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STABLE ODE-TYPE BLOWUP FOR SOME QUASILINEAR WAVE EQUATIONS WITH DERIVATIVE-QUADRATIC NONLINEARITIES

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We prove a constructive stable ODE-type blowup-result for open sets of solutions to a family of quasilinear wave equations in three spatial dimensions. The blowup is driven by a Riccati-type derivative-quadratic semilinear term, and the singularity is more severe than a shock in that the solution itself blows up like the log of the distance to the blowup-time. We assume that the quasilinear terms satisfy certain structural assumptions, which in particular ensure that the "elliptic part" of the wave operator vanishes precisely at the singular points. The initial data are compactly supported and can be small or large in L^∞ in an absolute sense, but we assume that their spatial derivatives satisfy a nonlinear smallness condition relative to the size of the time derivative. The first main idea of the proof is to construct a quasilinear integrating factor, which allows us to reformulate the wave equation as a first-order system whose solutions remain regular, all the way up to the singularity. Using the integrating factor, we construct quasilinear vector fields adapted to the nonlinear flow. The second main idea is to exploit some crucial monotonic terms in various estimates, especially the energy estimates, that feature the integrating factor. The availability of the monotonicity is tied to our size assumptions on the initial data and on the structure of the nonlinear terms. The third main idea is to propagate the relative smallness of the spatial derivatives all the way up to the singularity so that the solution behaves, in many ways, like an ODE solution. As a corollary of our main results, we show that there are quasilinear wave equations that exhibit two distinct kinds of blowup: the formation of shocks for one nontrivial set of data, and ODE-type blowup for another nontrivial set.

1. Introduction

A fundamental issue surrounding the study of solutions to nonlinear hyperbolic PDEs is that singularities can form in finite time, starting from smooth initial data. For a given singularity-forming solution, perhaps

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the most basic question one can ask is whether or not the singularity formation is stable under perturbations of its initial data. Our main result provides a constructive, affirmative answer to this question for some solutions to a class of quasilinear wave equations. Specifically, in three spatial dimensions, we provide a sharp proof of stable ODE-type blowup for solutions corresponding to an *open* set (in a suitable Sobolev topology in which there are no radial weights in the norms) of initial data for a class of quasilinear wave equations that are well-modeled by

$$-\partial_t^2\Phi + \frac{1}{1+\partial_t\Phi}\Delta\Phi = -(\partial_t\Phi)^2. \tag{1.1}$$

It is only for concreteness that we restrict our attention to three spatial dimensions; our approach can be applied to any number of spatial dimensions, with only slight modifications needed. See Sections 1B and 2A for a precise description of the class of equations that we treat, Section 1C1 for a summary of our main results, and Section 7 for the detailed statement of our main theorem.

There are many results on stable breakdown for solutions to wave equations, some of which we review in Section 1D. The "theory" of stable breakdown is quite fragmented in that the techniques that have been employed vary wildly between different classes of equations. In particular, the techniques that have been developed do not seem to apply to the equations that we study in this article. This will become more clear after we describe the main ideas of our proof (see Section 1C) and review prior works on stable breakdown. Although ODE-type blowup is arguably the simplest blowup scenario, there do not seem to be any prior constructive stable blowup-results of this type for scalar wave equations with derivative-quadratic nonlinearities, in any number of spatial dimensions. We mention, however, that in [Rodnianski and Speck 2018b], we proved, using rather different techniques specialized to Einstein's equations, a singularity formation result for Einstein's equations that can be interpreted as a stable ODE-type blowup-result for the first derivatives[1] of a solution to a quasilinear elliptic-hyperbolic system with derivative-quadratic nonlinearities.

Our proof of stable blowup is based on constructing a dynamic integrating factor $\mathcal{I}$, depending on Φ, that allows us to transform the problem of proving singularity formation into the problem of showing that a certain renormalized first-order system (involving Φ and $\mathcal{I}$) has regular solutions that exist for a sufficiently long time. This strategy of "renormalizing" the problem has been employed in other contexts, such as [Rodnianski and Sterbenz 2010; Raphaël and Rodnianski 2012] on stable blowup for equivariant wave maps, [Collot 2018] on the existence of singularity-forming solutions to energy-supercritical semilinear wave equations, and [Alinhac 1999a; 1999b; 2001; Christodoulou 2007; Christodoulou and Miao 2014; Speck 2016; 2019a; 2019b; 2018; Speck et al. 2016; Miao and Yu 2017; Miao 2018; Luk and Speck 2016; 2018] on stable shock formation in multiple spatial dimensions; later we will further describe these works. We stress that the issue of constructing an appropriately renormalized system lies at the heart of the difficulty of understanding the singular dynamics, and that the details behind the renormalization vary considerably between the different works. However, there is a common theme tying together many of

[1] For the solutions studied in [Rodnianski and Speck 2018b], relative to a geometrically defined coordinate system, the second fundamental form of the metric blows up, while the metric components do not; this can be viewed as the blowup of the first derivatives of the metric.

these singularity formation results (and also others that we describe in Section 1D): the idea of studying the dynamics with the help of a modulation parameter, which solves a corresponding modulation equation. Roughly, a modulation equation is an ODE-type equation that is coupled to the PDE of interest (i.e., an ODE with "PDE source terms"), and the role of the modulation parameter is to describe the "singular portion" of the PDE dynamics as being driven by the ODE-type equation (that is, the modulation equation). In the present article, the modulation parameter is $\mathcal{I}$, and it solves a transport equation (see (1C.1)) that allows us, in the renormalized system (see Proposition 2.5), to cancel the singularity-driving Riccati term $-(\partial_t \Phi)^2$ on the right-hand side of (1.1).

As in works cited in the previous paragraph, in the present article, the renormalized system that we construct becomes degenerate as Φ blows up, and for this reason, it is difficult to close the energy estimates. To achieve this, we crucially rely on some hidden monotonicity, which becomes active near the singularity and whose availability is tied to the monotonicity of the background ODE solutions (whose perturbations we study) and to our assumptions on the factor multiplying the Laplacian, e.g., $1/(1+\partial_t \Phi)$ in the model equation (1.1).

For the solutions featured in our main results, Φ itself blows up in L^∞. As of present, there is no known exhaustive classification of the kinds of singularities that can form in general solutions to quasilinear wave equations. Thus, in principle, for a different set of initial data compared to the set treated in our main theorem, other kinds of singularities could form in solutions to the equations under study. We highlight that the blowup of Φ is a much more drastic singularity compared to the formation of a shock, which for wave equations whose quasilinear terms are of the form $f(\partial\Phi)\cdot\partial^2\Phi$ (as in (1.1)), is a particular kind of singularity in which certain second-order derivatives of Φ blow up in L^∞ due to the intersection of a family of characteristic hypersurfaces, while Φ and $\partial\Phi$ remain bounded in L^∞. The blowup of Φ for the solutions under study here is philosophically important because it dashes any hope of uniquely weakly continuing the solution past the singularity, at least in a sense analogous to what might be achievable in the case of shock singularities in multiple spatial dimensions; see Remark 1.5 for further discussion on the blowup of Φ and Section 1D for further discussion on the formation of shock singularities in the context of multiple spatial dimensions and for discussion of the problem of (weakly) continuing[2] the solution past a shock.

There are a variety of singularity formation results for semilinear wave equations (whose nonlinearities satisfy appropriate assumptions) in which solutions exhibit a "universal blowup-profile" near the singularity, which roughly means that the solutions $\Phi(t,\underline{x})$ are asymptotic to $\kappa(t)f(\lambda(t)(\underline{x}-\underline{x}_0))$ as t approaches the blowup-time, where $f=f(\underline{x})$ is the universal profile, $\underline{x}_0$ is the spatial blowup-point, and κ and λ are functions that blow up as the singular time is approached; see Section 1D for a discussion of some of these results. In contrast, our proof does not rely on exhibiting a universal blowup-profile near

[2]The most significant weak continuation result in more than one spatial dimension is Christodoulou's recent solution [2019] of the restricted shock development problem in compressible fluid mechanics, which, roughly speaking, is a local well-posedness result for weak solutions and their corresponding hypersurfaces of discontinuity, starting from the first shock, whose formation from smooth initial conditions was described in detail in his breakthrough work for relativistic fluids [Christodoulou 2007] and in the follow-up work [Christodoulou and Miao 2014] on nonrelativistic compressible fluids. The term "restricted" means that the jump in entropy across the shock hypersurface was ignored. See Section 1D for further discussion.

singularities, and moreover, *we expect that there should not be any such universal blow-up profile for the set of singularity-forming solutions exhibited by our main theorem* (although we do not prove this). Let us briefly explain the reasons behind our expectation. Our proof is based on constructing the dynamic function $\mathcal{I} > 0$ mentioned above, whose evolution equation is coupled to Φ, such that $\mathcal{I}$ and $\mathcal{I}\partial_t \Phi$ remain rather smooth all the way up to the singularity. To prove that a singularity develops, we show that $\mathcal{I}$ vanishes in finite time in a region where $\mathcal{I}\partial_t \Phi$ is strictly positive; this shows that $\partial_t \Phi$ blows up "like $F(\underline{x})/\mathcal{I}(t, \underline{x})$", where the regular function F is positive in regions where $\mathcal{I}$ is small, while the blowup of Φ itself then follows from simple arguments given in Remark 1.5. Our proof allows for the possibility that at the time of first blowup, $\mathcal{I}$ and $\mathcal{I}\partial_t \Phi$ (where $\mathcal{I}\partial_t \Phi = F$ along the constant-time hypersurface of first blowup) could be arbitrary smooth functions of the spatial variables, aside from the facts that $\partial_t \mathcal{I} = -\mathcal{I}\partial_t \Phi$ (see (1C.1)), that $\mathcal{I}\partial_t \Phi$ is nonzero at points where $\mathcal{I}$ vanishes, and that the spatial derivatives of $\mathcal{I}$ and $\mathcal{I}\partial_t \Phi$ are small (the latter two features are consequences of our assumptions on the initial data, which we state in Section 3A); it is because of this *arbitrariness* that a universal blowup-profile is not featured in our proofs. See the last point of Theorem 1.2 and Remark 1.4 for further discussion of these points. We also remark that for related reasons, the proofs of the shock formation results described in Section 1D do not make any reference to universal blowup-profiles; see the next two paragraphs for further discussion.

We expect that for the solutions treated by our main theorem, generically, the constant-time hypersurface of first blowup will feature only isolated singularities; i.e., we expect that generic solutions have the property that the zeros of $\mathcal{I}$ within the constant-time hypersurface of first blowup are isolated.[3] For such isolated singularities, it should be possible to provide a sharper description of the blowup (compared to the description outlined in the previous paragraph) in a neighborhood of each singularity, essentially through Taylor expansions of the smooth functions $\mathcal{I}$ and $\mathcal{I}\partial_t \Phi$ at each blowup-point. Although (for brevity) we do not pursue this issue in the present article, deriving such a sharp description would be an important precursor step to studying the boundary of the maximal development of the initial data; see also Remark 1.3. This style of analysis (involving, in particular, Taylor expansions) was carried out in [Christodoulou 2007, Chapter 15] in three spatial dimensions in his sharp proof of shock formation for the wave equations of irrotational relativistic fluid mechanics, which are quasilinear. In particular, Christodoulou's work yielded a sharp description of a portion of the boundary of the maximal development in a neighborhood of nondegenerate shock singularities, which is a class of shock singularities that includes many that are isolated in the constant-time hypersurface of first blowup; see [Christodoulou 2007, Chapter 15] for the detailed nondegeneracy assumptions made in his study of the boundary of the maximal development. Christodoulou's results substantially extended those of Alinhac [1999a; 1999b; 2001], who, for various quasilinear wave equations in multiple spatial dimensions, gave a precise description of the formation of nondegenerate shock singularities, but without uncovering the structure of the boundary of the maximal development; see Section 1D for further discussion.

The arguments given in [Christodoulou 2007, Chapter 15] suggest (although they do not definitively prove) that there should not be any universal blowup-profile that captures the asymptotics of solutions

[3] We believe this because $\mathcal{I}$, when viewed as a function of the spatial variables at fixed first blowup time, has a minimum wherever it vanishes, and we believe that these minima should generically be nondegenerate critical points.

to the wave equations that he studied near shock singularities, even singularities that are isolated in the constant-time hypersurface of first blowup. The reason is that, much like in the present work, the blowup of the solution was shown in [Christodoulou 2007] to occur precisely along the zero level set of a smooth "dynamically constructed" function,[4] somewhat analogous to $\mathcal{I}$, and the arguments of [Christodoulou 2007, Chapter 15] allowed for the possibility that the function is arbitrary (aside from satisfying Christodoulou's nondegeneracy assumptions, which one might expect to generically hold). It is interesting to contrast this freedom against the rigidity that occurs in the study of blowup for semilinear heat equations with focusing power-law nonlinearities in multiple spatial dimensions, i.e., $\partial_t \Phi = \Delta\Phi + |\Phi|^{p-1}\Phi$ with $p > 1$: there is a classical result [Merle and Zaag 1997] showing that, under suitable assumptions on p, there exists an open set of initial data whose corresponding solutions form an isolated singularity in finite time and exhibit a universal blowup-profile. This work can be viewed as an extension of [Bricmont and Kupiainen 1994], in which the authors constructed (without proving stability) an infinite number of families of solutions containing isolated singularities within the constant-time hypersurface of first blowup such that the solutions exhibit universal blowup-profiles near the singularities, where each family corresponds to a distinct profile.

As a corollary of our main results, we show (see Section 1E) that there are quasilinear wave equations that exhibit *two distinct kinds of blowup*: ODE-type blowup for one nontrivial (but not necessarily open) set of initial data, and the formation of a shock for a different nontrivial set of data. We view this as a parable highlighting two key phenomena that would have to be accounted for in any sufficiently broad theory of singularity formation in solutions to quasilinear wave equations; i.e., in principle, a quasilinear wave equation can admit radically different types of singularity-forming solutions. The phenomenon of distinct types of singularities is well known for certain *semilinear* equations, but this issue has not been substantially explored for quasilinear equations. For example, for the nonlinear Schrödinger equation with a suitable L^2-supercritical semilinear term, there is a stable regime of self-similar blowup [Merle et al. 2010], while other radially symmetric solutions exhibit collapsing ring singularities [Merle et al. 2014]. In Section 1D, we describe some known results for semilinear wave equations exhibiting distinct types of singularity-forming solutions, but we mention already that there can be type-I blowup, in which the L^∞ norm of the solution itself blows up, as well as type-II blowup, in which the solution remains bounded in an appropriate Sobolev norm. We also highlight that in the quasilinear case, the phenomenon of distinct singularity types can be exhibited in the much simpler setting of quasilinear transport equations. For example, the inhomogeneous Burgers equation $\partial_t \Psi + \Psi \partial_x \Psi = \Psi^2$ admits the T-parametrized family of spatially homogeneous singularity-forming solutions $\Psi_{(\mathrm{ODE});T} := (T-t)^{-1}$ as well as solutions that form shocks, i.e., $\partial_x \Psi$ blows up but Ψ remains bounded (at least up to the singularity in $\partial_x \Psi$).

The precise algebraic details of the weight $1/(1+\partial_t \Phi)$ in front of the Laplacian term in (1.1) are not important for our proof. What is important is that the weight decays at an appropriate rate as $\partial_t \Phi \to \infty$, that is, as the singularity forms; see Section 2A for our assumptions on the weight. As we will explain,

[4]The function appearing in [Christodoulou 2007], denoted by μ, measures the inverse density of a family of characteristic hypersurfaces. It is a three-space-dimensional analog of the function, also denoted by μ, that we use in Section 1E to study shock formation in one spatial dimension.

this decay yields a friction-type spacetime integral that is important for closing the energy estimates, and it also helps us to prove that spatial derivative terms remain small relative to the time derivative terms, up to the singularity. The problem of providing a sharp description of blowup for solutions to derivative-quadratic semilinear wave equations, such as $-\partial_t^2\Phi + \Delta\Phi = -(\partial_t\Phi)^2$, remains open, even though [John 1981] showed, via proof by contradiction, that all nontrivial, smooth, compactly supported solutions to the equation $-\partial_t^2\Phi + \Delta\Phi = -(\partial_t\Phi)^2$ in three spatial dimensions blow up in finite time.

Our results show, in part due to the weight in front of the Laplacian, that the spatial-derivative-involving nonlinearities in (1.1) (and the other equations that we study) exhibit a subcritical[5] blowup-rate relative to the pure time derivative terms. However, as we explain below, this *subcritical behavior does not seem detectable relative to the standard partial derivatives* ∂_α; to detect the behavior, we will use a combination of "quasilinear vector field derivatives" $\mathcal{I}\partial_\alpha$ and standard derivatives ∂_α, where $\mathcal{I}$ is the "quasilinear integrating factor" mentioned above. As we have already alluded to above, our proof is based on showing that $\mathcal{I}\partial_\alpha\Phi$ remains bounded up to the singularity and that the singularity formation coincides with the vanishing of $\mathcal{I}$. In total, our approach allows us to treat the equations under study as quasilinear perturbations of the Riccati ODE $\frac{d^2}{dt^2}\Phi = \left(\frac{d}{dt}\Phi\right)^2$. By "perturbation of the Riccati ODE", we mean in particular that the singularity formation is similar to the one that occurs in the following T-parametrized family of ODE solutions to (1.1):

$$\Phi_{(\mathrm{ODE});T}(t) := \ln((T-t)^{-1}), \tag{1.2}$$

where $T \in \mathbb{R}$ is the blowup-time. Our methods are tailored to the quadratic term on the right-hand side of (1.1) in that they do not apply, at least in their current form, to semilinear terms of type $(\partial_t\Phi)^p$ for $p \neq 2$. However, derivative-quadratic terms are of particular interest in view of the fact that they commonly arise in nonlinear field theories (though the derivative-quadratic terms in such theories are often not Riccati-type, like the one featured on the right-hand side of (1.1)).

1A. ***Paper outline.*** • In the remainder of Section 1, we summarize our results, outline their proofs, place our work in context by discussing prior works on the breakdown of solutions, discuss a corollary (see Section 1E) of our main results, and summarize our notation.

• In Section 2, we define the quantities that play a role in our analysis and derive various evolution equations.

• In Section 3, we state our assumptions on the initial data and state bootstrap assumptions that are useful for studying the solution.

• In Section 4, we derive energy identities.

• In Section 5, which is the main section of the article, we derive a priori estimates that in particular yield strict improvements of the bootstrap assumptions.

[5]In contrast, for the semilinear equation $-\partial_t^2\Phi + \Delta\Phi = -(\partial_t\Phi)^2$, our approach suggests, but does not prove, that the blowup-rate for the Laplacian term $\Delta\Phi$ might be critical with respect to the expected blowup-rate for the other two terms in the equation, i.e., that all terms might blow up at the same rate.

- In Section 6, we state a standard local well-posedness result and continuation criteria for the equations under study.
- In Section 7, we prove the main theorem.

1B. ***The class of wave equations under study.*** Our main theorem concerns the following Cauchy problem for a quasilinear wave equation in three spatial dimensions:

$$-\partial_t^2\Phi + \mathscr{W}(\partial_t\Phi)\Delta\Phi = -(\partial_t\Phi)^2, \tag{1B.1a}$$

$$(\partial_t\Phi|_{\Sigma_0}, \partial_1\Phi|_{\Sigma_0}, \partial_2\Phi|_{\Sigma_0}, \partial_3\Phi|_{\Sigma_0}) = (\mathring{\Psi}_0, \mathring{\Psi}_1, \mathring{\Psi}_2, \mathring{\Psi}_3), \tag{1B.1b}$$

where throughout, Σ_t denotes the hypersurface of constant time t. We assume that $\partial_i\mathring{\Psi}_j = \partial_j\mathring{\Psi}_i$ for $i, j = 1, 2, 3$, which by Poincaré's lemma is equivalent to the existence of a function $\mathring{\Phi}$ on $\mathbb{R}^3$ such that $\mathring{\Psi}_i = \partial_i\mathring{\Phi}$ for $i = 1, 2, 3$; see Remark 1.1. Our use of the notation "$\mathring{\Psi}_\alpha$" for the data functions is tied to our use of the "renormalized solutions variables" Ψ_α that we will use in studying solutions; see Definition 2.3.

Remark 1.1 (viewing (1B.1a) as an equation in $\partial\Phi$). Since Φ itself is not featured in (1B.1a) (only its derivatives appear), we only need to prescribe the first derivatives of Φ along Σ_0 (subject to the constraint $\partial_i\mathring{\Psi}_j = \partial_j\mathring{\Psi}_i$ for $i, j = 1, 2, 3$ mentioned above) in order to solve for $\{\partial_\alpha\Phi\}_{\alpha=0,1,2,3}$. This is relevant in that we do not bother to derive estimates for Φ itself (see, however, Remark 1.5).

In (1B.1a), $\Delta := \sum_{a=1}^{3}\partial_a^2$ is the standard Euclidean Laplacian on $\mathbb{R}^3$ and $\mathscr{W} = \mathscr{W}(\partial_t\Phi)$ is a nonlinear "weight function" satisfying certain technical conditions stated below, specifically (2A.1)–(2A.5). Prototypical examples of weights satisfying (2A.1)–(2A.5) are the functions

$$\mathscr{W}(y) = \frac{1}{1+y^M} \quad \text{or} \quad \mathscr{W}(y) = \frac{1}{(1+y)^M}, \tag{1B.2}$$

where $M \geq 1$ is an integer, and the function

$$\mathscr{W}(y) = \exp(-y). \tag{1B.3}$$

1C. ***Rough summary of the results and discussion of the proof.***

1C1. *Rough summary of the results.* We now briefly summarize the main results; see Theorem 7.1 for precise statements.

Theorem 1.2 (stable ODE-type blowup, rough version). *Under suitable assumptions* (*stated in Section 2A*) *on the weight* $\mathscr{W}(\partial_t\Phi)$, *there exists an open set of compactly supported initial data* $\{\mathring{\Psi}_\alpha\}_{\alpha=0,1,2,3}$ (*see* (1B.1b)) *for* (1B.1a), *with* $\mathring{\Psi}_\alpha \in H^5(\mathbb{R}^3)$, *such that the solution blows up in finite time in a manner similar to the ODE solutions* $\Phi_{(\text{ODE});T}$ *from* (1.2). *In particular*, *there exists a time* $0 < T_{(\text{Lifespan})} < \infty$ *such that* $\|\partial_t\Phi\|_{L^\infty(\Sigma_t)}$ *and*[6] $\|\Phi\|_{L^\infty(\Sigma_t)}$ *blow up as* $t \uparrow T_{(\text{Lifespan})}$. *The data functions* $\{\mathring{\Psi}_\alpha\}_{\alpha=0,1,2,3}$ *are allowed to be large or small as measured by a Sobolev norm without radial weights*, *but* $\{\mathring{\Psi}_a\}_{a=1,2,3}$, $\nabla\mathring{\Psi}_0$, *and their spatial derivatives up to top order must satisfy a nonlinear smallness condition relative to*[7] $\max_{\Sigma_0}[\mathring{\Psi}_0]_+$.

[6]More precisely, one can conclude that $\|\Phi\|_{L^\infty(\Sigma_t)}$ blows up at $t = T_{(\text{Lifespan})}$ if the initial datum for Φ itself is prescribed; Remark 1.1.

[7]Here and throughout, $[f]_+ := \max\{f, 0\}$.

Moreover, let the integrating factor $\mathcal{I}$ be the solution to the initial value problem

$$\partial_t \mathcal{I} = -\mathcal{I}\partial_t \Phi, \quad \mathcal{I}|_{\Sigma_0} = 1. \tag{1C.1}$$

Then $\mathcal{I}$, the variables

$$\Psi_\alpha := \mathcal{I}\partial_\alpha \Phi, \tag{1C.2}$$

and their partial derivatives with respect to the Cartesian coordinates remain regular all the way up to time $T_{(\text{Lifespan})}$, except possibly at the top derivative level due to the vanishing of $\mathscr{W}(\partial_t \Phi)$ (which appears as a weight in the energies) as $\partial_t \Phi \uparrow \infty$. Moreover,

Along $\Sigma_{T_{(\text{Lifespan})}}$, the set of points at which a singularity forms is exactly

$$\Sigma^{\text{Blowup}}_{T_{(\text{Lifespan})}} := \{(T_{(\text{Lifespan})}, \underline{x}) \mid \mathcal{I}(T_{(\text{Lifespan})}, \underline{x}) = 0\}.$$

More precisely, there exists a data-dependent function[8] *F on $\mathbb{R}^3$ such that, for any real number $N < 5$, we have $F \in H^N(\mathbb{R}^3)$, such that $\lim_{t \uparrow T_{(\text{Lifespan})}} \|[\mathcal{I}\partial_t \Phi](t, \cdot) - F\|_{H^N(\mathbb{R}^3)} = 0$, and such that*

$$\min_{\Sigma^{\text{Blowup}}_{T_{(\text{Lifespan})}}} F > 0.$$

In particular, in view of (1C.1), *we see that, for spatial points $\underline{x}$ with $(T_{(\text{Lifespan})}, \underline{x}) \in \Sigma^{\text{Blowup}}_{T_{(\text{Lifespan})}}$, $\mathcal{I}(t, \underline{x})$ vanishes* ***linearly***[9] *as $t \uparrow T_{(\text{Lifespan})}$, and $\partial_t \Phi(t, \underline{x})$ blows up like $F(\underline{x})/\mathcal{I}(t, \underline{x})$ as $t \uparrow T_{(\text{Lifespan})}$.*

Remark 1.3 (maximal development). We anticipate that the sharp results of Theorem 1.2 should be useful for obtaining detailed information about the solution not just up to the first singular time, but also up to the boundary of the maximal development.[10] In the context of shock formation for irrotational solutions to the compressible Euler equations, [Christodoulou 2007, Chapter 15] used similar sharp estimates to follow the solution up to boundary. Broadly similar results were obtained in [Merle and Zaag 2012], in which, in the case of one spatial dimension, they gave a sharp description of the boundary of the maximal development for *any* singularity-forming solution to the semilinear focusing wave equation $-\partial_t^2 \Psi + \partial_x^2 \Psi = -|\Psi|^{p-1}\Psi$ with $p > 1$ and showed in particular that characteristic points on the boundary are isolated. In related work [Merle and Zaag 2015], in $n \geq 1$ spatial dimensions, the authors studied the focusing wave equation $-\partial_t^2 \Psi + \Delta\Psi = -|\Psi|^{p-1}\Psi$ with data $(\Psi|_{t=0}, \partial_t \Psi|_{t=0}) \in H^1 \times L^2$ in the subconformal case, i.e., the case $1 < p$ for $n = 1$ and $1 < p < (n+3)/(n-1)$ for $n \geq 2$. The authors showed that a subset[11] of the noncharacteristic portion of the blowup-surface is open, C^1, and stable under perturbations of the data.

[8] In Theorem 7.1, we did not explicitly mention the function F, but the existence of F and its properties follow easily from the results stated in the theorem, in particular from (7.6a) and (7.7).

[9] That is, for $(T_{(\text{Lifespan})}, \underline{x}) \in \Sigma^{\text{Blowup}}_{T_{(\text{Lifespan})}}$, we have $-\infty < \partial_t \mathcal{I}(T_{(\text{Lifespan})}, \underline{x}) < 0$.

[10] The maximal development of the data is, roughly, the largest possible classical solution that is uniquely determined by the data. Readers can consult [Sbierski 2016; Wong 2013] for further discussion.

[11] The subset is the collection of noncharacteristic points on the boundary of the maximal development such that near that point, the singularity-forming solution has the asymptotic profile of a Lorentz transform of a member of a certain family of equilibria.

Remark 1.4 (remarks on the set of points where blowup occurs). Theorem 1.2 shows in particular that at time $T_{(\text{Lifespan})}$, the singularity occurs precisely along the zero level set of the smooth[12] function $\mathcal{I}(T_{(\text{Lifespan})}, \cdot)$ on $\mathbb{R}^3$. Since any closed subset of $\mathbb{R}^3$ can be realized as the 0 level set of a smooth function, Theorem 1.2 allows for the possibility that any compact[13] subset of $\Sigma_{T_{(\text{Lifespan})}}$ could in principle be the set of points at which some initially smooth solution first blows up. We conjecture that this is in fact the case, that is, that, for any time $T_{(\text{Lifespan})} \in \mathbb{R}$ and any compact subset $\mathfrak{K}$ of $\Sigma_{T_{(\text{Lifespan})}}$, there is a solution Ψ to (1B.1a) that is smooth in a slab of the form $[T_{(\text{Lifespan})} - \delta, T_{(\text{Lifespan})}) \times \mathbb{R}^3$ (for some $\delta > 0$) and that blows up along $\{T_{(\text{Lifespan})}\} \times \mathfrak{K}$ (which is a subset of $\Sigma_{T_{(\text{Lifespan})}}$). One might approach proving this conjecture by adopting a "backwards approach", that is, by prescribing smooth initial data (more precisely, initial data belonging to a sufficiently high-order Sobolev space) for $\mathcal{I}$ and Ψ_α along $\Sigma_{T_{(\text{Lifespan})}}$ such that $\mathfrak{K} = \{\underline{x} \in \mathbb{R}^3 \mid \mathcal{I}(T_{(\text{Lifespan})}, \underline{x}) = 0\}$ and[14] $\Psi_0(T_{(\text{Lifespan})}, \underline{x}) > 0$ for $\underline{x} \in \mathfrak{K}$, and then trying to solve (2B.1) and the equations of Proposition 2.5 backwards, that is, for $t < T_{(\text{Lifespan})}$. Such an approach has successfully been employed in related contexts, for example in the construction of singular solutions to the Einstein equations of general relativity [Beyer and LeFloch 2010]. We also recall the aforementioned works [Bricmont and Kupiainen 1994; Merle and Zaag 1997] on focusing semilinear heat equations, in which families of singular solutions (with, however, singularities that are *isolated*) exhibiting prescribed asymptotic behavior were constructed.

Remark 1.5 (the blowup of Φ). We now make some remarks on the blowup of Φ itself since, as we highlighted in Remark 1.1, one does not need to prescribe the initial datum of Φ itself (and since in the rest of the paper we do not assume that initial data for Φ itself are prescribed). If one does prescribe its initial data, then the results of Theorem 7.1 can easily be used to show that Φ itself blows up at time $T_{(\text{Lifespan})}$ (such a result is not stated in Theorem 7.1). To deduce the blowup for Φ, one can first use (1C.1) and the fundamental theorem of calculus to deduce that $\ln \mathcal{I}(t, \underline{x}) + \Phi(t, \underline{x}) = \Phi(0, \underline{x})$, where $\Phi(0, \cdot)$ is a regular function that by assumption satisfies $\|\Phi(0, \cdot)\|_{L^\infty} < \infty$. Since the singularity formation for $\partial_t \Phi$ yielded by Theorem 7.1 coincides with the vanishing of $\mathcal{I}$ for the first time at $t = T_{(\text{Lifespan})}$, it follows that $\lim_{t \uparrow T_{(\text{Lifespan})}} \sup_{s \in [0,t)} \|\Phi\|_{L^\infty(\Sigma_s)} = \infty$, as is claimed in Theorem 1.2.

1C2. *The main ideas behind the proof of Theorem 7.1.* The initial data that we consider are such that the spatial derivatives of Φ up to top order are initially small relative to $\partial_t \Phi$. We also assume that the spatial derivatives of $\partial_t \Phi$ up to top order are initially small. The smallness assumptions that we need to close the proof are nonlinear in nature,[15] for reasons described just below (1C.5); see Section 3C for our precise smallness assumptions and Section 3D for a simple proof that such data exist. In our analysis, we propagate certain aspects of this smallness all the way up to the singularity. As we mentioned earlier, this

[12] More precisely, in view of our assumptions on the initial data, we have $\mathcal{I}(T_{(\text{Lifespan})}, \cdot) - 1 \in H^5(\mathbb{R}^3)$; see (7.6c).

[13] Since we are assuming that the initial data for $\partial_\alpha \Phi$ are compact, it is easy to show that the zero level set of $\mathcal{I}(T_{(\text{Lifespan})}, \cdot)$ is also compact; by virtue of finite speed of propagation for the wave equation satisfied by Ψ, one can show that there exists a compact subset K of $\mathbb{R}^3$ such that if $t \in [0, T_{(\text{Lifespan})}]$ and $\underline{x} \notin K$, then $\mathcal{I}(t, \underline{x}) = 1$.

[14] Note that by (1C.1)–(1C.2), this latter condition implies that $\partial_t \mathcal{I}(T_{(\text{Lifespan})}, \underline{x}) < 0$ for $\underline{x} \in \mathfrak{K}$.

[15] In particular, our smallness assumptions on the data (1B.1b) are *not* generally invariant under rescalings of the form $(\mathring{\Psi}_0, \mathring{\Psi}_1, \mathring{\Psi}_2, \mathring{\Psi}_3) \to \lambda^{-1}(\mathring{\Psi}_0, \mathring{\Psi}_1, \mathring{\Psi}_2, \mathring{\Psi}_3)$ if λ is too large.

allows us to effectively treat (1B.1a) as a perturbation of the Riccati ODE $\frac{d^2}{dt^2}\Phi = \left(\frac{d}{dt}\Phi\right)^2$. We again stress that the vanishing of the coefficient $\mathscr{W}(\partial_t\Phi)$ of the Laplacian term in (1B.1a) as the singularity forms is important for our estimates, in particular for showing that spatial derivative terms remain relatively small.

A key point is that it does not seem possible to follow the solution all the way to the singularity by studying the wave equation in the form (1B.1a). To caricature the situation, let us pretend that the singularity occurs at $t = 1$. Our proof shows, roughly, that, for $k \geq 1$, $\|\partial^k\Phi\|_{L^\infty(\Sigma_t)}$ blows up like $c_k(1-t)^{-k}$, where c_k is a data-dependent constant and ∂^k denotes k-th-order Cartesian coordinate partial derivatives. This means, in particular, that commuting (1B.1a) with more and more spatial derivatives makes the singularity strength of the nonlinear terms worse and worse, which is a serious obstacle to closing nonlinear estimates. For this reason, as our statement of Theorem 1.2 already makes clear, our proof is fundamentally based on the solution to (1C.1), that is, the integrating factor $\mathcal{I}$ solving the transport equation $\partial_t\mathcal{I} = -\mathcal{I}\partial_t\Phi$ with initial conditions $\mathcal{I}|_{\Sigma_0} = 1$. We again stress that the finite-time blowup $\partial_t\Phi \uparrow \infty$ would follow from the finite-time vanishing of $\mathcal{I}$ and thus, to prove that a singularity forms, *we will show that $\mathcal{I}$ vanishes in finite time*. Using $\mathcal{I}$, we are able to transform the wave equation into a "renormalized system" that is equivalent to (1B.1a) up to the singularity. We then analyze the renormalized system and show that the weighted derivatives $\{\Psi_\alpha := \mathcal{I}\partial_\alpha\Phi\}_{\alpha=0,1,2,3}$, $\mathcal{I}$, and the *Cartesian* spatial partial derivatives of these quantities remain bounded, in appropriate norms (some with $\mathcal{I}$ weights), all the way up to the singularity. In particular, our proof relies on a combination of the derivatives $\{\mathcal{I}\partial_\alpha\}_{\alpha=0,1,2,3}$ and $\{\partial_\alpha\}_{\alpha=0,1,2,3}$, where the weighted derivatives $\mathcal{I}\partial_\alpha$ act first. Here and throughout, $\partial_0 = \partial_t$ and $\{\partial_i\}_{i=1,2,3}$ are the standard Cartesian coordinate spatial partial derivatives.

In Proposition 2.5, we derive the renormalized system of equations satisfied by $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$. Here we only note that the system is first-order hyperbolic and that a *seemingly dangerous factor of $\mathcal{I}^{-1}$ appears in the equations* (recall that $\mathcal{I}$ vanishes at the singularity). However, the factor $\mathcal{I}^{-1}$ is multiplied by the weight $\mathscr{W} = \mathscr{W}(\mathcal{I}^{-1}\Psi_0)$ from (1B.1a), and due to our assumptions on $\mathscr{W}$, we are able to show that the product $\mathcal{I}^{-1}\mathscr{W}(\mathcal{I}^{-1}\Psi_0)$ (which is equal to $\mathcal{I}^{-1}\mathscr{W}(\partial_t\Phi)$) remains uniformly bounded up to the singularity. Moreover, the spatial derivatives of the product $\mathcal{I}^{-1}\mathscr{W}(\mathcal{I}^{-1}\Psi_0)$ also are controllable up to the singularity; it is in this sense that the equations satisfied by $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$ can be viewed as a "renormalization" (i.e., a "regularization") of the original problem. The proof (see Lemma 5.6) of these bounds for the product $\mathcal{I}^{-1}\mathscr{W}(\mathcal{I}^{-1}\Psi_0)$ constitutes the most technical analysis of the article and is based on separately treating regions where $\mathcal{I}$ is large and $\mathcal{I}$ is small.

To prove that $\partial_t\Phi$ blows up, we derive, in an appropriate *localized* region of spacetime, a pointwise bound for Ψ_0 of the form $\Psi_0 \gtrsim 1$. In view of the evolution equation (1C.1) for $\mathcal{I}$, we see that such a bound is strong enough to drive $\mathcal{I}$ to 0 in finite time (note that the right-hand side of the evolution equation in (1C.1) can be expressed as $-\Psi_0$). To prove that $\Psi_0 \gtrsim 1$, we of course rely on the size assumptions described in the first paragraph of this subsubsection, which in particular include the assumption that $\Psi_0|_{\Sigma_0} \gtrsim 1$ (in a localized region). If we caricature the situation by assuming the estimate[16] $\Psi_0 \sim \delta$ for some $\delta > 0$, then it follows from the evolution equation for $\mathcal{I}$ that $\mathcal{I} \sim 1 - \delta t$, $\partial_t\Phi \sim (1-\delta t)^{-1}$, $\ln\mathcal{I} + \Phi \sim \text{data}$, and thus $\Phi \sim \ln(1-\delta t)^{-1} + \text{data}$, where data is a smooth function determined by the

[16]Here we use the notation "$A \sim B$" to imprecisely indicate that A is well-approximated by B.

initial data. Note that $\ln(1-\delta t)^{-1}$ is one of the ODE blowup solutions (1.2). It is in this sense that our results yield the stability of ODE-type blowup.

In reality, to close the proof sketch described above, we must overcome several major difficulties. The first is that the blowup-time is not known in advance. However, we are able to make a good approximate guess for it, which is sufficient for closing a bootstrap argument. We now describe what we mean by this. The discussion in the previous paragraph suggests that the (future) blowup-time is approximately $1/\mathring{A}_*$, where $\mathring{A}_* := \max_{\Sigma_0}[\mathring{\Psi}_0]_+$ (where $\mathring{A}_* > 0$ by assumption). Indeed, if we set all spatial derivative terms equal to zero in (1B.1a), then the blowup-time is precisely $1/\mathring{A}_*$. Our main theorem confirms that, for data with small spatial derivatives, the blowup-time is a small perturbation of $1/\mathring{A}_*$. This is conceptually important in that it enables us to use a bootstrap argument in which we only aim to control the solution for times less than $2/\mathring{A}_*$; the factor of 2 gives us a sufficient margin of error to show that the singularity does form, and it allows us, in most cases, to soak factors of $1/\mathring{A}_*$ into the constants "C" in our estimates; see Section 5A for further discussion on our conventions for constants.

The second and main difficulty that we encounter in our proof is that we need to derive energy estimates for $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$ that hold up to the singularity and, at the same time, to control the integrating factor $\mathcal{I}$; most of our work in this paper is towards this goal. Our energies are roughly of the following form, where $V = (V_0, V_1, V_2, V_3)$ should be thought of as some k-th Cartesian spatial derivative of $(\Psi_0, \Psi_1, \Psi_2, \Psi_3)$:

$$\mathbb{E}[V] = \mathbb{E}[V](t) := \int_{\Sigma_t} \left\{ V_0^2 + \sum_{a=1}^{3} \mathscr{W}(\mathcal{I}^{-1}\Psi_0) V_a^2 \right\} d\underline{x}. \tag{1C.3}$$

For the data under study, $\mathbb{E}[V](0)$ is small. Since $\mathcal{I}$ is small near the singularity and Ψ_0 is order-unity, our assumptions on $\mathscr{W}$ imply that the factor $\mathscr{W}(\mathcal{I}^{-1}\Psi_0)$ on the right-hand side of (1C.3) is small near the singularity; i.e., the energy provides only weak control over $\{V_a\}_{a=1,2,3}$. This makes it difficult to control certain terms in the energy identities, which arise from commutator error terms (that are generated upon commuting the evolution equations for $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$ with spatial derivatives) and from the basic integration by parts argument that we use to derive the energy identities. To control the most difficult error integrals, we exploit the following spacetime integral, which also appears in the energy identities (roughly it is generated when ∂_t falls on the weight $\mathscr{W}(\mathcal{I}^{-1}\Psi_0)$ on the right-hand side of (1C.3)):

$$\mathfrak{I}[V](t) := \sum_{a=1}^{3} \int_{s=0}^{t} \int_{\Sigma_s} (\mathcal{I}^{-1}\Psi_0)^2 \mathscr{W}'(\mathcal{I}^{-1}\Psi_0)(V_a)^2 \, d\underline{x}\, ds, \tag{1C.4}$$

where $\mathscr{W}'(y) = \frac{d}{dy}\mathscr{W}(y)$. A good model scenario to keep in mind is the case $\mathscr{W} = 1/(1+\partial_t\Phi)$ in regions where $\partial_t\Phi$ is large (and thus the energy (1C.3) is weak), in which case $\mathscr{W}' = -1/(1+\partial_t\Phi)^2$, and the factor $(\mathcal{I}^{-1}\Psi_0)^2\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)$ on the right-hand side of (1C.4) can be expressed as $-(\partial_t\Phi)^2/(1+\partial_t\Phi)^2$. In view of our assumptions on $\mathscr{W}$, the term $\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)$ has a *quantitatively negative* sign in the difficult regions where $\mathcal{I}$ is small (which is equivalent to the largeness of $\partial_t\Phi$). More precisely, (1C.4) has a *friction-type* sign. This is important because the difficult error integrals mentioned above can be bounded by $\lesssim \varepsilon\mathfrak{I}[V](t)$, where, roughly, ε is the small L^∞ size of the solution's spatial derivatives. For this reason, the integral (1C.4) can be used to absorb the difficult error integrals. In total, this allows us to prove a priori energy estimates, roughly of the form

$$\mathbb{E}[V](t) + \mathfrak{I}[V](t) \leq \text{data} \times C \exp(Ct), \tag{1C.5}$$

where "data" is, roughly, the small size of the spatial derivatives of the initial data. For our proof to close, the right-hand side of (1C.5) must be sufficiently small. Thanks to our bootstrap assumption that $t < 2/\mathring{A}_*$, it suffices to choose that initial data so that data $\times\, C \exp(C/\mathring{A}_*)$ is sufficiently small. This is one example of the *nonlinear smallness* of the spatial derivatives, relative to $\mathring{A}_*$, that we impose to close the proof. In reality, to make this procedure work, we must separately treat regions where $\mathcal{I}$ is small and $\mathcal{I}$ is large; see Proposition 5.8 and its proof for the details. We stress that absorbing the difficult error integrals into the friction integral (1C.4) *is crucial for showing that the energies remain bounded up to the vanishing of* $\mathcal{I}$, which is in turn central for our approach. In the model case $\mathscr{W}(\partial_t \Phi) = 1/(1 + \partial_t \Phi)$, if we had instead tried to directly control the difficult error integrals by the energy, then we would have obtained the inequality $\mathbb{E}[V](t) \leq C \int_{s=0}^{t} \|\partial_t \Phi\|_{L^\infty(\Sigma_s)} \mathbb{E}[V](s)\, ds + \cdots$. Since $\|\partial_t \Phi\|_{L^\infty(\Sigma_s)}$ goes to infinity at a nonintegrable rate[17] as the blowup-time is approached, this would have led to a priori energy estimates allowing for the possibility that the energies blow up at the singularity, which would have completely invalidated our philosophy of obtaining nonsingular estimates for the $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$. We also highlight that the regularity theory of $\mathcal{I}$ is somewhat subtle at top order: our proof requires that we show that $\mathcal{I}$ and Φ have the same degree of differentiability and that the estimates for $\mathcal{I}$ do not involve any dangerous factors of $\mathcal{I}^{-1}$; these features are not immediately apparent from (1C.1).

The circle of ideas tied to the "regularization approach" that we have taken here seems to be new in the context of proving the stability of ODE-type blowup for a quasilinear wave equation. However, our approach has some parallels with the known proofs of stable shock formation in multiple spatial dimensions, which we describe in Section 1D. In those problems, the crux of the proofs also involves quasilinear integrating factors that "hide" the singularity. In shock formation problems, the integrating factor (which, earlier in the introduction, we referred to as a "modulation parameter") is tied to nonlinear geometric optics,[18] and its top-order regularity theory is very difficult, at least in multiple spatial dimensions (much more so than the top-order regularity theory of the integrating factor $\mathcal{I}$ employed in the present article). The proofs of shock formation also crucially rely on friction-type spacetime integrals, in analogy with (1C.4), that are available because the integrating factor has a negative derivative (in an appropriate direction) in regions near the singularity. However, in multidimensional shock formation problems, the top-order energy identities feature dangerous terms, analogous to terms of strength $\mathcal{I}^{-1}$, which lead to a priori energy estimates allowing for the possibility that the high-order energies might blow up like $\mathcal{I}^{-P}$ for some large universal constant P. This makes it difficult to derive the nonsingular estimates at the lower derivative levels, which are central for closing the proof. In contrast, in our work here, the difficult factors of $\mathcal{I}^{-1}$ are always multiplied by the term $\mathscr{W}$, which effectively ameliorates them, making it easier

[17] With a bit of additional effort, Theorem 7.1 could be sharpened to show that $\|\partial_t \Phi\|_{L^\infty(\Sigma_t)}$ blows up like $c/(T_{(\text{Lifespan})} - t)$, where c is a positive data-dependent constant.

[18] In the shock formation problems described in Section 1D, the integrating factor is the inverse foliation density of a family of characteristic hypersurfaces, which are the level sets of an eikonal function. We also encounter the inverse foliation density (which we denote by μ) in Section 1E, where we derive a shock formation result in one spatial dimension.

to close the energy estimates. On the other hand, the singularities that form in the solutions from our main results are much more severe in that Φ and $\partial_t \Phi$ blow up; in contrast, in the shock formation results (see Section 1D) for equations whose principal part is similar to that of (1.1), $|\Phi|$ and $\max_{\alpha=0,1,2,3} |\partial_\alpha \Phi|$ remain bounded up to the singularity, while $\max_{\alpha,\beta=0,1,2,3} |\partial_\alpha \partial_\beta \Phi|$ blows up in finite time. Our approach to proving Theorem 1.2 also has some parallels with Kichenassamy's stable blowup-results [1996] for semilinear wave equations with exponential nonlinearities, but we delay further discussion of this point until the next subsection.

1D. ***Our results in the context of prior breakdown-results.*** There are many prior breakdown-results for solutions to various hyperbolic equations, especially of wave type. Here we give a nonexhaustive account of some of these works, which is meant to give the reader some feel for the kinds of results that are known and how they compare with/contrast against our main results. In particular, we aim to expose how the proof techniques vary considerably between different types of breakdown-results. We separate the results into seven classes.

(1) (proofs of blowup by contradiction) For various hyperbolic systems, there are proofs of blowup by contradiction, based on showing that, for smooth solutions, certain spatially averaged quantities satisfy ordinary differential inequalities that force them to blow up in finite time, contradicting the assumed smoothness. Notable contributions of this type are [John 1979; 1981] on several classes of nonlinear wave equations with signed nonlinearities and Sideris' proof [1984a] of blowup for various hyperbolic systems, for semilinear wave equations in higher dimensions [Sideris 1984b] (which improved upon Kato's result [1980]), and for the compressible Euler equations in three spatial dimensions [Sideris 1985]. See also [Guo and Tahvildar-Zadeh 1999] for similar results in the case of the relativistic Euler and Euler–Maxwell equations. None of these results yield constructive information about the nature of the blowup, nor do they apply to the wave equations under study here.

(2) (blowup for semilinear wave equations with power-law nonlinearities) There are many interesting constructive blowup-results, in various spatial dimensions, for focusing semilinear wave equations of the form $\square_m \Phi = -|\Phi|^{p-1}\Phi$, where $\square_m := -\partial_t^2 + \Delta$ is the wave operator of the Minkowski metric m. A notable difference between these works and our work here is that these works relied on a careful analysis of the spectrum of a linearized operator. We now discuss some specific examples.

In one spatial dimension, sharp results are known about the structure of singularities. For example, [Merle and Zaag 2007] showed that, for $p \in (1, \infty)$, for general initial data, there is a one-parameter family of functions that serve as the blowup-profiles relative to self-similar variables at noncharacteristic points belonging to the boundary of the maximal development; see also Remark 1.3.

In more than one spatial dimension, much less is known. Under the assumption of radial symmetry, Donninger [2010] proved the nonlinear stability of the ODE blowup solutions $\Phi_{(\mathrm{ODE});T} := c_p(T-t)^{-2/(p-1)}$, where[19] $p = 3, 5, 7, \dots$. More precisely, using "similarity coordinates", he proved stability only in the interior of the backward light cone emanating from the singularity. In three spatial dimensions, in the

[19] Actually, for convenience, Donninger [2010] considered the semilinear term $-\Phi^p$. However, as he noted there, his work could be extended to apply to the term $-|\Phi|^{p-1}\Phi$ for $p > 1$.

subcritical cases $p \in (1, 3]$, [Donninger and Schörkhuber 2012] proved an asymptotic stability result for $\Phi_{(\mathrm{ODE});T}$ under radially symmetric perturbations of the data in the energy space, again only in the interior of the backward light cone emanating from the singularity. The result [Donninger and Schörkhuber 2012] sharpened (in the near-ODE case) the works [Merle and Zaag 2003; 2005], which showed that *all* solutions that form a singularity cannot blow up faster than the ODE rate $(T-t)^{-2/(p-1)}$, but which did not yield any information about the profile of the solution near the singularity. Donninger and Schörkhuber [2014] extended their stability results (still within radial symmetry) to the supercritical cases $p > 3$, but they assumed additional regularity on the initial data (which they believed to be essential for closing the proof). In [Donninger and Schörkhuber 2016], the authors proved results similar to those of [Donninger and Schörkhuber 2014], but without the assumption of radial symmetry. See also [Chatzikaleas and Donninger 2019] for extensions of the results of [Donninger and Schörkhuber 2016] to the case of the cubic wave equations in spatial dimensions belonging to the set $\{5, 7, 9, 11, 13\}$. In [Donninger 2017], in three spatial dimensions under the assumption of radial symmetry, Donninger established Strichartz estimates for solutions to wave equations featuring a self-similar potential, and as an application, in the critical case $p = 5$, he showed the stability of the ODE blowup in the interior of the backward light cone emanating from the singularity. In $n \geq 1$ spatial dimensions without symmetry assumptions, under the assumptions $1 < p$ if $n = 1$ and $1 < p < 1 + 4/(n-1)$ if $n \geq 2$, [Merle and Zaag 2016] used similarity coordinates to show that near a set of equilibria, solutions are either nonglobal, converge to 0, or converge to an explicit equilibrium solution. The authors also tied the various possibilities to the nature of the boundary of the maximal development (i.e., whether or not a certain point on the boundary is characteristic can affect which of the possibilities can occur there).

In three spatial dimensions, in the critical case $p = 5$, there are many blowup-results tied to the ground state solution $W(r) := (1 + r^2/3)^{-1/2}$. For solutions with (conserved) energy below that of the ground state, [Kenig and Merle 2008] established a sharp dichotomy showing that solutions blow up in finite time to the past and future if $\|\Phi\|_{\dot{H}^1(\Sigma_0)} > \|W\|_{\dot{H}^1(\Sigma_0)}$, while they exist globally and scatter if $\|\Phi\|_{\dot{H}^1(\Sigma_0)} < \|W\|_{\dot{H}^1(\Sigma_0)}$. For the same equation, the authors of [Krieger et al. 2009] proved the existence of radially symmetric "slow" type-II blowup solutions $\Phi(t, r) = \lambda^{1/2}(t) W(\lambda(t) r) + w(t, r)$, where w is a small error term, $\lambda(t) := t^{-1-\nu}$, $\nu > \frac{1}{2}$, and the singularity occurs at $t = 0$. In this context, a type-II singularity is such that the solution remains uniformly bounded in the energy space (which is critical) up to the time of first blowup. The results were extended to $\nu > 0$ in [Krieger and Schlag 2014]. In [Donninger et al. 2014], the results were extended to cases in which $\lambda(t)$ does not behave like a power law. Hillairet and Raphaël [2012] constructed type-II blowup solutions for the critical focusing wave equation in four spatial dimensions. Jendrej [2017] treated the case of five spatial dimensions. For the radial critical focusing wave equation in three spatial dimensions, [Duyckaerts et al. 2011] yielded that if the blowup-time T is finite and if the quantitative type-II condition $\sup_{t\in[0,T)}\{\|\partial_t\Phi\|^2_{L^2(\Sigma_t)} + \|\nabla\Phi\|^2_{L^2(\Sigma_t)}\} \leq \|\nabla W\|^2_{L^2} + \eta_0$ holds, where W is the ground state and $\eta_0 > 0$ is a small constant, then the blowup asymptotics are of the type exhibited by the solutions constructed in [Krieger et al. 2009]. The results were extended to the nonradial case in three and five spatial dimensions in [Duyckaerts et al. 2012b]. Similar results were obtained in the case of four spatial dimensions in [Côte et al. 2018] in the radial case. In [Duyckaerts et al. 2013], the authors gave a detailed description of the possible large-time behaviors of all finite-energy radial solutions to the focusing critical wave equation in three spatial dimensions, extending the work [Duyckaerts et al. 2012a], where information along a sequence of times was obtained. For $n \in \{3, 4, 5\}$ spatial dimensions, [Jendrej 2016] proved an upper bound for the blowup-rate $\lambda(t)$ for type-II blowup solutions whose asymptotics are $\Phi(t, r) = [\lambda(t)]^{(n-2)/2} W(\lambda(t) r) + w(t, r)$, assuming that w is sufficiently regular. In 11 or more spatial dimensions, [Collot 2018] considered a range of p-values that are energy-supercritical. For each sufficiently large integer ℓ, he constructed a codimension-$(\ell-1)$ Lipschitz manifold of spherically symmetric solutions that blow up like

$$\frac{1}{\lambda^{2/(p-1)}}(t) Q\left(\frac{r}{\lambda(t)}\right),$$

where $Q(r)$ is the ground state profile, $\lambda \sim c(T-t)^{\ell/\alpha}$, and $\alpha > 2$ is a constant that depends on p and the number of spatial dimensions.

(3) (constructive blowup-results for wave maps) There are related blowup-results for some wave maps whose targets admit a nontrivial harmonic map. For example, for the critical case of the wave maps equation $\square_m \Phi = \Phi(|\partial_t \Phi|^2 - |\nabla \Phi|^2)$, where $\Phi : \mathbb{R}^{1+2} \to \mathbb{S}^2$, under the equivariant symmetry assumption $\Phi(t, r, \theta) = (k\theta, \phi(t, r))$, where the first and second entries on the right-hand side are Euler angles parametrizing $\mathbb{S}^2$ and $k \in \mathbb{Z}_+$, there are blowup-results tied to the ground state $Q(r) := 2 \arctan(r^k)$. Rodnianski and Sterbenz [2010] gave a sharp description of *stable* blowup when $k \geq 4$. They showed that (under the symmetry assumptions) there is an open set of data, with energy slightly larger than the ground state, such that the corresponding solutions blow up at a time $T < \infty$. Moreover, the asymptotics can be described as $\phi(t, r) = Q(r/\lambda(t)) + q(t, r)$, where $\lambda(t) \to 0$ as $t \uparrow T$, $\lambda(t) \gtrsim (T-t)/|\ln(T-t)|^{1/4}$, and $(q, \partial_t q)$ is small in $\dot{H}^1 \times L^2$. In particular, Q is the universal blowup-profile. As in the works cited above involving $\lambda(t)$, a key point of the proof is to derive and analyze an appropriate *modulation equation*, that is, the ODE (which is coupled to the PDE) that governs the evolution of $\lambda(t)$. The function λ is somewhat analogous to the integrating factor $\mathcal{I}$ that we use in our work here. Raphaël and Rodnianski [2012] extended the results to all cases $k \geq 1$, proving *stable* blowup with

$$\lambda(t) = c_k(1 + o(1)) \frac{T-t}{|\ln(T-t)|^{1/(2k-2)}}$$

as $t \uparrow T$ in the cases $k \geq 2$, and, in the case $k = 1$, $\lambda(t) = (T-t)\exp(-\sqrt{\ln|T-t|} + O(1))$ as $t \uparrow T$. In [Krieger et al. 2008], in the case $k = 1$, the authors proved the existence of a continuum of related solutions (believed to be nongeneric) exhibiting the blowup-rates $\lambda(t) = (T-t)^{\nu}$, where $\nu > \frac{3}{2}$. The results were extended to $\nu > 1$ in [Gao and Krieger 2015]. In [Côte et al. 2015], in the equivariance class $k = 1$, the authors proved that within the subclass of degree-0 maps (i.e., in radial coordinates (t, r), one assumes $\phi(0, 0) = \phi(0, \infty) = 0$), there exist solutions with energy above but arbitrarily close to twice the energy of the ground state that blow up in finite time. Within the subclass of degree-1 maps (i.e., $\phi(0, 0) = 0$ and $\phi(0, \infty) = \pi$), for maps with energy bigger than that of the ground state but less than three times the energy of the ground state, the authors show that if a singularity forms, then the solution has asymptotics whose blowup-profiles are the same as those from the works [Krieger et al. 2008; Rodnianski and Sterbenz 2010; Raphaël and Rodnianski 2012]. For equivariant wave maps $\Phi : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{S}^2$, in the class $k = 1$, [Shahshahani 2016] proved the existence of a continuum of blowup solutions. In [Donninger 2011; Donninger et al. 2012], in the supercritical context of equivariant wave maps from $\mathbb{R}^{1+3}$ into $\mathbb{S}^3$, the authors proved the stability of self-similar blowup solutions $\phi_T(t, r) := 2\arctan(r/(T - r))$. More precisely, those results relied on some mode stability results that were later proved in [Costin et al. 2016]; see also [Costin et al. 2017] for similar results in a more general context.

(4) (blowup for semilinear wave equations with exponential nonlinearities) For the focusing semilinear wave equation $\Box_m \Phi = -e^{\Phi}$ in three spatial dimensions, [Kichenassamy 1996] proved that the singular solution $\ln(2/t^2)$ is stable under perturbations of the data along the constant-time hypersurface $\{t = -1\}$. Moreover, he showed that the blowup-surface is of the form $\{t = f(x)\}$, where $f(x)$ loses Sobolev regularity compared to the initial data. It would be interesting to see if our main results could be extended to show a similar result for the equations under study here. More precisely, we conjecture that a portion of the blowup-surface (which would be a subset of the boundary of the maximal development mentioned in Remark 1.3) is $\{\mathcal{I} = 0\}$ for the solutions under study here. Kichenassamy's work has one key feature in common with ours: he devised a reformulation of the wave equation in which no singularity was visible, in his case by constructing a singular change of coordinates adapted to the singularity; this is broadly similar to the approach taken by authors who have proved shock formation results, as we describe just below. However, unlike the "forwards approach" that we take in this article, Kichenassamy used a "backwards approach" in which he first solved problems in which the singularity was *prescribed* along blowup-surfaces and then showed that the map from the singularity to the Cauchy data along $\{t = -1\}$ is invertible; see also Remark 1.4 concerning backwards approaches. His proof of the invertibility of the map from the singularity data to the Cauchy data was based on studying appropriately linearized versions of the equations and on using Fuchsian techniques. The linearized equations exhibited derivative loss, and Kichenassamy used a Nash–Moser approach to handle the loss.

(5) (shock formation for quasilinear equations) Roughly speaking, a shock singularity is a "mild" type of singularity such that the solution remains bounded but one of its derivatives blows up. More precisely, shock singularities are tied to the intersection of a family of characteristics (which, in one spatial dimension, are curves), and the blowup occurs only for derivatives of the solution in directions transversal to the characteristics. There are many classical shock formation results in one spatial dimension, based on the method of characteristics, with important contributions coming from [Riemann 1860; Lax 1964; 1972; 1973; John 1974], among others. Even in one spatial dimension, the field is still active, as is evidenced by the recent work [Christodoulou and Perez 2016], which significantly sharpened [John 1974], giving a complete description of shock formation for electromagnetic plane waves in nonlinear crystals.

Shock singularities are of interest in particular because, at least for systems with suitable structure, there is hope of uniquely continuing the solution past the shock in a weak sense, subject to appropriate selection criteria (typically tied to Rankine–Hugoniot jump conditions across the shock curve, that is, the curve across which the solution exhibits a jump discontinuity). In one spatial dimension, there is an adequate rigorous mathematical theory, at least for strictly hyperbolic quasilinear PDE systems, that is able to accommodate the formation of shocks, the evolution of the solution after shocks, and subsequent interactions of the shocks. We stress that the one-dimensional theory fundamentally relies on the availability of PDE estimates in the space of functions of bounded variation. We refer readers to the comprehensive work [Dafermos 2000] for a detailed discussion of the theory of shock waves in one spatial dimension.

We now turn our attention to the case of more than one spatial dimension. As of the present, in more than one spatial dimension, there is no broad, rigorous well-posedness theory for solutions to quasilinear hyperbolic PDE systems that is able to accommodate the formation of shocks, the evolution of the solution after shocks, and subsequent interactions of the shocks. The main difficulty is that, in view of the fundamental result [Rauch 1986], bounded variation estimates typically fail for quasilinear hyperbolic systems. This prevents one from being able to directly extend the theory described in the previous paragraph to the case of multiple spatial dimensions. For this reason, in multiple spatial dimensions, it seems that one is forced to work with Sobolev spaces and to derive energy estimates up to top order, a task that has proven to be exceptionally difficult in the neighborhood of a shock singularity. Although in multiple spatial dimensions the theory of what happens *after* shocks form is in its infancy,[20] the rigorous theory of the *formation* of a shock, starting from smooth initial conditions, has undergone dramatic advancements in recent years. We refer readers to the survey article [Holzegel et al. 2016] for discussion concerning the history of the subject and for an overview of some recent shock formation results in the context of quasilinear wave equations in three spatial dimensions.

In more than one spatial dimension, the basic picture of what a shock singularity is remains unchanged from the case of one spatial dimension: it is a singularity such that the solution remains bounded but one of its derivatives blows up, and the blowup is tied to the intersection of a family of characteristic hypersurfaces, as in the case of one spatial dimension. In the case of more than one spatial dimension, the characteristic hypersurfaces are levels sets of a solution u to the eikonal equation, which is a (typically) fully nonlinear transport-type equation in u whose coefficients depend on the solution to the original PDE of interest; see the next paragraph for further discussion on eikonal functions in the context of quasilinear wave equations. The use of eikonal functions to analyze solutions to quasilinear hyperbolic PDEs is tantamount to the implementation of nonlinear geometric optics; again, we refer the reader to [Holzegel et al. 2016] for further discussion of eikonal functions and their role in the study of quasilinear wave equations.

We now summarize some important results in the theory of shock formation in more than one spatial dimension. Alinhac [1999a; 1999b; 2001] obtained the first results on shock formation without symmetry assumptions in more than one spatial dimension. The main new difficulty compared to the case of one spatial dimension is that the method of characteristics (implemented via eikonal functions) must be supplemented with energy estimates, which leads to enormous technical complications. Alinhac's work

[20] We mention, however, that Majda [1981; 1983a; 1983b] has solved, in appropriate Sobolev spaces, the *shock front problem*. That is, he proved a local existence result starting from an initial discontinuity given across a smooth hypersurface contained in the Cauchy hypersurface. The data must satisfy suitable jump conditions, entropy conditions, and higher-order compatibility conditions. This is different than [Christodoulou 2019] on the restricted shock development problem, which we describe below. A key difference is that [Christodoulou 2007; 2019] together describe the *emergence* of the shock hypersurface from an initially smooth solution, whereas Majda *prescribed* the initial singularity.

applied to small-data solutions to a class of scalar quasilinear wave equations of the form

$$(g^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = 0 \tag{1D.1}$$

that fail to satisfy the null condition. In (1D.1), $g = g(\partial\Phi)$ is a Lorentzian metric that depends on the solution. Alinhac showed that, for a set of "nondegenerate" small data, Φ and $\partial\Phi$ remain bounded, while $\partial^2\Phi$ blows up in finite time due to the intersection of the characteristics. As we alluded to above, his proof fundamentally relied on nonlinear geometric optics, that is, on an eikonal function, which in the case of scalar quasilinear wave equations is a solution to the eikonal equation

$$(g^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha u\partial_\beta u = 0, \tag{1D.2}$$

supplemented with appropriate initial data. The level sets of u are characteristic hypersurfaces[21] for (1D.1). As it turns out, the intersection of the level sets of u is tied to the formation of a singularity in the solution to (1D.1), just as in the case of one spatial dimension. Much like in the present work, the main estimates in Alinhac's proof did not concern singularities; the crux of his proof was to construct a system of geometric coordinates, one of which is u, and to prove that relative to them, the solution remains very smooth, except possibly at the very high derivative levels. He then showed that a singularity forms in the standard second-order derivatives $\partial^2\Phi$ as a consequence of a finite-time degeneracy between the geometric coordinates and the standard coordinates; roughly, the level sets of u intersect and cause the blowup, much like in the classical example of the blowup of solutions to Burgers' equation. The main challenge in the proof is that to derive energy estimates relative to the geometric coordinates, one must control the eikonal function, whose top-order regularity properties are difficult to obtain; naive estimates lead to the loss of a derivative.

The intricate regularity properties of eikonal functions had previously been understood in certain problems for quasilinear wave equations in which singularities did not form. For example, the first quasilinear wave problem in which the regularity properties of eikonal functions were fully exploited was the celebrated proof [Christodoulou and Klainerman 1993] of the stability of Minkowski spacetime as a solution to the Einstein vacuum equations. Eikonal functions have also played a central role in proofs of low-regularity well-posedness for quasilinear wave equations [Klainerman and Rodnianski 2003; 2005; Smith and Tataru 2005; Klainerman et al. 2015]. However, unlike in these problems, in the problem of shock formation, the top-order geometric energy estimates feature a degenerate weight that vanishes as the shock forms, which leads to a priori estimates allowing for the possibility that the high-order energies might blow up; note that this possible geometric energy blowup is distinct from the formation of the shock, which happens at the low derivative levels relative to the standard coordinates. The "degenerate weight" mentioned above is the inverse foliation density[22] of the level sets of u. The inverse foliation density vanishes when the characteristics intersect, and it is in some ways analogous to the integrating factor $\mathcal{I}$ that we use in our work here. Alinhac closed his singular top-order energy estimates with a Nash–Moser iteration scheme that was adapted to the singularity and that handled the issue of

[21] In the context of wave equations, characteristic hypersurfaces are often referred to as "null hypersurfaces" due to their intimate connection to the Lorentzian notion of a null vector field.

[22] In Section 1E, we encounter the inverse foliation density (we denote it by μ), although we do not need to derive energy estimates in that subsection since we consider only the case of one spatial dimension there.

the regularity theory of u in a different way than [Christodoulou and Klainerman 1993; Klainerman and Rodnianski 2003; 2005; Smith and Tataru 2005; Klainerman et al. 2015]. He then used a "descent scheme" to show that the top-order geometric energy blowup does not propagate down too far to the lower derivative levels. Consequently, the solution remains highly differentiable relative to the geometric coordinates. The solution's high degree of smoothness relative to the geometric coordinates is not just an interesting artifact of the approach, but rather is fundamental to all aspects of the proof.

Due to his reliance on the Nash–Moser iteration scheme, Alinhac's proof applied only to "nondegenerate" initial data such that the first singularity is isolated in the constant-time hypersurface of first blowup, and his framework breaks down precisely at the time of first blowup. For this reason, his approach is inadequate for following the solution to the boundary of the maximal development of the data (see footnote 10), which intersects the future of the first singular time. The breakthrough work [Christodoulou 2007] overcame this drawback and significantly sharpened Alinhac's results for the subset of quasilinear wave equations that arise in the study of irrotational relativistic fluid mechanics. More precisely, Christodoulou's proof yielded a sharp description of the solution up to the boundary of the maximal development. This information was essential even for setting up the shock development problem, which, roughly speaking, is the problem of uniquely extending the solution past the singularity in a weak sense, subject to appropriate jump conditions. We note that the shock development problem in relativistic fluid mechanics was solved in spherical symmetry in [Christodoulou and Lisibach 2016] and, in the recent breakthrough work [Christodoulou 2019] for the nonrelativistic compressible Euler equations and the relativistic Euler equations without symmetry assumptions in a restricted case (known as the restricted shock development problem) such that the jump in entropy across the shock hypersurface is ignored.

The wave equations studied in [Christodoulou 2007] form a subclass of the ones (1D.1) studied by Alinhac. They enjoy special properties that Christodoulou used in his proofs, notably an Euler–Lagrange formulation such that the Lagrangian is invariant under a symmetry group. The main technical improvement afforded by Christodoulou's framework is that in closing the energy estimates, he avoided using a Nash–Moser iteration scheme and instead used an approach similar to the one employed in the aforementioned works [Christodoulou and Klainerman 1993; Klainerman and Rodnianski 2003]. This approach is more robust and is capable of accommodating solutions such that the blowup occurs along a hypersurface, which, in the problem of shock formation, is what typically occurs along a portion of the boundary of the maximal development.[23] Christodoulou's results have since been extended in many directions, including to apply to other wave equations [Christodoulou and Miao 2014; Speck 2016], different sets of initial data [Speck et al. 2016; Miao and Yu 2017; Miao 2018], the compressible Euler equations with nonzero vorticity [Luk and Speck 2016; 2018; Speck 2019b], systems of wave equations with multiple speeds [Speck 2018], and quasilinear systems in which a solution to a transport equation forms a shock [Speck 2019a]. Some of the earlier extensions are explained in detail in the survey article [Holzegel et al. 2016].

(6) (breakdown-results for Einstein's equations) The Einstein equations of general relativity have many remarkable properties and as such, it is not surprising that there are stable breakdown-results that are

[23]Roughly, a portion of the boundary is equal to the zero level set of the inverse foliation density.

specialized to these equations. Here we simply highlight the following constructive results in three spatial dimensions without symmetry assumptions: Christodoulou's breakthrough results [2009] on the formation of trapped surfaces and the stable singularity formation results [Rodnianski and Speck 2018a; 2018b; Luk 2018]. The work [Rodnianski and Speck 2018b] can be viewed as a stable ODE-type blowup-result for Einstein's equations in which the wave speed became infinite at the singularity. Note that in contrast, for (1B.1a), the wave speed vanishes when $\partial_t \Phi$ blows up.

(7) (finite-time degeneration of hyperbolicity) In [Speck 2017], we studied the wave equations

$$-\partial_t^2 \Psi + (1+\Psi)^P \Delta \Psi = 0$$

in three spatial dimensions for $P = 1, 2$. We showed that there exists an open set of initial data such that Ψ is initially small but $1+\Psi$ vanishes in finite time, corresponding to a breakdown in the hyperbolicity of the equation, but without any singularity forming. The difficult part of the proof is closing the energy estimates in regions where $1+\Psi$ is small. The proof has some features in common with the proof of the main results of this paper. For example, the proof relies on monotonicity tied to the sign of $\partial_t \Psi$ and the small size of $\nabla \Psi$. In particular, this leads to the availability of a friction-type integral in the energy identities, analogous to the one (1C.4), which is crucially important for controlling certain error terms.

1E. ***Different kinds of singularity formation within the same quasilinear hyperbolic system.*** In this subsection, we show that there are quasilinear wave equations, closely related to the wave equation (1B.1a), that can exhibit two distinct kinds of blowup: ODE-type blowup for one set of data, and the formation of shocks for another set. The ODE-type blowup is provided by our main results, so in this subsection, most of the discussion is centered on shock formation. Our discussion is based on ideas and techniques found in [Christodoulou 2007; Speck 2016].

To initiate the discussion, we define

$$\Phi_0 := \partial_t \Phi. \tag{1E.1}$$

For convenience, we will restrict our discussion to the specific weight

$$\mathscr{W} = \frac{1}{1+\partial_t \Phi} = \frac{1}{1+\Phi_0},$$

though similar results hold for any weight that satisfies the assumptions of Section 2A. To proceed, we differentiate (1B.1a) with ∂_t to obtain the following closed equation in Φ_0:

$$\partial_t^2 \Phi_0 - \frac{1}{1+\Phi_0} \Delta \Phi_0 = -\frac{1}{1+\Phi_0} (\partial_t \Phi_0)^2 + \frac{2\Phi_0}{1+\Phi_0} \partial_t \Phi_0 + \frac{3\Phi_0^2}{1+\Phi_0} \partial_t \Phi_0. \tag{1E.2}$$

In the remainder of our discussion of shock formation, we will only consider plane-symmetric solutions, that is, solutions that depend only on t and x^1. Note that in (1E.2), $\Delta = \partial_1^2$ for plane-symmetric solutions.

To study plane-symmetric solutions to (1E.2), we will use the characteristic vector fields

$$L := \partial_t + \frac{1}{\sqrt{1+\Phi_0}} \partial_1, \quad \underline{L} := \partial_t - \frac{1}{\sqrt{1+\Phi_0}} \partial_1. \tag{1E.3}$$

We next define the characteristic coordinate u to be the solution to the following initial value problem for a transport equation:

$$Lu = 0, \quad u|_{\Sigma_0} = 1 - x^1. \tag{1E.4}$$

Above and throughout, $Xf := X^\alpha \partial_\alpha f$ denotes the derivative of the scalar function f in the direction of the vector field X. Our choice of the vector field L in (1E.4) is adapted to the problem of detecting "right-moving" shock formation. Similar results could be proved in the "left-moving" case; one could analyze such solutions by constructing a characteristic coordinate $\underline{u}$ that solves $\underline{L}\underline{u} = 0$. We view u as a new coordinate adapted to the characteristics, and we will use the "geometric" coordinate system (t, u) when analyzing solutions. In particular, the level sets of u are characteristic for (1E.2). In fact, it is not difficult to see that as a consequence of (1E.4), u is a solution to the eikonal equation (1D.2), where the inverse metric $(g^{-1})^{\alpha\beta}$ is determined by the principal-order coefficients in the wave equation (1E.2).

We now define $\Sigma_t^{u'}$, relative to the geometric coordinates, to be the subset $\Sigma_t^{u'} := \{(t, u) \mid 0 \le u \le u'\}$. Note that Σ_0^1 can be identified with an orientation-reversed version of the unit x^1 interval [0, 1]. Associated to u, we define the *inverse foliation density* $\mu > 0$ by

$$\mu := \frac{1}{\partial_t u}. \tag{1E.5}$$

Then $1/\mu$ is a measure of the density of the level sets of u, and $\mu = 0$ corresponds to the intersection of the characteristics, that is, the formation of a shock. From (1E.4), it follows that $\mu|_{\Sigma_0} = \sqrt{1 + \Phi_0} = 1 + \mathcal{O}(\Phi_0)$ (for Φ_0 small). One can check that from the above definitions, we have, in addition to (1E.4), the identities $Lt = 1$, $\mu\underline{L}t = \mu$, and $\mu\underline{L}u = 2$. In particular, $L = \frac{d}{dt}$ along the integral curves of L and $\mu\underline{L} = 2\frac{d}{du}$ along the integral curves of $\mu\underline{L}$.

Next, we differentiate (1B.1a) and (1E.4) with ∂_t and carry out tedious but straightforward calculations to obtain the following system in Φ_0 and μ:

$$L(\mu\underline{L}\Phi_0) = -\frac{1}{2(1+\Phi_0)}(L\Phi_0)(\mu\underline{L}\Phi_0) + \mu\frac{\Phi_0}{1+\Phi_0}\left\{1+\tfrac{3}{2}\Phi_0\right\}L\Phi_0 + \frac{\Phi_0}{1+\Phi_0}\left\{1+\tfrac{3}{2}\Phi_0\right\}(\mu\underline{L}\Phi_0), \tag{1E.6a}$$

$$\begin{aligned}\mu\underline{L}L\Phi_0 = &-\frac{\mu}{4(1+\Phi_0)}(L\Phi_0)^2 - \frac{3}{4(1+\Phi_0)}(L\Phi_0)(\mu\underline{L}\Phi_0)\\ &+\mu\frac{\Phi_0}{1+\Phi_0}\left\{1+\tfrac{3}{2}\Phi_0\right\}L\Phi_0 + \frac{\Phi_0}{1+\Phi_0}\left\{1+\tfrac{3}{2}\Phi_0\right\}(\mu\underline{L}\Phi_0),\end{aligned} \tag{1E.6b}$$

$$L\mu = \frac{1}{4(1+\Phi_0)}\mu L\Phi_0 + \frac{1}{4(1+\Phi_0)}(\mu\underline{L}\Phi_0). \tag{1E.6c}$$

For convenience, we will prove shock formation only for a restricted class of initial data supported in Σ_0^1; as can easily be inferred from our proof, the shock-forming solutions that we will construct are stable under plane-symmetric perturbations, and our approach could be applied to a much larger set of plane-symmetric initial data. Specifically, we will prove shock formation for solutions corresponding to initial data such that

$$\sup_{\Sigma_0^1} |\Phi_0| \le \varepsilon, \quad L\Phi_0|_{\Sigma_0} = 0, \quad \sup_{\Sigma_0^1} |\underline{L}\Phi_0| = 4, \tag{1E.7}$$

such that $\underline{L}\Phi_0|_{\Sigma_0^1}$ is *negative* at some maximum of $|\underline{L}\Phi_0|$ on Σ_0^1, and such that ε is small. The negativity of $\underline{L}\Phi_0$ will drive the vanishing of μ. To show the existence of such data, it is convenient to refer to the Cartesian coordinate x^1. Specifically, we fix a smooth nontrivial function $f = f(x^1)$ supported in Σ_0^1 and set $\Phi_0|_{\Sigma_0^1}(x^1) := \kappa f(\lambda x^1)$, where κ and λ are real parameters. Note that $\partial_1\Phi_0|_{\Sigma_0}(x^1) = \kappa\lambda f'(\lambda x^1)$. We then set

$$\partial_t\Phi_0|_{\Sigma_0^1} := -\frac{1}{\sqrt{1+\Phi_0|_{\Sigma_0^1}}}\partial_1\Phi_0|_{\Sigma_0^1},$$

which implies that $L\Phi_0|_{\Sigma_0^1} = 0$ and

$$\underline{L}\Phi_0|_{\Sigma_0^1}(x^1) = -2\kappa\lambda\frac{1}{\sqrt{1+\kappa f(\lambda x^1)}}f'(\lambda x^1).$$

We now choose $|\kappa|$ sufficiently small and λ sufficiently large, which allows us to achieve (1E.7) with $\varepsilon > 0$ arbitrarily small. Moreover, by adjusting the sign of κ, we can ensure that $\underline{L}\Phi_0|_{\Sigma_0^1}$ is negative at some maximum of $|\underline{L}\Phi_0|$ on Σ_0^1. We also note that from domain of dependence considerations, it follows that in terms of the geometric coordinates, solutions with data supported in Σ_0^1 vanish when $u \le 0$, and that the level set $\{u = 0\}$ can be described in Cartesian coordinates as $\{(t, x^1) \mid 1 - x^1 + t = 0\}$.

To derive estimates, we make the following bootstrap assumptions on any region of classical existence such that $0 \le t \le 2$ and $0 \le u \le 1$:

$$0 < \mu \le 3, \quad |\Phi_0| \le \sqrt{\varepsilon}, \quad |L\Phi_0| \le \sqrt{\varepsilon}, \quad |\underline{L}\Phi_0| \le 5. \tag{1E.8}$$

Note also that the solution satisfies $\Phi_0(t, u = 0) = 0$ and that the assumptions (1E.8) are consistent with the initial data when ε is small.

We now derive estimates. In the rest of the subsection, we will silently assume that $\varepsilon > 0$ is sufficiently small. We define

$$Q(t,u) := \sup_{(t',u')\in[0,t]\times[0,u]} \{|\Phi_0|(t',u') + |L\Phi_0|(t',u')\}. \tag{1E.9}$$

Note that $Q(0,u) \lesssim \varepsilon$, while our data-support assumptions and finite speed of propagation imply that $Q(t, 0) = 0$. Using the evolution equation (1E.6b), the bootstrap assumptions, the fact that $L = \frac{d}{dt}$ along the integral curves of L, and the fact that $\mu\underline{L} = 2\frac{d}{du}$ along the integral curves of $\mu\underline{L}$, we deduce $Q(t,u) \le CQ(0,u) + c\int_{t'=0}^{t} Q(t',u)\,dt' + c\int_{u'=0}^{u} Q(t,u')\,du'$. From this estimate and Gronwall's inequality (in two variables), we deduce that there are constants $C > 0$ and $c' > c$ such that, for $0 \le t \le 2$ and $0 \le u \le 1$, we have

$$Q(t,u) \le CQ(0,u)e^{c't}e^{c'u} \le CQ(0,u)e^{3c'} \le Ce^{3c'}\varepsilon \lesssim \varepsilon. \tag{1E.10}$$

Using the estimate (1E.10) and the bootstrap assumptions for μ and $\mu\underline{L}\Phi_0$ to control the terms on the right-hand side of (1E.6a), we deduce $|L(\mu\underline{L}\Phi_0)| \lesssim \varepsilon$. Integrating this estimate along the integral curves of L and using that $\mu(0,u) = 1 + \mathcal{O}(\varepsilon)$, we find that, for $0 \le t \le 2$ and $0 \le u \le 1$, we have $[\mu\underline{L}\Phi_0](t,u) = [\mu\underline{L}\Phi_0](0,u) + \mathcal{O}(\varepsilon) = \underline{L}\Phi_0(0,u) + \mathcal{O}(\varepsilon)$. Inserting this information into (1E.6c) and using (1E.10), we deduce $L\mu = \frac{1}{4}\underline{L}\Phi_0(0,u) + \mathcal{O}(\varepsilon)$. Integrating in time and using the initial condition $\mu(0,u) = 1 + \mathcal{O}(\varepsilon)$, we deduce that $\mu(t,u) = 1 + \frac{1}{4}\underline{L}\Phi_0(0,u)t + \mathcal{O}(\varepsilon) = 1 + \frac{1}{4}[\mu\underline{L}\Phi_0](t,u)t + \mathcal{O}(\varepsilon)$.

We now note that if ε is sufficiently small, then the above estimates yield strict improvements of the bootstrap assumptions (1E.8). By a standard continuity argument in t and u, this justifies the bootstrap assumptions and shows that the solution exists on regions of the form $0 \le t \le 2$ and $0 \le u \le 1$, as long as μ remains positive; the positivity of μ and the above estimates guarantee that $|\Phi_0| + \max_{\alpha=0,1} |\partial_\alpha \Phi_0|$ is finite. Moreover, since (by construction) $\sup_{\Sigma_0^1} |\underline{L}\Phi_0| = 4$ and since there is a value $u_* \in (0, 1)$ such that $\underline{L}\Phi_0(0, u_*) = -4$, the above estimates for $\mu\underline{L}\Phi_0$ and μ guarantee that $\min_{\Sigma_t^1} \mu = 1 + \mathcal{O}(\varepsilon) - t$ and that at points $(t, u) \in [0, 2] \times [0, 1]$ of classical existence with $\mu(t, u) \le \frac{1}{4}$, we have $\mu\underline{L}\Phi_0(t, u) \le -1$. It follows that $\min_{\Sigma_t^1} \mu$ cannot remain positive for times larger than $1 + \mathcal{O}(\varepsilon)$ and that

$$\min_{\Sigma_t^1} \mu \le \tfrac{1}{4} \quad \Longrightarrow \quad \sup_{\Sigma_t^1} |\underline{L}\Phi_0| \ge \frac{1}{\min_{\Sigma_t^1} \mu}.$$

In total, these arguments yield that $\sup_{\Sigma_t^1} |\underline{L}\Phi_0|$ blows up at some time $t_{(\text{Shock})} = 1 + \mathcal{O}(\varepsilon)$, while $|\Phi_0|$ and $|L\Phi_0|$ remain uniformly bounded by $\lesssim \varepsilon$. We have thus shown that a shock forms. Readers can consult [Holzegel et al. 2016] for further discussion of the style of proof of shock formation given here and extensions to the case of more than one spatial dimension.

We now revisit the solutions from our main results under the weight $\mathscr{W}(\partial_t\Phi) := 1/(1 + \partial_t\Phi)$. Notice that, for such solutions, Φ_0 also solves (1E.2) but is such that $|\Phi_0|$ blows up at the singularity. This is *different blowup behavior* compared to the shock-forming solutions to (1E.2) constructed above, in which $|\Phi_0|$ remained bounded. Notice also that our main theorem requires, roughly, that $\Phi_0|_{\Sigma_0}$ should not be too small, which is in contrast to the initial data for the shock-forming formation solutions described above. To close this subsection, we clarify that it could be, in principle, that the ODE-type blowup solutions that we have constructed are *unstable* when viewed as solutions to (1E.2), even though they are stable solutions of the original wave equation (1B.1a). The key point is that to solve (1E.2) (viewed as a wave equation for Φ_0), we need to prescribe the data functions $\Phi_0|_{\Sigma_0}$ and $\partial_t\Phi_0|_{\Sigma_0}$, whereas for the ODE-type blowup solutions we have constructed, we can freely prescribe (in plane symmetry) only $\Phi_0|_{\Sigma_0}$; the quantity $\partial_t\Phi_0|_{\Sigma_0}$ is not "free", but rather is uniquely determined from $\Phi_0|_{\Sigma_0}$ and $\partial_1\Phi|_{\Sigma_0}$ via the wave equation (1B.1a). Put differently, the ODE-type blowup solutions that we have constructed yield "special" solutions to (1E.2) that are constrained by the fact that Φ_0 is the time derivative of a solution to the original wave equation (1B.1a). In contrast, we expect that the methods of [Speck et al. 2016] could be used to show that the plane-symmetric shock-forming solutions to (1E.2) that we constructed in this subsection are stable under perturbations that break the plane symmetry.

1F. *Notation.* In this subsection, we summarize some notation that we use throughout.

- $\{x^\alpha\}_{\alpha=0,1,2,3}$ are the standard Cartesian coordinates on $\mathbb{R}^{1+3} = \mathbb{R}\times\mathbb{R}^3$ and $\partial_\alpha := \frac{\partial}{\partial x^\alpha}$ are the corresponding coordinate partial derivative vector fields; $x^0 \in \mathbb{R}$ is the time coordinate and $\underline{x} := (x^1, x^2, x^3) \in \mathbb{R}^3$ are the spatial coordinates.
- We often use the alternative notation $x^0 = t$ and $\partial_0 = \partial_t$.
- $\Sigma_t := \{(t, \underline{x}) \mid \underline{x} \in \mathbb{R}^3\}$ is the standard flat hypersurface of constant time.

- Greek "spacetime" indices such as α vary over 0, 1, 2, 3, while Latin "spatial" indices such as a vary over 1, 2, 3. We use primed indices, such as a', in the same way that we use their nonprimed counterparts. We use Einstein's summation convention in that repeated indices are summed over their respective ranges.
- We sometimes omit the arguments of functions appearing in pointwise inequalities. For example, we sometimes write $|f| \leq C\mathring{\epsilon}$ instead of $|f(t, \underline{x})| \leq C\mathring{\epsilon}$.
- $\nabla^k\Psi$ denotes the array comprising all k-th-order derivatives of Ψ with respect to the Cartesian spatial coordinate vector fields. We often use the alternative notation $\nabla\Psi$ in place of $\nabla^1\Psi$. For example, $\nabla^1\Psi = \nabla\Psi := (\partial_1\Psi, \partial_2\Psi, \partial_3\Psi)$.
- $|\nabla^{\leq k}\Psi| := \sum_{k'=0}^{k} |\nabla^{k'}\Psi|$.
- $|\nabla^{[a,b]}\Psi| := \sum_{k'=a}^{b} |\nabla^{k'}\Psi|$.
- $H^N(\Sigma_t)$ denotes the standard Sobolev space of functions on Σ_t. If $N \geq 0$ is an integer, then the corresponding norm is

$$\|f\|_{H^N(\Sigma_t)} := \left\{ \sum_{a_1+a_2+a_3 \leq N} \int_{\underline{x}\in\mathbb{R}^3} |\partial_1^{a_1}\partial_2^{a_2}\partial_3^{a_3} f(t, \underline{x})|^2 \, d\underline{x} \right\}^{1/2}.$$

In the case $N = 0$, we use the standard notation "L^2" in place of "H^0".

- Above and throughout, $d\underline{x} := dx^1\, dx^2\, dx^3$ denotes standard Lebesgue measure on Σ_t.
- $L^\infty(\Sigma_t)$ denotes the standard Lebesgue space of functions on Σ_t with corresponding norm $\|f\|_{L^\infty(\Sigma_t)} := \operatorname{ess\,sup}_{\underline{x}\in\mathbb{R}^3} |f(t, \underline{x})|$.
- If A and B are two quantities, then we often write $A \lesssim B$ to indicate that "there exists a constant $C > 0$ such that $A \leq CB$".
- We sometimes write $A = \mathcal{O}(B)$ to signify that there exists a constant $C > 0$ such that $|A| \leq C|B|$.
- Explicit and implicit constants are allowed to depend on the data-size parameters $\mathring{A}$ and $\mathring{A}_*^{-1}$ from Section 3A, in a manner that we more fully explain in Section 5A.

2. Mathematical setup and the evolution equations

In this section, we state our assumptions on the nonlinearities, define the quantities that we will study in the rest of the paper, and derive evolution equations.

2A. ***Assumptions on the weight.*** Let $\mathscr{W}$ be the scalar function from (1B.1a). We assume that there are constants $C_k > 0$ such that

$$\mathscr{W}(y) > 0, \quad y \in \left(-\tfrac{1}{2}, \infty\right), \tag{2A.1}$$

$$\mathscr{W}(0) = 1, \tag{2A.2}$$

$$\mathscr{W}'(y) \leq 0, \quad y \in [0, \infty), \tag{2A.3}$$

$$\left| \left\{ (1+y)^2 \frac{d}{dy} \right\}^k [(1+y)\mathscr{W}(y)] \right| \leq C_k, \quad 0 \leq k \leq 5, \; y \in \left(-\tfrac{1}{2}, \infty\right). \tag{2A.4}$$

We also assume that there is a constant $\alpha > 0$ such that

$$\mathscr{W}(y) \le \alpha |\mathscr{W}'(y)|^{1/2}, \quad y \in [1, \infty). \tag{2A.5}$$

Note that (2A.1), (2A.3), and (2A.5) imply in particular that

$$\mathscr{W}'(y) < 0, \quad y \in [1, \infty). \tag{2A.6}$$

2B. *The integrating factor and the renormalized solution variables.*

2B1. *Definitions.* As we described in Section 1C2, our analysis fundamentally relies on the following integrating factor.

Definition 2.1 (the integrating factor). Let Φ be the solution to the wave equation (1B.1a). We define $\mathcal{I} = \mathcal{I}(t, \underline{x})$ to be the solution to the transport equation

$$\partial_t \mathcal{I} = -\mathcal{I} \partial_t \Phi, \quad \mathcal{I}|_{\Sigma_0} = 1. \tag{2B.1}$$

Moreover, we define

$$\mathcal{I}_\star(t) := \min_{\Sigma_t} \mathcal{I}. \tag{2B.2}$$

Remark 2.2 (the vanishing of $\mathcal{I}$ implies singularity formation). It is straightforward to see from (2B.1) that if $\mathcal{I}(T, \underline{x}) = 0$ for some $T > 0$ and for one or more $\underline{x} \in \mathbb{R}^3$, then at such values of $\underline{x}$ we have $\lim_{t \uparrow T} \sup_{s \in [0,t)} \partial_t \Phi(s, \underline{x}) = \infty$. In fact, it follows that $\int_{s=0}^{t} |\partial_t \Phi(s, \underline{x})| \, ds = \infty$.

Most of our effort will go towards analyzing the following "renormalized" solution variables. We will show that they remain regular up to the singularity.

Definition 2.3 (renormalized solution variables). Let Φ be the solution to the wave equation (1B.1a) and let $\mathcal{I}$ be as in Definition 2.1. For $\alpha = 0, 1, 2, 3$, we define

$$\Psi_\alpha := \mathcal{I} \partial_\alpha \Phi. \tag{2B.3}$$

2B2. *A crucial identity for $\mathcal{I}$ and the $\mathcal{I}$-weighted evolution equations.* Our main goal in this subsubsection is to derive evolution equations for the renormalized solution variables; see Proposition 2.5. As a preliminary step, we first provide a lemma that shows that $\partial_i \mathcal{I}$ can be controlled in terms Ψ_i and the initial data, and that no singular factors of $\mathcal{I}^{-1}$ appear in the relationship. Though simple, the lemma is crucial for the top-order regularity theory of $\mathcal{I}$.

Lemma 2.4 (identity for the spatial derivatives of the integrating factor). *The following identity holds for* $i = 1, 2, 3$, *where* $\{\mathring{\Psi}_a\}_{a=1,2,3}$ *are the wave equation initial data from* (1B.1b):

$$\partial_i \mathcal{I} = -\Psi_i + \mathcal{I} \mathring{\Psi}_i. \tag{2B.4}$$

Proof. Dividing (2B.1) by $\mathcal{I}$ and then applying ∂_i, we compute that

$$\partial_t \left\{ \frac{\partial_i \mathcal{I} + \Psi_i}{\mathcal{I}} \right\} = 0. \tag{2B.5}$$

Integrating (2B.5) with respect to time and using the initial conditions $\mathcal{I}|_{\Sigma_0} = 1$ and $\Psi_i|_{\Sigma_0} = \mathring{\Psi}_i$, we arrive at (2B.4). □

We now derive the main evolution equations that we will study in the remainder of the paper.

Proposition 2.5 (the renormalized first-order system: $\mathcal{I}$-weighted evolution equations). *For solutions to* (1B.1a)–(1B.1b), *the renormalized solution variables of Definition 2.3 satisfy the system*

$$\partial_t \Psi_0 = \mathscr{W}(\mathcal{I}^{-1}\Psi_0)\sum_{a=1}^{3}\partial_a\Psi_a + \mathcal{I}^{-1}\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\sum_{a=1}^{3}(\Psi_a)^2 - \mathscr{W}(\mathcal{I}^{-1}\Psi_0)\sum_{a=1}^{3}\mathring{\Psi}_a\Psi_a, \tag{2B.6a}$$

$$\partial_t \Psi_i = \partial_i \Psi_0 - \mathring{\Psi}_i\Psi_0. \tag{2B.6b}$$

Proof. We first prove (2B.6a). From (1B.1a) and (2B.1), we deduce

$$\partial_t(\mathcal{I}\partial_t\Phi) = \mathcal{I}\mathscr{W}(\partial_t\Phi)\Delta\Phi = \mathscr{W}(\partial_t\Phi)\sum_{a=1}^{3}\partial_a(\mathcal{I}\partial_a\Phi) - \mathscr{W}(\partial_t\Phi)\sum_{a=1}^{3}(\partial_a\mathcal{I})\partial_a\Phi.$$

Using (2B.4) to substitute for $\partial_a\mathcal{I}$ and appealing to Definition 2.3, we arrive at the desired (2B.6a).

To prove (2B.6b), we first use Definition 2.3 and the symmetry property $\partial_t\partial_i\Phi = \partial_i\partial_t\Phi$ to obtain $\partial_t\Psi_i = (\partial_t \ln\mathcal{I})\Psi_i + \partial_i\Psi_0 - (\partial_i \ln\mathcal{I})\Psi_0$. Using (2B.1) to replace $\partial_t \ln\mathcal{I}$ with $-\mathcal{I}^{-1}\Psi_0$ and (2B.4) to replace $-\partial_i \ln\mathcal{I}$ with $\mathcal{I}^{-1}\Psi_i - \mathring{\Psi}_i$, we arrive at (2B.6b). □

3. Assumptions on the initial data and bootstrap assumptions

In this section, we state our size assumptions on the initial data for the wave equation (1B.1a), i.e., for $(\partial_t\Phi|_{\Sigma_0}, \partial_1\Phi|_{\Sigma_0}, \partial_2\Phi|_{\Sigma_0}, \partial_3\Phi|_{\Sigma_0}) = (\mathring{\Psi}_0, \mathring{\Psi}_1, \mathring{\Psi}_2, \mathring{\Psi}_3)$, and formulate bootstrap assumptions that are convenient for studying the solution. We also precisely describe the smallness assumptions that we need to close our estimates and show the existence of initial data that satisfy the smallness assumptions.

3A. ***Assumptions on the initial data.*** We assume that the initial data are compactly supported and satisfy the following size assumptions for $i = 1, 2, 3$:

$$\|\nabla^{\leq 2}\mathring{\Psi}_i\|_{L^\infty(\Sigma_0)} + \|\nabla^{[1,3]}\mathring{\Psi}_0\|_{L^\infty(\Sigma_0)} + \|\mathring{\Psi}_i\|_{H^5(\Sigma_0)} + \mathring{\epsilon}^{3/2}\|\nabla\mathring{\Psi}_0\|_{L^2(\Sigma_0)} + \|\nabla^2\mathring{\Psi}_0\|_{H^3(\Sigma_0)} \leq \mathring{\epsilon}, \tag{3A.1a}$$

$$\|\mathring{\Psi}_0\|_{L^\infty(\Sigma_0)} \leq \mathring{A}, \tag{3A.1b}$$

$$-\tfrac{1}{4} \leq \min_{\Sigma_0}\mathring{\Psi}_0, \tag{3A.1c}$$

where $\mathring{\epsilon} > 0$ and $\mathring{A} > 0$ are two data-size parameters that we will discuss below (roughly, $\mathring{\epsilon}$ will have to be small for our proofs to close). Roughly, in our analysis, we will propagate the above size assumptions during the solution's classical lifespan, except for the top-order spatial derivatives of Ψ_i; we are not able to control these top-order derivatives uniformly in the norm $\|\cdot\|_{L^2(\Sigma_t)}$ because of the presence of the weight $\mathscr{W}$ in our energy, which can go to 0 as the singularity forms (see Definition 4.2).

We now introduce the crucial parameter $\mathring{A}_*$ that controls the time of first blowup; our analysis shows that, for $\mathring{\epsilon}$ sufficiently small, the time of first blowup is $\{1 + \mathcal{O}(\mathring{\epsilon})\}\mathring{A}_*^{-1}$; see also Remark 3.2.

Definition 3.1 (the parameter that controls the time of first blowup). We define the data-dependent parameter $\mathring{A}_*$ as follows:

$$\mathring{A}_* := \max_{\Sigma_0} [\mathring{\Psi}_0]_+, \tag{3A.2}$$

where $[\mathring{\Psi}_0]_+ := \max\{\mathring{\Psi}_0, 0\}$.

Our main results concern solutions such that $\mathring{A}_* > 0$, so we will assume in the rest of the article that this is the case.

Remark 3.2 (the relevance of $\mathring{A}_*$). The solutions that we study are such that[24] $\partial_t \mathcal{I} = -\Psi_0$ and $\partial_t \Psi_0 \sim 0$ (throughout the evolution). Hence, by the fundamental theorem of calculus, we have $\Psi_0(t, \underline{x}) \sim \mathring{\Psi}_0(\underline{x})$ and $\mathcal{I}(t, \underline{x}) \sim 1 - t\mathring{\Psi}_0(\underline{x})$. From this last expression, we see that $\mathcal{I}$ is expected to vanish for the first time at approximately $t = \mathring{A}_*^{-1}$ which, since $\partial_t \mathcal{I} = -\mathcal{I}\partial_t \Phi$, implies the blowup of $\partial_t \Phi$ (see Remark 2.2). See Lemmas 5.1 and 5.2 for the precise statements.

3B. ***Bootstrap assumptions.*** To prove our main results, we find it convenient to rely on a set of bootstrap assumptions, which we provide in this subsection.

- *The size of* $T_{(\mathrm{Boot})}$. We assume that $T_{(\mathrm{Boot})}$ is a bootstrap time with

$$0 < T_{(\mathrm{Boot})} \leq 2\mathring{A}_*^{-1}, \tag{3B.1}$$

where $\mathring{A} > 0$ is the data-size parameter from Definition 3.1. The assumption (3B.1) gives us a sufficient margin of error to prove that finite-time blowup occurs (see Remark 3.2).

- *Blowup has not yet occurred.* Recall that, for the solutions under study, the vanishing of $\mathcal{I}$ will coincide with the formation of a singularity in $\partial_t \Phi$. For this reason, we assume that, for $t \in [0, T_{(\mathrm{Boot})})$, we have

$$\mathcal{I}_\star(t) > 0, \tag{3B.2}$$

where $\mathcal{I}_\star$ is defined in (2B.2).

- *The solution is contained in the regime of hyperbolicity.*[25] We assume that, for $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times \mathbb{R}^3$,

$$\frac{\Psi_0(t, \underline{x})}{\mathcal{I}(t, \underline{x})} > -\frac{1}{2}. \tag{3B.3}$$

- *Smallness of* $\mathcal{I}$ *implies largeness of* Ψ_0. We assume that, for $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times \mathbb{R}^3$,

$$\mathcal{I}(t, \underline{x}) \leq \tfrac{1}{8} \quad \Longrightarrow \quad \Psi_0(t, \underline{x}) \geq \tfrac{1}{8}\mathring{A}_*. \tag{3B.4}$$

- L^∞ *bootstrap assumptions.* We assume that, for $t \in [0, T_{(\mathrm{Boot})})$, we have

$$\|\Psi_0\|_{L^\infty(\Sigma_t)} \leq \mathring{A} + \varepsilon, \tag{3B.5a}$$

$$\|\nabla^{[1,3]}\Psi_0\|_{L^\infty(\Sigma_t)} \leq \varepsilon, \tag{3B.5b}$$

[24] Here "$A \sim B$" imprecisely indicates that A is well-approximated by B.

[25] In particular, the assumptions of Section 2A guarantee that $\mathscr{W}(\mathcal{I}^{-1}\Psi_0) > 0$ whenever (3B.3) holds. This inequality is needed in order to guarantee that (1B.1a) is a wave equation.

$$\|\nabla^{\leq 2}\Psi_i\|_{L^\infty(\Sigma_t)} \leq \varepsilon, \tag{3B.5c}$$

$$\|\mathcal{I}\|_{L^\infty(\Sigma_t)} \leq 1 + 2\mathring{A}_*^{-1}\mathring{A} + \varepsilon, \tag{3B.5d}$$

where $\varepsilon > 0$ is a small bootstrap parameter; we describe our smallness assumptions in the next subsection.

Remark 3.3 (the solution remains compactly supported in space). From the bootstrap assumptions and the assumptions of Section 2A on $\mathscr{W}$, we see that the wave speed $\{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\}^{1/2}$ associated to (1B.1a) remains uniformly bounded from above by a positive constant on the slab $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times \mathbb{R}^3$. It follows that there exists a large, data-dependent ball $B \subset \mathbb{R}^3$ such that $\Psi_\alpha(t, \underline{x})$ and $\mathcal{I} - 1$ vanish for $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times B^c$.

3C. *Smallness assumptions.* For the rest of the article, when we say that "A is small relative to B", we mean that $B > 0$ and that there exists a continuous increasing function $f : (0, \infty) \to (0, \infty)$ such that $A < f(B)$. For brevity, we typically do not specify the form of f.

In the rest of the article, we make the following relative smallness assumptions. We continually adjust the required smallness in order to close the estimates.

- The bootstrap parameter ε from Section 3B is small relative to 1 (i.e., in an absolute sense, without regard for the other parameters).
- ε is small relative to $\mathring{A}^{-1}$, where $\mathring{A}$ is the data-size parameter from (3A.1b).
- ε is small relative to the data-size parameter $\mathring{A}_*$ from (3A.2).
- We assume that

$$\varepsilon^{4/3} \leq \mathring{\epsilon} \leq \varepsilon, \tag{3C.1}$$

where $\mathring{\epsilon}$ is the data-smallness parameter from (3A.1a).

The first two of the above assumptions will allow us to control error terms that, roughly speaking, are of size $\varepsilon\mathring{A}^k$ for some integer $k \geq 0$. The third assumption is relevant because the expected blowup-time is approximately $\mathring{A}_*^{-1}$ (see Remark 3.2); the assumption will allow us to show that various error products, specifically ones featuring a small factor ε, remain small for $t \leq 2\mathring{A}_*^{-1}$, which is plenty of time for us to show that $\mathcal{I}$ vanishes and $\partial_t\Phi$ blows up. (3C.1) is convenient for closing our bootstrap argument.

3D. *Existence of initial data satisfying the smallness assumptions.* It is easy to construct initial data such that the parameters $\mathring{\epsilon}$, $\mathring{A}$, and $\mathring{A}_*$ satisfy the size assumptions stated in Section 3C. For example, we can start with *any* smooth compactly supported data $(\mathring{\Psi}_0, \mathring{\Psi}_1, \mathring{\Psi}_2, \mathring{\Psi}_3)$ such that $\max_{\Sigma_0} \mathring{\Psi}_0 > 0$ and $-\frac{1}{4} \leq \min_{\Sigma_0} \mathring{\Psi}_0$. We then consider the one-parameter family (for $i = 1, 2, 3$)

$$({}^{(\lambda)}\mathring{\Psi}_0(\underline{x}), {}^{(\lambda)}\mathring{\Psi}_i(\underline{x})) := (\mathring{\Psi}_0(\lambda^{-1}\underline{x}), \lambda^{-1}\mathring{\Psi}_i(\underline{x})).$$

It is straightforward to check that, for $\lambda > 0$ sufficiently large, all of the size assumptions of Section 3C are satisfied by the rescaled data (where, roughly speaking, the role of $\mathring{\epsilon}$ is played by λ^{-1}), as is (3A.1c).

The proof relies on the simple scaling identities

$$\begin{aligned}
\nabla^{k^{(\lambda)}}\mathring{\Psi}_0(\underline{x}) &= \lambda^{-k}(\nabla^k\mathring{\Psi}_0)(\lambda^{-1}\underline{x}),\\
\nabla^{k^{(\lambda)}}\mathring{\Psi}_i(\underline{x}) &= \lambda^{-1}(\nabla^k\mathring{\Psi}_i)(\underline{x}),\\
\|\nabla^{k^{(\lambda)}}\mathring{\Psi}_0\|_{L^2(\Sigma_0)} &= \lambda^{3/2-k}\|\nabla^k\mathring{\Psi}_0\|_{L^2(\Sigma_0)},\\
\|\nabla^{k^{(\lambda)}}\mathring{\Psi}_i\|_{L^2(\Sigma_0)} &= \lambda^{-1}\|\nabla^k\mathring{\Psi}_i\|_{L^2(\Sigma_0)}.
\end{aligned}$$

4. Energy identities

In this section, we define the energies that we use to control the solution in L^2 up to top order. We then derive energy identities.

4A. *Definitions.* The following energy functional serves as a building block for our energies.

Definition 4.1 (basic energy functional). To any array-valued function $V = V(t,\underline{x}) := (V_0, V_1, V_2, V_3)$, we associate the following energy, where $\mathcal{I}$ is as in Definition 2.1 and Ψ_0 is as in Definition 2.3:

$$\mathbb{E}[V] = \mathbb{E}[V](t) := \int_{\Sigma_t}\left\{V_0^2 + \sum_{a=1}^{3}\mathscr{W}(\mathcal{I}^{-1}\Psi_0)(V_a)^2\right\}d\underline{x}. \tag{4A.1}$$

We now define $\mathbb{Q}_{(\mathring{\epsilon})}(t)$, which is the main L^2-type quantity that we use to control the solution up to top order.

Definition 4.2 (the L^2-controlling quantity). Let $\mathring{\epsilon} > 0$ be the data-size parameter from Section 3A. We define the L^2-controlling quantity $\mathbb{Q}_{(\mathring{\epsilon})}$ as follows:

$$\begin{aligned}
\mathbb{Q}_{(\mathring{\epsilon})}(t) := \sum_{k=2}^{5}\int_{\Sigma_t}\left\{|\nabla^k\Psi_0|^2 + \sum_{a=1}^{3}\mathscr{W}(\mathcal{I}^{-1}\Psi_0)|\nabla^k\Psi_a|^2\right\}d\underline{x}\\
+\sum_{k=1}^{4}\int_{\Sigma_t}|\nabla^k\Psi_a|^2\,d\underline{x} + \mathring{\epsilon}^3\int_{\Sigma_t}\left\{|\nabla\Psi_0|^2 + \sum_{a=1}^{3}(\Psi_a)^2\right\}d\underline{x}.
\end{aligned}\tag{4A.2}$$

Remark 4.3 (the $\mathring{\epsilon}$-weight in the definition of $\mathbb{Q}_{(\mathring{\epsilon})}$). Our main a priori energy estimate shows that $\mathbb{Q}_{(\mathring{\epsilon})}(t) \lesssim \mathring{\epsilon}^2$ up to the singularity. The small coefficient of $\mathring{\epsilon}^3$ in front of the last integral on the right-hand side of (4A.2) is needed to ensure the $\mathcal{O}(\mathring{\epsilon}^2)$ smallness of $\mathbb{Q}_{(\mathring{\epsilon})}$. However, the small coefficient of $\mathring{\epsilon}^3$ implies that $\mathbb{Q}_{(\mathring{\epsilon})}(t)$ provides only weak L^2-control of $\nabla\Psi_0$ and Ψ_a; i.e., their L^2 norms can be as large as $\mathcal{O}(\mathring{\epsilon}^{-1/2})$. We clarify that the possible $\mathcal{O}(\mathring{\epsilon}^{-1/2})$-largeness of $\nabla\Psi_0$ is consistent with the construction of initial data described in Section 3D, where the largeness comes from the scaling identity

$$\|\nabla^{(\lambda)}\mathring{\Psi}_0\|_{L^2(\Sigma_0)} = \lambda^{1/2}\|\nabla\mathring{\Psi}_0\|_{L^2(\Sigma_0)},$$

and the large parameter λ can be viewed, roughly, as a size-$\mathcal{O}(\mathring{\epsilon}^{-1})$ quantity. Despite the possible $\mathcal{O}(\mathring{\epsilon}^{-1/2})$-largeness of $\nabla\Psi_0$ and Ψ_a in the norm $\|\cdot\|_{L^2(\Sigma_t)}$, we will nonetheless be able to show, through a separate argument, the following crucial bounds: $\nabla\Psi_0$ and Ψ_a are bounded in the norm $\|\cdot\|_{L^\infty(\Sigma_t)}$ by $\lesssim \mathring{\epsilon}$, up to the singularity; see Proposition 5.8.

4B. ***Basic energy identity.*** We aim to derive an energy identity for the L^2-controlling quantity $\mathbb{Q}_{(\mathring{\epsilon})}$ defined in (4A.2). As a preliminary step, in this subsection, we derive a standard energy identity for the building-block energy defined in (4A.1).

Lemma 4.4 (basic energy identity). *Let $\mathcal{I}$ be as in Definition 2.1, and assume that $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$ is a solution to* (2B.6a)–(2B.6b) *with initial data $\{\mathring{\Psi}_\alpha\}_{\alpha=0,1,2,3}$. Let $\mathbb{E}[V](t)$ be the building-block energy defined in* (4A.1)*. Let $\{F_\alpha\}_{\alpha=0,1,2,3}$ be scalar functions. Then for spatially compactly supported solutions $V := (V_0, V_1, V_2, V_3)$ to the inhomogeneous linear PDE system*

$$\partial_t V_0 = \sum_{a=1}^{3} \mathscr{W}(\mathcal{I}^{-1}\Psi_0)\partial_a V_a + F_0, \tag{4B.1a}$$

$$\partial_t V_i = \partial_i V_0 + F_i, \tag{4B.1b}$$

the following energy identity holds:

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}[V](t) = {} & \sum_{a=1}^{3} \int_{\Sigma_t} (\mathcal{I}^{-1}\Psi_0)^2 \mathscr{W}'(\mathcal{I}^{-1}\Psi_0)(V_a)^2 \, d\underline{x} \\
& + \sum_{a=1}^{3}\sum_{b=1}^{3} \int_{\Sigma_t} \mathcal{I}^{-1}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)(\partial_a\Psi_a)(V_b)^2 \, d\underline{x} \\
& + \sum_{a=1}^{3}\sum_{b=1}^{3} \int_{\Sigma_t} \mathcal{I}^{-2}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)(\Psi_a)^2(V_b)^2 \, d\underline{x} \\
& - \sum_{a=1}^{3}\sum_{b=1}^{3} \int_{\Sigma_t} \mathcal{I}^{-1}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\mathring{\Psi}_a\Psi_a(V_b)^2 \, d\underline{x} \\
& - 2\sum_{a=1}^{3} \int_{\Sigma_t} \mathcal{I}^{-1}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)(\partial_a\Psi_0)V_a V_0 \, d\underline{x} \\
& - 2\sum_{a=1}^{3} \int_{\Sigma_t} \mathcal{I}^{-2}\Psi_0\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\Psi_a V_a V_0 \, d\underline{x} \\
& + 2\sum_{a=1}^{3} \int_{\Sigma_t} \mathcal{I}^{-1}\Psi_0\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathring{\Psi}_a V_a V_0 \, d\underline{x} \\
& + 2\int_{\Sigma_t} V_0 F_0 \, d\underline{x} + 2\sum_{a=1}^{3} \int_{\Sigma_t} \mathscr{W}(\mathcal{I}^{-1}\Psi_0) V_a F_a \, d\underline{x}.
\end{aligned} \tag{4B.2}$$

Proof. First, using (2B.1) and (2B.6a), we compute that

$$\begin{aligned}
\partial_t\{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\} &= \mathcal{I}^{-1}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)(\partial_t\Psi_0) + (\mathcal{I}^{-1}\Psi_0)^2\mathscr{W}'(\mathcal{I}^{-1}\Psi_0) \\
&= \sum_{a=1}^{3}\mathcal{I}^{-1}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)(\partial_a\Psi_a) + \sum_{a=1}^{3}\mathcal{I}^{-2}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)(\Psi_a)^2 \\
&\quad - \sum_{a=1}^{3}\mathcal{I}^{-1}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\mathring{\Psi}_a\Psi_a + (\mathcal{I}^{-1}\Psi_0)^2\mathscr{W}'(\mathcal{I}^{-1}\Psi_0).
\end{aligned} \tag{4B.3}$$

Next, taking the time derivative of (4A.1), using (4B.3), and using (4B.1a)–(4B.1b) to substitute for $\partial_t V_\alpha$, we obtain

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}[V](t) &= 2\sum_{a=1}^{3}\int_{\Sigma_t}\{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)V_0\partial_a V_a + \mathscr{W}(\mathcal{I}^{-1}\Psi_0)V_a\partial_a V_0\}\,d\underline{x} \\
&\quad + 2\int_{\Sigma_t}\left\{V_0F_0 + \sum_{a=1}^{3}\mathscr{W}(\mathcal{I}^{-1}\Psi_0)V_aF_a\right\}d\underline{x} \\
&\quad + \sum_{a=1}^{3}\sum_{b=1}^{3}\int_{\Sigma_t}\mathcal{I}^{-1}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)(\partial_a\Psi_a)(V_b)^2\,d\underline{x} \\
&\quad + \sum_{a=1}^{3}\sum_{b=1}^{3}\int_{\Sigma_t}\mathcal{I}^{-2}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)(\Psi_a)^2(V_b)^2\,d\underline{x} \\
&\quad - \sum_{a=1}^{3}\sum_{b=1}^{3}\int_{\Sigma_t}\mathcal{I}^{-1}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\mathring{\Psi}_a\Psi_a(V_b)^2\,d\underline{x} \\
&\quad + \sum_{a=1}^{3}\int_{\Sigma_t}(\mathcal{I}^{-1}\Psi_0)^2\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)(V_a)^2\,d\underline{x}. \qquad \text{(4B.4)}
\end{aligned}$$

Integrating by parts in the first integral on the right-hand side of (4B.4) and using the identity (2B.4), we obtain

$$\begin{aligned}
&2\sum_{a=1}^{3}\int_{\Sigma_t}\{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)V_0\partial_a V_a + \mathscr{W}(\mathcal{I}^{-1}\Psi_0)V_a\partial_a V_0\}\,d\underline{x} \\
&\quad = -2\sum_{a=1}^{3}\int_{\Sigma_t}\mathcal{I}^{-1}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)(\partial_a\Psi_0)V_aV_0\,d\underline{x} \\
&\qquad -2\sum_{a=1}^{3}\int_{\Sigma_t}\mathcal{I}^{-2}\Psi_0\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\Psi_aV_aV_0\,d\underline{x} + 2\sum_{a=1}^{3}\int_{\Sigma_t}\mathcal{I}^{-1}\Psi_0\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathring{\Psi}_aV_aV_0\,d\underline{x}. \qquad \text{(4B.5)}
\end{aligned}$$

Using (4B.5) to substitute for the first integral on the right-hand side of (4B.4), we arrive at (4B.2). □

4C. ***Integral identity for the fundamental L^2-controlling quantity.*** With the help of Lemma 4.4, we now derive an energy identity for the controlling quantity $\mathbb{Q}_{(\mathring{\varepsilon})}$.

Lemma 4.5 (integral identity for the L^2-controlling quantity). *Consider the following inhomogeneous PDE system, obtained by commuting* (2B.6a)–(2B.6b) *with the k-th-order spatial derivative operator* ∇^k (*where $\mathcal{I}$ is as in Definition 2.1 and the precise form of the inhomogeneous terms $F_\alpha^{(k)}$ in* (4C.1a)–(4C.1b) *is not important for the purposes of this lemma*):

$$\partial_t\nabla^k\Psi_0 = \mathscr{W}(\mathcal{I}^{-1}\Psi_0)\sum_{a=1}^{3}\partial_a\nabla^k\Psi_a + F_0^{(k)}, \qquad \text{(4C.1a)}$$

$$\partial_t\nabla^k\Psi_i = \partial_i\nabla^k\Psi_0 + F_i^{(k)}. \qquad \text{(4C.1b)}$$

Then for solutions $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$ *to* (2B.6a)–(2B.6b), *the* L^2*-controlling quantity* $\mathbb{Q}_{(\mathring{\epsilon})}$ *of Definition 4.2 satisfies the following integral identity*:

$$
\begin{aligned}
\mathbb{Q}_{(\mathring{\epsilon})}(t) = \mathbb{Q}_{(\mathring{\epsilon})}(0) &+ \sum_{k=2}^{5}\sum_{a=1}^{3}\int_{s=0}^{t}\int_{\Sigma_s}(\mathcal{I}^{-1}\Psi_0)^2\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)|\nabla^k\Psi_a|^2\,d\underline{x}\,ds \\
&+ \sum_{k=2}^{5}\sum_{a=1}^{3}\sum_{b=1}^{3}\int_{s=0}^{t}\int_{\Sigma_s}\mathcal{I}^{-1}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)(\partial_a\Psi_a)|\nabla^k\Psi_b|^2\,d\underline{x}\,ds \\
&+ \sum_{k=2}^{5}\sum_{a=1}^{3}\sum_{b=1}^{3}\int_{s=0}^{t}\int_{\Sigma_s}\mathcal{I}^{-2}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)(\Psi_a)^2|\nabla^k\Psi_b|^2\,d\underline{x}\,ds \\
&- \sum_{k=2}^{5}\sum_{a=1}^{3}\sum_{b=1}^{3}\int_{s=0}^{t}\int_{\Sigma_s}\mathcal{I}^{-1}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\mathring{\Psi}_a\Psi_a|\nabla^k\Psi_b|^2\,d\underline{x}\,ds \\
&- 2\sum_{k=2}^{5}\sum_{a=1}^{3}\int_{s=0}^{t}\int_{\Sigma_s}\mathcal{I}^{-1}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)(\partial_a\Psi_0)\nabla^k\Psi_a\cdot\nabla^k\Psi_0\,d\underline{x}\,ds \\
&- 2\sum_{k=2}^{5}\sum_{a=1}^{3}\int_{s=0}^{t}\int_{\Sigma_s}\mathcal{I}^{-2}\Psi_0\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\Psi_a\nabla^k\Psi_a\cdot\nabla^k\Psi_0\,d\underline{x}\,ds \\
&+ 2\sum_{k=2}^{5}\sum_{a=1}^{3}\int_{s=0}^{t}\int_{\Sigma_s}\mathcal{I}^{-1}\Psi_0\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathring{\Psi}_a\nabla^k\Psi_a\cdot\nabla^k\Psi_0\,d\underline{x}\,ds \\
&+ 2\sum_{k=1}^{4}\sum_{a=1}^{3}\int_{s=0}^{t}\int_{\Sigma_s}\nabla^k\Psi_a\cdot\partial_a\nabla^k\Psi_0\,d\underline{x}\,ds \\
&+ 2\sum_{k=2}^{5}\int_{s=0}^{t}\int_{\Sigma_s}\nabla^k\Psi_0\cdot F_0^{(k)}\,d\underline{x}\,ds + 2\sum_{k=2}^{5}\sum_{a=1}^{3}\int_{s=0}^{t}\int_{\Sigma_s}\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\nabla^k\Psi_a\cdot F_a^{(k)}\,d\underline{x}\,ds \\
&+ 2\sum_{k=1}^{4}\sum_{a=1}^{3}\int_{s=0}^{t}\int_{\Sigma_s}\nabla^k\Psi_a\cdot F_a^{(k)}\,d\underline{x}\,ds \\
&+ 2\mathring{\epsilon}^3\sum_{a=1}^{3}\int_{s=0}^{t}\int_{\Sigma_s}\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\nabla\Psi_0\cdot\partial_a\nabla\Psi_a\,d\underline{x}\,ds + 2\mathring{\epsilon}^3\int_{s=0}^{t}\int_{\Sigma_s}\nabla\Psi_0\cdot F_0^{(1)}\,d\underline{x}\,ds \\
&+ 2\mathring{\epsilon}^3\sum_{a=1}^{3}\int_{s=0}^{t}\int_{\Sigma_s}\Psi_a\partial_a\Psi_0\,d\underline{x}\,ds - 2\mathring{\epsilon}^3\sum_{a=1}^{3}\int_{s=0}^{t}\int_{\Sigma_s}\Psi_0\Psi_a\mathring{\Psi}_a\,d\underline{x}\,ds. \qquad (4C.2)
\end{aligned}
$$

Proof. We take the time derivative of both sides of (4A.2). The time derivative of the first line of the right-hand side of (4A.2) is given by (4B.2), where the role of (V_0, V_1, V_2, V_3) in (4B.2) is played by $(\nabla^k\Psi_0, \nabla^k\Psi_1, \nabla^k\Psi_2, \nabla^k\Psi_3)$ and the role of the inhomogeneous terms F_α on the right-hand side of (4B.2) is played by the terms $F_\alpha^{(k)}$ from (4C.1a)–(4C.1b). Moreover, with the help of (2B.6b) and (4C.1a)–(4C.1b), we compute that the time derivatives of the terms on the second line of the right-hand side of (4A.2) are equal to

$$2\sum_{k=1}^{4}\sum_{a=1}^{3}\int_{\Sigma_t}\nabla^k\Psi_a\cdot\partial_a\nabla^k\Psi_0\,d\underline{x}+2\sum_{k=1}^{4}\sum_{a=1}^{3}\int_{\Sigma_t}\nabla^k\Psi_a\cdot F_a^{(k)}\,d\underline{x}$$
$$+2\mathring{\epsilon}^3\sum_{a=1}^{3}\int_{\Sigma_t}\{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\nabla\Psi_0\cdot\partial_a\nabla\Psi_a+\nabla\Psi_0\cdot F_0^{(1)}+\Psi_a\partial_a\Psi_0-\Psi_0\Psi_a\mathring{\Psi}_a\}\,d\underline{x}. \quad (4C.3)$$

Combining these calculations, we deduce that

$$\begin{aligned}\frac{d}{dt}\mathbb{Q}_{(\mathring{\epsilon})}(t)&=\sum_{k=2}^{5}\sum_{a=1}^{3}\int_{\Sigma_t}(\mathcal{I}^{-1}\Psi_0)^2\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)|\nabla^k\Psi_a|^2\,d\underline{x}\\&+\sum_{k=2}^{5}\sum_{a=1}^{3}\sum_{b=1}^{3}\int_{\Sigma_t}\mathcal{I}^{-1}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)(\partial_a\Psi_a)|\nabla^k\Psi_b|^2\,d\underline{x}\\&+\sum_{k=2}^{5}\sum_{a=1}^{3}\sum_{b=1}^{3}\int_{\Sigma_t}\mathcal{I}^{-2}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)(\Psi_a)^2|\nabla^k\Psi_b|^2\,d\underline{x}\\&-\sum_{k=2}^{5}\sum_{a=1}^{3}\sum_{b=1}^{3}\int_{\Sigma_t}\mathcal{I}^{-1}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\mathring{\Psi}_a\Psi_a|\nabla^k\Psi_b|^2\,d\underline{x}\\&-2\sum_{k=2}^{5}\sum_{a=1}^{3}\int_{\Sigma_t}\mathcal{I}^{-1}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)(\partial_a\Psi_0)\nabla^k\Psi_a\cdot\nabla^k\Psi_0\,d\underline{x}\\&-2\sum_{k=2}^{5}\sum_{a=1}^{3}\int_{\Sigma_t}\mathcal{I}^{-2}\Psi_0\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\Psi_a\nabla^k\Psi_a\cdot\nabla^k\Psi_0\,d\underline{x}\\&+2\sum_{k=2}^{5}\sum_{a=1}^{3}\int_{\Sigma_t}\mathcal{I}^{-1}\Psi_0\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)\mathring{\Psi}_a\nabla^k\Psi_a\cdot\nabla^k\Psi_0\,d\underline{x}\\&+2\sum_{k=2}^{5}\int_{\Sigma_t}\nabla^k\Psi_0\cdot F_0^{(k)}\,d\underline{x}+2\sum_{a=1}^{3}\int_{\Sigma_t}\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\nabla^k\Psi_a\cdot F_a^{(k)}\,d\underline{x}+(4C.3). \quad (4C.4)\end{aligned}$$

Integrating (4C.4) from time 0 to time t, we arrive at the desired identity (4C.2). □

5. A priori estimates

In this section, we use the data-size and bootstrap assumptions of Section 3 and the energy identities of Section 4 to derive a priori estimates for solutions to (2B.1) and to the renormalized equations of Proposition 2.5.

5A. ***Conventions for constants.*** In our estimates, the explicit constants $C > 0$ and $c > 0$ are free to vary from line to line. *These explicit constants, and implicit ones as well, are allowed to depend on the data-size parameters* $\mathring{A}$ *and* $\mathring{A}_*^{-1}$ *from Section 3A*. However, the constants can be chosen to be independent of the parameters $\mathring{\epsilon}$ and ε whenever $\mathring{\epsilon}$ and ε are sufficiently small relative to $\mathring{A}^{-1}$ and $\mathring{A}_*$ in the sense described in Section 3C. For example, under our conventions, we have $\mathring{A}_*^{-2}\varepsilon = \mathcal{O}(\varepsilon)$.

5B. ***Pointwise estimates tied to the integrating factor.*** In this subsection, we derive pointwise estimates that are important for analyzing $\mathcal{I}$.

We start by deriving sharp estimates for Ψ_0. The proof is based on separately considering regions where $\mathcal{I}$ is small and $\mathcal{I}$ is large. In Lemma 5.2, we will use these estimates to derive further information about the behavior of Ψ_0 in regions where $\mathcal{I}$ is small (i.e., near the singularity), which is crucial for closing the energy estimates.

Lemma 5.1 (pointwise estimates for Ψ_0). *Under the data-size assumptions of Section 3A, the bootstrap assumptions of Section 3B, and the smallness assumptions of Section 3C, the following pointwise estimates hold for* $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times \mathbb{R}^3$:

$$\Psi_0(t, \underline{x}) = \mathring{\Psi}_0(\underline{x}) + \mathcal{O}(\varepsilon), \tag{5B.1}$$

where $\mathring{\Psi}_0(\underline{x}) = \Psi_0(0, \underline{x})$.

In addition,

$$-\tfrac{5}{16} \leq \min_{\Sigma_t} \Psi_0. \tag{5B.2}$$

Proof. We first prove (5B.1). We will show that $|\partial_t \Psi_0(t, \underline{x})| \lesssim \varepsilon$. Then from this estimate and the fundamental theorem of calculus, we obtain the desired bound (5B.1). To prove the bound $|\partial_t \Psi_0(t, \underline{x})| \lesssim \varepsilon$, we first consider points $(t, \underline{x})$ such that $\mathcal{I}(t, \underline{x}) > \frac{1}{8}$. Then all factors of $\mathcal{I}^{-1}$ in the evolution equation (2B.6a) can be bounded by $\lesssim 1$. For this reason, the desired bound follows as a straightforward consequence of (2B.6a), the bootstrap assumptions, the data-size assumptions (3A.1a), and the assumptions of Section 2A on $\mathscr{W}$.

To finish the proof of (5B.1), it remains to show that $|\partial_t \Psi_0(t, \underline{x})| \lesssim \varepsilon$ at points $(t, \underline{x})$ such that $0 < \mathcal{I}(t, \underline{x}) \leq \frac{1}{8}$. From the bootstrap assumption (3B.4), we deduce that $1 \lesssim \Psi_0(t, \underline{x})$ at such points. From this bound, the bootstrap assumptions, the data-size assumptions (3A.1a), and the assumptions of Section 2A on $\mathscr{W}$, we deduce the following bound for some factors on the right-hand side of (2B.6a) at the spacetime points under consideration:

$$|\mathcal{I}^{-1}\mathscr{W}(\mathcal{I}^{-1}\Psi_0)| = \Psi_0^{-1}|(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)| \lesssim |(\mathcal{I}^{-1}\Psi_0)\mathscr{W}(\mathcal{I}^{-1}\Psi_0)| \lesssim 1.$$

With the help of this bound, the desired estimate $|\partial_t \Psi_0(t, \underline{x})| \lesssim \varepsilon$ follows as a straightforward consequence of (2B.6a), the bootstrap assumptions, the data-size assumptions (3A.1a), and the assumptions of Section 2A on $\mathscr{W}$. We have therefore proved (5B.1).

The bound (5B.2) then follows from (3A.1c) and (5B.1). □

In the next lemma, we derive sharp estimates for $\mathcal{I}$. The estimates are important for closing the energy estimates up to the singularity and for precisely tying the vanishing of $\mathcal{I}$ to the blowup of $\partial_t \Phi$.

Lemma 5.2 (crucial estimates for the integrating factor). *Under the data-size assumptions of Section 3A, the bootstrap assumptions of Section 3B, and the smallness assumptions of Section 3C, the following* *estimates hold for* $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times \mathbb{R}^3$:

$$\mathcal{I}(t, \underline{x}) = 1 - t\mathring{\Psi}_0(\underline{x}) + \mathcal{O}(\varepsilon), \tag{5B.3a}$$

$$\mathcal{I}_\star(t) = 1 - t\mathring{A}_* + \mathcal{O}(\varepsilon), \tag{5B.3b}$$

where $\mathring{\Psi}_0(\underline{x}) = \Psi_0(0, \underline{x})$, $\mathcal{I}_\star$ *is defined in* (2B.2), *and* $\mathring{A}_* > 0$ *is the data-size parameter from Definition 3.1.*

Moreover, the following implications hold for $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times \mathbb{R}^3$:

$$\mathcal{I}(t, \underline{x}) \le \tfrac{1}{4} \min\{1, \mathring{A}_*\} \quad \Longrightarrow \quad \frac{\Psi_0(t, \underline{x})}{\mathcal{I}(t, \underline{x})} \ge 1, \tag{5B.4a}$$

$$\mathcal{I}(t, \underline{x}) \le \tfrac{1}{4} \quad \Longrightarrow \quad \Psi_0(t, \underline{x}) \ge \tfrac{1}{4}\mathring{A}_*. \tag{5B.4b}$$

Finally, the following implications hold for $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times \mathbb{R}^3$:

$$\Psi(t, \underline{x}) \le 0 \implies \mathcal{I}(t, \underline{x}) \ge 1 - C\varepsilon \qquad \text{and} \qquad \Psi(t, \underline{x}) \le 0 \implies \frac{\Psi_0(t, \underline{x})}{\mathcal{I}(t, \underline{x})} \ge -\tfrac{3}{8}. \tag{5B.5}$$

Remark 5.3 (improvement of a bootstrap assumption). Note in particular that the estimate (5B.4b) provides a strict improvement of the bootstrap assumption (3B.4).

Remark 5.4 (the significance of (5B.5)). Note that (5B.5) is a strict improvement of the bootstrap assumption (3B.3) and that (5B.5) implies that $-\frac{3}{8} \le \partial_t \Phi(t, \underline{x})$ for $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times \mathbb{R}^3$. In view of the assumption (2A.1) for $\mathscr{W}$, we conclude that on $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times \mathbb{R}^3$, the wave speed $\{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\}^{1/2}$ is positive and uniformly bounded from above by a positive constant. In the rest of article, we often silently use this fact. Note, however, that from (2A.4) with $k = 0$, it follows that the wave speed vanishes when $\partial_t \Phi \to \infty$.

Proof. From (2B.1) and the estimate (5B.1), we deduce $\partial_t \mathcal{I}(t, \underline{x}) = -\mathring{\Psi}_0(\underline{x}) + \mathcal{O}(\varepsilon)$. Integrating in time and using the initial condition (2B.1), we find that $\mathcal{I}(t, \underline{x}) = 1 - t\mathring{\Psi}_0(\underline{x}) + \mathcal{O}(\varepsilon)$, which is (5B.3a).

Equation (5B.3b) follows as a simple consequence of (5B.3a) and definitions (2B.2) and (3A.2).

To prove (5B.4a), we first consider the case $\mathring{A}_* \ge 1$. From (5B.3a) and (5B.1), we deduce that $\mathcal{I}(t, \underline{x}) = 1 - t\Psi_0(t, \underline{x}) + \mathcal{O}(\varepsilon)$. It follows that if $\mathcal{I}(t, \underline{x}) \le \frac{1}{4}$, then $t\Psi_0(t, \underline{x}) \ge \frac{1}{2}$. Since $0 \le t \le 2\mathring{A}_*^{-1} \le 2$, we deduce that $\Psi_0(t, \underline{x})/\mathcal{I}(t, \underline{x}) \ge 1$, which is the desired conclusion. Next, we consider the case $\mathring{A}_* < 1$. Using (5B.3a) and (5B.1), we deduce that $\mathcal{I}(t, \underline{x}) = 1 - t\Psi_0(t, \underline{x}) + \mathcal{O}(\varepsilon)$. It follows that if $\mathcal{I}(t, \underline{x}) \le \frac{1}{4}\mathring{A}_*$, then $t\Psi_0(t, \underline{x}) \ge 1 - \frac{1}{2}\mathring{A}_*$. Since $0 \le t \le 2\mathring{A}_*^{-1}$, we deduce that

$$\frac{\Psi_0(t, \underline{x})}{\mathcal{I}(t, \underline{x})} \ge 2\big\{1 - \tfrac{1}{2}\mathring{A}_*\big\} = 2 - \mathring{A}_*,$$

which, in view of our assumption $\mathring{A}_* < 1$, is > 1. This completes our proof of (5B.4a).

The implication (5B.4b) can be proved using arguments similar to the ones that we used to prove (5B.4a), and we therefore omit the details.

Next, we note that when $\Psi_0(t, \underline{x}) \le 0$, the estimate $\mathcal{I}(t, \underline{x}) = 1 - t\Psi_0(t, \underline{x}) + \mathcal{O}(\varepsilon)$ proved above implies that $\mathcal{I}(t, \underline{x}) \ge 1 - C\varepsilon$, which yields the first implication stated in (5B.5). To obtain the second implication stated in (5B.5), we use the first implication and the estimate (5B.2). □

In the next lemma, we derive some simple pointwise estimates showing that the spatial derivatives of $\mathcal{I}$ up to top order can be controlled in terms of the spatial derivatives of $\{\Psi_a\}_{a=1,2,3}$.

Lemma 5.5 (estimates for the derivatives of the integrating factor). *Under the data-size assumptions of Section 3A, the bootstrap assumptions of Section 3B, and the smallness assumptions of Section 3C, the following pointwise estimates hold for* $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times \mathbb{R}^3$:

$$|\nabla \mathcal{I}| \lesssim \sum_{a=1}^{3} |\Psi_a| + \sum_{a=1}^{3} |\mathring{\Psi}_a|. \tag{5B.6a}$$

Moreover, for $2 \le k \le 6$, *the following estimate holds*:

$$|\nabla^k \mathcal{I}| \lesssim \sum_{a=1}^{3} |\nabla^{[1,k-1]} \Psi_a| + \sum_{a=1}^{3} |\nabla^{[1,k-1]} \mathring{\Psi}_a| + \varepsilon \sum_{a=1}^{3} |\mathring{\Psi}_a|. \tag{5B.6b}$$

Finally, the following estimate holds for $t \in [0, T_{(\mathrm{Boot})})$:

$$\|\nabla^{[1,3]} \mathcal{I}\|_{L^\infty(\Sigma_t)} \lesssim \varepsilon. \tag{5B.7}$$

Proof. The estimate (5B.6a) is straightforward consequence of (2B.4) and the bootstrap assumptions. Similarly, the estimate (5B.6b) is straightforward to derive via induction in k with the help of (2B.4), the bootstrap assumptions, the data-size assumptions (3A.1a), and (3C.1). Inequality (5B.7) then follows from (5B.6a)–(5B.6b), the bootstrap assumptions, the data-size assumptions (3A.1a), and (3C.1). □

5C. *Pointwise estimates involving the weight.* In the next lemma, we derive precise pointwise estimates for quantities that involve the weight function $\mathscr{W}$. The detailed information is important for closing the energy estimates and for showing that the spatial derivatives of $\mathscr{W} = \mathscr{W}(\partial_t \Phi) = \mathscr{W}(\mathcal{I}^{-1}\Psi_0)$ are controllable. Some of the analysis is delicate in that $\partial_t \Phi$ and its derivatives are allowed to be arbitrarily large (i.e., the estimates hold uniformly, arbitrarily close to the singularity).

Lemma 5.6 (pointwise estimates involving the weight $\mathscr{W}$). *Let* $\mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}}$ *be the characteristic function of the spacetime subset* $\{(t, \underline{x}) \mid 0 < \mathcal{I}(t, \underline{x}) \le \frac{1}{4}\min\{1, \mathring{A}_*\}\}$. *Under the data-size assumptions of Section 3A, the bootstrap assumptions of Section 3B, and the smallness assumptions of Section 3C, the following pointwise estimates hold for* $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times \mathbb{R}^3$:

$$\mathscr{W}(\mathcal{I}^{-1}\Psi_0) \lesssim 1, \tag{5C.1a}$$

$$|\nabla\{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\}| \lesssim \varepsilon \mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}}\{\mathcal{I}^{-2}|\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)|\}^{1/2} + \varepsilon\{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\}^{1/2} \tag{5C.1b}$$

$$\lesssim \varepsilon. \tag{5C.1c}$$

In addition, for $2 \le k \le 5$, *the following estimates hold*:

$$|\nabla^k\{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\}| \lesssim |\nabla^{[1,k]}\Psi_0| + \sum_{a=1}^{3} |\nabla^{\le k-1}\Psi_a| + \sum_{a=1}^{3} |\nabla^{\le k-1}\mathring{\Psi}_a|. \tag{5C.2}$$

Furthermore, the following estimates hold:

$$\mathcal{I}^{-1}\mathscr{W}(\mathcal{I}^{-1}\Psi_0) \lesssim \mathbf{1}_{\{0<\mathcal{I}\leq(1/4)\min\{1,\mathring{A}_*\}\}}\{\mathcal{I}^{-2}|\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)|\}^{1/2} + \{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\}^{1/2} \tag{5C.3a}$$

$$\lesssim 1. \tag{5C.3b}$$

Moreover, for $1 \leq k \leq 5$, *the following estimates hold*:

$$|\nabla^k\{\mathcal{I}^{-1}\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\}| \lesssim |\nabla^{[1,k]}\Psi_0| + \sum_{a=1}^{3}|\nabla^{\leq k-1}\Psi_a| + \sum_{a=1}^{3}|\nabla^{\leq k-1}\mathring{\Psi}_a|. \tag{5C.4}$$

Finally, for $0 \leq P \leq 2$, *the following estimates hold*:

$$\big|\mathcal{I}^{-2}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0) + \mathbf{1}_{\{0<\mathcal{I}\leq(1/4)\min\{1,\mathring{A}_*\}\}}\mathcal{I}^{-2}|\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)|\big| \lesssim \mathscr{W}(\mathcal{I}^{-1}\Psi_0), \tag{5C.5a}$$

$$|\mathcal{I}^{-P}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)| \lesssim 1. \tag{5C.5b}$$

Proof. Throughout this proof, we set

$$y = y(t,\underline{x}) := \frac{\Psi_0(t,\underline{x})}{\mathcal{I}(t,\underline{x})}.$$

Also, we silently use the observations of Remark 5.4.

Proof of (5C.1a): This bound is a trivial consequence of our assumption (2A.4) on $\mathscr{W}$.

Proof of (5C.1b) and (5C.1c): We first prove (5C.1b) at spacetime points $(t,\underline{x})$ such that $\mathcal{I}(t,\underline{x}) > \frac{1}{4}\min\{1,\mathring{A}_*\}$. This is the easy case because $\mathcal{I}^{-1} < 4\max\{1,\mathring{A}_*^{-1}\} \leq C$, and we therefore do not have to concern ourselves with the possibility of small denominators. Specifically, using the identity (2B.4), the bootstrap assumptions, the data-size assumptions (3A.1a), and the assumptions of Section 2A, we deduce that when $\mathcal{I}(t,\underline{x}) > \frac{1}{4}\min\{1,\mathring{A}_*\}$, we have

$$|\nabla\{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\}| \lesssim |\nabla\Psi_0| + \sum_{a=1}^{3}|\Psi_a| + \sum_{a=1}^{3}|\mathring{\Psi}_a| \lesssim \varepsilon. \tag{5C.6}$$

Next, we use the bootstrap assumptions and the assumptions of Section 2A on $\mathscr{W}$, including the uniform positivity and boundedness of $\mathscr{W}(y)$ on intervals of the form $y \in \big[-\frac{3}{8}, C\big]$, to obtain

$$\mathbf{1}_{\{\mathcal{I}>(1/4)\min\{1,\mathring{A}_*\}\}} \lesssim \mathscr{W}(\mathcal{I}^{-1}\Psi_0) \lesssim \{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\}^{1/2}.$$

It follows that the right-hand side of (5C.6) is $\lesssim$ the second term on the right-hand side of (5C.1b) as desired.

We now prove (5C.1b) at points $(t,\underline{x})$ such that $0 < \mathcal{I}(t,\underline{x}) \leq \frac{1}{4}\min\{1,\mathring{A}_*\}$. Along the way, we will prove some additional estimates that we will use later on. We start by defining the following weighted differential operator, which acts on functions $f = f(y)$: $D_Y f := y^2\frac{d}{dy}f$. Note that the chain rule implies

$$\nabla\mathscr{W}(y) = -D_Y\mathscr{W}(y)\nabla(y^{-1}). \tag{5C.7}$$

We therefore inductively deduce that, for $1 \leq k \leq 5$, we have

$$|\nabla^k \mathscr{W}(y)| \lesssim \sum_{n=1}^{k} |D_Y^n \mathscr{W}(y)| \Bigg\{ \sum_{\substack{\sum_{i=1}^n k_i = k \\ k_i \geq 1}} \prod_{i=1}^{n} |\nabla^{k_i}(y^{-1})| \Bigg\}. \tag{5C.8}$$

The case $k = 1$ in (5C.8) yields $|\nabla \mathscr{W}(\mathcal{I}^{-1}\Psi_0)| \lesssim (\mathcal{I}^{-1}\Psi_0)^2 |\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)| |\nabla\{\mathcal{I}\Psi_0^{-1}\}|$. Also using the identity (2B.4), the bootstrap assumptions, the data-size assumptions (3A.1a), (3C.1), the assumptions of Section 2A, and the crucially important estimate (5B.4b) (which implies that $\Psi_0^{-1} \lesssim 1$), we deduce that when $\mathcal{I}(t, \underline{x}) \leq \frac{1}{4}\min\{1, \mathring{A}_*\}$, we have $|\nabla\{\mathcal{I}\Psi_0^{-1}\}| \lesssim \varepsilon$ and thus

$$\begin{aligned} |\nabla \mathscr{W}(\mathcal{I}^{-1}\Psi_0)| &\lesssim \varepsilon (\mathcal{I}^{-1}\Psi_0)^2 |\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)| \lesssim \varepsilon \{(\mathcal{I}^{-1}\Psi_0)^2 |\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)|\}^{1/2} \\ &\lesssim \varepsilon \{\mathcal{I}^{-2} |\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)|\}^{1/2}, \end{aligned} \tag{5C.9}$$

which is $\lesssim$ the first term on the right-hand side of (5C.1b) as desired. This finishes the proof of (5C.1b). We clarify that to derive the next-to-last inequality in (5C.9), in which we bounded $(\mathcal{I}^{-1}\Psi_0)^2|\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)|$ by its square root, we used (2A.4) to deduce $y^2|\mathscr{W}'(y)| \lesssim 1$.

We now prove (5C.1c). From Remark 5.4, the assumptions of Section 2A on $\mathscr{W}$, and (5B.4b), we deduce that

$$\begin{aligned} \mathbf{1}_{\{0<\mathcal{I}\leq(1/4)\min\{1,\mathring{A}_*\}\}} \{\mathcal{I}^{-2}|\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)|\}^{1/2} &\lesssim \mathbf{1}_{\{0<\mathcal{I}\leq(1/4)\min\{1,\mathring{A}_*\}\}} \{(\mathcal{I}^{-2}\Psi_0^2)|\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)|\}^{1/2} \\ &\lesssim 1 \end{aligned} \tag{5C.10}$$

and that $\{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\}^{1/2} \lesssim 1$. That is, the non-$\varepsilon$ factors on the right-hand side of (5C.1b) are $\lesssim 1$. This yields (5C.1c).

Proof of (5C.2): The proof is similar to that of (5C.1b), but slightly simpler. Note that $k \in [2, 5]$ by assumption in this estimate. We first prove the estimate at points $(t, \underline{x})$ such that $\mathcal{I}(t, \underline{x}) > \frac{1}{4}\min\{1, \mathring{A}_*\}$. This is the easy case because $\mathcal{I}^{-1} < 4\max\{1, \mathring{A}_*^{-1}\} \leq C$, and we therefore do not have to concern ourselves with the possibility of small denominators. Specifically, using the identity (2B.4), the bootstrap assumptions, the data-size assumptions (3A.1a), (3C.1), and the assumptions of Section 2A, we deduce that when $\mathcal{I}(t, \underline{x}) > \frac{1}{4}\min\{1, \mathring{A}_*\}$, we have

$$|\nabla^k\{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\}| \lesssim |\nabla^{[1,k]}\Psi_0| + \sum_{a=1}^{3} |\nabla^{\leq k-1}\Psi_a| + \sum_{a=1}^{3} |\nabla^{\leq k-1}\mathring{\Psi}_a|, \tag{5C.11}$$

which is $\lesssim$ the right-hand side of (5C.2) as desired.

It remains for us to prove (5C.2) at points $(t, \underline{x})$ such that $0 < \mathcal{I}(t, \underline{x}) \leq \frac{1}{4}\min\{1, \mathring{A}_*\}$. Note that the estimate (5C.8) holds and that by (2A.4) and Remark 5.4, we have the following bound[26] for the factors of $D_Y^n\mathscr{W}(y)$ on the right-hand side of (5C.8): $|D_Y^n\mathscr{W}(y)| \lesssim 1$. From this bound, (5C.8), the bootstrap assumptions, and the data-size assumptions (3A.1a), we see that the desired bound (5C.2) will follow

[26]In obtaining this bound, it is helpful to note that $D_Y f = -\frac{d}{dz} f$, where $z := 1/y$.

once we show that the following bound holds when $2 \le k \le 5$ and $\mathcal{I}(t, \underline{x}) \le \frac{1}{4}\min\{1, \mathring{A}_*\}$:

$$|\nabla^k(y^{-1})| \lesssim |\nabla^{[1,k]}\Psi_0| + \sum_{a=1}^{3}|\nabla^{\le k-1}\Psi_a| + \sum_{a=1}^{3}|\nabla^{\le k-1}\mathring{\Psi}_a|. \tag{5C.12}$$

To prove (5C.12), we first note that (5B.4b) implies that $1 \lesssim \Psi_0(t, \underline{x})$ in the present context. Thus, the left-hand side of (5C.11), which is equal to $|\nabla^k(\mathcal{I}/\Psi_0)|$, is the k-th derivative of a ratio with a denominator uniformly bounded from below away from 0, and the desired estimate (5C.2) follows as a straightforward consequence of the identity (2B.4), the data-size assumptions (3A.1a), and the bootstrap assumptions.

Proof of (5C.3a), (5C.3b), and (5C.4): These estimates can be proved using arguments similar to the ones we used to prove (5C.1b) and (5C.2), based on separately considering the cases $\mathcal{I}(t, \underline{x}) > \frac{1}{4}\min\{1, \mathring{A}_*\}$ and $0 < \mathcal{I}(t, \underline{x}) \le \frac{1}{4}\min\{1, \mathring{A}_*\}$ and using the assumptions of Section 2A. We omit the details, noting only that we can write $\mathcal{I}^{-1}\mathscr{W}(\mathcal{I}^{-1}\Psi_0) = \Psi_0^{-1}y\mathscr{W}(y)$ and that the assumptions of Section 2A (especially (2A.5)), (5B.4a), (5B.4b), and Remark 5.4 imply that we have the following key estimates, relevant for the more difficult case $0 < \mathcal{I}(t, \underline{x}) \le \frac{1}{4}\min\{1, \mathring{A}_*\}$:

$$\begin{aligned}\mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}}\{\Psi_0^{-1}y\mathscr{W}(y)\} &\lesssim \mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}}\{y^2|\mathscr{W}'(y)|\}^{1/2} \\ &\lesssim \mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}}\{\mathcal{I}^{-2}|\mathscr{W}'(y)|\}^{1/2}\end{aligned} \tag{5C.13}$$

and, for $n \le 5$, $|D_Y^n(y\mathscr{W}(y))| \lesssim 1$ (footnote 26 is also relevant for obtaining this latter bound).

Proof of (5C.5a): We first note that by (2A.6) and (5B.4a), we have $\mathscr{W}'(\mathcal{I}^{-1}\Psi_0) < 0$ at points $(t, \underline{x})$ such that $\mathcal{I}(t, \underline{x}) \le \frac{1}{4}\min\{1, \mathring{A}_*\}$. From this fact and the identity $1 = \mathbf{1}_{\{\mathcal{I}>(1/4)\min\{1,\mathring{A}_*\}\}} + \mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}}$, it follows that

$$\begin{aligned}\text{LHS (5C.5a)} &= |\mathbf{1}_{\{\mathcal{I}>(1/4)\min\{1,\mathring{A}_*\}\}}\mathcal{I}^{-2}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)| \\ &\lesssim \mathbf{1}_{\{\mathcal{I}>(1/4)\min\{1,\mathring{A}_*\}\}}|\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)|.\end{aligned} \tag{5C.14}$$

Also using the bound $|\mathscr{W}'(y)| \lesssim 1$, which is a simple consequence of (2A.4), we find that the left-hand side of (5C.5a) is $\lesssim \mathbf{1}_{\{\mathcal{I}>(1/4)\min\{1,\mathring{A}_*\}\}}$. Next, we recall the estimate $\mathbf{1}_{\{\mathcal{I}>(1/4)\min\{1,\mathring{A}_*\}\}} \lesssim \mathscr{W}(\mathcal{I}^{-1}\Psi_0)$ that we derived in our proof of (5C.1b). Combining the above estimates, we conclude the desired bound (5C.5a).

Proof of (5C.5b): We first prove (5C.5b) at points $(t, \underline{x})$ such that $\mathcal{I}(t, \underline{x}) > \frac{1}{4}\min\{1, \mathring{A}_*\}$. Using the bootstrap assumptions and the assumptions of Section 2A on $\mathscr{W}$, we deduce, in view of Remark 5.4, that $|\mathcal{I}^{-P}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)| \lesssim |\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)| \lesssim 1$ as desired.

It remains for us to prove (5C.5b) at points $(t, \underline{x})$ such that $0 < \mathcal{I}(t, \underline{x}) \le \frac{1}{4}\min\{1, \mathring{A}_*\}$. Using (5B.4b), we see that $1 \lesssim \Psi_0(t, \underline{x})$ at such points, and it follows that $|\mathcal{I}^{-P}\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)| \lesssim |\{\mathcal{I}^{-1}\Psi_0\}^P\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)|$. Using the assumptions of Section 2A on $\mathscr{W}$ and the assumption $P \in [0, 2]$, we deduce, in view of Remark 5.4, that the right-hand side of the previous expression is $\lesssim 1$ as desired. This finishes the proof of (5C.5b) and completes the proof of the lemma. □

5D. ***Pointwise estimates for the inhomogeneous terms in the commuted evolution equations.*** With the estimates of Lemma 5.6 in hand, we are now ready to derive pointwise estimates for the inhomogeneous terms in the ∇^k-commuted evolution equations.

Lemma 5.7 (pointwise estimates for the inhomogeneous terms in the ∇^k-commuted evolution equations). *Let $\mathcal{I}$ be as in Definition 2.1, and let $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$ be a solution to the system* (2B.6a)–(2B.6b). *Consider the following inhomogeneous PDE system,*[27] *obtained by commuting* (2B.6a)–(2B.6b) *with* ∇^k:

$$\partial_t \nabla^k \Psi_0 = \mathscr{W}(\mathcal{I}^{-1}\Psi_0) \sum_{a=1}^{3} \partial_a \nabla^k \Psi_a + F_0^{(k)}, \tag{5D.1a}$$

$$\partial_t \nabla^k \Psi_i = \partial_i \nabla^k \Psi_0 + F_i^{(k)}. \tag{5D.1b}$$

Under the data-size assumptions of Section 3A, the bootstrap assumptions of Section 3B, and the smallness assumptions of Section 3C, for $k = 2, 3, 4, 5$ and $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times \mathbb{R}^3$, the following estimate holds:

$$|F_0^{(k)}| \lesssim \varepsilon |\nabla^{[2,k]} \Psi_0| + \varepsilon \mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}} \sum_{a=1}^{3} \{\mathcal{I}^{-2} |\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)|\}^{1/2} |\nabla^k \Psi_a| \\ + \varepsilon \sum_{a=1}^{3} \{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\}^{1/2} |\nabla^k \Psi_a| + \sum_{a=1}^{3} |\nabla^{[1,k-1]} \Psi_a| + \varepsilon^2 \sum_{a=1}^{3} |\Psi_a| + \sum_{a=1}^{3} |\nabla^{\le k} \mathring{\Psi}_a|. \tag{5D.2}$$

Moreover, for $k = 0, 1, 2, 3, 4$, the following estimate holds:

$$|F_0^{(k)}| \lesssim \underbrace{\varepsilon |\nabla^{[2,k]} \Psi_0|}_{\textit{absent if } k \le 1} + \underbrace{\sum_{a=1}^{3} |\nabla^{[1,k]} \Psi_a|}_{\textit{absent if } k=0} + \underbrace{\varepsilon^2 \sum_{a=1}^{3} |\Psi_a|}_{\textit{absent if } k=0} + \underbrace{\varepsilon \sum_{a=1}^{3} |\Psi_a|}_{\textit{absent if } k\ge 1} + \sum_{a=1}^{3} |\nabla^{\le k} \mathring{\Psi}_a|. \tag{5D.3}$$

Finally, for $k = 0, 1, 2, 3, 4, 5$, the following estimate holds:

$$\sum_{a=1}^{3} |F_a^{(k)}| \lesssim \underbrace{\varepsilon |\nabla^{[2,k]} \Psi_0|}_{\textit{absent if } k=0,1} + \sum_{a=1}^{3} |\nabla^{\le k} \mathring{\Psi}_a|. \tag{5D.4}$$

Proof. The estimate (5D.4) follows in a straightforward fashion from commuting (2B.6b) with ∇^k and using the bootstrap assumptions, the data-size assumptions (3A.1a), and (3C.1).

To prove (5D.2), we first commute (2B.6a) with ∇^k to obtain (5D.1a). The only products in $F_0^{(k)}$ that are difficult to bound are those that feature a factor in which k total derivatives fall on Ψ_a, specifically the products $\sum_{a=1}^{3} \{\nabla[\mathscr{W}(\mathcal{I}^{-1}\Psi_0)]\} \partial_a \nabla^{k-1} \Psi_a$, $\sum_{a=1}^{3} \mathcal{I}^{-1} \mathscr{W}(\mathcal{I}^{-1}\Psi_0) \Psi_a \nabla^k \Psi_a$, and $\sum_{a=1}^{3} \mathscr{W}(\mathcal{I}^{-1}\Psi_0) \mathring{\Psi}_a \nabla^k \Psi_a$. To bound the first of these, we use the estimate (5C.1b), which implies that the product is bounded by the sum of the second and third terms on the right-hand side of (5D.2) as desired. To handle the second and third products, we use (5C.1a), (5C.3a), the bootstrap assumptions, the data-size assumptions (3A.1a), and (3C.1) to bound them in magnitude by

$$\lesssim \varepsilon \mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}} \sum_{a=1}^{3} \{\mathcal{I}^{-2} |\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)|\}^{1/2} |\nabla^k \Psi_a| + \varepsilon \sum_{a=1}^{3} \{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\}^{1/2} |\nabla^k \Psi_a|,$$

which is in turn bounded by the sum of the second and third terms on the right-hand side of (5D.2) as desired. The remaining terms in $F_0^{(k)}$ feature $\le k-1$ derivatives of Ψ_a. These terms can easily seen to

[27] We do not bother to state the precise form of the inhomogeneous terms $F_\alpha^{(k)}$ here.

be $\lesssim$ the right-hand side of (5D.2) with the help of the estimates (5C.1a), (5C.1c), (5C.2), (5C.3b), and (5C.4), the bootstrap assumptions, the data-size assumptions (3A.1a), and (3C.1).

The estimate (5D.3) is easier to prove and can be obtained in a similar fashion with the help of the estimates (5C.1a), (5C.1c), (5C.2), (5C.3b), (5C.4), the bootstrap assumptions, the data-size assumptions (3A.1a), and (3C.1). □

5E. *The main a priori estimates.* We now derive the main result of this section: a priori estimates that hold up to top order and that in particular yield a strict improvement of the bootstrap assumptions. These are the main ingredients in the proof of our main theorem.

Proposition 5.8 (the main a priori estimates). *Let $\mathcal{I}$ be the integrating factor from Definition 2.1, and let $\mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}}$ be the characteristic function of the spacetime subset*

$$\left\{(t,\underline{x})\;\middle|\;0<\mathcal{I}(t,\underline{x})\le\tfrac{1}{4}\min\{1,\mathring{A}_*\}\right\}.$$

There exists a constant $C>0$ such that under the data-size assumptions of Section 3A, the bootstrap assumptions of Section 3B, and the smallness assumptions of Section 3C, for solutions $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$ to the system (2B.6a)–(2B.6b), *the L^2-controlling quantity $\mathbb{Q}_{(\mathring{\epsilon})}$ of Definition 4.2 satisfies the following estimate for $t\in[0,T_{(\mathrm{Boot})})$*:

$$\mathbb{Q}_{(\mathring{\epsilon})}(t)+\frac{1}{20}\mathring{A}_*^2\sum_{k=2}^{5}\sum_{a=1}^{3}\int_{s=0}^{t}\int_{\Sigma_s}\mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}}\mathcal{I}^{-2}|\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)||\nabla^k\Psi_a|^2\,d\underline{x}\,ds\le C\mathring{\epsilon}^2. \quad (5E.1)$$

In addition the following estimates hold for $t\in[0,T_{(\mathrm{Boot})})$ and $i=1,2,3$:

$$\mathring{\epsilon}\|\partial_t\Psi_0\|^2_{L^2(\Sigma_t)}+\|\nabla\partial_t\Psi_0\|^2_{H^3(\Sigma_t)}\le C\mathring{\epsilon}^2, \quad (5E.2a)$$

$$\mathring{\epsilon}^3\|\partial_t\Psi_i\|^2_{L^2(\Sigma_t)}+\|\nabla\partial_t\Psi_i\|^2_{H^3(\Sigma_t)}\le C\mathring{\epsilon}^2. \quad (5E.2b)$$

Moreover, $\mathcal{I}$ satisfies the following estimate for $t\in[0,T_{(\mathrm{Boot})})$:

$$\mathring{\epsilon}^3\|\nabla\mathcal{I}\|^2_{L^2(\Sigma_t)}+\|\nabla^{[2,5]}\mathcal{I}\|^2_{L^2(\Sigma_t)} +\int_{s=0}^{t}\int_{\Sigma_s}\mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}}\mathcal{I}^{-2}|\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)||\nabla^6\mathcal{I}|^2\,d\underline{x}\,ds\le C\mathring{\epsilon}^2. \quad (5E.3)$$

Finally, we have the following estimates for $t\in[0,T_{(\mathrm{Boot})})$, which in particular yield strict improvements of the bootstrap assumptions (3B.5a)–(3B.5d) *whenever $C\mathring{\epsilon}<\varepsilon$*:

$$\|\Psi_0\|_{L^\infty(\Sigma_t)}\le\mathring{A}+C\mathring{\epsilon}, \quad (5E.4a)$$

$$\|\nabla^{[1,3]}\Psi_0\|_{L^\infty(\Sigma_t)}\le C\mathring{\epsilon}, \quad (5E.4b)$$

$$\|\nabla^{\le 2}\Psi_i\|_{L^\infty(\Sigma_t)}\le C\mathring{\epsilon}, \quad (5E.4c)$$

$$\|\mathcal{I}\|_{L^\infty(\Sigma_t)}\le 1+2\mathring{A}_*^{-1}\mathring{A}+C\mathring{\epsilon}, \quad (5E.4d)$$

$$\|\nabla^{[1,3]}\mathcal{I}\|_{L^\infty(\Sigma_t)}\le C\mathring{\epsilon}. \quad (5E.4e)$$

Proof. Proof of (5E.1): The main step is to derive the following estimate:

$$\begin{aligned}
&\mathbb{Q}_{(\mathring{\epsilon})}(t) + \tfrac{1}{16}\mathring{A}_*^2 \sum_{k=2}^{5}\sum_{a=1}^{3} \int_{s=0}^{t}\int_{\Sigma_s} \mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}} \mathcal{I}^{-2} |\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)| |\nabla^k \Psi_a|^2 \, d\underline{x}\, ds \\
&\quad \le C\mathring{\epsilon}^2 + C\varepsilon \sum_{k=2}^{5}\sum_{a=1}^{3} \int_{s=0}^{t}\int_{\Sigma_s} \mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}} \mathcal{I}^{-2} |\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)| |\nabla^k \Psi_a|^2 \, d\underline{x}\, ds \\
&\qquad + C\int_{s=0}^{t} \mathbb{Q}_{(\mathring{\epsilon})}(s)\, ds. \qquad (5E.5)
\end{aligned}$$

Once we have shown (5E.5), we can absorb the second term on the right-hand side of (5E.5) into the second term on the left-hand side of (5E.5), which, for ε sufficiently small, at most reduces the coefficient of $\frac{1}{16}\mathring{A}_*^2$ in front of the second term on the left to the value of $\frac{1}{20}\mathring{A}_*^2$, as is stated on the left-hand side of (5E.1). We then use Gronwall's inequality and the assumption $0 < t < T_{(\mathrm{Boot})} \le 2\mathring{A}_*^{-1}$ to conclude that the left-hand side of (5E.1) is $\le C\exp(Ct)\mathring{\epsilon}^2 \le C\exp(C\mathring{A}_*^{-1})\mathring{\epsilon}^2 \le C\mathring{\epsilon}^2$ as desired.

To prove (5E.5), we must bound the terms on the right-hand side of (4C.2). As a first step, we note the following bound for the first term on the right-hand side of (4C.2): $\mathbb{Q}_{(\mathring{\epsilon})}(0) \le C\mathring{\epsilon}^2$, an estimate that follows as a straightforward consequence of definition (4A.2), the data-size assumptions (3A.1a)–(3A.1c), the initial condition $\mathcal{I}|_{\Sigma_0} = 1$ stated in (2B.1), and the assumptions of Section 2A on $\mathscr{W}$.

Next, we treat the spacetime integral on the first line of the right-hand side of (4C.2). Using (5B.4b), (5C.5a), and the bootstrap assumption (3B.5a) for $\|\Psi_0\|_{L^\infty(\Sigma_t)}$, we can express the integral as the negative integral

$$-\sum_{k=2}^{5}\sum_{a=1}^{3} \int_{s=0}^{t}\int_{\Sigma_s} \mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}} (\mathcal{I}^{-1}\Psi_0)^2 |\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)| |\nabla^k \Psi_a|^2 \, d\underline{x}\, ds,$$

which is bounded from above by the negative "favorable integral"

$$-\tfrac{1}{16}\mathring{A}_*^2 \sum_{k=2}^{5}\sum_{a=1}^{3} \int_{s=0}^{t}\int_{\Sigma_s} \mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}} \mathcal{I}^{-2} |\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)| |\nabla^k \Psi_a|^2 \, d\underline{x}\, ds,$$

plus an error integral that is bounded in magnitude by

$$\lesssim \sum_{k=2}^{5}\sum_{a=1}^{3} \int_{s=0}^{t}\int_{\Sigma_s} \mathscr{W}(\mathcal{I}^{-1}\Psi_0) |\nabla^k \Psi_a|^2 \, d\underline{x}\, ds.$$

We can therefore bring the favorable integral over to the left-hand side of (5E.5), where it appears with a "+" sign. Moreover, from Definition 4.2, we deduce that the error integral is bounded by the last term on the right-hand side of (5E.5) as desired.

We now bound the spacetime integrals on lines two to four of the right-hand side of (4C.2). Using the estimate (5C.5b), the bootstrap assumptions, the data-size assumptions (3A.1a), and (3C.1), we deduce that all three integrands are bounded in magnitude by $\lesssim \sum_{k=2}^{5}\sum_{a=1}^{3} \mathscr{W}(\mathcal{I}^{-1}\Psi_0)|\nabla^k\Psi_a|^2$. From Definition 4.2, we conclude that the corresponding error integrals are bounded by the last term on the right-hand side of (5E.5) as desired. Using similar reasoning and Young's inequality, we bound the last two spacetime integrals on the right-hand side of (4C.2) by $\le$ the right-hand side of (5E.5) as desired.

We now bound the spacetime integrals on lines five to seven of the right-hand side of (4C.2). Using the estimates (5C.5a) and (5C.5b), the bootstrap assumptions, the data-size assumptions (3A.1a), (3C.1), and Young's inequality, we deduce that all three integrands are bounded in magnitude by

$$\lesssim \varepsilon \mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}} \sum_{k=2}^{5}\sum_{a=1}^{3} \mathcal{I}^{-2}|\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)||\nabla^k\Psi_a|^2 + \varepsilon \sum_{k=2}^{5}\sum_{a=1}^{3} \mathscr{W}(\mathcal{I}^{-1}\Psi_0)|\nabla^k\Psi_a|^2 + \sum_{k=2}^{5}|\nabla^k\Psi_0|^2.$$

Appealing to Definition 4.2, we conclude that the corresponding error integrals are bounded in magnitude by $\le$ the right-hand side of (5E.5) as desired.

We now bound the spacetime integral on line eight of the right-hand side of (4C.2), in which the integrand is $2\sum_{k=1}^{4}\sum_{a=1}^{3}\nabla^k\Psi_a\cdot\partial_a\nabla^k\Psi_0$. Using Young's inequality, we bound this integrand by $\lesssim |\nabla^{[2,5]}\Psi_0|^2 + \sum_{a=1}^{3}|\nabla^{[1,4]}\Psi_a|^2$. From Definition 4.2, we conclude that the integral of the right-hand side of this expression over the spacetime domain $(s,\underline{x})\in[0,t]\times\mathbb{R}^3$ is bounded by the last term on the right-hand side of (5E.5) as desired.

We now bound the first spacetime integral on line nine of the right-hand side of (4C.2), in which the integrand is $2\sum_{k=2}^{5}\nabla^k\Psi_0\cdot F_0^{(k)}$. Using Young's inequality, (5D.2), and (3C.1), we pointwise bound this integrand in magnitude by

$$\lesssim |\nabla^{[2,5]}\Psi_0|^2 + \varepsilon \mathbf{1}_{\{0<\mathcal{I}\le(1/4)\min\{1,\mathring{A}_*\}\}} \sum_{a=1}^{3}\mathcal{I}^{-2}|\mathscr{W}'(\mathcal{I}^{-1}\Psi_0)||\nabla^{[2,5]}\Psi_a|^2 \\ + \sum_{a=1}^{3}\mathscr{W}(\mathcal{I}^{-1}\Psi_0)|\nabla^{[2,5]}\Psi_a|^2 + \sum_{a=1}^{3}|\nabla^{[1,4]}\Psi_a|^2 + \mathring{\epsilon}^3\sum_{a=1}^{3}|\Psi_a|^2 + \sum_{a=1}^{3}|\nabla^{\le 5}\mathring{\Psi}_a|^2. \quad (5E.6)$$

From Definition 4.2 and the data-size assumptions (3A.1a), we conclude that the integral of the right-hand side of (5E.6) over the spacetime domain $(s,\underline{x})\in[0,t]\times\mathbb{R}^3$ is $\le$ the right-hand side of (5E.5) as desired.

We now bound the second spacetime integral on line nine of the right-hand side of (4C.2), in which the integrand is $2\sum_{k=2}^{5}\sum_{a=1}^{3}\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\nabla^k\Psi_a\cdot F_a^{(k)}$. Using Young's inequality, (5C.1a), and (5D.4), we pointwise bound this integrand in magnitude by $\lesssim |\nabla^{[2,5]}\Psi_0|^2 + \sum_{a=1}^{3}\mathscr{W}(\mathcal{I}^{-1}\Psi_0)|\nabla^{[2,5]}\Psi_a|^2 + \sum_{a=1}^{3}|\nabla^{\le 5}\mathring{\Psi}_a|^2$. From Definition 4.2 and the data-size assumptions (3A.1a), we conclude that the integral of the right-hand side of this expression over the spacetime domain $(s,\underline{x})\in[0,t]\times\mathbb{R}^3$ is $\le$ the right-hand side of (5E.5) as desired.

We now bound the spacetime integral on line ten of the right-hand side of (4C.2), in which the integrand is $2\sum_{k=1}^{4}\sum_{a=1}^{3}\nabla^k\Psi_a\cdot F_a^{(k)}$. Using Young's inequality and (5D.4), we pointwise bound this integrand in magnitude by $\lesssim |\nabla^{[2,4]}\Psi_0|^2 + \sum_{a=1}^{3}|\nabla^{[1,4]}\Psi_a|^2 + \sum_{a=1}^{3}|\nabla^{\le 4}\mathring{\Psi}_a|^2$. From Definition 4.2 and the data-size assumptions (3A.1a), we conclude that the integral of the right-hand side of this expression over the spacetime domain $(s,\underline{x})\in[0,t]\times\mathbb{R}^3$ is $\le$ the right-hand side of (5E.5) as desired.

We now bound the first spacetime integral on line eleven of the right-hand side of (4C.2), in which the integrand is $2\mathring{\epsilon}^3\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\nabla\Psi_0\cdot\sum_{a=1}^{3}\partial_a\nabla\Psi_a$. Using the estimate (5C.1a) and Young's inequality, we bound this integrand by $\lesssim \mathring{\epsilon}^3|\nabla\Psi_0|^2 + \sum_{a=1}^{3}|\nabla^2\Psi_a|^2$. From Definition 4.2, we conclude that the integral of the right-hand side of this expression over the spacetime domain $(s,\underline{x})\in[0,t]\times\mathbb{R}^3$ is bounded by the last term on the right-hand side of (5E.5) as desired.

Finally, we bound the second spacetime integral on line eleven of the right-hand side of (4C.2), in which the integrand is $2\mathring{\epsilon}^3 \nabla\Psi_0 \cdot F_0^{(1)}$. Using Young's inequality and (5D.3), we pointwise bound this integrand in magnitude by

$$\lesssim \mathring{\epsilon}^3 |\nabla\Psi_0|^2 + \sum_{a=1}^{3} |\nabla\Psi_a|^2 + \mathring{\epsilon}^3 \sum_{a=1}^{3} |\Psi_a|^2 + \sum_{a=1}^{3} |\nabla^{\leq 1}\mathring{\Psi}_a|^2. \tag{5E.7}$$

From Definition 4.2 and the data-size assumptions (3A.1a), we conclude that the integral of the right-hand side of (5E.7) over the spacetime domain $(s, \underline{x}) \in [0, t] \times \mathbb{R}^3$ is $\leq$ the right-hand side of (5E.5) as desired. This completes our proof of (5E.5) and therefore finishes the proof of (5E.1).

Proof of (5E.4b) and (5E.4c): In view of Definition 4.2, we see that the estimates $\|\nabla^{[2,3]}\Psi_0\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}$ and $\|\nabla^{[1,2]}\Psi_i\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}$ follow from (5E.1) and Sobolev embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. To bound $\|\nabla\Psi_0\|_{L^\infty(\Sigma_t)}$, we first use (5D.1a), the bootstrap assumptions, the data-size assumptions (3A.1a), the estimates (5C.1a) and (5D.3), inequality (3C.1), and the already-proven bound $\|\nabla^{[1,2]}\Psi_i\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}$ to obtain $|\partial_t \nabla\Psi_0| \lesssim \varepsilon^2 + \mathring{\epsilon} + \sum_{a=1}^{3} |\nabla^{[1,2]}\Psi_a| \lesssim \mathring{\epsilon}$. From this bound, the fundamental theorem of calculus, and the data-size assumptions (3A.1a), we find that $|\nabla\Psi_0| \lesssim \mathring{\epsilon} + \int_{s=0}^{t} \mathring{\epsilon}\, ds \lesssim \mathring{\epsilon}$. This implies $\|\nabla\Psi_0\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}$, which completes the proof of (5E.4b). Similarly, from (2B.6b), the bootstrap assumptions, the data-size assumptions (3A.1a), and the already-proven bound $\|\nabla\Psi_0\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}$, we deduce $\sum_{a=1}^{3} |\partial_t \Psi_a| \lesssim \mathring{\epsilon}$. From this bound, the fundamental theorem of calculus, and the data-size assumption (3A.1a), we find that $\sum_{a=1}^{3} |\Psi_a| \lesssim \mathring{\epsilon}$, which implies $\sum_{a=1}^{3} \|\Psi_a\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}$, thereby completing the proof of (5E.4c).

Proof of (5E.4a): We first use (2B.6a), the estimates (5C.1a) and (5C.3b), the bootstrap assumptions, the data-size assumptions (3A.1a), and the already-proven bound $\|\nabla^{\leq 1}\Psi_i\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}$ to obtain $|\partial_t \Psi_0| \lesssim \mathring{\epsilon}$. From this bound, the fundamental theorem of calculus, the data-size assumption (3A.1b), and the fact that $0 < t \leq 2\mathring{A}_*^{-1}$, we find that $\|\Psi_0\|_{L^\infty(\Sigma_t)} \leq \|\mathring{\Psi}_0\|_{L^\infty(\Sigma_0)} + C\mathring{\epsilon} \leq \mathring{A} + C\mathring{\epsilon}$, which is the desired bound (5E.4a).

Proof of (5E.4d) and (5E.4e): We repeat the proof of (5B.3a), but using the bootstrap assumption (3B.5d) and the estimates (5E.4a)–(5E.4c) instead of using the full set of bootstrap assumptions. We find that $\mathcal{I}(t, \underline{x}) = 1 - t\mathring{\Psi}_0(\underline{x}) + \mathcal{O}(\mathring{\epsilon})$. From this estimate, the fact that $0 < t < 2\mathring{A}_*^{-1}$, and the data-size assumption (3A.1b), we conclude the desired bound (5E.4d). Similarly, to prove (5E.4e), we repeat the proof of (5B.7), but using the estimates (5E.4a)–(5E.4d) instead of the bootstrap assumptions.

Proof of (5E.3): The estimate (5E.3) follows as a straightforward consequence of the pointwise estimates (5B.6a)–(5B.6b), the weight estimate (5C.5b), the energy estimate (5E.1), and the data-size assumptions (3A.1a).

Proof of (5E.2a) and (5E.2b): To prove (5E.2a), we first use (5D.1a) and the estimate (5C.1a) to deduce that

$$\begin{aligned} &\mathring{\epsilon}\|\partial_t\Psi_0\|_{L^2(\Sigma_t)}^2 + \|\nabla\partial_t\Psi_0\|_{H^3(\Sigma_t)}^2 \\ &\lesssim \sum_{k=2}^{5}\sum_{a=1}^{3} \|\{\mathscr{W}(\mathcal{I}^{-1}\Psi_0)\}^{1/2}\nabla^k\Psi_a\|_{L^2(\Sigma_t)}^2 + \sum_{a=1}^{3}\|\nabla\Psi_a\|_{L^2(\Sigma_t)}^2 + \mathring{\epsilon}\|F_0^{(0)}\|_{L^2(\Sigma_t)}^2 + \sum_{k=1}^{4}\|F_0^{(k)}\|_{L^2(\Sigma_t)}^2. \end{aligned} \tag{5E.8}$$

Next, we recall that the already-proven estimates (5E.4a)–(5E.4d) imply that the bootstrap assumptions (3B.5a)–(3B.5d) hold with $C\mathring{\epsilon}$ in place of ε. It follows that the pointwise estimate (5D.3) holds with $C\mathring{\epsilon}$ in place of ε. From this fact, Definition 4.2, the energy estimate (5E.1), and the data-size assumptions (3A.1a), we deduce that the right-hand side of (5E.8) is $\lesssim \mathring{\epsilon}^2$, which is the desired bound (5E.2a).

The estimate (5E.2b) can be proved using similar arguments based on the evolution equation (5D.1b) and the pointwise estimate (5D.4), and we omit the details. □

6. Local well-posedness and continuation criteria

In this section, we provide a proposition that yields standard well-posedness results and continuation criteria pertaining to the quantities $\{\partial_\alpha \Phi\}_{\alpha=0,1,2,3}$, $\mathcal{I}$, and $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$.

Proposition 6.1. *Let $N \geq 3$ be an integer and let $(\partial_t \Phi|_{\Sigma_0}, \partial_1 \Phi|_{\Sigma_0}, \partial_2 \Phi|_{\Sigma_0}, \partial_3 \Phi|_{\Sigma_0}) = (\mathring{\Psi}_0, \mathring{\Psi}_1, \mathring{\Psi}_2, \mathring{\Psi}_3)$ be initial data for the wave equation* (1B.1a) *satisfying $\mathring{\Psi}_\alpha \in H^N(\mathbb{R}^3)$, $\alpha = 0, 1, 2, 3$, and with $\partial_i \mathring{\Psi}_j = \partial_j \mathring{\Psi}_i$ for $i, j = 1, 2, 3$ (see Remark 1.1). Let $\mathcal{H} := \left(-\frac{1}{2}, \infty\right)$ denote the regime of hyperbolicity, and note that the following holds*: (1B.1a) *is a nondegenerate*[28] *wave equation at points $(t, \underline{x})$ such that $\partial_t \Phi(t, \underline{x}) \in \mathcal{H}$ (see* (2A.1) *for justification of this assertion). Assume that $\mathring{\Psi}_0(\mathbb{R}^3)$ is contained in a compact subset $\mathfrak{K}$ of $\mathcal{H}$. Let $\mathcal{I}$, $\mathcal{I}_\star$, and $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$ be the quantities defined in Definitions 2.1 and 2.3. Then there exist a compact set $\mathfrak{K}'$ of $\mathcal{H}$ containing $\mathfrak{K}$ in its interior and a time $T > 0$, depending on $\mathfrak{K}$ and $\sum_{\alpha=0}^{3} \|\mathring{\Psi}_\alpha\|_{H^N}$, such that a unique*[29] *classical solution to* (1B.1a) *exists on $[0, T) \times \mathbb{R}^3$, such that $\partial_t \Phi([0, T) \times \mathbb{R}^3) \subset \mathfrak{K}'$, and such that the following regularity properties hold for $\alpha = 0, 1, 2, 3$*:

$$\partial_\alpha \Phi \in C([0, T), H^N). \tag{6.1}$$

In addition, the solution depends continuously on the data.

Let $T_{(\mathrm{Lifespan})}$ be the supremum of all times $T > 0$ such that the classical solution to (1B.1a) *exists on $[0, T) \times \mathbb{R}^3$ and satisfies the above properties. Then either $T_{(\mathrm{Lifespan})} = \infty$, or $T_{(\mathrm{Lifespan})} < \infty$ and one of the following two breakdown scenarios must occur*:

(1) *There exists a sequence of points $\{(t_n, \underline{x}_n)\}_{n=1}^{\infty} \subset [0, T_{(\mathrm{Lifespan})}) \times \mathbb{R}^3$ such that $\partial_t \Phi(t_n, \underline{x}_n)$ escapes every compact subset of $\mathcal{H}$ as $n \to \infty$.*

(2) $\lim_{t \uparrow T_{(\mathrm{Lifespan})}} \sup_{s \in [0,t)} \sum_{\alpha=0}^{3} \|\partial_\alpha \partial_t \Phi\|_{L^\infty(\Sigma_s)} = \infty$.

Moreover, on $[0, T_{(\mathrm{Lifespan})}) \times \mathbb{R}^3$, $\mathcal{I}$ and $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$ are classical solutions to (2B.1) *and* (2B.6a)–(2B.6b) *such that*

$$\mathcal{I} - 1 \in C([0, T_{(\mathrm{Lifespan})}), H^{N+1}(\mathbb{R}^3)), \qquad \Psi_\alpha \in C([0, T_{(\mathrm{Lifespan})}), H^N(\mathbb{R}^3)). \tag{6.2}$$

Finally, the quantity $\mathcal{I}_\star$ defined in (2B.2) *satisfies the following estimates*:

$$0 < \mathcal{I}_\star(t) < \infty \quad \textit{for } t \in [0, T_{(\mathrm{Lifespan})}). \tag{6.3}$$

[28] By nondegenerate, we mean that relative to the Cartesian coordinates, the 4×4 matrix of components $g_{\alpha\beta}$ has signature $(-, +, +, +)$, where $g := -dt^2 + 1/(\mathscr{W}(\partial_t \Phi)) \sum_{a=1}^{3} (dx^a)^2$ is the metric corresponding to (1B.1a).

[29] More precisely, Φ is uniquely determined only up to a constant when only its first derivatives along Σ_0 are prescribed; see Remark 1.1.

Proof. The statements concerning Φ are standard and can be proved using the ideas found, for example, in [Speck 2008].

Next, we note that the evolution equation plus the initial condition for $\mathcal{I}$ stated in (2B.1), the fact that $\mathcal{I}(t,\cdot)-1$ is compactly supported in space (see Remark 3.3), and the fact that

$$\partial_t\Phi \in C([0,T_{(\text{Lifespan})}), H^N(\mathbb{R}^3)) \subset C([0,T_{(\text{Lifespan})}), C^1(\mathbb{R}^3))$$

(i.e., (6.1)) can be used to deduce (6.3). Similarly, from (2B.1), the identity (2B.4), the definition $\Psi_\alpha := \mathcal{I}\partial_\alpha\Phi$ (see Definition 2.3), (6.1), and the standard Sobolev–Moser calculus, it is straightforward to deduce (6.2). □

7. The main theorem

In this section, we state and prove our main stable blowup-result.

Theorem 7.1 (stable ODE-type blowup). *Assume that the weight function $\mathscr{W}$ appearing in the wave equation* (1B.1a) *satisfies the assumptions stated in Section 2A. Consider compactly supported initial data* $(\partial_t\Phi|_{\Sigma_0}, \partial_1\Phi|_{\Sigma_0}, \partial_2\Phi|_{\Sigma_0}, \partial_3\Phi|_{\Sigma_0}) = (\mathring{\Psi}_0, \mathring{\Psi}_1, \mathring{\Psi}_2, \mathring{\Psi}_3)$ *for the wave equation* (1B.1a) *with* $\partial_i\mathring{\Psi}_j = \partial_j\mathring{\Psi}_i$ *for* $i,j=1,2,3$ (*see Remark 1.1*) *that satisfy the data-size assumptions* (3A.1a)–(3A.1c) *involving the parameters* $\mathring{\epsilon}$ *and* $\mathring{A}$, *and let* $\mathring{A}_*$ *be the data-size parameter defined in* (3A.2). *Let* $\mathcal{I}$, $\mathcal{I}_\star$, *and* $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$ *be the quantities defined in Definitions 2.1 and 2.3. We define*

$$T_{(\text{Lifespan})} := \sup\{t>0 \mid \{\partial_\alpha\Phi\}_{\alpha=0,1,2,3} \text{ exist classically on } [0,t)\times\mathbb{R}^3\}. \tag{7.1}$$

If $\mathring{\epsilon}>0$, $\mathring{A}>0$, *and* $\mathring{A}_*>0$, *and if* $\mathring{\epsilon}$ *is small relative to* $\mathring{A}^{-1}$ *and* $\mathring{A}_*$ *in the sense explained in Section 3C, then the following conclusions hold.*

- *Characterization of the solution's classical lifespan*: *The solution's classical lifespan is characterized by* $\mathcal{I}_\star$ *as follows*:

$$T_{(\text{Lifespan})} = \sup\Big\{t>0 \,\Big|\, \inf_{s\in[0,t)} \mathcal{I}_\star(s) > 0\Big\}. \tag{7.2}$$

Moreover,

$$\mathcal{I}(t,\underline{x}) > 0 \quad \text{for } (t,\underline{x}) \in [0,T_{(\text{Lifespan})})\times\mathbb{R}^3, \tag{7.3a}$$

$$\lim_{t\uparrow T_{(\text{Lifespan})}} \mathcal{I}_\star(t) = 0. \tag{7.3b}$$

In addition, *the following estimate holds*:

$$T_{(\text{Lifespan})} = \mathring{A}_*^{-1}\{1+\mathcal{O}(\mathring{\epsilon})\}. \tag{7.4}$$

- *Regularity properties of* Ψ_α *and* $\mathcal{I}$ *on* $[0,T_{(\text{Lifespan})})\times\mathbb{R}^3$: *On the slab* $[0,T_{(\text{Lifespan})})\times\mathbb{R}^3$, *the solution satisfies the energy bounds* (5E.1)–(5E.3), *the* L^∞ *estimates* (5E.4a)–(5E.4e), (5B.1)–(5B.2), *and* (5B.3a)–(5B.5) (*with* $C\mathring{\epsilon}$ *on the right-hand side in place of* ε *in these equations*). *Moreover*, $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$ *and* $\mathcal{I}$ *enjoy the following regularity*:

$$\Psi_0 \in C([0,T_{(\text{Lifespan})}), H^5(\mathbb{R}^3)) \cap L^\infty([0,T_{(\text{Lifespan})}), H^5(\mathbb{R}^3)), \tag{7.5a}$$

$$\Psi_i \in C([0,T_{(\text{Lifespan})}), H^5(\mathbb{R}^3)) \cap L^\infty([0,T_{(\text{Lifespan})}), H^4(\mathbb{R}^3)), \tag{7.5b}$$

$$\mathcal{I}-1 \in C([0,T_{(\text{Lifespan})}), H^6(\mathbb{R}^3)) \cap L^\infty([0,T_{(\text{Lifespan})}), H^5(\mathbb{R}^3)). \tag{7.5c}$$

- *Regularity properties of* Ψ_α *and* $\mathcal{I}$ *on* $[0, T_{(\text{Lifespan})}] \times \mathbb{R}^3$: Ψ_α *and* $\mathcal{I}$ *do not blow up at time* $T_{(\text{Lifespan})}$, *but rather continuously extend to* $[0, T_{(\text{Lifespan})}] \times \mathbb{R}^3$ *as functions that enjoy the following regularity, where* N *is any real number with* $N < 5$:

$$\Psi_0 \in L^\infty([0, T_{(\text{Lifespan})}], H^5(\mathbb{R}^3)) \cap C([0, T_{(\text{Lifespan})}], H^N(\mathbb{R}^3)), \tag{7.6a}$$

$$\Psi_i \in C([0, T_{(\text{Lifespan})}], H^4(\mathbb{R}^3)), \tag{7.6b}$$

$$\mathcal{I} - 1 \in C([0, T_{(\text{Lifespan})}], H^5(\mathbb{R}^3)). \tag{7.6c}$$

- *Description of the vanishing of* $\mathcal{I}$ *and the blowup of* $\partial_t \Phi$: *For* $(t, \underline{x}) \in [0, T_{(\text{Lifespan})}) \times \mathbb{R}^3$, *we have*

$$\mathcal{I}(t, \underline{x}) \le \tfrac{1}{4} \quad \Longrightarrow \quad \partial_t \Phi(t, \underline{x}) \ge \frac{\mathring{A}_*}{4\mathcal{I}(t, \underline{x})}. \tag{7.7}$$

Let

$$\Sigma_{T_{(\text{Lifespan})}}^{\text{Blowup}} := \{(T_{(\text{Lifespan})}, \underline{x}) \mid \mathcal{I}(T_{(\text{Lifespan})}, \underline{x}) = 0\}. \tag{7.8}$$

Then if $(T_{(\text{Lifespan})}, \underline{x}) \in \Sigma_{T_{(\text{Lifespan})}}^{\text{Blowup}}$, *we have*[30]

$$\lim_{t \uparrow T_{(\text{Lifespan})}} \partial_t \Phi(t, \underline{x}) = \infty. \tag{7.9}$$

Finally, if $(T_{(\text{Lifespan})}, \underline{x}) \notin \Sigma_{T_{(\text{Lifespan})}}^{\text{Blowup}}$, *then there exists an open ball* $B_{\underline{x}} \subset \mathbb{R}^3$ *centered at* $\underline{x}$ *such that, for* $\alpha = 0, 1, 2, 3$, *we have* $\partial_\alpha \Phi \in C([0, T_{(\text{Lifespan})}], H^5(B_{\underline{x}}))$.

Proof. Let $C_* > 1$ be a constant; we will enlarge C_* as needed throughout the proof. Let $T_{(\text{Max})}$ be the supremum of the set of real numbers T with $0 \le T \le 2\mathring{A}_*^{-1}$ such that the following properties hold:

- $\{\partial_\alpha \Phi\}_{\alpha=0,1,2,3}$ is a classical solution to (1B.1a) on $[0, T) \times \mathbb{R}^3$ (see Remark 1.1) satisfying the properties stated in Proposition 6.1 (with $N = 5$ in the proposition).
- $\mathcal{I}$ is a classical solution to (2B.1) on $[0, T) \times \mathbb{R}^3$ satisfying the properties stated in Proposition 6.1 (with $N = 5$ in the proposition).
- $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$ are classical solutions to (2B.6a)–(2B.6b) for $(t, \underline{x}) \in [0, T) \times \mathbb{R}^3$ such that the properties stated in Proposition 6.1 hold (with $N = 5$ in the proposition).
- The bootstrap assumptions (3B.3) and (3B.4) hold for $(t, \underline{x}) \in [0, T) \times \mathbb{R}^3$.
- The L^∞ bootstrap assumptions (3B.5a)–(3B.5d) hold for $t \in [0, T)$ with $\varepsilon := C_* \mathring{\epsilon}$.
- $\inf\{\mathcal{I}_\star(t) \mid t \in [0, T)\} > 0$, where $\mathcal{I}_\star$ is defined in (2B.2). Note that this implies that the bootstrap assumption (3B.2) holds for $t \in [0, T)$.

Throughout the rest of the proof, we will assume that $\mathring{\epsilon}$ is sufficiently small and that C_* is sufficiently large without explicitly mentioning it every time. Next, we note that the hypotheses of Proposition 6.1 hold with $N = 5$. Hence, by Proposition 6.1 and Sobolev embedding, we have $T_{(\text{Max})} > 0$.

[30] See also Remark 1.5 concerning the blowup of Φ itself, if initial data for Φ itself are prescribed.

We will now show that $T_{(\text{Max})} = T_{(\text{Lifespan})}$, where $T_{(\text{Lifespan})}$ is defined by (7.1). We first note that clearly, $T_{(\text{Max})} \leq T_{(\text{Lifespan})}$. To proceed, we assume for the sake of deriving a contradiction that

$$\inf_{s\in[0,T_{(\text{Max})})} \mathcal{I}_{\star}(s) > 0. \tag{7.10}$$

Then, in view of Definitions 2.1 and 2.3, (2B.6a)–(2B.6b), the bootstrap assumptions, the assumptions of Section 2A on $\mathscr{W}$, the data-size assumptions (3A.1a), and (3C.1), we see that (7.10) implies

$$\lim_{t\uparrow T_{(\text{Max})}} \sup_{s\in[0,t)} \left\{ \|\partial_t \Phi\|_{L^{\infty}(\Sigma_s)} + \sum_{\alpha=0}^{3} \|\partial_\alpha \partial_t \Phi\|_{L^{\infty}(\Sigma_s)} \right\} < \infty.$$

Also taking into account Remark 5.4, we see that neither of the two breakdown scenarios of Proposition 6.1 occur on $[0, T_{(\text{Max})}) \times \mathbb{R}^3$. Moreover, by Proposition 5.8, if C_* is large enough, then the bootstrap assumption inequalities (3B.5a)–(3B.5d) hold in a strict sense (that is, with "$\leq$" replaced by "$<$") on $[0, T_{(\text{Max})}) \times \mathbb{R}^3$. Moreover, all estimates proved prior to Proposition 5.8 hold with ε replaced by $C\mathring{\epsilon}$, and we will use this fact in the rest of the proof without mentioning it again. Furthermore, (5B.5) and (5B.4b) respectively yield strict improvements of the bootstrap assumptions (3B.3) and (3B.4) for $(t, \underline{x}) \in [0, T_{(\text{Max})}) \times \mathbb{R}^3$. Next, we note that the estimate (5B.3b) implies that $\mathcal{I}_{\star}(t)$ cannot remain positive for t larger than $\mathring{A}_*^{-1}\{1 + \mathcal{O}(\mathring{\epsilon})\}$. From this fact, it follows that $T_{(\text{Max})} < 2\mathring{A}_*^{-1}$. Combining these facts and appealing to Proposition 6.1, we deduce that $\{\partial_\alpha \Phi\}_{\alpha=0,1,2,3}$, $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$, and $\mathcal{I}$ extend as classical solutions to a region of the form $[0, T_{(\text{Max})} + \Delta) \times \mathbb{R}^3$ for some $\Delta > 0$ with $T_{(\text{Max})} + \Delta < 2\mathring{A}_*^{-1}$ (on which these variables enjoy the regularity properties guaranteed by Proposition 6.1), such that $\inf_{s\in[0,T_{(\text{Max})}+\Delta)} \mathcal{I}_{\star}(s) > 0$, and such that the bootstrap assumptions (3B.3)–(3B.5d) hold for $(t, \underline{x}) \in [0, T_{(\text{Max})} + \Delta) \times \mathbb{R}^3$. In total, this contradicts the definition of $T_{(\text{Max})}$. Therefore, (7.10) is impossible, and it follows that

$$T_{(\text{Max})} = \sup\left\{t > 0 \,\middle|\, \inf_{s\in[0,t)} \mathcal{I}_{\star}(s) > 0\right\}, \tag{7.11}$$

that

$$\inf_{s\in[0,T_{(\text{Max})})} \mathcal{I}_{\star}(s) = 0, \tag{7.12}$$

and that the estimates (5E.1)–(5E.3) and (5E.4a)–(5E.4e) hold for $t \in [0, T_{(\text{Max})})$.

Next, we note that the estimate (7.7) follows from (5B.4b). Then, using (7.7) and (7.12), we see that $\lim_{t\uparrow T_{(\text{Max})}} \sup_{s\in[0,t)} \|\partial_t \Phi\|_{L^{\infty}(\Sigma_s)} = \infty$, that is, that $\partial_t \Phi$ blows up at time $T_{(\text{Max})}$. We have therefore shown that $T_{(\text{Max})} = T_{(\text{Lifespan})}$ and that $T_{(\text{Lifespan})}$ is characterized by (7.2). Moreover, from the estimate (5B.3b), we find that $\mathcal{I}_{\star}$ vanishes for the first time at $T_{(\text{Lifespan})} = \mathring{A}_*^{-1}\{1+\mathcal{O}(\mathring{\epsilon})\}$, which in total yields (7.3a) and (7.4).

In the rest of this proof, we sometimes silently use that $\Psi_0 \in L^{\infty}([0, T_{(\text{Lifespan})}), L^2(\mathbb{R}^3))$ and $\mathcal{I} - 1 \in L^{\infty}([0, T_{(\text{Lifespan})}), L^2(\mathbb{R}^3))$. These facts do not follow from the energy estimates (5E.1) and (5E.3), but instead follow from (5E.4a), (5E.4d), and the compactly supported (in space) nature of Ψ_0 and $\mathcal{I} - 1$. Next, we easily conclude from the definition (4A.2) of $\mathbb{Q}_{(\mathring{\epsilon})}(t)$ and the fact that the estimate (5E.1) holds on $[0, T_{(\text{Lifespan})})$ that $\Psi_0 \in L^{\infty}([0, T_{(\text{Lifespan})}), H^5(\mathbb{R}^3))$ (as is stated in (7.5a)) and that $\Psi_i \in L^{\infty}([0, T_{(\text{Lifespan})}), H^4(\mathbb{R}^3))$ (as is stated in (7.5b)). The same reasoning yields that $\Psi_0 \in L^{\infty}([0, T_{(\text{Lifespan})}], H^5(\mathbb{R}^3))$ (as is stated in (7.6a)), where the open time interval is replaced with

$[0, T_{(\text{Lifespan})}]$. The fact that $\Psi_\alpha \in C([0, T_{(\text{Lifespan})}), H^5(\mathbb{R}^3))$ (as is stated in (7.5a)–(7.5b)) is a standard result that can be proved using energy-based arguments (similar to the ones we used to prove (5E.1)) and standard facts from functional analysis. We omit the details and instead refer the reader to [Speck 2008, Section 2.7.5]. We clarify that in proving this "soft result", it is important that, for fixed $t \in [0, T_{(\text{Lifespan})})$, we have $\min_{[0,t]\times\mathbb{R}^3} \mathcal{I} > 0$, which, in view of Remark 5.4, implies in particular that the weight $\mathscr{W}(\mathcal{I}^{-1}\Psi_0)$ on the right-hand side of (4A.2) is bounded from above and from below by t-dependent positive constants on $[0, t] \times \mathbb{R}^3$ (and thus the energy estimates are nondegenerate away from $\Sigma_{T_{(\text{Lifespan})}}$). Through similar reasoning based on (2B.1) (which states that $\partial_t \mathcal{I} = -\Psi_0$), the identity (2B.4), and the estimate (5E.3), we deduce that $\mathcal{I} - 1 \in C([0, T_{(\text{Lifespan})}), H^6(\mathbb{R}^3)) \cap L^\infty([0, T_{(\text{Lifespan})}), H^5(\mathbb{R}^3))$. We have therefore proved (7.5a)–(7.5c).

We will now prove (7.6a)–(7.6c). We first note that the estimates (5E.1) and (5E.2a)–(5E.2b) and (2B.1) imply that $\partial_t \Psi_\alpha \in L^\infty([0, T_{(\text{Lifespan})}), H^4(\mathbb{R}^3))$ and $\partial_t \mathcal{I} \in L^\infty([0, T_{(\text{Lifespan})}), H^5(\mathbb{R}^3))$. Hence, from the fundamental theorem of calculus, the initial conditions (2B.1) and (3A.1a)–(3A.1b), and the completeness of the Sobolev spaces $H^M(\mathbb{R}^3)$, we obtain $\Psi_\alpha \in C([0, T_{(\text{Lifespan})}], H^4(\mathbb{R}^3))$ and $\mathcal{I} - 1 \in C([0, T_{(\text{Lifespan})}], H^5(\mathbb{R}^3))$. In particular, we have shown (7.6b)–(7.6c). Moreover, (7.6c) and Sobolev embedding together yield that $\mathcal{I} \in C([0, T_{(\text{Lifespan})}], C(\mathbb{R}^3))$ and thus $\mathcal{I}_\star \in C([0, T_{(\text{Lifespan})}], [0, \infty))$. Since we have already shown that $T_{(\text{Lifespan})}$ is equal to the right-hand side of (7.11) and shown (7.12), it follows that $\mathcal{I}_\star(T_{(\text{Lifespan})}) = 0$ and that $\lim_{t \uparrow T_{(\text{Lifespan})}} \mathcal{I}_\star(t) = 0$, that is, that (7.3b) holds. To obtain that, for any real number $N < 5$, we have $\Psi_0 \in C([0, T_{(\text{Lifespan})}], H^N(\mathbb{R}^3))$ (as is stated in (7.6a)), we interpolate between[31] L^2 and H^5 and use the already-shown facts $\Psi_0 \in L^\infty([0, T_{(\text{Lifespan})}], H^5(\mathbb{R}^3)) \cap C([0, T_{(\text{Lifespan})}], H^4(\mathbb{R}^3))$. We have therefore proved (7.6a).

The desired localized blowup-result (7.9) for points in $\Sigma^{\text{Blowup}}_{T_{(\text{Lifespan})}}$ (where $\Sigma^{\text{Blowup}}_{T_{(\text{Lifespan})}}$ is defined in (7.8)) now follows from (7.7) and the continuous extension property $\mathcal{I} \in C([0, T_{(\text{Lifespan})}], C(\mathbb{R}^3))$ mentioned in the previous paragraph.

Finally, we will show that if $(T_{(\text{Lifespan})}, \underline{x}) \notin \Sigma^{\text{Blowup}}_{T_{(\text{Lifespan})}}$, then there exists an open ball $B_{\underline{x}} \subset \mathbb{R}^3$ centered at $\underline{x}$ such that, for $\alpha = 0, 1, 2, 3$, we have $\partial_\alpha \Phi \in C([0, T_{(\text{Lifespan})}], H^5(B_{\underline{x}}))$. To proceed, we first note that if $(T_{(\text{Lifespan})}, \underline{x}) \notin \Sigma^{\text{Blowup}}_{T_{(\text{Lifespan})}}$, then the results proved above imply that there exist a $\delta_{\underline{x}} > 0$ and a radius $r_{\underline{x}} > 0$ such that, with $B_{\underline{x};r_{\underline{x}}} \subset \mathbb{R}^3$ denoting the open ball of radius $r_{\underline{x}}$ centered at $\underline{x}$, we have $\mathcal{I}(t, \underline{y}) > 0$ for $(t, \underline{y}) \in [T_{(\text{Lifespan})} - \delta_{\underline{x}}, T_{(\text{Lifespan})}] \times \bar{B}_{\underline{x};r_{\underline{x}}}$ (where $\bar{B}_{\underline{x};r_{\underline{x}}}$ denotes the closure of $B_{\underline{x};r_{\underline{x}}}$) and such that, for $t \in [T_{(\text{Lifespan})} - \delta_{\underline{x}}, T_{(\text{Lifespan})})$, we have $\|\Psi_\alpha\|_{H^5(\{t\}\times B_{\underline{x};r_{\underline{x}}})} < \infty$ and $\|\mathcal{I} - 1\|_{H^6(\{t\}\times B_{\underline{x};r_{\underline{x}}})} < \infty$. Hence, since the wave speed of the system is uniformly bounded from above on $[T_{(\text{Lifespan})} - \delta_{\underline{x}}, T_{(\text{Lifespan})}] \times \bar{B}_{\underline{x};r_{\underline{x}}}$ (see Remark 3.3), since $\mathcal{I}$ is uniformly bounded from above and from below by positive ($\underline{x}$-dependent) constants on $[T_{(\text{Lifespan})} - \delta_{\underline{x}}, T_{(\text{Lifespan})}] \times \bar{B}_{\underline{x};r_{\underline{x}}}$, and since the estimates (7.5a)–(7.5c) and (7.6a)–(7.6c) hold, we can derive Sobolev estimates (using energy-based arguments) similar to the ones that we derived three paragraphs above, but localized in space,[32] for

[31]Here, we mean the following standard inequality: if $f \in H^5(\Sigma_t)$ and $0 \le N \le 5$, then there exists a constant $C_N > 0$ such that $\|f\|_{H^N(\Sigma_t)} \le C_N \|f\|^{1-N/5}_{L^2(\Sigma_t)} \|f\|^{N/5}_{H^5(\Sigma_t)}$.

[32]For example, for $\sigma > 0$ chosen sufficiently large, for t near $T_{(\text{Lifespan})}$, and for $s \in [t, T_{(\text{Lifespan})}]$, one can view the state of the solution on $\{t\} \times B_{\underline{x};r_{\underline{x}}}$ as an "initial" condition and use energy identities to obtain Sobolev estimates on $\{s\} \times B_{\underline{x};r_{\underline{x}} - \sigma s} \subset \Sigma_s$.

(2B.1) and (2B.6a)–(2B.6b), starting from initial conditions on $\{t\} \times B_{\underline{x};r_{\underline{x}}}$ for some t sufficiently close to $T_{(\text{Lifespan})}$. This yields the existence of an open ball $B_{\underline{x}} \subset B_{\underline{x};r_{\underline{x}}}$ centered at $\underline{x}$ such that the following regularity properties hold: $\Psi_\alpha \in C([0, T_{(\text{Lifespan})}], H^5(B_{\underline{x}}))$ and $\mathcal{I} - 1 \in C([0, T_{(\text{Lifespan})}], H^6(B_{\underline{x}}))$. We clarify that in deriving these spatially localized energy estimates on the *closed* time interval $[0, T_{(\text{Lifespan})}]$, we have exploited the following crucially important consequence of the bounds noted above and the assumptions of Section 2A on $\mathscr{W}$: the weight $\mathscr{W}(\mathcal{I}^{-1}\Psi_0)$ (which appears, for example, on the right-hand side of (4A.2)) is strictly positive on the domain $[T_{(\text{Lifespan})} - \delta_{\underline{x}}, T_{(\text{Lifespan})}] \times \bar{B}_{\underline{x}}$. Finally, from the regularity properties of $\{\Psi_\alpha\}_{\alpha=0,1,2,3}$ and $\mathcal{I}$ mentioned above, the positivity of $\mathcal{I}$ on $[T_{(\text{Lifespan})} - \delta_{\underline{x}}, T_{(\text{Lifespan})}] \times \bar{B}_{\underline{x}}$, and the standard Sobolev–Moser calculus, we conclude, in view of Definition 2.3, the desired result $\partial_\alpha \Phi \in C([0, T_{(\text{Lifespan})}], H^5(B_{\underline{x}}))$. We have therefore proved the theorem. □

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