ANALYSIS & PDE

Volume 13 No. 2 2020

NGOC CUONG NGUYEN

ON THE HÖLDER CONTINUOUS SUBSOLUTION PROBLEM FOR THE COMPLEX MONGE–AMPÈRE EQUATION, II

msp

ANALYSIS AND PDE
Vol. 13, No. 2, 2020
dx.doi.org/10.2140/apde.2020.13.435

ON THE HÖLDER CONTINUOUS SUBSOLUTION PROBLEM FOR THE COMPLEX MONGE–AMPÈRE EQUATION, II

NGOC CUONG NGUYEN

We solve the Dirichlet problem for the complex Monge–Ampère equation on a strictly pseudoconvex domain with the right-hand side being a positive Borel measure which is dominated by the Monge–Ampère measure of a Hölder continuous plurisubharmonic function. If the boundary data is continuous, then the solution is continuous. If the boundary data is Hölder continuous, then the solution is also Hölder continuous. In particular, the answer to a question of A. Zeriahi is always affirmative.

1. Introduction

The Hölder regularity of plurisubharmonic solutions to the complex Monge–Ampère equation in a strictly pseudoconvex domain has a long history. First, Bedford and Taylor [1976] obtained Hölder continuous solutions for the Dirichlet problem of the equation assuming the right-hand side is Hölder continuous. Later, this result was extended to a larger class of measures by Guedj, Kołodziej and Zeriahi [Guedj et al. 2008]; namely, the measures have L^p density with respect to the Lebesgue measure with some extra assumptions on the density near the boundary and the boundary data. The extra assumptions are removed in other subsequent works [Baracco et al. 2016; Charabati 2015]. On the other hand, the complex Monge–Ampère operator of a Hölder continuous plurisubharmonic function is not necessary absolutely continuous with respect to the Lebesgue measure. Examples of such measures are Hausdorff measures due to Charabati [2017], and the volume form of a smooth real hypersurface of codimension 1 by Pham [2010]. (See also [Vu 2018] for a generalization to generic CR manifolds of arbitrary codimension.) So far, these results give only sufficient conditions on the measures such that the solution to the equation is Hölder continuous. In [Nguyen 2018] we gave a necessary and sufficient condition for a measure whose Monge–Ampère potential is Hölder continuous. This result is partly inspired by a global result due to Dinh and Nguyên [2014]. However, to use the result in [Nguyen 2018] we require the Hölder continuous subsolution have zero value on the boundary and its total Monge–Ampère mass be finite, which cannot be true for a general Hölder continuous plurisubharmonic function (Remark 1.1). In this paper we will remove these restrictions.

There are several motivations to study the Hölder regularity of solutions. First, it is a basic question in pluripotential theory to characterize measures for which the complex Monge–Ampère equation admits bounded, continuous and Hölder continuous solutions [Kołodziej 2013]. Next, Dinh, Nguyên and Sibony [Dinh et al. 2010] showed that the Monge–Ampère measure of a Hölder continuous plurisubharmonic

MSC2010: 32U40, 35J96, 53C55.
Keywords: Dirichlet problem, weak solutions, Hölder continuous, Monge–Ampère, subsolution problem.

function is locally moderate, which is a very useful generalization of Skoda's theorem. We refer the reader to [Dinh et al. 2010] for its application in complex dynamics. Their work leads to the interesting open problem of whether the converse holds. The question in the toric setting has been studied recently in [Coman et al. 2018, Theorem 4.4]. In this case the problem reduces to a real Monge–Ampère equation on a convex polytope. Hölder continuity is also studied with regard to the extremal functions arising in (pluri-)potential theory. In fact the Hölder continuity of the so-called relative extremal function u_K and the Siciak–Zahariuta extremal function V_K [Siciak 1997; 2000] of a compact set $K \subset \mathbb{C}^n$ is proven to be equivalent to a Markov-type inequality in multivariate interpolation theory [Baran and Bialas-Ciez 2014; Pawłucki and Pleśniak 1986] (see [Dinh et al. 2017; Vu 2018] for analogous results in the compact Kähler manifold setting). We believe that our work will be useful to study the Hölder continuity of the above extremal functions. For example we hope the techniques developed here can be used to simplify the proof in [Vu 2018].

In this paper we continue our research, initiated in [Nguyen 2018], which focuses on the Dirichlet problem for the complex Monge–Ampère equation in a bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$, provided a Hölder continuous subsolution exists. Let $\varphi \in \mathrm{PSH}(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$ for some $0 < \alpha \leq 1$. Assume also that

$$\varphi = 0 \quad \text{on } \partial\Omega.$$

We consider the set

$$\mathcal{M}(\varphi, \Omega) := \{\mu \text{ is positive Borel measure} : \mu \leq (dd^c\varphi)^n \text{ in } \Omega\}.$$

We also say that φ is a Hölder continuous subsolution to measures in $\mathcal{M}(\varphi, \Omega)$. Given ψ a Hölder continuous function on the boundary $\partial\Omega$ and a measure μ in $\mathcal{M}(\varphi, \Omega)$ we look for a real-valued function u satisfying

$$u \in \mathrm{PSH} \cap L^\infty(\Omega), \qquad (dd^c u)^n = \mu \quad \text{in } \Omega, \qquad \lim_{z \to x} u(z) = \psi(x) \quad \text{for } x \in \partial\Omega, \tag{1-1}$$

and

$$u \in C^{0,\alpha'}(\bar{\Omega}) \quad \text{for some } 0 < \alpha' \leq 1. \tag{1-2}$$

The Dirichlet problem (1-1) was solved by Kołodziej [1995] provided that there exists a bounded plurisubharmonic subsolution. In our setting, the Hölder continuity of ψ on $\partial\Omega$ and of φ on $\bar{\Omega}$ are necessary in order to solve the Dirichlet problem (1-1), (1-2). In [Nguyen 2018] this problem is solved under the extra assumptions

$$\psi \equiv 0 \quad \text{and} \quad \int_\Omega (dd^c\varphi)^n < +\infty.$$

We will see now that these assumptions are not generic.

Remark 1.1. Let ρ be a defining function for a smoothly bounded pseudoconvex domain Ω. Then, $-|\rho|^\alpha$ for $0 < \alpha < 1$ is Hölder continuous on $\bar{\Omega}$ and its Monge–Ampère measure is

$$\alpha^n |\rho|^{n(\alpha-1)} (dd^c\rho)^n + n\alpha^n (1-\alpha) |\rho|^{n(\alpha-1)-1} d\rho \wedge d^c\rho \wedge (dd^c\rho)^{n-1}.$$

Therefore, for a neighborhood $U_x \subset \mathbb{C}^n$ of a strictly pseudoconvex point $x \in \partial\Omega$,

$$\int_{U_x \cap \Omega} [dd^c(-|\rho|^\alpha)]^n = +\infty.$$

In particular, $\int_\Omega [dd^c(-|\rho|^\alpha)]^n = +\infty$. Furthermore, let v be a plurisubharmonic function on Ω and Hölder continuous on $\overline{\Omega}$ satisfying

$$\int_\Omega (dd^c v)^n < +\infty.$$

A typical way to modify the value of v on $\partial\Omega$ is to add it to an envelope

$$h(z) = \sup\{w(z) : w \in \mathrm{PSH}(\Omega) \cap C^0(\overline{\Omega}) : w_{|\partial\Omega} \le -v\}. \tag{1-3}$$

However, we cannot guarantee that the function $v + h$ has finite Monge–Ampère mass on Ω. Thus, removing the above assumptions is desirable for applications.

The first main result of this paper is as follows.

Theorem A. *Let $\psi \in C^0(\partial\Omega)$ and $\mu \in \mathcal{M}(\varphi, \Omega)$. Then, there exists a unique solution $u \in C^0(\overline{\Omega})$ to the Dirichlet problem* (1-1).

This theorem is closely related to a question of S. Kołodziej [Dinew et al. 2016, Question 14], where he asked if one could prove Theorem A when the subsolution φ is only *continuous*? The question is still open in general. Very recently in [Kołodziej and Nguyen 2018b] we showed that Theorem A still holds true for the subsolution φ satisfying a Dini-type continuity condition.

The next result gives a necessary and sufficient condition under which a positive Borel measure admits a Hölder continuous plurisubharmonic potential. In particular, the answer to a question of A. Zeriahi [Dinew et al. 2016, Question 17] is affirmative.

Theorem B. *Assume that ψ is Hölder continuous and $\mu \in \mathcal{M}(\varphi, \Omega)$. Then, the Dirichlet problem* (1-1), (1-2) *is solvable.*

This theorem can be seen as the local version of [Demailly et al. 2014], where the compact Kähler manifold setting was considered (see also [Kołodziej and Nguyen 2018a] for the Hermitian manifold case and the notion of subsolution in the compact manifold setting there). Now, we can say that the complex Monge–Ampère equation on a compact Kähler (Hermitian) manifold admits a Hölder continuous solution if and only if it can be written locally as Monge–Ampère operators of Hölder continuous plurisubharmonic functions. The result has been also generalized to the complex Hessian equation [Kołodziej and Nguyen 2019]. Given Hölder continuous plurisubharmonic functions $u_1, \dots, u_n$ in Ω, it follows by the theorem that there exists a Hölder continuous plurisubharmonic function such that

$$(dd^c u)^n = dd^c u_1 \wedge \cdots \wedge dd^c u_n.$$

We also obtain the convexity of the set of Monge–Ampère measures of Hölder continuous plurisubharmonic functions in Ω. Another important consequence is the so-called L^p property given in Corollary C below. In particular, our result covers the main findings in [Baracco et al. 2016, Charabati 2015; 2017], where the L^p property with respect to the Lebesgue measure was considered.

Corollary C. *Let $\mu \in \mathcal{M}(\varphi, \Omega)$ and $f \in L^p(\Omega, d\mu)$, $p > 1$, a nonnegative function. Suppose that φ is a Hölder continuous plurisubharmonic function on a neighborhood of $\bar{\Omega}$. Then, $f\mu \in \mathcal{M}(\tilde{\varphi}, \Omega)$ for a Hölder continuous plurisubharmonic function $\tilde{\varphi}$ in Ω.*

2. Preliminaries

In this section we will recall results that are needed in the proofs of Theorems A and B and Corollary C. If there is no other indication, then the notation in this section will be used for the rest of the paper.

Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let $\rho \in C^2(\bar{\Omega})$ be a strictly plurisubharmonic defining function for Ω. Namely,

$$\Omega = \{\rho < 0\} \qquad \text{and} \qquad d\rho \neq 0 \quad \text{on } \partial\Omega. \tag{2-1}$$

Let us denote by $\beta = dd^c|z|^2$ the standard Kähler form in $\mathbb{C}^n$. Without loss of generality we may assume

$$dd^c\rho \geq \beta \quad \text{on } \bar{\Omega}. \tag{2-2}$$

Throughout the paper the Hölder continuous subsolution φ and the associated set of measures $\mathcal{M}(\varphi, \Omega)$ are defined as in the Introduction.

The following estimate will be very useful for us. For simplicity we write

$$\|\cdot\|_\infty := \sup_\Omega |\cdot| \quad \text{and} \quad \|\cdot\|_p := \int_\Omega |\cdot|^p \, dV_{2n}, \tag{2-3}$$

for the Lebesgue L^p-norm for $p \geq 1$.

Lemma 2.1 [Błocki 1993]. *Let $v_1, \dots, v_n, v, h \in \mathrm{PSH} \cap L^\infty(\Omega)$ be such that $v_i \leq 0$ for $i = 1, \dots, n$, and $v \leq h$. Assume that $\lim_{z\to\partial\Omega}[h(z) - v(z)] = 0$. Then, for an integer $1 \leq k \leq n$,*

$$\int_\Omega (h - v)^k \, dd^c v_1 \wedge \cdots \wedge dd^c v_n \leq k! \, \|v_1\|_\infty \cdots \|v_k\|_\infty \int_\Omega (dd^c v)^k \wedge dd^c v_{k+1} \wedge \cdots \wedge dd^c v_n. \tag{2-4}$$

Consider also the Cegrell class

$$\mathcal{E}_0 = \left\{ v \in \mathrm{PSH} \cap L^\infty(\Omega) \;\middle|\; \lim_{x\to z} v(x) = 0 \text{ for all } z \in \partial\Omega \text{ and } \int_\Omega (dd^c v)^n < +\infty \right\}. \tag{2-5}$$

The Cegrell inequality in this class reads:

Lemma 2.2 [Cegrell 2004]. *Let $v_1, \dots, v_n \in \mathcal{E}_0$. Then,*

$$\int_\Omega dd^c v_1 \wedge \cdots \wedge dd^c v_n \leq \left(\int_\Omega (dd^c v_1)^n \right)^{1/n} \cdots \left(\int_\Omega (dd^c v_n)^n \right)^{1/n}. \tag{2-6}$$

We need also to work with a subclass of the Cegrell class:

$$\mathcal{E}_0' := \left\{ v \in \mathcal{E}_0 : \int_\Omega (dd^c v)^n \leq 1 \right\}. \tag{2-7}$$

The decay of the volume of sublevel sets of functions in the class $\mathcal{E}_0'$ is equivalent to the volume-capacity inequality. This inequality plays a crucial role in the capacity method due to Kołodziej to obtain the *a priori and stability estimates* for weak solutions of the complex Monge–Ampère equation. Here the capacity is the Bedford–Taylor capacity and it is defined as follows. For a Borel set $E \subset \Omega$,

$$\operatorname{cap}(E, \Omega) := \sup\left\{ \int_E (dd^c w)^n : w \in \mathrm{PSH}(\Omega),\ 0 \le w \le 1 \right\}. \tag{2-8}$$

In what follows we shall write $\operatorname{cap}(E)$ instead of $\operatorname{cap}(E, \Omega)$ for simplicity as the domain Ω is already fixed.

3. Proof of Theorem A

In this section we shall prove the following result.

Proposition 3.1. *Assume that $\mu \in \mathcal{M}(\varphi, \Omega)$. Then, there exist uniform constants $\alpha_0, C > 0$ depending only on φ, Ω such that, for every compact set $K \subset \Omega$,*

$$\mu(K) \le C \operatorname{cap}(K) \exp\left(\frac{-\alpha_0}{[\operatorname{cap}(K)]^{1/n}} \right). \tag{3-1}$$

Notice that under the assumption $\int_\Omega (dd^c\varphi)^n < +\infty$ a similar inequality, without the factor $\operatorname{cap}(K)$ on the right-hand side, was proven in [Nguyen 2018].

Remark 3.2. Theorem A will follow immediately from the proposition and [Kołodziej 2005, Theorem 5.9] as μ belongs to the class $\mathcal{F}(A, h)$ with $h = e^{\alpha_0 x}$ and a uniform $A > 0$.

We will need the following two lemmas. The first one tells us how fast the Monge–Ampère mass of $(dd^c\varphi)^n$ on large sublevel sets goes to infinity.

Lemma 3.3. *Let $v \in \mathcal{E}_0'$. Then, there exists a uniform constant C such that, for $s > 0$,*

$$\int_{\{v<-s\}} (dd^c\varphi)^n \le \frac{C\|\varphi\|_\infty^n}{s^n}. \tag{3-2}$$

This estimate should be compared with [Kołodziej 2005, Lemma 4.1]. If φ is a C^2-smooth function on $\overline{\Omega}$, then exponential decay as s tends to $+\infty$ has been obtained there. Although in our case, we are more interested in the situation when s tends to 0.

Proof. Set $v_s := \max\{v, -s\}$. Then, $v_s = v$ on a neighborhood of $\partial\Omega$. Moreover,

$$v_{s/2} - v \ge \frac{s}{2} \quad \text{on } \{v < -s\} \Subset \Omega. \tag{3-3}$$

Therefore,

$$\int_{\{v<-s\}} (dd^c\varphi)^n \le \left(\frac{2}{s}\right)^n \int_\Omega (v_{s/2} - v)^n (dd^c\varphi)^n \le \frac{2^n n!}{s^n} \|\varphi\|_\infty^n \int_\Omega (dd^c v)^n, \tag{3-4}$$

where the second inequality follows from Lemma 2.1. □

On the other hand the volume with respect to the measure $(dd^c\varphi)^n$ of small sublevel sets of functions in $\mathcal{E}_0'$ decays exponentially fast to zero. The Hölder continuity of φ is crucially important to prove such an estimate.

Lemma 3.4. *There exist uniform constants $\tau > 0$ and $C > 0$ such that, for $v \in \mathcal{E}_0'$ and $s \geq 2$,*

$$\int_{\{v<-s\}} (dd^c\varphi)^n \leq Ce^{-\tau s}. \tag{3-5}$$

Proof. We follow ideas of Dinh, Nguyên and Sibony [Dinh et al. 2010]. With the same notation as in the proof of Lemma 3.3 we have for $s \geq 2$

$$\int_{\{v<-s\}} (dd^c\varphi)^n \leq \frac{2}{s}\int_\Omega (v_{s/2}-v)(dd^c\varphi)^n \leq \int_\Omega (v_{s/2}-v)(dd^c\varphi)^n. \tag{3-6}$$

Let us define

$$S_k := (dd^c\varphi)^k \wedge \beta^{n-k},$$

where $\beta = dd^c|z|^2$ and $0 \leq k \leq n$ is an integer. Our first goal is to show that there exist $\alpha_k > 0$ and $C > 0$ (independent of v and s) such that, for $v \in \mathcal{E}_0'$ and $s \geq 1$,

$$\int_\Omega (v_s - v)S_k \leq C\left(\int_\Omega (v_s - v)\, dV_{2n}\right)^{\alpha_k}, \tag{3-7}$$

where $v_s = \max\{v, -s\}$. Indeed, without loss of generality we may assume that

$$0 < \|v_s - v\|_1 < \tfrac{1}{100}. \tag{3-8}$$

Otherwise, if $\|v_s - v\|_1 = 0$, then the inequality trivially holds. If $\|v_s - v\|_1 \geq \frac{1}{100}$, then we have, using $s \geq 1$, $v \leq 0$, and Lemma 2.1, that

$$\int_\Omega (v_s - v)S_k = \int_{\{v<-s\}} (-s-v)S_k \leq \int_\Omega (-v)^k S_k \leq C\|\varphi\|_\infty^k. \tag{3-9}$$

This implies the inequality.

Next, under the assumption (3-8) we prove the inequality by induction in k. The case $k = 0$ is obvious. Assume that for every integer $m \leq k$ we have

$$\int_\Omega (v_s - v)S_m \leq C\left(\int_\Omega (v_s - v)\, dV_{2n}\right)^{\alpha_m}. \tag{3-10}$$

Then, we need to show that there exists $0 < \alpha_{k+1} \leq 1$ such that

$$\int_\Omega (v_s - v)S_{k+1} \leq C\left(\int_\Omega (v_s - v)\, dV_{2n}\right)^{\alpha_{k+1}}. \tag{3-11}$$

For simplicity we write

$$S := (dd^c\varphi)^k \wedge \beta^{n-k-1}. \tag{3-12}$$

Let us still write φ to be a Hölder continuous extension of φ onto a neighborhood U of $\overline{\Omega}$. Consider the convolution of φ with the standard smoothing kernel χ, i.e., $\chi \in C_c^\infty(\mathbb{C}^n)$ a radial function such that $\chi(z) \geq 0$, $\operatorname{supp}\chi \Subset B(0,1)$ and $\int_{\mathbb{C}^n} \chi(z)\, dV_{2n} = 1$. Namely, for $z \in U$ and $\delta > 0$ small,

$$\varphi * \chi_t(z) = \int_{B(0,1)} \varphi(z - tz')\chi(z')\, dV_{2n}(z') = \frac{1}{t^{2n}}\int_{B(z,t)} \varphi(z')\chi\left(\frac{z-z'}{t}\right) dV_{2n}(z'). \tag{3-13}$$

Observe that

$$|\varphi * \chi_t(z) - \varphi(z)| \le \int_{B(0,1)} |\varphi(z - tz') - \varphi(z)|\chi(z')\, dV_{2n}(z') \le Ct^\alpha \tag{3-14}$$

and

$$\left|\frac{\partial^2 \varphi * \chi_t}{\partial z_j \partial \bar{z}_k}(z)\right| \le \frac{C\|\varphi\|_\infty}{t^2}. \tag{3-15}$$

We first have

$$\begin{aligned}\int_\Omega (v_s - v)\, dd^c\varphi \wedge S &\le \left|\int_\Omega (v_s - v)\, dd^c\varphi * \chi_t \wedge S\right| + \left|\int_\Omega (v_s - v)\, dd^c(\varphi * \chi_t - \varphi) \wedge S\right| \\ &=: I_1 + I_2. \end{aligned}\tag{3-16}$$

It follows from (3-15) that

$$I_1 \le \frac{C\|\varphi\|_\infty}{t^2} \int_\Omega (v_s - v) S \wedge \beta = \frac{C\|\varphi\|_\infty}{t^2} \int_\Omega (v_s - v) S_k. \tag{3-17}$$

Hence,

$$I_1 \le \frac{C\|\varphi\|_\infty}{t^2} \|v_s - v\|_1^{\alpha_k}. \tag{3-18}$$

We turn to the estimate of the second integral I_2. By integration by parts

$$\begin{aligned}\int_\Omega (v_s - v) dd^c(\varphi * \chi_t - \varphi) \wedge S &= \int_\Omega (\varphi * \chi_t - \varphi) dd^c(v_s - v) \wedge S \\ &= \int_{\{v < -s/2\}} (\varphi * \chi_t - \varphi)\, dd^c(v_s - v) \wedge S, \end{aligned}\tag{3-19}$$

as $v_s = v$ on $\{v \ge -s\}$. Hence,

$$I_2 \le \int_{\{v<-s/2\}} |\varphi * \chi_t - \varphi| (dd^c v + dd^c v_s) \wedge S \le Ct^\alpha \int_{\{v<-s/2\}} (dd^c v + dd^c v_s) \wedge S. \tag{3-20}$$

For the first term of the integral on the right-hand side we have

$$\begin{aligned}\int_{\{v<-s/2\}} dd^c v \wedge S &\le \left(\frac{4}{s}\right)^k \int_{\{v<-s/4\}} (v_{s/4} - v)^k\, dd^c v \wedge S \\ &\le \frac{C}{s^k} \int_\Omega (v_{s/4} - v)^k\, dd^c v \wedge (dd^c\varphi)^k \wedge \beta^{n-k-1}. \end{aligned}\tag{3-21}$$

Applying Lemma 2.1 we conclude that

$$\int_\Omega (v_{s/4} - v)^k\, dd^c v \wedge (dd^c\varphi)^k \wedge \beta^{n-k-1} \le C\|\varphi\|_\infty^k \int_\Omega (dd^c v)^{k+1} \wedge \beta^{n-k-1}. \tag{3-22}$$

Using $dd^c\rho \ge \beta$ (see Section 2) and Cegrell's inequality we get

$$\begin{aligned}\int_\Omega (dd^c v)^{k+1} \wedge \beta^{n-k-1} &\le \int_\Omega (dd^c v)^{k+1} \wedge (dd^c\rho)^{n-k-1} \\ &\le \left(\int_\Omega (dd^c v)^n\right)^{(k+1)/n} \left(\int_\Omega (dd^c\rho)^n\right)^{(n-k-1)/n}. \end{aligned}\tag{3-23}$$

Combining (3-21), (3-22) and (3-23) we have, for $s \geq 1$,

$$\int_{\{v<-s/2\}} dd^c v \wedge S \leq C\|\varphi\|_\infty^k. \tag{3-24}$$

Notice that $v_s \in \mathcal{E}_0'$. The same arguments as above imply that for $s \geq 1$

$$\int_{\{v<-s/2\}} dd^c v_s \wedge S \leq C\|\varphi\|_\infty^k. \tag{3-25}$$

Thus, altogether we have

$$I_1 + I_2 \leq \frac{C\|\varphi\|_\infty}{t^2}\|v_s - v\|_1^{\alpha_k} + C\|\varphi\|_\infty^k t^\alpha. \tag{3-26}$$

If we choose

$$t = \|v_s - v\|_1^{\alpha_k/3}, \quad \alpha_{k+1} = \frac{\alpha\alpha_k}{3}, \tag{3-27}$$

then the proof of (3-7) is completed.

We now conclude the proof of the lemma. It follows from [Nguyen 2018, equation (2.26)] and [Kołodziej 2005, Lemma 4.1] that

$$\int_\Omega (v_s - v)\, dV_{2n} \leq Ce^{-\tau_0 s},$$

where $\tau_0 > 0$ and $C > 0$ are uniform constants independent of v and s. Combining this with (3-6) and the inequality (3-7) for $k = n$ the lemma follows. □

Remark. The referee has suggested that the exponent 2 in the denominator of (3-15) can be improved. Thus, the inequality (3-26) can be improved too, so the final choice of α_n will be better.

We are ready to prove the main result of this section.

Proof of Proposition 3.1. Let us define $\nu := (dd^c\varphi)^n$. First, we show that for $v \in \mathcal{E}_0'$ there exist uniform constants $\alpha_1, C > 0$ such that

$$\nu(v < -s) \leq \frac{Ce^{-\alpha_1 s}}{s^n} \quad \text{for all } s > 0. \tag{3-28}$$

Indeed, there are two possibilities: either $s \geq 2$ or $s < 2$. If $s \geq 2$, then the inequality follows from Lemma 3.4 as

$$s^n e^{-\tau s/2} \leq \left(\frac{2n}{\tau}\right)^n e^{-n}.$$

(We can take $\alpha_1 = \tau/2$). Otherwise, if $0 < s < 2$, we have $e^{-\alpha_1 s} \geq C$. Then, the desired inequality follows from Lemma 3.3.

To complete the proof of the proposition we use an argument which is inspired by the proofs in [Åhag et al. 2009]. Let $K \subset \Omega$ be compact. Since ν is dominated by a Monge–Ampère measure of a bounded plurisubharmonic function, it vanishes on pluripolar sets. Hence, we may assume that K is nonpluripolar.

Let h_K^* be the relative extremal function of K with respect to Ω. Since $K \subset \Omega$ is compact, it is well known that

$$\lim_{\zeta \to \partial\Omega} h_K^*(\zeta) = 0.$$

By [Bedford and Taylor 1982, Proposition 5.3] we have

$$\tau^n := \mathrm{cap}(K, \Omega) = \int_\Omega (dd^c h_K^*)^n > 0.$$

Let $0 < x < 1$. Since the function $w := h_K^*/\tau$ satisfies assumptions of the inequality (3-28), we have

$$\nu(h_K^* < -1 + x) = \nu\left(w < \frac{-1+x}{\tau}\right) \le C \frac{\tau^n}{\alpha_1^n (1-x)^n} \exp\left(-\frac{\alpha_1(1-x)}{\tau}\right).$$

Letting $x \to 0^+$, we obtain

$$\nu(h_K^* \le -1) \le \frac{C}{\alpha_1^n} \mathrm{cap}(K, \Omega) \exp\left(\frac{-\alpha_1}{[\mathrm{cap}(K, \Omega)]^{1/n}}\right). \tag{3-29}$$

Since $h_K = h_K^*$ outside a pluripolar set, we have

$$\nu(K) \le \nu(h_K = -1) = \nu(h_K^* = -1) \le \nu(h_K^* \le -1). \tag{3-30}$$

We combine (3-29) and (3-30) to finish the proof. □

4. Proof of Theorem B

In this section we will prove the Hölder continuity of the solution obtained in Theorem A provided furthermore that the boundary data ψ is Hölder continuous. Notice that the zero boundary values of the subsolution φ are not essential. We can modify them by adding an appropriate envelope, similar to (4-5), because no condition has been imposed on the total mass of the subsolution.

By Theorem A there exists a unique continuous solution to the Dirichlet problem (1-1), namely, $u \in \mathrm{PSH}(\Omega) \cap C^0(\bar{\Omega})$ solving

$$(dd^c u)^n = \mu, \qquad u(z) = \psi(z) \quad \text{for all } z \in \partial\Omega. \tag{4-1}$$

We are going to show that $u \in C^{0,\alpha'}(\bar{\Omega})$ for some exponent $0 < \alpha' \le 1$.

Outline of the proof. Let us sketch the proof of Theorem B. Overall we follow the steps in the proof of [Nguyen 2018], which in turns followed [Guedj et al. 2008]. Though, we need to consider the problem on an increasing exhaustive sequence of relatively compact domains in Ω. Define for $\delta > 0$ small

$$\Omega_\delta := \{z \in \Omega : \mathrm{dist}(z, \partial\Omega) > \delta\}, \tag{4-2}$$

and for $z \in \Omega_\delta$ we define

$$u_\delta(z) := \sup_{|\zeta| \le \delta} u(z + \zeta), \quad \hat{u}_\delta(z) := \frac{1}{\sigma_{2n}\delta^{2n}} \int_{|\zeta| \le \delta} u(z+\zeta)\, dV_{2n}(\zeta), \tag{4-3}$$

where σ_{2n} is the volume of the unit ball.

Then, we wish to show that

$$\sup_{\Omega_\delta}(\hat{u}_\delta - u) \lesssim \delta^{\varpi}$$

for some $0 < \varpi \le 1$, where $\lesssim$ means that the inequality holds up to an absolute constant. Thanks to the Hölder continuity of the boundary data we can extend $\hat{u}_\delta$ to $\tilde{u}$ by a gluing process such that the new function is plurisubharmonic on Ω and equal to u outside Ω_ε for some (small) $\varepsilon > \delta$. Moreover, we shall still have

$$\sup_{\Omega_\delta}(\hat{u}_\delta - u) \le \sup_{\Omega}(\tilde{u} - u) + C\varepsilon^{\alpha},$$

where α is the Hölder exponent of the boundary data ψ. Next, we shall show that

$$\int_{\Omega_\varepsilon} (dd^c\varphi)^n \lesssim \frac{1}{\varepsilon^n}.$$

This estimate enables us to invoke the results of [Nguyen 2018]. It gives a precise quantitative estimate $\sup_\Omega(\tilde{u} - u)$ in terms of δ and ε. Finally, we can choose $\varepsilon = \delta^{\varpi'}$ with $\varpi' > 0$ so small that our desired inequality holds.

We now proceed to give details of the argument. For the remaining part of the proof we fix a small $\delta_0 > 0$ and consider two parameters δ, ε such that

$$0 < \delta \le \varepsilon < \delta_0. \tag{4-4}$$

We may assume that $\psi \in C^{0,2\alpha}(\partial\Omega)$, where $0 < \alpha \le \frac{1}{2}$ (decreasing α if necessary) is the Hölder exponent of the subsolution φ. Then, we define

$$h(z) = \sup\{v(z) \in \mathrm{PSH}(\Omega) \cap C^0(\overline{\Omega}) : h_{|\partial\Omega} \le \psi\}. \tag{4-5}$$

It is well known [Bedford and Taylor 1976, Theorem 6.2] that $h \in \mathrm{PSH}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ and $h = \psi$ on $\partial\Omega$, which is also the solution of the homogeneous Monge–Ampère equation in Ω. Hence, we may assume

$$\psi \in \mathrm{PSH}(\Omega) \cap C^{0,\alpha}(\overline{\Omega}) \quad \text{and} \quad (dd^c\psi)^n \equiv 0. \tag{4-6}$$

Thanks to the comparison principle [Bedford and Taylor 1982] we get

$$\psi + \varphi \le u \le \psi \quad \text{on } \overline{\Omega}. \tag{4-7}$$

Lemma 4.1. *We have, for* $z \in \overline{\Omega}_\delta \setminus \Omega_\varepsilon$,

$$u_\delta(z) \le u(z) + C\varepsilon^{\alpha}. \tag{4-8}$$

In particular,

$$\sup_{\Omega_\delta}(\hat{u}_\delta - u) \le \sup_{\Omega_\varepsilon}(\hat{u}_\delta - u) + C\varepsilon^{\alpha}. \tag{4-9}$$

Remark 4.2. It is important to keep in mind that the uniform constants $C > 0$ which appear in the lemma, and many times below, are independent of δ and ε.

Proof. Fix a point $z \in \overline{\Omega}_\delta \setminus \Omega_\varepsilon$. Since u is continuous, there is $\zeta_1 \in \mathbb{C}^n$ with $|\zeta_1| \le \delta$ such that

$$u_\delta(z) = u(z + \zeta_1). \tag{4-10}$$

Moreover, there exists $\zeta_2 \in \mathbb{C}^n$ with $|\zeta_2| \leq \varepsilon$ such that $z+\zeta_2 \in \partial\Omega$. Using this and (4-7) we get

$$\begin{aligned} u_\delta(z) - u(z) &\leq \psi(z+\zeta_1) - [\psi(z)+\varphi(z)] \\ &= [\psi(z+\zeta_1)-\psi(z)] + [\varphi(z)-\varphi(z+\zeta_2)] \\ &\leq C_1|\zeta_1|^\alpha + C_2|\zeta_2|^\alpha, \end{aligned} \tag{4-11}$$

where $C_1 = \|\psi\|_{C^{0,\alpha}},\ C_2 = \|\varphi\|_{C^{0,\alpha}}$. Since $\delta \leq \varepsilon$, we conclude the proof of the first part.

To prove the second part, we observe that $u \leq \hat{u}_\delta \leq u_\delta$. Therefore,

$$\sup_{\Omega_\delta}(\hat{u}_\delta - u) \leq \sup_{\Omega_\varepsilon}(\hat{u}_\delta - u) + \sup_{\Omega_\delta\setminus\Omega_\varepsilon}(u_\delta - u). \tag{4-12}$$

Combining this with the first part we get the second part. □

The lemma above tells us that to obtain the Hölder continuity of the solution u it is enough to get the estimate on the domain Ω_ε for ε a small constant, which is comparable to a small positive power of δ. To achieve our goal we will work on the domain Ω_ε and keep track of the (negative) exponent of ε.

Recall that

$$\Omega_\varepsilon = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) > \varepsilon\}. \tag{4-13}$$

We define

$$D_\varepsilon := \{\rho(z) < -\varepsilon\}, \tag{4-14}$$

where ρ is the defining function of Ω as in (2-1). The following lemma is very similar to Lemma 3.3. The main observation is that the domains D_ε and Ω_ε are comparable.

Lemma 4.3. *Let $1 \leq k \leq n$ be an integer. Let $v \in \mathrm{PSH} \cap L^\infty(\Omega)$. Then,*

$$\int_{\Omega_\varepsilon} (dd^c v)^k \wedge \beta^{n-k} \leq \frac{C\|v\|_\infty^k}{\varepsilon^k}, \tag{4-15}$$

where C is independent of ε.

Proof. Observe that, from Hopf's lemma,

$$|\rho(z)| \geq c_0 \operatorname{dist}(z, \partial\Omega) \tag{4-16}$$

for a uniform constant $0 < c_0 \leq 1$. Therefore,

$$\Omega_\varepsilon \subset \{\rho(z) < -c_0\varepsilon\}. \tag{4-17}$$

Since $\max\{\rho, -\varepsilon'/2\} - \rho \geq \varepsilon'/2$ with $\varepsilon' = c_0\varepsilon$ on the latter set, it follows that

$$\begin{aligned} \int_{\Omega_\varepsilon} (dd^c v)^k \wedge \beta^{n-k} &\leq \left(\frac{2}{\varepsilon'}\right)^k \int_\Omega \left(\max\left\{\rho, -\frac{\varepsilon'}{2}\right\} - \rho\right)^k (dd^c v)^k \wedge \beta^{n-k} \\ &\leq \frac{C\|v\|_\infty^k}{\varepsilon^k} \int_\Omega (dd^c\rho)^k \wedge \beta^{n-k}, \end{aligned} \tag{4-18}$$

where we used Lemma 2.1 for the second inequality. The last integral is bounded by the C^2-smoothness of ρ on $\overline{\Omega}$. □

We will now approximate the subsolution φ. Let us define

$$\varphi_\varepsilon := \max\left\{\varphi - \varepsilon, \frac{A\rho}{\varepsilon}\right\}, \tag{4-19}$$

where $A := 1 + \|\varphi\|_\infty$.

Lemma 4.4. *We have*

$$\int_\Omega (dd^c\varphi_\varepsilon)^n \le \frac{CA^n}{\varepsilon^n}. \tag{4-20}$$

Moreover,

$$\mathbf{1}_{D_\varepsilon} \cdot \mu \le (dd^c\varphi_\varepsilon)^n \tag{4-21}$$

as two measures, where D_ε is defined in (4-14).

Proof. To estimate the Monge–Ampère mass of φ_ε we use [Bedford and Taylor 1982, Corollary 4.3], which is a consequence of the comparison principle. Since $A\rho/\varepsilon \le \varphi_\varepsilon \le 0$ and the functions have zero values on the boundary,

$$\int_\Omega (dd^c\varphi_\varepsilon)^n \le \frac{A^n}{\varepsilon^n}\int_\Omega (dd^c\rho)^n. \tag{4-22}$$

The last integral is finite as ρ is C^2 on a neighborhood of the closure of Ω. Furthermore, since $\varphi_\varepsilon(z) = \varphi(z) - \varepsilon$ on $D_\varepsilon = \{\rho < -\varepsilon\}$ when $0 < \varepsilon < 1$, it is clear that

$$\mathbf{1}_{D_\varepsilon} \cdot \mu \le (dd^c\varphi_\varepsilon)^n. \qquad \square$$

Remark 4.5. Using the same argument, we also get that for an integer $1 \le k \le n$

$$\int_\Omega (dd^c\varphi_\varepsilon)^k \wedge \beta^{n-k} \le \frac{CA^k}{\varepsilon^k}. \tag{4-23}$$

We obtain now the volume-capacity inequality for the approximation sequence.

Corollary 4.6. *There exist uniform constants $\alpha_1 > 0$ and $C > 0$ which are independent of ε such that, for every compact set $K \subset \Omega$,*

$$\int_K (dd^c\varphi_\varepsilon)^n \le \frac{C}{\varepsilon^n} \cdot \mathrm{cap}(K) \cdot \exp\left(\frac{-\alpha_1}{[\mathrm{cap}(K)]^{1/n}}\right). \tag{4-24}$$

In particular, for a fixed $\tau > 0$, there is a constant $C(\tau) > 0$ such that, for every compact set $K \subset \Omega$,

$$\int_K (dd^c\varphi_\varepsilon)^n \le \frac{C(\tau)}{\varepsilon^n}[\mathrm{cap}(K)]^{1+\tau}. \tag{4-25}$$

Proof. This is the analogue of Proposition 3.1 with φ replaced by φ_ε, and thus the proof is the same as that of the proposition. Here we need to take into account three facts:

$$\|\varphi_\varepsilon\|_\infty \le \frac{C}{\varepsilon} \quad \text{and} \quad \|\varphi_\varepsilon\|_{C^{0,\alpha}(\bar\Omega)} \le \frac{C}{\varepsilon}, \tag{4-26}$$

and, for an integer $1 \le k \le n$ (Remark 4.5),

$$\int_\Omega (dd^c\varphi_\varepsilon)^k \wedge \beta^{n-k} \le \frac{C}{\varepsilon^k}. \tag{4-27}$$

This explains why we need an extra factor C/ε^n on the right-hand side of the inequality. $\square$

Next, we have the following stability estimate for the Monge–Ampère equation similar to [Guedj et al. 2008, Theorem 1.1]. However, it also takes into account the possibility of infinite total mass of the measure on the right-hand side.

Proposition 4.7. *Let u be the solution of* (4-1) *and Ω_ε be defined by* (4-13). *Let $v \in \mathrm{PSH} \cap L^\infty(\Omega)$ be such that $v = u$ on $\Omega \setminus \Omega_\varepsilon$. Then, there is $0 < \alpha_2 \le 1$ such that*

$$\sup_\Omega (v - u) \le \frac{C}{\varepsilon^n} \left(\int_\Omega \max\{v - u, 0\}\, d\mu \right)^{\alpha_2}. \tag{4-28}$$

Proof. Without loss of generality we may assume that $\sup_\Omega (v - u) > 0$. Set

$$s_0 := \inf_\Omega (u - v). \tag{4-29}$$

We know that for $0 < s < |s_0|$

$$U(s) := \{u < v + s_0 + s\} \Subset \Omega_\varepsilon. \tag{4-30}$$

Lemma 4.8. *Fix $\tau > 0$. Suppose $0 < s, t < |s_0|/2$. Then,*

$$t^n \operatorname{cap}(U(s)) \le \frac{C(\tau)}{\varepsilon^n} [\operatorname{cap}(U(s + t))]^{1+\tau}. \tag{4-31}$$

Proof of Lemma 4.8. Let $0 \le w \le 1$ be a plurisubharmonic function in Ω. We have the chain of inequalities

$$\begin{aligned} t^n \int_{U(s)} (dd^c w)^n &= \int_{\{u < v + s_0 + s\}} [dd^c (tw)]^n \\ &\le \int_{\{u < v + s_0 + s + tw\}} [dd^c (v + tw)]^n \le \int_{\{u < v + s_0 + s + tw\}} (dd^c u)^n, \end{aligned} \tag{4-32}$$

where we used the comparison principle [Bedford and Taylor 1982, Theorem 4.1] for the last inequality. Since $\{u < v + s_0 + s + tw\} \subset U(s + t)$ and w is arbitrary, we get

$$t^n \operatorname{cap}(U(s)) \le \int_{U(s+t)} d\mu. \tag{4-33}$$

If we define $\varepsilon' := c_0 \varepsilon$, where c_0 is the constant in (4-16), then

$$\mathbf{1}_{D_{\varepsilon'}} \cdot d\mu \le (dd^c \varphi_{\varepsilon'})^n$$

as two measures. Since $U(s + t) \subset \Omega_\varepsilon \subset D_{\varepsilon'}$, it follows that

$$\int_{U(s+t)} d\mu \le \int_{U(s+t)} (dd^c \varphi_{\varepsilon'})^n \le \frac{C(\tau)}{(c_0 \varepsilon)^n} [\operatorname{cap}(U(s + t))]^{1+\tau}, \tag{4-34}$$

where the last inequality follows from Corollary 4.6. The proof of the lemma is complete. □

Now together with Lemma 4.8, the rest of the proof of the proposition is the same as in [Guedj et al. 2008, Theorem 1.1] (see also [Kołodziej and Nguyen 2016, Theorem 3.11]). □

The following result is a variation of Lemma 2.7 in [Nguyen 2018], where we considered the Hölder continuity of a measure ν on $\mathcal{E}'_0$, though the situation now is different as $\nu(\Omega)$ is no longer finite.

Theorem 4.9. *Let u be the solution of* (4-1) *and Ω_ε be defined by* (4-13). *Let $v \in \mathrm{PSH} \cap L^\infty(\Omega)$ be such that $v = u$ on $\Omega \setminus \Omega_\varepsilon$. Then, there exists $0 < \alpha_3 \leq 1$ such that*

$$\int_\Omega |v-u|\, d\mu \leq \frac{C}{\varepsilon^{n+1}} \left(\int_\Omega |v-u|\, dV_{2n} \right)^{\alpha_3}. \tag{4-35}$$

Proof. This is a variation of the inequality (3-7) with

$$S_{k,\varepsilon} := (dd^c \varphi_\varepsilon)^k \wedge \beta^{n-k}, \tag{4-36}$$

where $\varphi_\varepsilon = \max\{\varphi - \varepsilon, A\rho/\varepsilon\}$ and $0 \leq k \leq n$ is an integer. Since $\mu \leq S_{n,\varepsilon}$ on Ω_ε, it is enough to show that there is $0 < \tau \leq 1$ satisfying

$$\int_\Omega (v-u) S_{n,\varepsilon} \leq \frac{C}{\varepsilon^{n+1}} \|v-u\|_1^\tau \tag{4-37}$$

for $v \geq u$ on Ω. (In the general case we use the identity

$$|v-u| = (\max\{v,u\} - u) + (\max\{v,u\} - v)$$

and apply twice the inequality (4-37) to get the theorem.)

Now we can repeat the inductive arguments of the proof of (3-7) with v, u and φ_ε in the places of v_s, v and φ, respectively. However, there are differences as follows. First, v, u are no longer in $\mathcal{E}_0'$. Second, if φ is extended as in the proof of Lemma 3.4, then $\varphi_\varepsilon = \max\{\varphi - \varepsilon, A\rho/\varepsilon\}$ is also defined on the neighborhood U of $\bar{\Omega}$, and

$$\|\varphi_\varepsilon\|_{C^{0,\alpha}(U)} \leq \frac{C}{\varepsilon}.$$

Taking into account above differences, to pass from the k-th step to the $(k+1)$-th step we need the following inequality, corresponding to (3-16) (with $S_\varepsilon := (dd^c\varphi_\varepsilon)^k \wedge \beta^{n-k-1}$):

$$\begin{aligned} \int_\Omega (v-u)\, dd^c\varphi_\varepsilon \wedge S_\varepsilon &\leq \left| \int_\Omega (v-u)\, dd^c \varphi_\varepsilon * \chi_t \wedge S_\varepsilon \right| + \left| \int_\Omega (v-u)\, dd^c(\varphi_\varepsilon * \chi_t - \varphi_\varepsilon) \wedge S_\varepsilon \right| \\ &=: I_{1,\varepsilon} + I_{2,\varepsilon}. \end{aligned} \tag{4-38}$$

Since

$$\left| \frac{\partial^2 \varphi_\varepsilon * \chi_t}{\partial z_j \partial \bar{z}_k}(z) \right| \leq \frac{C\|\varphi\|_\infty}{\varepsilon t^2}, \tag{4-39}$$

and by the induction hypothesis at the k-th step, there exists $0 < \tau_k \leq 1$ such that

$$\int_\Omega (v-u) S_\varepsilon \wedge \beta \leq \frac{C}{\varepsilon^{k+1}} \|v-u\|_1^{\tau_k},$$

we conclude that

$$I_{1,\varepsilon} \leq \frac{C\|\varphi\|_\infty}{\varepsilon\, t^2} \int_\Omega (v-u) S_\varepsilon \wedge \beta \leq \frac{C\|\varphi\|_\infty}{\varepsilon^{k+2}\, t^2} \|v-u\|_1^{\tau_k}. \tag{4-40}$$

Similarly to (3-19), by integration by parts, $u = v$ on $\Omega \setminus \Omega_\varepsilon$, and

$$|\varphi_\varepsilon * \chi_t(z) - \varphi_\varepsilon(z)| \leq \frac{Ct^\alpha}{\varepsilon},$$

it follows that

$$I_{2,\varepsilon} \le \frac{Ct^\alpha}{\varepsilon} \int_{\Omega_\varepsilon} (dd^c v + dd^c u) \wedge S_\varepsilon. \tag{4-41}$$

At this point as u, v do not belong to $\mathcal{E}'_0$ we need to use a different argument to bound $I_{2,\varepsilon}$. Namely, similarly to Lemma 4.3, we have

$$\int_{\Omega_\varepsilon} (dd^c u + dd^c v) \wedge S_\varepsilon \le \frac{C\|u+v\|_\infty (1+\|\varphi\|_\infty)^k}{\varepsilon^{k+1}}. \tag{4-42}$$

Indeed, we first have

$$\begin{aligned}\int_{\Omega_\varepsilon} dd^c(u+v) \wedge (dd^c\varphi_\varepsilon)^k \wedge \beta^{n-k-1} &\le \frac{2}{\varepsilon'} \int_\Omega \left(\max\left\{\rho, -\frac{\varepsilon'}{2}\right\} - \rho\right) \wedge dd^c(u+v) \wedge (dd^c\varphi_\varepsilon)^k \wedge \beta^{n-k-1} \\ &\le \frac{C}{\varepsilon} \|u+v\|_\infty \int_\Omega (dd^c\rho) \wedge (dd^c\varphi_\varepsilon)^k \wedge \beta^{n-k-1},\end{aligned}$$

where $\varepsilon' = c_0\varepsilon$ with c_0 defined by (4-16). The desired inequality (4-42) follows from Remark 4.5. Now, combining (4-41) and (4-42) we get

$$I_{2,\varepsilon} \le \frac{Ct^\alpha}{\varepsilon^{k+2}}. \tag{4-43}$$

Next, it is easy to see (from Lemma 4.4) that

$$\int_\Omega (v-u) S_n \le \frac{C\|u\|_\infty (1+\|\varphi\|_\infty)^n}{\varepsilon^n}.$$

Therefore, we can assume $0 < \|v-u\|_1 < 0.01$. Thanks to (4-40) and (4-43) we have

$$\int_\Omega (v-u)\, dd^c\varphi_\varepsilon \wedge S_\varepsilon \le \frac{C}{\varepsilon^{k+2}\, t^2} \|v-u\|_1^{\tau_k} + \frac{Ct^\alpha}{\varepsilon^{k+2}}.$$

If we choose $t = \|v-u\|_1^{\tau_k/3}$, $\tau_{k+1} = \alpha\tau_k/3$, then

$$\int_\Omega (v-u) S_\varepsilon \wedge dd^c\varphi_\varepsilon \le \frac{C}{\varepsilon^{k+2}} \|v-u\|_1^{\tau_{k+1}}.$$

Thus, the induction argument is completed, and the theorem follows. □

The last ingredient to prove Theorem B was proved first in [Baracco et al. 2016] (see also [Nguyen 2018, Lemma 2.12]). Here, the estimate is sharper and the proof is simpler too.

Lemma 4.10. *For $\delta > 0$ small we have*

$$\int_{\Omega_\delta} |\hat{u}_\delta - u|\, dV_{2n} \le C\delta. \tag{4-44}$$

Proof. First, we know from the classical Jensen formula (see, e.g., [Guedj et al. 2008, Lemma 4.3]) that

$$\int_{\Omega_{2\delta}} |\hat{u}_\delta - u| \le C\delta^2 \int_{\Omega_\delta} \Delta u(z). \tag{4-45}$$

Again, it follows from Lemma 4.3 applied for $k = 1$ and $\delta = \varepsilon$, that

$$\int_{\Omega_\delta} \Delta u(z) \le \frac{C}{\delta}. \tag{4-46}$$

Therefore,

$$\int_{\Omega_\delta} |\hat{u}_\delta - u|\, dV_{2n} \le \int_{\Omega_{2\delta}} |\hat{u}_\delta - u|\, dV_{2n} + \|u\|_\infty \int_{\Omega_\delta \setminus \Omega_{2\delta}} dV_{2n} \le C\delta. \tag{4-47}$$

This is the required inequality. □

We are ready to prove the Hölder continuity of the solution.

End of proof of Theorem B. Let us fix δ such that $0 < \delta < \delta_0$ small and let ε be such that $\delta \le \varepsilon < \delta_0$, which is to be determined later. Thanks to Lemma 4.1 and $\hat{u}_\delta \le u_\delta$ we have $\hat{u}_\delta - C\varepsilon^\alpha \le u$ on $\partial\Omega_\varepsilon$. Therefore, the function

$$\tilde{u} := \begin{cases} \max\{\hat{u}_\delta - C\varepsilon^\alpha, u\} & \text{on } \Omega_\varepsilon, \\ u & \text{on } \Omega \setminus \Omega_\varepsilon \end{cases} \tag{4-48}$$

belongs to $\mathrm{PSH}(\Omega) \cap C^0(\overline{\Omega})$. Notice that $\tilde{u} \ge u$ in Ω, and

$$\tilde{u} = u \quad \text{on } \Omega \setminus \Omega_\varepsilon. \tag{4-49}$$

Again, by the second part of Lemma 4.1 we have

$$\begin{aligned} \sup_{\Omega_\delta}(\hat{u}_\delta - u) &\le \sup_{\Omega_\varepsilon}(\hat{u}_\delta - u) + C\varepsilon^\alpha \\ &\le \sup_{\Omega}(\tilde{u} - u) + C\varepsilon^\alpha + C\varepsilon^\alpha. \end{aligned} \tag{4-50}$$

By the stability estimate (Proposition 4.7) there exists $0 < \alpha_2 \le 1$ such that

$$\begin{aligned} \sup_{\Omega}(\tilde{u} - u) &\le \frac{C}{\varepsilon^n} \left(\int_\Omega \max\{\tilde{u} - u, 0\}\, d\mu \right)^{\alpha_2} \\ &\le \frac{C}{\varepsilon^n} \left(\int_\Omega |\tilde{u} - u|\, d\mu \right)^{\alpha_2}, \end{aligned} \tag{4-51}$$

where we used the fact that $\tilde{u} = u$ outside Ω_ε. Using Theorem 4.9, there is $0 < \alpha_3 \le 1$ such that

$$\begin{aligned} \sup_{\Omega}(\tilde{u} - u) &\le \frac{C}{\varepsilon^{n+(n+1)\alpha_2}} \left(\int_\Omega |\tilde{u} - u|\, dV_{2n} \right)^{\alpha_2\alpha_3} \\ &\le \frac{C}{\varepsilon^{2n+1}} \left(\int_{\Omega_\delta} |\hat{u}_\delta - u|\, dV_{2n} \right)^{\alpha_2\alpha_3}, \end{aligned} \tag{4-52}$$

where we used $0 \le \tilde{u} - u \le \mathbf{1}_{\Omega_\varepsilon} \cdot (\hat{u}_\delta - u)$ and $\Omega_\varepsilon \subset \Omega_\delta$ for the second inequality. It follows from (4-50), (4-52), and Lemma 4.10 that

$$\sup_{\Omega_\delta}(\hat{u}_\delta - u) \le C\varepsilon^\alpha + \frac{C\delta^{\alpha_2\alpha_3}}{\varepsilon^{2n+1}}. \tag{4-53}$$

Now, we choose $\alpha_4 = \alpha\alpha_2\alpha_3/(2n + 1 + \alpha)$ and

$$\varepsilon = \delta^{\alpha_2\alpha_3/(2n+1+\alpha)}.$$

Then, $\sup_{\Omega_\delta}(\hat{u}_\delta - u) \le C\delta^{\alpha_4}$. Finally, thanks to [Guedj et al. 2008, Lemma 4.2] we infer that $\sup_{\Omega_\delta}(u_\delta - u) \le C\delta^{\alpha_4}$. The proof of the theorem is finished. □

Remark 4.11. In the above proof the Hölder exponent of the solution u is $\alpha' = \alpha_4 = \alpha\alpha_2\alpha_3/(2n+1+\alpha)$, where we can take $0 < \alpha_2 < 1/(n+1)$ and $\alpha_3 = \alpha^n/3^n$ by [Guedj et al. 2008] and Theorem 4.9 respectively. In our opinion it is far from being optimal. If we assume that the subsolution φ is merely continuous, then we do not know if the inequality (4-51) holds true. Therefore, it seems to be hard to improve the proof above to get the answer for the subsolution problem in the continuous category.

5. Proof of Corollary C

Let $\mu \in \mathcal{M}(\varphi, \Omega)$ and $0 \le f \in L^p(\Omega, d\mu)$ with $p > 1$. We wish to show that there exists $\tilde{\varphi} \in \mathrm{PSH}(\Omega) \cap C^{0,\tilde{\alpha}}(\bar{\Omega})$, with $0 < \tilde{\alpha} \le 1$, such that

$$f\,d\mu \in \mathcal{M}(\tilde{\varphi}, \Omega). \tag{5-1}$$

The proof of the corollary is similar to that of Theorem B with the aid of the following two lemmas.

Lemma 5.1. *Fix a constant $\tau > 0$. Then, there exists a uniform constant $C(\tau)$ such that, for every compact set $K \subset \Omega$,*

$$\int_K f\,d\mu \le C(\tau)[\mathrm{cap}(K)]^{1+\tau}. \tag{5-2}$$

Proof. Hölder's inequality and Proposition 3.1 give us

$$\int_K f\,d\mu \le \|f\|_{L^p(\Omega,d\mu)}[\mu(K)]^{(p-1)/p} \le C\left[\mathrm{cap}(K)\cdot\exp\left(\frac{-\alpha_0}{[\mathrm{cap}(K)]^{1/n}}\right)\right]^{(p-1)/p}. \tag{5-3}$$

Let $0 < a, b, c < 1$ be fixed. The following elementary inequality holds for $x > 0$:

$$x^a \exp\left(-\frac{c}{x^b}\right) \le C(\tau)x^{1+\tau},$$

where $C(\tau) = C(\tau, a, b, c)$ depends only on τ, a, b, c. Thus, the desired inequality follows. □

Thanks to the lemma and [Kołodziej 2005, Theorem 5.9] we can solve the Monge–Ampère equation

$$u \in \mathrm{PSH}(\Omega) \cap C^0(\bar{\Omega}), \quad (dd^c u)^n = f\,d\mu, \quad u_{|\partial\Omega} = 0. \tag{5-4}$$

Moreover, the above lemma will enable us to have the stability estimate (Proposition 4.7). The next lemma is also a consequence of the generalized Hölder inequality which was proved in [Nguyen 2018, Corollary 2.14].

Lemma 5.2. *Let $v \in \mathrm{PSH}(\Omega) \cap C^0(\bar{\Omega})$ be such that $v \ge u$ in Ω and $v = u$ near $\partial\Omega$. Then, there exist uniform constants $C > 0$ and $0 < \tilde{\alpha}_3 < 1$ such that*

$$\int_\Omega (v-u) f\,d\mu \le C\|v-u\|_{L^1(d\mu)}^{\tilde{\alpha}_3}. \tag{5-5}$$

Next, we use the extendability assumption of φ to get a result similar to Lemma 4.1 in the current setting. Namely, let $\widetilde{\Omega}$ be a strictly pseudoconvex neighborhood of $\bar{\Omega}$ such that $\varphi \in \mathrm{PSH}(\widetilde{\Omega})$ and Hölder continuous on the closure of $\widetilde{\Omega}$. Thanks to the results in [Nguyen 2018] there exists $v \in \mathrm{PSH}(\widetilde{\Omega})$ and Hölder continuous in $\widetilde{\Omega}$ satisfying

$$(dd^c v)^n = \mathbf{1}_\Omega f\, d\mu \quad \text{in } \widetilde{\Omega}, \qquad v = 0 \quad \text{on } \partial\widetilde{\Omega}.$$

Consider h to be the maximal pluriharmonic extension into Ω of $(-v)_{|\partial\Omega}$ which is Hölder continuous on $\partial\Omega$ (see (1-3)). So h is also Hölder continuous on $\bar{\Omega}$. Then, by the comparison principle,

$$v + h \le u \le 0 \quad \text{on } \bar{\Omega}.$$

From this we easily deduce the desired estimate near the boundary for u.

Now the rest of the proof goes exactly as in the proof of Theorem B. Namely, the inequality (4-51) holds for the measure $f\, d\mu$, next use Lemma 5.2 and Theorem 4.9 to get the inequality (4-52). Then we get the Hölder continuity of u. Notice that the Hölder exponent is worse by a factor of $\tilde{\alpha}_3$. Thus, $f\, d\mu \in \mathcal{M}(u, \Omega)$.

Acknowledgement

I am very grateful to Sławomir Kołodziej for many useful discussions. I would like to thank Kang-Tae Kim for his generous support and encouragement. The author is supported by the NRF Grant 2011-0030044 (SRC-GAIA) of The Republic of Korea. I also thank the referee who read the manuscript very carefully and gave many helpful comments and suggestions.

References

[Åhag et al. 2009] P. Åhag, U. Cegrell, S. Kołodziej, H. H. Phạm, and A. Zeriahi, "Partial pluricomplex energy and integrability exponents of plurisubharmonic functions", *Adv. Math.* **222**:6 (2009), 2036–2058. MR Zbl

[Baracco et al. 2016] L. Baracco, T. V. Khanh, S. Pinton, and G. Zampieri, "Hölder regularity of the solution to the complex Monge–Ampère equation with L^p density", *Calc. Var. Partial Differential Equations* **55**:4 (2016), art. id. 74. MR Zbl

[Baran and Bialas-Ciez 2014] M. Baran and L. Bialas-Ciez, "Hölder continuity of the Green function and Markov brothers' inequality", *Constr. Approx.* **40**:1 (2014), 121–140. MR Zbl

[Bedford and Taylor 1976] E. Bedford and B. A. Taylor, "The Dirichlet problem for a complex Monge–Ampère equation", *Invent. Math.* **37**:1 (1976), 1–44. MR Zbl

[Bedford and Taylor 1982] E. Bedford and B. A. Taylor, "A new capacity for plurisubharmonic functions", *Acta Math.* **149**:1-2 (1982), 1–40. MR Zbl

[Błocki 1993] Z. Błocki, "Estimates for the complex Monge–Ampère operator", *Bull. Polish Acad. Sci. Math.* **41**:2 (1993), 151–157. MR Zbl

[Cegrell 2004] U. Cegrell, "The general definition of the complex Monge–Ampère operator", *Ann. Inst. Fourier* (*Grenoble*) **54**:1 (2004), 159–179. MR Zbl

[Charabati 2015] M. Charabati, "Hölder regularity for solutions to complex Monge–Ampère equations", *Ann. Polon. Math.* **113**:2 (2015), 109–127. MR Zbl

[Charabati 2017] M. Charabati, "Regularity of solutions to the Dirichlet problem for Monge–Ampère equations", *Indiana Univ. Math. J.* **66**:6 (2017), 2187–2204. MR Zbl

[Coman et al. 2018] D. Coman, V. Guedj, S. Sahin, and A. Zeriahi, "Toric pluripotential theory", preprint, 2018. arXiv

[Demailly et al. 2014] J.-P. Demailly, S. Dinew, V. Guedj, S. Kołodziej, H. H. Pham, and A. Zeriahi, "Hölder continuous solutions to Monge–Ampère equations", *J. Eur. Math. Soc.* **16**:4 (2014), 619–647. MR Zbl

[Dinew et al. 2016] S. Dinew, V. Guedj, and A. Zeriahi, "Open problems in pluripotential theory", *Complex Var. Elliptic Equ.* **61**:7 (2016), 902–930. MR Zbl

[Dinh and Nguyên 2014] T.-C. Dinh and V.-A. Nguyên, "Characterization of Monge–Ampère measures with Hölder continuous potentials", *J. Funct. Anal.* **266**:1 (2014), 67–84. MR Zbl

[Dinh et al. 2010] T.-C. Dinh, V.-A. Nguyên, and N. Sibony, "Exponential estimates for plurisubharmonic functions and stochastic dynamics", *J. Differential Geom.* **84**:3 (2010), 465–488. MR Zbl

[Dinh et al. 2017] T.-C. Dinh, X. Ma, and V.-A. Nguyên, "Equidistribution speed for Fekete points associated with an ample line bundle", *Ann. Sci. Éc. Norm. Sup.* (4) **50**:3 (2017), 545–578. MR Zbl

[Guedj et al. 2008] V. Guedj, S. Kołodziej, and A. Zeriahi, "Hölder continuous solutions to Monge–Ampére equations", *Bull. Lond. Math. Soc.* **40**:6 (2008), 1070–1080. MR Zbl

[Kołodziej 1995] S. Kołodziej, "The range of the complex Monge–Ampère operator, II", *Indiana Univ. Math. J.* **44**:3 (1995), 765–782. MR Zbl

[Kołodziej 2005] S. Kołodziej, *The complex Monge–Ampère equation and pluripotential theory*, Mem. Amer. Math. Soc. **840**, 2005. MR Zbl

[Kołodziej 2013] S. Kołodziej, "Weak solutions to the complex Monge–Ampère equation", pp. 415–428 in *European Congress of Mathematics*, edited by R. Latała et al., Eur. Math. Soc., Zürich, 2013. MR Zbl

[Kołodziej and Nguyen 2016] S. Kołodziej and N. C. Nguyen, "Weak solutions of complex Hessian equations on compact Hermitian manifolds", *Compos. Math.* **152**:11 (2016), 2221–2248. MR Zbl

[Kołodziej and Nguyen 2018a] S. Kołodziej and N. C. Nguyen, "Hölder continuous solutions of the Monge–Ampère equation on compact Hermitian manifolds", *Ann. Inst. Fourier* (*Grenoble*) **68**:7 (2018), 2951–2964. MR Zbl

[Kołodziej and Nguyen 2018b] S. Kołodziej and N. C. Nguyen, "A remark on the continuous subsolution problem for the complex Monge–Ampère equation", preprint, 2018. arXiv

[Kołodziej and Nguyen 2019] S. Kołodziej and N. C. Nguyen, "An inequality between complex Hessian measures of Hölder continuous $m-$subharmonic functions and capacity", 2019, available at http://tinyurl.com/kolnguyen. To appear in *Geometric analysis*, Progr. Math., Birkhäuser.

[Nguyen 2018] N. C. Nguyen, "On the Hölder continuous subsolution problem for the complex Monge–Ampère equation", *Calc. Var. Partial Differential Equations* **57**:1 (2018), art. id. 8. MR Zbl

[Pawłucki and Pleśniak 1986] W. Pawłucki and W. Pleśniak, "Markov's inequality and C^∞ functions on sets with polynomial cusps", *Math. Ann.* **275**:3 (1986), 467–480. MR Zbl

[Pham 2010] P. H. Hiep, "Hölder continuity of solutions to the Monge–Ampère equations on compact Kähler manifolds", *Ann. Inst. Fourier* (*Grenoble*) **60**:5 (2010), 1857–1869. MR Zbl

[Siciak 1997] J. Siciak, "Wiener's type sufficient conditions in $\mathbb{C}^N$", *Univ. Iagel. Acta Math.* 35 (1997), 47–74. MR Zbl

[Siciak 2000] J. Siciak, "Wiener's type sufficient conditions for regularity in $\mathbb{C}^N$", pp. 39–46 in *Complex analysis and geometry* (Paris, 1997), edited by P. Dolbeault et al., Progr. Math. **188**, Birkhäuser, Basel, 2000. MR Zbl

[Vu 2018] D.-V. Vu, "Complex Monge–Ampère equation for measures supported on real submanifolds", *Math. Ann.* **372**:1-2 (2018), 321–367. MR Zbl

Received 22 Mar 2018. Revised 20 Nov 2018. Accepted 23 Feb 2019.

NGOC CUONG NGUYEN: nguyen.ngoc.cuong@im.uj.edu.pl, cuongnn@kaist.ac.kr
Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland

and

Department of Mathematics and Center for Geometry and its Applications, Pohang University of Science and Technology, Pohang, South Korea
Current address: *Department of Mathematical Sciences, KAIST, Daejeon, South Korea*

mathematical sciences publishers msp

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2020 is US $340/year for the electronic version, and $550/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 13 No. 2 2020

2157-5045(2020)13:2;1-C