap \pi \,\mathrm{Fil}^{r-2} D_S = \pi \,\mathrm{Fil}^{r-1} D_S$ is equivalent to showing that the right vertical arrow in (2-12) is injective. Note that, by using Lemma 2.35 and Remark 2.36, we have that $\mathrm{gr}^{r-2} D_S = \bigoplus_{i+j=r-2} \mathrm{gr}^i S \otimes_R \mathrm{gr}^j D$ and similarly for $\mathrm{gr}^{r-1} D_S$. Therefore, the right vertical arrow in (2-12) induces R-linear maps $\mathrm{gr}^i S \otimes_R \mathrm{gr}^j D \xrightarrow{\pi} \mathrm{gr}^{i-1} S \otimes_R \mathrm{gr}^j D$ for $i + j = r - 2$. As $\mathrm{gr}^j D$ is a flat R-module and the preceding map is R-linear, it is enough to show that the map $\mathrm{gr}^i S \xrightarrow{\pi} \mathrm{gr}^{i+1} S$, induced from the multiplication-by-π map $\mathrm{Fil}^i S \xrightarrow{\pi} \mathrm{Fil}^{i+1} S$, is injective. This follows from Lemma 2.28. Hence we obtain that the right vertical arrow in (2-12) is injective; in particular, $\mathrm{Fil}^r D_S \cap \pi D_S = \pi \,\mathrm{Fil}^{r-1} D_S$ for each $r \in \mathbb{N}$. □

Now, let us assume that D is finite free over $R[1/p]$, we have $\mathrm{Fil}^0 D = D$ and there exists a finite free R-submodule $M \subset D$ such that $M[1/p] = D$. Let S and S' be as in Lemma 2.37 and equip M_S and $M_{S'}$ with induced filtrations; i.e., $\mathrm{Fil}^r M_S := M_S \cap \mathrm{Fil}^r D_S \subset D_S$ and $\mathrm{Fil}^r M_{S'} := M_{S'} \cap \mathrm{Fil}^r D_{S'} \subset D_{S'}$. As M is free over R, the natural map $M_S \to M_{S'}$ is injective and we note the following.

Lemma 2.40. *For each $r \in \mathbb{N}$, we have $\mathrm{Fil}^r M_S = \mathrm{Fil}^r M_{S'} \cap M_S$ as submodules of $M_{S'}$. Moreover, if $S = E_{R,\varpi}^{[u,v]}$, then we have $\mathrm{Fil}^r M_S \cap \pi M_S = \pi \,\mathrm{Fil}^{r-1} M_S$ as submodules of M_S.*

Proof. The first claim follows from the definition of filtration on M_S and $M_{S'}$ and Lemma 2.37. For the second claim, by Lemma 2.39, we have $\mathrm{Fil}^r M_S \cap \pi M_S = \pi \,\mathrm{Fil}^{r-1} D_S \cap \pi M_S = \pi \,\mathrm{Fil}^{r-1} M_S$. □

2.8.3. *Poincaré lemma.* In the notation of Definition 2.23, let us set $A = \boldsymbol{A}^{\star}_{R,\varpi}$, $B = R^{\star}_{\varpi}$ and $E = E^{\star}_{R,\varpi}$ for $\star \in \{\mathrm{PD}, [u], [u, v]\}$. Let

$$\omega_0 := \frac{d[X_0^\flat]}{1+[X_0^\flat]} \quad \text{and} \quad \omega_i := \frac{d[X_i^\flat]}{[X_i^\flat]} \quad \text{for } 1 \leqslant i \leqslant d.$$

Set

$$\Omega^1 := \bigoplus_{i=1}^{d} \mathbb{Z}\omega_i \quad \text{and} \quad \Omega^k := \bigwedge^k \Omega^1 \quad \text{for all } k \in \mathbb{N}.$$

Then, we have $\Omega^k_{E/B} = E \otimes_{\mathbb{Z}} \Omega^k$ and, from Remark 2.25 (iv), note that, for $r \in \mathbb{Z}$, we have the filtered de Rham complex of E relative to B:

$$\mathrm{Fil}^r \,\Omega^\bullet_{E/B} := \mathrm{Fil}^r E \to \mathrm{Fil}^{r-1} E \otimes_{\mathbb{Z}} \Omega^1 \to \mathrm{Fil}^{r-2} E \otimes_{\mathbb{Z}} \Omega^2 \to \cdots .$$

From the discussion before Lemma 2.40, let M be a finite free R-module such that $M[1/p] = \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$, where V is a positive crystalline representation of G_R. Moreover, we set $M_B := B \otimes_R M$, equipped with a filtration induced from the tensor product filtration on $M_B[1/p]$, and we similarly set $M_E := E \otimes_R M$, equipped with a filtration induced from the tensor product filtration on $M_E[1/p]$. Furthermore, the B-linear differential operator on E induces a quasinilpotent integrable connection $\partial : M_E \to M_E \otimes_E \Omega^1_{E/B}$ satisfying Griffiths transversality with respect to the filtration (since $\partial(\mathrm{Fil}^r E) \subset \mathrm{Fil}^{r-1} E$). In particular, for each $r \in \mathbb{Z}$, we have the following filtered de Rham complex:

$$\begin{aligned} \mathrm{Fil}^r M_E \otimes \Omega^\bullet_{E/B} &:= \mathrm{Fil}^r M_E \to \mathrm{Fil}^{r-1} M_E \otimes_E \Omega^1_{E/B} \to \mathrm{Fil}^{r-2} M_E \otimes_E \Omega^2_{E/B} \to \cdots \\ &= \mathrm{Fil}^r M_E \to \mathrm{Fil}^{r-1} M_E \otimes_{\mathbb{Z}} \Omega^1 \to \mathrm{Fil}^{r-2} M_E \otimes_{\mathbb{Z}} \Omega^2 \to \cdots . \end{aligned}$$

Using the equality $M_B = M_E^{\partial=0}$ and Lemma 2.40, let us note that

$$\mathrm{Fil}^r\, M_B = \mathrm{Fil}^r\, M_E \cap M_E^{\partial=0} = (\mathrm{Fil}^r\, M_E)^{\partial=0},$$

and we obtain the following filtered Poincaré lemma.

Lemma 2.41. *The natural map* $\mathrm{Fil}^r\, M_B \to \mathrm{Fil}^r\, M_E \otimes \Omega^{\bullet}_{E/B}$ *is a quasi-isomorphism.*

Proof. We have a natural injection $\epsilon : \mathrm{Fil}^r\, M_B \to \mathrm{Fil}^r\, M_E$, so we give a contracting (B-linear) homotopy. Note that M is a finite free R-module, so we may choose $\{f_1, \dots, f_h\}$ as an R-basis of M. Now define a B-linear map $h^0 : M_E \to M_B$ by $\sum_{j=1}^h a_j f_j \mapsto \sum_{j=1}^h a_{j,0} f_j$, where a_j is in E and $a_{j,0}$ is the projection to the 0-th coordinate (see Remark 2.25 (iii), where 0 corresponds to the coordinate $(0, \dots, 0)$). Moreover, note that, after inverting p and using the tensor product filtration on $M_E[1/p]$, we get that h^0 induces a $B[1/p]$-linear map

$$h^0 : \mathrm{Fil}^r\, M_E[1/p] \to \mathrm{Fil}^r\, M_B[1/p].$$

In particular, we obtain an induced B-linear map

$$h^0 : \mathrm{Fil}^r\, M_E \to M_B \cap \mathrm{Fil}^r\, M_B[1/p] = \mathrm{Fil}^r\, M_B.$$

It is clear that we have $h^0\epsilon = \mathrm{id}$.

Next, for $q > 0$, define a B-linear map

$$h^q : M_E \otimes_{\mathbb{Z}} \Omega^q \to M_E \otimes_{\mathbb{Z}} \Omega^{q-1}$$

given by the formula

$$h^q\Big(f_j a_j \prod_{i=0}^d (V_i - 1)^{[k_i]} V_{i_1}\omega_{i_1} \wedge \cdots \wedge V_{i_q}\omega_{i_q}\Big) = f_j a_j \prod_{i=0}^d (V_i - 1)^{[k_i + \delta_{ji_1}]} V_{i_2}\omega_{i_2} \wedge \cdots \wedge V_{i_q}\omega_{i_q} \quad \text{if } k_j = 0$$

and 0 otherwise (here δ denotes the Kronecker delta function). Moreover, note that, after inverting p and using the tensor product filtration on $M_E[1/p]$, we get that h^q induces a $B[1/p]$-linear map

$$h^q : \mathrm{Fil}^{r-q}\, M_E[1/p] \otimes_{\mathbb{Z}} \Omega^q \to \mathrm{Fil}^{r-q+1}\, M_E[1/p] \otimes_{\mathbb{Z}} \Omega^{q-1}.$$

In particular, we obtain an induced B-linear map

$$h^q : \mathrm{Fil}^{r-q}\, M_E \otimes_{\mathbb{Z}} \Omega^q \to \mathrm{Fil}^{r-q+1}\, M_E \otimes_{\mathbb{Z}} \Omega^{q-1}.$$

It is easy to see $\epsilon h^0 + h^1 d = \mathrm{id}$ and $dh^q + h^{q+1} d = \mathrm{id}$. Hence we obtain the desired B-linear homotopy, proving the claim. □

3. Finite-height p-adic representations

In this section, we will recall the notion of relative Wach modules from [Abhinandan 2025] and prove some lemmas that will be used later. We will use the setup and notation of Section 2.1; in particular, we fix some $m \in \mathbb{N}_{\geqslant 1}$.

Notation. For an algebra S admitting a Frobenius endomorphism φ and an S-module M admitting a Frobenius-semilinear endomorphism $\varphi : M \to M$, we will denote by $\varphi^*(M) \subset M$ the S-submodule generated by the image of φ.

3.1. ***Relative Wach modules.*** Set $q := \varphi(\pi)/\pi$ in $\boldsymbol{A}_R^+$, and let T be a free $\mathbb{Z}_p$-representation of G_R. Then, note that we have an $\boldsymbol{A}_R^+$-submodule $\boldsymbol{D}^+(T) := (\boldsymbol{A}^+ \otimes_{\mathbb{Q}_p} T)^{H_R} \subset \boldsymbol{D}(T)$, equipped with induced commuting actions of (φ, Γ_R).

Definition 3.1 [Abhinandan 2025, Definition 4.8]. A $\mathbb{Z}_p$-representation T is said to be *positive* and of *finite q-height* if there exists a finite projective $\boldsymbol{A}_R^+$-submodule $\boldsymbol{N}(T) \subset \boldsymbol{D}^+(T)$ of rank $\mathrm{rk}_{\mathbb{Z}_p} T$, stable under the action of φ and Γ_R and satisfying the following conditions:

(i) The natural $\boldsymbol{A}_R$-linear map $\boldsymbol{A}_R \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T) \xrightarrow{\sim} \boldsymbol{D}(T)$ is a (φ, Γ_R)-equivariant isomorphism, where $\boldsymbol{N}(T)$ is equipped with the induced action of (φ, Γ_R).

(ii) The $\boldsymbol{A}_R^+$-module $\boldsymbol{N}(T)/\varphi^*(\boldsymbol{N}(T))$ is killed by q^s for some $s \in \mathbb{N}$.

(iii) The induced action of Γ_R on $\boldsymbol{N}(T)/\pi\boldsymbol{N}(T)$ is trivial.

(iv) There exists $R' \subset \overline{R}$ finite étale over R such that $\boldsymbol{A}_{R'}^+ \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)$ is free over $\boldsymbol{A}_{R'}^+$.

The *height* of T is defined to be the smallest $s \in \mathbb{N}$ satisfying (ii) above. Furthermore, a positive finite q-height p-adic representation V of G_R is a representation admitting a positive finite q-height $\mathbb{Z}_p$-lattice $T \subset V$, and we set $\boldsymbol{N}(V) := \boldsymbol{N}(T)[1/p]$, satisfying properties analogous to (i)–(iv) above. The height of V is defined to be the height of T. For $k \in \mathbb{Z}$, let

$$T(k) := T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(k) \quad \text{and} \quad V(k) := T(k)[1/p],$$

define

$$\boldsymbol{N}(T(k)) := \frac{1}{\pi^k}\boldsymbol{N}(T)(k) \quad \text{and} \quad \boldsymbol{N}(V(k)) := \frac{1}{\pi^k}\boldsymbol{N}(V)(k)$$

and define the height of $T(k)$ to be (height of T) $- k$. We call $T(k)$ and $V(k)$ representations of *finite q-height.*

For general properties of Wach modules, we refer the reader to [Abhinandan 2025, §4.2]. Let us note that there is a natural filtration on Wach modules attached to finite q-height representations.

Definition 3.2. Let V be a finite q-height representation of G_R. For each $r \in \mathbb{Z}$, set

$$\mathrm{Fil}^r\, \boldsymbol{N}(V) := \{x \in \boldsymbol{N}(V) \mid \varphi(x) \in q^r \boldsymbol{N}(V)\} \quad \text{and} \quad \mathrm{Fil}^r\, \boldsymbol{N}(T) := \mathrm{Fil}^r\, \boldsymbol{N}(V) \cap \boldsymbol{N}(T) \subset \boldsymbol{N}(V).$$

Lemma 3.3. *We have* $\mathrm{Fil}^r\, \boldsymbol{N}(T) = \{x \in \boldsymbol{N}(T) \mid \varphi(x) \in q^r \boldsymbol{N}(T)\}$. *Moreover, we have*

$$\mathrm{Fil}^r\, \boldsymbol{N}(T(k)) = \pi^{-k}\, \mathrm{Fil}^{r+k}\, \boldsymbol{N}(T)(k) \quad \textit{and} \quad \mathrm{Fil}^r\, \boldsymbol{N}(V(k)) = \pi^{-k}\, \mathrm{Fil}^{r+k}\, \boldsymbol{N}(V)(k).$$

Proof. The first claim is true because $q^r \boldsymbol{N}(V) \cap \boldsymbol{N}(T) = (q^r \boldsymbol{B}_R^+ \cap \boldsymbol{A}_R^+) \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T) = q^r \boldsymbol{N}(T)$. To show the second claim, let $\pi^{-k} x \otimes \epsilon^{\otimes k}$ be an element of $\mathrm{Fil}^r\, \pi^{-k}\boldsymbol{N}(T)(k)$, with $x \in \boldsymbol{N}(T)$ and $\epsilon^{\otimes k}$ a $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(k)$. By assumption, $\varphi(\pi^{-k} x \otimes \epsilon^{\otimes k}) = (q\pi)^{-k}\varphi(x) \otimes \epsilon^{\otimes k}$ belongs to $q^r \pi^{-k}\boldsymbol{N}(T)(k)$. Therefore, we see that $\varphi(x)$ belongs to $q^{r+k}\boldsymbol{N}(T)$; i.e., x is in $\mathrm{Fil}^{r+k}\, \boldsymbol{N}(T)$. The converse is obvious. □

Remark 3.4. Set $\mathrm{Fil}^r\, \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) := \xi^r \boldsymbol{A}_{\mathrm{inf}}(\bar{R})$ and $\mathrm{Fil}^r\, \boldsymbol{A} := \boldsymbol{A} \cap \mathrm{Fil}^r\, \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \subset \boldsymbol{A}_{\mathrm{inf}}(\bar{R})$ for each $r \in \mathbb{N}$. If T is a positive finite q-height $\mathbb{Z}_p$-representation of G_R, then, from [Abhinandan 2025, Lemma 4.53], for the filtration on Wach modules as in Definition 3.2, we have

$$\mathrm{Fil}^r\, \boldsymbol{N}(T) = \boldsymbol{N}(T) \cap \mathrm{Fil}^r\, \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} T = \boldsymbol{N}(T) \cap \mathrm{Fil}^r\, \boldsymbol{A} \otimes_{\mathbb{Z}_p} T \subset \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} T \quad \text{for each } r \in \mathbb{N}.$$

The operator ψ defined in Section 2.4 commutes with the action of G_R, so, by linearity, it extends to a map $\psi : \boldsymbol{D}(T) \to \boldsymbol{D}(T)$ and from Proposition 2.4 we get that $\psi(\boldsymbol{D}^+(T)) \subset \boldsymbol{D}^+(T)$.

Lemma 3.5. *Let T be positive finite q-height $\mathbb{Z}_p$-representation of G_R of height s. Then, for $k \geqslant s$, we have $\psi(\boldsymbol{N}(T(k))) \subset \boldsymbol{N}(T(k))$.*

Proof. Note that we have $q^s \boldsymbol{N}(T) \subset \varphi^*(\boldsymbol{N}(T))$. So, for $k \geqslant s$ and x in $\boldsymbol{N}(T(k))$, we must have that $\varphi(\pi^k)x = (q\pi)^k x$ is in $\varphi^*(\boldsymbol{N}(T)(k))$. Therefore, $\psi(x)$ belongs to $(1/\pi^k)\boldsymbol{N}(T)(k) = \boldsymbol{N}(T(k))$. □

3.2. *Wach modules and crystalline representations.* From [Abhinandan 2025, §4.3.1], we have an R-algebra $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\bar{R})$ equipped with a Frobenius endomorphism φ, a continuous action of Γ_R, a Γ_R-stable filtration and an $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$-linear integrable connection satisfying Griffiths transversality with respect to the filtration and commuting with the action of φ and Γ_R.

Theorem 3.6 [Abhinandan 2025, Theorem 4.24, Proposition 4.27, and Corollary 4.26]. *Let V be a finite q-height representation of G_R; then V is crystalline. Moreover, if V is positive, then we have an isomorphism of $R[1/p]$-modules*

$$M[1/p] := (\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V))^{\Gamma_R} \xrightarrow{\sim} \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V),$$

compatible with respective Frobenii, filtrations and connections. Furthermore, we have the natural $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$-linear isomorphisms

$$\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V) \xleftarrow{\sim} \mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M[1/p] \xrightarrow{\sim} \mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V), \tag{3-1}$$

compatible with the respective Frobenii, filtrations, connections and the actions of Γ_R.

Remark 3.7. In Theorem 3.6, the $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$-module $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)$ is equipped with the following structures: a Frobenius endomorphism, given as $\varphi \otimes \varphi$; an $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$-linear connection, given by the natural $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$-linear differential operator $\partial_R \otimes 1$ (see Remark 2.27 for notation); an action of Γ_R, where any g in Γ_R acts as $g \otimes g$; and an $\mathbb{N}$-indexed decreasing filtration given as the tensor product filtration, i.e.,

$$\mathrm{Fil}^r(\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)) = \sum_{i+j=r} \mathrm{Fil}^i\, \mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \mathrm{Fil}^j\, \boldsymbol{N}(V),$$

which is well defined because each term of the summation is an $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$-submodule of $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)$ (using that, as $\boldsymbol{A}_R^+$-modules, $\boldsymbol{N}(V)$ and $\mathrm{Fil}^j\, \boldsymbol{N}(V)$ are finite projective, see [Abhinandan 2023, Section 5.2]). The module $M[1/p]$ is equipped with induced structures; in particular, the filtration on $M[1/p]$ is given as $\mathrm{Fil}^r\, M[1/p] = (\mathrm{Fil}^r(\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)))^{\Gamma_R}$ and its compatibility with the Hodge filtration on $\mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$

follows from [Abhinandan 2025, §4.5.1]. Then, in (3-1), the middle and right-hand terms are equipped with the following structures: a Frobenius endomorphism, given as $\varphi \otimes \varphi$; an $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$-linear connection, given as $\partial_R \otimes 1 + 1 \otimes \partial_D$, where ∂_D is the connection on $\mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$ (see Section 2.3); an action of Γ_R, where any g in Γ_R acts as $g \otimes 1$; and an $\mathbb{N}$-indexed decreasing filtration given as the tensor product filtration (see Lemma 2.35), where we use the filtration on $M[1/p]$ as above and the Hodge filtration on $\mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$. As the respective connections on $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$ and $\mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$ satisfy Griffiths transversality with respect to their respective filtrations, therefore, it follows that the connection on $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$ also satisfies Griffiths transversality with respect to the tensor product filtration. Then, by the compatibility of the isomorphisms in (3-1) with connections and filtrations, we see that the respective connection on each term of (3-1) satisfies Griffiths transversality with respect to the filtration. Finally, note that the left-hand isomorphism in (3-1) is given as $ab \otimes x \leftarrow\!\shortmid a \otimes b \otimes x$.

The proof of Theorem 3.6 depends on the following important observation.

Lemma 3.8 [Abhinandan 2025, Proposition 4.27]. *Let V be a positive finite q-height representation of G_R such that the $\boldsymbol{A}_R^+$-module $\boldsymbol{N}(T)$ is finite free of rank* $\dim_{\mathbb{Q}_p} V$. *Then there exists a finite free R-module*

$$M_0 \subset M := (\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T))^{\Gamma_R},$$

stable under the Frobenius and such that $M_0[1/p] = M[1/p] \xrightarrow{\sim} \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$ are free $R[1/p]$-modules of rank $\dim_{\mathbb{Q}_p} V$.

Proposition 3.9. *Let V be a positive finite q-height representation of G_R of height s such that $\boldsymbol{N}(T)$ is free over $\boldsymbol{A}_R^+$. Let*

$$M_0 \subset M := (\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T))^{\Gamma_R}$$

be the free R-module obtained in Lemma 3.8. Then the R-module $M_0/\varphi^*(M_0)$ is killed by p^{ms}.

Proof. In order to prove the claim, we will use — without recalling constructions and notation — the proof of [Abhinandan 2025, Proposition 4.28]. Let $\boldsymbol{f} = \{f_1, \ldots, f_h\}$ be an $\boldsymbol{A}_R^+$-basis of $\boldsymbol{N}(T)$. Then, from Lemma 3.8 and the proof of [Abhinandan 2025, Proposition 4.28], we have that M_0 is a free R-module with basis $\boldsymbol{g} = \{g_1, \ldots, g_h\}$, where $\boldsymbol{g} = \varphi^m(\boldsymbol{f})\varphi^m(A)$ for some A in $\mathrm{GL}(h, \mathcal{O}\hat{S}_m^{\mathrm{PD}})$. It is easy to see that M_0 is independent of the choice of the $\boldsymbol{A}_R^+$-basis of $\boldsymbol{N}(T)$. Note that we have $q = \varphi(\pi)/\pi = p\varphi(\pi/t)(t/\pi)$, and since π/t is a unit in $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$ (see Lemma 2.18), we therefore obtain that q and p are associates in $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$. Furthermore, $\boldsymbol{N}(T)/\varphi^*(\boldsymbol{N}(T))$ is killed by q^s, where s is the height of V. So

$$(\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T))/\varphi^{m,*}(\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T))$$

is killed by p^{ms}, where we write

$$\varphi^{m,*}(\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)) = \bigoplus_{i=1}^{h} \mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \varphi^m(f_i).$$

Now, recall that $\det A$ is a unit in $\mathcal{O}\hat{S}_m^{\mathrm{PD}}$ [Abhinandan 2025, Lemma 4.43]; therefore, $\varphi^m(\det A)$ is a unit in $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$ and $\varphi^m(A)$ is invertible over $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$; in particular, we have the isomorphism

$$\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0 \xrightarrow{\sim} \varphi^{m,*}(\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)).$$

So, we get that the cokernel of the natural inclusion $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0 \subset \mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)$ is killed by p^{ms}. Moreover, the observation above also implies that the cokernel of the composition

$$\varphi^{m,*}(\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0) \subset \mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0 \xrightarrow{\sim} \varphi^{m,*}(\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T))$$

is killed by p^{ms}. In other words, we get that

$$p^{ms}(\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0) \subset \varphi^{m,*}(\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0) \subset \varphi^{*}(\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0).$$

Finally, as the action of the Frobenius is Γ_R-equivariant, by taking Γ_R-invariants, we therefore get that $p^{ms}M_0 \subset \varphi^*(M_0)$; i.e., $M_0/\varphi^*(M_0)$ is killed by p^{ms}. □

Remark 3.10. From the proof of Proposition 3.9, note that we have an inclusion

$$p^s(\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)) \subset \varphi^*(\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)).$$

Since the action of Frobenius is Γ_R-equivariant, by taking Γ_R-invariants of the preceding inclusion, we therefore get that $p^sM \subset \varphi^*(M)$. Moreover, from Lemma 3.8 and Proposition 3.9, as $M_0 \subset M$, it also follows that the cokernel of the composition

$$\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \to \mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)$$

is killed by p^{ms} (in fact, the cokernel is killed by p^s, see Remark 3.12).

Remark 3.11. Using Theorem 3.6, we equip $M \subset M[1/p]$ with a p-adically quasinilpotent integrable connection $\partial : M \to M \otimes_R \Omega_R^1$ and an induced filtration compatible with the tensor product filtration on $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)$ (see [Abhinandan 2025, §4.5.1]); the connection satisfies Griffiths transversality with respect to the filtration. Furthermore, using the explicit description of M_0 in Proposition 3.9, we obtain an induced filtration on M_0 and an induced p-adically quasinilpotent integrable connection $\partial : M_0 \to M_0 \otimes_R \Omega_R^1$ satisfying Griffiths transversality with respect to the filtration.

Remark 3.12. Note that we fixed $m \in \mathbb{N}_{\geqslant 1}$ in the beginning and the R-modules obtained above depend on this choice. In particular, let $1 \leqslant m \leqslant m'$, with $\varpi = \zeta_{p^m} - 1$ and $\varpi' = \zeta_{p^{m'}} - 1$. Then, we have an inclusion $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\boldsymbol{A}_{R,\varpi'}^{\mathrm{PD}}$, and we obtain

$$M = (\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T))^{\Gamma_R} \subset (\mathcal{O}\boldsymbol{A}_{R,\varpi'}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T))^{\Gamma_R} = M'.$$

As the cokernel of $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \to \mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)$ is killed by p^{ms} (see Remark 3.10) and

$$\mathcal{O}\boldsymbol{A}_{R,\varpi'}^{\mathrm{PD}} \otimes_R M \subset \mathcal{O}\boldsymbol{A}_{R,\varpi'}^{\mathrm{PD}} \otimes_R M',$$

the cokernel of $\mathcal{O}\boldsymbol{A}^{\mathrm{PD}}_{R,\varpi'} \otimes_R M' \to \mathcal{O}\boldsymbol{A}^{\mathrm{PD}}_{R,\varpi'} \otimes_{\boldsymbol{A}^+_R} \boldsymbol{N}(T)$ is also killed by p^{ms}. In particular, taking $m = 1$, we see that the cokernel of

$$\mathcal{O}\boldsymbol{A}^{\mathrm{PD}}_{R,\varpi'} \otimes_R M' \to \mathcal{O}\boldsymbol{A}^{\mathrm{PD}}_{R,\varpi'} \otimes_{\boldsymbol{A}^+_R} \boldsymbol{N}(T)$$

is always killed by p^s. Finally, let M_0 and M'_0 be R-modules obtained for m and m', respectively, in Lemma 3.8. Then we have that $\varphi^{m'-m}(M'_0) \subset M_0$.

3.3. ***Filtrations and a Poincaré lemma.*** Let T be a positive finite q-height $\mathbb{Z}_p$-representation of G_R, and set $V = T[1/p]$. Let $\boldsymbol{N}(T)$ denote the associated Wach module over $\boldsymbol{A}^+_R$, and set

$$M := (\mathcal{O}\boldsymbol{A}^{\mathrm{PD}}_{R,\varpi} \otimes_{\boldsymbol{A}^+_R} \boldsymbol{N}(T))^{\Gamma_R}$$

as a finitely generated p-torsion free R-module. Now consider the diagram

$$\begin{array}{ccc} \mathcal{B} \otimes_{R[1/p]} M[1/p] & \xrightarrow[\sim]{\alpha} & \mathcal{B} \otimes_{\boldsymbol{B}^+_R} \boldsymbol{N}(V) \\ \wr\downarrow & & \wr\downarrow \beta \\ \mathcal{B} \otimes_{R[1/p]} \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V) & \xrightarrow[\sim]{\alpha_{\mathcal{B}}} & \mathcal{B} \otimes_{\mathbb{Q}_p} V \end{array} \tag{3-2}$$

where $\boldsymbol{B}^+_R = \boldsymbol{A}^+_R[1/p]$ and the maps are as follows: the right vertical arrow is the $\mathcal{B}$-linear extension of the natural inclusion

$$\boldsymbol{N}(V) \subset \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Q}_p} V \subset \mathcal{B} \otimes_{\mathbb{Q}_p} V;$$

the top horizontal arrow is the extension along $\mathcal{O}\boldsymbol{A}^{\mathrm{PD}}_{R,\varpi} \to \mathcal{B}$ of the $\mathcal{O}\boldsymbol{A}^{\mathrm{PD}}_{R,\varpi}$-linear isomorphism

$$\mathcal{O}\boldsymbol{A}^{\mathrm{PD}}_{R,\varpi} \otimes_R M[1/p] \xrightarrow{\sim} \mathcal{O}\boldsymbol{A}^{\mathrm{PD}}_{R,\varpi} \otimes_{\boldsymbol{A}^+_R} \boldsymbol{N}(V)$$

(see the first isomorphism in (3-1) of Theorem 3.6); the left vertical arrow is the extension along $R[1/p] \to \mathcal{B}$ of the $R[1/p]$-linear isomorphism

$$M[1/p] \xrightarrow{\sim} \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$$

(see the second isomorphism in (3-1) of Theorem 3.6), and it is compatible with the respective filtrations; and the bottom horizontal arrow is the filtration-compatible $\mathcal{B}$-linear isomorphism from (2-6) (see Lemma 2.38). The diagram commutes by definition, and the right vertical arrow is an isomorphism because the other three arrows are isomorphisms; see [Abhinandan 2025, §4.5] for a similar diagram over $\mathcal{O}\boldsymbol{B}_{\mathrm{cris}}(\bar{R})$. Using the right vertical arrow of diagram (3-2), for each $r \in \mathbb{Z}$, we set

$$\mathrm{Fil}^r(\mathcal{B} \otimes_{\boldsymbol{B}^+_R} \boldsymbol{N}(V)) := \beta^{-1}(\mathrm{Fil}^r \mathcal{B} \otimes_{\mathbb{Q}_p} V). \tag{3-3}$$

In (3-2), by the compatibility of the left vertical arrow and the bottom horizontal arrow with the respective filtrations, an easy diagram chase shows that, for each $r \in \mathbb{Z}$, the top horizontal arrow induces the isomorphism

$$\alpha : \mathrm{Fil}^r(\mathcal{B} \otimes_{R[1/p]} M[1/p]) \xrightarrow{\sim} \mathrm{Fil}^r(\mathcal{B} \otimes_{\boldsymbol{B}^+_R} \boldsymbol{N}(V)). \tag{3-4}$$

3.3.1. *Filtration on scalar extensions of Wach modules.* Let S be a ring such that $\boldsymbol{A}_R^+ \subset S \subset \mathcal{B}$ and p is not invertible in S. Set $N_S := S \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)$ and note that we have a natural embedding $N_S \to \mathcal{B} \otimes_{\boldsymbol{B}_R^+} \boldsymbol{N}(V)$. We equip N_S with the induced filtration; i.e., for each $r \in \mathbb{Z}$, using (3-3), set

$$\mathrm{Fil}^r N_S := N_S \cap \mathrm{Fil}^r(\mathcal{B} \otimes_{\boldsymbol{B}_R^+} \boldsymbol{N}(V)) \subset \mathcal{B} \otimes_{\boldsymbol{B}_R^+} \boldsymbol{N}(V). \tag{3-5}$$

Similarly, we set

$$\mathrm{Fil}^r N_S[1/p] := N_S[1/p] \cap \mathrm{Fil}^r(\mathcal{B} \otimes_{\boldsymbol{B}_R^+} \boldsymbol{N}(V))$$

for each $r \in \mathbb{Z}$, and it is clear that

$$\mathrm{Fil}^r N_S = N_S \cap \mathrm{Fil}^r N_S[1/p].$$

Remark 3.13. Let S and S' be such that $S \subset S' \subset \mathcal{B}$ and p is not invertible in S'. Then, from the definition of the respective filtrations on N_S and $N_{S'}$ in (3-5), it is clear that

$$\mathrm{Fil}^r N_S = N_S \cap \mathrm{Fil}^r N_{S'} \subset N_{S'}.$$

Lemma 3.14. *Let $S \subset E_{\overline{R}}^{[u,v]}$ be a G_R-stable $\boldsymbol{A}_R^+$-subalgebra. Then, the filtration on N_S in (3-5) is stable under the natural action of G_R on N_S.*

Proof. Let us consider the commutative diagram

$$\begin{array}{ccc} E_{\overline{R}}^{[u,v]} \otimes_R M[1/p] & \xrightarrow[\sim]{\alpha} & E_{\overline{R}}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V) \\ \downarrow & & \downarrow \\ \mathcal{B} \otimes_{R[1/p]} M[1/p] & \xrightarrow[\sim]{\alpha} & \mathcal{B} \otimes_{\boldsymbol{B}_R^+} \boldsymbol{N}(V) \end{array} \tag{3-6}$$

where the bottom horizontal arrow is the top horizontal isomorphism of (3-2); the top horizontal arrow is the extension of the $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$-linear isomorphism

$$\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M[1/p] \xrightarrow{\sim} \mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)$$

(see the first isomorphism in (3-1) of Theorem 3.6) along the G_R-equivariant map $\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \to E_{\overline{R}}^{[u,v]}$ (see Remark 2.27) and compatible with the respective Frobenii, $\boldsymbol{A}_{\overline{R}}^{[u,v]}$-linear connections and the actions of G_R; and the vertical maps are extensions of scalars along the map $E_{\overline{R}}^{[u,v]} \to \mathcal{B}$ (see Lemma 2.31). Now, by using the definition of filtrations on each term (see (2-7) and (3-5)) and the isomorphism in (3-4), the top horizontal arrow induces the following $E_{\overline{R}}^{[u,v]}$-linear isomorphism for each $r \in \mathbb{Z}$:

$$\alpha : \mathrm{Fil}^r(E_{\overline{R}}^{[u,v]} \otimes_R M[1/p]) \xrightarrow{\sim} \mathrm{Fil}^r(E_{\overline{R}}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)). \tag{3-7}$$

As the source of (3-7) is stable under the natural action of G_R on $E_{\overline{R}}^{[u,v]} \otimes_R M[1/p]$ and the top horizontal arrow of (3-6) is G_R-equivariant, it therefore follows that the target of (3-7) is stable under the natural action of G_R on $E_{\overline{R}}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)$. Finally, note that we have the G_R-equivariant inclusion $S \subset E_{\overline{R}}^{[u,v]}$, so, by using Remark 3.13, we obtain that $\mathrm{Fil}^r N_S$ is stable under the natural action of G_R on N_S. ☐

Remark 3.15. Let S be any ring out of

$$\mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\bar{R}), \quad E^{\star}_{R,\varpi} \text{ for } \star \in \{\mathrm{PD}, [u], [u,v]\}, \quad \text{or} \quad E^{\star}_{\bar{R}} \text{ for } \star \in \{\mathrm{PD}, [u], [u,v]\}.$$

Then, by Lemma 3.14, we get that, for each $r \in \mathbb{Z}$, the isomorphism in (3-7) induces a G_R-equivariant S-linear isomorphism

$$\alpha : \mathrm{Fil}^r(S \otimes_R M[1/p]) \xrightarrow{\sim} \mathrm{Fil}^r(S \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)). \tag{3-8}$$

In particular, as the connection on $S \otimes_R M[1/p]$ satisfies Griffiths transversality with respect to the filtration, similar to Remark 3.7, it therefore follows that the connection on $S \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)$ satisfies Griffiths transversality with respect to the filtration in (3-5).

Remark 3.16. Let $E = E^{\star}_{R,\varpi}$ or $E^{\star}_{\bar{R}}$ for $\star \in \{\mathrm{PD}, [u], [u,v]\}$. We claim that

$$\mathrm{Fil}^r(E \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)) = \sum_{i+j=r} \mathrm{Fil}^i E \cdot \mathrm{Fil}^j \boldsymbol{N}(V),$$

where $\mathrm{Fil}^i E \cdot \mathrm{Fil}^j \boldsymbol{N}(V)$ denotes the image of $\mathrm{Fil}^i E \otimes_{\boldsymbol{A}_R^+} \mathrm{Fil}^j \boldsymbol{N}(V) \to E \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)$. Indeed, using Lemma 2.31, Remark 3.4 and (3-4), it easily follows that

$$\mathrm{Fil}^i E \cdot \mathrm{Fil}^j \boldsymbol{N}(V) \subset \mathrm{Fil}^r(\mathcal{B} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V));$$

in particular, from (3-5), we deduce that

$$\sum_{i+j=r} \mathrm{Fil}^i E \cdot \mathrm{Fil}^j \boldsymbol{N}(V) \subset \mathrm{Fil}^r(E \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)).$$

To show the reverse inclusion, recall that the isomorphism $\mathrm{Fil}^r M[1/p] \xrightarrow{\sim} \mathrm{Fil}^r \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$ is a finite projective $R[1/p]$-module (see Theorem 3.6 and [Brinon 2008, Proposition 8.3.2]), in particular flat as an R-module, and the natural map $\mathrm{Fil}^i E \otimes_R \mathrm{Fil}^j M[1/p] \to E \otimes_R M[1/p]$ is injective by Lemma 2.35 for each $i, j \in \mathbb{N}$; we denote the image as $\mathrm{Fil}^i E \cdot \mathrm{Fil}^j M[1/p]$ and note that

$$\mathrm{Fil}^r(E \otimes_R M[1/p]) = \sum_{i+j=r} \mathrm{Fil}^i E \otimes_R \mathrm{Fil}^j M[1/p] = \sum_{i+j=k} \mathrm{Fil}^i E \cdot \mathrm{Fil}^j M[1/p].$$

Now, since the isomorphism $E \otimes_R M[1/p] \xrightarrow{\sim} E \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)$ is given by the natural multiplication map and the filtration on $M[1/p]$ is given as the tensor product filtration (see Remark 3.7), we therefore obtain that the natural map

$$\sum_{i+j=k} \mathrm{Fil}^i E \cdot \mathrm{Fil}^j M[1/p] \to \sum_{i+j=r} \mathrm{Fil}^i E \cdot \mathrm{Fil}^j \boldsymbol{N}(V)$$

is injective. But, from (3-8), we have the isomorphism $\mathrm{Fil}^r(E \otimes_R M[1/p]) \xrightarrow{\sim} \mathrm{Fil}^r(E \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V))$. Hence it follows that $\mathrm{Fil}^r(E \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)) = \sum_{i+j=r} \mathrm{Fil}^i E \cdot \mathrm{Fil}^j \boldsymbol{N}(V)$.

Next, let $S = \boldsymbol{A}^{\star}_{R,\varpi}$ for $\star \in \{+, \mathrm{PD}, [u], [u,v], (0,v]+\}$ or $E^{\star}_{R,\varpi}$ for $\star \in \{\mathrm{PD}, [u], [u,v]\}$, and set $N_S := S \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)$. Then, we have the following.

Lemma 3.17. *For each $r \in \mathbb{Z}$, we have that* $\mathrm{Fil}^r N_S \cap \pi N_S = \pi \,\mathrm{Fil}^{r-1} N_S$.

Proof. Note that the claim is clear for $r \leqslant 0$, so let $r \geqslant 1$. Let $S' = E_{R,\varpi}^{[u,v]}$ and, using the definition of the filtration on $N_{S'}[1/p]$ in (3-5), the S'-linear isomorphism in (3-7) and Lemma 2.39, note that

$$\begin{aligned}\mathrm{Fil}^r N_{S'}[1/p] \cap \pi N_{S'}[1/p] &= \alpha(\mathrm{Fil}^r(S' \otimes_R M[1/p])) \cap \alpha(\pi S' \otimes_R M[1/p]) \\ &= \alpha(\mathrm{Fil}^r(S' \otimes_R M[1/p]) \cap \pi(S' \otimes_R M[1/p])) \\ &= \alpha(\pi\, \mathrm{Fil}^{r-1}(S' \otimes_R M[1/p])) = \pi\, \mathrm{Fil}^{r-1} N_{S'}[1/p].\end{aligned}$$

In particular, we get that

$$\mathrm{Fil}^r N_{S'} \cap \pi N_{S'} = \pi\, \mathrm{Fil}^{r-1} N_{S'}[1/p] \cap \pi N_{S'} = \pi\, \mathrm{Fil}^{r-1} N_{S'}.$$

Now, by using the definition of the filtration on N_S in (3-5), Remark 3.13 and the equality above, we get

$$\mathrm{Fil}^r N_S \cap \pi N_S \subset \pi\, \mathrm{Fil}^{r-1} N_{S'} \cap \pi N_S = \pi\, \mathrm{Fil}^{r-1} N_S.$$

The other inclusion, i.e., $\pi\, \mathrm{Fil}^{r-1} N_S \subset \mathrm{Fil}^r N_S \cap \pi N_S$, is obvious. □

Lemma 3.18. *For each $r \in \mathbb{Z}$, we have*

$$\mathrm{Fil}^r N_S[1/p] = \sum_{i+j=r} \mathrm{Fil}^i S \cdot \mathrm{Fil}^j \boldsymbol{N}(V),$$

where $\mathrm{Fil}^i S \cdot \mathrm{Fil}^j \boldsymbol{N}(V)$ *denotes the image of* $\mathrm{Fil}^i S \otimes_{A_R^+} \mathrm{Fil}^j \boldsymbol{N}(V) \to N_S[1/p]$.

Proof. The claim for $E_{R,\varpi}^{\star}$ was shown in Remark 3.16. For $\boldsymbol{A}_{R,\varpi}^{\star}$, the claim for $\star \in \{\mathrm{PD}, [u], [u,v]\}$ follows from the proof of Lemma 3.21 (see Remark 3.22), and, for $\boldsymbol{A}_{R,\varpi}^{+}$, the claim follows from Lemma 3.19. So, it remains to show the claim for $\boldsymbol{A}_{R,\varpi}^{(0,v]+}$. Let

$$S = \boldsymbol{A}_{R,\varpi}^{(0,v]+}, \quad A = \boldsymbol{A}_{R,\varpi}^{+}, \quad B = \boldsymbol{A}_{R,\varpi}^{[u]}, \quad C = \boldsymbol{A}_{R,\varpi}^{[u,v]} \quad \text{and} \quad N[1/p] = \boldsymbol{N}(V).$$

By definition, we have $C = S + B$, and the ideal $\mathrm{Fil}^i C$ is topologically generated by $(\mathrm{Fil}^i S + \mathrm{Fil}^i B)C$ for all $i \in \mathbb{N}$ (see Remark 2.8). Moreover, from Remark 3.22, we have

$$\mathrm{Fil}^r N_B[1/p] = \sum_{i+j=r} \mathrm{Fil}^i B \cdot \mathrm{Fil}^j N[1/p] \quad \text{and} \quad \mathrm{Fil}^r N_C[1/p] = \sum_{i+j=r} \mathrm{Fil}^i C \cdot \mathrm{Fil}^j N[1/p].$$

So, by setting $M := \sum_{i+j=r} \mathrm{Fil}^i S \cdot \mathrm{Fil}^j N[1/p]$, we see that

$$\mathrm{Fil}^r N_C[1/p] = \sum_{i+j=r} \mathrm{Fil}^i C \cdot \mathrm{Fil}^j N[1/p] = M + \mathrm{Fil}^r N_B[1/p] = \mathrm{Fil}^r N_S[1/p] + \mathrm{Fil}^r N_B[1/p].$$

Now, consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & M + \mathrm{Fil}^r N_B[1/p] & \longrightarrow & (\mathrm{Fil}^r N_B[1/p])/(M \cap \mathrm{Fil}^r N_B[1/p]) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Fil}^r N_S[1/p] & \longrightarrow & \mathrm{Fil}^r N_C[1/p] & \longrightarrow & (\mathrm{Fil}^r N_C[1/p])/(\mathrm{Fil}^r N_S[1/p]) & \longrightarrow & 0 \end{array}$$

where the left vertical arrow is injective (by an argument similar to the first part of Remark 3.16). To get the claim, it is enough to show that the right vertical arrow is bijective. Note that

$$\begin{aligned}(\mathrm{Fil}^r\, N_C[1/p])/(\mathrm{Fil}^r\, N_S[1/p]) &= (\mathrm{Fil}^r\, N_S[1/p] + \mathrm{Fil}^r\, N_B[1/p])/(\mathrm{Fil}^r\, N_S[1/p])\\ &= (\mathrm{Fil}^r\, N_B[1/p])/(\mathrm{Fil}^r\, N_S[1/p] \cap \mathrm{Fil}^r\, N_B[1/p]).\end{aligned}$$

It is clear that $M \cap \mathrm{Fil}^r\, N_B[1/p] \subset \mathrm{Fil}^r\, N_S[1/p] \cap \mathrm{Fil}^r\, N_B[1/p]$, and we claim that the reverse inclusion also holds. Indeed, as $N[1/p]$ is a finite projective $\boldsymbol{A}_R^+[1/p]$-module and $A = S \cap B \subset C$, we therefore get that $N_A[1/p] = N_S[1/p] \cap N_B[1/p] \subset N_C[1/p]$. Then, it follows that

$$\mathrm{Fil}^r\, N_S[1/p] \cap \mathrm{Fil}^r\, N_B[1/p] \subset N_S[1/p] \cap N_B[1/p] = N_A[1/p];$$

in particular, we see that

$$\mathrm{Fil}^r\, N_S[1/p] \cap \mathrm{Fil}^r\, N_B[1/p] = \mathrm{Fil}^r\, N_A[1/p] \cap \mathrm{Fil}^r\, N_B[1/p] \subset M \cap \mathrm{Fil}^r\, N_B[1/p],$$

where the equality follows from Remark 3.13 and the inclusion follows by using the description of $\mathrm{Fil}^r\, N_A[1/p]$ from Lemma 3.19. So, we obtain that the right vertical arrow in the diagram is bijective, and hence the left vertical arrow is bijective as well; i.e.,

$$\mathrm{Fil}^r\, N_S[1/p] = \sum_{i+j=r} \mathrm{Fil}^i\, S \cdot \mathrm{Fil}^j\, \boldsymbol{N}(V). \qquad \square$$

Set

$$\mathrm{Fil}^i\, \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) := \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \cap \mathrm{Fil}^i\, \boldsymbol{A}_{\mathrm{cris}}(\bar{R}) = \xi^i \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \subset \boldsymbol{A}_{\mathrm{cris}}(\bar{R}) \quad \text{for } i \in \mathbb{Z}.$$

Lemma 3.19. *For $S = \boldsymbol{A}_{R,\varpi}^+$ and any $r \in \mathbb{Z}$, we have*

$$\mathrm{Fil}^r\, N_S[1/p] = (\mathrm{Fil}^r\, \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} V) \cap N_S[1/p] = \sum_{i+j=r} \mathrm{Fil}^i\, \boldsymbol{A}_{R,\varpi}^+ \cdot \mathrm{Fil}^j\, \boldsymbol{N}(V).$$

Proof. The first equality is obvious from the definition of the filtration on $N_S[1/p]$ in (3-5) and Remark 3.13. For the second equality, we will show a stronger claim: $\mathrm{Fil}^r\, N_S = \sum_{i+j=r} \mathrm{Fil}^i\, \boldsymbol{A}_{R,\varpi}^+ \cdot \mathrm{Fil}^j\, \boldsymbol{N}(T)$. From the first equality, note that

$$\mathrm{Fil}^r\, N_S = (\mathrm{Fil}^r\, \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} V) \cap N_S = (\mathrm{Fil}^r\, \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} T) \cap N_S.$$

Let us set $F^r N_S := \sum_{i+j=r} \mathrm{Fil}^i\, \boldsymbol{A}_{R,\varpi}^+ \cdot \mathrm{Fil}^j\, \boldsymbol{N}(T)$ for each $r \in \mathbb{N}$; note that the inclusion $F^r N_S \subset \mathrm{Fil}^r\, N_S$ is obvious. To prove the reverse inclusion, we will simplify the claim a bit. Note that the natural map $\boldsymbol{A}_{R,\varpi}^+ \otimes_{\boldsymbol{A}_R^+} \mathrm{Fil}^r\, \boldsymbol{N}(T) \to N_S$ is injective because the morphism $\boldsymbol{A}_R^+ \to \boldsymbol{A}_{R,\varpi}^+$ is flat. It follows that

$$F^r N_S = \sum_{i+j=r} \mathrm{Fil}^i\, \boldsymbol{A}_{R,\varpi}^+ \otimes_{\boldsymbol{A}_R^+} \mathrm{Fil}^j\, \boldsymbol{N}(T) = \xi F^{r-1} N_S + \boldsymbol{A}_{R,\varpi}^+ \otimes_{\boldsymbol{A}_R^+} \mathrm{Fil}^r\, \boldsymbol{N}(T).$$

Now, to show the inclusion $\mathrm{Fil}^r\, N_S \subset F^r N_S$, we will proceed by induction on $r \in \mathbb{N}$. The case $r = 0$ is trivial, so assume that $r \geqslant 1$ and the claim holds for all $k \leqslant r-1$. Let us note that, inside $\boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} T$, we have

$$\mathrm{Fil}^r\, N_S \cap \xi\, \mathrm{Fil}^{r-2}\, N_S = (\xi^r \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} T) \cap N_S \cap (\xi^{r-1} \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} T) \cap \xi N_S = \xi\, \mathrm{Fil}^{r-1}\, N_S.$$

Therefore, it follows that the natural inclusion $\mathrm{Fil}^r N_S \subset \mathrm{Fil}^{r-1} N_S$ induces an injective map

$$(\mathrm{Fil}^r N_S)/(\xi\, \mathrm{Fil}^{r-1} N_S) \to (\mathrm{Fil}^{r-1} N_S)/(\xi\, \mathrm{Fil}^{r-2} N_S),$$

where we have

$$(\mathrm{Fil}^{r-1} N_S)/(\xi\, \mathrm{Fil}^{r-2} N_S) = (\boldsymbol{A}_{R,\varpi}^+ \otimes_{\boldsymbol{A}_R^+} \mathrm{Fil}^{r-1} \boldsymbol{N}(T))/((\boldsymbol{A}_{R,\varpi}^+ \otimes_{\boldsymbol{A}_R^+} \mathrm{Fil}^{r-1} \boldsymbol{N}(T)) \cap (\xi\, \mathrm{Fil}^{r-2} N_S)).$$

In particular, given any element x in $\mathrm{Fil}^r N_S$, we can write

$$x = \xi y + z \quad \text{for some } y \in \mathrm{Fil}^{r-1} N_S = F^{r-1} N_S \text{ and } z \in \boldsymbol{A}_{R,\varpi}^+ \otimes_{\boldsymbol{A}_R^+} \mathrm{Fil}^{r-1} \boldsymbol{N}(T).$$

To obtain the claim, it is enough to show that z is an element of $F^r N_S$.

Note that we have

$$\mathrm{Fil}^r N_S = (\xi^r \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} T) \cap N_S,$$

so we see that $z = x - \xi y = \xi^r z'$, for some $z' \in \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} T$. Recall that we have $\boldsymbol{A}_{R,\varpi}^+ = \boldsymbol{A}_R^+[\pi_m]$, where $\pi_m = \varphi^{-m}(\pi)$. It follows that any element $a \in \boldsymbol{A}_{R,\varpi}^+$ has a unique presentation as

$$a = \sum_{i=0}^{e} a_i (1+\pi_m)^{i/p},$$

with $a_i \in \boldsymbol{A}_R^+$ and $e = p^{m-1}(p-1)$. Let us write $z = \sum_j f_j n_j$ for some $f_j \in \boldsymbol{A}_{R,\varpi}^+$ and $n_j \in \mathrm{Fil}^{r-1} \boldsymbol{N}(T)$. Then, expressing each f_j as above, i.e., in terms of the powers of $1+\pi_m$, and rearranging the sum for z in terms of the powers of $1+\pi_m$, we get that $z = \sum_{i=0}^{e} z_i (1+\pi_m)^{i/p}$ for some $z_i \in \mathrm{Fil}^{r-1} \boldsymbol{N}(T)$ (obtained from elements n_j above). Now, by using Remark 3.4, we can write each z_i as $\xi^{r-1} w_i$ for some $w_i \in \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} T$. Plugging the values of z and z_i into the equality $z = \sum_{i=0}^{e} z_i (1+\pi_m)^{i/p}$ and noting that $\boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} T$ is ξ-torsion free, we get that $\xi z' = \sum_{i=0}^{e} w_i (1+\pi_m)^{i/p}$. Reducing the latter equality modulo $\xi \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} T$, we obtain the equality $\sum_{i=0}^{e} w_i \zeta_{p^m}^{i/p} = 0 \bmod \xi$ in $\mathbb{C}^+(\bar{R}) \otimes_{\mathbb{Z}_p} T$, which is possible only if $w_0 = w_1 = \cdots = w_e \bmod \xi \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} T$. So we write

$$\xi z' = \xi w_0 + \sum_{i=1}^{e} (w_i - w_0)(1+\pi_m)^{i/p},$$

with $w_i - w_0 \in \xi \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} T$ for each $1 \leqslant i \leqslant e$. In particular, we get that

$$z = \xi^r z' = \xi^r w_0 + \sum_{i=1}^{e} \xi^{r-1}(w_i - w_0)(1+\pi_m)^{i/p} = \xi z_0 + \sum_{i=1}^{e} (z_i - z_0)(1+\pi_m)^{i/p}.$$

Note that $z_0 \in \mathrm{Fil}^{r-1} \boldsymbol{N}(T)$ and

$$z_i - z_0 = \xi^{r-1}(w_i - w_0) \in (\xi^r \boldsymbol{A}_{\mathrm{inf}}(\bar{R}) \otimes_{\mathbb{Z}_p} T) \cap \mathrm{Fil}^{r-1} \boldsymbol{N}(T) = \mathrm{Fil}^r \boldsymbol{N}(T)$$

(see Remark 3.4) for each $1 \leqslant i \leqslant e$. Therefore, it follows that

$$z \in \xi\, \mathrm{Fil}^{r-1} N_S + \boldsymbol{A}_{R,\varpi}^+ \otimes_{\boldsymbol{A}_R^+} \mathrm{Fil}^r \boldsymbol{N}(T) = F^r N_S.$$

This allows us to conclude. □

Next, let $k \in \mathbb{Z}$ and consider the p-adic representation $V(k)$ of G_R. Using (3-5) and Lemma 3.14, we define a Γ_R-stable filtration on $E_{R,\varpi}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V(k))$ as follows:

$$\mathrm{Fil}^r(E_{R,\varpi}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V(k))) := \pi^{-k}\,\mathrm{Fil}^{r+k}(E_{R,\varpi}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V))(k). \tag{3-9}$$

From the explicit description of the filtration in Remark 3.16 and by using Lemma 3.3, it follows that

$$\mathrm{Fil}^r(E_{R,\varpi}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V(k))) = \sum_{i+j=r} \mathrm{Fil}^i\, E_{R,\varpi}^{[u,v]} \cdot \mathrm{Fil}^j\, \boldsymbol{N}(V(k)).$$

Furthermore, let $S \subset E_{R,\varpi}^{[u,v]}$ be as above (see before Lemma 3.17). Then, we note that we have a natural embedding $S \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T(k)) \to E_{R,\varpi}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V(k))$, and we equip the former with an induced Γ_R-stable filtration; i.e., for each $r \in \mathbb{Z}$, set

$$\mathrm{Fil}^r(S \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T(k))) := (S \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T(k))) \cap \mathrm{Fil}^r(E_{R,\varpi}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V(k))) \subset E_{R,\varpi}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V(k)). \tag{3-10}$$

Using (3-9) and Remark 3.13, it is easy to see the following.

Lemma 3.20. *For each $r \in \mathbb{Z}$, we have*

$$\mathrm{Fil}^r(S \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T(k))) = \pi^{-k}\,\mathrm{Fil}^{r+k}(S \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T))(k).$$

3.3.2. *Filtered Poincaré lemma.* In the notation of Section 2.8.3, let us set $A = \boldsymbol{A}_{R,\varpi}^{\star}$ (resp. $\boldsymbol{A}_{\bar{R}}^{\star}$), $B = R_{\varpi}^{\star}$ and $E = E_{R,\varpi}^{\star}$ (resp. $E_{\bar{R}}^{\star}$) for $\star \in \{\mathrm{PD}, [u], [u,v]\}$. Let

$$\omega_0 := \frac{dX_0}{1+X_0} \quad \text{and} \quad \omega_i := \frac{dX_i}{X_i} \quad \text{for } 1 \leqslant i \leqslant d.$$

Set

$$\Omega^1 := \bigoplus_{i=1}^{d} \mathbb{Z}\omega_i \quad \text{and} \quad \Omega^k := \bigwedge^k \Omega^1.$$

Then, we have $\Omega_{E/A}^k = E \otimes_{\mathbb{Z}} \Omega^k$ and, from Remark 2.25 (iv), for $r \in \mathbb{Z}$, we have the filtered de Rham complex of E relative to A:

$$\mathrm{Fil}^r\,\Omega_{E/A}^{\bullet} := \mathrm{Fil}^r\,E \to \mathrm{Fil}^{r-1}\,E \otimes_{\mathbb{Z}} \Omega^1 \to \mathrm{Fil}^{r-2}\,E \otimes_{\mathbb{Z}} \Omega^2 \to \cdots.$$

Let T be a positive finite q-height $\mathbb{Z}_p$-representation of G_R as above and assume that $\boldsymbol{N}(T)$ is finite free over $\boldsymbol{A}_R^+$. Let us set $N_A := A \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)$, equipped with a filtration as in (3-5), and similarly set $N_E := E \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)$, equipped with a filtration as in (3-5). Note that the A-linear differential operator on E induces a quasinilpotent integrable connection $\partial : N_E \to N_E \otimes_E \Omega_{E/A}^1$ satisfying Griffiths transversality with respect to the filtration (since the same is true after inverting p, see Remark 3.15). In particular, for each $r \in \mathbb{Z}$, we have the filtered de Rham complex

$$\begin{aligned}\mathrm{Fil}^r\,N_E \otimes \Omega_{E/A}^{\bullet} &:= \mathrm{Fil}^r\,N_E \to \mathrm{Fil}^{r-1}\,N_E \otimes_E \Omega_{E/A}^1 \to \mathrm{Fil}^{r-2}\,N_E \otimes_E \Omega_{E/A}^2 \to \cdots \\ &= \mathrm{Fil}^r\,N_E \to \mathrm{Fil}^{r-1}\,N_E \otimes_{\mathbb{Z}} \Omega^1 \to \mathrm{Fil}^{r-2}\,N_E \otimes_{\mathbb{Z}} \Omega^2 \to \cdots.\end{aligned}$$

Using the equality $N_A = N_E^{\partial=0}$ and (3-5), we note that $\mathrm{Fil}^r\,N_A = \mathrm{Fil}^r\,N_E \cap N_E^{\partial=0} = (\mathrm{Fil}^r\,N_E)^{\partial=0}$. Then we have the following filtered Poincaré lemma.

Lemma 3.21. *The natural map* $\mathrm{Fil}^r N_A \to \mathrm{Fil}^r N_E \otimes \Omega^{\bullet}_{E/A}$ *is a quasi-isomorphism.*

Proof. The claim follows by employing an argument similar to the proof of Lemma 2.41, where we use the description of filtration on $N_E[1/p]$ from Remark 3.16. We omit the details. □

Remark 3.22. From the proof of Lemma 3.21, using the map $h^0 : \mathrm{Fil}^r N_E[1/p] \to \mathrm{Fil}^r N_A[1/p]$, it follows that, for any $r \in \mathbb{Z}$, we have

$$\mathrm{Fil}^r N_A[1/p] = \sum_{i+j=r} \mathrm{Fil}^i A \cdot \mathrm{Fil}^j \boldsymbol{N}(V),$$

where $\mathrm{Fil}^i A \cdot \mathrm{Fil}^j \boldsymbol{N}(V)$ denotes the image of $\mathrm{Fil}^i A \otimes_{\boldsymbol{A}_R^+} \mathrm{Fil}^j \boldsymbol{N}(V) \to A \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)$.

3.4. *Relative Fontaine–Laffaille modules.* In this subsection we will consider the category of relative Fontaine–Laffaille modules $\mathrm{MF}_{[0,s],\mathrm{free}}(R, \Phi, \partial)$ defined in [Tsuji 2020, §4] as a full subcategory of the abelian category $\mathfrak{MF}^{\nabla}_{[0,s]}(R)$ introduced in [Faltings 1989, §II]. Let $s \in \mathbb{N}$ such that $s \leqslant p-2$.

Definition 3.23. Define the category of *free relative Fontaine–Laffaille* modules of level $[0, s]$, denoted by $\mathrm{MF}_{[0,s],\mathrm{free}}(R, \Phi, \partial)$, as follows:

An object with weights/level in the interval $[0, s]$ is a quadruple $(M, \mathrm{Fil}^{\bullet} M, \partial, \Phi)$ such that:

(i) M is a free R-module of finite rank. It is equipped with a decreasing filtration $\{\mathrm{Fil}^k M\}_{k\in\mathbb{Z}}$ by finite R-submodules, with $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{s+1} M = 0$ and such that $\mathrm{gr}^k_{\mathrm{Fil}} M$ is a finite free R-module for all $k \in \mathbb{Z}$.

(ii) The connection $\partial : M \to M \otimes_R \Omega^1_R$ is quasinilpotent and integrable and satisfies Griffiths transversality with respect to the filtration; i.e., $\partial(\mathrm{Fil}^k M) \subset \mathrm{Fil}^{k-1} M \otimes_R \Omega^1_R$ for all $k \in \mathbb{Z}$.

(iii) Let $(\varphi^*(M), \varphi^*(\partial))$ denote the pullback of (M, ∂) by $\varphi : R \to R$ and equip it with a decreasing filtration

$$\mathrm{Fil}^k_p(\varphi^*(M)) = \sum_{i\in\mathbb{N}} (p^i/i!)\varphi^*(\mathrm{Fil}^{k-i} M) \quad \text{for } k \in \mathbb{Z}.$$

Suppose that there is an R-linear morphism $\Phi : \varphi^*(M) \to M$ such that Φ is compatible with connections, $\Phi(\mathrm{Fil}^k_p(\varphi^*(M))) \subset p^k M$ for $0 \leqslant k \leqslant s$ and

$$\sum_{k=0}^{s} p^{-k}\Phi(\mathrm{Fil}^k_p(\varphi^*(M))) = M.$$

Denote the composition $M \to \varphi^*(M) \xrightarrow{\Phi} M$ by φ.

A morphism between two objects of the category $\mathrm{MF}_{[0,s],\mathrm{free}}(R, \Phi, \partial)$ is a continuous R-linear map compatible with the homomorphism Φ and the connection ∂ on each side.

Remark 3.24. In Definition 3.23 (iii), note that $\varphi^*(M)$ denotes the R-module $R \otimes_{\varphi,R} M$ on which the O_F-linear connection is given by the formula $\varphi^*(\partial)(a \otimes x) = da \otimes x + a \otimes \partial(x)$ for any a in R and x in M. Furthermore, compatibility of the R-linear morphism $\Phi : \varphi^*(M) \to M$ with connections means that, for any a in R and x in M, we must have $\partial \circ \Phi(a \otimes x) = \Phi \circ \varphi^*(\partial)(a \otimes x)$.

To an object M in $\mathrm{MF}_{[0,s],\mathrm{free}}(R, \varphi, \mathrm{Fil})$, we can functorially associate a $\mathbb{Z}_p$-module as

$$T^*_{\mathrm{cris}}(M) := \mathrm{Hom}_{R,\mathrm{Fil},\varphi,\partial}(M, \mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\bar{R})),$$

i.e., R-linear maps from M to $\mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\bar{R})$, compatible with the respective Frobenii, filtrations and connections. Set

$$T_{\mathrm{cris}}(M) := \mathrm{Hom}_{\mathbb{Z}_p}(T^*_{\mathrm{cris}}(M), \mathbb{Z}_p),$$

and note that it is a finite free $\mathbb{Z}_p$-module of rank $\mathrm{rk}_R\, M$, admitting a continuous action of G_R. By [Faltings 1989; Tsuji 2020], the p-adic representation $V_{\mathrm{cris}}(M) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_{\mathrm{cris}}(M)$ is crystalline with Hodge–Tate weights in the interval $[-s, 0]$.

Theorem 3.25 [Abhinandan 2025, Theorem 5.4]. *For a free relative Fontaine–Laffaille module M over R of level $[0, s]$, the associated p-adic representation*

$$V_{\mathrm{cris}}(M) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_{\mathrm{cris}}(M)$$

of G_R is a positive finite q-height representation (in the sense of Definition 3.1).

Remark 3.26. (i) The results of [Abhinandan 2025] are shown for $s = p-2$. However, all the arguments can be adapted almost verbatim (by replacing $p-2$ everywhere by any $0 \leqslant s \leqslant p-2$).

(ii) Let M be a free relative Fontaine–Laffaille module over R of level $[0, s]$, and let $T = T_{\mathrm{cris}}(M)$ be its associated $\mathbb{Z}_p$-representation of G_R. Then, from Theorem 3.25, we have a free relative Wach module $\boldsymbol{N}(T)$ over $\boldsymbol{A}_R^+$ associated to T. Moreover, by combining [Abhinandan 2025, Propositions 5.23 & 5.27] and the proof of [Abhinandan 2025, Theorem 5.4], we have a natural isomorphism

$$\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \xrightarrow{\sim} \mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T),$$

compatible with the respective Frobenii, filtrations, connections and the actions of Γ_R.

(iii) From the proof of [Abhinandan 2025, Theorem 5.4], one can observe that $M/\Phi(\varphi^*(M))$ is p^s-torsion and s equals the maximum among the absolute values of Hodge–Tate weights of $V_{\mathrm{cris}}(M)$.

Remark 3.27. In Definition 3.23, we considered finite free R-modules. For the R/p^n-module M/p^n, the associated $\mathbb{Z}/p^n$-representation of G_R is given as

$$T_{\mathrm{cris}}(M/p^n) = T_{\mathrm{cris}}(M)/p^n.$$

Moreover, we associate a Wach module to $T/p^n = T_{\mathrm{cris}}(M)/p^n$ as $\boldsymbol{N}(T/p^n) := \boldsymbol{N}(T)/p^n$ and we have a natural isomorphism

$$\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}/p^n \otimes_{\boldsymbol{A}_R^+/p^n} \boldsymbol{N}(T/p^n) \xrightarrow{\sim} \mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}/p^n \otimes_{R/p^n} M/p^n$$

compatible with the respective Frobenii, filtrations, connections and the actions of Γ_R; see [Abhinandan 2025, §5.3].

4. Galois cohomology complexes

In this section, we will describe Koszul complexes computing the cohomology for the action of Γ_R and $\operatorname{Lie}\Gamma_R$ on certain modules.

4.1. ***Relative Fontaine–Herr complex.*** From Section 2.4, recall that we have an equivalence between $\mathbb{Z}_p$-representations of G_R and étale (φ, Γ_R)-modules over $\boldsymbol{A}_R$, so it is natural to expect that the continuous G_R-cohomology groups of a $\mathbb{Z}_p$-representation T could be computed using its associated étale (φ, Γ_R)-module $\boldsymbol{D}(T)$. Below, we will consider the continuous cohomology (for the weak topology) of étale (φ, Γ_R)-modules over $\boldsymbol{A}_R$ and $\boldsymbol{A}_R^\dagger$ (see Section 2.4).

Definition 4.1. Let D be an étale (φ, Γ_R)-module over $\boldsymbol{A}_R$ or $\boldsymbol{A}_R^\dagger$. In the derived category of abelian groups, let $\mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, D)$ denote the complex of continuous cochains with values in D.

Theorem 4.2 [Andreatta and Iovita 2008, Theorems 3.3 and 7.10.6; Herr 1998]. *Let $T \in \mathrm{Rep}_{\mathbb{Z}_p}(G_R)$, and let $\boldsymbol{D}(T)$ and $\boldsymbol{D}^\dagger(T)$ be the associated étale (φ, Γ_R)-module over $\boldsymbol{A}_R$ and $\boldsymbol{A}_R^\dagger$, respectively. Then we have natural quasi-isomorphisms*

$$\big[\mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \boldsymbol{D}(T)) \xrightarrow{1-\varphi} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \boldsymbol{D}(T))\big] \simeq \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, T),$$
$$\big[\mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \boldsymbol{D}^\dagger(T)) \xrightarrow{1-\varphi} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \boldsymbol{D}^\dagger(T))\big] \simeq \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, T).$$

Remark 4.3. Theorem 4.2 is also valid for $S = R[\varpi]$, where $\varpi = \zeta_{p^m} - 1$, and we replace G_R by $G_S \triangleleft G_R$, Γ_R by $\Gamma_S = \Gamma_R' \rtimes \Gamma_K \triangleleft \Gamma_R$ and consider complexes in terms of étale (φ, Γ_S)-modules over respective period rings $\boldsymbol{A}_{R,\varpi}$ and $\boldsymbol{A}_{R,\varpi}^\dagger$ (defined in an obvious way).

4.2. ***Koszul complexes.*** Recall that $K = F(\zeta_{p^m})$ for $m \in \mathbb{N}_{\geqslant 1}$. Let $S = R[\varpi]$ for $\varpi = \zeta_{p^m} - 1$. From Section 2.4, recall that $S_\infty[1/p] = R_\infty[1/p]$ is a Galois extension of $S[1/p]$, with Galois group

$$\Gamma_S = \Gamma_R' \rtimes \Gamma_K \triangleleft \Gamma_R.$$

Also recall that we fixed topological generators $\{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ of Γ_S such that $\{\gamma_1, \ldots, \gamma_d\}$ are topological generators of $\Gamma_S' := \Gamma_R'$ and γ_0 is a lift (to Γ_S) of a topological generator of Γ_K. Furthermore, χ denotes the p-adic cyclotomic character, and recall that $c = \chi(\gamma_0) = \exp(p^m)$.

In this subsection, we will recall the definition of Koszul complexes from [Colmez and Nizioł 2017, §4.2] computing continuous Γ_S-cohomology of topological modules admitting a continuous action of Γ_S, in particular, étale (φ, Γ_S)-modules (see Remark 4.3). Let $\tau_i = \gamma_i - 1$ for $1 \leqslant i \leqslant d$, and set

$$K(\tau_i) : 0 \to \mathbb{Z}_p[\![\tau_i]\!] \xrightarrow{\tau_i} \mathbb{Z}_p[\![\tau_i]\!] \to 0,$$

where the middle map is multiplication by τ_i and the right-hand term is placed in degree 0.

Definition 4.4. Define

$$K(\tau_1, \ldots, \tau_d) := K(\tau_1) \widehat{\otimes}_{\mathbb{Z}_p} K(\tau_2) \widehat{\otimes}_{\mathbb{Z}_p} \cdots \widehat{\otimes}_{\mathbb{Z}_p} K(\tau_d)$$

to be the *Koszul complex* associated to $(\tau_1, \ldots, \tau_d)$.

Remark 4.5. The degree q term in the complex $K(\tau_1, \dots, \tau_d)$ (Definition 4.4) equals the exterior power $\bigwedge_A^q A^d$, where $A = \mathbb{Z}_p[[\tau_1, \dots, \tau_d]] \xrightarrow{\sim} \mathbb{Z}_p[[\Gamma_S']]$ is an isomorphism; the last term denotes the Iwasawa algebra of Γ_S'. The differential $d_{q-1}^1 : \bigwedge_A^q A^d \to \bigwedge_A^{q-1} A^d$ is given as

$$d_{q-1}^1(e_{i_1\cdots i_q}) = \sum_{k=1}^{q} (-1)^{k+1} e_{i_1\cdots\hat{i}_k\cdots i_q} \tau_{i_k}$$

in the standard basis $\{e_{i_1\cdots i_q} \mid 1 \leqslant i_1 < \cdots < i_q \leqslant d\}$ of $\bigwedge_A^q A^d$. In the category of topological A-modules, the augmentation map $A \to \mathbb{Z}_p$ makes $K(\tau_1, \dots, \tau_d)$ into a resolution of $\mathbb{Z}_p$. Explicitly, we have that

$$K(\tau_1, \dots, \tau_d) = 0 \to A^{I_d'} \xrightarrow{d_{d-1}^1} \cdots \xrightarrow{d_1^1} A^{I_1'} \xrightarrow{d_0^1} A \to 0,$$

where $A^{I_q'} = \bigoplus_{I_q'} A$ for $I_q' = \{(i_1, \dots, i_q) \mid 1 \leqslant i_1 < \cdots < i_q \leqslant d\}$ and the differentials are as described above. Similarly, for $c = \chi(\gamma_0)$, we can define the Koszul complex $K(\tau_1^c, \dots, \tau_d^c)$, where $\tau_i^c := \gamma_i^c - 1$.

Definition 4.6. Let $\Lambda := \mathbb{Z}_p[[\Gamma_S]]$, and define the complex

$$K(\Lambda) := 0 \to \Lambda^{I_d'} \xrightarrow{d_{d-1}^1} \cdots \xrightarrow{d_1^1} \Lambda^{I_d'} \xrightarrow{d_0^1} \Lambda \to 0,$$

where we have $\Lambda^{I_q'} = \bigoplus_{I_q'} \Lambda$ and the indexing sets I_q' were described in Remark 4.5. From [Morita 2008, Lemma 4.3], we have an isomorphism of complexes

$$\lim_m \mathbb{Z}_p[\Gamma_K/(\Gamma_K)^{p^m}] \otimes_{\mathbb{Z}_p} K(\tau_1, \dots, \tau_d) \xrightarrow{\sim} K(\Lambda).$$

Similarly, one can obtain $K^c(\Lambda)$ from $K(\tau_1^c, \dots, \tau_d^c)$. Both $K(\Lambda)$ and $K^c(\Lambda)$ are resolutions of $\mathbb{Z}_p[[\Gamma_K]]$ in the category of topological left Λ-modules.

Example 4.7. For $d = 2$, the complex $K(\Lambda)$ in Definition 4.6 is given as

$$0 \to \Lambda \xrightarrow{d_1^1} \Lambda \oplus \Lambda \xrightarrow{d_0^1} \Lambda \to 0,$$

where $d_1^1(x) = (-x\tau_2, x\tau_1)$ and $d_0^1(y, z) = y\tau_1 + z\tau_2$.

Definition 4.8. Define a map $\tau_0 : K^c(\Lambda) \to K(\Lambda)$ by setting, in each degree,

$$\tau_0^0 = \gamma_0 - 1 \quad \text{and} \quad \tau_0^q : (a_{i_1\cdots i_q}) \mapsto (a_{i_1\cdots i_q}(\gamma_0 - \delta_{i_1\cdots i_q}))$$

for $1 \leqslant q \leqslant d$, $1 \leqslant i_1 < \cdots < i_q \leqslant d$ and $\delta_{i_1\cdots i_q} = \delta_{i_q} \cdots \delta_{i_1}$, with $\delta_{i_j} = (\gamma_{i_j}^c - 1)(\gamma_{i_j} - 1)^{-1}$.

Let M be a topological $\mathbb{Z}_p$-module admitting a continuous action of Γ_S.

Definition 4.9. Define the Γ_S'-*Koszul complexes* of M by setting $\mathrm{Kos}(\Gamma_S', M) := \mathrm{Hom}_{\Lambda,\mathrm{cont}}(K(\Lambda), M)$ and $\mathrm{Kos}^c(\Gamma_S', M) := \mathrm{Hom}_{\Lambda,\mathrm{cont}}(K^c(\Lambda), M)$. Moreover, define the Γ_S-*Koszul complex* of M as

$$\mathrm{Kos}(\Gamma_S, M) := \big[\mathrm{Kos}(\Gamma_S', M) \xrightarrow{\tau_0} \mathrm{Kos}^c(\Gamma_S', M)\big].$$

Proposition 4.10 [Colmez and Nizioł 2017, §4.2; Lazard 1965]. *There exists a natural quasi-isomorphism of complexes* $\mathrm{Kos}(\Gamma_S, M) \simeq \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_S, M)$.

Definition 4.11. Let D be an étale (φ, Γ_S)-module over $\boldsymbol{A}_{R,\varpi}$, and set

$$\mathrm{Kos}(\varphi, \Gamma_S, D) := \left[\begin{array}{ccc} \mathrm{Kos}(\Gamma'_S, D) & \xrightarrow{1-\varphi} & \mathrm{Kos}(\Gamma'_S, D) \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_0} \\ \mathrm{Kos}^c(\Gamma'_S, D) & \xrightarrow{1-\varphi} & \mathrm{Kos}^c(\Gamma'_S, D) \end{array}\right].$$

Note that, from Proposition 4.10 and Definition 4.11, we have a natural quasi-isomorphism of complexes $\mathrm{Kos}(\varphi, \Gamma_S, D) \simeq \big[\mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_S, D) \xrightarrow{1-\varphi} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_S, D)\big]$, so we conclude the following.

Proposition 4.12. *Let T be in $\mathrm{Rep}_{\mathbb{Z}_p}(G_S)$ and $D_\varpi(T)$ be the associated étale (φ, Γ_S)-module over $\boldsymbol{A}_{R,\varpi}$. Then we have a natural quasi-isomorphism of complexes* $\mathrm{Kos}(\varphi, \Gamma_S, D_\varpi(T)) \simeq \mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T)$.

4.3. ***Lie algebra cohomology.*** In this subsection we will fix constants $u, v \in \mathbb{R}$ such that

$$\frac{p-1}{p} \leqslant u \leqslant \frac{v}{p} < 1 < v;$$

for example, one can take $u = (p-1)/p$ and $v = p-1$.

4.3.1. *Convergence of operators.* From Section 2.7, recall that we have rings $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$, $\boldsymbol{A}_{R,\varpi}^{[u]}$ and $\boldsymbol{A}_{R,\varpi}^{[u,v]}$ equipped with a continuous action of $\Gamma_S \lhd \Gamma_R$.

Lemma 4.13. *For $i \in \{0, 1, \dots, d\}$, the operators*

$$\nabla_i := \log \gamma_i = \sum_{k \in \mathbb{N}} \frac{(-1)^k (\gamma_i - 1)^{k+1}}{k+1}$$

converge as a series of operators on $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$, $\boldsymbol{A}_{R,\varpi}^{[u]}$ and $\boldsymbol{A}_{R,\varpi}^{[u,v]}$.

Proof. From Lemma 2.21, note that

$$(\gamma_0 - 1)(p^m, \pi_m^{p^m})^k \boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \subset (p^m, \pi_m^{p^m})^{k+1} \boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$$

for all $k \geqslant 0$. Using the fact that $\gamma_0 - 1$ acts as a twisted derivation, we see that, for any x in $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$, the expression $(\gamma_0 - 1)^k x$ belongs to $(p^m, \pi_m^{p^m})^k \boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$. Therefore, to check that the series

$$\nabla_0(x) = \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_0 - 1)^{k+1}(x)}{k+1}$$

converges in $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$, it is enough to show that, for a fixed $0 \leqslant j \leqslant k$, the p-adic valuation of

$$\left\lfloor \frac{p^m j}{e} \right\rfloor ! \, \frac{p^{m(k-j)}}{k}$$

goes to $+\infty$ as $k \to +\infty$, which follows from an elementary computation. In particular, we have that $\nabla_0(x)$ converges in $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$.

Now, let us consider γ_i for $i \in \{1, \dots, d\}$. Again, from Lemma 2.21, note that

$$(\gamma_i - 1)(p^m, \pi_m^{p^m})^k \boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \subset (p^m, \pi_m^{p^m})^{k+1} \boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$$

for all $k \geqslant 0$. Using the fact that $\gamma_i - 1$ acts as a twisted derivation, we conclude that, for any x in $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$, the expression $(\gamma_i - 1)^k x$ belongs to $(p^m, \pi_m^{p^m})^k \boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$. Therefore, using an estimate similar the case of γ_0, we conclude that the series

$$\nabla_i(x) = \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_i - 1)^{k+1}(x)}{k+1}$$

converges in $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$. The case of $\boldsymbol{A}_{R,\varpi}^{[u]}$ and $\boldsymbol{A}_{R,\varpi}^{[u,v]}$ follow from similar arguments (using Lemma 2.22 for $\boldsymbol{A}_{R,\varpi}^{[u,v]}$). This allows us to conclude. □

Next, note that formally we can write

$$\frac{\log(1+X)}{X} = 1 + a_1 X + a_2 X^2 + a_3 X^3 + \cdots, \qquad \frac{X}{\log(1+X)} = 1 + b_1 X + b_2 X^2 + b_3 X^3 + \cdots,$$

where $\upsilon_p(a_k) \geqslant -k/(p-1)$ for all $k \geqslant 1$, and therefore $\upsilon_p(b_k) \geqslant -k/(p-1)$ for all $k \geqslant 1$. Setting $X = \gamma_i - 1$ for $i \in \{0, 1, \ldots, d\}$, we make the following claim.

Lemma 4.14. *For $i \in \{0, 1, \ldots, d\}$, the operators*

$$\frac{\nabla_i}{\gamma_i - 1} = \frac{\log \gamma_i}{\gamma_i - 1} \quad \textit{and} \quad \frac{\gamma_i - 1}{\nabla_i} = \frac{\gamma_i - 1}{\log \gamma_i}$$

converge as series of operators on $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$, $\boldsymbol{A}_{R,\varpi}^{[u]}$ and $\boldsymbol{A}_{R,\varpi}^{[u,v]}$.

Proof. We will only show that these series converge on $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$; the case of $\boldsymbol{A}_{R,\varpi}^{[u]}$ and $\boldsymbol{A}_{R,\varpi}^{[u,v]}$ follow similarly (using Lemma 2.22 for $\boldsymbol{A}_{R,\varpi}^{[u,v]}$). Note that we have

$$\upsilon_p(a_k) \geqslant \frac{-k}{p-1} \text{ and } \upsilon_p(b_k) \geqslant \frac{-k}{p-1} \quad \text{for all } k \geqslant 1,$$

so it is enough to show the convergence of $(\gamma_i - 1)/\log \gamma_i$. Now, from Lemma 2.21, we have, for $k \geqslant 1$,

$$(\gamma_i - 1)(p^m, \pi_m^{p^m})^k \boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \subset (p^m, \pi_m^{p^m})^{k+1} \boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}.$$

Since $\gamma_i - 1$ acts as a twisted derivation, for any x in $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$, from the proof of Lemma 4.13, we have that $(\gamma_i - 1)^k x$ belongs to $(p^m, \pi_m^{p^m})^k \boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$. Therefore, to check that the series

$$\sum_{k \in \mathbb{N}} (-1)^k b_k (\gamma_i - 1)^k x$$

converges in $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$, it is enough to show that, for a fixed $0 \leqslant j \leqslant k$, the p-adic valuation of

$$b_k p^{m(k-j)} \left\lfloor \frac{p^m j}{e} \right\rfloor !$$

goes to $+\infty$ as $k \to +\infty$, which follows from an elementary computation. So, we get that the series $(\gamma_i - 1)/\log \gamma_i$ converges on $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$. This concludes our proof. □

4.3.2. *Koszul Complexes for* $\mathrm{Lie}\,\Gamma_S$. For $0 \leqslant i \leqslant d$, let ∇_i denote the operators defined as above. The Lie algebra $\mathrm{Lie}\,\Gamma'_S$ of the p-adic Lie group Γ'_S is a finite free $\mathbb{Z}_p$-module of rank d; i.e., $\mathrm{Lie}\,\Gamma'_S = \mathbb{Z}_p[\nabla_i]_{1\leqslant i\leqslant d}$ and the Lie algebra $\mathrm{Lie}\,\Gamma_S$ of the p-adic Lie group Γ_S is a finite free $\mathbb{Z}_p$-module of rank $d+1$; i.e., $\mathrm{Lie}\,\Gamma_S = \mathbb{Z}_p[\nabla_i]_{0\leqslant i\leqslant d}$. Moreover, we have

$$\begin{aligned} [\nabla_i, \nabla_j] &= \nabla_i \circ \nabla_j - \nabla_j \circ \nabla_i = 0 && \text{for } 1 \leqslant i, j \leqslant d, \\ [\nabla_0, \nabla_i] &= \nabla_0 \circ \nabla_i - \nabla_i \circ \nabla_0 = p^m \nabla_i && \text{for } 1 \leqslant i \leqslant d. \end{aligned}$$

In particular, $\mathrm{Lie}\,\Gamma'_S$ is commutative as a $\mathbb{Z}_p$-algebra, however $\mathrm{Lie}\,\Gamma_S$ is noncommutative. Let M be a topological $\mathbb{Z}_p$-module admitting a continuous action of $\mathrm{Lie}\,\Gamma_S$.

Definition 4.15. Define the complex $\mathrm{Kos}(\mathrm{Lie}\,\Gamma'_S, M) := M \to M^{I'_1} \to \cdots \to M^{I'_d}$ with differentials dual to those in Remark 4.5 (with τ_i replaced by ∇_i).

Consider a morphism of complexes $\nabla_0 : \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_S, M) \to \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_S, M)$ defined on the q-th term as $\nabla_0 - qp^m : M^{I'_q} \to M^{I'_q}$.

Definition 4.16. Define the $\mathrm{Lie}\,\Gamma_S$-*Koszul complex* with values in M as

$$\mathrm{Kos}(\mathrm{Lie}\,\Gamma_S, M) := \big[\mathrm{Kos}(\mathrm{Lie}\,\Gamma'_S, M) \xrightarrow{\nabla_0} \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_S, M)\big].$$

Proposition 4.17 [Colmez and Nizioł 2017, §4.3; Lazard 1965]. *There exist natural quasi-isomorphisms of complexes*

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{cont}}(\mathrm{Lie}\,\Gamma'_S, M) &\simeq \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_S, M), \\ \mathrm{R}\Gamma_{\mathrm{cont}}(\mathrm{Lie}\,\Gamma_S, M) &\simeq \mathrm{Kos}(\mathrm{Lie}\,\Gamma_S, M). \end{aligned}$$

5. Syntomic complexes and finite-height representations

We will assume the setup of Section 2. Recall that we fixed some $m \in \mathbb{N}_{\geqslant 1}$ and, from Section 2.5, we have rings $R_\varpi^\star$ for $\star \in \{\ \ , +, \mathrm{PD}, [u], (0, v]+, [u, v]\}$. Unless otherwise stated, we will assume $u = (p-1)/p$ and $v = p-1$. Note that the p-adic completion of the module of differentials of R relative to $\mathbb{Z}$ is given as $\Omega^1_R = \bigoplus_{i=1}^d R\, d\log X_i$. Also, for $\star \in \{+, \mathrm{PD}, [u], [u, v]\}$, we have

$$\Omega^1_{R_\varpi^\star} = R_\varpi^\star \frac{dX_0}{1+X_0} \oplus \bigg(\bigoplus_{i=1}^d R_\varpi^\star\, d\log X_i\bigg).$$

5.1. ***Formulation of the main result.*** In Sections 5 and 6 we will work with the following class of representations.

Assumption 5.1. Let T be a positive finite q-height $\mathbb{Z}_p$-representation of G_R of height s, and set $V = T[1/p]$ (see Definition 3.1). Assume that the Wach module $\boldsymbol{N}(T)$ is free of rank $\mathrm{rk}_{\mathbb{Z}_p} T$ over $\boldsymbol{A}_R^+$ and $M \subset \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$ is a free R-submodule of rank $\mathrm{rk}_{\mathbb{Z}_p} T$ such that M is stable under the induced Frobenius, $M[1/p] = \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$ and the induced connection over M is p-adically quasinilpotent, integrable and satisfies Griffiths transversality with respect to the induced filtration. Furthermore, assume that $p^s M \subset \varphi^*(M)$ and there is a natural map

$$\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \to \mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)$$

compatible with the respective Frobenii, filtrations, connections and actions of Γ_R, and such that it is a p^N-isomorphism with $N = n(T, e) \in \mathbb{N}$ for $e = [K : F] = p^{m-1}(p-1)$.

Example 5.2. Following are some cases in which Assumption 5.1 is satisfied:

(i) Assuming that $\boldsymbol{N}(T)$ is a free $\boldsymbol{A}_R^+$-module, from Proposition 3.9 and Remark 3.11, we have that the R-module $M := M_0$ (in the notation of the proposition) satisfies Assumption 5.1 with $m = 1$ and $n(T, e) = s$.

(ii) Let $M = (\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T))^{\Gamma_R}$ with an additional assumption that it is free over R of rank $\mathrm{rk}_{\mathbb{Z}_p} T$. Then, the module M depends on T and $m \in \mathbb{N}_{\geqslant 1}$ (see Remark 3.12), and it satisfies Assumption 5.1 with $n(T, e) = s$ (see Remarks 3.10–3.12).

(iii) For our intended global applications to relative Fontaine–Laffaille modules, we note that, for representations arising from finite free relative Fontaine–Laffaille modules of level $[0, s]$ with $s \leqslant p-2$ as in Section 3.4, the conditions of Assumption 5.1 are automatically satisfied, with M being the relative Fontaine–Laffaille module and $n(T, e) = 0$ (see Remark 3.26).

Let us first consider the case of $S = R[\varpi]$. From Section 2.5 we have the divided power ring $R_\varpi^{\mathrm{PD}} \twoheadrightarrow S$, and we have a finite free R_ϖ^{PD}-module $M_\varpi^{\mathrm{PD}} := R_\varpi^{\mathrm{PD}} \otimes_R M$ equipped with a Frobenius-semilinear endomorphism φ given by the diagonal action on each component of the tensor product, and a filtration $\{\mathrm{Fil}^k M_\varpi^{\mathrm{PD}}\}_{k \in \mathbb{N}}$ induced from the tensor product filtration on $M_\varpi^{\mathrm{PD}}[1/p]$ (see the discussion before Lemma 2.40). Moreover, the O_F-linear integrable connection on M and the continuous O_F-linear de Rham differential operator on R_ϖ^{PD} induce an O_F-linear integrable connection $\partial : M_\varpi^{\mathrm{PD}} \to M_\varpi^{\mathrm{PD}} \otimes_{R_\varpi^{\mathrm{PD}}} \Omega^1_{R_\varpi^{\mathrm{PD}}}$ defined by sending $a \otimes x \mapsto a \otimes \partial_M(x) + x da$. It is easy to see that the connection ∂ on M_ϖ^{PD} satisfies Griffiths transversality with respect to the filtration since the same is true for the connection on M and the differential operator on R_ϖ^{PD}. In particular, we have the filtered de Rham complex

$$\mathrm{Fil}^r \mathcal{D}^\bullet_{S,M} := \mathrm{Fil}^r M_\varpi^{\mathrm{PD}} \to \mathrm{Fil}^{r-1} M_\varpi^{\mathrm{PD}} \otimes_{R_\varpi^{\mathrm{PD}}} \Omega^1_{R_\varpi^{\mathrm{PD}}} \to \cdots. \tag{5-1}$$

Fix a basis of $\Omega^1_{R_\varpi^{\mathrm{PD}}}$ as $\{dX_0/(1+X_0), dX_1/X_1, \ldots, dX_d/X_d\}$. We will next equip $\Omega^1_{R_\varpi^{\mathrm{PD}}}$ with an action of Frobenius. Let $j \in \mathbb{N}$ and $I_j = \{0 \leqslant i_1 < \cdots < i_j \leqslant d\}$. For $\boldsymbol{i} = (i_1, \ldots, i_j) \in I_j$, set

$$\omega_{\boldsymbol{i}} := \frac{dX_0}{1+X_0} \wedge \frac{dX_{i_2}}{X_{i_2}} \wedge \cdots \wedge \frac{dX_{i_j}}{X_{i_j}} \quad \text{if } i_1 = 0 \qquad \text{and} \qquad \omega_{\boldsymbol{i}} := \frac{dX_{i_1}}{X_{i_1}} \wedge \cdots \wedge \frac{dX_{i_j}}{X_{i_j}} \quad \text{otherwise.}$$

Define the operators φ and ψ on $\Omega^j_{R_\varpi^{\mathrm{PD}}}$ by the formulas

$$\varphi\Big(\sum_{\boldsymbol{i} \in I_j} x_{\boldsymbol{i}} \omega_{\boldsymbol{i}}\Big) = \sum_{\boldsymbol{i} \in I_j} \varphi(x_{\boldsymbol{i}}) \omega_{\boldsymbol{i}} \quad \text{and} \quad \psi\Big(\sum_{\boldsymbol{i} \in I_j} x_{\boldsymbol{i}} \omega_{\boldsymbol{i}}\Big) = \sum_{\boldsymbol{i} \in I_j} \psi(x_{\boldsymbol{i}}) \omega_{\boldsymbol{i}}. \tag{5-2}$$

Remark 5.3. Note that (5-2) is not the natural definition of Frobenius, since we have $d(\varphi(x)) = p\varphi(dx)$ in (5-2). But in order to define ψ integrally, we need to divide the usual Frobenius on $\Omega^1_{R^\star_\varpi}$ by powers of p. Recall that, with the usual definition of Frobenius, we have $\varphi\partial = \partial\varphi$ over $M \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$; see Section 2.3. However, using (5-2) for Ω^1_R as well, we see that, for any $f \in M$, we now have

$$\partial_M(\varphi(f)) = \sum_{i=1}^{d} \partial_i(\varphi(f))\omega_i = \sum p\varphi(\partial_i(f))\omega_i = p\varphi(\partial_M(f)).$$

Definition 5.4. Let $r \in \mathbb{N}$, and consider the complex $\mathrm{Fil}^r\,\mathcal{D}^\bullet_{S,M}$ as above. For $n \in \mathbb{N}$, let $S_n = S \otimes \mathbb{Z}/p^n$ and $M_n = M \otimes \mathbb{Z}/p^n$. Define the *syntomic complex* and the *syntomic cohomology* of S with coefficients in M as

$$\mathrm{Syn}(S, M, r) := \left[\mathrm{Fil}^r\,\mathcal{D}^\bullet_{S,M} \xrightarrow{p^r - p^\bullet\varphi} \mathcal{D}^\bullet_{S,M}\right], \qquad H^*_{\mathrm{syn}}(S, M, r) := H^*(\mathrm{Syn}(S, M, r)),$$
$$\mathrm{Syn}(S, M, r)_n := \mathrm{Syn}(S, M, r) \otimes \mathbb{Z}/p^n, \qquad H^*_{\mathrm{syn}}(S_n, M_n, r) := H^*(\mathrm{Syn}(S, M, r)_n).$$

Our main local result is as follows.

Theorem 5.5. *Consider the setting of Assumption 5.1, and let $r \in \mathbb{Z}$ such that $r \geqslant s+1$. Then there exist p^N-quasi-isomorphisms*

$$\alpha_r^{\mathcal{L}\mathrm{az}} : \tau_{\leqslant r-s-1}\,\mathrm{Syn}(S, M, r) \simeq \tau_{\leqslant r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T(r)),$$
$$\alpha_{r,n}^{\mathcal{L}\mathrm{az}} : \tau_{\leqslant r-s-1}\,\mathrm{Syn}(S, M, r)_n \simeq \tau_{\leqslant r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T/p^n(r)),$$

where $N = N(T, e, r) \in \mathbb{N}$ depends on the representation T, $e = [K : F]$ and the twist r.

Remark 5.6. For M as in Example 5.2 (ii), note that, in Theorem 5.5, the constant N can precisely be given as $N = 14r + 9s + 2$; see Section 6.1.

Remark 5.7. Almost all statements and proofs in Sections 5 and 6 are true for $m \geqslant 1$. However, for some lemmas in Sections 6.5 and 6.6, we need to assume that $m \geqslant 2$. So from now on, the reader may safely assume that $m \geqslant 2$ in Sections 5 and 6 and obtain Theorem 5.5 for $m = 1$ using the Galois descent of Lemma 6.21.

Using Theorem 5.5, we can obtain a similar statement over R. Recall that R is smooth over O_F and, for $r \in \mathbb{Z}$, we have the filtered de Rham complex

$$\mathrm{Fil}^r\,\mathcal{D}^\bullet_{R,M} := \mathrm{Fil}^r\,M \to \mathrm{Fil}^{r-1}\,M \otimes_R \Omega^1_R \to \mathrm{Fil}^{r-2}\,M \otimes_R \Omega^2_R \to \cdots. \tag{5-3}$$

Remark 5.8. One can also consider the formulation of a filtered de Rham complex for M as in (5-1). In that case one considers a surjection $R^+_\varpi \twoheadrightarrow R$ via the map $X_0 \mapsto 0$. By writing down the corresponding de Rham complex one readily sees that it is quasi-isomorphic to $\mathcal{D}^\bullet_{R,M}$.

Using (5-3), similar to Definition 5.4, one can define the syntomic complex of R with coefficients in M. Then using Theorem 5.5 for $\varpi = \zeta_{p^2} - 1$ (in particular, Example 5.2 (ii) for $m = 2$), Corollary 6.20 and Galois descent in Lemma 6.21 for $e = p(p-1)$), we obtain the following.

Corollary 5.9. *Consider the setting of Assumption 5.1, and let $r \in \mathbb{Z}$ such that $r \geqslant s+1$. Then there exist p^N-quasi-isomorphisms*

$$\tau_{\leqslant r-s-1} \operatorname{Syn}(R, M, r) \simeq \tau_{\leqslant r-s-1} \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, T(r)),$$
$$\tau_{\leqslant r-s-1} \operatorname{Syn}(R, M, r)_n \simeq \tau_{\leqslant r-s-1} \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, T/p^n(r)),$$

where $N = N(p, r, s) \in \mathbb{N}$ depends on the prime p, twist r and height s of T.

Remark 5.10. For M as in Example 5.2 (ii), note that, in Corollary 5.9, the constant N can precisely be given as $N = 18r + 9s + 3p(p-1) + 2$; see Section 6.1.

In Theorem 5.5 we only prove the p-adic case. The modulo p^n case follows in a similar manner. The complete proof is divided in two main steps: First, we will modify the syntomic complexes with coefficients in M to relate it to a "differential" Koszul complex with coefficients in $\boldsymbol{N}(T)$; see Proposition 5.28. Next, we will modify the Koszul complex from the first step to obtain a Koszul complex computing the continuous G_S-cohomology of $T(r)$; see Theorem 5.5 and Proposition 6.1. The key to the connection between these two steps will be provided by the comparison isomorphism in Theorem 3.6 and a filtered Poincaré lemma. In the rest of Section 5, we will show the first step. The second step will be worked out in Section 6.

5.2. *Syntomic complexes with coefficients.* For $\star \in \{[u], [u, v], [u, v/p]\}$, define a finite free $R_\varpi^\star$-module $M_\varpi^\star := R_\varpi^\star \otimes_R M$. Via the diagonal action of Frobenius on each component, define Frobenius-semilinear operators $\varphi : M_\varpi^{[u]} \to M_\varpi^{[u]}$ and $\varphi : M_\varpi^{[u,v]} \to M_\varpi^{[u,v/p]}$. Equip $M_\varpi^\star$ with a filtration $\{\mathrm{Fil}^k M_\varpi^\star\}_{k \in \mathbb{N}}$ induced from the tensor product filtration on $M_\varpi^\star[1/p]$; see the discussion before Lemma 2.40. Furthermore, the O_F-linear integrable connection on M and the continuous O_F-linear de Rham differential operator on $R_\varpi^\star$ induce an O_F-linear integrable connection on M_ϖ which satisfies Griffiths transversality with respect to the filtration since the same is true for the connection on M and the differential operator on $R_\varpi^\star$. In particular, we have the filtered de Rham complex

$$\mathrm{Fil}^r \mathscr{D}^\bullet_{R_\varpi^\star, M} := \mathrm{Fil}^r M_\varpi^\star \to \mathrm{Fil}^{r-1} M_\varpi^\star \otimes \Omega^1_{R_\varpi^\star} \to \mathrm{Fil}^{r-2} M_\varpi^\star \otimes \Omega^2_{R_\varpi^\star} \to \cdots. \tag{5-4}$$

Moreover, for $\star \in \{[u], [u, v], [u, v/p]\}$, we define operators φ and ψ on $\Omega^j_{R_\varpi^\star}$ as in (5-2). From (5-4), for $\star \in \{[u], [u, v]\}$, denote by $\mathscr{D}^\bullet_{R_\varpi^\star, M}$ the source de Rham complex and, for $\star \in \{[u], [u, v/p]\}$, denote by $\mathscr{E}^\bullet_{R_\varpi^\star, M}$ the target de Rham complex.

Definition 5.11. Define $\operatorname{Syn}(M_\varpi^\star, r) := \big[\mathrm{Fil}^r \mathscr{D}^\bullet_{R_\varpi^\star, M} \xrightarrow{p^r - p^\bullet\varphi} \mathscr{E}^\bullet_{R_\varpi^\star, M}\big]$.

5.3. *Change of the disk of convergence.* In this section, we denote the syntomic complex $\operatorname{Syn}(S, M, r)$ in Definition 5.4 by $\operatorname{Syn}(M_\varpi^{\mathrm{PD}}, r)$.

Proposition 5.12. *For $1/(p-1) \leqslant u \leqslant 1$, the natural morphism between syntomic complexes*

$$\operatorname{Syn}(M_\varpi^{\mathrm{PD}}, r) \to \operatorname{Syn}(M_\varpi^{[u]}, r),$$

induced by the inclusion $M_\varpi^{\mathrm{PD}} \subset M_\varpi^{[u]}$, is a p^{2r}-isomorphism.

The proposition follows from the next lemma by setting $k = r$.

Lemma 5.13. *Let $j, k \in \mathbb{N}$. If $1/(p-1) \leqslant u \leqslant 1$, the following map is a p^{k+r}-isomorphism*:

$$p^k - p^j\varphi : \mathrm{Fil}^r\, M_\varpi^{[u]} \otimes \Omega^j_{R_\varpi^{[u]}} / \mathrm{Fil}^r\, M_\varpi^{\mathrm{PD}} \otimes \Omega^j_{R_\varpi^{\mathrm{PD}}} \to M_\varpi^{[u]} \otimes \Omega^j_{R_\varpi^{[u]}} / M_\varpi^{\mathrm{PD}} \otimes \Omega^j_{R_\varpi^{\mathrm{PD}}}.$$

Proof. The proof is motivated by [Colmez and Nizioł 2017, Lemma 3.2]. Note that we can decompose everything in the basis of the $\omega_{\boldsymbol{i}}$, where $\boldsymbol{i} \in I_j = \{0 \leqslant i_1 < \cdots < i_j \leqslant d\}$. Then by the definition of Frobenius on $\omega_{\boldsymbol{i}}$, we are reduced to showing that

$$p^k - p^j\varphi : \mathrm{Fil}^r\, M_\varpi^{[u]} / \mathrm{Fil}^r\, M_\varpi^{\mathrm{PD}} \to M_\varpi^{[u]} / M_\varpi^{\mathrm{PD}}$$

is a p^{k+r}-isomorphism. Since $\varphi(R_\varpi^{[u]}) \subset R_\varpi^{[u/p]} \subset R_\varpi^{\mathrm{PD}}$ for $1/(p-1) \leqslant u \leqslant 1$, we therefore have $M_\varpi^{\mathrm{PD}} \subset M_\varpi^{[u]}$ and $\varphi(M_\varpi^{[u]}) \subset M_\varpi^{\mathrm{PD}}$.

For p^k-injectivity, recall that $\mathrm{Fil}^r\, M_\varpi^{[u]} = M_\varpi^{[u]} \cap \mathrm{Fil}^r\, M_\varpi^{\mathrm{PD}}$ (see Lemma 2.40), so, for any x in $\mathrm{Fil}^r\, M_\varpi^{[u]}$, it suffices to show that if $(p^k - p^j\varphi)x \in M_\varpi^{\mathrm{PD}}$ then $p^k x \in M_\varpi^{\mathrm{PD}}$. Since we can write

$$p^k x = (p^k - p^j\varphi)x + p^j\varphi(x) \quad \text{and} \quad \varphi(M_\varpi^{[u]}) \subset M_\varpi^{\mathrm{PD}},$$

we therefore get $p^k x \in M_\varpi^{\mathrm{PD}}$. Next, let us show the p^{k+r}-surjectivity. Let $\{f_1, \dots, f_h\}$ be an R-basis of M and take $x = \sum_{i=1}^h a_i \otimes f_i \in M_\varpi^{[u]}$. Let $N = ke/(u(p-1))$; then from the definition of $R_\varpi^{[u]}$ we can write

$$a_i = a_{i1} + a_{i2}, \quad \text{with } a_{i2} \in R_{\varpi,N}^{[u]} \text{ and } a_{i1} \in p^{-\lfloor Nu/e \rfloor} R_\varpi^+ \subset p^{-k} R_\varpi^{\mathrm{PD}},$$

where we write $R_{\varpi,N}^{[u]}$ as in the notation of Lemma 2.11 (it consists of power series in X_0 involving terms X_0^s for $s \geqslant N$). Now let

$$x_1 = \sum_{i=1}^h a_{i1} \otimes f_i \quad \text{and} \quad x_2 = \sum_{i=1}^h a_{i2} \otimes f_i,$$

so that $x = x_1 + x_2$. By Lemma 2.11 and the fact that M is stable under φ, it follows that $(1 - p^{j-k}\varphi)$ is bijective on $R_{\varpi,N}^{[u]} \otimes_R M$ (note that the series of operators $\sum_{i\in\mathbb{N}} p^{(j-k)i}\varphi^i$ converge as an inverse to $1 - p^{j-k}\varphi$ on $R_{\varpi,N}^{[u]} \otimes_R M$). In particular, we can write $x_2 = (1 - p^{j-k}\varphi)z$ for some $z = \sum_{i=1}^h b_i \otimes f_i \in M_\varpi^{[u]}$. Also, by Lemma 2.9 we can write

$$b_i = b_{i1} + b_{i2}, \quad \text{with } b_{i1} \in \mathrm{Fil}^r\, R_\varpi^{[u]} \text{ and } b_{i2} \in p^{-\lfloor ru \rfloor} R_\varpi^+.$$

By setting

$$z_1 = \sum_{i=1}^h b_{i1} \otimes f_i \in \mathrm{Fil}^r\, M_\varpi^{[u]} \quad \text{and} \quad z_2 = \sum_{i=1}^h b_{i2} \otimes f_i \in p^{-r} M_\varpi^{\mathrm{PD}},$$

we obtain $(1 - p^{j-k}\varphi)z_2 = p^{-k}(p^k - p^j\varphi)z_2 \in p^{-k-r} M_\varpi^{\mathrm{PD}}$. Using the preceding observation in the expression for x, we get

$$x - (1 - p^{j-k}\varphi)z_1 = x_1 + (1 - p^{j-k}\varphi)z_2 \in p^{-k} M_\varpi^{\mathrm{PD}} + p^{-k-r} M_\varpi^{\mathrm{PD}} \subset p^{-k-r} M_\varpi^{\mathrm{PD}}.$$

Therefore, we obtain $x \in p^{-k-r} M_\varpi^{\mathrm{PD}} + p^{-k}(p^k - p^j\varphi)\, \mathrm{Fil}^r\, M_\varpi^{[u]}$, allowing us to conclude. □

5.4. ***Change of the annulus of convergence.*** We will consider the base change of the syntomic complex from R_ϖ^{PD} to $R_\varpi^{[u,v]}$.

Proposition 5.14. *For $pu \leqslant v$, there exists a p^{2r+4s}-quasi-isomorphism*

$$\tau_{\leqslant r-s-1}\operatorname{Syn}(M_\varpi^{[u]}, r) \simeq \tau_{\leqslant r-s-1}\operatorname{Syn}(M_\varpi^{[u,v]}, r);$$

i.e., we have p^{2r+4s}-isomorphisms $H^k_{\mathrm{syn}}(M_\varpi^{[u]}, r) \simeq H^k_{\mathrm{syn}}(M_\varpi^{[u,v]}, r)$ for $0 \leqslant k \leqslant r-s-1$.

Proof. The claim follows by combining the results from Lemmas 5.15, 5.16 and 5.18. □

To prove the claim in Proposition 5.14, we will pass to the corresponding (quasi-isomorphic) ψ-complex. Recall that we have the isomorphism $\varphi^*(\mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)) \xrightarrow{\sim} \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$. Let $\boldsymbol{f} = \{f_1, \dots, f_h\}$ denote an R-basis of M. Then $\boldsymbol{f}$ and $\varphi(\boldsymbol{f})$ form two different basis of $\mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$ over $R[1/p]$. So, we can write $\boldsymbol{f} = \varphi(\boldsymbol{f})X$, where $X = (x_{ij}) \in (h, R[1/p])$. For our choice of M (see Assumption 5.1) and using Theorem 3.6 and Proposition 3.9, we have $x_{ij} \in p^{-s}R$, where $1 \leqslant i, j \leqslant h$ and s is the height of V. Define

$$\psi : M^{[u]} = R_\varpi^{[u]} \otimes_R M \to p^{-s} R_\varpi^{[pu]} \otimes_R M, \quad \boldsymbol{f}\boldsymbol{y}^\intercal \mapsto \boldsymbol{f}\psi(X\boldsymbol{y}^\intercal),$$

where we consider the operator ψ on $R_\varpi^{[u]}$ defined in Section 2.6. It is easy to show that this map is well defined, i.e., independent of the choice of a basis for M. Using the operator ψ on $M_\varpi^{[u]}$ as above and on $\Omega^\bullet_{R_\varpi^{[u]}}$ as in (5-2), define the complex

$$\operatorname{Syn}^\psi(M_\varpi^{[u]}, r) := \left[\operatorname{Fil}^r M_\varpi^{[u]} \otimes \Omega^\bullet_{R_\varpi^{[u]}} \xrightarrow{p^{r+s}\psi - p^{\bullet+s}} M_\varpi^{[pu]} \otimes \Omega^\bullet_{R_\varpi^{[pu]}}\right].$$

Lemma 5.15. *The commutative diagram*

$$\begin{array}{ccc}
\operatorname{Fil}^r M_\varpi^{[u]} \otimes \Omega^\bullet_{R_\varpi^{[u]}} & \xrightarrow{p^r - p^\bullet\varphi} & M_\varpi^{[u]} \otimes \Omega^\bullet_{R_\varpi^{[u]}} \\
\downarrow \mathrm{id} & & \downarrow p^s\psi \\
\operatorname{Fil}^r M_\varpi^{[u]} \otimes \Omega^\bullet_{R_\varpi^{[u]}} & \xrightarrow{p^{r+s}\psi - p^{\bullet+s}} & M_\varpi^{[pu]} \otimes \Omega^\bullet_{R_\varpi^{[pu]}}
\end{array}$$

defines a p^{2s}-quasi-isomorphism from $\operatorname{Syn}(M_\varpi^{[u]}, r)$ to $\operatorname{Syn}^\psi(M_\varpi^{[u]}, r)$.

Proof. First, we will look at the cokernel complex which is the cokernel of the right vertical arrow. By definition, we have that $\psi(M_\varpi^{[u]}) \subset p^{-s}M_\varpi^{[pu]}$; in particular, $p^s\psi(M_\varpi^{[u]}) \subset M_\varpi^{[pu]}$. Moreover, note that the operator $\psi : R_\varpi^{[u]} \to R_\varpi^{[pu]}$ is surjective and $p^sM \subset \varphi^*(M)$; see Assumption 5.1. Therefore,

$$M_\varpi^{[pu]} = R_\varpi^{[pu]} \otimes_R M \subset \psi(R_\varpi^{[u]} \otimes_R \varphi^*(M)) \subset \psi(M_\varpi^{[u]}).$$

Hence $p^s\psi(M_\varpi^{[u]})$ is p^s-isomorphic to $M_\varpi^{[pu]}$ and the cokernel complex is killed by p^s.

Next, for the kernel complex, we proceed as follows: let $M = \bigoplus_{j=1}^h Rf_j$; therefore $M_\varpi^{[u]} = \bigoplus_{j=1}^h R_\varpi^{[u]} f_j$. Recall that $M/\varphi^*(M)$ is killed by p^s, so we have a p^s-isomorphism

$$\bigoplus_{j=1}^h R_\varpi^{[u]}\varphi(f_j) \xrightarrow{\sim} M_\varpi^{[u]}.$$

Note that an element $y = \sum_{j=1}^{h} y_j\varphi(f_j)$ is in $\big(\bigoplus_{j=1}^{h} R_{\varpi}^{[u]}\varphi(f_j)\big)^{\psi=0}$ if and only if y_j is in $(R_{\varpi}^{[u]})^{\psi=0}$. Indeed, $\psi(y)=0$ if and only if $\sum_{j=1}^{h}\psi(y_j)f_j = 0$, and since the f_j are linearly independent over $R[1/p]$, we see that $\psi(y)=0$ if and only if $\psi(y_j)=0$ for all $1\leqslant j\leqslant h$. In particular, we obtain a p^s-isomorphism

$$(M_{\varpi}^{[u]})^{\psi=0} \xleftarrow{\sim} \Big(\bigoplus_{j=1}^{h} R_{\varpi}^{[u]}\varphi(f_j)\Big)^{\psi=0} = \bigoplus_{j=1}^{h}(R_{\varpi}^{[u]})^{\psi=0}\varphi(f_j).$$

Using the definition of ψ on $\Omega^k_{R_{\varpi}^{[u]}}$ in the chosen basis of (5-2), it easily follows that

$$(M\otimes_R \Omega^k_{R_{\varpi}^{[u]}})^{\psi=0} = (M_{\varpi}^{[u]})^{\psi=0}\otimes_{\mathbb{Z}}\Omega^k.$$

Recall that, from Lemma 2.15 (ii), we have a decomposition

$$(R_{\varpi}^{[u]})^{\psi=0} = \bigoplus_{\alpha\neq 0} R_{\varpi,\alpha}^{[u]} = \bigoplus_{\alpha\neq 0} R_{\varpi}^{[u]}u_\alpha,$$

where $u_\alpha = (1+X_0)^{\alpha_0}X_1^{\alpha_1}\cdots X_d^{\alpha_d}$, where $\alpha=(\alpha_0,\dots,\alpha_d)$ is a $(d{+}1)$-tuple with $\alpha_i\in\{0,\dots,p-1\}$ for each $0\leqslant i\leqslant d$. Moreover, we have $\partial_i(u_\alpha)=\alpha_i u_\alpha$ for each $0\leqslant i\leqslant d$. In particular, $\partial_i(R_{\varpi,\alpha}^{[u]})\subset R_{\varpi,\alpha}^{[u]}$. Now, using the decomposition of $(R_{\varpi}^{[u]})^{\psi=0}$, we set $M_\alpha = \bigoplus_{j=1}^{h} R_{\varpi,\alpha}^{[u]}\varphi(f_j)$ and obtain that $(M_{\varpi}^{[u]})^{\psi=0}$ is p^s-isomorphic to $\bigoplus_{\alpha\neq 0} M_\alpha$. From the O_F-linear continuous de Rham differential operator on $R_{\varpi,\alpha}^{[u]}$ and the O_F-linear integrable connection on $M_{\varpi}^{[u]}$, we obtain an induced O_F-linear integrable connection

$$\partial : M_\alpha \to M_\alpha\otimes\Omega^1_{R_{\varpi,\alpha}^{[u]}} = M_\alpha\otimes_{\mathbb{Z}}\Omega^1.$$

Then the decomposition of $(M_{\varpi}^{[u]})^{\psi=0}$ shows that the kernel complex in the claim is p^s-isomorphic to the direct sum of the complexes

$$0\to M_\alpha\to M_\alpha\otimes\Omega^1\to M_\alpha\otimes\Omega^2\to\cdots, \tag{5-5}$$

where $\alpha\neq 0$. We will show that (5-5) is exact for each α; the idea of the proof is based on [Colmez and Nizioł 2017, Lemma 3.4]. Since everything is p-adically complete and p-torsion free, we only need to show the exactness of (5-5) modulo p. Note that, for $y=\sum_{j=1}^{h} y_j\varphi(f_j)\in M_\alpha$, we have

$$\partial\Big(\sum_{j=1}^{h} y_j\varphi(f_j)\Big) = \sum_{j=1}^{h} y_j\partial_M(\varphi(f_j)) + \varphi(f_j)\partial(y_j),$$

where ∂_M denotes the connection on M. Recall that from Remark 5.3 we have $\varphi\partial_M = p\partial_M\varphi$. So $\partial(y) - \sum_{i=1}^{h}\varphi(f_j)\partial(y_j)\in pM_\alpha$. Moreover, using Lemma 2.16, we have $\partial_i(y_j)-\alpha_i y_j\in pR_{\varpi,\alpha}^{[u]}$. So we get that the complex (5-5) has a very simple shape modulo p: if $d=0$, it is just $M_\alpha\xrightarrow{\alpha_0}M_\alpha$; if $d=1$, it is the complex $M_\alpha\xrightarrow{(\alpha_0,\alpha_1)}M_\alpha\oplus M_\alpha\xrightarrow{-\alpha_1+\alpha_0}M_\alpha$; for general d, it is the total complex attached to a $(d{+}1)$-dimensional cube with all vertices equal to M_α and arrows in the i-th direction equal to α_i. As one of the α_i is invertible by assumption, this implies that the cohomology of the total complex is 0 and (5-5) is exact for each α. This allows us to conclude. ☐

Following the definition of ψ over $M^{[u]}$ (see the discussion before Lemma 5.15), one can define an operator

$$\psi : R_{\varpi}^{[u,v]} \otimes_R M \to p^{-s} R_{\varpi}^{[pu,pv]} \otimes_R M$$

as a left inverse to φ and set

$$\mathrm{Syn}^{\psi}(M_{\varpi}^{[u,v]}, r) := \big[\mathrm{Fil}^r\, M_{\varpi}^{[u,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u,v]}} \xrightarrow{p^{r+s}\psi - p^{\bullet+s}} M_{\varpi}^{[pu,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[pu,v]}}\big].$$

Lemma 5.16. *For $u \leqslant 1 \leqslant v$, the natural morphism of complexes $\mathrm{Syn}^{\psi}(M_{\varpi}^{[u]}, r) \to \mathrm{Syn}^{\psi}(M_{\varpi}^{[u,v]}, r)$ is a p^{2r}-quasi-isomorphism in degrees $k \leqslant r - s - 1$.*

Proof. The map between the complexes is induced by the diagram

$$\begin{array}{ccc} \mathrm{Fil}^r\, M_{\varpi}^{[u]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u]}} & \xrightarrow{p^{r+s}\psi - p^{\bullet+s}} & M_{\varpi}^{[pu]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[pu]}} \\ \downarrow & & \downarrow \\ \mathrm{Fil}^r\, M_{\varpi}^{[u,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u,v]}} & \xrightarrow{p^{r+s}\psi - p^{\bullet+s}} & M_{\varpi}^{[pu,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[pu,v]}} \end{array}$$

where the vertical arrows are natural maps induced by the inclusion $R_{\varpi}^{[u]} \subset R_{\varpi}^{[u,v]}$. Therefore, it suffices to show that the mapping fibre

$$\big[\mathrm{Fil}^r\, M_{\varpi}^{[u,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u,v]}} / \mathrm{Fil}^r\, M_{\varpi}^{[u]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u]}} \xrightarrow{p^{r+s}\psi - p^{\bullet+s}} M_{\varpi}^{[pu,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[pu,v]}} / M_{\varpi}^{[pu]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[pu]}}\big]$$

is p^{2r}-acyclic. By Lemma 5.17, we can ignore the filtration, and by working in the basis $\{\omega_{\boldsymbol{i}}, \boldsymbol{i} \in I_k\}$ of Ω^k, it is enough to show that

$$p^{r+s}\psi - p^{k+s} : M_{\varpi}^{[u,v]}/M_{\varpi}^{[u]} \to M_{\varpi}^{[pu,v]}/M_{\varpi}^{[pu]}$$

is a p^r-isomorphism for $k \leqslant r - s - 1$. But note that $M_{\varpi}^{[u,v]}/M_{\varpi}^{[u]} \xrightarrow{\sim} M_{\varpi}^{[pu,v]}/M_{\varpi}^{[pu]}$ is an isomorphism; therefore, we see that $1 - p^i\psi$ is an endomorphism of this quotient for $i = r - k$. Moreover, for $i \geqslant s+1$, we get that $1 - p^i\psi$ is invertible on $M_{\varpi}^{[u,v]}/M_{\varpi}^{[u]}$ with the inverse given as $1 + p^{i-s}(p^s\psi) + p^{2(i-s)}(p^s\psi)^2 + \cdots$. Therefore, it follows that $p^{r+s}\psi - p^{k+s} = p^{k+s}(p^{r-k}\psi - 1)$ is a p^{k+s}-isomorphism. Since $k + s \leqslant r - 1$, we obtain that the complex in the claim is p^{2r}-acyclic. □

Lemma 5.17. *The natural map*

$$\mathrm{Fil}^r\, M_{\varpi}^{[u,v]}/\,\mathrm{Fil}^r\, M_{\varpi}^{[u]} \to M_{\varpi}^{[u,v]}/M_{\varpi}^{[u]}$$

is a p^r-isomorphism for $u \leqslant 1 \leqslant v$.

Proof. The map in the claim is injective by Lemma 2.40. For p^r-surjectivity, let $\{f_1, \ldots, f_h\}$ be an R-basis of M, and let $x = \sum_{i=1}^h b_i \otimes f_i \in R_{\varpi}^{[u,v]} \otimes_R M$. By [Colmez and Nizioł 2017, Lemma 3.5], we have a p^r-isomorphism

$$\mathrm{Fil}^r\, R_{\varpi}^{[u,v]}/\,\mathrm{Fil}^r\, R_{\varpi}^{[u]} \xrightarrow{\sim} R_{\varpi}^{[u,v]}/R_{\varpi}^{[u]},$$

so we can write $p^r b_i = b_{i1} + b_{i2}$, with $b_{i1} \in \mathrm{Fil}^r\, R_{\varpi}^{[u,v]}$ and $b_{i2} \in R_{\varpi}^{[u]}$. Since $\sum_{i=1}^h b_{i1} \otimes f_i \in \mathrm{Fil}^r\, M_{\varpi}^{[u,v]}$, we get the desired conclusion. □

Lemma 5.18. *The commutative diagram*

$$\begin{array}{ccc} \mathrm{Fil}^r\, M_{\varpi}^{[u,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u,v]}} & \xrightarrow{\;p^r - p^{\bullet}\varphi\;} & M_{\varpi}^{[u,v/p]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u,v/p]}} \\ \downarrow \mathrm{id} & & \downarrow p^s\psi \\ \mathrm{Fil}^r\, M_{\varpi}^{[u,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u,v]}} & \xrightarrow{\;p^{r+s}\psi - p^{\bullet+s}\;} & M_{\varpi}^{[pu,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[pu,v]}} \end{array}$$

defines a p^{2s}-quasi-isomorphism from $\mathrm{Syn}(M_{\varpi}^{[u,v]}, r)$ *to* $\mathrm{Syn}^{\psi}(M_{\varpi}^{[u,v]}, r)$.

Proof. The claim follows in manner similar to the proof of Lemma 5.15 by replacing $M_{\varpi}^{[u]}$ with $M_{\varpi}^{[u,v]}$ and $R_{\varpi}^{[u]}$ with $R_{\varpi}^{[u,v]}$. One only needs to note that Lemmas 2.15 (ii) and 2.16 are true for the ring $R_{\varpi}^{[u,v]}$ as well. We omit the proof. □

5.5. *Differential Koszul Complex.* Our next goal is to relate the syntomic complex $\mathrm{Syn}(M_{\varpi}^{[u,v]}, r)$ in Section 5.4 to a complex with coefficients in the Wach module $\boldsymbol{N}(T)$ from Assumption 5.1; see Proposition 5.28. Before stating the result we will verify some results in order to define the latter complex.

Let $\Omega^1_{\boldsymbol{A}_{R,\varpi}^{[u,v]}}$ denote the p-adic completion of the module of differentials of $\boldsymbol{A}_{R,\varpi}^{[u,v]}$ relative to $\mathbb{Z}$. Via the isomorphism $\iota_{\mathrm{cycl}} : R_{\varpi}^{[u,v]} \xrightarrow{\sim} \boldsymbol{A}_{R,\varpi}^{[u,v]}$, we choose a basis $\{\omega_0, \omega_1, \ldots, \omega_d\}$ of $\Omega^1_{\boldsymbol{A}_{R,\varpi}^{[u,v]}}$ obtained as the image of $\{dX_0/(1+X_0), dX_1/X_1, \ldots, dX_d/X_d\}$ under ι_{cycl} (see Section 2.5); in particular, we have the differential operators ∂_i over $\boldsymbol{A}_{R,\varpi}^{[u,v]}$ for $0 \leqslant i \leqslant d$. Moreover, from Definition 2.7, $\boldsymbol{A}_{R,\varpi}^{[u,v]}$ is endowed with a filtration and we have the filtered de Rham complex $\mathrm{Fil}^r\, \Omega^{\bullet}_{\boldsymbol{A}_{R,\varpi}^{[u,v]}}$. The differential operators ∂_i are related to the infinitesimal action of Γ_R by the relation $\nabla_i := \log \gamma_i = t\partial_i$ for $0 \leqslant i \leqslant d$, where

$$\log \gamma_i = \sum_{k \in \mathbb{N}} \frac{(-1)^k(\gamma_i - 1)^{k+1}}{k+1}.$$

Let us set

$$N_{\varpi}^{[u,v]}(T) := \boldsymbol{A}_{R,\varpi}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)$$

and equip it with a Γ_R-stable filtration as in (3-5). Recall that for an indeterminate X we have formal expressions

$$\frac{\log(1+X)}{X} \quad \text{and} \quad \frac{X}{\log(1+X)}$$

(see before Lemma 4.14).

Lemma 5.19. *For $i \in \{0, 1, \ldots, d\}$, the operators*

$$\nabla_i = \log \gamma_i, \qquad \frac{\nabla_i}{\gamma_i - 1} = \frac{\log \gamma_i}{\gamma_i - 1} \quad \textit{and} \quad \frac{\gamma_i - 1}{\nabla_i} = \frac{\gamma_i - 1}{\log \gamma_i}$$

converge as a series of operators on $N_{\varpi}^{[u,v]}(T)$. The same is true for $\boldsymbol{A}_{R,\varpi}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T(r))$ for any $r \in \mathbb{Z}$, and $\mathrm{Fil}^k\, N_{\varpi}^{[u,v]}(T(r))$ *for any $k \in \mathbb{Z}$.*

Proof. We will only show the claim for the operator ∇_i; the claim for the convergence of operators $\nabla_i/(\gamma_i - 1)$ and $(\gamma_i - 1)/\nabla_i$ follows in a manner similar to Lemma 4.14. For $0 \leqslant i \leqslant d$, we have that

$\gamma_i - 1$ acts as a twisted derivation; i.e., for any $a \in \boldsymbol{A}_{R,\varpi}^{[u,v]}$ and $x \in \boldsymbol{N}(T)$, we have

$$(\gamma_i - 1)(ax) = (\gamma_i - 1)a \cdot x + \gamma_i(a)(\gamma_i - 1)x.$$

Note that the action of Γ_R is trivial on $\boldsymbol{N}(T)/\pi\boldsymbol{N}(T)$. So, using Lemma 2.22 and the preceding discussion, we have

$$(\gamma_i - 1)(p^m, \pi_m^{p^m})^k N_{\varpi}^{[u,v]}(T) \subset (p^m, \pi_m^{p^m})^{k+1} N_{\varpi}^{[u,v]}(T).$$

Now, similar to the proof of Lemma 4.13, for $k \geqslant 0$, it follows that

$$(\gamma_i - 1)^k N_{\varpi}^{[u,v]}(T) \subset (p^m, \pi_m^{p^m})^k N_{\varpi}^{[u,v]}(T).$$

The same estimation of the p-adic valuation of the coefficients as in the proof Lemma 4.13 helps us conclude that $\log\gamma_i$ converges as a series of operators on $N_{\varpi}^{[u,v]}(T)$.

Next, from Lemma 3.20, recall that

$$\operatorname{Fil}^k N_{\varpi}^{[u,v]}(T(r)) = \pi^{-r} \operatorname{Fil}^{r+k} N_{\varpi}^{[u,v]}(T)(r).$$

As t/π is a unit in $\boldsymbol{A}_{R,\varpi}^{[u,v]}$ (see Lemma 2.18) and the action of Γ_S is trivial on $t^{-r} \otimes \epsilon^{\otimes r}$, where $\epsilon^{\otimes r}$ denotes a $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(r)$, it is therefore enough to show that ∇_i converges on $\operatorname{Fil}^k N_{\varpi}^{[u,v]}(T)$ for all $k \in \mathbb{N}$. Now, recall that from Remark 3.15 we have a Γ_R-equivariant isomorphism of $E_{R,\varpi}^{[u,v]}$-modules

$$\alpha : \operatorname{Fil}^r(E_{R,\varpi}^{[u,v]} \otimes_R M[1/p]) \xrightarrow{\sim} \operatorname{Fil}^k(E_{R,\varpi}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V))$$

(see (3-8)). Moreover, note that ∇_i converges on $E_{R,\varpi}^{[u,v]}$ since it converges on $\boldsymbol{A}_{R,\varpi}^{[u,v]}$ (see Lemma 4.13) and Γ_R acts trivially on $R_{\varpi}^{[u,v]}$. So, using that the filtration on $E_{R,\varpi}^{[u,v]} \otimes_R M[1/p]$ is given as the tensor product filtration (see Lemma 2.35), the action of Γ_S is trivial on $M[1/p]$ and the ideal $\operatorname{Fil}^j E_{R,\varpi}^{[u,v]}$ is closed in $E_{R,\varpi}^{[u,v]}$ for all $j \in \mathbb{N}$ (see Remark 2.25 (ii)), it follows that ∇_i converges on $\operatorname{Fil}^r(E_{R,\varpi}^{[u,v]} \otimes_R M[1/p])$ and, since α is Γ_R-equivariant, ∇_i also converges on $\operatorname{Fil}^k(E_{R,\varpi}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V))$. Combining the two discussions above, it follows that

$$\nabla_i(\operatorname{Fil}^k N_{\varpi}^{[u,v]}(T)) \subset \operatorname{Fil}^k(E_{R,\varpi}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(V)) \cap N_{\varpi}^{[u,v]}(T) = \operatorname{Fil}^k N_{\varpi}^{[u,v]}(T)$$

(see Remark 3.13). A similar argument shows that the operators $\nabla_i/(\gamma_i - 1)$ and $(\gamma_i - 1)/\nabla_i$ also converge on $\operatorname{Fil}^k N_{\varpi}^{[u,v]}(T)$. This allows us to conclude. □

Lemma 5.20. *For the filtered modules and operators ∇_i defined above and $0 \leqslant i \leqslant d$, we have*

$$\nabla_i(\operatorname{Fil}^k N_{\varpi}^{[u,v]}(T)) \subset \pi \operatorname{Fil}^{k-1} N_{\varpi}^{[u,v]}(T) = t \operatorname{Fil}^{k-1} N_{\varpi}^{[u,v]}(T).$$

Proof. Note that the action of Γ_R is trivial on $N_{\varpi}^{[u,v]}(T)/\pi N_{\varpi}^{[u,v]}(T)$. So, using Lemma 5.19, we infer that

$$\nabla_i(\operatorname{Fil}^k N_{\varpi}^{[u,v]}(T)) \subset \operatorname{Fil}^k N_{\varpi}^{[u,v]}(T) \cap \pi N_{\varpi}^{[u,v]}(T) = \pi \operatorname{Fil}^{k-1} N_{\varpi}^{[u,v]}(T),$$

where the last equality follows from Lemma 3.17. As t/π is a unit in $E_{R,\varpi}^{[u,v]}$ (see Lemma 2.18), we can also write $\nabla_i(\operatorname{Fil}^k N_{\varpi}^{[u,v]}(T)) \subset t \operatorname{Fil}^{k-1} N_{\varpi}^{[u,v]}(T)$. □

For $0 \leqslant i \leqslant d$, it is easy to see that we have $\nabla_i = \log \gamma_i = \lim_{n \to +\infty} (\gamma_i^{p^n} - 1)/p^n$, from which one can easily show that ∇_i satisfies a Leibniz rule (see the proof of [Morrow and Tsuji 2020, Theorem 4.2] for a similar argument). Now using Lemma 5.19 we define differential operators ∂_i over $N_{\varpi}^{[u,v]}(T)$ as $\partial_i := \nabla_i / t = (\log \gamma_i)/t$. In the basis $\{\omega_0, \ldots, \omega_d\}$ of $\Omega^1_{\boldsymbol{A}_{R,\varpi}^{[u,v]}}$, we set $\partial = (\partial_0, \ldots, \partial_d)$ and obtain a connection $\partial : N_{\varpi}^{[u,v]}(T) \to N_{\varpi}^{[u,v]}(T) \otimes \Omega^1_{\boldsymbol{A}_{R,\varpi}^{[u,v]}}$ by sending $ax \mapsto a\partial(x) + x \otimes da$.

Lemma 5.21. *The connection ∂ on $N_{\varpi}^{[u,v]}(T)$ is integrable, and satisfies a Leibniz rule and Griffiths transversality with respect to the filtration; i.e., $\partial_i(\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) \subset \mathrm{Fil}^{k-1} N_{\varpi}^{[u,v]}(T)$ for $0 \leqslant i \leqslant d$.*

Proof. From Section 4.3.2, recall that $[\nabla_i, \nabla_j] = 0$ for $1 \leqslant i, j \leqslant d$ and $[\nabla_0, \nabla_i] = p^m \nabla_i$ for $1 \leqslant i \leqslant d$. It follows that, over $N_{\varpi}^{[u,v]}(T)$, we have a composition of operators

$$t^2(\partial_i \circ \partial_j - \partial_j \circ \partial_i) = t\partial_i(t\partial_j) - t\partial_j(t\partial_i) = \nabla_i \circ \nabla_j - \nabla_j \circ \nabla_i = 0 \quad \text{for } 1 \leqslant i, j \leqslant d.$$

Next, for $1 \leqslant i \leqslant d$, we have

$$\begin{aligned} \nabla_0 \circ \nabla_i - \nabla_i \circ \nabla_0 = t\partial_0 \circ (t\partial_i) - t\partial_i \circ (t\partial_0) &= tp^m \partial_i + t^2 \partial_0 \circ \partial_i - t^2 \partial_i \circ \partial_0 \\ &= p^m \nabla_i + t^2(\partial_0 \circ \partial_i - \partial_i \circ \partial_0). \end{aligned}$$

In particular, $\partial_0 \circ \partial_i - \partial_i \circ \partial_0 = 0$. Since $\partial \circ \partial = (\partial_i \circ \partial_j)_{i,j}$ for $0 \leqslant i \leqslant j \leqslant d$ and $N_{\varpi}^{[u,v]}(T)$ is t-torsion free, we conclude that the connection ∂ is integrable. Moreover, it is clear that ∂ satisfies a Leibniz rule, and it satisfies Griffiths transversality because we have

$$\partial_i(\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) = t^{-1}\nabla_i(\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) \subset \mathrm{Fil}^{k-1} N_{\varpi}^{[u,v]}(T)$$

using Lemma 5.20. □

Let $S = \boldsymbol{A}_{R,\varpi}^{[u,v]}$. Then, from Lemma 5.21, we have the filtered de Rham complex $\mathrm{Fil}^r N_{\varpi}^{[u,v]}(T) \otimes \Omega^\bullet_S$. In the chosen basis $\{\omega_1, \ldots, \omega_d\}$ of Ω^1_S, an element of $\Omega^q_S = \bigwedge^q \Omega^1_S$ can be expressed as $\sum_{\boldsymbol{i}} x_{\boldsymbol{i}} \omega_{\boldsymbol{i}}$ in a unique manner, where $x_{\boldsymbol{i}} \in S$ and $\omega_{\boldsymbol{i}} = \omega_{i_1} \wedge \cdots \wedge \omega_{i_q}$ for $\boldsymbol{i} = (i_1, \ldots, i_q) \in I_q = \{0 \leqslant i_1 < \cdots < i_q \leqslant d\}$. In this case, the map involving differential operators becomes

$$(\partial_i) : (\mathrm{Fil}^{k-q} N_{\varpi}^{[u,v]}(T))^{I_q} \to (\mathrm{Fil}^{k-q-1} N_{\varpi}^{[u,v]}(T))^{I_{q+1}} \quad \text{for } 0 \leqslant i \leqslant d.$$

Definition 5.22. Define the ∂-*Koszul complex* for $\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)$ as

$$\mathrm{Kos}(\partial_A, \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) : \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T) \xrightarrow{(\partial_i)} (\mathrm{Fil}^{k-1} N_{\varpi}^{[u,v]}(T))^{I_1} \to \cdots.$$

Remark 5.23. (i) By definition, it follows that we have a natural isomorphism between complexes

$$\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T) \otimes \Omega^\bullet_{\boldsymbol{A}_{R,\varpi}^{[u,v]}} \xrightarrow{\sim} \mathrm{Kos}(\partial_A, \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)).$$

(ii) Let $I'_q = \{(i_1, \ldots, i_q)$ such that $1 \leqslant i_1 < \cdots < i_q \leqslant d\}$ and $\partial' = (\partial_1, \ldots, \partial_d)$. Set

$$\mathrm{Kos}(\partial'_A, \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) : \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T) \xrightarrow{(\partial_i)} (\mathrm{Fil}^{k-1} N_{\varpi}^{[u,v]}(T))^{I'_1} \to \cdots,$$

and note that $\mathrm{Kos}(\partial_A, \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) = \big[\mathrm{Kos}(\partial'_A, \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) \xrightarrow{\partial_0} \mathrm{Kos}(\partial'_A, \mathrm{Fil}^{k-1} N_{\varpi}^{[u,v]}(T))\big]$.

(iii) Computations carried out in this section are true over the ring $\boldsymbol{A}_{R,\varpi}^{[u,v/p]}$ as well.

5.6. *Poincaré lemma.* For $\star \in \{\mathrm{PD}, [u], [u, v]\}$, from Definition 2.24, Remark 2.25 and Lemma 2.26, recall that we have rings $E^{\star}_{R,\varpi}$ equipped with a filtration, Frobenius φ sending

$$E^{\mathrm{PD}}_{R,\varpi} \to E^{\mathrm{PD}}_{R,\varpi}, \quad E^{[u]}_{R,\varpi} \to E^{[u]}_{R,\varpi} \quad \text{and} \quad E^{[u,v]}_{R,\varpi} \to E^{[u,v/p]}_{R,\varpi},$$

and an action of G_R which commutes with the Frobenius. Moreover, from Remark 2.27, we have a subring $\mathcal{O}\boldsymbol{A}^{\mathrm{PD}}_{R,\varpi} \subset \mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\bar{R})$ equipped with induced structures, and we have a natural embedding $\mathcal{O}\boldsymbol{A}^{\mathrm{PD}}_{R,\varpi} \subset \boldsymbol{E}^{\mathrm{PD}}_{R,\varpi}$ compatible with the respective Frobenii, filtrations, $\boldsymbol{A}^{\mathrm{PD}}_{R,\varpi}$-linear connections and actions of Γ_R.

From Assumption 5.1, we have a natural map

$$\mathcal{O}\boldsymbol{A}^{\mathrm{PD}}_{R,\varpi} \otimes_R M \to \mathcal{O}\boldsymbol{A}^{\mathrm{PD}}_{R,\varpi} \otimes_R \boldsymbol{N}(T),$$

which is a $p^{n(T,e)}$-isomorphism compatible with the respective Frobenii, filtrations, connections and the actions of Γ_R. Recall that $M^{[u,v]}_{\varpi} = R^{[u,v]}_{\varpi} \otimes_R M$ and $N^{[u,v]}_{\varpi}(T) = \boldsymbol{A}^{[u,v]}_{R,\varpi} \otimes_{\boldsymbol{A}^+_R} \boldsymbol{N}(T)$, and after extension of scalars we have a map

$$E^{[u,v]}_{R,\varpi} \otimes_{R^{[u,v]}_{\varpi}} M^{[u,v]}_{\varpi} \to E^{[u,v]}_{R,\varpi} \otimes_{\boldsymbol{A}^{[u,v]}_{R,\varpi}} N^{[u,v]}_{\varpi}(T),$$

which is a $p^{n(T,e)}$-isomorphism compatible with the respective Frobenii, connections and the actions of Γ_R. Moreover, in the $p^{n(T,e)}$-isomorphism above, the left-hand term is equipped with a filtration as described in the discussion before Lemma 2.40, and the right-hand term is equipped with a filtration as in (3-5), which is compatible with the filtration on the left-hand term by definition.

Let $R_1 := \boldsymbol{A}^{[u,v]}_{R,\varpi}$, $R_2 := R^{[u,v]}_{\varpi}$ and $R_3 := E^{[u,v]}_{R,\varpi}$. Set $X_{0,1} := \pi_m$ and $X_{0,2} := X_0$, and set $X_{i,1} := [X_i^{\flat}]$ and $X_{i,2} := X_i$ for $1 \leqslant i \leqslant d$. For $j = 1, 2$, set

$$\Omega^1_j := \mathbb{Z}\frac{dX_{0,j}}{1 + X_{0,j}} \bigoplus_{i=1}^{d} \mathbb{Z}\frac{dX_{i,j}}{X_{i,j}} \quad \text{and} \quad \Omega^1_3 := \Omega^1_1 \oplus \Omega^1_2.$$

For $j = 1, 2, 3$, let $\Omega^k_j = \bigwedge^k \Omega_j$. Therefore, we see that $\Omega^k_{R_j} = R_j \otimes \Omega^k_j$. Recall that from (5-4) we have the filtered de Rham complex $\mathrm{Fil}^r M^{[u,v]}_{\varpi} \otimes \Omega^{\bullet}_1$. Set $\Delta_2 := E^{[u,v]}_{R,\varpi} \otimes_{R^{[u,v]}_{\varpi}} M^{[u,v]}_{\varpi}$ equipped with a filtration as described in the discussion before Lemma 2.40. Using the O_F-linear de Rham differential operator

$$\partial_{R_3} : \mathrm{Fil}^r E^{[u,v]}_{R,\varpi} \to \mathrm{Fil}^{r-1} E^{[u,v]}_{R,\varpi} \otimes_{\mathbb{Z}} \Omega^1_3$$

and the O_F-linear integrable connection

$$\partial_{R_2} : \mathrm{Fil}^r M^{[u,v]}_{\varpi} \to \mathrm{Fil}^{r-1} M^{[u,v]}_{\varpi} \otimes_{\mathbb{Z}} \Omega^1_2,$$

we obtain an O_F-linear integrable connection on Δ_2 as

$$\partial_{R_3} : \Delta_2 \to \Delta_2 \otimes_{\mathbb{Z}} \Omega^1_3, \quad ax \mapsto a\partial_{R_2}(x) + \partial_{R_3}(a)x.$$

Moreover, the connection ∂_{R_3} on Δ_2 satisfies Griffiths transversality with respect to the filtration, i.e., $\partial_{R_3} : \mathrm{Fil}^r \Delta_2 \to \mathrm{Fil}^{r-1} \Delta_2 \otimes_{\mathbb{Z}} \Omega^1_3$, since the same is true for the differential operator on $E^{[u,v]}_{R,\varpi}$ and the connection on $M^{[u,v]}_{\varpi}$. In particular, we have the filtered de Rham complex $\mathrm{Fil}^r \Delta_2 \otimes \Omega^{\bullet}_3$.

Lemma 5.24. *The natural map* $\mathrm{Fil}^r\, M_{\varpi}^{[u,v]} \otimes \Omega_2^{\bullet} \to \mathrm{Fil}^r\, \Delta_2 \otimes \Omega_3^{\bullet}$ *is a quasi-isomorphism.*

Proof. In the notation of Section 2.8.3, note that $A = R_1,\ B = R_2$ and $E = R_3$. Moreover, by definition, it is clear that $\mathrm{Fil}^r\, M_{\varpi}^{[u,v]} = (\mathrm{Fil}^r\, \Delta_2)^{\partial_{R_1}=0}$. Therefore, using Lemma 2.41, we obtain the claim. □

Similar to above and using the discussion of Section 5.5, it is easy to see that, for $R_1 = A_{R,\varpi}^{[u,v]}$, we have a filtered de Rham complex $\mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T) \otimes \Omega_1^{\bullet}$. Let $\Delta_1 := E_{R,\varpi}^{[u,v]} \otimes_{A_{R,\varpi}^{[u,v]}} N_{\varpi}^{[u,v]}(T)$ be equipped with the filtration described in (3-5). Then, similar to the case of Δ_2, we have a filtered de Rham complex $\mathrm{Fil}^r\, \Delta_1 \otimes \Omega_3^{\bullet}$ and, similar to Lemma 5.24, we obtain the following.

Lemma 5.25. *The natural map* $\mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T) \otimes \Omega_1^{\bullet} \to \mathrm{Fil}^r\, \Delta_1 \otimes \Omega_3^{\bullet}$ *is a quasi-isomorphism.*

Proof. In the notation of Section 3.3.2, note that $A = R_1,\ B = R_2$ and $E = R_3$. Using the equality $N_{\varpi}^{[u,v]}(T) = \Delta_1^{\partial=0}$ and (3-5), we note that

$$\mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T) = \mathrm{Fil}^r\, \Delta_1 \cap \Delta_1^{\partial=0} = (\mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T))^{\partial=0}.$$

Therefore, using Lemma 3.21, we obtain the claim. □

Remark 5.26. Statements analogous to Lemmas 5.24 and 5.25 for $R_{\varpi}^{[u,v/p]}$ and $A_{R,\varpi}^{[u,v/p]}$ (instead of $R_{\varpi}^{[u,v]}$ and $A_{R,\varpi}^{[u,v]}$), respectively, are also true.

Definition 5.27. Consider $N_{\varpi}^{[u,v]}(T)$ as above, and note that it is equipped with a Frobenius-semilinear morphism

$$\varphi : N_{\varpi}^{[u,v]}(T) \to N_{\varpi}^{[u,v/p]}(T).$$

Using Definition 5.22 and Remark 5.23, set

$$\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T)) := \begin{bmatrix} \mathrm{Kos}(\partial'_A, \mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T)) & \xrightarrow{p^r - p^{\bullet}\varphi} & \mathrm{Kos}(\partial'_A, N_{\varpi}^{[u,v/p]}(T)) \\ \downarrow \partial_0 & & \downarrow \partial_0 \\ \mathrm{Kos}(\partial'_A, \mathrm{Fil}^{r-1}\, N_{\varpi}^{[u,v]}(T)) & \xrightarrow{p^r - p^{\bullet+1}\varphi} & \mathrm{Kos}(\partial'_A, N_{\varpi}^{[u,v/p](T)}) \end{bmatrix}.$$

Proposition 5.28. *There exists a natural* $p^{2n(T,e)}$*-quasi-isomorphism between complexes* $\mathrm{Syn}(M_{\varpi}^{[u,v]}, r)$ *and* $\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T))$, *where* $n(T,e) \in \mathbb{N}$ *as in Assumption 5.1.*

Proof. Note that, using Lemma 5.24 with

$$R_1 = R_{\varpi}^{[u,v]}, \quad R_3 = E_{R,\varpi}^{[u,v]}, \quad \Delta_1 = E_{R,\varpi}^{[u,v]} \otimes_{R_{\varpi}^{[u,v]}} M_{\varpi}^{[u,v]} \quad \text{and} \quad \Delta'_1 = E_{R,\varpi}^{[u,v/p]} \otimes_{R_{\varpi}^{[u,v/p]}} M_{\varpi}^{[u,v/p]},$$

we have natural quasi-isomorphisms of complexes

$$\mathrm{Syn}(M_{\varpi}^{[u,v]}, r) \simeq \big[\mathrm{Fil}^r\, M_{\varpi}^{[u,v]} \otimes \Omega_1^{\bullet} \xrightarrow{p^r - p^{\bullet}\varphi} M_{\varpi}^{[u,v/p]} \otimes \Omega_1^{\bullet}\big] \simeq \big[\mathrm{Fil}^r\, \Delta_1 \otimes \Omega_3^{\bullet} \xrightarrow{p^r - p^{\bullet}\varphi} \Delta'_1 \otimes \Omega_3^{\bullet}\big].$$

Next, using Lemma 5.24 with

$$R_2 = A_{R,\varpi}^{[u,v]}, \quad R_3 = E_{R,\varpi}^{[u,v]}, \quad \Delta_2 = E_{R,\varpi}^{[u,v]} \otimes_{A_{R,\varpi}^{[u,v]}} N_{\varpi}^{[u,v]}(T) \quad \text{and} \quad \Delta'_2 = E_{R,\varpi}^{[u,v/p]} \otimes_{A_{R,\varpi}^{[u,v/p]}} N_{\varpi}^{[u,v/p]},$$

together with Remark 5.23, note that we have natural quasi-isomorphisms of complexes

$$\begin{aligned}\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) &\simeq \big[\mathrm{Fil}^r N_{\varpi}^{[u,v]}(T) \otimes \Omega_2^{\bullet} \xrightarrow{p^r - p^{\bullet}\varphi} \mathrm{Fil}^r N_{\varpi}^{[u,v/p]} \otimes \Omega_2^{\bullet}\big] \\ &\simeq \big[\mathrm{Fil}^r \Delta_2 \otimes \Omega_3^{\bullet} \xrightarrow{p^r - p^{\bullet}\varphi} \Delta_2' \otimes \Omega_3^{\bullet}\big].\end{aligned}$$

Finally, using the $p^{n(T,e)}$-isomorphism

$$E_{R,\varpi}^{[u,v]} \otimes_{R_{\varpi}^{[u,v]}} M_{\varpi}^{[u,v]} \xrightarrow{\sim} E_{R,\varpi}^{[u,v]} \otimes_{\boldsymbol{A}_{R,\varpi}^{[u,v]}} N_{\varpi}^{[u,v]}(T)$$

from Assumption 5.1, we have $p^{n(T,e)}$-isomorphisms $\mathrm{Fil}^r \Delta_1 \simeq \mathrm{Fil}^r \Delta_2$ and $\Delta_1' \simeq \Delta_2'$. Hence, from the discussion above, we obtain a natural $p^{2n(T,e)}$-quasi-isomorphism of complexes

$$\mathrm{Syn}(M_{\varpi}^{[u,v]}, r) \simeq \mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)). \qquad \square$$

6. Syntomic complexes and (φ, Γ)-modules

In this section, we will work under the setup of Assumption 5.1 and carry out the second step of the proof of Theorem 5.5. Recall that we have a finite free $\boldsymbol{A}_{R,\varpi}^{[u,v]}$-module

$$N_{\varpi}^{[u,v]}(T) = \boldsymbol{A}_{R,\varpi}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)$$

equipped with a Γ_R-stable filtration as in (3-5) and, from Definition 5.27, we have the complex

$$\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)).$$

Let $S = R[\varpi]$. From the theory of étale (φ, Γ_S)-modules in Section 2.4, we have

$$D_{\varpi}(T(r)) = \boldsymbol{A}_{R,\varpi} \otimes_{\boldsymbol{A}_R} \boldsymbol{D}(T(r)),$$

and from Definition 4.11 we have the complex $\mathrm{Kos}(\varphi, \Gamma_S, D_{\varpi}(T(r)))$. In this section, our goal is to show the following.

Proposition 6.1. *There exist natural p^N-quasi-isomorphisms of complexes*

$$\tau_{\leqslant r}\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) \simeq \tau_{\leqslant r}\mathrm{Kos}(\varphi, \Gamma_S, D_{\varpi}(T(r))),$$

where $N = N(r, s) \in \mathbb{N}$ depends only on the height s of the representation T and twist r.

6.1. ***Proof of Theorem 5.5.*** Note that, by combining Propositions 5.12 and 5.14, we have a natural p^{4r+4s}-quasi-isomorphism of complexes

$$\tau_{\leqslant r-s-1}\, \mathrm{Syn}(M_{\varpi}^{\mathrm{PD}}, r) \simeq \tau_{\leqslant r-s-1}\, \mathrm{Syn}(M_{\varpi}^{[u,v]}, r).$$

Next, from Proposition 5.28, we have a natural $p^{2n(T,e)}$-quasi-isomorphism of complexes

$$\mathrm{Syn}(M_{\varpi}^{[u,v]}, r) \simeq \mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)).$$

Furthermore, by Proposition 6.1, we have a natural $p^{10r+3s+2}$-quasi-isomorphism of complexes

$$\tau_{\leqslant r}\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) \simeq \tau_{\leqslant r}\mathrm{Kos}(\varphi, \Gamma_S, D_{\varpi}(T(r))),$$

where $\tau_{\leqslant}$ denotes the canonical truncation (for the explicit constant, see the proof of Proposition 6.1 at the end of Section 6.6). Finally, by Proposition 4.10 and Theorem 4.2, we have a natural quasi-isomorphism of complexes

$$\mathrm{Kos}(\varphi, \Gamma_S, D_{\varpi}(T(r))) \simeq \mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T(r)).$$

Combining all these statements gives us the desired conclusion with $N = 2n(T,e) + 14r + 7s + 2$. □

In the rest of this section, we will prove Proposition 6.1.

6.2. ***From differential forms to the infinitesimal action of*** $\mathbf{\Gamma_S}$**.** Note that Lemma 5.19 describes the action of $\mathrm{Lie}\,\Gamma_S$ on $\mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T)$. Then, for the Lie subgroup $\Gamma'_S \subset \Gamma_S$ (see Section 2.4 for notation), using Definition 4.15, we have the complex $\mathrm{Kos}(\mathrm{Lie}\,\Gamma'_S, \mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T))$ and we consider its subcomplex, i.e., a complex made of submodules in each degree stable under the differentials of the complex, as follows:

$$\begin{aligned}&\mathcal{K}(\mathrm{Lie}\,\Gamma'_S, \mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T))\\ &\qquad := \mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T) \xrightarrow{(\nabla_i)} (t\,\mathrm{Fil}^{r-1}\, N_{\varpi}^{[u,v]}(T))^{I'_1} \to \cdots \to (t^k\,\mathrm{Fil}^{r-k}\, N_{\varpi}^{[u,v]}(T))^{I'_k} \to \cdots.\end{aligned}$$

Using the same differentials, we can define a complex $\mathcal{K}(\mathrm{Lie}\,\Gamma'_S, t\,\mathrm{Fil}^{r-1}\, N_{\varpi}^{[u,v]}(T))$ as a subcomplex of $\mathrm{Kos}(\mathrm{Lie}\,\Gamma'_S, \mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T))$. Now consider a morphism of complexes

$$\nabla_0 : \mathcal{K}(\mathrm{Lie}\,\Gamma'_S, \mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T)) \to \mathcal{K}(\mathrm{Lie}\,\Gamma'_S, t\,\mathrm{Fil}^{r-1}\, N_{\varpi}^{[u,v]}(T))$$

given as $\nabla_0 = \log \gamma_0$ in degree 0 and as

$$\nabla_0 - kp^m : (t^k\,\mathrm{Fil}^{r-k}\, N_{\varpi}^{[u,v]}(T(r)))^{I'_k} \to (t^{k+1}\,\mathrm{Fil}^{r-k-1}\, N_{\varpi}^{[u,v]}(T(r)))^{I'_k}$$

on the k-th term of the definition above for $1 \leqslant k \leqslant d$. The morphism of complexes is well defined because we have $\nabla_0\nabla_i - \nabla_i\nabla_0 = p^m\nabla_i$ for $1 \leqslant i \leqslant d$ (see Section 4.3.2 and the discussion after Definition 4.15). Write the total complex of the diagram thus obtained as $\mathcal{K}(\mathrm{Lie}\,\Gamma_S, \mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T))$, which is a subcomplex of $\mathrm{Kos}(\mathrm{Lie}\,\Gamma_S, \mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T))$ by definition. Similarly, we can define complexes $\mathcal{K}(\mathrm{Lie}\,\Gamma'_S, N_{\varpi}^{[u,v/p]}(T))$ and $\mathcal{K}(\mathrm{Lie}\,\Gamma'_S, tN_{\varpi}^{[u,v/p]}(T))$ and a map ∇_0 from the former to the latter complex.

Recall that, from Definition 5.27, we have the Koszul complex $\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T))$. Note that $\nabla_i = t\partial_i$ for all $0 \leqslant i \leqslant d$ (see Section 5.5). So we consider a morphism of complexes

$$\mathrm{Kos}(\partial'_A, \mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T)) \to \mathcal{K}(\mathrm{Lie}\,\Gamma'_S, \mathrm{Fil}^r\, N_{\varpi}^{[u,v]}(T))$$

given by the identity map in degree 0 and multiplication by t^k on the k-th term of the definition above; i.e.,

$$(\mathrm{Fil}^{r-k}\, N_{\varpi}^{[u,v]}(T(r)))^{I'_k} \xrightarrow{\times t^k} (t^k\,\mathrm{Fil}^{r-k}\, N_{\varpi}^{[u,v]}(T(r)))^{I'_k} \quad \text{for } 1 \leqslant k \leqslant d.$$

It is clear that the map thus defined is bijective, i.e., we obtain an isomorphism of complexes. Similarly, multiplying by powers of t as above, we obtain an isomorphism of complexes

$$\mathrm{Kos}(\partial'_A, \mathrm{Fil}^{r-1}\, N_{\varpi}^{[u,v]}(T)) \xrightarrow{\sim} \mathcal{K}(\mathrm{Lie}\,\Gamma'_S, t\,\mathrm{Fil}^{r-1}\, N_{\varpi}^{[u,v]}(T)).$$

Furthermore, one can do a similar construction for $N_{\varpi}^{[u,v/p]}(T)$ to obtain isomorphism of complexes

$$\mathrm{Kos}(\partial'_A, N_{\varpi}^{[u,v/p]}(T)) \xrightarrow{\sim} \mathcal{K}(\mathrm{Lie}\,\Gamma'_S, N_{\varpi}^{[u,v/p]}(T)),$$
$$\mathrm{Kos}(\partial'_A, N_{\varpi}^{[u,v/p]}(T)) \xrightarrow{\sim} \mathcal{K}(\mathrm{Lie}\,\Gamma'_S, tN_{\varpi}^{[u,v/p]}(T)).$$

As each term of these complexes admits a Frobenius-semilinear morphism

$$\varphi : t^j\,\mathrm{Fil}^{r-j}\,N_{\varpi}^{[u,v]}(T) \to t^j N_{\varpi}^{[u,v/p]}(T),$$

we obtain the following morphism of complexes (see Definition 5.27 for the source complex):

$$\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r\,N_{\varpi}^{[u,v]}(T)) \to \begin{bmatrix} \mathcal{K}(\mathrm{Lie}\,\Gamma'_S, \mathrm{Fil}^r\,N_{\varpi}^{[u,v]}(T)) & \xrightarrow{p^r-\varphi} & \mathcal{K}(\mathrm{Lie}\,\Gamma'_S, N_{\varpi}^{[u,v/p]}(T)) \\ \big\downarrow \nabla_0 & & \big\downarrow \nabla_0 \\ \mathcal{K}(\mathrm{Lie}\,\Gamma'_S, t\,\mathrm{Fil}^{r-1}\,N_{\varpi}^{[u,v]}(T)) & \xrightarrow{p^r-\varphi} & \mathcal{K}(\mathrm{Lie}\,\Gamma'_S, tN_{\varpi}^{[u,v/p]}(T)) \end{bmatrix}.$$

From the discussion above, we have the following.

Lemma 6.2. *The morphism of complexes described above is an isomorphism.*

Recall that s is the height of T and we fixed some $r \geqslant s+1$. Set $N_{\varpi}^{[u,v]}(T(r)) := \boldsymbol{A}_{R,\varpi}^{[u,v]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T(r))$ and equip it with the natural action of Γ_R and a Γ_R-stable filtration as in (3-10). Then, from Lemma 5.19, recall that the operators ∇_i are well defined over $\mathrm{Fil}^k\,N_{\varpi}^{[u,v]}(T(r))$ for $0 \leqslant i \leqslant d$. Using these operators, we consider a subcomplex of the Koszul complex $\mathrm{Kos}(\mathrm{Lie}\,\Gamma'_S, \mathrm{Fil}^0\,N_{\varpi}^{[u,v]}(T(r)))$ (Definition 4.15) as follows:

$$\begin{aligned}&\mathcal{K}(\mathrm{Lie}\,\Gamma'_S, \mathrm{Fil}^0\,N_{\varpi}^{[u,v]}(T(r))) \\ &\qquad := \mathrm{Fil}^0\,N_{\varpi}^{[u,v]}(T(r)) \xrightarrow{(\nabla_i)} (t\,\mathrm{Fil}^{-1}\,N_{\varpi}^{[u,v]}(T(r)))^{I'_1} \to \cdots \to (t^k\,\mathrm{Fil}^{-k}\,N_{\varpi}^{[u,v]}(T(r)))^{I'_k} \to \cdots.\end{aligned}$$

Similarly, we can define a complex $\mathcal{K}(\mathrm{Lie}\,\Gamma'_S, t\,\mathrm{Fil}^{-1}\,N_{\varpi}^{[u,v]}(T(r)))$ as a subcomplex of the Koszul complex $\mathrm{Kos}(\mathrm{Lie}\,\Gamma'_S, \mathrm{Fil}^0\,N_{\varpi}^{[u,v]}(T(r)))$. Moreover, similar to the discussion before Lemma 6.2, we can define a morphism of complexes

$$\nabla_0 : \mathcal{K}(\mathrm{Lie}\,\Gamma'_S, \mathrm{Fil}^0\,N_{\varpi}^{[u,v]}(T(r))) \to \mathcal{K}(\mathrm{Lie}\,\Gamma'_S, t\,\mathrm{Fil}^{-1}\,N_{\varpi}^{[u,v]}(T(r))).$$

The associated total complex, written as $\mathcal{K}(\mathrm{Lie}\,\Gamma_S, \mathrm{Fil}^r\,N_{\varpi}^{[u,v]}(T))$, is a subcomplex of the Koszul complex $\mathrm{Kos}(\mathrm{Lie}\,\Gamma_S, \mathrm{Fil}^0\,N_{\varpi}^{[u,v]}(T(r)))$. Furthermore, by a similar construction, we can define the complexes $\mathcal{K}(\mathrm{Lie}\,\Gamma'_S, N_{\varpi}^{[u,v/p]}(T(r)))$ and $\mathcal{K}(\mathrm{Lie}\,\Gamma'_S, tN_{\varpi}^{[u,v/p]}(T(r)))$ and a morphism ∇_0 from the former to the latter.

Next, from Lemma 3.20, recall that $\mathrm{Fil}^k\,N_{\varpi}^{[u,v]}(T(r)) = \pi^{-r}\,\mathrm{Fil}^{k+r}\,N_{\varpi}^{[u,v]}(T)(r)$ for each $k \in \mathbb{Z}$. Let ϵ^{-r} denote a $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(-r)$; then we see that

$$(t^r \otimes \epsilon^{-r})\,\mathrm{Fil}^k\,N_{\varpi}^{[u,v]}(T(r)) = (t/\pi)^r\,\mathrm{Fil}^{r+k}\,N_{\varpi}^{[u,v]}(T) = \mathrm{Fil}^{r+k}\,N_{\varpi}^{[u,v]}(T),$$

where the last equality follows since t/π is a unit in $\boldsymbol{A}_{R,\varpi}^{[u,v]}$ (see Lemma 2.18). Now, consider a morphism of complexes

$$\mathcal{K}(\operatorname{Lie}\Gamma_S', \operatorname{Fil}^0 N_\varpi^{[u,v]}(T(r))) \to \mathcal{K}(\operatorname{Lie}\Gamma_S', \operatorname{Fil}^r N_\varpi^{[u,v]}(T))$$

given as multiplication by $t^r \otimes \epsilon^{-r}$ in each degree; in particular, it is given as

$$(t^k \operatorname{Fil}^{-k} N_\varpi^{[u,v]}(T(r)))^{I_k'} \xrightarrow{\times(t^r\otimes\epsilon^{-r})} (t^k \operatorname{Fil}^{r-k} N_\varpi^{[u,v]}(T))^{I_k'}$$

on the k-th term of the definition above for $1 \leqslant k \leqslant d$. Note that the map thus defined is bijective on each term by the preceding discussion. Similarly, we have

$$(t^r \otimes \epsilon^{-r}) N_\varpi^{[u,v/p]}(T(r)) = (t/\pi)^r N_\varpi^{[u,v/p]}(T) = N_\varpi^{[u,v/p]}(T),$$

which yields an isomorphism of complexes

$$\mathcal{K}(\operatorname{Lie}\Gamma_S', N_\varpi^{[u,v/p]}(T(r))) \xrightarrow{\sim} \mathcal{K}(\operatorname{Lie}\Gamma_S', t N_\varpi^{[u,v/p]}(T(r))).$$

Putting these together, we obtain the following.

Lemma 6.3. *The morphism of complexes below, given as multiplication by $t^r \otimes \epsilon^{-r}$ on each term, is an isomorphism*:

$$\begin{bmatrix} \mathcal{K}(\operatorname{Lie}\Gamma_S', \operatorname{Fil}^0 N_\varpi^{[u,v]}(T(r))) & \xrightarrow{p^r(1-\varphi)} & \mathcal{K}(\operatorname{Lie}\Gamma_S', N_\varpi^{[u,v/p]}(T(r))) \\ \downarrow \nabla_0 & & \downarrow \nabla_0 \\ \mathcal{K}(\operatorname{Lie}\Gamma_S', t \operatorname{Fil}^{-1} N_\varpi^{[u,v]}(T(r))) & \xrightarrow{p^r(1-\varphi)} & \mathcal{K}(\operatorname{Lie}\Gamma_S', t N_\varpi^{[u,v/p]}(T(r))) \end{bmatrix}$$

$$\xrightarrow{\sim} \begin{bmatrix} \mathcal{K}(\operatorname{Lie}\Gamma_S', \operatorname{Fil}^r N_\varpi^{[u,v]}(T)) & \xrightarrow{p^r-\varphi} & \mathcal{K}(\operatorname{Lie}\Gamma_S', N_\varpi^{[u,v/p]}(T)) \\ \downarrow \nabla_0 & & \downarrow \nabla_0 \\ \mathcal{K}(\operatorname{Lie}\Gamma_S', t \operatorname{Fil}^{r-1} N_\varpi^{[u,v]}(T)) & \xrightarrow{p^r-\varphi} & \mathcal{K}(\operatorname{Lie}\Gamma_S', t N_\varpi^{[u,v/p]}(T)) \end{bmatrix}.$$

In order to change from "$\operatorname{Lie}\Gamma_S$-Koszul complexes" to "$\Gamma_S$-Koszul complexes", we modify the source complex in Lemma 6.3 to define $\mathcal{K}(\varphi, \operatorname{Lie}\Gamma_S, N_\varpi^{[u,v]}(T(r)))$ as follows:

$$\begin{bmatrix} \mathcal{K}(\operatorname{Lie}\Gamma_S', \operatorname{Fil}^0 N_\varpi^{[u,v]}(T(r))) & \xrightarrow{1-\varphi} & \mathcal{K}(\operatorname{Lie}\Gamma_S', N_\varpi^{[u,v/p]}(T(r))) \\ \downarrow \nabla_0 & & \downarrow \nabla_0 \\ \mathcal{K}(\operatorname{Lie}\Gamma_S', t \operatorname{Fil}^{-1} N_\varpi^{[u,v]}(T(r))) & \xrightarrow{1-\varphi} & \mathcal{K}(\operatorname{Lie}\Gamma_S', t N_\varpi^{[u,v/p]}(T(r))) \end{bmatrix}$$

By definition, the complex $\mathcal{K}(\varphi, \operatorname{Lie}\Gamma_S, N_\varpi^{[u,v]}(T(r)))$ is p^{4r}-isomorphic to the source complex in Lemma 6.3. Combining this with Lemmas 6.2 and 6.3, we get the following.

Proposition 6.4. *There exists a natural p^{4r}-quasi-isomorphism of complexes*

$$\operatorname{Kos}(\varphi, \partial_A, \operatorname{Fil}^r N_\varpi^{[u,v]}(T)) \simeq \mathcal{K}(\varphi, \operatorname{Lie}\Gamma_S, N_\varpi^{[u,v]}(T(r))).$$

6.3. ***From the infinitesimal action of Γ_S to the continuous action of Γ_S.*** In this subsection, we will study Koszul complexes involving operators $\gamma_i - 1$ over $N_{\varpi}^{[u,v]}(T(r))$. Note that

$$(\gamma_i - 1)\operatorname{Fil}^k N_{\varpi}^{[u,v]}(T(r)) \subset \operatorname{Fil}^k N_{\varpi}^{[u,v]}(T(r)) \cap \pi N_{\varpi}^{[u,v]}(T(r)) = \pi \operatorname{Fil}^{k-1} N_{\varpi}^{[u,v]}(T(r)),$$

where the last equality follows from Lemmas 3.17 and 3.20. Define a subcomplex of the Koszul complex $\operatorname{Kos}(\Gamma'_S, \operatorname{Fil}^0 N_{\varpi}^{[u,v]}(T(r)))$ (see Definition 4.9) as follows:

$$\begin{aligned}&\mathcal{K}(\Gamma'_S, \operatorname{Fil}^0 N_{\varpi}^{[u,v]}(T(r)))\\ &\qquad := \operatorname{Fil}^0 N_{\varpi}^{[u,v]}(T(r)) \xrightarrow{(\tau_i)} (\pi \operatorname{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r)))^{I'_1} \to (\pi^2 \operatorname{Fil}^{-2} N_{\varpi}^{[u,v]}(T(r)))^{I'_2} \to \cdots.\end{aligned}$$

Similarly, we can define a complex $\mathcal{K}^c(\Gamma'_S, \pi \operatorname{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r)))$ as a subcomplex of the Koszul complex $\operatorname{Kos}^c(\Gamma'_S, \operatorname{Fil}^0 N_{\varpi}^{[u,v]}(T(r)))$ (see Definition 4.9), where $c = \chi(\gamma_0) = \exp(p^m)$. Consider a morphism of complexes

$$\tau_0 : \mathcal{K}(\Gamma'_S, \operatorname{Fil}^0 N_{\varpi}^{[u,v]}(T(r))) \to \mathcal{K}^c(\Gamma'_S, \pi \operatorname{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r))),$$

which is given as $\gamma_0 - 1$ in degree 0 and as

$$\tau_0^k : (\pi^k \operatorname{Fil}^{-k} N_{\varpi}^{[u,v]}(T(r)))^{I'_k} \to (\pi^{k+1} \operatorname{Fil}^{-k-1} N_{\varpi}^{[u,v]}(T(r)))^{I'_k}$$

on the k-th term of the definition above for $1 \leqslant k \leqslant d$ (see Definitions 4.8 and 4.9). Denote the total complex of the diagram thus obtained by $\mathcal{K}(\Gamma_S, \operatorname{Fil}^0 N_{\varpi}^{[u,v]}(T(r)))$, which is a subcomplex of the Koszul complex $\operatorname{Kos}(\Gamma_S, \operatorname{Fil}^0 N_{\varpi}^{[u,v]}(T(r)))$. In a similar manner, we can define complexes $\mathcal{K}(\Gamma'_S, N_{\varpi}^{[u,v/p]}(T(r)))$ and $\mathcal{K}^c(\Gamma'_S, \pi N_{\varpi}^{[u,v/p]}(T(r)))$ and a map τ_0 from the former to the latter complex.

Recall that t/π is a unit in $\boldsymbol{A}_{R,\varpi}^{[u,v]}$ (see Lemma 2.18); therefore, we see that

$$t^k \operatorname{Fil}^{-k} N_{\varpi}^{[u,v]}(T(r)) = \pi^k \operatorname{Fil}^{-k} N_{\varpi}^{[u,v]}(T(r)) \quad \text{for all } k \in \mathbb{Z}.$$

Now, define a morphism of complexes

$$\beta : \mathcal{K}(\Gamma'_S, \operatorname{Fil}^0 N_{\varpi}^{[u,v]}(T(r))) \to \mathcal{K}(\operatorname{Lie}\Gamma'_S, \operatorname{Fil}^0 N_{\varpi}^{[u,v]}(T(r))),$$

which is the identity in degree 0 and given as

$$\begin{aligned}\beta_k : (t^k \operatorname{Fil}^{-k} N_{\varpi}^{[u,v]}(T(r)))^{I'_k} &\to (t^k \operatorname{Fil}^{-k} N_{\varpi}^{[u,v]}(T(r)))^{I'_k}\\ (a_{i_1\cdots i_k}) &\mapsto (\nabla_{i_k} \cdots \nabla_{i_1} \tau_{i_1}^{-1} \cdots \tau_{i_k}^{-1}(a_{i_1\cdots i_k}))\end{aligned}$$

on the k-th term of the definition above for $1 \leqslant k \leqslant d$. Similarly, define a morphism of complexes

$$\beta^c : \mathcal{K}^c(\Gamma'_S, t \operatorname{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r))) \to \mathcal{K}^c(\operatorname{Lie}\Gamma'_S, t \operatorname{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r))),$$

which is given as $\beta_0^c = \nabla_0 \tau_0^{-1}$ in degree 0 and as

$$\begin{aligned}\beta_k^c : (t^{k+1} \operatorname{Fil}^{-k-1} N_{\varpi}^{[u,v]}(T(r)))^{I'_k} &\to (t^{k+1} \operatorname{Fil}^{-k-1} N_{\varpi}^{[u,v]}(T(r)))^{I'_k}\\ (a_{i_1\cdots i_k}) &\mapsto (\nabla_{i_k} \cdots \nabla_{i_1} \nabla_0 \tau_0^{-1} \tau_{i_1}^{c,-1} \cdots \tau_{i_k}^{c,-1}(a_{i_1\cdots i_k}))\end{aligned}$$

on the k-th term of the definition above for $1 \leqslant k \leqslant d$. Similarly, one can define the maps β and β^c for the $\boldsymbol{A}_{R,\varpi}^{[u,v/p]}$-module $N_{\varpi}^{[u,v/p]}$, giving morphisms of complexes

$$\beta : \mathcal{K}(\Gamma'_S, N_{\varpi}^{[u,v/p]}(T(r))) \to \mathcal{K}(\operatorname{Lie}\Gamma'_S, N_{\varpi}^{[u,v/p]}(T(r))),$$
$$\beta^c : \mathcal{K}^c(\Gamma'_S, tN_{\varpi}^{[u,v/p]}(T(r))) \to \mathcal{K}^c(\operatorname{Lie}\Gamma'_S, tN_{\varpi}^{[u,v/p]}(T(r))).$$

For each $j \in \mathbb{N}$, we have that $t^j \operatorname{Fil}^{-j} N_{\varpi}^{[u,v]}(T(r)) \subset N_{\varpi}^{[u,v]}(T(r))$, and the induced Frobenius gives

$$\varphi(t^j \operatorname{Fil}^{-j} N_{\varpi}^{[u,v]}(T(r))) = \varphi(t^{j-r} \operatorname{Fil}^{r-j} N_{\varpi}^{[u,v]}(T)(r)) \subset t^j N_{\varpi}^{[u,v/p]}(T(r)),$$

where we have used Lemma 3.20 and the fact that t/π is a unit in $\boldsymbol{A}_{R,\varpi}^{[u,v]}$; see Lemma 2.18. Using the Frobenius morphism and the morphism of complexes described above, we obtain an induced morphism of complexes

$$\left[\begin{array}{ccc} \mathcal{K}(\Gamma'_S, \operatorname{Fil}^0 N_{\varpi}^{[u,v]}(T(r))) & \xrightarrow{1-\varphi} & \mathcal{K}(\Gamma'_S, N_{\varpi}^{[u,v/p]}(T(r))) \\ \downarrow \tau_0 & & \downarrow \tau_0 \\ \mathcal{K}^c(\Gamma'_S, t\operatorname{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r))) & \xrightarrow{1-\varphi} & \mathcal{K}^c(\Gamma'_S, tN_{\varpi}^{[u,v/p]}(T(r))) \end{array}\right] \xrightarrow{(\beta,\beta^c)} \mathcal{K}(\varphi, \operatorname{Lie}\Gamma_S, N_{\varpi}^{[u,v]}(T(r))).$$

We denote the complex on the left by $\mathcal{K}(\varphi, \Gamma_S, N_{\varpi}^{[u,v]}(T(r)))$ and write the map as

$$\mathscr{L} = (\beta, \beta^c) : \mathcal{K}(\varphi, \Gamma_S, N_{\varpi}^{[u,v]}(T(r))) \to \mathcal{K}(\varphi, \operatorname{Lie}\Gamma_S, N_{\varpi}^{[u,v]}(T(r))).$$

Proposition 6.5. *The morphism of complexes $\mathscr{L}$ described above is an isomorphism.*

Proof. The proof follows in essentially the same manner as [Colmez and Nizioł 2017, Lemma 4.6]. One needs to use Lemmas 2.22, 4.14 and 5.19 instead of [Colmez and Nizioł 2017, Lemma 2.34] in the proof. We omit the details. ☐

6.4. ***Change of the annulus of convergence: Part I.*** In this subsection, we will pass from the analytic ring $\boldsymbol{A}_{R,\varpi}^{[u,v]}$ to the overconvergent ring $\boldsymbol{A}_{R,\varpi}^{(0,v]+}$ and also twist our module by $\mathbb{Z}_p(r)$. Let us start by setting $N_{\varpi}^{(0,v]+}(T(r)) := \boldsymbol{A}_{R,\varpi}^{(0,v]+} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T(r))$ and equipping it with the natural action of Γ_R and a Γ_R-stable filtration as in (3-10). Define a subcomplex of the Koszul complex $\operatorname{Kos}(\Gamma'_S, \operatorname{Fil}^0 N_{\varpi}^{(0,v]+}(T(r)))$ (see Definition 4.9) as follows:

$$\mathcal{K}(\Gamma'_S, \operatorname{Fil}^0 N_{\varpi}^{(0,v]+}(T(r)))$$
$$:= \operatorname{Fil}^0 N_{\varpi}^{(0,v]+}(T(r)) \xrightarrow{(\tau_i)} (\pi \operatorname{Fil}^{-1} N_{\varpi}^{(0,v]+}(T(r)))^{I'_1} \to (\pi^2 \operatorname{Fil}^{-2} N_{\varpi}^{(0,v]+}(T(r)))^{I'_2} \to \cdots.$$

Similarly, we can define a complex $\mathcal{K}^c(\Gamma'_S, \pi \operatorname{Fil}^{-1} N_{\varpi}^{(0,v]+}(T(r)))$ as a subcomplex of the Koszul complex $\operatorname{Kos}^c(\Gamma'_S, \operatorname{Fil}^0 N_{\varpi}^{(0,v]+}(T(r)))$; see Definition 4.9. Now, consider a morphism of complexes

$$\tau_0 : \mathcal{K}(\Gamma'_S, \operatorname{Fil}^0 N_{\varpi}^{(0,v]+}(T(r))) \to \mathcal{K}^c(\Gamma'_S, \pi \operatorname{Fil}^{-1} N_{\varpi}^{(0,v]+}(T(r))),$$

which is given as $\gamma_0 - 1$ in degree 0 and as

$$\tau_0^k : (\pi^k \operatorname{Fil}^{-k} N_{\varpi}^{(0,v]+}(T(r)))^{I'_2} \to (\pi^k \operatorname{Fil}^{-k-1} N_{\varpi}^{(0,v]+}(T(r)))^{I'_2}$$

on the k-th term of the definition above for $1 \leqslant k \leqslant d$ (see Definitions 4.8 and 4.9). Write the total complex of the diagram thus obtained as $\mathcal{K}(\Gamma_S, \mathrm{Fil}^0 N_{\varpi}^{(0,v]+}(T(r)))$, a subcomplex of the Koszul complex $\mathrm{Kos}(\Gamma_S, \mathrm{Fil}^0 N_{\varpi}^{(0,v]+}(T(r)))$. In a similar manner, we can define the complexes $\mathcal{K}(\Gamma'_S, N_{\varpi}^{(0,v/p]+}(T(r)))$ and $\mathcal{K}^c(\Gamma'_S, \pi N_{\varpi}^{(0,v/p]+}(T(r)))$ and a map τ_0 from the former to the latter complex.

For each $j \in \mathbb{N}$, we have that $\pi^j \mathrm{Fil}^{-j} N_{\varpi}^{(0,v]+}(T(r)) \subset N_{\varpi}^{(0,v]+}(T(r))$, and the induced Frobenius gives

$$\varphi(\pi^j \mathrm{Fil}^{-j} N_{\varpi}^{(0,v]+}(T(r))) = \varphi(\pi^{j-r} \mathrm{Fil}^{r-j} N_{\varpi}^{(0,v]+}(T)(r)) \subset \pi^j N_{\varpi}^{(0,v/p]+}(T(r)),$$

where the equality follows from Lemma 3.20. So we define the complex

$$\mathcal{K}(\varphi, \Gamma_S, N_{\varpi}^{(0,v]+}(T(r))) := \left[\begin{array}{ccc} \mathcal{K}(\Gamma'_S, \mathrm{Fil}^0 N_{\varpi}^{(0,v]+}(T(r))) & \xrightarrow{1-\varphi} & \mathcal{K}(\Gamma'_S, N_{\varpi}^{(0,v/p]+}(T(r))) \\ \downarrow \tau_0 & & \downarrow \tau_0 \\ \mathcal{K}^c(\Gamma'_S, \pi \mathrm{Fil}^{-1} N_{\varpi}^{(0,v]+}(T(r))) & \xrightarrow{1-\varphi} & \mathcal{K}^c(\Gamma'_S, \pi N_{\varpi}^{(0,v/p]+}(T(r))) \end{array}\right].$$

Proposition 6.6. *The natural morphism of complexes*

$$\mathcal{K}(\varphi, \Gamma_S, N_{\varpi}^{(0,v]+}(T(r))) \to \mathcal{K}(\varphi, \Gamma_S, N_{\varpi}^{[u,v]}(T(r))),$$

induced by the inclusion $N_{\varpi}^{(0,v]+}(T(r)) \subset N_{\varpi}^{[u,v]}(T(r))$, is a p^{3r}-quasi-isomorphism.

Proof. The map in the claim is injective on each term, so we need to show that the cokernel complex is killed by p^{3r}. In the cokernel complex, for $k \in \mathbb{N}$, we have maps

$$1-\varphi : \pi^k \mathrm{Fil}^{-k} N_{\varpi}^{[u,v]}(T(r))/\pi^k \mathrm{Fil}^{-k} N_{\varpi}^{(0,v]+}(T(r)) \to \pi^k N_{\varpi}^{[u,v/p]}(T(r))/\pi^k N_{\varpi}^{(0,v/p]+}(T(r)), \quad (6\text{-}1)$$

and it is enough to show that these are p^{3r}-bijective. Let us set

$$N_{\varpi}^{(0,v]+}(T) := \boldsymbol{A}_{R,\varpi}^{(0,v]+} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T),$$
$$N_{\varpi}^{(0,v]+}(T)(r) := N_{\varpi}^{(0,v]+}(T) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r),$$
$$N_{\varpi}^{[u,v]}(T)(r) := N_{\varpi}^{[u,v]}(T) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r),$$

equipped with the filtration as in (3-5) (up to twisting the filtered pieces by $\mathbb{Z}_p(r)$ in the latter cases). Moreover, for any $k \in \mathbb{N}$, by Lemma 3.20, we have that

$$\pi^k \mathrm{Fil}^{-k} N_{\varpi}^{(0,v]+}(T(r)) = \pi^{k-r} \mathrm{Fil}^{r-k} N_{\varpi}^{(0,v]+}(T)(r),$$
$$\pi^k \mathrm{Fil}^{-k} N_{\varpi}^{[u,v]}(T(r)) = \pi^{k-r} \mathrm{Fil}^{r-k} N_{\varpi}^{[u,v]}(T)(r).$$

So, for $n = r-k$, we can rewrite (6-1) as

$$1-\varphi : \pi^{-n} \mathrm{Fil}^n N_{\varpi}^{[u,v]}(T)/\pi^{-n} \mathrm{Fil}^n N_{\varpi}^{(0,v]+}(T) \to \pi^{-n} N_{\varpi}^{[u,v/p]}(T)/\pi^{-n} N_{\varpi}^{(0,v/p]+}(T). \quad (6\text{-}2)$$

Note that the twist has disappeared since φ acts trivially on it. For $n \leqslant 0$, the claim follows from Lemma 6.7. For $n > 0$, we first claim that the following natural map is p^n-bijective:

$$\pi_1^{-n} N_{\varpi}^{[u,v]}(T)/\pi_1^{-n} N_{\varpi}^{(0,v]+}(T) \to \pi^{-n} \mathrm{Fil}^n N_{\varpi}^{[u,v]}(T)/\pi^{-n} \mathrm{Fil}^n N_{\varpi}^{(0,v]+}(T). \quad (6\text{-}3)$$

Indeed, recall that $\xi = \pi/\pi_1$ and, from (3-5) and Lemma 3.18, it is clear that

$$\xi^n N_{\varpi}^{(0,v]+}(T) \subset \mathrm{Fil}^n N_{\varpi}^{(0,v]+}(T);$$

in particular, we have

$$N_{\varpi}^{(0,v]+}(T) \subset N_{\varpi}^{[u,v]}(T) \cap \xi^{-n} \mathrm{Fil}^n N_{\varpi}^{(0,v]+}(T) = (\boldsymbol{A}_{R,\varpi}^{[u,v]} \cap \xi^{-n} \boldsymbol{A}_{R,\varpi}^{(0,v]+}) \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T) = N_{\varpi}^{(0,v]+}(T),$$

where the first equality follows because $\boldsymbol{N}(T)$ is free over $\boldsymbol{A}_R^+$ and the second equality follows because

$$\xi^n \boldsymbol{A}_{R,\varpi}^{[u,v]} \cap \boldsymbol{A}_{R,\varpi}^{(0,v]+} \subset \mathrm{Fil}^n \boldsymbol{A}_{R,\varpi}^{(0,v]+} = \xi^n \boldsymbol{A}_{R,\varpi}^{(0,v]+}$$

(see Definition 2.7 and Remark 2.8). In particular, we see that

$$\pi_1^{-n} N_{\varpi}^{[u,v]}(T) \cap \pi^{-n} \mathrm{Fil}^n N_{\varpi}^{(0,v]+}(T) = \pi_1^{-n} N_{\varpi}^{(0,v]+}(T);$$

i.e., (6-3) is injective. Next, to show the p^n-surjectivity of (6-3), write $\boldsymbol{A}_{R,\varpi}^{[u,v]} = \boldsymbol{A}_{R,\varpi}^{[u]} + \boldsymbol{A}_{R,\varpi}^{(0,v]+}$ and set

$$N_{\varpi}^{[u]}(T) := \boldsymbol{A}_{R,\varpi}^{[u]} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T) \quad \text{and} \quad N_{\varpi}^{+}(T) := \boldsymbol{A}_{R,\varpi}^{+} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)$$

equipped with the induced filtration as in (3-5). Then, to obtain the p^n-surjectivity of (6-3), it is enough to show that the natural map

$$\pi_1^{-n} N_{\varpi}^{[u]}(T) + \pi^{-n} \mathrm{Fil}^n N_{\varpi}^{+}(T) \to \pi^{-n} \mathrm{Fil}^n N_{\varpi}^{[u]}(T)$$

is p^n-surjective, or equivalently, that the natural map

$$\xi^n N_{\varpi}^{[u]}(T) + \mathrm{Fil}^n N_{\varpi}^{+}(T) \to \mathrm{Fil}^n N_{\varpi}^{[u]}(T)$$

is p^n-surjective. To show the latter claim, let $\{e_1, \ldots, e_h\}$ be an $\boldsymbol{A}_R^+$-basis of $\boldsymbol{N}(T)$, take $x \in \mathrm{Fil}^n N_{\varpi}^{[u]}(T)$ and write $x = \sum_{i=1}^h a_i e_i$, with $a_i \in \boldsymbol{A}_{R,\varpi}^{[u]}$. Note that from Lemma 2.9 we can write $a_i = a_{i1} + a_{i2}$, with $a_{i1} \in \mathrm{Fil}^n \boldsymbol{A}_{R,\varpi}^{[u]} \subset p^{-n}\xi^n \boldsymbol{A}_{R,\varpi}^{[u]}$ (see Remark 2.8) and $a_{i2} \in p^{-\lfloor nu \rfloor} \boldsymbol{A}_{R,\varpi}^{+}$. So we see that

$$x_1 = \sum_{i=1}^h a_{i1} e_i \in p^{-n}\xi^n N_{\varpi}^{[u]}(T),$$

$$x_2 = \sum_{i=1}^h a_{i2} e_i = x - x_1 \in p^{-\lfloor nu \rfloor} N_{\varpi}^{+}(T) \cap \mathrm{Fil}^n N_{\varpi}^{[u]}(T) \subset N_{\varpi}^{[u]}(T)[1/p].$$

Now, as we have $u = (p-1)/p < 1$, it follows that $p^n x_2$ is in $N_{\varpi}^{+}(T) \cap \mathrm{Fil}^n N_{\varpi}^{[u]}(T) = \mathrm{Fil}^n N_{\varpi}^{+}(T)$ (see Lemma 3.17); i.e., $p^n x = p^n x_1 + p^n x_2$ is in $\xi^n N_{\varpi}^{[u]}(T) + \mathrm{Fil}^n N_{\varpi}^{+}(T)$. In particular, we get that (6-3) is p^n-bijective, and therefore (6-2) is p^n-isomorphic to

$$1 - \varphi : \pi_1^{-n} N_{\varpi}^{[u,v]}(T)/\pi_1^{-n} N_{\varpi}^{(0,v]+}(T) \to \pi^{-n} N_{\varpi}^{[u,v/p]}(T)/\pi^{-n} N_{\varpi}^{(0,v/p]+}(T).$$

Recall that we have $v = p-1$, so by Lemma 2.20 (iii) it follows that π divides p in $\boldsymbol{A}_{R,\varpi}^{(0,v/p]+}$ and π_1 divides p in $\boldsymbol{A}_{R,\varpi}^{(0,v]+}$; therefore, (6-2) is p^{2n}-isomorphic to the map

$$1 - \varphi : N_{\varpi}^{[u,v]}(T)/N_{\varpi}^{(0,v]+}(T) \to N_{\varpi}^{[u,v/p]}(T)/N_{\varpi}^{(0,v/p]+}(T).$$

Now, from Lemma 6.7, the map above is bijective (note that Frobenius has no effect on twist). Therefore, we conclude that (6-1) is p^{3n}-bijective. As $n = r - k \leqslant r$, it follows that the cokernel complex of the map in the claim of the lemma is killed by p^{3r}. This allows us to conclude. ☐

Lemma 6.7. *For each $k \in \mathbb{N}$, the following natural map is bijective*:

$$1 - \varphi : \pi^k N_{\varpi}^{[u,v]}(T)/\pi^k N_{\varpi}^{(0,v]+}(T) \xrightarrow{\sim} \pi^k N_{\varpi}^{[u,v/p]}(T)/\pi^k N_{\varpi}^{(0,v/p]+}(T),$$

Proof. For $k = 0$, using a basis of $\boldsymbol{N}(T)$, one first shows that the natural map

$$N_{\varpi}^{[u,v]}(T)/N_{\varpi}^{(0,v]+}(T) \to N_{\varpi}^{[u,v/p]}(T)/N_{\varpi}^{(0,v/p]+}(T)$$

is bijective; in particular, $1-\varphi$ is an endomorphism of $N_{\varpi}^{[u,v]}(T)/N_{\varpi}^{(0,v]+}(T)$. Then, following the strategy of [Colmez and Nizioł 2017, Lemma 4.8], one shows that, on the preceding quotient, $1+\varphi+\varphi^2+\cdots$ converges as an inverse to $1-\varphi$. We omit the details. For $k > 0$, note that φ preserves the quotient $\pi^k N_{\varpi}^{[u,v]}(T)/\pi^k N_{\varpi}^{(0,v]+}(T)$. So, from the case $k = 0$, it follows that $1+\varphi+\varphi^2+\cdots$ converges on the preceding quotient as well. ☐

6.5. ***Change of the annulus of convergence*: *Part II.*** In this subsection, we will change the ring of coefficients from $\boldsymbol{A}_{R,\varpi}^{(0,v]+}$ to $\boldsymbol{A}_{R,\varpi}^{(0,v/p]+}$ by replacing φ with its left inverse ψ (under the assumption that $m \geqslant 2$).

6.5.1. *From (φ, Γ_S)-complexes to (ψ, Γ_S)-complexes.* From Proposition 2.4, recall that we have the left inverse ψ of the Frobenius endomorphism on $\boldsymbol{A}$, satisfying $\psi(\boldsymbol{A}) \subset \boldsymbol{A}$. This induces an operator

$$\psi : \boldsymbol{A}_{R,\varpi}^{(0,v/p]+} \to \boldsymbol{A}_{R,\varpi}^{(0,v]+},$$

which commutes with the action of Γ_R; in particular, we have $\psi(\boldsymbol{A}_{R,\varpi}^{(0,v]+}) \subset \boldsymbol{A}_{R,\varpi}^{(0,v]+}$. Equivalently, one can also define the operator ψ by first identifying

$$\iota_{\mathrm{cycl}} : R_{\varpi}^{(0,v/p]+} \xrightarrow{\sim} \boldsymbol{A}_{R,\varpi}^{(0,v/p]+}$$

and then considering the left inverse of the cyclotomic Frobenius over $R_{\varpi}^{(0,v/p]+}$; see Sections 2.6 and 2.7.

Next, from Lemma 3.5, recall that the operator ψ extends to $\boldsymbol{N}(T(r))$ and we have $\psi(\boldsymbol{N}(T(r))) \subset \boldsymbol{N}(T(r))$. By extending scalars to $\boldsymbol{A}_{R,\varpi}^{(0,v]+}$ and from the discussion above, we see that

$$\psi(N_{\varpi}^{(0,v]+}(T(r))) \subset \psi(N_{\varpi}^{(0,v/p]+}(T(r))) \subset N_{\varpi}^{(0,v]+}(T(r)).$$

Moreover, using the description of the filtration on $N_{\varpi}^{(0,v]+}(T)$ from Lemma 3.18, it follows that, for $0 \leqslant k \leqslant r$, we have

$$\varphi(\mathrm{Fil}^{r-k} N_{\varpi}^{(0,v]+}(T)) \subset q^{r-k} N_{\varpi}^{(0,v/p]+}(T).$$

Upon multiplying the terms of the preceding inclusion by $\varphi(\pi^{k-r})$ and twisting by $\mathbb{Z}_p(r)$, we get

$$\varphi(\pi^{k-r}\,\mathrm{Fil}^{r-k} N_{\varpi}^{(0,v]+}(T)(r)) \subset \pi^{k-r} N_{\varpi}^{(0,v/p]+}(T)(r).$$

In particular, by using Lemma 3.20, we note that $\pi^k \operatorname{Fil}^{-k} N_{\varpi}^{(0,v]+}(T(r)) \subset \psi(\pi^k N_{\varpi}^{(0,v/p]+}(T(r)))$, and since $\operatorname{Fil}^{-k} N_{\varpi}^{(0,v]+}(T(r)) \subset N_{\varpi}^{(0,v/p]+}(T(r))$, it follows that

$$(\psi - 1)(\pi^k \operatorname{Fil}^{-k} N_{\varpi}^{(0,v]+}(T(r))) \subset \psi(\pi^k N_{\varpi}^{(0,v/p]+}(T(r))).$$

Set

$$\mathcal{K}(\Gamma'_S, N_\psi) := \psi\big(\mathcal{K}(\Gamma'_S, N_{\varpi}^{(0,v/p]+}(T(r)))\big),$$
$$\mathcal{K}^c(\Gamma'_S, N_\psi) := \psi\big(\mathcal{K}^c(\Gamma'_S, N_{\varpi}^{(0,v/p]+}(T(r)))\big).$$

From Section 6.4, recall that we defined maps

$$\tau_0 : \mathcal{K}(\Gamma'_S, \operatorname{Fil}^0 N_{\varpi}^{(0,v]+}(T(r))) \to \mathcal{K}^c(\Gamma'_S, \pi \operatorname{Fil}^{-1} N_{\varpi}^{(0,v]+}(T(r))),$$
$$\tau_0 : \psi\big(\mathcal{K}(\Gamma'_S, N_{\varpi}^{(0,v/p]+}(T(r)))\big) \to \psi\big(\mathcal{K}^c(\Gamma'_S, N_{\varpi}^{(0,v/p]+}(T(r)))\big).$$

As ψ commutes with the action of Γ_S, from the latter map, we therefore obtain an induced morphism $\tau_0 : \mathcal{K}(\Gamma'_S, N_\psi) \to \mathcal{K}^c(\Gamma'_S, N_\psi)$. Now, using the discussion above, note that we have a well-defined map between source complexes of the maps τ_0 above, given as $\psi - 1 : \mathcal{K}(\Gamma'_S, \operatorname{Fil}^0 N_{\varpi}^{(0,v]+}(T(r))) \to \mathcal{K}(\Gamma'_S, N_\psi)$, and similarly for the target complexes of τ_0. Therefore, similar to the complex $\mathcal{K}(\varphi, \Gamma_S, N^{(0,v]+}(T(r)))$ in Section 6.4, we define the following complex:

$$\mathcal{K}(\psi, \Gamma_S, N_{\varpi}^{(0,v]+}(T(r))) := \left[\begin{array}{ccc} \mathcal{K}(\Gamma'_S, \operatorname{Fil}^0 N_{\varpi}^{(0,v]+}(T(r))) & \xrightarrow{\psi - 1} & \mathcal{K}(\Gamma'_S, N_\psi) \\ \downarrow \tau_0 & & \downarrow \tau_0 \\ \mathcal{K}^c(\Gamma'_S, \pi \operatorname{Fil}^{-1} N_{\varpi}^{(0,v]+}(T(r))) & \xrightarrow{\psi - 1} & \mathcal{K}^c(\Gamma'_S, N_\psi)) \end{array}\right].$$

Proposition 6.8. *The morphism*

$$\tau_{\leqslant r}\mathcal{K}(\varphi, \Gamma_S, N_{\varpi}^{(0,v]+}(T(r))) \to \tau_{\leqslant r}\mathcal{K}(\psi, \Gamma_S, N_{\varpi}^{(0,v]+}(T(r))),$$

induced by the identity in the first column and ψ in the second column, is a p^{r+2}-quasi-isomorphism.

Proof. By definition, note that the map is surjective on each term, so we need to show that the kernel complex is p^{r+2}-acyclic. Since the map in the claim is the identity on the first column, the kernel complex can be written as

$$\tau_{\leqslant r}\big[\mathcal{K}(\Gamma'_S, (N_{\varpi}^{(0,v/p]+}(T(r)))^{\psi=0}) \xrightarrow{\tau_0} \mathcal{K}^c(\Gamma'_S, (\pi N_{\varpi}^{(0,v/p]+}(T(r)))^{\psi=0})\big].$$

Clearly the terms of the complex above are $\varphi(\boldsymbol{A}_{R,\varpi}^{(0,v]+})$-modules. We recall that $p/\pi \in \varphi(\boldsymbol{A}_{R,\varpi}^{(0,v]+})$ (since π_1 divides p in $\boldsymbol{A}_{R,\varpi}^{(0,v]+}$, see also Lemma 2.20 (ii) for $v = p - 1$), so we obtain that $(\pi^k N_{\varpi}^{(0,v/p]+}(T(r)))^{\psi=0}$ is p^{r-k}-isomorphic to $(N_{\varpi}^{(0,v/p]+}(T)(r))^{\psi=0}$ for $k \leqslant r$. In particular, the complex above is p^r-quasi-isomorphic to the complex

$$\tau_{\leqslant r}\big[\operatorname{Kos}(\Gamma'_S, (N_{\varpi}^{(0,v/p]+}(T)(r))^{\psi=0}) \xrightarrow{\tau_0} \operatorname{Kos}^c(\Gamma'_S, (N_{\varpi}^{(0,v/p]+}(T)(r))^{\psi=0})\big]. \tag{6-4}$$

We will show that the complex in (6-4) is p^2-acyclic, but to prove our claim we will need a simpler description of the $\varphi(\boldsymbol{A}_{R,\varpi}^{(0,v]+})$-module $(N_{\varpi}^{(0,v/p]+}(T))^{\psi=0}$.

Let $\{e_1, \ldots, e_h\}$ denote an $\boldsymbol{A}_R^+$-basis of $\boldsymbol{N}(T)$. Since the attached (φ, Γ_S)-module

$$D_\varpi(T) = \boldsymbol{A}_{R,\varpi} \otimes_{\boldsymbol{A}_R} \boldsymbol{D}(T)$$

over $\boldsymbol{A}_{R,\varpi}$ is étale, we see that $\{\varphi(e_1), \ldots, \varphi(e_h)\}$ is an $\boldsymbol{A}_{R,\varpi}$-basis of $D_\varpi(T)$. Now, let us note that $z = \sum_{j=1}^h z_j \varphi(e_j)$ is in $D_\varpi(T)^{\psi=0}$ if and only if $z_j \in (\boldsymbol{A}_{R,\varpi})^{\psi=0}$ for each $1 \leqslant j \leqslant h$. Indeed, $\psi(z) = 0$ if and only if $\sum_{j=1}^h \psi(z_j) e_j = 0$, and since the e_j are linearly independent over $\boldsymbol{A}_{R,\varpi}$, we see that $\psi(z) = 0$ if and only if $\psi(z_j) = 0$ for all $1 \leqslant j \leqslant h$. Next, using Lemma 2.15 (ii), note that we have a decomposition $\boldsymbol{A}_{R,\varpi}^{\psi=0} = \bigoplus_{\alpha \neq 0} \varphi(\boldsymbol{A}_{R,\varpi})[X^\flat]^\alpha$, where $[X^\flat]^\alpha = (1+\pi_m)^{\alpha_0}[X_1^\flat]^{\alpha_0} \cdots [X_d^\flat]^{\alpha_d}$ and $\alpha = (\alpha_0, \ldots, \alpha_d)$ is a $(d+1)$-tuple with $\alpha_i \in \{0, \ldots, p-1\}$. Therefore, we see that

$$D_\varpi(T)^{\psi=0} = \Big(\sum_{j=1}^h \boldsymbol{A}_{R,\varpi}\varphi(e_j)\Big)^{\psi=0} = \bigoplus_{\alpha\neq 0}\sum_{j=1}^h \varphi(\boldsymbol{A}_{R,\varpi} e_j)[X^\flat]^\alpha = \bigoplus_{\alpha\neq 0}\varphi(D_\varpi(T))[X^\flat]^\alpha.$$

Note that inside $D_\varpi(T)$ we have $(N_\varpi^{(0,v/p]+}(T))^{\psi=0} = D_\varpi(T)^{\psi=0} \cap N_\varpi^{(0,v/p]+}(T)$. Using the decomposition above, we set $N[X^\flat]^\alpha := \varphi(D_\varpi(T))[X^\flat]^\alpha \cap N_\varpi^{(0,v/p]+}(T)$ for $\alpha \neq 0$, where the intersection is taken inside $D_\varpi(T)^{\psi=0}$. Note that $\varphi(\boldsymbol{A}_{R,\varpi}^{(0,v]+}) \subset \varphi(\boldsymbol{A}_{R,\varpi}) \cap \boldsymbol{A}_{R,\varpi}^{(0,v/p]+}$. Therefore, it follows that $N := N[X^\flat]^\alpha[X^\flat]^{-\alpha}$ is a $\varphi(\boldsymbol{A}_{R,\varpi}^{(0,v]+})$-module contained in $N_\varpi^{(0,v/p]+}(T)$, stable under the action of Γ_S and independent of α. Indeed, for the last part note that, for $\alpha \neq \alpha'$, we have $\sum_{i=1}^h \varphi(x_i e_i)[X^\flat]^\alpha \in N[X^\flat]^\alpha$ if and only if $\sum_{i=1}^h \varphi(x_i e_i)[X^\flat]^{\alpha'} \in N[X^\flat]^{\alpha'}$. In conclusion, we get the equalities

$$(N_\varpi^{(0,v/p]+}(T))^{\psi=0} = \bigoplus_{\alpha\neq 0} N[X^\flat]^\alpha = \bigoplus_{\alpha\neq 0}\varphi(N_\varpi^{(0,v]+})[X^\flat]^\alpha,$$

where the last equality is a result of the following.

Lemma 6.9. *For $v = p-1$, let $x \in D_\varpi(T)$ such that $\varphi(x) \in N_\varpi^{(0,v/p]+}(T)$. Then $x \in N_\varpi^{(0,v]+}(T)$. In particular, we have $N = \varphi(N_\varpi^{(0,v]+}(T))$.*

Proof. Let $N_\varpi^+(T) = \boldsymbol{A}_{R,\varpi}^+ \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)$, and note that

$$D_\varpi(T)/p = (N_\varpi^+(T)/p)[1/\pi_m] \quad \text{and} \quad N_\varpi^{(0,v]+}(T) = \sum_{n\in\mathbb{N}} p^n \pi_m^{-\lfloor ne/v\rfloor} N_\varpi^+(T)$$

(since $\boldsymbol{N}(T)$ is finite free over $\boldsymbol{A}_R^+$). Then the proof of [Colmez and Nizioł 2017, Lemma 2.14] can easily be adapted to obtain the claim. We omit the details. □

Remark 6.10. From Lemma 6.9, we have $N = \varphi(N_\varpi^{(0,v]+}(T))$. Then, for any $i \in \{0, \ldots, d\}$, using Lemma 2.22 (i), note that $(\gamma_i - 1)\boldsymbol{A}_{R,\varpi}^{(0,v]+} \subset \pi \boldsymbol{A}_{R,\varpi}^{(0,v]+}$ and, from Definition 3.1, note that $(\gamma_i - 1)\boldsymbol{N}(T) \subset \pi \boldsymbol{N}(T)$. Since φ commutes with the action of Γ_S, we conclude that $(\gamma_i - 1)N \subset \varphi(\pi)N$.

From the discussion above, it follows that the complex in (6-4) is isomorphic to the complex

$$\tau_{\leqslant r}\bigoplus_{\alpha\neq 0}\big[\mathrm{Kos}(\Gamma_S', N(r)[X^\flat]^\alpha) \xrightarrow{\tau_0} \mathrm{Kos}^c(\Gamma_S', N(r)[X^\flat]^\alpha)\big]. \tag{6-5}$$

Lemma 6.11. *The complex described in (6-5) is p^2-acyclic.*

Proof. Our proof is motivated by the proof of [Colmez and Nizioł 2017, Lemma 4.10]. One can treat the terms of (6-5) corresponding to each α separately. The case of $\alpha_k \neq 0$, for some $k \neq 0$, follows similar to the proof of [Colmez and Nizioł 2017, Lemma 4.10], where one shows that both the complexes $\mathrm{Kos}(\Gamma'_S, N(r)[X^\flat]^\alpha)$ and $\mathrm{Kos}^c(\Gamma'_S, N(r)[X^\flat]^\alpha)$ are p-acyclic by using the facts that $(\gamma_k - 1)N \subset \varphi(\pi)N$ (see Remark 6.10) and π divides p in $\varphi(\boldsymbol{A}_{R,\varpi}^{(0,v]+})$ (since π_1 divides p in $\boldsymbol{A}_{R,\varpi}^{(0,v]+}$, see Lemma 2.20 (ii) for $v = p-1$). We omit the details.

Now, let $\alpha_k = 0$ for all $k \neq 0$ and $\alpha_0 \neq 0$. To prove that the complex in (6-5) is p-acyclic, we will show that $\tau_0 : \mathrm{Kos} \to \mathrm{Kos}^c$ is injective and the cokernel complex is killed by p. This amounts to showing the same statement for the map

$$\gamma_0 - \delta_{i_1} \cdots \delta_{i_q} : N[X^\flat]^\alpha(r) \to N[X^\flat]^\alpha(r), \quad \delta_{i_j} = \frac{\gamma_{i_j}^c - 1}{\gamma_{i_j} - 1}. \tag{6-6}$$

Let $n = p^{-m}(c-1)\alpha_0 \in \mathbb{Z}_p^\times$, $F = c^r(1+\pi)^n\gamma_0 - \delta_{i_1}\cdots\delta_{i_q}$ and $\epsilon^{\otimes r}$ be a $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(r)$. Then

$$(\gamma_0 - \delta_{i_1}\cdots\delta_{i_q})(x[X^\flat]^\alpha \otimes \epsilon^{\otimes r}) = F(x) \cdot [X^\flat]^\alpha \otimes \epsilon^{\otimes r}$$

for any $x \in N$. Moreover, we have that $c^r - 1$ is divisible by p^m, $(1+\pi)^n = 1 + n\pi \bmod \pi^2$ and $\delta_{i_j} - 1 \in (\gamma_{i_j} - 1)\mathbb{Z}_p[\![\gamma_{i_j} - 1]\!]$. Therefore, we can write $\pi^{-1}F$ in the form $\pi^{-1}F = n + \pi^{-1}F'$, with $F' \in (p^m, \pi^2, \gamma_0 - 1, \ldots, \gamma_d - 1)\mathbb{Z}_p[\![\pi, \Gamma_S]\!]$. Now, let $f = p/\pi \in \varphi(\boldsymbol{A}_R^{(0,v]+})$ and note that $\pi^{-1}p^m x = \pi^{m-1}f^m x$ is in $\pi^{m-1}N$. Moreover, we have $(\gamma_j - 1)N \subset \varphi(\pi)N$ for $0 \leqslant j \leqslant d$ (see Remark 6.10) and $\varphi(\pi)/\pi^2 \in \varphi(\boldsymbol{A}_{R,\varpi}^{(0,v]+})$ (since π_1 divides p in $\boldsymbol{A}_{R,\varpi}^{(0,v]+}$, see Lemma 2.20 (ii) for $v = p-1$). Furthermore, $\pi_m^{p^m}$ divides π and p in $\varphi(\boldsymbol{A}_{R,\varpi}^{(0,v]+})$ (see Lemma 2.20 (ii) for $v = p-1$). So we get that $\pi^{-1}F'(x) \in \pi_m^{p^m}N$ (since we assumed $m \geqslant 2$). In particular, we see that $\pi^{-1}F' = 0$ on $\pi_m^a N/\pi_m^{a+b}N$ for all $a \in \mathbb{N}$ and $b = p^m$. Hence $\pi^{-1}F$ induces multiplication by n on $\pi_m^a N/\pi_m^{a+b}N$ for all $a \in \mathbb{N}$, which implies that it is an isomorphism on N. From the preceding discussion, we conclude that the map in (6-6) is injective and its image is contained in $\pi N[X^\flat]^\alpha(r)$. But, as π divides p in $\varphi(\boldsymbol{A}_{R,\varpi}^{(0,v]+})$, we therefore get that the cokernel of (6-6) is killed by p, as claimed. □

Using Lemma 6.11, we conclude that the natural morphism of complexes in the claim of Proposition 6.8 is a p^{r+2}-quasi-isomorphism. □

6.5.2. *Changing the overconvergence radius.* Recall that $m \geqslant 2$, and let $\ell = p^{m-1}$. Then, we have the inclusions

$$\psi(\pi_m^{-\ell}\boldsymbol{A}_{R,\varpi}^{(0,v]+}) \subset \psi(\pi_m^{-\ell}\boldsymbol{A}_{R,\varpi}^{(0,v/p]+}) \subset \pi_m^{-p^{m-2}}\boldsymbol{A}_{R,\varpi}^{(0,v]+} \subset \pi_m^{-\ell}\boldsymbol{A}_{R,\varpi}^{(0,v/p]+}$$

from Proposition 2.17 (i). In other words, $\pi_m^{-\ell}\boldsymbol{A}_{R,\varpi}^{(0,v]+}$ is stable under ψ. Set

$$D_\varpi^{(0,v]+}(T(r)) := \boldsymbol{A}_{R,\varpi}^{(0,v]+} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{D}^+(T(r)),$$

and note that it is stable under the action of Γ_S. Next, from Lemma 2.15, we have $\psi(\boldsymbol{A}_{R,\varpi}^{(0,v/p]+}) = \boldsymbol{A}_{R,\varpi}^{(0,v]+}$, and, for $v = p-1$, using Lemma 2.20 (iii), we have that $\pi_m^{-p\ell}\pi$ is a unit in $\boldsymbol{A}_{R,\varpi}^{(0,v/p]+}$. Hence, utilising Proposition 2.17, it follows that $\psi(\pi^{-r}\boldsymbol{A}_{R,\varpi}^{(0,v/p]+}) = \pi_1^{-r}\boldsymbol{A}_{R,\varpi}^{(0,v]+}$, and therefore

$$\psi(\pi^{-r}D_\varpi^{(0,v/p]+}(T(r))) \subset \pi_1^{-r}D_\varpi^{(0,v]+}(T(r)).$$

Moreover, since $\psi(\boldsymbol{N}(T)) \subset \boldsymbol{D}^+(T)$, from the discussion above, we see that

$$\psi(N_{\varpi}^{(0,v/p]+}(T(r))) \subset \psi(\pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))) \subset \pi_1^{-r} D_{\varpi}^{(0,v]+}(T(r)).$$

Furthermore, for $k \in \mathbb{N}$ and $k \leqslant r$, it follows that we have $\pi^k N_{\varpi}^{(0,v/p]+}(T(r)) \subset \pi^{k-r} D_{\varpi}^{(0,v/p]+}(T(r))$ and

$$\psi(\pi^k N_{\varpi}^{(0,v/p]+}(T(r))) \subset \pi_1^{k-r} D_{\varpi}^{(0,v]+}(T(r)) \subset \pi^{k-r} D_{\varpi}^{(0,v/p]+}(T(r)).$$

By replacing v by v/p in Section 6.4, we define a complex $\mathcal{K}(\Gamma'_S, N_{\varpi}^{(0,v/p]+}(T(r)))$ as follows:

$$N_{\varpi}^{(0,v/p]+}(T(r)) \xrightarrow{(\tau_i)} (\pi N_{\varpi}^{(0,v/p]+}(T(r)))^{I'_1} \to (\pi^2 N_{\varpi}^{(0,v/p]+}(T(r)))^{I'_2} \to \cdots .$$

Similarly, we define a complex $\mathcal{K}^c(\Gamma'_S, N_{\varpi}^{(0,v/p]+}(T(r)))$ and a map τ_0 from the former to the latter complex. Note that, from the discussion above and the inclusion $N_{\varpi}^{(0,v/p]+}(T(r)) \subset \pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))$, we have $(\psi - 1)(\pi^k N_{\varpi}^{(0,v/p]+}(T(r))) \subset \pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))$. So we define the complex

$$\mathcal{K}(\psi, \Gamma_S, N_{\varpi}^{(0,v/p]+}(T(r))) := \left[\begin{array}{ccc} \mathcal{K}(\Gamma'_S, N_{\varpi}^{(0,v/p]+}(T(r))) & \xrightarrow{\psi-1} & \mathrm{Kos}(\Gamma'_S, \pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))) \\ \downarrow \tau_0 & & \downarrow \tau_0 \\ \mathcal{K}^c(\Gamma'_S, \pi N_{\varpi}^{(0,v/p]+}(T(r))) & \xrightarrow{\psi-1} & \mathrm{Kos}^c(\Gamma'_S, \pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))) \end{array}\right].$$

Lemma 6.12. *The morphism of complexes*

$$\tau_{\leqslant r}\mathcal{K}(\psi, \Gamma_S, N_{\varpi}^{(0,v]+}(T(r))) \to \tau_{\leqslant r}\mathcal{K}(\psi, \Gamma_S, N_{\varpi}^{(0,v/p]+}(T(r))),$$

induced by the inclusions

$$N_{\varpi}^{(0,v]+}(T(r)) \subset N_{\varpi}^{(0,v/p]+}(T(r)) \quad \textit{and} \quad \psi(N_{\varpi}^{(0,v/p]+}(T(r))) \subset \pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r)),$$

is a p^{r+2s}-quasi-isomorphism.

Proof. As the map in the claim is injective on each term, we need to show that the cokernel complex is killed by p^{r+2s}. For $k \in \mathbb{N}$ and $k \leqslant r$, in the cokernel complex, we have maps

$$\psi - 1 : \pi^{k-r} N_{\varpi}^{(0,v/p]+}(T)/\pi^{k-r}\,\mathrm{Fil}^{r-k} N_{\varpi}^{(0,v]+}(T) \to \pi^{-r} D_{\varpi}^{(0,v/p]+}(T)/\psi(\pi^{k-r} N_{\varpi}^{(0,v/p]+}(T)), \quad (6\text{-}7)$$

and to prove the claim it is enough to show that (6-7) is p^{r+2s}-bijective (the twist (r) has disappeared because ψ acts trivially on it). First, we will show the p^{r+s}-surjectivity. Recall that we have $\pi^s \boldsymbol{D}^+(T) \subset \boldsymbol{N}(T) \subset \boldsymbol{D}^+(T)$ (see [Abhinandan 2025, Corollary 4.11]), and, by extending scalars to $\boldsymbol{A}_{R,\varpi}^{(0,v/p]+}$ and dividing out by π^r, we see that $\pi^{s-r} D_{\varpi}^{(0,v/p]+}(T) \subset \pi^{-r} N_{\varpi}^{(0,v/p]+}(T)$. So, it follows that $\pi^{-r} D_{\varpi}^{(0,v/p]+}(T)/\pi^{k-r} N_{\varpi}^{(0,v/p]+}(T)$ is killed by π^{k+s}, and, since π divides p in $\boldsymbol{A}_{R,\varpi}^{(0,v/p]+}$ (see Lemma 2.20 for $v = p-1$), we get that the preceding quotient is killed by p^{k+s}. Note that the quotient $\pi^{-r} D_{\varpi}^{(0,v/p]+}(T)/\pi^{k-r} N_{\varpi}^{(0,v/p]+}(T)$ surjects onto the cokernel of (6-7). Hence, for $k \leqslant r$, we see that the cokernel of (6-7) is killed by p^{r+s} (this also shows that the truncation in degree $\leqslant r$ is necessary in order to bound the power of p).

Next, to show the p^s-injectivity of (6-7), let $x \in N_{\varpi}^{(0,v/p]+}(T)$ such that there is a $y \in N_{\varpi}^{(0,v/p]+}(T)$ satisfying $(\psi-1)(\pi^{k-r}x) = \psi(\pi^{k-r}y)$ or, equivalently, we have that $x = \xi^{r-k}\psi(x-y)$ belongs to $\xi^{r-k}\psi(N_{\varpi}^{(0,v/p]+}(T))$. Note that

$$\psi(N_{\varpi}^{(0,v/p]+}(T)) \subset \psi(D_{\varpi}^{(0,v/p]+}(T)) \subset D_{\varpi}^{(0,v]+},$$

so we see that $\varphi(x) \in D_{\varpi}^{(0,v/p]+}$. Moreover, from the discussion above, we know that the natural inclusion $N_{\varpi}^{(0,v/p]+}(T) \subset D_{\varpi}^{(0,v/p]+}(T)$ is p^s-surjective. Therefore, it follows that $\varphi(p^s x) = p^s\varphi(x)$ is in $N_{\varpi}^{(0,v/p]+}(T)$; in particular, we see that

$$\psi(\varphi(p^s x)) = \psi(p^s q^{r-k}(x-y));$$

i.e., $\varphi(p^s x) - q^{r-k}p^s(x-y)$ is in $(N_{\varpi}^{(0,v/p]+}(T))^{\psi=0}$. From the description of $(N_{\varpi}^{(0,v/p]+}(T))^{\psi=0}$ before Lemma 6.9, we can write

$$\varphi(p^s x) = p^s q^{r-k}(x-y) + \sum_{\alpha\neq 0}\varphi(x_\alpha)[X^\flat]^\alpha \quad \text{for some } x_\alpha \in N_{\varpi}^{(0,v]+}(T).$$

In particular, we see that $\varphi(p^s x)$ is in $N_{\varpi}^{(0,v/p]+}(T)$ and, from Lemma 6.9, we get that $p^s x$ is in $N_{\varpi}^{(0,v]+}(T)$. Furthermore, as we have $\psi(N_{\varpi}^{(0,v/p]+}(T)) \subset D_{\varpi}^{(0,v]+}(T)$, we see that $p^s x$ is in

$$N_{\varpi}^{(0,v]+}(T) \cap \xi^{r-k}D_{\varpi}^{(0,v]+}(T) \subset N_{\varpi}^{(0,v]+}(T) \cap (\mathrm{Fil}^{r-k} \boldsymbol{A}_{\bar{R}}^{(0,v]+} \otimes_{\mathbb{Z}_p} V) \subset \mathrm{Fil}^{r-k} N_{\varpi}^{(0,v]+}(T),$$

where the last inclusion follows from the definition of the filtration on $N_{\varpi}^{(0,v]+}(T)$ in (3-5). In particular, we have shown that $p^s\pi^{k-r}x$ belongs to $\pi^{k-r}\,\mathrm{Fil}^{k-r} N_{\varpi}^{(0,v]+}(T)$, and hence (6-7) is p^s-injective. This allows us to conclude. □

From the discussion before Lemma 6.12, recall that we have inclusions

$$\psi(\pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r))) \subset \pi_1^{-r}D_{\varpi}^{(0,v]+}(T(r)) \subset \pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r)).$$

Using the constructions in Section 4, we define the complex

$$\mathrm{Kos}(\psi,\Gamma_S,D_{\varpi}^{(0,v/p]+}(T(r))) := \left[\begin{array}{ccc} \mathrm{Kos}(\Gamma_S',\pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r))) & \xrightarrow{\psi-1} & \mathrm{Kos}(\Gamma_S',\pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r))) \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_0} \\ \mathrm{Kos}^c(\Gamma_S',\pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r))) & \xrightarrow{\psi-1} & \mathrm{Kos}^c(\Gamma_S',\pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r))) \end{array}\right].$$

Lemma 6.13. *The morphism of complexes*

$$\tau_{\leqslant r}\mathcal{K}(\psi,\Gamma_S,N_{\varpi}^{(0,v/p]+}(T(r))) \to \tau_{\leqslant r}\mathrm{Kos}(\psi,\Gamma_S,D_{\varpi}^{(0,v/p]+}(T(r))),$$

induced by the inclusion

$$N_{\varpi}^{(0,v/p]+}(T(r)) \subset \pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r)),$$

is a p^{r+s}-quasi-isomorphism.

Proof. Note that, for the map of truncated complexes, the cokernel complex consists of $\boldsymbol{A}_{R,\varpi}^{(0,v/p]+}$-modules, given as

$$\pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))/\pi^k N_{\varpi}^{(0,v/p]+}(T(r)) \quad \text{for } k \leqslant r.$$

Recall that $\pi^s \boldsymbol{D}^+(T) \subset \boldsymbol{N}(T) \subset \boldsymbol{D}^+(T)$ (see [Abhinandan 2025, Corollary 4.11]), and, by extending scalars to $\boldsymbol{A}_{R,\varpi}^{(0,v/p]+}$, dividing out by π^r and twisting by $\mathbb{Z}_p(r)$, we see that

$$\pi^{s-r} D_{\varpi}^{(0,v/p]+}(T(r)) \subset N_{\varpi}^{(0,v/p]+}(T(r)).$$

It follows that the quotient $\pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))/\pi^k N_{\varpi}^{(0,v/p]+}(T(r))$ is killed by π^{k+s}, and, since π divides p in $\boldsymbol{A}_{R,\varpi}^{(0,v/p]+}$ (see Lemma 2.20 for $v = p-1$), we get that the preceding quotient is killed by p^{k+s}. Since $k \leqslant r$, we hence conclude that the cokernel complex is p^{r+s}-acyclic. ☐

6.6. *Change of the disk of convergence.* In this subsection, we will relate complexes in previous subsections to the Koszul complex computing continuous G_S-cohomology of $T(r)$. Recall that, in Section 2.4.5, we defined an operator $\psi : D_{\varpi}(T(r)) \to D_{\varpi}(T(r))$ as a left inverse of φ. Using this operator, we define the complex

$$\operatorname{Kos}(\psi, \Gamma_S, D_{\varpi}(T(r))) := \begin{bmatrix} \operatorname{Kos}(\Gamma'_S, D_{\varpi}(T(r))) & \xrightarrow{\psi-1} & \operatorname{Kos}(\Gamma'_S, D_{\varpi}(T(r))) \\ \big\downarrow \tau_0 & & \big\downarrow \tau_0 \\ \operatorname{Kos}^c(\Gamma'_S, D_{\varpi}(T(r))) & \xrightarrow{\psi-1} & \operatorname{Kos}^c(\Gamma'_S, D_{\varpi}(T(r))) \end{bmatrix}.$$

Lemma 6.14. *The natural morphism of complexes*

$$\operatorname{Kos}(\psi, \Gamma_S, D_{\varpi}^{(0,v/p]+}(T(r))) \to \operatorname{Kos}(\psi, \Gamma_S, D_{\varpi}(T(r))),$$

induced by the inclusion $\pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r)) \subset D_{\varpi}(T(r))$, is a quasi-isomorphism.

Proof. The map in the claim is injective on each term, so we examine the cokernel complex. Write

$$D_{\varpi}(T(r)) = D_{\varpi}^{(0,v/p]+}(T(r))[1/\pi_m]^{\wedge},$$

where $^{\wedge}$ denotes the p-adic completion. By Lemma 2.15, we have

$$\psi(\boldsymbol{A}_{R,\varpi}^{(0,v/p]+}) = \boldsymbol{A}_{R,\varpi}^{(0,v]+} \subset \boldsymbol{A}_{R,\varpi}^{(0,v/p]+},$$

and, for $\ell = p^{m-1}$, by Lemma 2.20 (iii), we have that $\pi_m^{-p\ell}\pi$ is a unit in $\boldsymbol{A}_{R,\varpi}^{(0,v/p]+}$. So, for $k \geqslant 1$, we get that

$$\psi(\pi_m^{-p^k \ell r} \boldsymbol{A}_{R,\varpi}^{(0,v/p]+}) \subset \pi_m^{-p^{k-1}\ell r} \boldsymbol{A}_{R,\varpi}^{(0,v/p]+}$$

(see Proposition 2.17). Moreover, recall that

$$\psi(D_{\varpi}^{(0,v/p]+}(T(r))) \subset D_{\varpi}^{(0,v/p]+}(T(r)).$$

Coupling this with the observation above, we get that

$$\psi(\pi_m^{-p^k \ell r} D_{\varpi}^{(0,v/p]+}(T(r))) \subset \pi_m^{-p^{k-1}\ell r} D_{\varpi}^{(0,v/p]+}(T(r)).$$

Therefore, it follows that the natural map

$$\psi : D_{\varpi}(T(r))/\pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r)) \rightarrow D_{\varpi}(T(r))/\pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))$$

is (pointwise) topologically nilpotent and $1-\psi$ is bijective over this quotient. Therefore, we obtain that the following complexes are acyclic:

$$\big[\mathrm{Kos}(\Gamma_S', D_{\varpi}(T(r))/\pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))) \xrightarrow{\psi-1} \mathrm{Kos}(\Gamma_S', D_{\varpi}(T(r))/\pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r)))\big],$$
$$\big[\mathrm{Kos}^c(\Gamma_S', D_{\varpi}(T(r))/\pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))) \xrightarrow{\psi-1} \mathrm{Kos}^c(\Gamma_S', D_{\varpi}(T(r))/\pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r)))\big].$$

Hence we conclude that the cokernel complex of the map in the claim is acyclic. □

Recall that we have the complex $\mathrm{Kos}(\varphi, \Gamma_S, D_{\varpi}(T(r)))$ from Definition 4.11, and we make the following claim.

Proposition 6.15. *The natural morphism of complexes*

$$\mathrm{Kos}(\varphi, \Gamma_S, D_{\varpi}(T(r))) \rightarrow \mathrm{Kos}(\psi, \Gamma_S, D_{\varpi}(T(r))),$$

induced by the identity on the first column and ψ on the second column, is a quasi-isomorphism.

Proof. Notice that the map ψ is surjective on $D_{\varpi}(T(r))$, so the cokernel complex is 0. To obtain the acyclicity of the kernel complex, we need to show that the complex

$$\big[\mathrm{Kos}(\Gamma_S', D_{\varpi}(T(r))^{\psi=0}) \xrightarrow{\tau_0} \mathrm{Kos}(\Gamma_S', D_{\varpi}(T(r))^{\psi=0})\big]$$

is acyclic. To show our claim, we will analyse the module $D_{\varpi}(T(r))^{\psi=0}$. Let $\{e_1, \ldots, e_h\}$ denote an $\boldsymbol{A}_R^+$-basis $\boldsymbol{N}(T)$ and set $f_i = e_i \otimes \epsilon^{\otimes r}$ for each $1 \leqslant i \leqslant h$, where $\epsilon^{\otimes r}$ is a $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(r)$. Since we have the isomorphism

$$\boldsymbol{A}_R \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)(r) \xrightarrow{\sim} \boldsymbol{D}(T)(r) = \boldsymbol{D}(T(r)),$$

it therefore follows that $\{f_1, \ldots, f_h\}$ is an $\boldsymbol{A}_R$-basis of $\boldsymbol{D}(T(r))$. Furthermore, as

$$D_{\varpi}(T(r)) = \boldsymbol{A}_{R,\varpi} \otimes_{\boldsymbol{A}_R} \boldsymbol{D}(T(r))$$

is an étale (φ, Γ_R)-module over $\boldsymbol{A}_{R,\varpi}$, we see that $\{\varphi(f_1), \ldots, \varphi(f_h)\}$ is an $\boldsymbol{A}_{R,\varpi}$-basis of $D_{\varpi}(T(r))$. In this basis, we have that $z = \sum_{j=1}^{h} z_j \varphi(f_j)$ is in $D_{\varpi}(T(r))^{\psi=0}$ if and only if z_j is in $\boldsymbol{A}_{R,\varpi}^{\psi=0}$ for each $1 \leqslant j \leqslant h$. Indeed, $\psi(z) = 0$ if and only if

$$\sum_{j=1}^{h} \psi(z_j \varphi(f_j)) = \sum_{j=1}^{h} \psi(z_j) f_j = 0,$$

and, since the f_j are linearly independent over $\boldsymbol{A}_{R,\varpi}$, we see that $\psi(z) = 0$ if and only if $\psi(z_j) = 0$ for all $1 \leqslant j \leqslant h$.

Next, from Proposition 2.17, we have a decomposition

$$\boldsymbol{A}_{R,\varpi}^{\psi=0} = \bigoplus_{\alpha} \varphi(\boldsymbol{A}_{R,\varpi})[X^{\flat}]^{\alpha},$$

where $[X^\flat]^\alpha = (1+\pi_m)^{\alpha_0}[X_1^\flat]^{\alpha_0} \cdots [X_d^\flat]^{\alpha_d}$ and $\alpha = (\alpha_0, \ldots, \alpha_d)$ is a $(d+1)$-tuple with $\alpha_i \in \{0, \ldots, p-1\}$. Therefore, we get

$$(D_\varpi(T(r)))^{\psi=0} = \left(\sum_{i=1}^{h} \boldsymbol{A}_{R,\varpi} f_j\right)^{\psi=0} = \bigoplus_{\alpha\neq 0} \sum_{i=1}^{h} \varphi(\boldsymbol{A}_{R,\varpi} f_j)[X^\flat]^\alpha.$$

Note that the last term identifies with

$$\bigoplus_{\alpha\neq 0} \sum_{i=1}^{h} \varphi(D_\varpi(T))(r)[X^\flat]^\alpha.$$

So, we obtain that the kernel complex of the map in the claim is isomorphic to the complex

$$\bigoplus_{\alpha\neq 0}\big[\mathrm{Kos}(\Gamma'_S, \varphi(D_\varpi(T))(r)[X^\flat]^\alpha) \xrightarrow{\tau_0} \mathrm{Kos}^c(\Gamma'_S, \varphi(D_\varpi(T))(r)[X^\flat]^\alpha)\big]. \tag{6-8}$$

Lemma 6.16. *The complex described in (6-8) is acyclic.*

Proof. The proof follows in a manner similar to Lemma 6.11, where one notes that it is enough to show the claim modulo p, and, for the latter, one uses the fact that $D_\varpi(T)/p = (N_\varpi^+(T)/p)[1/\pi_m]$ for $N_\varpi^+(T) = \boldsymbol{A}_{R,\varpi}^+ \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)$. We omit the details to avoid repetition. □

Using Lemma 6.16, we conclude that the natural morphism of complexes in the claim of Proposition 6.15 is a quasi-isomorphism. □

Proof of Proposition 6.1. Recall that s is the height of the representation T and r is the twist (see Assumption 5.1). Note that, from Proposition 6.4, we have a natural p^{4r}-quasi-isomorphism of complexes $\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_\varpi^{[u,v]}(T)) \simeq \mathcal{K}(\varphi, \mathrm{Lie}\,\Gamma_S, N_\varpi^{[u,v]}(T(r)))$. Then, in Proposition 6.5, we replace the infinitesimal action of Γ_S with the continuous action of Γ_S and obtain a natural isomorphism of complexes $\mathcal{K}(\varphi, \mathrm{Lie}\,\Gamma_S, N_\varpi^{[u,v]}(T(r))) \simeq \mathcal{K}(\varphi, \Gamma_S, N_\varpi^{[u,v]}(T(r)))$. Furthermore, in Proposition 6.6, we switch from analytic coefficient rings to overconvergent coefficient rings to obtain a natural p^{3r}-quasi-isomorphism of complexes $\mathcal{K}(\varphi, \Gamma_S, N_\varpi^{[u,v]}(T(r))) \simeq \mathcal{K}(\varphi, \Gamma_S, N_\varpi^{(0,v]+}(T(r)))$. Next, in Proposition 6.8 and Lemmas 6.12 and 6.13, we change the overconvergence radius to obtain a $p^{3r+3s+2}$-quasi-isomorphism of complexes $\tau_{\leqslant r}\mathcal{K}(\varphi, \Gamma_S, N_\varpi^{(0,v]+}(T(r))) \simeq \tau_{\leqslant r}\mathrm{Kos}(\psi, \Gamma_S, D_\varpi^{(0,v/p]+}(T(r)))$, where $\tau_\leqslant$ denotes the canonical truncation. Finally, in Lemma 6.14 and Proposition 6.15, we change the disk of convergence to obtain natural quasi-isomorphisms of complexes

$$\mathrm{Kos}(\psi, \Gamma_S, D_\varpi^{(0,v/p]+}(T(r))) \simeq \mathrm{Kos}(\psi, \Gamma_S, D_\varpi(T(r))) \simeq \mathrm{Kos}(\varphi, \Gamma_S, D_\varpi(T(r))).$$

Combining these statements, we get the claim of Proposition 6.1 with $N = 10r + 3s + 2$. □

6.7. ***Comparison with the Fontaine–Messing period map.*** The aim of this subsection is to show that the comparison map from $\mathrm{Syn}(S, M, r)$ to $R\Gamma_{\mathrm{cont}}(G_S, (T(r)))$ in Theorem 5.5 coincides with the Fontaine–Messing period map. We will follow the strategy in [Colmez and Nizioł 2017, §4.7]. Recall that we have $S = R[\varpi]$, $\bar{S} = \bar{R} \subset \overline{\mathrm{Fr}(R)}$ and $S_\infty = R_\infty \subset \overline{\mathrm{Fr}(R)}$. Note that, by Definition 2.24, we have rings $E_{\bar{S}}^\star := E_{\bar{R}}^\star$ for $\star \in \{\mathrm{PD}, [u], [u, v]\}$ equipped with a Frobenius, a filtration and an action of $G_S \triangleleft G_R$.

Let us recall that T is a positive finite q-height $\mathbb{Z}_p$-representation of G_R as in Assumption 5.1 and $V = T[1/p]$. Note that, by tensoring the fundamental exact sequence in (2-2) with T, we get the following p^r-exact sequence:

$$0 \to T(r)' \to \mathrm{Fil}^r \boldsymbol{A}_{\mathrm{cris}}(\bar{S}) \otimes_{\mathbb{Z}_p} T \xrightarrow{p^r - \varphi} \boldsymbol{A}_{\mathrm{cris}}(\bar{S}) \otimes_{\mathbb{Z}_p} T \to 0. \tag{6-9}$$

Next, from Assumption 5.1, we have a finite free R-module $M \subset \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$ such that $M[1/p] = \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V)$. Moreover, we have a natural injective map

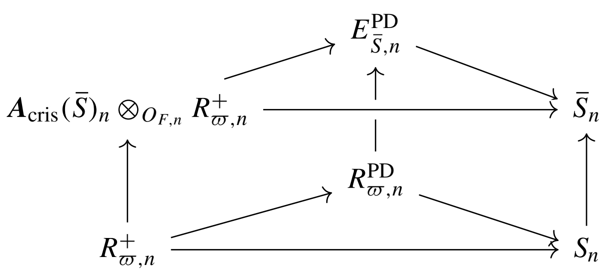
$$\mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \to \mathcal{O}\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T)$$

compatible with the respective Frobenii, filtrations, $\boldsymbol{A}_{R,\varpi}^{\mathrm{PD}}$-linear connections and actions of Γ_R. Additionally, by definition, we have a natural inclusion $\boldsymbol{A}^+ \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T) \subset \boldsymbol{A}^+ \otimes_{\mathbb{Z}_p} T$ compatible with the respective Frobenii and actions of G_R. Extending scalars to $\mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\bar{S})$ in both the maps and composing them, we obtain the top horizontal arrow in the following diagram:

$$\begin{array}{ccc} \mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\bar{S}) \otimes_R M & \longrightarrow & \mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\bar{S}) \otimes_{\mathbb{Z}_p} T \\ \downarrow & & \downarrow \\ \mathcal{O}\boldsymbol{B}_{\mathrm{cris}}(\bar{S}) \otimes_R \mathcal{O}\boldsymbol{D}_{\mathrm{cris}}(V) & \xrightarrow{\sim} & \mathcal{O}\boldsymbol{B}_{\mathrm{cris}}(\bar{S}) \otimes_{\mathbb{Q}_p} V \end{array} \tag{6-10}$$

where the vertical arrows are natural inclusions and the lower horizontal arrow is a natural isomorphism (since V is crystalline) compatible with the respective Frobenii, filtrations, actions of G_R and $\boldsymbol{B}_{\mathrm{cris}}(\bar{S})$-linear connections satisfying Griffiths transversality with respect to the filtrations; see [Brinon 2008, Proposition 8.4.3]. The diagram commutes by definition (see [Abhinandan 2025, §4.5] for a similar diagram), and it follows that the top horizontal arrow is injective. Now, recall that the filtration on the bottom left object is given by the tensor product filtration (see Lemma 2.35 and Remark 2.36) and the filtration on the bottom right object is induced by the natural filtration on $\mathcal{O}\boldsymbol{B}_{\mathrm{cris}}(\bar{S})$. As the filtration on the objects in the top row are induced from the filtration on the objects in the bottom row of their respective columns (see the discussion before Lemma 2.40 for the top left corner), it therefore follows that the filtration on $\mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\bar{S}) \otimes_R M$ matches with the induced filtration from $\mathcal{O}\boldsymbol{A}_{\mathrm{cris}}(\bar{S}) \otimes_{\mathbb{Z}_p} T$.

Now, we consider the commutative diagram



where the subscript n denotes the reduction modulo p^n, the bottom horizontal arrow is induced by $X_0 \mapsto \varpi$ and the top horizontal arrow is the extension of the θ-map by the bottom horizontal arrow.

Using the rings discussed above, we will define the local Fontaine–Messing period map. Set

$$\Omega_{E^{\mathrm{PD}}_{\bar{S},n}} := E^{\mathrm{PD}}_{\bar{S},n} \otimes_{R^+_{\varpi,n}} \Omega_{R^+_{\varpi,n}}, \qquad \Delta^{\mathrm{PD}} := E^{\mathrm{PD}}_{\bar{S}} \otimes_R M \quad \text{and} \quad \Delta^{\mathrm{PD}}_n = \Delta^{\mathrm{PD}}/p^n,$$

and equip them with the induced filtration, Frobenius, G_S-action and $\mathbf{A}_{\mathrm{cris}}(\bar{S})_n$-linear integrable connection ∂ satisfying Griffiths transversality with respect to the filtration. In particular, for $r \in \mathbb{Z}$, we have the filtered de Rham complex

$$\mathrm{Fil}^r \mathcal{D}^\bullet_{\bar{S},M,n} : \mathrm{Fil}^r \Delta^{\mathrm{PD}}_n \to \mathrm{Fil}^{r-1} \Delta^{\mathrm{PD}}_n \otimes_{R^+_{\varpi,n}} \Omega^1_{R^+_{\varpi,n}} \to \mathrm{Fil}^{r-2} \Delta^{\mathrm{PD}}_n \otimes_{R^+_{\varpi,n}} \Omega^2_{R^+_{\varpi,n}} \to \cdots.$$

Let us note that, by extending the diagram (6-10) along the natural inclusion $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\bar{S}) \subset E^{\mathrm{PD}}_{\bar{S}}$ (see Remark 2.27), we obtain an $E^{\mathrm{PD}}_{\bar{S}}$-linear injective map $E^{\mathrm{PD}}_{\bar{S}} \otimes_R M \to E^{\mathrm{PD}}_{\bar{S}} \otimes_{\mathbb{Z}_p} T$ compatible with the respective Frobenii, filtrations, $\mathbf{A}_{\mathrm{cris}}(\bar{S})$-linear connections and actions of G_R. Then, for each $r \in \mathbb{Z}$, by reducing the induced map on the r-th filtered part modulo p^n and taking horizontal sections for the $\mathbf{A}_{\mathrm{cris}}(\bar{S})_n$-linear connections, we obtain a natural map

$$(\mathrm{Fil}^r \Delta^{\mathrm{PD}}_n)^{\partial=0} = (\mathrm{Fil}^r (E^{\mathrm{PD}}_{\bar{S},n} \otimes_R M))^{\partial=0} \to (\mathrm{Fil}^r E^{\mathrm{PD}}_{\bar{S},n} \otimes_{\mathbb{Z}_p} T)^{\partial=0} = \mathrm{Fil}^r \mathbf{A}_{\mathrm{cris}}(\bar{S})_n \otimes_{\mathbb{Z}_p} T. \tag{6-11}$$

In particular, from the discussion above and the filtered Poincaré Lemma 3.21, we get a natural map

$$\mathrm{Fil}^r \mathcal{D}^\bullet_{\bar{S},M,n} \xleftarrow{\sim} (\mathrm{Fil}^r \Delta^{\mathrm{PD}}_n)^{\partial=0} \to \mathrm{Fil}^r \mathbf{A}_{\mathrm{cris}}(\bar{S})_n \otimes_{\mathbb{Z}_p} T. \tag{6-12}$$

Notation. For a G_S-module D, let $C(G_S, D)$ denote the complex of continuous cochains of G_S with values in D.

Definition 6.17. Define the syntomic complex with coefficients in M as

$$\mathrm{Syn}(\bar{S}, M, r)_n := \big[\mathrm{Fil}^r \mathcal{D}^\bullet_{\bar{S},M,n} \xrightarrow{p^r - p^\bullet \varphi} \mathcal{D}^\bullet_{\bar{S},M,n}\big]. \tag{6-13}$$

Define the Fontaine–Messing period map

$$\tilde{\alpha}^{\mathrm{FM}}_{r,n,S} : \mathrm{Syn}(S, M, r)_n \to C(G_S, T/p^n(r)') \tag{6-14}$$

as the composition

$$\begin{aligned}\mathrm{Syn}(S, M, r)_n = \big[\mathrm{Fil}^r \mathcal{D}^\bullet_{S,M,n} \xrightarrow{p^r - p^\bullet \varphi} \mathcal{D}^\bullet_{S,M,n}\big] &\to C\big(G_S, \big[\mathrm{Fil}^r \mathcal{D}^\bullet_{\bar{S},M,n} \xrightarrow{p^r - p^\bullet \varphi} \mathcal{D}^\bullet_{\bar{S},M,n}\big]\big) \\ &\to C\big(G_S, \big[\mathrm{Fil}^r \mathbf{A}_{\mathrm{cris}}(\bar{S})_n \otimes T \xrightarrow{p^r - \varphi} \mathbf{A}_{\mathrm{cris}}(\bar{S})_n \otimes T\big]\big) \xleftarrow{\sim} C(G_S, T/p^n(r)'),\end{aligned}$$

where the second right arrow is induced by (6-12) and the only left arrow is a p^r-quasi-isomorphism as noted in (6-9).

Remark 6.18. The definition of the Fontaine–Messing period map in (6-14) can also be given for R: we use the ring $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\bar{R})$ instead of $E^{\mathrm{PD}}_{\bar{S}}$ and set $\Delta^{\mathrm{PD}} = \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\bar{R}) \otimes_R M$. Then the map in (6-12) gets replaced by the map

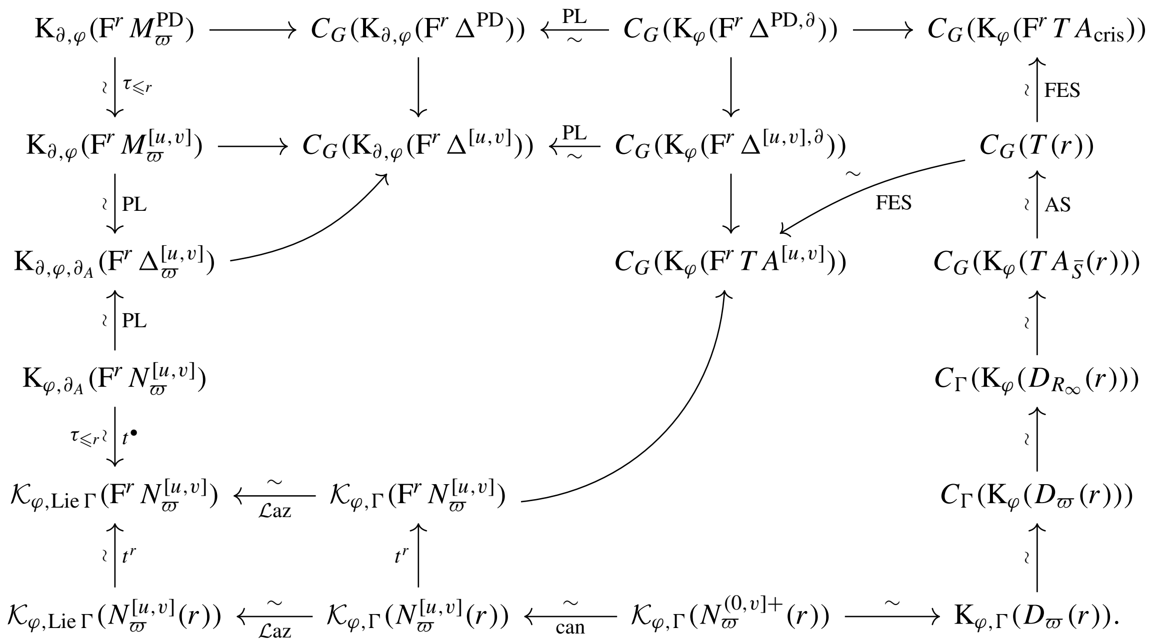
$$\mathrm{Fil}^r \mathcal{D}^\bullet_{\bar{R},M,n} \xrightarrow{\sim} \mathrm{Fil}^r \mathbf{A}_{\mathrm{cris}}(\bar{R})_n \otimes T$$

(where the filtered de Rham complex is obtained similar to the modulo p^n version of the complex $\mathrm{Fil}^r \mathscr{D}^\bullet_{R,M}$ in (5-3)). The definition of $\mathrm{Syn}(\bar{R}, M, r)_n$ follows naturally, and, since the fundamental exact sequence is G_R-equivariant, we obtain the Fontaine–Messing period map

$$\tilde{\alpha}^{\mathrm{FM}}_{r,n,R} : \mathrm{Syn}(R, M, r)_n \to C(G_R, T/p^n(r)').$$

Theorem 6.19. *The map $\tilde{\alpha}^{\mathrm{FM}}_{r,n,S}$ in (6-14) is $p^{N(T,e,r)}$-equal to $\alpha^{\mathcal{L}\mathrm{az}}_{r,n}$ from Theorem 5.5.*

Proof. The p-power equality of the two maps follows from the diagram below (where we only show the p-adic version to simplify notation). The objects and morphisms are described after the diagram. Note that we have $\mathrm{K}_{\partial,\varphi}(\mathrm{F}^r M^{\mathrm{PD}}_{\varpi}) = \mathrm{Syn}(S, M, r)$, and the map $\tilde{\alpha}^{\mathrm{FM}}_{r,S}$ is obtained by composing the arrows in the top row (note that $C_G(T(r))$ is p^r-isomorphic to $C_G(T(r)'))$. Furthermore, the map $\alpha^{\mathcal{L}\mathrm{az}}_r$ is obtained by composing the maps in the outer left vertical, bottom horizontal and right vertical boundary. The isomorphisms in the diagram indicate a p-power quasi-isomorphism between complexes. Finally, a simple diagram chase gives us the claim. ☐



In the diagram, we take

$$\Delta^{\mathrm{PD}} = E^{\mathrm{PD}}_{\bar{S}} \otimes_R M, \quad \Delta^{\mathrm{PD},\partial} = (\Delta^{\mathrm{PD}})^{\partial=0}, \quad T A_{\mathrm{cris}} = \boldsymbol{A}_{\mathrm{cris}}(\bar{S}) \otimes_{\mathbb{Z}_p} T,$$
$$\Delta^{[u,v]} = E^{[u,v]}_{\bar{S}} \otimes_R M, \quad \Delta^{[u,v],\partial} = (\Delta^{[u,v]})^{\partial=0}, \quad T A^{[u,v]} = \boldsymbol{A}^{[u,v]}_{\bar{S}} \otimes_{\mathbb{Z}_p} T, \quad \Delta^{[u,v]}_{\varpi} = E^{[u,v]}_{R,\varpi} \otimes_R M$$

(see Definition 2.24) and also

$$T A_{\bar{S}}(r) = \boldsymbol{A}_{\bar{S}} \otimes_{\mathbb{Z}_p} T(r), \qquad D_{\varpi}(r) = D_{\varpi}(T(r)),$$
$$N^{\star}_{\varpi}(r) = N^{\star}_{\varpi}(T(r)), \qquad D_{R_\infty}(r) = \boldsymbol{A}_{S_\infty} \otimes_{\boldsymbol{A}_{R,\varpi}} D_{\varpi}(r).$$

Moreover, $G = G_S$ and $\Gamma = \Gamma_S$ with C_G and C_Γ denoting the complex of continuous cochains for G and Γ, respectively. The letter "K" denotes the Koszul complex with subscripts: ∂ denotes the operators $((1+X_0)\partial/\partial X_0, \ldots, X_d\partial/\partial X_d)$; the subscript Γ denotes the operators $(\gamma_0 - 1, \ldots, \gamma_d - 1)$ for our choice of topological generators of Γ; the subscript Lie Γ denotes the operators $(\nabla_0, \ldots, \nabla_d)$, with $\nabla_i = \log \gamma_i$; and the subscript ∂_A denotes $((1+X_0)\partial/\partial X_0, X_1\partial/\partial X_1, \ldots, X_d\partial/\partial X_d)$ as operators on $\boldsymbol{A}_R^{[u,v]}$ and $E_R^{[u,v]}$ via the isomorphism $\iota_{\text{cycl}} : R_\varpi^{[u,v]} \xrightarrow{\sim} \boldsymbol{A}_{R,\varpi}^{[u,v]}$. The letter "$\mathcal{K}$" denotes a certain subcomplex of the Koszul complex (see Sections 6.2–6.5).

Next, let us describe the maps between the rows. FES denotes a map coming from the fundamental exact sequences in (2-2) and (2-5). AS denotes a map originating from the Artin–Schreier theory in (2-4). PL denotes maps coming from the filtered Poincaré lemma of Section 2.8. In the first column, going from the first row to the second row is induced by the inclusion $R_\varpi^{\text{PD}} \subset R_\varpi^{[u,v]}$. The leftmost slanted vertical map from the third to the second row is induced by the inclusion $E_{R,\varpi}^{[u,v]} \subset E_{\bar{S}}^{[u,v]}$. From the second to the third row, the map in the third column is induced similar to (6-11). The leftmost vertical map from the second to the third row is the content of Lemma 5.24, and the leftmost vertical map from the fourth to the third row is the content of Lemma 5.25; the composition being the content of Proposition 5.28. The rightmost vertical map from the fourth to the third row is the inflation map from Γ_S to G_S using the inclusion $\boldsymbol{A}_{S_\infty} \subset \boldsymbol{A}_{\bar{S}}$ (one could use almost étale descent to obtain the quasi-isomorphism), and the rightmost vertical map from the fifth to the fourth row uses the inclusion $\boldsymbol{A}_{R,\varpi} \subset \boldsymbol{A}_{S_\infty}$ (the quasi-isomorphism is obtained by decompletion techniques). The leftmost vertical arrow from the fourth to the fifth row is given by multiplication by suitable powers of t as in Lemma 6.2, and the rightmost vertical arrow from the sixth to the fifth row is the comparison between the complex computing the continuous cohomology of Γ_S and the Koszul complex as in Section 4.2. The inclusions

$$\boldsymbol{A}_{R,\varpi}^{+} \subset \boldsymbol{A}_{\text{inf}}(\bar{S}) \subset \boldsymbol{A}_{\bar{S}}^{[u,v]} \quad \text{and} \quad \boldsymbol{A}_{\text{inf}}(\bar{S}) \otimes_{\boldsymbol{A}_R^+} \boldsymbol{N}(T) \subset \boldsymbol{A}_{\text{inf}}(\bar{S}) \otimes_{\mathbb{Z}_p} T$$

induce the slanted vertical arrow from the fifth to the third row.

Finally, let us describe the maps between the columns. The top two maps from the first to the second column are induced by the respective inclusions $R_\varpi^{\text{PD}} \subset E_{\bar{S}}^{\text{PD}}$ and $R_\varpi^{[u,v]} \subset E_{\bar{S}}^{[u,v]}$. The bottom two maps $\mathcal{L}$az between the first and the second column are Lazard isomorphisms discussed in Section 6.2. The bottom map from the third to the second column is induced canonically from the inclusion $\boldsymbol{A}_{R,\varpi}^{(0,v]+} \subset \boldsymbol{A}_{R,\varpi}^{[u,v]}$. From the third to the fourth column, the top horizontal map is induced similar to (6-11) and the bottom horizontal map is induced by the inclusion $\boldsymbol{A}_{R,\varpi}^{(0,v]+} \subset \boldsymbol{A}_{R,\varpi}$ (see Sections 6.5 and 6.6).

Corollary 6.20. *The morphism of complexes $\tilde{\alpha}_{r,n,R}^{\text{FM}}$ in Remark 6.18 is a $p^{N(p,r,s)}$-quasi-isomorphism.*

Proof. Let $m = 2$, i.e., $K = F(\zeta_{p^2} - 1)$ and $e = p(p-1)$. Then, over $S = O_K \otimes_{O_F} R$, we know that the local Fontaine–Messing period map $\tilde{\alpha}_{r,n,S}^{\text{FM}}$ is p^N-isomorphic to the Lazard map $\alpha_{r,n}^{\mathcal{L}\text{az}}$ from Theorem 6.19. Moreover, the Lazard map $\alpha_{r,n}^{\mathcal{L}\text{az}}$ is a p^N-quasi-isomorphism by Theorem 5.5. Since we fixed m, it follows that $N = 2n(T, e) + 14r + 7s + 2$ only depends on p, r and s (see Section 6.1 for the explicit constant). Next, to descend to R, we note that the Fontaine–Messing period map is $G = \text{Gal}(F(\zeta_{p^2})/F)$-equivariant;

i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Syn}(R, M, r)_n & \xrightarrow{\tilde{\alpha}^{\mathrm{FM}}_{r,n,R}} & C(G_R, T/p^n(r)') \\ \downarrow & & \downarrow\wr \\ R\Gamma(G, \mathrm{Syn}(S, M, r)_n) & \xrightarrow{\tilde{\alpha}^{\mathrm{FM}}_{r,n,S}} & R\Gamma(G, C(G_S, T/p^n(r)')) \end{array}$$

where the right vertical map is a quasi-isomorphism. So, from the Galois descent argument in Lemma 6.21 (for $e = p(p-1)$), it follows that the left vertical arrow is a $p^{4r+3p(p-1)}$-quasi-isomorphism. Hence we obtain that the morphism of complexes $\tilde{\alpha}^{\mathrm{FM}}_{r,n,R}$ in Remark 6.18 is a $p^{N(p,r,s)}$-quasi-isomorphism for $N(p, r, s) = 2N + 4r + 3p(p-1)$. □

6.8. *Galois descent.* Let $e = [K : F] = p^{m-1}(p-1)$, $G = \mathrm{Gal}(K/F)$ and $S = O_K \otimes_{O_F} R$. For notational convenience, we will use crystalline and syntomic complexes as in Section 7.2. We view the R-module M in Assumption 5.1 as an object in $\mathrm{CR}(R/O_F, \mathrm{Fil}, \varphi)$, i.e., a filtered crystal equipped with Frobenius (see Remark 7.3 and Definition 7.4).

Lemma 6.21. *The following natural map is a p^{4r+3e}-quasi-isomorphism*:

$$R\Gamma_{\mathrm{syn}}(R, M, r) \to R\Gamma(G, R\Gamma_{\mathrm{syn}}(S, M, r)).$$

Proof. The claim may be shown by adapting the arguments provided in the proof of [Colmez and Nizioł 2017, Lemma 5.9] to the current setting. □

7. Crystals and syntomic cohomology

7.1. *Filtered Frobenius crystals.* Let κ be a perfect field of characteristic p, and set $O_F = W(\kappa)$ and $F = \mathrm{Fr}\, O_F$. Furthermore, let K be a finite extension of F such that $K \cap F^{\mathrm{ur}} = F$, and let O_K denote its ring of integers.

Notation. Hereafter, we use letters $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, etc. to denote schemes as well as p-adic formal schemes.

Let $\mathfrak{X}$ be a (p-adic formal) scheme over O_K with X its (rigid) generic fibre and $\mathfrak{X}_\kappa$ its special fibre. Set $\Sigma = \mathrm{Spec}\, O_F$ ($\Sigma = \mathrm{Spf}\, O_F$), and, for $n \in \mathbb{N}$, let $\mathfrak{X}_n = \mathfrak{X} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n$ and $\Sigma_n = \mathrm{Spec}(O_F/p^n)$. Consider the big (étale) crystalline site $\mathrm{CRIS}(\mathfrak{X}_n/\Sigma_n)$ with the PD-ideal $(p(O_F/p^n), [\ \ ])$ and the category of crystals of $\mathcal{O}_{\mathfrak{X}_n/\Sigma_n}$-modules; see [Bauer 1992, Corollary 1.15 and Proposition 1.17; Berthelot 1974, §III.4.2; Berthelot et al. 1982, §1.1.18, §1.1.19]. Set $\mathrm{CR}(\mathfrak{X}_n/\Sigma_n)$ to be the full subcategory of finite locally free crystals. The homomorphisms $\mathfrak{X}_n \to \mathfrak{X}_{n+1}$ and $\Sigma_n \to \Sigma_{n+1}$ induce a pullback functor

$$i^*_{n,n+1} : \mathrm{CR}(\mathfrak{X}_{n+1}/\Sigma_{n+1}) \to \mathrm{CR}(\mathfrak{X}_n/\Sigma_n).$$

Similarly, define the big crystalline site $\mathrm{CRIS}(\mathfrak{X}_1/\Sigma_n)$ and the category of finite locally free crystals $\mathrm{CR}(\mathfrak{X}_1/\Sigma_n)$. Note that the natural pullback functor

$$i^*_n : \mathrm{CR}(\mathfrak{X}_n/\Sigma_n) \to \mathrm{CR}(\mathfrak{X}_1/\Sigma_n)$$

induces an equivalence of categories by [Berthelot 1974, Chapitre IV, Théorèm 1.4.1].

Definition 7.1. A finite locally free crystal on $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ is the data $\mathcal{F} = (\mathcal{F}_n)_{n\geqslant 1}$, where $\mathcal{F}_n$ is an object of $\mathrm{CR}(\mathfrak{X}_n/\Sigma_n)$ and we have isomorphisms $i^*_{n,n+1}(\mathcal{F}_{n+1}) \xrightarrow{\sim} \mathcal{F}_n$. A morphism between two crystals $\mathcal{F}$ and $\mathcal{G}$ on $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ is a collection of morphisms $\mathcal{F}_n \to \mathcal{G}_n$ for each $n \geqslant 1$ compatible with the pullback isomorphisms. Denote the category of such objects by $\mathrm{CR}(\mathfrak{X}/\Sigma)$. A finite locally free crystal on $\mathrm{CRIS}(\mathfrak{X}_1/\Sigma)$ is defined similarly and the pullback functor

$$i^* : \mathrm{CR}(\mathfrak{X}/\Sigma) \to \mathrm{CR}(\mathfrak{X}_1/\Sigma)$$

induces an equivalence of categories.

Consider the category of filtered crystals on $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ in the sense of [Tsuji 2020, Definition 16] (for the relation between this category and Ogus' book [1994], see [Tsuji 2020, Remark 19]). Take $\mathrm{CR}(\mathfrak{X}_n/\Sigma_n, \mathrm{Fil})$ to be the full subcategory of finite locally free filtered crystals on $\mathrm{CRIS}(\mathfrak{X}_n/\Sigma_n)$. We have the natural pullback functor

$$i^*_{n,n+1} : \mathrm{CR}(\mathfrak{X}_{n+1}/\Sigma_{n+1}, \mathrm{Fil}) \to \mathrm{CR}(\mathfrak{X}_n/\Sigma_n, \mathrm{Fil}).$$

Definition 7.2. A finite locally free filtered crystal on $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ is the data $(\mathcal{F}_n)_{n\geqslant 1}$ in $\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil})$ such that the isomorphisms $i^*_{n,n+1}(\mathcal{F}_{n+1}) \xrightarrow{\sim} \mathcal{F}_n$ are compatible with filtration. A morphism between two filtered crystals is defined in an obvious way, and we denote this category by $\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil})$.

Remark 7.3. Let R denote the p-adic completion of an étale algebra over $O_F[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$, let $\mathrm{MIC}(R)$ be the category of finite projective R-modules equipped with an integrable connection and let

$$\mathrm{MIC}_{\mathrm{conv}}(R) \subset \mathrm{MIC}(R)$$

denote the full subcategory of modules whose connection is p-adically quasinilpotent. Let $\mathfrak{X} = \mathrm{Spf}\, R$. Then from [Berthelot 1974, Chapitre IV, Théorèm 1.6.5] and [Morrow and Tsuji 2020, Lemma 1.9] we obtain an equivalence of categories $\mathrm{CR}(\mathfrak{X}/\Sigma) \xrightarrow{\sim} \mathrm{MIC}_{\mathrm{conv}}(R)$. This equivalence restricts to an equivalence $\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil}) \xrightarrow{\sim} \mathrm{MIC}_{\mathrm{conv}}(R, \mathrm{Fil})$.

Finally, we will consider crystals equipped with a Frobenius structure. The Frobenius endomorphism of O_F and the absolute Frobenius on $\mathfrak{X}_1$ induce Frobenius pullbacks

$$F^*_{\mathfrak{X}_1} : \mathrm{CR}(\mathfrak{X}_1/\Sigma_n) \to \mathrm{CR}(\mathfrak{X}_1/\Sigma_n),$$
$$F^*_{\mathfrak{X}_1} : \mathrm{CR}(\mathfrak{X}_1/\Sigma) \to \mathrm{CR}(\mathfrak{X}_1/\Sigma).$$

Recall that we have the natural pullback functor $i^* : \mathrm{CR}(\mathfrak{X}/\Sigma) \to \mathrm{CR}(\mathfrak{X}_1/\Sigma)$.

Definition 7.4. A *Frobenius structure* on a finite locally free crystal $\mathcal{F}$ on $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ is a morphism $\varphi_{\mathcal{F}} : F^*_{\mathfrak{X}_1} i^*\mathcal{F} \to i^*\mathcal{F}$ such that it becomes an isomorphism in the isogeny category $\mathrm{CR}(\mathfrak{X}/\Sigma)_{\mathbb{Q}}$. A morphism between two crystals with Frobenius structure is taken to be a morphism in $\mathrm{CR}(\mathfrak{X}/\Sigma)$ compatible with respective Frobenius structures. Denote the category of finite locally free crystals (resp. filtered crystals) equipped with a Frobenius structure by $\mathrm{CR}(\mathfrak{X}/\Sigma, \varphi)$ (resp. $\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil}, \varphi)$).

7.2. ***Syntomic complex.*** We will let $\mathfrak{X}$ be a smooth (p-adic formal) scheme over O_K, let $\Sigma = \operatorname{Spec} O_F$ ($\Sigma = \operatorname{Spf} O_F$), and let $\mathcal{F}$ be an object of $\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil}, \varphi)$, i.e., a finite locally free filtered crystal on $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ equipped with a Frobenius structure. In this subsection, we will define the syntomic cohomology of $\mathfrak{X}$ with coefficients in $\mathcal{F}$.

Let $u_{\mathfrak{X}_n/\Sigma_n} : (\mathfrak{X}_n/\Sigma_n)_{\mathrm{cris}} \to \mathfrak{X}_{n,\text{ét}}$ denote the projection from the crystalline topos to the étale topos. In the following, we regard sheaves on $\mathfrak{X}_{n,\text{ét}}$ as sheaves on $\mathfrak{X}_{\kappa,\text{ét}}$. For $r \geqslant 0$, we have filtered crystalline cohomology complexes of $\mathcal{F}$

$$\mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathrm{Fil}^r \mathcal{F})_n := \mathrm{R}\Gamma(\mathfrak{X}_{n,\text{ét}}, \mathrm{R}u_{\mathfrak{X}_n/\Sigma_n *} \mathrm{Fil}^r \mathcal{F}_n),$$
$$\mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathrm{Fil}^r \mathcal{F}) := \operatorname{holim}_n \mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathrm{Fil}^r \mathcal{F})_n.$$

Definition 7.5. Define the modulo p^n and the completed syntomic complex with coefficients as

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X}, \mathcal{F}, r)_n := \big[\mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathrm{Fil}^r \mathcal{F})_n \xrightarrow{p^r - \varphi} \mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathcal{F})_n\big],$$
$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X}, \mathcal{F}, r) := \operatorname{holim}_n \mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X}, \mathcal{F}, r)_n.$$

The mapping fibres are taken in the derived ∞-category of abelian groups.

Remark 7.6. In the derived category $D^+(\mathfrak{X}_{\kappa,\text{ét}}, \mathbb{Z}/p^n)$, we have quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X}, \mathcal{F}, r)_n \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X}, \mathcal{F}, r) \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^n,$$
$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X}, \mathcal{F}, r)_n \simeq \big[\mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathcal{F})_n \xrightarrow{(p^r - \varphi, \mathrm{can})} \mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathcal{F})_n \oplus \mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathcal{F}/\mathrm{Fil}^r \mathcal{F})_n\big].$$

Definition 7.7. Define $\mathscr{F}_{n,\text{ét},\mathfrak{X}}$ to be the étale sheafification of $(\mathfrak{U} \to \mathfrak{X}) \mapsto \mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{U}, \mathcal{F})_n$ and $\mathrm{Fil}^r \mathscr{F}_{n,\text{ét},\mathfrak{X}}$ to be the étale sheafification of $(\mathfrak{U} \to \mathfrak{X}) \mapsto \mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{U}, \mathrm{Fil}^r \mathcal{F})_n$ for $\mathfrak{U} \to \mathfrak{X}$ any étale map. Similarly, define $\mathscr{S}_{n,\text{ét}}(\mathcal{F}, r)_{\mathfrak{X}}$ to be the étale sheafification of $(\mathfrak{U} \to \mathfrak{X}) \mapsto \mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{U}, \mathcal{F}, r)_n$.

Lemma 7.8. *In the setting above, we have*

$$\mathscr{S}_{n,\text{ét}}(\mathcal{F}, r)_{\mathfrak{X}} = \big[\mathrm{Fil}^r \mathscr{F}_{n,\text{ét},\mathfrak{X}} \xrightarrow{p^r - \varphi} \mathscr{F}_{n,\text{ét},\mathfrak{X}}\big] \quad \text{and} \quad \mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X}, \mathcal{F}, r)_n = \mathrm{R}\Gamma(\mathfrak{X}_{\kappa,\text{ét}}, \mathscr{S}_{n,\text{ét}}(\mathcal{F}, r)_{\mathfrak{X}}).$$

Remark 7.9. The syntomic cohomology with coefficients can also be described using hypercoverings; for example, see [Tsuji 1996, §2.6; 1999, §2.1].

Notation. In the rest of this article, we will denote the modulo p^n (resp. completed) syntomic complex with coefficients in $\mathcal{F}$ by $\mathscr{S}_n(\mathcal{F}, r)_{\mathfrak{X}}$ (resp. $\mathscr{S}(\mathcal{F}, r)_{\mathfrak{X}}$).

8. p-adic nearby cycles

In this section, we give some global applications of the computations done in previous sections.

8.1. ***Fontaine–Laffaille modules.*** Let R denote the p-adic completion of an étale algebra over

$$O_F[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$$

for some $d \in \mathbb{N}$ satisfying Assumption 2.1, and let $s \in \mathbb{N}$ such that $s \leqslant p - 2$. In Section 3.4, we defined the category $\mathrm{MF}_{[0,s],\mathrm{free}}(R, \Phi, \partial)$ of *free relative Fontaine–Laffaille* modules of level $[0, s]$.

Let us now globalise the definition above. Let $\mathfrak{X}$ be a smooth (p-adic formal) scheme defined over O_F. Consider a covering $\{\mathfrak{U}_i\}_{i\in I}$ of $\mathfrak{X}$ with $\mathfrak{U}_i = \operatorname{Spec} A_i$ ($\mathfrak{U}_i = \operatorname{Spf} A_i$) such that the p-adic completions $\hat{A}_i$ satisfy Assumption 2.1 for each $i \in I$. We fix lifts of Frobenius modulo p as $\varphi_i : \hat{A}_i \to \hat{A}_i$.

Remark 8.1. In Section 3.4, we fixed a lifting φ of the absolute Frobenius on R/p. However, for another lift φ', the categories

$$\mathrm{MF}_{[0,s],\mathrm{free}}(R, \Phi, \partial) \quad \text{and} \quad \mathrm{MF}_{[0,s],\mathrm{free}}(R, \Phi', \partial)$$

are naturally equivalent; see [Faltings 1989, Theorem 2.3; Tsuji 2020, Remark 33]. In particular, there is a well-defined isomorphism $\alpha_{\varphi,\varphi'} : \varphi^* M \xrightarrow{\sim} \varphi'^* M$ compatible with connections.

Definition 8.2. Define $\mathrm{MF}_{[0,s],\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$ to be the category of finite locally free filtered $\mathcal{O}_{\mathfrak{X}}$-modules $\mathcal{M}$ equipped with a p-adically quasinilpotent integrable connection satisfying Griffiths transversality with respect to filtration, and such that there exists a covering $\{\mathfrak{U}_i\}_{i\in I}$ of $\mathfrak{X}$ as above with $\mathcal{M}_{\mathfrak{U}_i} \in \mathrm{MF}_{[0,s],\mathrm{free}}(\hat{A}_i, \Phi, \partial)$ for all $i \in I$, and on $\mathfrak{U}_{ij}$ the two structures glue well under $\alpha_{\varphi_i,\varphi_j}$.

Remark 8.3. Let $\Sigma = \operatorname{Spec} O_F$ or $\Sigma = \operatorname{Spf} O_F$; then the category $\mathrm{MF}_{[0,s],\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$ is a full subcategory of $\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil}, \varphi)$ described in Definition 7.4.

Remark 8.4. To any object of $\mathrm{MF}_{[0,s],\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$, [Faltings 1989, Theorem 2.6*] associated a compatible system of étale sheaves on $\mathrm{Sp}(\hat{A}_i[1/p])$ (see the functor T_{cris} in Section 3.4). These sheaves can be expressed in terms of certain finite étale coverings of $\mathrm{Sp}(\hat{A}_i[1/p])$. Extending these by normalisation to $\mathrm{Spec}(\hat{A}_i)$, the resulting coverings glue to give a covering of the smooth formal O_F-scheme $\mathfrak{X}'$ associated to $\mathfrak{X}$. For $\mathfrak{X}$ a smooth p-adic formal scheme, note that $\mathfrak{X} = \mathfrak{X}'$, and this gives us an étale $\mathbb{Z}_p$-local system on the rigid generic fibre X of $\mathfrak{X}$, which we denote by $\mathbb{L}$. On the other hand, for $\mathfrak{X}$ a smooth scheme, if $\mathfrak{X}$ is proper then, this covering is finite and algebraic and we obtain an étale $\mathbb{Z}_p$-local system $\mathbb{L}$ on $X = \mathfrak{X} \otimes_{O_F} F$, or if $\mathfrak{X}$ is an open subscheme of a proper semistable scheme $\mathfrak{Y}$ over O_F to which $\mathcal{M}$ extends, i.e., there exists a (log) Fontaine–Laffaille module $\mathcal{N}$ over $\mathfrak{Y}$ (in the sense of [Tsuji 1996, Definition 2.3.6 & Remark 2.3.13]) such that $\mathcal{M} = \mathcal{N}|_{\mathfrak{X}}$, then the étale local system $\mathbb{L}$ on $X = \mathfrak{X} \otimes_{O_F} F$ is again well defined; see [Tsuji 1996, p. 63 & Appendix]. By [loc. cit.], note that, in the case of schemes, the preceding assumptions are necessary to obtain the étale local system $\mathbb{L}$ on X.

8.2. *Fontaine–Messing period map.* Let $\Sigma = \operatorname{Spec} O_F$ or $\Sigma = \operatorname{Spf} O_F$, and let K be a finite extension of F such that $K \cap F^{\mathrm{ur}} = F$. Take $0 \leqslant s \leqslant p-2$ and $r \geqslant s+1$.

8.2.1. *The case of schemes.* Let $\mathfrak{X}$ be a smooth scheme over O_F with $i : \mathfrak{X}_{\kappa,\text{ét}} \to \mathfrak{X}_{\text{ét}}$ and $j : X_{\text{ét}} \to \mathfrak{X}_{\text{ét}}$ the natural morphism of sites. Take $\mathcal{M}$ in $\mathrm{MF}_{[0,s],\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$, and let $\mathbb{L}$ denote the associated $\mathbb{Z}_p$-local system on the generic fibre X (note that, from Remark 8.4, we need to additionally assume that $\mathfrak{X}$ is a proper scheme or an open subscheme of a proper scheme to which $\mathcal{M}$ extends, however, constructions in this subsection are independent of these assumptions). From [Abhinandan 2025, §5.3], the $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ corresponds to a finite locally free filtered crystal in $\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil}, \varphi)$ equipped with a Frobenius structure, and (by abuse of notation) we denote this crystal again by $\mathcal{M}$.

To describe the Fontaine–Messing period map one can almost verbatim adapt the methods from [Tsuji 1996, §5; 1999, §3.1]. One first constructs a local version of the map and then uses hypercoverings to globalise. Below we will describe the technical inputs needed for the construction of the Fontaine–Messing map; for the actual construction the reader should refer to [loc. cit.]. We focus on the local setup first; i.e., let $\mathfrak{X}$ be an affine smooth scheme over O_F. Let $\mathfrak{Y} = \mathfrak{X} \otimes_{O_F} O_K$, and choose an embedding $\mathfrak{Y} \hookrightarrow \mathfrak{Z}$ such that $\mathfrak{Z}$ is an affine smooth scheme over O_F. Then $\mathfrak{Y}$ can be covered by affine étale $\mathfrak{Y}$-schemes $\mathfrak{U} = \operatorname{Spec} A$, with $A = O_K \otimes_{O_F} B$ and B an étale algebra over $O_F[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ such that its p-adic completion $\hat{B}$ satisfies Assumption 2.1. Let Y (resp. U) denote the generic fibre of $\mathfrak{Y}$ (resp. $\mathfrak{U}$), i.e., $Y = \mathfrak{Y} \otimes_{O_K} K$ (resp. $U = \mathfrak{U} \otimes_{O_K} K$).

Remark 8.5. Take A as above, let A^h denote the p-adic henselisation of A, let $\overline{A^h}$ denote the union of finite A^h-subalgebras $S \subset \overline{\operatorname{Fr} A^h}$ such that $S[1/p]$ is étale over $A^h[1/p]$ and set

$$G_{A^h} = \operatorname{Gal}(\overline{A^h}[1/p]/A^h[1/p]).$$

Then, by Elkik's approximation theorem [1973, Corollary p. 579], we have a natural isomorphism of Galois groups $G_{A^h} \simeq G_{\hat{A}}$. Therefore, any discrete $G_{\hat{A}}$-module can be regarded as a locally constant sheaf on the étale site of the generic fibre $U^h = \mathfrak{U}^h \otimes_{O_K} K$, where $\mathfrak{U}^h = \operatorname{Spec} A^h$.

Remark 8.6. Note that we have henselian versions of the fundamental exact sequences in (2-2) and (6-9), where one replaces $\bar{\hat{A}}$ by $\overline{A^h}$ and $G_{\hat{A}}$ with G_{A^h}. In particular, similar to (6-13), one obtains a syntomic complex $\operatorname{Syn}(\overline{A^h}, \mathcal{M}_{\mathfrak{U}}, r)_n$ of discrete G_{A^h}-modules which we denote by $\overline{\mathcal{S}}_n(\mathcal{M}, r)_{\mathfrak{U}}$. Note that from Remark 8.5 the complex of G_{A^h}-modules $\overline{\mathcal{S}}_n(\mathcal{M}, r)_{\mathfrak{U}}$ can be regarded as a complex of locally constant sheaves on $U^h_{\text{ét}}$, and we obtain a morphism

$$\Gamma(\mathfrak{U}, i_* \mathcal{S}_n(\mathcal{M}, r)_{\mathfrak{Y}}) \to \Gamma(U^h, \overline{\mathcal{S}}_n(\mathcal{M}, r)_{\mathfrak{U}})$$

and a natural map

$$\mathrm{R}\Gamma(G_{\hat{A}}, T_{\text{cris}}(\mathcal{M}_{\mathfrak{U}})/p^n(r)) \to \mathrm{R}\Gamma_{\text{ét}}(U^h, \mathbb{L}/p^n(r)_U). \tag{8-1}$$

Using Remarks 8.5 and 8.6 together with the Poincaré Lemma 3.21, the fundamental exact sequence (see (2-2), (6-9) and (6-12)) and (8-1), note that, from the construction in [Tsuji 1996, §5; 1999, §3.1], one obtains a natural morphism in $D^+(\mathfrak{Y}_{\text{ét}}, \mathbb{Z}/p^n)$:

$$\mathcal{S}_n(\mathcal{M}, r)_{\mathfrak{Y}} \to i^* \mathrm{R} j_* \mathbb{L}/p^n(r)'_Y. \tag{8-2}$$

Next, let $\mathfrak{X}$ be a smooth scheme over O_F, set $\mathfrak{Y} = \mathfrak{X} \otimes_{O_F} O_K$ and let Y denote its generic fibre. To globalise the construction above, one considers an étale hypercovering $\mathfrak{U}^\bullet$ of $\mathfrak{X}$ and chooses a morphism of simplicial schemes $i^\bullet : \mathfrak{U}^\bullet \to \mathfrak{Z}^\bullet$ such that, for each $s \in \mathbb{N}$, the morphism i^s is an immersion of schemes, $\mathfrak{Z}^s$ is smooth over O_F and there exist compatible liftings of Frobenius $F_{\mathfrak{Z}^\bullet} := \{F_{\mathfrak{Z}^\bullet_n} : \mathfrak{Z}^\bullet_n \to \mathfrak{Z}^\bullet_n\}$. Then, using the local description above and the theory of hypercoverings, from the construction in [Tsuji 1996, §5; 1999, §3.1], we obtain a natural morphism in $D^+(\mathfrak{Y}_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$ (independent of choices by [loc. cit.]):

$$\alpha^{\text{FM}}_{r,n,\mathfrak{Y}} : \mathcal{S}_n(\mathcal{M}, r)_{\mathfrak{Y}} \to i^* \mathrm{R} j_* \mathbb{L}/p^n(r)'_Y.$$

8.2.2. *The case of formal schemes.* The definition of the Fontaine–Messing period map for p-adic formal schemes follows in a manner similar to that of schemes, with certain key differences which we point out below. Let $\mathfrak{X}$ be a smooth p-adic formal scheme over O_F, and set $\mathfrak{Y} = \mathfrak{X} \otimes_{O_F} O_K$. In this case, an affine étale formal scheme over $\mathfrak{Y}$ can be covered by affine formal schemes $\mathfrak{U} = \operatorname{Spf} S$, with $S = O_K \otimes_{O_F} R$ and R as in Assumption 2.1. For such local models, we consider the p-adically completed version of the Fontaine–Messing period map described in (8-2). Finally, to obtain the global version, one proceeds in exactly the same manner as in the case of schemes (with a hypercovering $(\mathfrak{U}^\bullet, \mathfrak{Z}^\bullet, F_{\mathfrak{Z}^\bullet})$, where each $\mathfrak{U}^s$ is of the form described above).

Remark 8.7. Note that, in the cyclotomic case, i.e., $K = F(\zeta_{p^m})$ for $m \in \mathbb{N}$, the map described in (8-2) coincides with the composition of the map $\tilde{\alpha}^{\mathrm{FM}}_{r,n,S}$ described in Section 6.7 with the quasi-isomorphism

$$C(G_S, T/p^n(r)') \xrightarrow{\sim} \mathrm{R}\Gamma_{\text{ét}}(U, \mathbb{L}/p^n(r)')$$

obtained by applying the $K(\pi, 1)$-lemma for p-coefficients; see [Colmez and Nizioł 2017, §5.4.1; Scholze 2013, Theorem 4.9].

8.3. ***A global result.*** To state the main global result, let $\mathfrak{X}$ be a smooth (p-adic formal) scheme defined over O_F (for $\mathfrak{X}$ a scheme, assume that it is proper or an open subscheme of a proper semistable scheme defined over O_F). Let $\mathcal{M}$ be an object of the category $\mathrm{MF}_{[0,s],\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$, i.e., a relative Fontaine–Laffaille module of level $[0, s]$ for $0 \leqslant s \leqslant p-2$ (for $\mathfrak{X}$ an open scheme, further assume that $\mathcal{M}$ extends to the compactification of $\mathfrak{X}$, see Remark 8.4). Let $\mathbb{L}$ denote the associated $\mathbb{Z}_p$-local system on the (rigid) generic fibre X of $\mathfrak{X}$. Then, we show the following.

Theorem 8.8. *For $r \geqslant s+1$ and $0 \leqslant k \leqslant r-s-1$, the Fontaine–Messing period map*

$$\alpha^{\mathrm{FM}}_{r,n,\mathfrak{X}} : \mathcal{H}^k(\mathscr{S}_n(\mathcal{M}, r)_{\mathfrak{X}}) \to i^*\mathrm{R}^k j_* \mathbb{L}/p^n(r)'_X \tag{8-3}$$

is a p^N-isomorphism, where $N = N(p, r, s) \in \mathbb{N}$ depends on p, r and s but not on $\mathfrak{X}$ or n.

Proof for schemes. By the definition of the Fontaine–Messing period map in Section 8.2, we see that it is enough to show the p-power quasi-isomorphism locally (provided the power of p does not depend on the local model). Let A be an O_F-algebra such that its p-adic completion $\hat{A}$ satisfies Assumption 2.1, $\mathfrak{U} = \operatorname{Spec} A$ and $M := \mathcal{M}_{\mathfrak{U}}$. Note that we have

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}}, r)_n = \mathrm{Syn}(\hat{A}, M, r)_n \quad \text{and} \quad \mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}}, r) = \mathrm{Syn}(\hat{A}, M, r).$$

The Fontaine–Messing period map

$$\alpha^{\mathrm{FM}}_{r,n,\mathfrak{U}} : \mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}}, r)_n \to \mathrm{R}\Gamma_{\text{ét}}(U^h, \mathbb{L}/p^n(r)'_{U^h})$$

is the same as the composition of the henselian version of the map $\tilde{\alpha}^{\mathrm{FM}}_{r,n}$ with the natural map in (8-1),

$$C(G_{A^h}, T/p^n(r)') \to \mathrm{R}\Gamma_{\text{ét}}(U^h, \mathbb{L}/p^n(r)'_{U^h})$$

(see Remarks 6.18 and 8.7 for the p-adically completed version). Note that the henselian version of the map $\tilde{\alpha}^{\mathrm{FM}}_{r,n}$ is obtained by replacing $\overline{\hat{A}}$ by $\overline{A^h}$ and $G_{\hat{A}}$ with G_{A^h}. We set

$$\mathrm{Syn}(A, M, r) := \mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}}, r).$$

Let $k \leqslant r - s - 1$. Our claim is that the map

$$\alpha^{\mathrm{FM}}_{r,n,A} : H^k(\mathrm{Syn}(A, M, r)_n) \xrightarrow{\tilde{\alpha}^{\mathrm{FM}}_{r,n}} H^k(G_{A^h}, T/p^n(r)') \to H^k(U^h_{\text{ét}}, \mathbb{L}/p^n(r)'_{U^h})$$

is an isomorphism (up to some power of p). To show the claim, we will pass to the p-adic completion of A. Let $\mathcal{U} := \mathrm{Sp}(\hat{A}[1/p])$, and consider the following commutative diagram:

$$\begin{array}{ccccc} H^k(\mathrm{Syn}(A, M, r)_n) & \xrightarrow{\tilde{\alpha}^{\mathrm{FM}}_{r,n,A}} & H^k(G_{A^h}, T/p^n(r)') & \longrightarrow & H^k(U^h_{\text{ét}}, \mathbb{L}/p^n(r)'_{U^h}) \\ \| & & \downarrow\wr & & \downarrow\wr \\ H^k(\mathrm{Syn}(\hat{A}, M, r)_n) & \xrightarrow[\sim]{\tilde{\alpha}^{\mathrm{FM}}_{r,n,\hat{A}}} & H^k(G_{\hat{A}}, T/p^n(r)') & \xrightarrow{\sim} & H^k(\mathcal{U}_{\text{ét}}, \mathbb{L}/p^n(r)'_{\mathcal{U}}) \end{array}$$

The middle vertical arrow is an isomorphism because the two Galois groups are equal by Elkik's approximation theorem [1973, Corollary p. 579] (see Remark 8.5). The right vertical arrow is an isomorphism due to Gabber [1994, Theorem 1]. The bottom left horizontal arrow is a p^N-isomorphism for $N = N(p, r, s) \in \mathbb{N}$ as shown in the case of formal schemes below (for $R = \hat{A}$); in particular, the top left horizontal arrow is also a p^N-isomorphism. The bottom right horizontal arrow is an isomorphism by a $K(\pi, 1)$-lemma due to Scholze [2013, Theorem 4.9], and therefore the top right horizontal arrow is also an isomorphism. Hence it follows that the composition of the top two horizontal arrows, i.e., $\alpha^{\mathrm{FM}}_{r,n,A}$ is a p^N-isomorphism. □

Proof for formal schemes. By the definition of the Fontaine–Messing period map in Section 8.2, we see that it is enough to show the p-power quasi-isomorphism locally (provided the power of p does not depend on the local model). Let R be an O_F-algebra satisfying Assumption 2.1, $\mathfrak{U} = \mathrm{Spf}\, R$ and $M := \mathcal{M}_{\mathfrak{U}}$. We have that the Fontaine–Messing period map

$$\alpha^{\mathrm{FM}}_{r,n,R} : H^k(\mathrm{Syn}(R, M, r)_n) \to H^k(G_R, T/p^n(r)') \xrightarrow{\sim} H^k(U_{\text{ét}}, \mathbb{L}/p^n(r)'_U)$$

is the same as the composition of the map $\tilde{\alpha}^{\mathrm{FM}}_{r,n,R}$ (see Remarks 6.18 and 8.7) with the natural isomorphism

$$H^k(G_R, T/p^n(r)') \xrightarrow{\sim} H^k(U_{\text{ét}}, \mathbb{L}/p^n(r)'_U);$$

see the $K(\pi, 1)$-lemma of [Scholze 2013, Theorem 4.9].

Finally, to show the isomorphism in degrees $0 \leqslant k \leqslant r - s - 1$, we use Corollary 6.20 with Example 5.2 (iii) for Fontaine–Laffaille modules. To compute $N = N(p, r, s) \in \mathbb{N}$, we combine the constants obtained in the proof of Theorem 5.5, Corollary 6.20 (i.e., Lemma 6.21 for $e = p(p - 1)$) and Example 5.2 (iii) to obtain that $N = 32r + 14s + 3p(p - 1) + 4$. In particular, N does not depend on n or the local model $\mathfrak{U}$. This allows us to conclude. □

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