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The aim of this paper is to present an abstract and general approach to jump inequalities in harmonic analysis. Our principal conclusion is the refinement of  $r$ -variational estimates, previously known for  $r > 2$ , to endpoint results for the jump quasiseminorm corresponding to  $r = 2$ . This is applied to the dimension-free results recently obtained by the first two authors in collaboration with Bourgain, and Wróbel, and also to operators of Radon type treated by Jones, Seeger, and Wright.

### 1. Introduction

Variational and jump inequalities in harmonic analysis, probability, and ergodic theory have been studied extensively since [Bourgain 1989], where a variational version of the Hardy–Littlewood maximal function was introduced. The purpose of this paper is to formulate general sufficient conditions that allow us to deal with variational and jump inequalities for a wide class of operators. Our approach will be based on certain bootstrap arguments. As an application we extend the known  $L^p$  estimates for  $r$ -variations for  $r > 2$  (see definition (1.2)) to endpoint assertions for the jump quasiseminorm  $J_2^p$  (see definition (1.3)), which corresponds to  $r = 2$ . In this way our results will extend previously recently obtained assertions in [Bourgain et al. 2018; 2019] for dimension-free estimates given for  $r > 2$ , as well as a number of results in [Jones et al. 2008] for operators of Radon type.

We recall the notation for jump quasiseminorms from [Mirek et al. 2018b]. For any  $\lambda > 0$  and  $\mathbb{I} \subset \mathbb{R}$  the  $\lambda$ -jump counting function of a function  $f : \mathbb{I} \rightarrow \mathbb{C}$  is defined by

$$N_\lambda(f) := N_\lambda(f(t) : t \in \mathbb{I}) \\ := \sup\{J \in \mathbb{N} : \text{there exists } t_0 < \dots < t_J, t_j \in \mathbb{I}, \text{ such that } \min_{0 < j \leq J} |f(t_j) - f(t_{j-1})| \geq \lambda\} \quad (1.1)$$

and the  $r$ -variation seminorm by

$$V^r(f) := V^r(f(t) : t \in \mathbb{I}) := \begin{cases} \sup_{J \in \mathbb{N}} \sup_{\substack{t_0 < \dots < t_J \\ t_j \in \mathbb{I}}} \left( \sum_{j=1}^J |f(t_j) - f(t_{j-1})|^r \right)^{\frac{1}{r}}, & 0 < r < \infty, \\ \sup_{\substack{t_0 < t_1 \\ t_j \in \mathbb{I}}} |f(t_1) - f(t_0)|, & r = \infty, \end{cases} \quad (1.2)$$

where the former supremum is taken over all finite increasing sequences in  $\mathbb{I}$ .

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Throughout the article  $(X, \mathcal{B}, \mathfrak{m})$  denotes a  $\sigma$ -finite measure space. For a function  $f : X \times \mathbb{I} \rightarrow \mathbb{C}$  the jump quasiseminorm on  $L^p(X)$  for  $1 < p < \infty$  is defined by

$$\begin{aligned} J_2^p(f) &:= J_2^p(f : X \times \mathbb{I} \rightarrow \mathbb{C}) := J_2^p((f(\cdot, t))_{t \in \mathbb{I}}) := J_2^p((f(\cdot, t))_{t \in \mathbb{I}} : X \rightarrow \mathbb{C}) \\ &:= \sup_{\lambda > 0} \left\| \lambda N_\lambda(f(\cdot, t) : t \in \mathbb{I}) \right\|_{L^p}^{\frac{1}{2}}. \end{aligned} \quad (1.3)$$

In this connection by [Mirek et al. 2018b, Lemma 2.12] we note that

$$\|V^r(f)\|_{L^{p,\infty}} \lesssim_{p,r} J_2^p(f) \leq \|V^2(f)\|_{L^p} \quad (1.4)$$

for  $r > 2$ , and the first inequality fails for  $r = 2$ .

We now briefly list our main results:

- (1) The extension to the jump quasiseminorm  $J_2^p$  of dimension-free estimates for maximal averages over convex sets, as given by Theorems 1.9, 1.11 and 1.14 below.
- (2) The corresponding extension to  $J_2^p$  of the previous dimension-free estimates for cubes in the discrete setting; see Theorem 1.18.
- (3) The general  $J_2^p$  results for operators of Radon type (both averages and singular integrals) in Theorems 1.22 and 1.30, related to the previous results in [Jones et al. 2008].

Underlying the proofs of all these results will be the basic facts about the jump quantity  $J_2^p$  obtained in our recent paper [Mirek et al. 2018b], and the bootstrap arguments in Section 2 of the present paper. The reader might compare the methods in Section 2 with related arguments in [Bourgain et al. 2018, Section 2.2] as well as [Nagel et al. 1978; Duoandikoetxea and Rubio de Francia 1986; Carbery 1986], and Christ's observation included in [Carbery 1988]. The techniques in Section 2 will be carried out in the following framework. We assume that we are given a measure space  $(X, \mathcal{B}, \mathfrak{m})$  which is endowed with a sequence of linear operators  $(S_j)_{j \in \mathbb{Z}}$  acting on  $L^1(X) + L^\infty(X)$  that play the role of the Littlewood–Paley operators. Namely, the following conditions are satisfied:

- (1) The family  $(S_j)_{j \in \mathbb{Z}}$  is a resolution of the identity on  $L^2(X)$ ; i.e., the identity

$$\sum_{j \in \mathbb{Z}} S_j = \text{Id} \quad (1.5)$$

holds in the strong operator topology on  $L^2(X)$ .

- (2) For every  $1 < p < \infty$  we have

$$\left\| \left( \sum_{j \in \mathbb{Z}} |S_j f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \|f\|_{L^p}, \quad f \in L^p(X). \quad (1.6)$$

Suppose now we have a family of linear operators  $(T_t)_{t \in \mathbb{I}}$  acting on  $L^1(X) + L^\infty(X)$ , where the index set  $\mathbb{I}$  is a countable subset of  $(0, \infty)$ . We assume that  $\mathbb{I} \subseteq (0, \infty)$  to make our exposition consistent with the results in the literature. One of our aims is to understand what kind of conditions have to be imposed on the family  $(T_t)_{t \in \mathbb{I}}$ , in terms of its interactions with the Littlewood–Paley operators  $(S_j)_{j \in \mathbb{Z}}$

to obtain the inequality

$$J_2^p((T_t f)_{t \in \mathbb{I}} : X \rightarrow \mathbb{C}) \lesssim \|f\|_{L^p} \tag{1.7}$$

in some range of  $p$ 's. We accomplish this task in Section 2 by proving Theorems 2.14 and 2.39 for positive operators<sup>1</sup> by certain bootstrap arguments, and Theorem 2.28 for general operators. Our approach will be based on extension of ideas from [Duoandikoetxea and Rubio de Francia 1986; Bourgain et al. 2018] to a more abstract setting.

As mentioned above it has been very well known since [Bourgain 1989] that  $r$ -variational estimates (and consequently maximal estimates, see (1.2)) can be deduced from jump inequalities. Namely, a priori jump estimates (1.7) in an open range of  $p \in (1, \infty)$  imply

$$\|V^r(T_t f : t \in \mathbb{I})\|_{L^p} \lesssim_{p,r} \|f\|_{L^p}$$

in the same range of  $p$ 's and for all  $r \in (2, \infty]$ . This follows from (1.4) and interpolation. Therefore, it is natural to say that the jump inequality in (2.2) is an endpoint for  $r$ -variations at  $r = 2$ . On the other hand, we also know that the range of  $r \in (2, \infty]$  in  $r$ -variational estimates, for many operators in harmonic analysis, is sharp due to the sharp estimates in Lépingle's inequality for martingales; see [Mirek et al. 2018b].

Here and later we write  $a \lesssim b$  if  $a \leq Cb$ , where the constant  $0 < C < \infty$  is allowed to depend on  $p$ , but not on the underlying abstract measure space  $X$  or function  $f$ . If  $C$  is allowed to depend on some additional parameters this will be indicated by adding a subscript to the symbol  $\lesssim$ .

**1A. Applications to dimension-free estimates.** An important application of the results from Section 2 will be bounds independent of the dimension in jump inequalities associated with the Hardy–Littlewood averaging operators. Let  $G \subset \mathbb{R}^d$  be a symmetric convex body, that is, a nonempty symmetric convex open bounded subset of  $\mathbb{R}^d$ . Define for  $t > 0$  and  $x \in \mathbb{R}^d$  the averaging operator

$$\mathcal{A}_t^G f(x) := |G|^{-1} \int_G f(x - ty) \, dy, \quad f \in L^1_{\text{loc}}(\mathbb{R}^d). \tag{1.8}$$

It follows from the spherical maximal theorem that, in the case that  $G$  is the Euclidean ball, the maximal operator  $\mathcal{A}_*^G f := \sup_{t>0} |\mathcal{A}_t^G f|$  corresponding to (1.8) is bounded on  $L^p(\mathbb{R}^d)$  for all  $p > 1$ , uniformly in  $d \in \mathbb{N}$  [Stein 1982]. This result was extended to arbitrary symmetric convex bodies  $G \subset \mathbb{R}^d$  in [Bourgain 1986a] (for  $p = 2$ ) and [Bourgain 1986b; Carbery 1986] (for  $p > \frac{3}{2}$ ). For unit balls  $G = B^q$  induced by  $\ell^q$  norms in  $\mathbb{R}^d$  the full range  $p > 1$  of dimension-free estimates was established in [Müller 1990] (for  $1 \leq q < \infty$ ) and [Bourgain 2014] (for cubes  $q = \infty$ ) with constants depending on  $q$ . In the latter case the product structure of the cubes is important; this result was recently extended to products of Euclidean balls of arbitrary dimensions [Sommer 2017].

Variational versions of most of the aforementioned dimension-free estimates were obtained in [Bourgain et al. 2018] for  $r > 2$ . In this article we give a shorter and more self-contained proof of the main results of that work and extend them to the endpoint  $r = 2$  by appealing to Theorems 2.14 and 2.39. A notable

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<sup>1</sup>A linear operator  $T$  is positive if  $Tf \geq 0$  for every  $f \geq 0$ .

simplification is that we do not use the maximal estimates as a black box. In particular, we reprove all dimension-free estimates for the maximal function  $\mathcal{A}_\star^G$ .

In view of (1.4) and by real interpolation, Theorem 1.9 extends [Bourgain et al. 2018, Theorem 1.2].

**Theorem 1.9.** *Let  $d \in \mathbb{N}$  and  $G \subset \mathbb{R}^d$  be a symmetric convex body. Then for every  $1 < p < \infty$  and  $f \in L^p(\mathbb{R}^d)$  we have*

$$J_2^p((\mathcal{A}_{2^k}^G f)_{k \in \mathbb{Z}} : \mathbb{R}^d \rightarrow \mathbb{C}) \lesssim \|f\|_{L^p}, \quad (1.10)$$

where the implicit constant is independent of  $d$  and  $G$ .

As a consequence of Theorem 1.9 and the decomposition into long and short jumps, see (2.2), Theorems 1.11 and 1.14 below extend Theorems 1.1 and 1.3 in [Bourgain et al. 2018], respectively. Hence Theorem 1.9 can be thought of as the main result of this paper, since inequalities (1.12) and (1.15) were obtained in [Bourgain et al. 2018]. However, we shall present a different approach to establish the estimates in (1.12) and (1.15).

**Theorem 1.11.** *Let  $G$  be as in Theorem 1.9. Then for every  $\frac{3}{2} < p < 4$  and  $f \in L^p(\mathbb{R}^d)$  we have*

$$\left\| \left( \sum_{k \in \mathbb{Z}} (V^2(\mathcal{A}_t^G f : t \in [2^k, 2^{k+1}]))^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \|f\|_{L^p}. \quad (1.12)$$

In particular,

$$J_2^p((\mathcal{A}_t^G f)_{t>0} : \mathbb{R}^d \rightarrow \mathbb{C}) \lesssim \|f\|_{L^p}, \quad (1.13)$$

where the implicit constants in (1.12) and (1.13) are independent of  $d$  and  $G$ .

**Theorem 1.14.** *Let  $d \in \mathbb{N}$  and  $G \subset \mathbb{R}^d$  be the unit ball induced by the  $\ell^q$  norm in  $\mathbb{R}^d$  for some  $1 \leq q \leq \infty$ . Then for every  $1 < p < \infty$  and  $f \in L^p(\mathbb{R}^d)$  we have*

$$\left\| \left( \sum_{k \in \mathbb{Z}} (V^2(\mathcal{A}_t^G f : t \in [2^k, 2^{k+1}]))^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim_q \|f\|_{L^p}. \quad (1.15)$$

In particular

$$J_2^p((\mathcal{A}_t^G f)_{t>0} : \mathbb{R}^d \rightarrow \mathbb{C}) \lesssim_q \|f\|_{L^p}, \quad (1.16)$$

where the implicit constants in (1.15) and (1.16) are independent of  $d$ .

The method of the present paper also allows us to provide estimates independent of the dimension in jump inequalities associated with the discrete averaging operator along cubes in  $\mathbb{Z}^d$ . For every  $x \in \mathbb{Z}^d$  and  $N \in \mathbb{N}$  let

$$A_N f(x) := \frac{1}{|Q_N \cap \mathbb{Z}^d|} \sum_{y \in Q_N \cap \mathbb{Z}^d} f(x-y), \quad f \in \ell^1(\mathbb{Z}^d), \quad (1.17)$$

be the discrete Hardy–Littlewood averaging operator, where  $Q_N = [-N, N]^d$ .

**Theorem 1.18.** *For every  $\frac{3}{2} < p < 4$  and  $f \in \ell^p(\mathbb{Z}^d)$  we have*

$$J_2^p((A_N f)_{N \in \mathbb{N}} : \mathbb{Z}^d \rightarrow \mathbb{C}) \lesssim \|f\|_{\ell^p}. \quad (1.19)$$

Moreover, if we consider only lacunary parameters, then (1.19) remains true for all  $1 < p < \infty$  and we have

$$J_2^p((A_{2^k} f)_{k \geq 0} : \mathbb{Z}^d \rightarrow \mathbb{C}) \lesssim \|f\|_{\ell^p}, \tag{1.20}$$

where the implicit constants in (1.19) and (1.20) are independent of  $d$ .

Theorem 1.18 provides the endpoint estimate at  $r = 2$  for the recent dimension-free estimates [Bourgain et al. 2019] for  $r$ -variations corresponding to operator (1.17).

The dimension-free results are proved in Section 3A by combining the results from Section 2 (Theorems 2.14 and 2.39) with the jump estimates for the Poisson semigroup from [Mirek et al. 2018b] and Fourier multiplier estimates from [Bourgain 1986a; 2014; Müller 1990].

**1B. Applications to operators of Radon type.** Another important class of operators which was extensively studied in [Jones et al. 2008] in the context of jump inequalities are operators of Radon type modeled on polynomial mappings.

Let  $P = (P_1, \dots, P_d) : \mathbb{R}^k \rightarrow \mathbb{R}^d$  be a polynomial mapping, where each component  $P_j : \mathbb{R}^k \rightarrow \mathbb{R}$  is a polynomial with  $k$  variables and real coefficients. We fix  $\Omega \subset \mathbb{R}^k$  a convex open bounded set containing the origin (not necessarily symmetric), and for every  $x \in \mathbb{R}^d$  and  $t > 0$  we define the Radon averaging operator

$$\mathcal{M}_t^P f(x) := \frac{1}{|\Omega_t|} \int_{\Omega_t} f(x - P(y)) \, dy, \tag{1.21}$$

where  $\Omega_t = \{x \in \mathbb{R}^k : t^{-1}x \in \Omega\}$ . Using Theorems 2.14 and 2.39 we easily deduce Theorem 1.22; see Section 3C.

**Theorem 1.22.** *For every  $1 < p < \infty$  and  $f \in L^p(\mathbb{R}^d)$  we have*

$$J_2^p((\mathcal{M}_t^P f)_{t>0} : \mathbb{R}^d \rightarrow \mathbb{C}) \lesssim_{d,p} \|f\|_{L^p}, \tag{1.23}$$

where the implicit constant is independent of the coefficients of  $P$ .

Before we formulate a corresponding result for truncated singular integrals we need to fix some definitions and notation. A *modulus of continuity* is a function  $\omega : [0, \infty) \rightarrow [0, \infty)$  with  $\omega(0) = 0$  that is subadditive in the sense that

$$u \leq t + s \implies \omega(u) \leq \omega(t) + \omega(s).$$

Substituting  $s = 0$  one sees that  $\omega(u) \leq \omega(t)$  for all  $0 \leq u \leq t$ . The basic example is  $\omega(t) = t^\theta$ , with  $\theta \in (0, 1)$ . Note that the composition and sum of two moduli of continuity is again a modulus of continuity. In particular, if  $\omega(t)$  is a modulus of continuity and  $\theta \in (0, 1)$ , then  $\omega(t)^\theta$  and  $\omega(t^\theta)$  are also moduli of continuity.

The *Dini norm* and the *log-Dini norm* of a modulus of continuity are defined respectively by setting

$$\|\omega\|_{\text{Dini}} := \int_0^1 \omega(t) \frac{dt}{t} \quad \text{and} \quad \|\omega\|_{\log\text{Dini}} := \int_0^1 \omega(t) \frac{|\log t| \, dt}{t}. \tag{1.24}$$

For any  $c > 0$  the integral can be equivalently (up to a  $c$ -dependent multiplicative constant) replaced by the sum over  $2^{-j/c}$  with  $j \in \mathbb{N}$ .

Finally, for every  $x \in \mathbb{R}^d$  and  $t > 0$  we will consider the truncated singular Radon transform

$$\mathcal{H}_t^P f(x) := \int_{\mathbb{R}^k \setminus \Omega_t} f(x - P(y))K(y) \, dy, \tag{1.25}$$

defined for every Schwartz function  $f$  in  $\mathbb{R}^d$ , where  $K : \mathbb{R}^k \setminus \{0\} \rightarrow \mathbb{C}$  is a kernel satisfying the following conditions:

- (1) The size condition: there exists a constant  $C_K > 0$  such that

$$|K(x)| \leq C_K |x|^{-k} \quad \text{for all } x \in \mathbb{R}^k. \tag{1.26}$$

- (2) The cancellation condition:

$$\int_{\Omega_R \setminus \Omega_r} K(y) \, dy = 0 \quad \text{for } 0 < r < R < \infty. \tag{1.27}$$

- (3) The smoothness condition:

$$\sup_{R > 0} \sup_{|y| \leq \frac{1}{2}Rt} \int_{R \leq |x| \leq 2R} |K(x) - K(x + y)| \, dx \leq \omega_K(t) \tag{1.28}$$

for every  $t \in (0, 1)$  with some modulus of continuity  $\omega_K$ .

In many applications it is easy to verify the somewhat stronger pointwise version of the smoothness estimate from (1.28). Namely,

$$|K(x) - K(x + y)| \leq \omega_K \left( \frac{|y|}{|x|} \right) |x|^{-k}, \quad \text{provided that } |y| \leq \frac{|x|}{2}, \tag{1.29}$$

for some modulus of continuity  $\omega_K$ . One can immediately see that condition (1.29) implies condition (1.28). Our next result establishes an analogue of the inequality (1.23) for the operators in (1.25).

**Theorem 1.30.** *Suppose that  $\|\omega_K^\theta\|_{\log\text{Dini}} + \|\omega_K^{\theta/2}\|_{\text{Dini}} < \infty$  for some  $\theta \in (0, 1]$ . Then for every  $p \in \{1 + \theta, (1 + \theta)'\}$  and  $f \in L^p(\mathbb{R}^d)$  we have*

$$J_2^P((\mathcal{H}_t^P f)_{t>0} : \mathbb{R}^d \rightarrow \mathbb{C}) \lesssim_{d,p} \|f\|_{L^p}, \tag{1.31}$$

where the implicit constant is independent of the coefficients of  $P$ . More precisely:

- (1) If  $\|\omega_K^\theta\|_{\log\text{Dini}} < \infty$ , then

$$J_2^P((\mathcal{H}_{2^k} f)_{k \in \mathbb{Z}} : \mathbb{R}^d \rightarrow \mathbb{C}) \lesssim \|f\|_{L^p}. \tag{1.32}$$

- (2) If  $\|\omega_K^{\theta/2}\|_{\text{Dini}} < \infty$ , then

$$\left\| \left( \sum_{k \in \mathbb{Z}} V^2(\mathcal{H}_t f : t \in [2^k, 2^{k+1}]) \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \|f\|_{L^p}. \tag{1.33}$$

The inequality (1.23) was proved in [Jones et al. 2008] for the averages  $\mathcal{M}_t^P$  over Euclidean balls. The inequality (1.31) was proved in that work for monomial curves, i.e., in the case

$$k = 1, \quad d = 2, \quad K(y) = y^{-1} \quad \text{and} \quad P(x) = (x, x^a), \quad \text{where } a > 1.$$

General polynomials were considered in [Mirek et al. 2017] (although jump estimates are not explicitly stated in that article they can also be obtained with minor modifications of the proofs). Multidimensional variants of  $\mathcal{H}_t^P$  were also studied in that work under stronger regularity conditions imposed on the kernel  $K$ . Inequalities (1.23) and (1.31) will be used to establish jump inequalities for the discrete analogues of (1.21) and (1.25) in [Mirek et al. 2018a].

Finally we provide van der Corput integral estimates in Lemma B.1 and Proposition B.2, which have the feature of permitting the handling of the oscillatory integrals with nonsmooth amplitudes. Their broader scope will be needed in the proof of Theorem 1.30.

## 2. An abstract approach to jump inequalities

**2A. Preliminaries.** Let  $(X, \mathcal{B}, \mathfrak{m})$  be a  $\sigma$ -finite measure space endowed with a sequence of linear Littlewood–Paley operators  $(S_j)_{j \in \mathbb{Z}}$  satisfying (1.5), (1.6). Assume that  $(T_t)_{t \in \mathbb{I}}$  is a family of linear operators acting on  $L^1(X) + L^\infty(X)$ , where the index set  $\mathbb{I}$  is a subset of  $(0, \infty)$ . Under suitable conditions imposed on the family  $(T_t)_{t \in \mathbb{I}}$  in terms of its interactions with the Littlewood–Paley operators  $(S_j)_{j \in \mathbb{Z}}$  as in the Introduction, we will study strong uniform jump inequalities

$$J_2^p((T_t f)_{t \in \mathbb{I}} : X \rightarrow \mathbb{C}) \lesssim \|f\|_{L^p} \tag{2.1}$$

in various ranges of  $p$ 's; see Theorems 2.14, 2.28, and 2.39.

To avoid further problems with measurability we will always assume that  $\mathbb{I}$  is countable. Usually  $\mathbb{I}$  is  $\mathbb{D} := \{2^n : n \in \mathbb{Z}\}$  the set of all dyadic numbers or  $\mathbb{I}$  is  $\mathbb{U} := \bigcup_{n \in \mathbb{Z}} 2^{-n} \mathbb{N}$  the set of nonnegative rational numbers whose denominators in reduced form are powers of 2. In practice, the countability assumption may be removed if for every  $f \in L^1(X) + L^\infty(X)$  the function  $\mathbb{I} \ni t \mapsto T_t f(x)$  is continuous for  $\mathfrak{m}$ -almost every  $x \in X$ . In our applications this will always be the case.

We recall the decomposition into long and short jumps from [Jones et al. 2008, Lemma 1.3], which tells that for every  $\lambda > 0$  we have

$$\lambda N_\lambda(T_t f(x) : t \in \mathbb{I})^{\frac{1}{2}} \lesssim \lambda N_{\frac{\lambda}{3}}(T_t f(x) : t \in \mathbb{D})^{\frac{1}{2}} + \left( \sum_{k \in \mathbb{Z}} (\lambda N_\lambda(T_t f(x) : t \in [2^k, 2^{k+1}) \cap \mathbb{I})^{\frac{1}{2}})^2 \right)^{\frac{1}{2}}. \tag{2.2}$$

In other words the  $\lambda$ -jump counting function can be dominated by the long jumps (the first term in (2.2) with  $t \in \mathbb{D}$ ) and the short jumps (the square function in (2.2)). Similar inequalities hold for the maximal function and for  $r$ -variations.

We deal with  $L^p$  bounds for the long jump counting function corresponding to  $T_t$  with  $t \in \mathbb{D}$  in two ways, similarly to [Duoandikoetxea and Rubio de Francia 1986]. The first approach is to find an approximating family of operators (see the family  $(P_k)_{k \in \mathbb{Z}}$  in Theorem 2.14) for which the bound in question is known and control a square function that dominates the error term; see (2.15) in Theorem 2.14. In our case this

method works for positive operators with martingales or related operators as the approximating family. The second approach is to express  $T_{2^k}$  as a telescoping sum

$$T_{2^k} f = \sum_{j \geq k} T_{2^j} f - T_{2^{j+1}} f = \sum_{j \geq k} B_j f \tag{2.3}$$

and try to deduce bounds in question from the behavior of  $B_j = T_{2^j} - T_{2^{j+1}}$ . This approach is needed if  $T_t$  is a truncated singular integral-type operator; see Theorem 2.28. Similar strategies also yield  $L^p$  bounds for maximal functions  $\sup_{k \in \mathbb{Z}} |T_{2^k} f(x)|$  or  $r$ -variations  $V^r(T_{2^k} f(x) : k \in \mathbb{Z})$ .

In order to deal with short jumps we note that the square function on the right-hand side of (2.2) is dominated by the square function associated with 2-variations, which in turn is controlled by a series of square functions

$$\begin{aligned} & \left( \sum_{k \in \mathbb{Z}} (V^2(T_t f(x) : t \in [2^k, 2^{k+1}) \cap \mathbb{I}))^2 \right)^{\frac{1}{2}} \\ & \leq \sqrt{2} \sum_{l \geq 0} \left( \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^l-1} |(T_{2^{k+2^{k-l}(m+1)}} - T_{2^{k+2^{k-l}m}}) f(x)|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{2.4}$$

The square function on the right-hand side of (2.4) gives rise to assumption (2.40). Inequality (2.4) follows from the next lemma with  $\mathfrak{g}(t) = T_{2^{k+t}} f(x)$  and  $r = 2$ .

**Lemma 2.5.** *Let  $r \in [1, \infty)$ ,  $k \in \mathbb{Z}$ , and a function  $\mathfrak{g} : [0, 2^k] \cap \mathbb{U} \rightarrow \mathbb{C}$  be given. Then*

$$V^r(\mathfrak{g}(t) : t \in [0, 2^k] \cap \mathbb{U}) \leq 2^{\frac{r-1}{r}} \sum_{l \geq 0} \left( \sum_{m=0}^{2^l-1} |\mathfrak{g}(2^{k-l}(m+1)) - \mathfrak{g}(2^{k-l}m)|^r \right)^{\frac{1}{r}}. \tag{2.6}$$

The variation norm on the left-hand side of (2.6) can be extended to all  $t \in [0, 2^k]$  if  $\mathfrak{g} : [0, 2^k] \rightarrow \mathbb{C}$  is continuous. Lemma 2.5 originates in [Lewko and Lewko 2012], where it was observed that the 2-variation norm of a sequence of length  $N$  can be controlled by the sum of  $\log N$  square functions and this observation was used to obtain a variational version of the Rademacher–Menshov theorem. Inequality (2.6), essentially in this form, was independently proved by the first author and Trojan [Mirek and Trojan 2016] and used to estimate  $r$ -variations for discrete Radon transforms. Lemma 2.5 has been used in several recent articles on  $r$ -variations, including [Bourgain et al. 2018]. For completeness we include a proof, which is shorter than the previous proofs.

*Proof of Lemma 2.5.* Due to monotonicity of  $r$ -variations it suffices to prove (2.6) with  $\mathbb{U}_N = \{u/2^N : u \in \mathbb{N} \text{ and } 0 \leq u \leq 2^{k+N}\}$  in place of  $[0, 2^k] \cap \mathbb{U}$ . Observe that

$$V^r(\mathfrak{g}(t) : t \in \mathbb{U}_N) = V^r\left(\mathfrak{g}\left(\frac{t}{2^N}\right) : t \in [0, 2^{k+N}] \cap \mathbb{Z}\right).$$

The proof will be completed if we show that

$$V^r(\mathfrak{g}(t) : t \in [0, 2^n] \cap \mathbb{Z}) \leq 2^{1-\frac{1}{r}} \sum_{l=0}^n \left( \sum_{m=0}^{2^{n-l}-1} |\mathfrak{g}(2^l(m+1)) - \mathfrak{g}(2^l m)|^r \right)^{\frac{1}{r}}. \tag{2.7}$$

Once (2.7) is established we apply it with  $g(t/2^N)$  in place of  $g(t)$  and  $n = k + N$  and obtain (2.6). We prove (2.7) by induction on  $n$ . The case  $n = 0$  is easy to verify. Let  $n \geq 1$  and suppose that the claim is known for  $n - 1$ . Let  $0 \leq t_0 < \dots < t_J < 2^n$  be an increasing sequence of integers. For  $j \in \{0, \dots, J\}$  let  $s_j \leq t_j \leq u_j$  be the closest smaller and larger even integer, respectively. Then

$$\begin{aligned} \left(\sum_{j=1}^J |g(t_j) - g(t_{j-1})|^r\right)^{\frac{1}{r}} &= \left(\sum_{j=1}^J |(g(t_j) - g(s_j)) + (g(s_j) - g(u_{j-1})) + (g(u_{j-1}) - g(t_{j-1}))|^r\right)^{\frac{1}{r}} \\ &\leq \left(\sum_{j=1}^J |g(s_j) - g(u_{j-1})|^r\right)^{\frac{1}{r}} + \left(\sum_{j=1}^J |(g(t_j) - g(s_j)) + (g(u_{j-1}) - g(t_{j-1}))|^r\right)^{\frac{1}{r}}. \end{aligned}$$

In the first term we notice that the sequence  $u_0 \leq s_1 \leq u_1 \leq \dots$  is monotonically increasing and takes values in  $2\mathbb{N}$ , so we can apply the induction hypothesis to the function  $g(2 \cdot)$ . In the second term we use the elementary inequality  $(a + b)^r \leq 2^{r-1}(a^r + b^r)$  and observe  $|t_j - s_j| \leq 1$ ,  $|t_{j-1} - u_{j-1}| \leq 1$ , and  $s_j \geq u_{j-1}$ , so that this is bounded by the  $l = 0$  summand in (2.7).  $\square$

**2B. Preparatory estimates.** We recall Lemma 2.8 that deduces a vector-valued inequality from a maximal one. Then we apply it to obtain Lemma 2.9.

**Lemma 2.8** [Duoandikoetxea and Rubio de Francia 1986, p. 544]. *Suppose that  $(X, \mathcal{B}, m)$  is a  $\sigma$ -finite measure space and  $(M_k)_{k \in \mathbb{J}}$  is a sequence of linear operators on  $L^1(X) + L^\infty(X)$  indexed by a countable set  $\mathbb{J}$ . The corresponding maximal operator is defined by*

$$M_{*, \mathbb{J}} f := \sup_{k \in \mathbb{J}} \sup_{|g| \leq |f|} |M_k g|,$$

where the supremum is taken in the lattice sense. Let  $q_0, q_1 \in [1, \infty]$  and  $0 \leq \theta \leq 1$  with  $\frac{1}{2} = (1 - \theta)/q_0$  and  $q_0 \leq q_1$ . Let  $q_\theta \in [q_0, q_1]$  be given by

$$\frac{1}{q_\theta} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1} = \frac{1}{2} + \frac{1 - q_0/2}{q_1}.$$

Then

$$\left\| \left( \sum_{k \in \mathbb{J}} |M_k g_k|^2 \right)^{\frac{1}{2}} \right\|_{L^{q_\theta}} \leq \left( \sup_{k \in \mathbb{J}} \|M_k\|_{L^{q_0} \rightarrow L^{q_0}} \right)^{1-\theta} \|M_{*, \mathbb{J}}\|_{L^{q_1} \rightarrow L^{q_1}}^\theta \left\| \left( \sum_{k \in \mathbb{J}} |g_k|^2 \right)^{\frac{1}{2}} \right\|_{L^{q_\theta}}.$$

*Proof.* Consider the operator  $\tilde{M}g := (M_k g_k)_{k \in \mathbb{J}}$  acting on sequences of functions  $g = (g_k)_{k \in \mathbb{J}}$  in  $L^1(X) + L^\infty(X)$ . By Fubini's theorem

$$\begin{aligned} \|\tilde{M}g\|_{L^{q_0}(\ell^{q_0})} &= \|\|M_k g_k\|_{L^{q_0}}\|_{\ell^{q_0}} \\ &\leq \left( \sup_{k \in \mathbb{J}} \|M_k\|_{L^{q_0} \rightarrow L^{q_0}} \right) \|g\|_{L^{q_0}(\ell^{q_0})} \\ &= \left( \sup_{k \in \mathbb{J}} \|M_k\|_{L^{q_0} \rightarrow L^{q_0}} \right) \|g\|_{L^{q_0}(\ell^{q_0})}. \end{aligned}$$

By definition of the maximal operator

$$\begin{aligned} \|\tilde{M}g\|_{L^{q_1}(\ell^\infty)} &= \left\| \sup_{k \in \mathbb{J}} |M_k g_k| \right\|_{L^{q_1}} \|M_{*,\mathbb{J}}(\sup_{k \in \mathbb{J}} |g_k|)\|_{L^{q_1}} \\ &\leq \|M_{*,\mathbb{J}}\|_{L^{q_1} \rightarrow L^{q_1}} \left\| \sup_{k \in \mathbb{J}} |g_k| \right\|_{L^{q_1}} = \|M_{*,\mathbb{J}}\|_{L^{q_1} \rightarrow L^{q_1}} \|g\|_{L^{q_1}(\ell^\infty)}. \end{aligned}$$

The claim for  $q_\theta \in [q_0, q_1]$  follows by complex interpolation between  $L^{q_0}(X; \ell^{q_0}(\mathbb{J}))$  and  $L^{q_1}(X; \ell^\infty(\mathbb{J}))$ . □

**Lemma 2.9.** *Suppose that  $(X, \mathcal{B}, \mathfrak{m})$  is a  $\sigma$ -finite measure space with a sequence of operators  $(S_k)_{k \in \mathbb{Z}}$  that satisfy the Littlewood–Paley inequality (1.6). Let  $1 \leq q_0 \leq q_1 \leq 2$  and  $L \in \mathbb{N}$  be a positive integer and let  $\mathbb{V}_L = \{(k, l) \in \mathbb{Z}^2 : 0 \leq l \leq L - 1\}$ . Let  $(M_{k,l})_{(k,l) \in \mathbb{V}_L}$  be a sequence of operators bounded on  $L^{q_1}(X)$  such that*

$$\left\| \left( \sum_{k \in \mathbb{Z}} \sum_{l=0}^{L-1} |M_{k,l} S_{k+j} f|^2 \right)^{\frac{1}{2}} \right\|_{L^2} \leq a_j \|f\|_{L^2}, \quad f \in L^2(X), \tag{2.10}$$

for some positive numbers  $(a_j)_{j \in \mathbb{Z}}$ . Then for  $p = q_1$  and for all  $f \in L^p(X)$  we have

$$\begin{aligned} &\left\| \left( \sum_{k \in \mathbb{Z}} \sum_{l=0}^{L-1} |M_{k,l} S_{k+j} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \\ &\lesssim L^{\frac{1}{2} \frac{2-q_1}{2-q_0}} \left( \sup_{(k,l) \in \mathbb{V}_L} \|M_{k,l}\|_{L^{q_0} \rightarrow L^{q_0}}^{\frac{q_0}{2} \frac{2-q_1}{2-q_0}} \right) \|M_{*,\mathbb{V}_L}\|_{L^{q_1} \rightarrow L^{q_1}}^{\frac{2-q_1}{2}} a_j^{\frac{q_1-q_0}{2-q_0}} \|f\|_{L^p}. \end{aligned} \tag{2.11}$$

If  $M_{k,l}$  are convolution operators on an abelian group  $\mathbb{G}$ , then (2.11) also holds for  $q_1 \leq p \leq q_1'$ . The implicit constants in the conclusion do not depend on the qualitative bounds that we assume for the operators  $M_{k,l}$  on  $L^{q_1}(X)$ .

*Proof.* First we show (2.11). In the case  $q_1 = 2$  this is identical to the hypothesis (2.10), so suppose  $q_1 < 2$ . Let  $\theta$  and  $q_\theta \in [q_0, q_1]$  be as in Lemma 2.8; then by that lemma and Littlewood–Paley inequality (1.6) we obtain

$$\begin{aligned} &\left\| \left( \sum_{k \in \mathbb{Z}} \sum_{l=0}^{L-1} |M_{k,l} S_{k+j} f|^2 \right)^{\frac{1}{2}} \right\|_{L^{q_\theta}} \\ &\lesssim \left( \sup_{(k,l) \in \mathbb{V}_L} \|M_{k,l}\|_{L^{q_0} \rightarrow L^{q_0}}^{1-\theta} \right) \|M_{*,\mathbb{V}_L}\|_{L^{q_1} \rightarrow L^{q_1}}^\theta \left\| \left( \sum_{k \in \mathbb{Z}} \sum_{l=0}^{L-1} |S_{k+j} f|^2 \right)^{\frac{1}{2}} \right\|_{L^{q_\theta}} \\ &\lesssim L^{\frac{1}{2}} \left( \sup_{(k,l) \in \mathbb{V}_L} \|M_{k,l}\|_{L^{q_0} \rightarrow L^{q_0}}^{1-\theta} \right) \|M_{*,\mathbb{V}_L}\|_{L^{q_1} \rightarrow L^{q_1}}^\theta \|f\|_{L^{q_\theta}}. \end{aligned} \tag{2.12}$$

Since  $q_\theta \leq q_1 < 2$ , there is a unique  $\nu \in (0, 1]$  such that

$$\frac{1}{q_1} = \frac{\nu}{q_\theta} + \frac{1-\nu}{2}.$$

Substituting the definition of  $q_\theta$  we obtain

$$\frac{1}{q_1} = \frac{\nu\theta}{q_1} + \frac{1}{2}.$$

It follows that

$$\begin{aligned} 1 - \theta &= \frac{q_0}{2}, & \theta &= \frac{2 - q_0}{2}, & \nu\theta &= \frac{2 - q_1}{2}, \\ \nu &= \frac{2 - q_1}{2 - q_0}, & \nu(1 - \theta) &= \frac{2 - q_1}{2 - q_0} \frac{q_0}{2}, & 1 - \nu &= \frac{q_1 - q_0}{2 - q_0}. \end{aligned}$$

Interpolating (2.12) with the hypothesis (2.10) gives the claim (2.11) for  $p = q_1$ .

If  $M_{k,l}$  are convolution operators, then by duality the first inequality in (2.12) also holds with  $q_\theta$  replaced by  $q'_\theta$ . Also,

$$\frac{1}{q'_1} = \frac{\nu}{q'_\theta} + \frac{1 - \nu}{2},$$

so the same argument as before also works for  $p = q'_1$ . The conclusion for  $q_1 < p < q'_1$  follows by complex interpolation. □

**2C. Long jumps for positive operators.** Suppose now we have a sequence of positive linear operators  $(A_k)_{k \in \mathbb{Z}}$  and an approximating family of linear operators  $(P_k)_{k \in \mathbb{Z}}$  both acting on  $L^1(X) + L^\infty(X)$  such that for every  $1 < p < \infty$  the maximal lattice operator

$$P_* f := \sup_{k \in \mathbb{Z}} \sup_{|g| \leq |f|} |P_k g|$$

satisfies the maximal estimate

$$\|P_*\|_{L^p \rightarrow L^p} \lesssim 1. \tag{2.13}$$

Theorem 2.14 will be based on a variant of the bootstrap argument discussed in the context of differentiation in lacunary directions in [Nagel et al. 1978]. These ideas were also used to provide  $L^p$  bounds for maximal Radon transforms in [Duoandikoetxea and Rubio de Francia 1986]. It was observed by Christ that the argument from [Nagel et al. 1978] can be formulated as an abstract principle, which was useful in many situations [Carbery 1988] and also in the context of dimension-free estimates [Carbery 1986].

**Theorem 2.14.** *Assume that  $(X, \mathcal{B}, \mathfrak{m})$  is a  $\sigma$ -finite measure space endowed with a sequence of linear operators  $(S_j)_{j \in \mathbb{Z}}$  satisfying (1.5) and (1.6). Given parameters  $1 \leq q_0 < q_1 \leq 2$ , let  $(A_k)_{k \in \mathbb{Z}}$  be a sequence of positive linear operators such that  $\sup_{k \in \mathbb{Z}} \|A_k\|_{L^{q_0} \rightarrow L^{q_0}} \lesssim 1$ . Suppose that the maximal function  $P_*$  satisfies (2.13) with  $p = q_1$  and*

$$\left\| \left( \sum_{k \in \mathbb{Z}} |(A_k - P_k)S_{k+j}f|^2 \right)^{\frac{1}{2}} \right\|_{L^2} \leq a_j \|f\|_{L^2}, \quad f \in L^2(X), \tag{2.15}$$

for some positive numbers  $(a_j)_{j \in \mathbb{Z}}$  satisfying

$$\mathbf{a} := \sum_{j \in \mathbb{Z}} a_j^{\frac{q_1 - q_0}{2 - q_0}} < \infty.$$

Then for all  $f \in L^p(X)$  with  $p = q_1$  we have

$$\left\| \left( \sum_{k \in \mathbb{Z}} |(A_k - P_k)f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim (1 + \mathbf{a}^{\frac{2}{q_1}}) \|f\|_{L^p}. \tag{2.16}$$

In particular

$$\|A_*\|_{L^p \rightarrow L^p} \lesssim 1 + \mathbf{a}^{\frac{2}{q_1}}. \tag{2.17}$$

If in addition we have the jump inequality

$$J_2^p((P_k f)_{k \in \mathbb{Z}} : X \rightarrow \mathbb{C}) \lesssim \|f\|_{L^p}, \tag{2.18}$$

then also

$$J_2^p((A_k f)_{k \in \mathbb{Z}} : X \rightarrow \mathbb{C}) \lesssim (1 + \mathbf{a}^{\frac{2}{q_1}}) \|f\|_{L^p}. \tag{2.19}$$

If  $A_k$  and  $P_k$  are convolution operators on an abelian group  $\mathbb{G}$ , all these implications also hold for  $q_1 \leq p \leq q'_1$ , and we have the vector-valued estimate

$$\left\| \left( \sum_{k \in \mathbb{Z}} |A_k f_k|^r \right)^{\frac{1}{r}} \right\|_{L^p} \lesssim \left\| \left( \sum_{k \in \mathbb{Z}} |f_k|^r \right)^{\frac{1}{r}} \right\|_{L^p} \tag{2.20}$$

in the same range  $q_1 \leq p \leq q'_1$  for all  $1 \leq r \leq \infty$ .

A few remarks concerning the assumptions in Theorem 2.14 are in order. In applications it is usually not difficult to verify the assumption (2.15). For general operators the most reasonable and efficient way is to apply  $TT^*$  methods. However, for convolution operators on  $\mathbb{G}$  assumption (2.15) can be verified using Fourier transform methods, which may be simpler than  $TT^*$  methods. Let us explain the second approach more precisely when  $\mathbb{G} = \mathbb{R}^d$ . We first have to fix some terminology.

Let  $A$  be a  $d \times d$  real matrix whose eigenvalues have positive real part. We set

$$t^A := \exp(A \log t) \quad \text{for } t > 0. \tag{2.21}$$

Let  $q$  be a smooth  $A$ -homogeneous quasinorm on  $\mathbb{R}^d$ , that is,  $q : \mathbb{R}^d \rightarrow [0, \infty)$  is a continuous function, smooth on  $\mathbb{R}^d \setminus \{0\}$ , and such that

- (1)  $q(x) = 0 \iff x = 0$ ;
- (2) there is  $C \geq 1$  such that for all  $x, y \in \mathbb{R}^d$  we have  $q(x + y) \leq C(q(x) + q(y))$ ;
- (3)  $q(t^A x) = tq(x)$  for all  $t > 0$  and  $x \in \mathbb{R}^d$ .

Let also  $q_*$  be a smooth (away from 0)  $A^*$ -homogeneous quasinorm, where  $A^*$  is the adjoint matrix to  $A$ . We only have to find a sequence of Littlewood–Paley projections associated with the quasinorm  $q_*$ . For this purpose let  $\phi_0 : [0, \infty) \rightarrow [0, \infty)$  be a smooth function such that  $0 \leq \phi_0 \leq \mathbf{1}_{[1/2, 2]}$  and its dilates  $\phi_j(x) := \phi_0(2^j x)$  satisfy

$$\sum_{j \in \mathbb{Z}} \phi_j^2 = \mathbf{1}_{(0, \infty)}. \tag{2.22}$$

For each  $j \in \mathbb{Z}$  we define the Littlewood–Paley operator  $\tilde{S}_j$  such that  $\widehat{\tilde{S}_j f} = \psi_j \hat{f}$  corresponds to a smooth function  $\psi_j(\xi) := \phi_j(q_*(\xi))$  on  $\mathbb{R}^d$ . By (2.22) we see that (1.5) holds for  $S_j = \tilde{S}_j^2$ . Moreover, by [Rivière 1971, Theorem II.1.5] we obtain the Littlewood–Paley inequality (1.6) for the operators  $S_j$  and  $\tilde{S}_j$ .

If  $(\Phi_t : t > 0)$  is a family of Schwartz functions such that  $\hat{\Phi}_t(\xi) = \hat{\Phi}(tq_*(\xi))$ , where  $\Phi$  is a nonnegative Schwartz function on  $\mathbb{R}^d$  with integral 1, then by [Jones et al. 2008, Theorem 1.1] we know that for every  $1 < p < \infty$  we have

$$J_2^p((\Phi_{2^k} * f)_{k \in \mathbb{Z}} : \mathbb{R}^d \rightarrow \mathbb{C}) \lesssim \|f\|_{L^p}, \quad f \in L^p(\mathbb{R}^d). \tag{2.23}$$

The maximal version of inequality (2.23) has been known for a long time and follows from the Hardy–Littlewood maximal theorem [Stein 1993]. Hence taking  $P_k f = \Phi_{2^k} * f$  for  $k \in \mathbb{Z}$ , we may assume that (2.18) is verified.

Suppose now we have a family  $(A_k)_{k \in \mathbb{Z}}$  of convolution operators  $A_k f = \mu_{2^k} * f$  corresponding to a family of probability measures  $(\mu_t : t > 0)$  on  $\mathbb{R}^d$  such that

$$|\hat{\mu}_t(\xi) - \hat{\mu}_t(0)| \leq \omega(tq_*(\xi)) \quad \text{if } tq_*(\xi) \leq 1, \tag{2.24}$$

$$|\hat{\mu}_t(\xi)| \leq \omega((tq_*(\xi))^{-1}) \quad \text{if } tq_*(\xi) \geq 1 \tag{2.25}$$

for some modulus of continuity  $\omega$ .

Theorem 2.14, taking into account all the facts mentioned above, yields

$$J_2^p((\mu_{2^k} * f)_{k \in \mathbb{Z}} : \mathbb{R}^d \rightarrow \mathbb{C}) \lesssim \|f\|_{L^p}, \quad f \in L^p(\mathbb{R}^d), \tag{2.26}$$

for  $p = q_1$  and  $q_0 = 1$  as long as

$$\mathbf{a} = \sum_{j \in \mathbb{Z}} \omega(2^{-|j|})^{\frac{q_1 - q_0}{2 - q_0}} < \infty,$$

since (2.15) can be easily verified with  $a_j = \omega(2^{-|j|})$  using (2.24), (2.25) and the properties of  $S_j$  and  $\Phi$ .

*Proof of Theorem 2.14.* We begin with the proof of (2.16). If  $q_1 = 2$  then we use (1.5) and (2.15) and we are done. We now assume that  $q_1 < 2$ . By the monotone convergence theorem it suffices to consider only finitely many  $M_k := A_k - P_k$ 's in (2.16), let us say those with  $|k| \leq K$ . Restrict all summations and suprema to  $|k| \leq K$  and let  $B$  be the smallest implicit constant for which (2.16) holds with  $p = q_1$ . In view of the qualitative boundedness hypothesis we obtain  $B < \infty$ , but the bound may depend on  $K$ . Our aim is to show that  $B \lesssim 1 + \mathbf{a}^{2/q_1}$ . There is nothing to do if  $B \lesssim 1$ . Therefore, we will assume that  $B \gtrsim 1$ , so by (1.5), (2.13) and (2.11) with  $L = 1$  and  $M_{k,0} := M_k$ , we obtain

$$\left\| \left( \sum_{|k| \leq K} |M_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \leq \sum_{j \in \mathbb{Z}} \left\| \left( \sum_{|k| \leq K} |M_k S_{k+j} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim (1 + \|M_*\|_{L^p \rightarrow L^p}^{\frac{2-q_1}{2}} \mathbf{a}) \|f\|_{L^p}.$$

By positivity we have  $|A_* f| \leq \sup_{|k| \leq K} A_k |f|$  and consequently we obtain

$$|A_* f| \leq \sup_{|k| \leq K} A_k |f| \leq \sup_{|k| \leq K} P_k |f| + \left( \sum_{|k| \leq K} |M_k |f|^2 \right)^{\frac{1}{2}}. \tag{2.27}$$

By (2.27) and (2.13) we get

$$\|M_*\|_{L^p \rightarrow L^p} \leq \|P_*\|_{L^p \rightarrow L^p} + \|A_*\|_{L^p \rightarrow L^p} \leq 2\|P_*\|_{L^p \rightarrow L^p} + B \lesssim 1 + B.$$

Taking into account these inequalities we have

$$\left\| \left( \sum_{|k| \leq K} |(A_k - P_k)f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} = \left\| \left( \sum_{|k| \leq K} |M_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim (1 + \mathbf{a}(1 + B)^{\frac{2-q_1}{2}}) \|f\|_{L^p}.$$

Taking the supremum over  $f$  gives

$$B \lesssim 1 + \mathbf{a}(1 + B)^{\frac{2-q_1}{2}} \lesssim (1 + \mathbf{a})B^{\frac{2-q_1}{2}},$$

since we have assumed  $B \gtrsim 1$ , and the conclusion (2.16) follows.

Once (2.16) is proven, in view of (2.27) we immediately obtain (2.17). In a similar way, if (2.18) holds, we deduce (2.19) from (2.16). Indeed,

$$\begin{aligned} J_2^p((A_k f)_{k \in \mathbb{Z}}) &\lesssim J_2^p((P_k f)_{k \in \mathbb{Z}}) + J_2^p((M_k f)_{k \in \mathbb{Z}}) \\ &\lesssim \|f\|_{L^p} + \|V^2(M_k f : k \in \mathbb{Z})\|_{L^p} \\ &\lesssim \|f\|_{L^p} + \left\| \left( \sum_{k \in \mathbb{Z}} |M_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p}. \end{aligned}$$

In the case of convolution operators we can run the above proof of (2.16) with  $p = q'_1$ , since in this case Lemma 2.9 tells that (2.11) also holds with  $p = q'_1$ . Once the estimate (2.16) is known for  $p = q_1, q'_1$ , by interpolation we extend it to  $q_1 \leq p \leq q'_1$ , and all other inequalities follow as before. Finally, the vector-valued estimate (2.20) with  $r = \infty$  is equivalent to the maximal estimate by positivity, with  $r = 1$  it follows by duality, and with  $1 < r < \infty$  by complex interpolation.  $\square$

**2D. Long jumps for nonpositive operators.** We now drop the positivity assumption and we will be working with general operators  $(B_k)_{k \in \mathbb{Z}}$  acting on  $L^1(X) + L^\infty(X)$ . This will require some knowledge about the maximal lattice operator  $B_*$  defined in (2.29) and about the sum of  $B_k$ 's over  $k \in \mathbb{Z}$ . No bootstrap argument seems to be available for nonpositive operators and therefore additional assumptions like (2.30) and (2.32) will be indispensable. The proof of Theorem 2.28 is based on the ideas from [Duoandikoetxea and Rubio de Francia 1986].

**Theorem 2.28.** *Assume that  $(X, \mathcal{B}, \mathfrak{m})$  is a  $\sigma$ -finite measure space endowed with a sequence of linear operators  $(S_j)_{j \in \mathbb{Z}}$  satisfying (1.5) and (1.6). Let  $1 \leq q_0 < q_1 \leq 2$  and let  $(B_k)_{k \in \mathbb{Z}}$  be a sequence of linear operators commuting with the sequence  $(S_j)_{j \in \mathbb{Z}}$  such that  $\sup_{k \in \mathbb{Z}} \|B_k\|_{L^{q_0} \rightarrow L^{q_0}} \lesssim 1$ . Suppose that the maximal lattice operator*

$$B_* f := \sup_{k \in \mathbb{Z}} \sup_{|g| \leq |f|} |B_k g| \tag{2.29}$$

satisfies

$$\|B_*\|_{L^{q_1} \rightarrow L^{q_1}} \lesssim 1. \tag{2.30}$$

We also assume

$$\left\| \left( \sum_{k \in \mathbb{Z}} |B_k S_{k+j} f|^2 \right)^{\frac{1}{2}} \right\|_{L^2} \leq a_j \|f\|_{L^2}, \quad f \in L^2(X), \tag{2.31}$$

for some positive numbers  $(a_j)_{j \in \mathbb{Z}}$ :

(1) Suppose that  $(B_k)_{k \in \mathbb{Z}}$  additionally satisfies

$$\left\| \sum_{k \in \mathbb{Z}} B_k \right\|_{L^{q_1} \rightarrow L^{q_1}} \lesssim 1. \tag{2.32}$$

Let  $P_k := \sum_{j>k} S_j$  and assume that the jump inequality (2.18) holds for the sequence  $(P_k)_{k \in \mathbb{Z}}$  with  $p = q_1$ . Then for all  $f \in L^p(X)$  with  $p = q_1$  we have

$$\begin{aligned} J_2^p \left( \left( \sum_{j \geq k} B_j f \right)_{k \in \mathbb{Z}} : X \rightarrow \mathbb{C} \right) \\ \lesssim \left( \left\| \sum_{k \in \mathbb{Z}} B_k \right\|_{L^{q_1} \rightarrow L^{q_1}} + \left( \sup_{k \in \mathbb{Z}} \|B_k\|_{L^{q_0} \rightarrow L^{q_0}} \right)^{\frac{q_0}{2} \frac{2-q_1}{2-q_0}} \|B_*\|_{L^{q_1} \rightarrow L^{q_1}}^{\frac{2-q_1}{2}} \tilde{\mathbf{a}} \right) \|f\|_{L^p}, \end{aligned} \tag{2.33}$$

where

$$\tilde{\mathbf{a}} := \sum_{j \in \mathbb{Z}} (|j| + 1) a_j^{\frac{q_1 - q_0}{2 - q_0}} < \infty.$$

(2) Suppose that there is a sequence of self-adjoint linear operators  $(\tilde{S}_j)_{j \in \mathbb{Z}}$  such that  $S_j = \tilde{S}_j^2$  for every  $j \in \mathbb{Z}$  and satisfying (1.6) and (2.31) with  $\tilde{S}_{k+j}$  in place of  $S_{k+j}$ . Then for every sequence  $(\varepsilon_k)_{k \in \mathbb{Z}}$  bounded by 1 and for all  $f \in L^p(X)$  with  $p = q_1$  we have

$$\left\| \sum_{k \in \mathbb{Z}} \varepsilon_k B_k f \right\|_{L^p} \lesssim \left( \sup_{k \in \mathbb{Z}} \|B_k\|_{L^{q_0} \rightarrow L^{q_0}} \right)^{\frac{q_0}{2} \frac{2-q_1}{2-q_0}} \|B_*\|_{L^{q_1} \rightarrow L^{q_1}}^{\frac{2-q_1}{2}} \mathbf{a} \|f\|_{L^p}, \tag{2.34}$$

where  $\mathbf{a}$  is as in Theorem 2.14.

In the case of convolution operators on an abelian group  $\mathbb{G}$  all these implications also hold for  $q_1 \leq p \leq q'_1$ .

In applications in harmonic analysis we will take  $B_k = T_{2^k} - T_{2^{k+1}}$  for  $k \in \mathbb{Z}$ , where  $T_t$  is a truncated singular integral operator of convolution type; see (2.3). This class of operators motivates, to a large extent, the assumptions in Theorem 2.28. In many cases they can be verified if we manage to find positive operators  $A_k$  such that  $|B_k f| \lesssim A_k |f|$  for every  $k \in \mathbb{Z}$  and  $f \in L^1(X) + L^\infty(X)$ . In practice,  $A_k$  is an averaging operator. We shall illustrate this more precisely by appealing to the discussion after Theorem 2.14.

Suppose that  $(B_k)_{k \in \mathbb{Z}}$  is a family of convolution operators  $B_k f = \sigma_{2^k} * f$  corresponding to a family of finite measures  $(\sigma_t : t > 0)$  on  $\mathbb{R}^d$  such that  $\sup_{t>0} \|\sigma_t\| < \infty$  and for every  $k \in \mathbb{Z}$  and  $t \in [2^k, 2^{k+1}]$  we have

$$|\hat{\sigma}_t(\xi)| \leq \omega(2^k \mathbf{q}_*(\xi)) \quad \text{if } 2^k \mathbf{q}_*(\xi) \leq 1, \tag{2.35}$$

$$|\hat{\sigma}_t(\xi)| \leq \omega((2^k \mathbf{q}_*(\xi))^{-1}) \quad \text{if } 2^k \mathbf{q}_*(\xi) \geq 1 \tag{2.36}$$

for some modulus of continuity  $\omega$ . Additionally, we assume that  $|\sigma_{2^k}| \lesssim \mu_{2^k}$  for some family of finite positive measures  $(\mu_t : t > 0)$  on  $\mathbb{R}^d$  such that  $\sup_{t>0} \|\mu_t\| < \infty$  and satisfying (2.24) and (2.25). In view of these assumptions and Theorem 2.14 we see that condition (2.30) holds, since  $|B_k f| \lesssim A_k |f|$ , where  $A_k f = \mu_{2^k} * f$ . Therefore,

$$\left\| \sum_{k \in \mathbb{Z}} B_k f \right\|_{L^p} \lesssim \mathbf{a} \|f\|_{L^p}$$

implies (2.32) with  $p = q_1$  and  $q_0 = 1$ , provided that

$$\mathbf{a} = \sum_{j \in \mathbb{Z}} \omega(2^{-|j|})^{\frac{q_1 - q_0}{2 - q_0}} < \infty,$$

since (2.31) can be verified with  $a_j = \omega(2^{-|j|})$  using (2.35), (2.36), and the properties of  $\tilde{S}_j$  associated with (2.22). Having proven (2.30) and (2.32) we see that (2.33) holds for the operators  $B_k f = \sigma_{2^k} * f$  with  $p = q_1$  and  $q_0 = 1$  as long as

$$\tilde{\mathbf{a}} = \sum_{j \in \mathbb{Z}} (|j| + 1) \omega(2^{-|j|})^{\frac{q_1 - q_0}{2 - q_0}} < \infty.$$

*Proof of Theorem 2.28.* In order to prove inequality (2.33) we employ the decomposition

$$\sum_{j \geq k} B_j = P_k \sum_{j \in \mathbb{Z}} B_j - \sum_{l > 0} \sum_{j < 0} S_{k+l} B_{k+j} + \sum_{l \leq 0} \sum_{j \geq 0} S_{k+l} B_{k+j}; \quad (2.37)$$

see [Duoandikoetxea and Rubio de Francia 1986, p. 548]. The  $J_2^p$  quasiseminorm of the first term on the right-hand side in (2.37) with  $p = q_1$  is bounded, due to (2.18), and (2.32), which ensures boundedness of the operator  $\sum_{j \in \mathbb{Z}} B_j$ .

The estimates for the second and the third terms are similar and we only consider the last term. We take the  $\ell^2$  norm with respect to the parameter  $k$  and estimate

$$\begin{aligned} & J_2^p \left( \left( \sum_{l \leq 0} \sum_{j \geq 0} B_{k+j} S_{k+l} f \right)_{k \in \mathbb{Z}} : X \rightarrow \mathbb{C} \right) \\ & \leq \left\| \left( \sum_{k \in \mathbb{Z}} \left| \sum_{l \leq 0} \sum_{j \geq 0} B_{k+j} S_{k+l} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \\ & = \left\| \left( \sum_{k \in \mathbb{Z}} \left| \sum_{m \geq 0} \sum_{n=k-m}^k B_{n+m} S_n f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \\ & \leq \sum_{m \geq 0} \left\| \left( \sum_{k \in \mathbb{Z}} \left| \sum_{n=k-m}^k B_{n+m} S_n f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \quad (\text{by the triangle inequality}) \\ & \leq \sum_{m \geq 0} (m+1)^{\frac{1}{2}} \left\| \left( \sum_{k \in \mathbb{Z}} \sum_{n=k-m}^k |B_{n+m} S_n f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \quad (\text{by Hölder's inequality}) \\ & = \sum_{m \geq 0} (m+1) \left\| \left( \sum_{n \in \mathbb{Z}} |B_{n+m} S_n f|^2 \right)^{\frac{1}{2}} \right\|_{L^p}. \end{aligned}$$

By (2.11), with  $L = 1$  and  $M_{k,0} := B_k$ , we obtain

$$\sum_{j \in \mathbb{Z}} (|j| + 1) \left\| \left( \sum_{k \in \mathbb{Z}} |B_k S_{k+j} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \left( \sup_{k \in \mathbb{Z}} \|B_k\|_{L^{q_0} \rightarrow L^{q_0}}^{\frac{q_0}{2} \frac{2-q_1}{2-q_0}} \right) \|B_*\|_{L^{q_1} \rightarrow L^{q_1}}^{\frac{2-q_1}{2}} \tilde{\mathbf{a}} \|f\|_{L^p}.$$

To prove the second part observe that for a sequence of functions  $(f_j)_{j \in \mathbb{Z}}$  in  $L^p(X; \ell^2(\mathbb{Z}))$  we have the inequality

$$\left\| \sum_{j \in \mathbb{Z}} \tilde{S}_j f_j \right\|_{L^p} \lesssim \left\| \left( \sum_{j \in \mathbb{Z}} |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p}, \tag{2.38}$$

which is the dual version of inequality (1.6) for the sequence  $(\tilde{S}_j)_{j \in \mathbb{Z}}$ . To prove (2.34) we will use (1.5) and (2.38). Indeed,

$$\begin{aligned} \left\| \sum_{k \in \mathbb{Z}} \varepsilon_k B_k f \right\|_{L^p} &\leq \sum_{j \in \mathbb{Z}} \left\| \sum_{k \in \mathbb{Z}} \varepsilon_k B_k S_{k+j} f \right\|_{L^p} && \text{(by (1.5))} \\ &= \sum_{j \in \mathbb{Z}} \left\| \sum_{k \in \mathbb{Z}} \tilde{S}_{k+j} (\varepsilon_k B_k \tilde{S}_{k+j} f) \right\|_{L^p} && \text{(since } S_j = \tilde{S}_j^2) \\ &\lesssim \sum_{j \in \mathbb{Z}} \left\| \left( \sum_{k \in \mathbb{Z}} |B_k \tilde{S}_{k+j} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} && \text{(by (2.38))} \\ &\lesssim \left( \sup_{k \in \mathbb{Z}} \|B_k\|_{L^{q_0} \rightarrow L^{q_0}}^{\frac{q_0}{2} \frac{2-q_1}{2-q_0}} \right) \|B_*\|_{L^{q_1} \rightarrow L^{q_1}}^{\frac{2-q_1}{2}} \mathbf{a} \|f\|_{L^p}, \end{aligned}$$

where in the last step we have used Lemma 2.9, with  $L = 1$  and  $M_{k,0} := B_k$ . □

**2E. Short variations.** We will work with a sequence of linear operators  $(A_t)_{t \in \mathbb{U}}$  (not necessarily positive) acting on  $L^1(X) + L^\infty(X)$ . However, positive operators will be distinguished in our proof and in this case we can also proceed as before using some bootstrap arguments.

For every  $k \in \mathbb{Z}$  and  $t \in [2^k, 2^{k+1}]$  we will use the notation

$$\Delta((A_s)_{s \in \mathbb{I}})_t f := \Delta(A_t) f := A_t f - A_{2^k} f.$$

**Theorem 2.39.** *Assume that  $(X, \mathcal{B}, \mathfrak{m})$  is a  $\sigma$ -finite measure space endowed with a sequence of linear operators  $(S_j)_{j \in \mathbb{Z}}$  satisfying (1.5) and (1.6). Let  $(A_t)_{t \in \mathbb{U}}$  be a family of linear operators such that the square function estimate*

$$\left\| \left( \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^l-1} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}) S_{j+k} f|^2 \right)^{\frac{1}{2}} \right\|_{L^2} \leq 2^{-\frac{l}{2}} a_{j,l} \|f\|_{L^2} \tag{2.40}$$

holds for all  $j \in \mathbb{Z}$  and  $l \in \mathbb{N}$  with some numbers  $a_{j,l} \geq 0$  such that for every  $0 < \varepsilon < \rho$  we have

$$\sum_{l \geq 0} \sum_{j \in \mathbb{Z}} 2^{-\varepsilon l} a_{j,l}^\rho < \infty. \tag{2.41}$$

(1) Let  $1 < q_0 < 2$  and  $4 < q_\infty < \infty$ , and suppose that for each  $q_0 \leq p \leq q_\infty$  the vector-valued estimate

$$\left\| \left( \sum_{k \in \mathbb{Z}} |A_{2^k(1+t)} f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \left\| \left( \sum_{k \in \mathbb{Z}} |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \tag{2.42}$$

holds uniformly in  $t \in \mathbb{U} \cap [0, 1]$ . Then for each

$$\frac{3}{1 + 1/q_0} < p < \frac{4}{1 + 2/q_\infty}$$

we have

$$\left\| \left( \sum_{k \in \mathbb{Z}} V^2(A_t f : t \in [2^k, 2^{k+1}] \cap \mathbb{U})^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \|f\|_{L^p}, \tag{2.43}$$

and for each  $4 \leq p < q_\infty$  and

$$r > \frac{p}{2} \frac{q_\infty - 2}{q_\infty - p}$$

we have

$$\left\| \left( \sum_{k \in \mathbb{Z}} V^r(A_t f : t \in [2^k, 2^{k+1}] \cap \mathbb{U})^r \right)^{\frac{1}{r}} \right\|_{L^p} \lesssim \|f\|_{L^p} \tag{2.44}$$

for all  $f \in L^p(X)$ .

(2) Let  $q_0 \in [1, 2)$  and  $\alpha \in [0, 1]$  be such that  $\alpha q_0 \leq 1$ . Suppose that we have the operator norm Hölder-type condition

$$\|A_{t+h} - A_t\|_{L^{q_0} \rightarrow L^{q_0}} \lesssim \left(\frac{h}{t}\right)^\alpha, \quad t, t+h \in \mathbb{U} \text{ and } h \in (0, 1]. \tag{2.45}$$

Then for every exponent  $q_1$  satisfying

$$q_0 \leq 2 - \frac{2 - q_0}{2 - \alpha q_0} < q_1 \leq 2 \tag{2.46}$$

and such that

$$\|\Delta((A_s)_{s \in \mathbb{U}})_{*, \mathbb{U}}\|_{L^{q_1} \rightarrow L^{q_1}} \lesssim 1 \tag{2.47}$$

we have for all  $f \in L^p(X)$  with  $p = q_1$  that the estimate (2.43) holds with the implicit constant which is a constant multiple of

$$\mathbf{a} := \sum_{l \geq 0} \sum_{j \in \mathbb{Z}} 2^{-\left(\alpha \frac{2-q_1}{2-q_0} \frac{q_0}{2} + \frac{1}{2} \frac{q_1-q_0}{2-q_0} - \frac{2-q_1}{2-q_0} \frac{1}{2}\right)l} a_{j,l}^{\frac{q_1-q_0}{2-q_0}} < \infty.$$

(3) Moreover, if  $(A_t)_{t \in \mathbb{U}}$  is a family of positive linear operators, then the condition (2.47) may be replaced by a weaker condition

$$\|A_{*, \mathbb{D}}\|_{L^{q_1} \rightarrow L^{q_1}} \lesssim 1 \tag{2.48}$$

and the estimate (2.43) holds as well with the implicit constant which is a constant multiple of  $1 + \mathbf{a}^{2/q_1}$ .

In the case of convolution operators on an abelian group  $\mathbb{G}$  the implication from (2.48) to (2.43) also holds with  $p$  replaced by  $p'$ .

Theorem 2.39 combined with the results formulated in the previous two paragraphs for dyadic scales will allow us to control, in view of (2.2), the cases for general scales. The first part of Theorem 2.39 gives (2.43) in a restricted range of  $p$ 's. If one asks for a larger range, a smoothness condition like in (2.45) must be assumed. Inequality (2.45) combined with maximal estimate (2.47) gives larger range of  $p$ 's in (2.43). If we work with a family of positive operators the condition (2.47) may be relaxed to (2.48) by some bootstrap argument. In the context of discussion after Theorem 2.14 and Theorem 2.28 let us look at a particular situation of (2) and prove (2.43).

Suppose that  $(A_t)_{t>0}$  is a family of convolution operators  $A_t f = \sigma_t * f$  corresponding to a family of finite measures  $(\sigma_t : t > 0)$  on  $\mathbb{R}^d$  such that  $\sup_{t>0} \|\sigma_t\| < \infty$  and satisfying (2.35) and (2.36). We assume that  $|\sigma_t| \lesssim \mu_t$  for some family of finite positive measures  $(\mu_t : t > 0)$  on  $\mathbb{R}^d$  such that  $\sup_{t>0} \|\mu_t\| < \infty$  and satisfying (2.24) and (2.25) to make sure that condition (2.47) holds. Additionally, let us assume that (2.45) holds with  $\alpha = 1$  and  $q_0 = 1, 2$ . By Plancherel's theorem, (2.35) and (2.36) we obtain

$$\|(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m})S_{j+k}f\|_{L^2} \lesssim \omega(2^{-|j|})\|S_{j+k}f\|_{L^2}. \tag{2.49}$$

Thus (2.45) with  $q_0 = 2$ ,  $t = 2^k + 2^{k-l}m$ ,  $h = 2^{k-l}$  combined with (2.49) imply

$$\|(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m})S_{j+k}f\|_{L^2} \lesssim \min(2^{-l}, \omega(2^{-|j|}))\|S_{j+k}f\|_{L^2}. \tag{2.50}$$

Consequently (2.40) holds with  $a_{j,l} = \min\{1, 2^l \omega(2^{-|j|})\}$  and Theorem 2.39 gives the desired conclusion as long as

$$a = \sum_{l \geq 0} \sum_{j \in \mathbb{Z}} 2^{-\frac{(q_1-1)l}{2}} (\min\{1, 2^l \omega(2^{-|j|})\})^{q_1-1} < \infty.$$

*Proof of Theorem 2.39(1).* By Minkowski's inequality for  $2 \leq s \leq q_\infty < \infty$  we have

$$\begin{aligned} & \left\| \left( \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^l-1} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m})f_k|^s \right)^{\frac{1}{s}} \right\|_{L^{q_\infty}} \\ &= \left\| \sum_{m=0}^{2^l-1} \sum_{k \in \mathbb{Z}} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m})f_k|^s \right\|_{L^{q_\infty/s}} \\ &\leq \sum_{m=0}^{2^l-1} \left\| \sum_{k \in \mathbb{Z}} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m})f_k|^s \right\|_{L^{q_\infty/s}} \\ &\leq 2^l \sup_{0 \leq m < 2^l} \left\| \left( \sum_{k \in \mathbb{Z}} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m})f_k|^s \right)^{\frac{1}{s}} \right\|_{L^{q_\infty}} \\ &\leq 2^{l+s} \sup_{0 \leq m \leq 2^l} \left\| \left( \sum_{k \in \mathbb{Z}} |A_{2^k+2^{k-l}m}f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^{q_\infty}}^s \\ &\lesssim 2^{l+s} \left\| \left( \sum_{k \in \mathbb{Z}} |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^{q_\infty}}^s, \end{aligned}$$

where we have applied (2.42) in the last step. Using this with  $f_k = S_{j+k} f$  and applying (1.6) we obtain

$$\left\| \left( \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^l-1} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}) S_{j+k} f|^s \right)^{\frac{1}{s}} \right\|_{L^{q_\infty}} \lesssim 2^{\frac{l}{s}} \|f\|_{L^{q_\infty}}$$

for all  $2 \leq s \leq q_\infty < \infty$ . By interpolation with (2.40) we obtain

$$\left\| \left( \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^l-1} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}) S_{j+k} f|^r \right)^{\frac{1}{r}} \right\|_{L^p} \lesssim 2^{-\frac{\theta l}{2} + \frac{(1-\theta)l}{s}} a_{j,l}^\theta \|f\|_{L^p}, \quad (2.51)$$

where  $0 < \theta \leq 1$  and

$$\frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{s} \quad \text{and} \quad \frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{q_\infty},$$

so

$$\theta = \frac{2q_\infty - p}{p q_\infty - 2}.$$

By Lemma 2.5, or more precisely by an analogue of inequality (2.4) with  $\ell^r$  norm in place of  $\ell^2$  norm, and by (2.51) we obtain

$$\left\| \left( \sum_{k \in \mathbb{Z}} V^r(A_t f : t \in [2^k, 2^{k+1}] \cap \cup) \right)^{\frac{1}{r}} \right\|_{L^p} \lesssim \sum_{l \geq 0} \sum_{j \in \mathbb{Z}} 2^{-\frac{\theta l}{2} + \frac{(1-\theta)l}{s}} a_{j,l}^\theta \|f\|_{L^p}. \quad (2.52)$$

In view of (2.41) with  $\varepsilon = \theta/2 - (1-\theta)/s$  and  $\rho = \theta$  this estimate is summable in  $l$  and  $j$ , provided that  $-\theta/2 + (1-\theta)/s < 0$ . In particular, for

$$2 \leq p < \frac{4}{1 + 2/q_\infty}$$

we use  $s = 2$ . For  $4 \leq p < q_\infty$  we use

$$s > \frac{q_\infty(p-2)}{q_\infty - p}$$

and then

$$r > \frac{p q_\infty - 2}{2 q_\infty - p}.$$

For  $q_0 \in (1, 2)$  by Minkowski's inequality and (2.42) we have

$$\begin{aligned} & \left\| \left( \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^l-1} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}) f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^{q_0}} \\ & \leq \sum_{m=0}^{2^l-1} \left\| \left( \sum_{k \in \mathbb{Z}} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}) f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^{q_0}} \\ & \leq 2^{l+1} \sup_{0 \leq m \leq 2^l} \left\| \left( \sum_{k \in \mathbb{Z}} |A_{2^k+2^{k-l}m} f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^{q_0}} \lesssim 2^l \left\| \left( \sum_{k \in \mathbb{Z}} |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^{q_0}}. \end{aligned}$$

Substituting  $f_k = S_{j+k} f$ , applying (1.6), and interpolating with (2.40) we obtain

$$\left\| \left( \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^l-1} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}) S_{j+k} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim 2^{-\frac{\theta l}{2} + (1-\theta)l} a_{j,l}^\theta \|f\|_{L^p}, \quad (2.53)$$

with

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{q_0},$$

for  $0 < \theta < 1$ . Hence

$$\theta = \frac{2}{p} \frac{p - q_0}{2 - q_0}$$

and in view of (2.41) with  $\varepsilon = \theta/2 - (1 - \theta)$  and  $\rho = \theta$  this estimate is summable in  $l$  and  $j$ , provided that  $-\theta/2 + (1 - \theta) < 0$ . The conclusion again follows from Lemma 2.5 and (2.53) like in (2.52) with

$$\frac{3}{1 + 1/q_0} < p \leq 2. \quad \square$$

*Proof of Theorem 2.39(2)–(3).* By the monotone convergence theorem we may restrict  $k$  in (2.43) to  $|k| \leq K_0$  and parameters  $l$  to the set

$$\mathbb{U}_{L_0}^k := \{u/2^{L_0} : u \in \mathbb{N} \text{ and } 2^{k+L_0} \leq u \leq 2^{k+L_0+1}\}$$

for some  $K_0 \in \mathbb{N}$  and  $L_0 \in \mathbb{Z}$  as long as we obtain estimates independent of  $K_0$  and  $L_0$ . Fix  $K_0, L_0$  and let  $\mathbb{U} := \bigcup_{|k| \leq K_0} \mathbb{U}_{L_0}^k$ . Let  $q_1$  satisfy (2.46); then invoking (1.5) and (2.11), with  $L = 2^l$ , we obtain

$$\begin{aligned} & \left\| \left( \sum_{|k| \leq K_0} \sum_{m=0}^{2^l-1} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}) f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \\ & \lesssim 2^{\frac{2-q_1}{2} \frac{l}{2}} \left( \sup_{\substack{|k| \leq K_0 \\ 0 \leq m < 2^l}} \|A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}\|_{L^{\frac{2-q_1}{2} q_0} \rightarrow L^{q_0}} \right) \|\Delta((A_s)_{s \in \mathbb{U}})_{*, \mathbb{U}}\|_{L^{\frac{2-q_1}{2}} \rightarrow L^{q_1}} \\ & \qquad \cdot \left( \sum_{j \in \mathbb{Z}} (2^{-\frac{l}{2}} a_{j,l})^{\frac{q_1-q_0}{2-q_0}} \right) \|f\|_{L^p} \\ & \lesssim 2^{\frac{2-q_1}{2} \frac{l}{2}} ((2^{-\alpha l})^{\frac{2-q_1}{2} \frac{q_0}{2}}) \|\Delta((A_s)_{s \in \mathbb{U}})_{*, \mathbb{U}}\|_{L^{q_1} \rightarrow L^{q_1}} 2^{-\frac{l}{2} \frac{q_1-q_0}{2-q_0}} \sum_{j \in \mathbb{Z}} a_{j,l}^{\frac{q_1-q_0}{2-q_0}} \|f\|_{L^p}. \end{aligned}$$

In order for the right-hand side to be summable in  $l$  we need

$$\frac{1}{2} \frac{2-q_1}{2-q_0} - \alpha \frac{2-q_1}{2-q_0} \frac{q_0}{2} - \frac{1}{2} \frac{q_1-q_0}{2-q_0} < 0 \iff (2-q_1) - \alpha(2-q_1)q_0 - (q_1-q_0) < 0.$$

It suffices to ensure

$$(2-q_1)(1-\alpha q_0) - (q_1-q_0) < 0 \iff q_1 > \frac{2(1-\alpha q_0) + q_0}{2-\alpha q_0} = 2 - \frac{2-q_0}{2-\alpha q_0},$$

and this is our hypothesis (2.46). Hence under this condition by Lemma 2.5 we conclude for general operators that

$$\begin{aligned} \left\| \left( \sum_{|k| \leq K_0} V^2(A_t f : t \in \cup_{L_0}^k)^2 \right)^{\frac{1}{2}} \right\|_{L^p} &\lesssim \sum_{l=0}^{K_0+L_0} \left\| \left( \sum_{|k| \leq K_0} \sum_{m=0}^{2^l-1} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}) f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \\ &\lesssim \|\Delta((A_s)_{s \in \cup})_{*, \mathbb{I}}\|_{L^{q_1} \rightarrow L^{q_1}}^{\frac{2-q_1}{2}} \mathbf{a} \|f\|_{L^p}, \end{aligned} \quad (2.54)$$

as desired. For positive operators crude estimates and interpolation show that

$$B := \|A_{*, \mathbb{I}}\|_{L^p \rightarrow L^p} < \infty,$$

with  $p = q_1$ , since  $\mathbb{I}$  is finite. Note that

$$\sup_{t \in \mathbb{I}} |A_t f(x)| \leq \sup_{t \in \mathbb{D}} |A_t f(x)| + \left( \sum_{k \in \mathbb{Z}} \sup_{t \in [2^k, 2^{k+1}) \cap \mathbb{I}} |(A_t - A_{2^k}) f(x)|^2 \right)^{\frac{1}{2}}. \quad (2.55)$$

Therefore, appealing to (2.55), (2.48) and (2.54) we obtain by a bootstrap argument that  $B \lesssim 1 + B^{(2-q_1)/2} \mathbf{a}$ , since

$$\|\Delta((A_s)_{s \in \cup})_{*, \mathbb{I}}\|_{L^{q_1} \rightarrow L^{q_1}}^{\frac{2-q_1}{2}} \lesssim B^{\frac{2-q_1}{2}}.$$

Hence,  $B \lesssim 1 + \mathbf{a}^{2/q_1}$ . In particular, the estimate (2.54) becomes uniform in  $\mathbb{I} \subset \cup$ , and this simultaneously implies (2.43).

In the case of convolution operators we may replace  $p = q_1$  by  $p = q_1'$  in Lemma 2.9 and all subsequent arguments.  $\square$

### 3. Applications

**3A. Dimension-free estimates for jumps in the continuous setting.** We begin by providing dimension-free endpoint estimates, for  $r = 2$ , in the main results of [Bourgain et al. 2018]. Let  $G \subset \mathbb{R}^d$  be a symmetric convex body. By the definition of the averaging operator (1.8) we have  $\mathcal{A}_t^G \tilde{U} = \tilde{U} \mathcal{A}_t^{U(G)}$ , where  $\tilde{U} f := f \circ U$  is the composition operator with an invertible linear map  $U : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . It follows that all estimates in Section 1 are not affected if  $G$  is replaced by  $U(G)$ .

By [Bourgain 1986a], after replacing  $G$  by its image under a suitable invertible linear transformation, we may assume that the normalized characteristic function  $\mu := |G|^{-1} \mathbf{1}_G$  satisfies

$$|\hat{\mu}(\xi)| \leq C |\xi|^{-1}, \quad (3.1)$$

$$|\hat{\mu}(\xi) - 1| \leq C |\xi|, \quad (3.2)$$

$$|\langle \xi, \nabla \hat{\mu}(\xi) \rangle| \leq C, \quad (3.3)$$

with the constant  $C$  independent of the dimension. In [Bourgain 1986a] these estimates were proved with  $|L(G)\xi|$  in place of  $|\xi|$  on the right-hand side, where  $L(G)$  is the isotropic constant corresponding to  $G$ . The above form is obtained by rescaling.

Then  $\mathcal{A}_t := \mathcal{A}_t^G$  is the convolution operator with  $\mu_t$  and  $\hat{\mu}_t(\xi) = \hat{\mu}(t\xi)$ . The Poisson semigroup is defined by

$$\widehat{\mathcal{P}_t f}(\xi) := p_t(\xi) \hat{f}(\xi), \quad \text{where } p_t(\xi) := e^{-2\pi t|\xi|}.$$

The associated Littlewood–Paley operators are given by  $S_k := \mathcal{P}_{2^k} - \mathcal{P}_{2^{k+1}}$ . Their Fourier symbols satisfy

$$|\widehat{S}_k(\xi)| \lesssim \min\{2^k|\xi|, 2^{-k}|\xi|^{-1}\}, \tag{3.4}$$

where  $\widehat{S}_k(\xi)$  is the multiplier associated with the operator  $S_k$ , i.e.,  $\widehat{S_k f}(\xi) = \widehat{S}_k(\xi) \hat{f}(\xi)$ . From now on, for simplicity of notation, we will use this convention. The symbols associated with the Poisson semigroup  $P_k := \mathcal{P}_{2^k}$  satisfy

$$|\widehat{P}_k(\xi) - 1| \lesssim |2^k \xi| \quad \text{and} \quad |\widehat{P}_k(\xi)| \lesssim 2^{-k} |\xi|^{-1}. \tag{3.5}$$

*Proof of Theorem 1.9.* We verify that the sequence  $(A_k)_{k \in \mathbb{Z}}$ , where  $A_k := \mathcal{A}_{2^k}$ , satisfies the hypotheses of Theorem 2.14 for every  $1 = q_0 < q_1 \leq 2$ .

The maximal inequality (2.13) and the Littlewood–Paley inequality (1.6) for the Poisson semigroup with constants independent of the dimension are well known [Stein 1970]. The jump estimate (2.18) was recently established in [Mirek et al. 2018b, Theorem 1.5].

It remains to verify condition (2.15) for the operators  $M_k := A_k - P_k$ . In view of (3.1), (3.2) and (3.5), we have

$$|\widehat{M}_k(\xi)| \lesssim \min\{|2^k \xi|^{-1}, |2^k \xi|\}.$$

For  $\xi \in \mathbb{R}^d \setminus \{0\}$  let  $k_0 \in \mathbb{Z}$  be such that  $\tilde{\xi} = 2^{k_0} \xi$  satisfies  $|\tilde{\xi}| \simeq 1$ . By (3.5) it follows that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |\widehat{M}_k(\xi) \widehat{S}_{k+j}(\xi)|^2 &\lesssim \sum_{k \in \mathbb{Z}} \min\{|2^k \xi|^{-1}, |2^k \xi|\}^2 \min\{|2^{k+j} \xi|^{-1}, |2^{k+j} \xi|\}^2 \\ &= \sum_{k \in \mathbb{Z}} \min\{|2^k \tilde{\xi}|^{-1}, |2^k \tilde{\xi}|\}^2 \min\{|2^{k+j} \tilde{\xi}|^{-1}, |2^{k+j} \tilde{\xi}|\}^2 \\ &\lesssim \sum_{k \in \mathbb{Z}} \min\{2^{-k}, 2^k\}^2 \min\{(2^{k+j})^{-1}, 2^{k+j}\}^2 \lesssim 2^{-\delta|j|} \end{aligned} \tag{3.6}$$

for  $\delta \in (0, 2)$  with the implicit constant independent of the dimension. By Plancherel’s theorem this shows that (2.15) holds with  $a_j \lesssim 2^{-\delta|j|/2}$ . □

*Proof of Theorem 1.11.* We will apply Theorem 2.39 with  $A_t := \mathcal{A}_t := \mathcal{A}_t^G$ . By a simple scaling we have  $\mathcal{A}_{2^k(1+t)} = \mathcal{A}_{2^k}^{(1+t)G}$ . Hence Theorem 2.14, with  $A_k = \mathcal{A}_{2^k}^{(1+t)G}$ , applies and we obtain the vector-valued inequality (2.20) for all  $1 < p < \infty$  and  $r = 2$ , which consequently guarantees (2.42). It remains to verify the hypothesis (2.40) of Theorem 2.39. We repeat the estimate [Bourgain et al. 2018, (4.23)]. By (3.3) for  $t > 0$  and  $h > 0$  we have

$$|\hat{\mu}((t+h)\xi) - \hat{\mu}(t\xi)| \leq \int_t^{t+h} |\langle \xi, \nabla \hat{\mu}(u\xi) \rangle| du \lesssim \int_t^{t+h} \frac{du}{u} \lesssim \frac{h}{t}. \tag{3.7}$$

By the Plancherel theorem this implies

$$\|\mathcal{A}_{t+h} - \mathcal{A}_t\|_{L^2 \rightarrow L^2} \lesssim \frac{h}{t}. \tag{3.8}$$

This allows us to estimate the square of the left-hand side of (2.40) by

$$\begin{aligned} \text{LHS (2.40)}^2 &= \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^l-1} \|(\mathcal{A}_{2^k+2^{k-l}(m+1)} - \mathcal{A}_{2^k+2^{k-l}m})S_{j+k}f\|_{L^2}^2 \\ &\lesssim \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^l-1} 2^{-2l} \|S_{j+k}f\|_{L^2}^2 \\ &= 2^{-l} \sum_{k \in \mathbb{Z}} \|S_{j+k}f\|_{L^2}^2 \lesssim 2^{-l} \|f\|_{L^2}^2. \end{aligned}$$

Secondly, by (3.1) and (3.2) for every  $0 \leq m < 2^l$  we have

$$|\hat{\mu}((2^k + 2^{k-l}(m+1))\xi) - \hat{\mu}((2^k + 2^{k-l}m)\xi)| \lesssim \min\{|2^k \xi|, |2^k \xi|^{-1}\}.$$

Arguing similarly to (3.6) we obtain

$$\text{LHS (2.40)}^2 \lesssim 2^l 2^{-\delta|j|} \|f\|_2^2.$$

Hence (2.40) holds with  $a_{j,l} = \min\{1, 2^l 2^{-\delta|j|/2}\}$ . □

*Proof of Theorem 1.14.* By Theorem 1.9 we have the hypothesis (2.48) of Theorem 2.39. The hypothesis (2.40) was verified in the proof of Theorem 1.11. The remaining hypothesis (2.45) is given by [Bourgain et al. 2018, Lemma 4.2], but we give a more direct proof.

Recall that  $B^q$  is the unit ball induced by  $\ell^q$  norm in  $\mathbb{R}^d$ . From [Müller 1990] (for  $1 \leq q < \infty$ ), and [Bourgain 2014] (for  $q = \infty$ ) we use the multiplier norm estimate

$$\|\tilde{m}\|_{M^p} \lesssim_{p,q,\alpha} 1, \quad \tilde{m} = (\xi \cdot \nabla)^\alpha \hat{\mu},$$

for  $\alpha \in (0, 1)$  and  $p \in (1, \infty)$  with implicit constant independent of the dimension. For a Lipschitz function  $h : (\frac{1}{2}, \infty) \rightarrow \mathbb{R}$  such that  $|h(t)| \lesssim |t|^{-1}$  and  $|h'(t)| \lesssim |t|^{-1}$  fractional differentiation can be inverted by fractional integration:

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} (u-t)^{\alpha-1} D^\alpha h(u) du, \quad t > \frac{1}{2};$$

see [Deleaval et al. 2018, Lemma 6.11]. In particular, for  $t > 1$  we obtain

$$h(t) - h(1) = \frac{1}{\Gamma(\alpha)} \int_1^{+\infty} ((u-t)_+^{\alpha-1} - (u-1)^{\alpha-1}) D^\alpha h(u) du,$$

where  $u_+ := \max(u, 0)$  denotes the positive part. In view of (3.1) and (3.3) this result can be applied to the function  $h(t) = \hat{\mu}(t\xi)$  for any  $\xi \in \mathbb{R}^d \setminus \{0\}$ . Observing  $D^\alpha h(u) = u^{-\alpha} \tilde{m}(u\xi)$  we obtain

$$\hat{\mu}(t\xi) - \hat{\mu}(\xi) = \frac{1}{\Gamma(\alpha)} \int_1^{+\infty} ((u-t)_+^{\alpha-1} - (u-1)_+^{\alpha-1}) u^{-\alpha} \tilde{m}(u\xi) du.$$

On the other hand we have

$$\int_1^{+\infty} |(u-t)_+^{\alpha-1} - (u-1)_+^{\alpha-1}| u^{-\alpha} du \lesssim_\alpha (t-1)^\alpha,$$

and for a Schwartz function  $f \in \mathcal{S}(\mathbb{R}^d)$  this implies

$$\begin{aligned} \|\mathcal{F}_\xi^{-1}((\hat{\mu}(t\xi) - \hat{\mu}(\xi))\hat{f}(\xi))\|_{L^p} &\leq \int_1^{+\infty} |(u-t)_+^{\alpha-1} - (u-1)_+^{\alpha-1}| u^{-\alpha} \cdot \|\mathcal{F}_\xi^{-1}((u\xi \cdot \nabla)^\alpha \hat{\mu})(\xi)\hat{f}(\xi)\|_{L^p} du \\ &\lesssim_\alpha (t-1)^\alpha \sup_{u>0} \|\mathcal{F}_\xi^{-1}((u\xi \cdot \nabla)^\alpha \hat{\mu})(u\xi)\hat{f}(\xi)\|_{L^p} \\ &\lesssim_\alpha (t-1)^\alpha \|((\xi \cdot \nabla)^\alpha \hat{\mu})(\xi)\|_{M^p} \|f\|_{L^p}, \end{aligned}$$

where we have used the Fourier inversion formula and Fubini’s theorem in the first step and scale invariance of the multiplier norm in the last step. Since the multiplier  $\hat{\mu}(t\xi) - \hat{\mu}(\xi)$  is (qualitatively) bounded on  $L^p$  with norm  $\leq 2$ , by density of Schwartz functions this implies

$$\|\hat{\mu}(t \cdot) - \hat{\mu}\|_{M^p} \lesssim_\alpha (t-1)^\alpha,$$

which by scaling implies the hypothesis (2.45). □

Finally we emphasize that once Theorem 1.9 is proved, alternative proofs of Theorems 1.11 and 1.14 follow by appealing to the short variational estimates given in [Bourgain et al. 2018].

**3B. Dimension-free estimates for jumps in the discrete setting.** We outline the proof of Theorem 1.18. The strategy is much the same as for the proofs of Theorems 1.9 and 1.11. Let

$$\mathfrak{m}_N(\xi) = \frac{1}{(2N+1)^d} \sum_{m \in Q_N} e^{2\pi i m \cdot \xi} \quad \text{for } \xi \in \mathbb{T}^d$$

be the multiplier corresponding to the operators  $A_N$  defined in (1.17). Here we remind the reader of the following estimates for  $\mathfrak{m}_N$  established recently in [Bourgain et al. 2019]. Namely there is a constant  $0 < C < \infty$  independent of the dimension such that for every  $N, N_1, N_2 \in \mathbb{N}$  and for every  $\xi \in \mathbb{T}^d \equiv [-\frac{1}{2}, \frac{1}{2}]^d$  we have

$$\begin{aligned} |\mathfrak{m}_N(\xi)| &\leq C(N|\xi|)^{-1}, \\ |\mathfrak{m}_N(\xi) - 1| &\leq CN|\xi|, \\ |\mathfrak{m}_{N_1}(\xi) - \mathfrak{m}_{N_2}(\xi)| &\leq C|N_1 - N_2| \max\{N_1^{-1}, N_2^{-1}\}, \end{aligned} \tag{3.9}$$

where  $|\cdot|$  denotes the Euclidean norm restricted to  $\mathbb{T}^d$ .

The discrete Poisson semigroup is defined by

$$\widehat{\mathcal{P}_t f}(\xi) := p_t(\xi)\hat{f}(\xi), \quad \text{where } p_t(\xi) := e^{-2\pi t|\xi|_{\sin}},$$

for every  $\xi \in \mathbb{T}^d$  and

$$|\xi|_{\sin} := \left( \sum_{j=1}^d (\sin(\pi \xi_j))^2 \right)^{\frac{1}{2}}.$$

We set  $P_k := \mathcal{P}_{2^k}$  and the associated Littlewood–Paley operators are given by  $S_k := \mathcal{P}_{2^k} - \mathcal{P}_{2^{k+1}}$ . The maximal inequality (2.13) and the Littlewood–Paley inequality (1.6) for the discrete Poisson semigroup with constants independent of the dimension follow from [Stein 1970]. The jump estimate (2.18) for discrete Poisson semigroup was recently proved in [Mirek et al. 2018b, Theorem 1.5]. Moreover, using  $|\xi| \leq |\xi|_{\sin} \leq \pi|\xi|$  for  $\xi \in \mathbb{T}^d$ , we see that the corresponding Fourier symbols  $\widehat{S}_k(\xi)$  and  $\widehat{P}_k(\xi)$  satisfy estimates (3.4) and (3.5) as well.

In order to prove (1.20) we have to verify that the sequence  $(A_k)_{k \in \mathbb{N}}$ , where  $A_k := A_{2^k}$ , satisfies the hypotheses of Theorem 2.14 for every  $1 = q_0 < q_1 \leq 2$ . Taking into account (3.9), (3.4) and (3.5) (associated with the discrete Poisson semigroup) it suffices to proceed as in the proof of Theorem 1.9. To prove (1.19) we argue as in the proof of Theorem 1.11.

**3C. Jump inequalities for the operators of Radon type.** In this section we prove Theorems 1.22 and 1.30. By the lifting procedure for the Radon transforms described in [Stein 1993, Chapter 11, Section 2.4] we can assume without loss of generality that our polynomial mapping  $P(x) := (x)^\Gamma$  is the canonical polynomial mapping for some  $\Gamma \subset \mathbb{N}_0^k \setminus \{0\}$  with lexicographical order, given by

$$\mathbb{R}^k \ni x = (x_1, \dots, x_k) \mapsto (x)^\Gamma := (x_1^{\gamma_1} \cdots x_k^{\gamma_k} : \gamma \in \Gamma) \in \mathbb{R}^\Gamma,$$

where  $\mathbb{R}^\Gamma := \mathbb{R}^{|\Gamma|}$  is identified with the space of all vectors whose coordinates are labeled by multi-indices  $\gamma = (\gamma_1, \dots, \gamma_k) \in \Gamma$ .

Throughout what follows  $A$  is the diagonal  $|\Gamma| \times |\Gamma|$  matrix such that  $(Ax)_\gamma = |\gamma| x_\gamma$  for every  $x \in \mathbb{R}^\Gamma$  and let  $q_*$  be the quasinorm associated with  $A^* = A$ , given by

$$q_*(\xi) = \max_{\gamma \in \Gamma} (|\xi_\gamma|^{\frac{1}{|\gamma|}}) \quad \text{for } \xi \in \mathbb{R}^\Gamma.$$

We shall later freely appeal, without explicit mention, to the discussions after Theorems 2.14, 2.28 and 2.39 with  $d = |\Gamma|$ ,  $A$  and  $q_*$  as above.

*Proof of Theorem 1.22.* Let  $\mathcal{M}_t := \mathcal{M}_t^P$ , where  $P(x) = (x)^\Gamma$ . Observe that  $\mathcal{M}_t$  is a convolution operator with a probability measure  $\mu_t$ , whose Fourier transform is defined by

$$\widehat{\mu}_t(\xi) := \frac{1}{|\Omega_t|} \int_{\Omega_t} e^{-2\pi i \xi \cdot (y)^\Gamma} dy \quad \text{for } \xi \in \mathbb{R}^\Gamma.$$

Condition (2.25) with  $\omega(t) = t^{1/d}$  follows from Proposition B.2 and Lemma A.1. It is not difficult to see that (2.24) also holds.

In order to prove (1.23) it suffices, in view of (2.2), to show inequality (2.19) with  $A_k := \mathcal{M}_{2^k}$  and inequality (2.43) with  $A_t := \mathcal{M}_t$  for every  $1 = q_0 < q_1 \leq 2$ . We have already seen that (2.26) holds; hence (2.19) holds and we are done. We now show (2.43). For this purpose note that (2.45) holds for all  $1 \leq q_0 < \infty$ . This combined with (2.24) and (2.25) permits us to prove (2.49) and (2.50), which imply (2.40) and Theorem 2.39 yields the conclusion.  $\square$

*Proof of Theorem 1.30.* Let  $\mathcal{H}_t := \mathcal{H}_t^P$ , where  $P(x) = (x)^\Gamma$ . Denote the Fourier multiplier corresponding to the truncated singular Radon transform by

$$\Psi_t(\xi) := \int_{\mathbb{R}^k \setminus \Omega_t} e^{-2\pi i \xi \cdot (y)^\Gamma} K(y) \, dy \quad \text{for } \xi \in \mathbb{R}^\Gamma. \tag{3.10}$$

For a fixed  $\kappa \in (0, 1)$  we claim

$$\begin{aligned} |\Psi_t(\xi) - \Psi_s(\xi)| &\lesssim_\kappa |t^A \xi|_\infty^{-\frac{1}{d}} + \omega_K(|t^A \xi|_\infty^{-\frac{1}{d}}) \\ &\lesssim (tq_*(\xi))^{-\frac{1}{d}} + \omega_K((tq_*(\xi))^{-\frac{1}{d}}) \quad \text{if } tq_*(\xi) \geq 1, \end{aligned} \tag{3.11}$$

for all  $s, t \in (0, \infty)$  such that  $\kappa t \leq s \leq t$ . Indeed, by Proposition B.2 we obtain

$$\begin{aligned} |\Psi_t(\xi) - \Psi_s(\xi)| &= \left| \int_{\Omega_t \setminus \Omega_s} e^{-2\pi i \xi \cdot (y)^\Gamma} K(y) \, dy \right| \\ &\lesssim \sup_{v \in \mathbb{R}^k : |v| \leq t \Lambda^{-\frac{1}{d}}} \int |(\mathbf{1}_{\Omega_t \setminus \Omega_s} K)(y) - (\mathbf{1}_{\Omega_t \setminus \Omega_s} K)(y - v)| \, dy, \end{aligned}$$

with

$$\Lambda = \sum_{\gamma \in \Gamma} t^{|\gamma|} |\xi_\gamma|.$$

The claim (3.11) clearly holds for  $\Lambda \leq 1$ . If  $\Lambda \geq 1$ , then for a fixed  $v$  we use (1.28) and the fact that  $\Omega_t \setminus \Omega_s \subseteq B(0, t) \setminus B(0, c_{\Omega} \kappa t)$  to estimate the contribution of  $y$  such that  $y, y - v \in \Omega_t \setminus \Omega_s$ . On the set of  $y$  such that exactly one of  $y, y - v$  is contained in  $\Omega_t \setminus \Omega_s$  we use (1.26); the measure of this set is bounded by a multiple of  $t^{k-1} |v|$  due to Lemma A.1. This finishes the proof of (3.11).

Additionally, we have

$$|\Psi_t(\xi) - \Psi_s(\xi)| \lesssim |t^A \xi|_\infty^{\frac{1}{d}} \lesssim (tq_*(\xi))^{\frac{1}{d}} + \omega_K((tq_*(\xi))^{\frac{1}{d}}) \quad \text{if } tq_*(\xi) \leq 1 \tag{3.12}$$

due to the cancellation condition (1.27) and (1.26).

To prove (1.31) we fix  $\theta \in (0, 1]$  and  $p \in \{1 + \theta, (1 + \theta)'\}$  and invoking (2.2) it suffices to prove inequalities (1.32) and (1.33). Inequality (1.32) will follow from (2.33) with  $q_0 = 1$ ,  $q_1 = 1 + \theta$ , and  $B_j := \mathcal{H}_{2^j} - \mathcal{H}_{2^{j+1}}$  upon expressing  $\mathcal{H}_{2^k}$  as a telescoping series like in (2.3). Inequality (1.33) will be a consequence of (2.43) with  $q_0 = 1$ ,  $q_1 = 1 + \theta$ , and  $A_t := \mathcal{H}_t$ . Let  $(\sigma_t : t > 0)$  be a family of measures defined by

$$\sigma_t * f(x) = \int_{\Omega_t \setminus \Omega_{2^k}} f(x - (y)^\Gamma) K(y) \, dy \quad \text{for every } t \in [2^k, 2^{k+1}], k \in \mathbb{Z}. \tag{3.13}$$

Estimates (3.11) and (3.12) allow us to verify (2.35) and (2.36) respectively with  $\omega(t) := t^{1/d} + \omega_K(t^{1/d})$ . Moreover  $|\sigma_{2^k}| \lesssim \mu_{2^k}$ , where  $\mu_t$  is the measure associated with the averaging operator  $\mathcal{M}_t$ . Hence the discussion after Theorem 2.28 guarantees that inequality (2.33) holds, since  $B_k f = \sigma_{2^{k+1}} * f$ . To prove (2.43) it suffices to note that (2.45) holds for all  $1 \leq q_0 < \infty$ . Moreover inequalities (2.49) and (2.50) remain true for  $A_t = \mathcal{H}_t$ . Then Theorem 2.39 completes the proof.  $\square$

### Appendix A: Neighborhoods of boundaries of convex sets

We will show how to control the measure of neighborhoods of the boundaries of convex sets. The proof of the lemma below is based on a simple Vitali covering argument.

**Lemma A.1.** *Let  $\Omega \subset \mathbb{R}^k$  be a bounded and convex set and let  $0 < s \lesssim \text{diam}(\Omega)$ . Then*

$$|\{x \in \mathbb{R}^k : \text{dist}(x, \partial\Omega) < s\}| \lesssim_k s \text{diam}(\Omega)^{k-1}.$$

*The implicit constant depends only on the dimension  $k$ , but not on the convex set  $\Omega$ .*

*Proof.* Let  $r = \text{diam} \Omega$ . By translation we may assume  $\Omega \subseteq B(0, r)$ , where  $B(y, s)$  denotes an open ball centered at  $y \in \mathbb{R}^k$  with radius  $s > 0$ . Notice

$$\{x \in \mathbb{R}^k : \text{dist}(x, \partial\Omega) < s\} \subseteq \bigcup_{y \in \partial\Omega} B(y, s).$$

By the Vitali covering lemma there exists a finite subset  $Y \subset \partial\Omega$  such that the balls  $B(y, s)$ , with  $y \in Y$ , are pairwise disjoint and

$$\left| \bigcup_{y \in \partial\Omega} B(y, s) \right| \lesssim \left| \bigcup_{y \in Y} B(y, s) \right|.$$

Consider the nearest-point projection  $P : \mathbb{R}^k \rightarrow \text{cl} \Omega$ , that is,  $P(x) = x'$ , where  $x' \in \text{cl} \Omega$  is the unique point such that  $|x - x'| = \text{dist}(x, \text{cl} \Omega)$ . It is well known that  $P$  is well-defined and contractive with respect to the Euclidean metric. The restriction of  $P$  to the sphere  $\partial B(0, r)$  defines a surjection  $P_\partial : \partial B(0, r) \rightarrow \partial\Omega$ . This follows from the fact that for every point  $x \in \partial\Omega$  there exists a linear functional  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}$  such that  $\phi(y) \leq \phi(x)$  for every  $y \in \text{cl} \Omega$ ; see, e.g., [Rockafellar 1970, Corollary 11.6.1]. For each  $y \in Y$  we choose  $z(y) \in \partial B(0, r)$  such that  $P_\partial(z(y)) = y$ . Then the balls  $B(z(y), s)$  are pairwise disjoint in view of the contractivity of  $P$  and contained in the set

$$\{x \in \mathbb{R}^k : r - s < |x| < r + s\},$$

which has measure  $\lesssim s(r + s)^{k-1}$ . But the union of the balls  $B(z(y), s)$  has the same measure as  $\bigcup_{y \in Y} B(y, s)$ , and the conclusion follows.  $\square$

### Appendix B: Estimates for oscillatory integrals

We present the following variant of van der Corput's oscillatory integral lemma with a rough amplitude function.

**Lemma B.1.** *Given an interval  $(a, b) \subset \mathbb{R}$  suppose that  $\phi : (a, b) \rightarrow \mathbb{R}$  is a smooth function such that  $|\phi^{(k)}(x)| \gtrsim \lambda$  for every  $x \in (a, b)$  with some  $\lambda > 0$ . Assume additionally that*

- either  $k \geq 2$ ,
- or  $k = 1$  and  $\phi'$  is monotonic.

Then for every locally integrable function  $\psi : \mathbb{R} \rightarrow \mathbb{C}$  we have

$$\left| \int_a^b e^{i\phi(x)} \psi(x) dx \right| \lesssim_k \inf_{a \leq x \leq b} \int_{x-\lambda^{-1/k}}^{x+\lambda^{-1/k}} |\psi(y)| dy + \lambda^{\frac{1}{k}} \int_{-\lambda^{-1/k}}^{\lambda^{-1/k}} \int_a^b |\psi(x) - \psi(x-y)| dx dy.$$

*Proof.* Let  $\eta$  be a smooth positive function with  $\text{supp } \eta \subseteq [-1, 1]$  and  $\int_{\mathbb{R}} \eta(x) dx = 1$ . Let  $\rho(x) := \psi * \lambda^{1/k} \eta(\lambda^{1/k} x)$ , and note that

$$|\psi(x) - \rho(x)| \leq \lambda^{\frac{1}{k}} \int_{\mathbb{R}} |\psi(x) - \psi(x-y)| |\eta(\lambda^{\frac{1}{k}} y)| dy.$$

Then we may replace  $\psi$  by  $\rho$  on the left-hand side of the conclusion. For every  $x_0 \in (a, b)$  by partial integration and the van der Corput lemma, see for example [Stein 1993, Section VIII.1.2], we have

$$\begin{aligned} \left| \int_a^b e^{i\phi(x)} \rho(x) dx \right| &= \left| \rho(x_0) \int_a^b e^{i\phi(x)} dx + \int_a^b e^{i\phi(x)} \int_{x_0}^x \rho'(y) dy dx \right| \\ &\leq \left| \rho(x_0) \int_a^b e^{i\phi(x)} dx \right| + \left| \int_a^{x_0} \rho'(y) \int_a^y e^{i\phi(x)} dx dy \right| + \left| \int_{x_0}^b \rho'(y) \int_y^b e^{i\phi(x)} dx dy \right| \\ &\lesssim \lambda^{-\frac{1}{k}} \left( |\rho(x_0)| + \int_a^b |\rho'(x)| dx \right). \end{aligned}$$

The latter term is estimated using

$$|\rho'(x)| = |(\psi(x) - \psi) * \lambda^{\frac{1}{k}} \eta(\lambda^{\frac{1}{k}} \cdot)'(x)| \lesssim \lambda^{\frac{2}{k}} \int_{\mathbb{R}} |\psi(x) - \psi(x-y)| |\eta'(\lambda^{\frac{1}{k}} y)| dy,$$

and the conclusion follows. □

We will also need a multidimensional version of Lemma B.1. As before  $B(y, s)$  denotes an open ball centered at  $y \in \mathbb{R}^k$  with radius  $s > 0$ .

**Proposition B.2** [Zorin-Kranich 2017]. *Given  $d, k \in \mathbb{N}$ , let  $P(x) = \sum_{1 \leq |\alpha| \leq d} \lambda_\alpha x^\alpha$  be a polynomial in  $k$  variables of degree at most  $d$  with real coefficients. Let  $R > 0$  and let  $\psi : \mathbb{R}^k \rightarrow \mathbb{C}$  be an integrable function supported in  $B(0, R/2)$ . Then*

$$\left| \int_{\mathbb{R}^k} e^{iP(x)} \psi(x) dx \right| \lesssim_{d,k} \sup_{v \in \mathbb{R}^k : |v| \leq R\Lambda^{-\frac{1}{d}}} \int_{\mathbb{R}^k} |\psi(x) - \psi(x-v)| dx,$$

where

$$\Lambda := \sum_{1 \leq |\alpha| \leq d} R^{|\alpha|} |\lambda_\alpha|.$$

We include the proof for completeness.

*Proof.* Changing the variables we have

$$\left| \int_{\mathbb{R}^k} e^{iP(x)} \psi(x) dx \right| = R^k \left| \int_{\mathbb{R}^k} e^{iP_R(x)} \psi_R(x) dx \right|,$$

where

$$P_R(x) = \sum_{1 \leq |\alpha| \leq d} R^{|\alpha|} \lambda_\alpha x^\alpha, \quad \psi_R(x) = \psi(Rx) \quad \text{and} \quad \text{supp } \psi_R \subseteq B(0, \frac{1}{2}).$$

Let us define

$$\beta = \sup_{v \in \mathbb{R}^k; |v| \leq \Lambda^{-\frac{1}{d}}} \int_{\mathbb{R}^k} |\psi_R(x) - \psi_R(x - v)| \, dx,$$

and observe that  $\|\psi_R\|_{L^1} \lesssim \beta \Lambda^{1/d}$ . So there is nothing to prove if  $\Lambda \lesssim 1$ . We assume that  $\Lambda \gtrsim 1$ . Let  $\eta$  be a nonnegative smooth bump function with integral 1, which is supported in the ball  $B(0, \frac{1}{2})$ . Then we define  $\rho(x) = \Lambda^{k/d} \eta(\Lambda^{1/d} x)$  and  $\phi(x) = \psi_R * \rho(x)$  and we note

$$\int_{\mathbb{R}^k} |\psi_R(x) - \phi(x)| \, dx \leq \Lambda^{\frac{k}{d}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} |\psi_R(x) - \psi_R(x - y)| \, dx \eta(\Lambda^{\frac{1}{d}} y) \, dy \lesssim \beta.$$

The proof will be completed if we show that

$$\left| \int_{\mathbb{R}^k} e^{iP_R(x)} \phi(x) \, dx \right| \lesssim_{d,k} \beta. \tag{B.3}$$

Since  $\phi$  is a smooth function supported in  $B(0, 1)$  we invoke [Stein and Wainger 2001, Lemma 2.2] to get the conclusion. Indeed, that result ensures that there exists a unit vector  $\xi \in \mathbb{R}^k$  and an integer  $m \in \mathbb{N}$  such that  $|(\xi \cdot \nabla)^m P_R| > c_{k,d} \Lambda$  on the unit ball  $B(0, 1)$  for some  $c_{k,d} > 0$ . We may assume, without loss of generality, that  $\xi = e_1 = (1, 0, \dots, 0) \in \mathbb{R}^k$ . Then by the van der Corput lemma, see for example [Stein 1993, Corollary, p. 334], we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^k} e^{iP_R(x)} \phi(x) \, dx \right| &\lesssim \Lambda^{-\frac{1}{d}} \int_{\mathbb{R}^{k-1} \cap B(0,1)} \left( |\phi(1, x')| + \int_{-1}^1 |\partial_1 \phi(x_1, x')| \, dx_1 \right) \, dx' \\ &\lesssim \Lambda^{-\frac{1}{d}} \|\nabla \phi\|_{L^1}, \end{aligned}$$

since  $\text{supp } \phi \subseteq B(0, 1)$  and  $\phi(1, x') = 0$  for every  $x' \in \mathbb{R}^{k-1} \cap B(0, 1)$ .

We now show that  $\|\nabla \phi\|_{L^1} \lesssim \Lambda^{1/d} \beta$ . Indeed, for every  $j \in \{1, \dots, k\}$  we have

$$\begin{aligned} \|\partial_j \phi\|_{L^1} &= \int_{\mathbb{R}^k} \left| \int_{\mathbb{R}^k} \psi_R(x - y) \partial_j \rho(y) \, dy \right| \, dx \\ &= \int_{\mathbb{R}^k} \left| \int_{\mathbb{R}^k} (\psi_R(x) - \psi_R(x - y)) \partial_j \rho(y) \, dy \right| \, dx \\ &\lesssim \Lambda^{\frac{k}{d} + \frac{1}{d}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} |\psi_R(x) - \psi_R(x - y)| |(\partial_j \eta)(\Lambda^{\frac{1}{d}} y)| \, dx \, dy \lesssim \Lambda^{\frac{1}{d}} \beta. \end{aligned}$$

This proves (B.3) and completes the proof of Proposition B.2. □

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