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ENERGY CONSERVATION FOR THE COMPRESSIBLE EULER AND NAVIER–STOKES EQUATIONS WITH VACUUM

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We consider the compressible isentropic Euler equations on $[0, T] \times \mathbb{T}^d$ with a pressure law $p \in C^{1,\gamma-1}$, where $1 \leq \gamma < 2$. This includes all physically relevant cases, e.g., the monoatomic gas. We investigate under what conditions on its regularity a weak solution conserves the energy. Previous results have crucially assumed that $p \in C^2$ in the range of the density; however, for realistic pressure laws this means that we must exclude the vacuum case. Here we improve these results by giving a number of sufficient conditions for the conservation of energy, even for solutions that may exhibit vacuum: firstly, by assuming the velocity to be a divergence-measure field; secondly, imposing extra integrability on $1/\rho$ near a vacuum; thirdly, assuming ρ to be quasinearly subharmonic near a vacuum; and finally, by assuming that u and ρ are Hölder continuous. We then extend these results to show global energy conservation for the domain $[0, T] \times \Omega$ where Ω is bounded with a C^2 boundary. We show that we can extend these results to the compressible Navier–Stokes equations, even with degenerate viscosity.

1. Introduction

In recent years some substantial effort has been directed towards investigating the relation between energy (or, more generally, entropy) conservation and regularity of weak solutions to a given physical system of equations.

Onsager's conjecture states that a weak solution of the (three-dimensional) incompressible Euler system will conserve energy if it is Hölder regular with exponent greater than $\frac{1}{3}$. Otherwise it is possible for solutions to exist where anomalous dissipation of energy occurs. First results towards energy conservation for weak solutions are due to Eyink [1994] and Constantin, E, and Titi [Constantin et al. 1994]; see also [Duchon and Robert 2000]. The sharpest results in optimal Besov spaces are due to Cheskidov et al. [2008] and Fjordholm and Wiedemann [2018]. Further, Bardos and Titi [2018], Bardos, Titi, and Wiedemann [Bardos et al. 2018], and Drivas and Nguyen [2018] have extended these results to consider solutions on a bounded domain.

Investigating the possibility of analogous statements for other systems has become another lively direction of research. Sufficient regularity conditions for the energy to be conserved were studied for a number of models: inhomogeneous incompressible Euler [Chen and Yu 2019] and Navier–Stokes [Leslie and Shvydkoy 2016], compressible Euler [Feireisl et al. 2017], the full Euler system [Drivas and Eyink 2018], compressible Navier–Stokes [Yu 2017], and Euler–Korteweg [Dębiec et al. 2018]. A general class of first-order conservation laws was considered in [Gwiazda et al. 2018], and in [Bardos et al. 2019] on bounded domains.

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Another direction of research was aimed towards the construction of $\left(\frac{1}{3}-\varepsilon\right)$-Hölder continuous solutions to the incompressible Euler system that do *not* conserve energy. With the application, and further refinements, of the method of convex integration this was achieved recently in [Isett 2018; Buckmaster et al. 2019]. Thus the famous conjecture of Lars Onsager for the incompressible Euler equations is fully resolved.

One of the major differences between incompressible and compressible fluid dynamics is the possible formation of *vacuum* in the latter case. This means that the density of the fluid may become zero in some region. More precisely, consider the isentropic compressible Euler system

$$\begin{aligned} \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)+\nabla p(\rho)&=0,\\ \partial_t\rho+\operatorname{div}(\rho u)&=0, \end{aligned} \tag{1-1}$$

where u denotes the velocity and ρ the density of the fluid. We will specify the constitutive pressure law $p=p(\rho)$ later. It is classically known that conservation laws like (1-1) may develop singularities (shocks) in finite time, which prohibits the use of a smooth notion of solution. Rather, one works with solutions in the sense of distributions, which may be very rough. Suppose now the density were initially bounded away from zero, $\rho^0\geq c>0$. If the solution were smooth, then from the continuity equation $\partial_t\rho+\operatorname{div}(\rho u)=0$ it would easily follow (see equation (7) in [DiPerna and Lions 1989]) that ρ remains bounded away from zero for all times. More precisely, this requires u to have bounded divergence. However, there seems to be no way to guarantee that the velocity component of a weak solution of (1-1) has bounded divergence, and thus it cannot be excluded that the solution spontaneously develops vacuum in finite time. In fact, to our knowledge it remains an outstanding open question whether this can actually occur for the compressible Euler or even Navier–Stokes equations.

The formation of vacuum constitutes a degeneracy that, in many situations, vastly complicates the mathematical analysis of compressible models. For instance, the compressible Euler equations cease to be strictly hyperbolic in vacuum regions. In the context of the current contribution, densities close to zero invalidate the methods and results from previous works like [Feireisl et al. 2017; Gwiazda et al. 2018; Bardos et al. 2019]: There, it is a crucial assumption that the nonlinearities depend on the dependent variables in a twice continuously differentiable fashion, in order to treat them like a quadratic expression in the commutator estimates. For the system (1-1), a typical and physically reasonable pressure law would be the polytropic one, i.e., $p(\rho)=\rho^\gamma$ with $\gamma>1$. The second derivative, however, is of order $\rho^{\gamma-2}$ and thus blows up at zero, at least if $\gamma<2$. But the regime $1<\gamma<2$ is precisely the relevant one $\big($for instance, a monoatomic gas has $\gamma=\frac{5}{3}\big)$.

The starting point of our current work is the result of Feireisl, Gwiazda, Świerczewska-Gwiazda, and Wiedemann [Feireisl et al. 2017] for the compressible Euler system, which we quote below. It gives sufficient conditions, in terms of Besov regularity of a weak solution, for energy conservation, but only as long as vacuum is excluded. In the presence of vacuum, the relevant commutator estimate involving the pressure completely breaks down, and it turns out that substantially new techniques are required to fix this. To our knowledge, the only other result on energy conservation for non-C^2 nonlinearities is the one on active scalar equations [Akramov and Wiedemann 2019], using however different techniques.

In the current article, we give a number of sufficient conditions to ensure energy conservation even after possible formation of vacuum.

First (Section 3), we consider the condition that the velocity be a so-called divergence-measure field; this notion is well known in geometric measure theory and hyperbolic conservation laws, but it may seem a bit unmotivated to consider in the present situation. However, justification comes from the compressible Navier–Stokes system, whose a priori estimates ensure this condition. We extensively discuss the ramifications of our result with respect to the Navier–Stokes equations in Section 3A, where we also compare it to recent work of Cheng Yu [2017].

In Section 4, we identify as a sufficient condition for energy conservation an estimate for the quotient between the density and its mollification; see (4-1). This, in itself, may seem rather artificial, and we go on to identify more natural conditions that will ensure (4-1) holds. Arguably, our strongest result is Corollary 4.4: under the slightly stronger assumption of Hölder (instead of Besov) regularity, but with the expected exponents, we can show energy conservation *no matter how the density behaves near vacuum*. It is surprising that this result is completely agnostic to the way that ρ approaches zero. It crucially relies on a new measure-theoretic observation (Lemma 4.3) that may be of independent interest.

If one does want to assume only Besov regularity, then one needs to make further assumptions on the density near vacuum; we show that energy is conserved provided the density descends into vacuum sufficiently fast (Corollary 4.7) or sufficiently slowly (Corollary 4.11).

Finally, in Section 5 we demonstrate how to extend our results, so far shown only under periodic boundary conditions, to the case of a bounded domain.

1A. ***The result of Feireisl et al.*** To formulate the local or global energy equality for (1-1) it is useful to define the so-called pressure potential by

$$P(\rho) = \rho \int_1^\rho \frac{p(r)}{r^2}\,\mathrm{d}r.$$

The following theorem was proven in [Feireisl et al. 2017, Theorem 4.1].

Theorem 1.1. *Let ρ, u be a solution of* (1-1) *in the sense of distributions. Assume*

$$u \in B_3^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B_3^{\beta,\infty}((0,T)\times\mathbb{T}^d), \quad 0 \le \underline{\rho} \le \rho \le \bar{\rho} \text{ a.e. in } (0,T)\times\mathbb{T}^d$$

for some constants $\underline{\rho}$, $\bar{\rho}$, and $0 \le \alpha, \beta \le 1$ such that

$$\beta > \max\bigl\{1-2\alpha, \tfrac{1}{2}(1-\alpha)\bigr\}.$$

Assume further that $p \in C^2[\underline{\rho}, \bar{\rho}]$, and in addition

$$p'(0) = 0 \quad \text{as soon as } \underline{\rho} = 0.$$

Then the energy is locally conserved; *i.e.*,

$$\partial_t\bigl(\tfrac{1}{2}\rho|u|^2 + P(\rho)\bigr) + \operatorname{div}\bigl[\bigl(\tfrac{1}{2}\rho|u|^2 + p(\rho) + P(\rho)\bigr)u\bigr] = 0 \tag{1-2}$$

in the sense of distributions on $(0,T)\times\mathbb{T}^d$.

Our aim in the current paper is to improve the above theorem by relaxing the C^2 assumption on the pressure. This will allow one, for instance, to apply the theorem in the physically relevant case of the isentropic pressure law $p(\rho)=\kappa\rho^\gamma$ with the adiabatic coefficient $\gamma\in(1,2)$, without excluding vacuum.

2. Preliminaries

2A. ***Function spaces.*** For $\Omega:=(0,T)\times\mathbb{T}^d$ we recall the Besov space $B_p^{\alpha,\infty}(\Omega)$, which is the space of tempered distributions w for which the norm

$$\|w\|_{B_p^{\alpha,\infty}(\Omega)}:=\|w\|_{L^p(\Omega)}+\sup_{\xi\in\Omega}\frac{\|w(\cdot+\xi)-w\|_{L^p(\Omega\cap(\Omega-\xi))}}{|\xi|^\alpha} \tag{2-1}$$

is finite. The above norm provides a control over shifts of the distribution w, making Besov spaces a convenient environment for our analysis, as it relies on convolutions with a mollifying kernel.

Let $\eta\in C_c^\infty(\mathbb{R}^N)$ be a positive, radial function of integral 1 with

$$\eta(x)=\begin{cases}1 & \text{for } |x|\le\frac{1}{3},\\ 0 & \text{for } |x|\ge 1,\end{cases}$$

and for $N=1+d$ set

$$\eta^\varepsilon(x)=\frac{1}{\varepsilon^N}\eta\left(\frac{x}{\varepsilon}\right).$$

We define the notation $w^\varepsilon:=\eta^\varepsilon * w$. For any function w, w^ε is well-defined on $\Omega^\varepsilon=\{x\in\Omega:\mathrm{d}(x,\partial\Omega)>\varepsilon\}$.

It is then easy to check that the definition of the Besov spaces implies

$$\|w^\varepsilon-w\|_{L^p(\Omega^\varepsilon)}\le C\varepsilon^\alpha\|w\|_{B_p^{\alpha,\infty}(\Omega)}$$

and

$$\|\nabla w^\varepsilon\|_{L^p(\Omega^\varepsilon)}\le C\varepsilon^{\alpha-1}\|w\|_{B_p^{\alpha,\infty}(\Omega)}.$$

By $\mathscr{M}(\Omega)$ we denote the space of signed Radon measures equipped with the total variation norm

$$\|\mu\|_{TV}:=\int_\Omega \mathrm{d}|\mu|.$$

2B. ***Derivation of the local energy equality.*** The starting point in the proof of Theorem 1.1, as well as all our results, is to mollify the Euler equations, then derive the local energy equality for the regularized quantities, and finally estimate commutator errors generated by nonlinear terms. As this strategy is a common part in the proofs of our theorems, we devote this section to the said derivation, omitting the details of passing to the limit under the assumptions of Theorem 1.1.

We begin by mollifying the momentum equation in time and space to obtain

$$\partial_t(\rho u)^\varepsilon+\operatorname{div}(\rho u\otimes u)^\varepsilon+\nabla p^\varepsilon(\rho)=0, \tag{2-2}$$

or, in terms of commutators

$$\begin{aligned}\partial_t(\rho^\varepsilon u^\varepsilon)&+\operatorname{div}((\rho u)^\varepsilon\otimes u^\varepsilon)+\nabla p(\rho^\varepsilon)\\ &=\partial_t(\rho^\varepsilon u^\varepsilon-(\rho u)^\varepsilon)+\operatorname{div}((\rho u)^\varepsilon\otimes u^\varepsilon-(\rho u\otimes u)^\varepsilon)+\nabla(p(\rho^\varepsilon)-p^\varepsilon(\rho)).\end{aligned} \tag{2-3}$$

Making use of the identity

$$\operatorname{div}((\rho u)^\varepsilon \otimes u^\varepsilon) = u^\varepsilon \operatorname{div}(\rho u)^\varepsilon + ((\rho u)^\varepsilon \cdot \nabla) u^\varepsilon,$$

we can see that multiplying (2-3) by u^ε yields

$$\rho^\varepsilon \partial_t\big(\tfrac{1}{2}|u^\varepsilon|^2\big) + ((\rho u)^\varepsilon \cdot \nabla)\tfrac{1}{2}|u^\varepsilon|^2 + \rho^\varepsilon u^\varepsilon\, \nabla(P'(\rho^\varepsilon)) = r_1^\varepsilon + r_2^\varepsilon + r_3^\varepsilon, \tag{2-4}$$

where

$$r_1^\varepsilon = \partial_t(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon) \cdot u^\varepsilon,$$
$$r_2^\varepsilon = \operatorname{div}((\rho u)^\varepsilon \otimes u^\varepsilon - (\rho u \otimes u)^\varepsilon) \cdot u^\varepsilon,$$
$$r_3^\varepsilon = \nabla(p(\rho^\varepsilon) - p^\varepsilon(\rho)) \cdot u^\varepsilon.$$

Using the mollified continuity equation

$$\partial_t \rho^\varepsilon + \operatorname{div}(\rho u)^\varepsilon = 0 \tag{2-5}$$

multiplied by $\frac{1}{2}|u^\varepsilon|^2$, we can rewrite (2-4) as

$$\partial_t\big(\tfrac{1}{2}\rho^\varepsilon |u^\varepsilon|^2\big) + \operatorname{div}\big((\rho u)^\varepsilon\, \tfrac{1}{2}|u^\varepsilon|^2\big) + \rho^\varepsilon u^\varepsilon\, \nabla(P'(\rho^\varepsilon)) = r_1^\varepsilon + r_2^\varepsilon + r_3^\varepsilon. \tag{2-6}$$

On the other hand writing (2-5) in the form

$$\partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = \operatorname{div}(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon)$$

and multiplying by $P'(\rho^\varepsilon)$ we get

$$\partial_t(P(\rho^\varepsilon)) + \operatorname{div}(\rho^\varepsilon u^\varepsilon) P'(\rho^\varepsilon) = \operatorname{div}(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon)\; P'(\rho^\varepsilon). \tag{2-7}$$

Combining (2-6) and (2-7) we obtain

$$\partial_t\big(\tfrac{1}{2}\rho^\varepsilon |u^\varepsilon|^2 + P(\rho^\varepsilon)\big) + \operatorname{div}\big((\rho u)^\varepsilon\, \tfrac{1}{2}|u^\varepsilon|^2 + \rho^\varepsilon u^\varepsilon P'(\rho^\varepsilon)\big) = r_1^\varepsilon + r_2^\varepsilon + r_3^\varepsilon + s^\varepsilon, \tag{2-8}$$

where we set

$$s^\varepsilon := \operatorname{div}(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon)\; P'(\rho^\varepsilon).$$

The proof of Theorem 4.1 in [Feireisl et al. 2017] shows that when ρ, u are Besov regular and p is of class C^2, the left-hand side of (2-8) converges to the left-hand side of (1-2) and each term on the right-hand side of (2-8) converges to zero, where each convergence is in the sense of distributions.

3. Energy conservation assuming the divergence of velocity is a bounded measure

Our first result establishes local energy conservation for weak solutions of (1-1) under the additional assumption that the velocity field u is a divergence-measure field.

Remark 3.1. See [Chen and Torres 2005] for details on the role of divergence-measure fields in the theory of hyperbolic conservation laws.

Theorem 3.2. *Let ρ, u be a solution of* (1-1) *in the sense of distributions. Assume*

$$u \in B_3^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B_3^{\beta,\infty}((0,T)\times\mathbb{T}^d), \quad 0 \le \underline{\rho} \le \rho \le \bar{\rho} \text{ a.e. in } (0,T)\times\mathbb{T}^d$$

for some constants $\underline{\rho}$, $\bar{\rho}$, and $0 \le \alpha, \beta \le 1$ such that

$$\beta > \max\bigl\{1-2\alpha, \tfrac{1}{2}(1-\alpha)\bigr\}.$$

Assume further that

$$\operatorname{div} u \in \mathscr{M}((0,T)\times\mathbb{T}^d) \quad \text{and} \quad p \in C[\underline{\rho}, \bar{\rho}].$$

Then the energy is locally conserved; *i.e.*,

$$\partial_t\bigl(\tfrac{1}{2}\rho|u|^2 + P(\rho)\bigr) + \operatorname{div}\bigl[\bigl(\tfrac{1}{2}\rho|u|^2 + p(\rho) + P(\rho)\bigr)u\bigr] = 0$$

in the sense of distributions on $(0,T)\times\mathbb{T}^d$.

Proof. Take a sequence $p^\delta \in C^2[\underline{\rho}, \bar{\rho}]$ that converges uniformly to $p \in C[\underline{\rho}, \bar{\rho}]$; that is, for each $\delta > 0$,

$$\|p - p^\delta\|_{L^\infty} \le \delta.$$

Then using p^δ in (2-2) we have

$$\partial_t(\rho u)^\varepsilon + \operatorname{div}(\rho u \otimes u)^\varepsilon + \nabla(p^\delta(\rho))^\varepsilon = \nabla[(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]. \tag{3-1}$$

Now the left-hand side of the last equality satisfies all the conditions of Theorem 1.1, so for each fixed $\delta > 0$ we have, in the limit as $\varepsilon \to 0$,

$$\partial_t\bigl(\tfrac{1}{2}\rho|u|^2 + P^\delta(\rho)\bigr) + \operatorname{div}\bigl[\bigl(\tfrac{1}{2}\rho|u|^2 + p^\delta(\rho) + P^\delta(\rho)\bigr)u\bigr], \tag{3-2}$$

where

$$P^\delta(\rho) := \rho\int_1^\rho \frac{p^\delta(r)}{r^2}\,\mathrm{d}r.$$

We will now show that (3-2) converges as $\delta \to 0$ in the sense of distributions on $(0,T)\times\mathbb{T}^d$ to

$$\partial_t\bigl(\tfrac{1}{2}\rho|u|^2 + P(\rho)\bigr) + \operatorname{div}\bigl[\bigl(\tfrac{1}{2}\rho|u|^2 + p(\rho) + P(\rho)\bigr)u\bigr].$$

Let $\varphi \in C_c^\infty((0,T)\times\mathbb{T}^d)$. From the choice of p^δ we have

$$\left|\int_0^T\int_{\mathbb{T}^d} \nabla\varphi\cdot(p^\delta(\rho) - p(\rho))u\,\mathrm{d}x\,\mathrm{d}t\right| \le C\|\varphi\|_{\mathscr{C}^1}\|p^\delta - p\|_{L^\infty}\|u\|_{L^3} \le C(\varphi, u)\delta.$$

For the terms containing $P^\delta(\rho)$ notice that

$$|P^\delta(\rho) - P(\rho)| \le \rho\int_1^\rho \frac{|p^\delta(r) - p(r)|}{r^2}\,\mathrm{d}r \le \|p^\delta - p\|_{L^\infty}\,\rho\left|\int_1^\rho \frac{1}{r^2}\,\mathrm{d}r\right| \le (1+\rho)\|p^\delta - p\|_{L^\infty}.$$

Hence we can estimate

$$\left|\int_0^T\int_{\mathbb{T}^d} \partial_t\varphi\,(P^\delta(\rho) - P(\rho))\,\mathrm{d}x\,\mathrm{d}t\right| \le C\|\varphi\|_{\mathscr{C}^1}(1 + \|\rho\|_{L^1})\delta \le C(\varphi)\delta,$$

and similarly for the divergence term. It follows that both terms of (3-2) containing P^δ converge as $\delta \to 0$ to the corresponding terms for P.

The final step of the proof is to consider the term coming into (2-8) from the right-hand side of (3-1). We need to show that

$$\nabla[(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)] \cdot u^\varepsilon$$

converges to zero in the sense of distributions on $(0, T) \times \mathbb{T}^d$ as first ε and then δ tend to zero. Multiplying by $\varphi \in C_c^\infty((0, T) \times \mathbb{T}^d)$, integrating over time and space, and integrating by parts we obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} \nabla[(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]\varphi u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\ = -\int_0^T \int_{\mathbb{T}^d} [(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]\varphi \operatorname{div} u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\mathbb{T}^d} [(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]\nabla\varphi \cdot u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3-3}$$

For the second term on the right-hand side of the last equality we see that

$$\begin{aligned}\left|\int_0^T \int_{\mathbb{T}^d} [(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]\nabla\varphi \cdot u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right| &= \left|\int_0^T \int_{\mathbb{T}^d} [p^\delta(\rho) - p(\rho)]^\varepsilon \nabla\varphi \cdot u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right| \\ &\le C\|\varphi\|_{C^1} \|(p^\delta - p)^\varepsilon\|_{L^\infty} \|u\|_{L^3} \\ &\le C\|\varphi\|_{C^1} \|p^\delta - p\|_{L^\infty} \|u\|_{L^3} \le C\delta.\end{aligned}$$

Finally, for the first term on the right-hand side of (3-3) we invoke the assumption that $\operatorname{div} u$ is a bounded Radon measure to see that

$$\begin{aligned}\left|\int_0^T \int_{\mathbb{T}^d} \varphi[(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)] \operatorname{div} u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right| &= \left|\int_0^T \int_{\mathbb{T}^d} \varphi[p^\delta(\rho) - p(\rho)]^\varepsilon (\operatorname{div} u)^\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right| \\ &\le \|\varphi\|_{C^0} \|(p^\delta - p)^\varepsilon\|_{L^\infty} \|(\operatorname{div} u)^\varepsilon\|_{L^1} \\ &\le \|\varphi\|_{C^0} \|p^\delta - p\|_{L^\infty} \|\operatorname{div} u\|_{TV} \le C\delta\end{aligned}$$

and so we are done. □

3A. ***Application to the compressible Navier–Stokes equations.*** When studying the result of Theorem 3.2 we see that the condition $\operatorname{div} u \in \mathscr{M}((0, T) \times \mathbb{T}^d)$ is quite a strong assumption for solutions to the compressible Euler equations; however, it is given for the compressible Navier–Stokes equations where one obtains a priori from the diffusion term that $u \in L^2(0, T; H^1)$. Therefore a natural question to ask is what happens when we consider the solutions to the compressible Navier–Stokes equations with vacuum, and how these results relate to the current results in [Yu 2017].

The compressible Navier–Stokes equations are given by

$$\begin{aligned}\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) &= \operatorname{div} \mathbb{S}(\nabla u), \\ \partial_t \rho + \operatorname{div}(\rho u) &= 0, \\ \mathbb{S}(\nabla u) &:= \mu\big(\nabla u + (\nabla u)^T - \tfrac{2}{3} \operatorname{div} u \mathbb{I}\big) + \nu \operatorname{div} u \mathbb{I},\end{aligned} \tag{3-4}$$

where we have the constants $\mu > 0$ and $\nu \geq 0$. Here we will use the main properties that $\mathbb{S}(\nabla u)$ is symmetric and positive definite. For degenerate viscosity, the momentum equation becomes, instead,

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = \operatorname{div}(\rho \mathbb{S}(\nabla u)). \tag{3-5}$$

Corollary 3.3. *Let ρ, u be a solution of* (3-4) *or* (3-5) *in the sense of distributions. Assume*

$$u \in B_3^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad u \in L^2(0,T;H^1(\mathbb{T}^d)), \quad \rho, \rho u \in B_3^{\beta,\infty}((0,T)\times\mathbb{T}^d),$$
$$0 \leq \underline{\rho} \leq \rho \leq \bar{\rho} \text{ a.e. in } (0,T)\times\mathbb{T}^d$$

for some constants $\underline{\rho}$, $\bar{\rho}$, and $0 \leq \alpha, \beta \leq 1$ such that

$$\beta > \max\{1-2\alpha, \tfrac{1}{2}(1-\alpha)\}. \tag{3-6}$$

Assume further that $p \in C[\underline{\rho}, \bar{\rho}]$. Then the energy is locally conserved; *i.e.*,

$$\partial_t\big(\tfrac{1}{2}\rho|u|^2 + P(\rho)\big) + \mathbb{S}(\nabla u) : \nabla u + \operatorname{div}\big[\big(\tfrac{1}{2}\rho|u|^2 + p(\rho) + P(\rho) + \mathbb{S}(\nabla u)\big)u\big] = 0 \tag{3-7}$$

for (3-4) *and*

$$\partial_t\big(\tfrac{1}{2}\rho|u|^2 + P(\rho)\big) + \rho\mathbb{S}(\nabla u) : \nabla u + \operatorname{div}\big[\big(\tfrac{1}{2}\rho|u|^2 + p(\rho) + P(\rho) + \rho\mathbb{S}(\nabla u)\big)u\big] = 0 \tag{3-8}$$

for (3-5), *in the sense of distributions on* $(0,T)\times\mathbb{T}^d$.

Remark 3.4. The condition $\operatorname{div} u \in \mathscr{M}$ is trivially satisfied if we assume that $u \in L^2(0,T;H^1)$ and so does not appear in the statement of Corollary 3.3.

Remark 3.5. For $d \leq 3$ we can use Besov embedding theorems, see [Bahouri et al. 2011], to observe that $H^1 \hookrightarrow B_2^{1,\infty} \hookrightarrow B_3^{2/3,\infty}$ and so assuming that $u \in B_3^{\alpha_1,\infty}(0,T;B_3^{\alpha_2,\infty})$ and $\rho, \rho u \in B_3^{\beta_1,\infty}(0,T;B_3^{\beta_2,\infty})$ we have the same assumptions on the pairs (α_1,β_1) and (α_2,β_2) as (3-6) but can assume that $\alpha_2 \geq \frac{2}{3}$ and remove the assumption that $u \in L^2(0,T;H^1)$.

Remark 3.6. We have assumed that the density ρ is bounded above to simplify the proof, though this is not necessary. Indeed, we can assume that for some $C > 0$, $p^\delta(r) = p(r)$ for $r \geq C$ and so still obtain uniform convergence of p^δ to p for unbounded density.

Proof. We only have to consider the extra term $\operatorname{div}\mathbb{S}(\nabla u)$ in the derivation of the local energy equality that we performed previously. We see that

$$-\int_0^T\int_{\mathbb{T}^d} \operatorname{div}\mathbb{S}(\nabla u^\varepsilon)\cdot u^\varepsilon\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{T}^d}\mathbb{S}(\nabla u^\varepsilon):\nabla u^\varepsilon\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{T}^d}(\mathbb{S}(\nabla u^\varepsilon)u^\varepsilon)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t$$

and so obtain (3-7). For (3-8) we perform the same calculation as above; however, with an extra ρ in the equation, the diffusion term is no longer linear and thus we pick up an extra commutator estimate

$$r_d^\varepsilon := \int_0^T\int_{\mathbb{T}^d}\operatorname{div}(\rho^\varepsilon\mathbb{S}(\nabla u^\varepsilon) - (\rho\mathbb{S}(\nabla u))^\varepsilon)\cdot\varphi u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

We can perform an integration by parts to obtain

$$|r_d^\varepsilon| \le \left| \int_0^T \int_{\mathbb{T}^d} [(\rho^\varepsilon \mathbb{S}(\nabla u^\varepsilon) - (\rho \mathbb{S}(\nabla u))^\varepsilon) u^\varepsilon] \cdot \nabla \varphi \,\mathrm{d}x \,\mathrm{d}t \right| + \left| \int_0^T \int_{\mathbb{T}^d} (\rho^\varepsilon \mathbb{S}(\nabla u^\varepsilon) - (\rho \mathbb{S}(\nabla u))^\varepsilon) : \nabla u^\varepsilon \varphi \,\mathrm{d}x \,\mathrm{d}t \right|. \tag{3-9}$$

Note the pointwise identity where for any two functions f, g we have

$$f^\varepsilon g^\varepsilon - (fg)^\varepsilon = (f^\varepsilon - f)(g^\varepsilon - g) - \int_{-\varepsilon}^{\varepsilon} \int_{\mathbb{T}^d} \eta^\varepsilon(\tau, \xi)(f(t-\tau, x-\xi) - f(t,x))(g(t-\tau, x-\xi) - g(t,x)) \,\mathrm{d}\xi \,\mathrm{d}\tau. \tag{3-10}$$

Applying this allows us to split the two terms on the right-hand side of (3-9) into four more terms which we can estimate. We focus on the first of these terms only, as the other terms produce the same estimates, after applying Fubini's theorem, as seen in [Feireisl et al. 2017]. We see that

$$\begin{aligned} |r_d^\varepsilon| &\le \left| \int_0^T \int_{\mathbb{T}^d} [(\rho^\varepsilon - \rho)(\mathbb{S}(\nabla u^\varepsilon) - \mathbb{S}(\nabla u)) u^\varepsilon] \cdot \nabla \varphi \,\mathrm{d}x \,\mathrm{d}t \right| + \left| \int_0^T \int_{\mathbb{T}^d} (\rho^\varepsilon - \rho)(\mathbb{S}(\nabla u^\varepsilon) - \mathbb{S}(\nabla u)) : \nabla u^\varepsilon \varphi \,\mathrm{d}x \,\mathrm{d}t \right| \\ &\le \|\varphi\|_{C^1} \|\rho\|_{L^\infty} \|u\|_{L^2} \|\mathbb{S}(\nabla u^\varepsilon) - \mathbb{S}(\nabla u)\|_{L^2} + \|\varphi\|_{C^0} \|\rho\|_{L^\infty} \|\nabla u\|_{L^2} \|\mathbb{S}(\nabla u^\varepsilon) - \mathbb{S}(\nabla u)\|_{L^2}. \end{aligned}$$

Using the a priori estimate that $u \in L^2(0, T; H^1)$ we see that $\|\mathbb{S}(\nabla u^\varepsilon) - \mathbb{S}(\nabla u)\|_{L^2} \to 0$ as $\varepsilon \to 0$ and thus $r_d^\varepsilon \to 0$ as $\varepsilon \to 0$. □

The work of Cheng Yu [2017] also studies energy conservation for the compressible Navier–Stokes systems where a vacuum could occur. The result in [Yu 2017] treats the case where $p(\rho) = \rho^\gamma$ for $\gamma > 1$ and thus where $p \in C^{1,\gamma-1}$, with strong assumptions of spacial regularity where

$$\sqrt{\rho} \nabla u \in L^2(0, T; L^2(\Omega)) \quad \text{and} \quad \frac{\nabla \rho}{\sqrt{\rho}} \in L^\infty(0, T; L^2(\Omega)),$$

among other assumptions. However, [Yu 2017] only assumes integrability in time. The condition $\nabla \rho / \sqrt{\rho} \in L^\infty(0, T; L^2(\Omega))$ restricts the allowable vacuum cases and will only allow vacuum on measure-zero sets with a nice approach to this set. The result presented here complements the result in [Yu 2017] as we show that by assuming some differential regularity in time for both ρ and u then we can weaken the spacial regularity assumptions and only need continuity of the pressure p. Specifically, we can have vacuum on measurable subsets of the domain where the approach to this set can be quite generic.

4. Energy conservation assuming Hölder continuity of the pressure

For the next result we fix $1 < \gamma < 2$ and we will assume that the pressure p is of class $C^{1,(\gamma-1)}$, thus relaxing the regularity assumption of Theorem 1.1. The expense of this relaxation is that we require $\alpha + \gamma\beta > 1$ where before we only needed $\alpha + 2\beta > 1$.

Theorem 4.1. *Let ρ, u be a solution of* (1-1) *in the sense of distributions. Assume*

$$u \in B_p^{\alpha,\infty}((0,T) \times \mathbb{T}^d), \quad \rho, \rho u \in B_q^{\beta,\infty}((0,T) \times \mathbb{T}^d), \quad 0 \le \underline{\rho} \le \rho \le \bar{\rho} \ \textit{a.e. in } (0,T) \times \mathbb{T}^d$$

for some constants $\underline{\rho}$, $\bar{\rho}$ *and* $0 \le \alpha, \beta \le 1$ *such that*

$$\frac{1}{p}+\frac{2}{q} \le 1, \quad \frac{2}{p}+\frac{1}{q} \le 1, \quad p, q \ge 2, \quad \alpha+\gamma\beta > 1, \quad \text{and} \quad 2\alpha+\beta > 1.$$

Define $\mathscr{B}_{\varepsilon^\beta} := \{x : 0 < \rho^\varepsilon(x) < \varepsilon^\beta \text{ and } \rho \neq 0\}$ *and assume that*

$$\left\| \frac{\rho^\varepsilon - \rho}{\rho^\varepsilon} \right\|_{L^q(\mathscr{B}_{\varepsilon^\beta})} \le C(\rho), \tag{4-1}$$

where C *does not depend on* ε. *Assume further that* $p \in C^{1,(\gamma-1)}([\underline{\rho}, \bar{\rho}])$, *and, in addition*

$$p'(0) = 0 \quad \text{as soon as } \underline{\rho} = 0.$$

Then the energy is locally conserved; *i.e.*, (1-2) *holds in the sense of distributions on* $(0, T) \times \mathbb{T}^d$.

A large part of the proof of this theorem is identical to the proof of Theorem 1.1. In particular we regularize the balance equations to derive an energy balance for the smooth functions ρ^ε and u^ε. Then we need to show that the corresponding commutator errors vanish in the limit $\varepsilon \to 0$. This is done in the same way as in [Feireisl et al. 2017], the only difference being in the terms involving the pressure. In particular, we will have to estimate an appropriate norm of the difference $p(\rho)^\varepsilon - p(\rho^\varepsilon)$. This will be done by means of the following lemma, which is an adaptation to our present case of the argument in [Feireisl et al. 2017, p. 10]; see also [Gwiazda et al. 2018, Lemma 3.1].

Lemma 4.2. *Let* $\gamma \in (1, 2)$ *and* $p \in C^{1,\gamma-1}([a, b])$. *If* $\rho \in B^{\beta,\infty}_{\gamma q}(\Omega; [a, b])$, *then*

$$\|p^\varepsilon(\rho) - p(\rho^\varepsilon)\|_{L^q} \le C\varepsilon^{\gamma\beta} \|\rho\|^\gamma_{B^{\beta,\infty}_{\gamma q}}.$$

Proof. First we note that by the fundamental theorem of calculus

$$\begin{aligned} p(s) - p(s_0) = \int_{s_0}^{s} p'(t)\,\mathrm{d}t &= \int_{s_0}^{s} p'(s_0)\,\mathrm{d}t + \int_{s_0}^{s} p'(t) - p'(s_0)\,\mathrm{d}t \\ &= p'(s_0)(s-s_0) + \int_{s_0}^{s} p'(t) - p'(s_0)\,\mathrm{d}t. \end{aligned}$$

Since $p' \in C^{0,\gamma-1}$, we have

$$\left| \int_{s_0}^{s} p'(t) - p'(s_0)\,\mathrm{d}t \right| \le \int_{s_0}^{s} |p'(t) - p'(s_0)|\,\mathrm{d}t \le C \int_{s_0}^{s} \mathrm{d}t \sup_{t\in[s_0,s]} |t-s_0|^{\gamma-1} \le C|s-s_0|^\gamma.$$

Thus,

$$|p(s) - p(s_0) - p'(s_0)(s-s_0)| \le C|s-s_0|^\gamma.$$

As the constant C is independent of s, s_0 we see that

$$|p(\rho^\varepsilon) - p(\rho) - p'(\rho)(\rho^\varepsilon - \rho)| \le C|\rho - \rho^\varepsilon|^\gamma, \tag{4-2}$$

and similarly,

$$|p(\rho(y)) - p(\rho(x)) - p'(\rho(x))(\rho(y) - \rho(x))| \le C|\rho(x) - \rho(y)|^\gamma. \tag{4-3}$$

Applying convolution against the function η^ε with respect to y in (4-3) and using Jensen's inequality we obtain

$$|p^\varepsilon(\rho) - p(\rho) - p'(\rho)(\rho^\varepsilon - \rho)| \le C|\rho - \rho(\,\cdot\,)|^\gamma *_y \eta^\varepsilon. \tag{4-4}$$

Combining (4-2) and (4-4) we get

$$|p^\varepsilon(\rho) - p(\rho^\varepsilon)| \le C|\rho - \rho^\varepsilon|^\gamma + C|\rho - \rho(\,\cdot\,)|^\gamma *_y \eta^\varepsilon. \tag{4-5}$$

Taking the L^q norm of both sides of (4-5) for the first term on the right-hand side we see that

$$C\||\rho - \rho^\varepsilon|^\gamma\|_{L^q} = C\|\rho - \rho^\varepsilon\|^\gamma_{L^{\gamma q}}.$$

Finally, for the L^q norm of (4-5) for the second term on the right-hand side by Jensen's inequality and Fubini's theorem we have

$$\begin{aligned} C\||\rho - \rho(\,\cdot\,)|^\gamma *_y \eta^\varepsilon\|_{L^q} &\le C\biggl(\iint |\rho(x) - \rho(x-y)|^{\gamma q}\,\mathrm{d}x\,\eta_\varepsilon(y)\,\mathrm{d}y\biggr)^{1/q} \\ &= C\biggl(\int \|\rho(\,\cdot\,) - \rho(\,\cdot\, - y)\|^{\gamma q}_{L^{\gamma q}}\eta_\varepsilon(y)\,\mathrm{d}y\biggr)^{1/q} \\ &\le C\sup_y |\eta_\varepsilon(y)|^{1/q}\biggl(\int_{\operatorname{supp}\eta_\varepsilon} \|\rho(\,\cdot\,) - \rho(\,\cdot\, - y)\|^{\gamma q}_{L^{\gamma q}}\,\mathrm{d}y\biggr)^{1/q} \\ &\le C\sup_{y\in\operatorname{supp}\eta_\varepsilon} \|\rho(\,\cdot\,) - \rho(\,\cdot\, - y)\|^{\gamma}_{L^{\gamma q}}. \end{aligned}$$

Finally, we use the definition of the Besov norm and (2-1) to write

$$\begin{aligned} \|p^\varepsilon(\rho) - p(\rho^\varepsilon)\|_{L^q} &\le C\bigl(\|\rho^\varepsilon - \rho\|^\gamma_{L^{\gamma q}} + \sup_{s\in\operatorname{supp}\eta^\varepsilon}\|\rho(\,\cdot\,) - \rho(\,\cdot\, - s)\|^\gamma_{L^{\gamma q}}\bigr) \\ &\le C\varepsilon^{\gamma\beta}\|\rho\|^\gamma_{B^{\beta,\infty}_{\gamma q}} + \sup_{s\in\operatorname{supp}\eta^\varepsilon}|s|^{\gamma\beta}\|\rho\|^\gamma_{B^{\beta,\infty}_{\gamma q}} \le C\varepsilon^{\gamma\beta}\|\rho\|^\gamma_{B^{\beta,\infty}_{\gamma q}}. \end{aligned}$$ □

Proof of Theorem 4.1. As remarked above the only novelty needed to establish the desired result is to estimate commutator errors due to nonlinearity of the pressure. Precisely, we need to show that the local versions of r_3^ε and s^ε, which we will denote by R^ε and S^ε, of (2-8) converge to zero as $\varepsilon \to 0$. For a test function $\varphi \in C_c^\infty((0,T)\times\mathbb{T}^d)$ we define

$$\begin{aligned} R^\varepsilon &:= \int_0^T\int_{\mathbb{T}^d} \nabla(p(\rho^\varepsilon) - p(\rho)^\varepsilon)\cdot\varphi u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t, \\ S^\varepsilon &:= \int_0^T\int_{\mathbb{T}^d} \varphi\,\operatorname{div}[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon]\,P'(\rho^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4-6}$$

Integrating (4-6) by parts and using Lemma 4.2 we obtain the estimate

$$\begin{aligned} |R^\varepsilon| &\le \|\varphi\|_{\mathscr{C}^1}\int_0^T\int_{\mathbb{T}^d} |p(\rho)^\varepsilon - p(\rho)^\varepsilon|(|\nabla u^\varepsilon| + |u^\varepsilon|)\,\mathrm{d}x\,\mathrm{d}t \\ &\le C\|\varphi\|_{\mathscr{C}^1}\|p(\rho^\varepsilon) - p(\rho)^\varepsilon\|_{L^{q/2}}(\|\nabla u^\varepsilon\|_{L^p} + \|u^\varepsilon\|_{L^p}) \\ &\le C(\varepsilon^{\gamma\beta+(\alpha-1)} + \varepsilon^{\gamma\beta+\alpha})\|\rho\|^\gamma_{B^{\beta,\infty}_{\gamma q/2}}\|u\|_{B^{\alpha,\infty}_p} \\ &\le C(\varepsilon^{\gamma\beta+(\alpha-1)} + \varepsilon^{\gamma\beta+\alpha})\|\rho\|^\gamma_{B^{\beta,\infty}_q}\|u\|_{B^{\alpha,\infty}_p}, \end{aligned}$$

where for the last inequality we used that $\frac{1}{2}\gamma q < q$, so we can embed $B^{\beta,\infty}_q$ into $B^{\beta,\infty}_{\gamma q/2}$.

We now investigate the term S^ε and see that we can integrate by parts to obtain

$$\begin{aligned}|S^\varepsilon| &= \left|\int_0^T\int_{\mathbb{T}^d} \varphi \operatorname{div}[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon]\, P'(\rho^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\le \int_0^T\int_{\mathbb{T}^d} |\nabla\varphi\cdot[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon]\, P'(\rho^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{T}^d} |\varphi[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon]\cdot\nabla P'(\rho^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4-7}$$

We make note of the pointwise identity (3-10) but with f and g replaced by ρ and u respectively, that is,

$$\begin{aligned}\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon &= (\rho^\varepsilon - \rho)(u^\varepsilon - u) \\ &\quad - \int_{-\varepsilon}^{\varepsilon}\int_{\mathbb{T}^d} \eta^\varepsilon(\tau,\xi)(\rho(t-\tau,x-\xi)-\rho(t,x))(u(t-\tau,x-\xi)-u(t,x))\,\mathrm{d}\xi\,\mathrm{d}\tau,\end{aligned}$$

and using (3-10) allows us to split first term on the right-hand side of (4-7) into two terms. Here again we focus on the first of these terms only, as the other one produces the same estimates, after applying Fubini's theorem, as seen in [Feireisl et al. 2017]. We see that

$$\int_0^T\int_{\mathbb{T}^d} |\nabla\varphi\cdot(\rho^\varepsilon-\rho)(u^\varepsilon-u)P'(\rho^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t \le \|\varphi\|_{C^1}\varepsilon^\beta\|\rho\|_{B^{\beta,\infty}_q}\varepsilon^\alpha\|u\|_{B^{\alpha,\infty}_p}\|P'(\rho^\varepsilon)\|_{L^\infty}.$$

We will now focus on the second term on the right-hand side of (4-7), namely,

$$\int_0^T\int_{\mathbb{T}^d} |\varphi[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon]\cdot\nabla P'(\rho^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t,$$

and by letting $y=(t,x)$ we split $(0,T)\times\mathbb{T}^d$ into two disjoint domains $\mathscr{A} := \{y : \rho^\varepsilon(y) = 0\}$ and $\mathscr{A}^c$ and see that trivially on $\mathscr{A}$ we have $\rho(y)=0$ a.e. For the integral over $\mathscr{A}$ we note that $\nabla P'(\rho^\varepsilon)$ is a distribution that may have a singular part but we see that $\varphi[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon]$ is smooth and equals zero on $\mathscr{A}$ and so any singular part vanishes. Thus we are left with

$$\int_{\mathscr{A}^c} |\varphi[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon]\nabla P'(\rho^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t,$$

and using again the identity (3-10) we obtain

$$\int_{\mathscr{A}^c} |\varphi[(\rho^\varepsilon-\rho)(u^\varepsilon-u)]\nabla P'(\rho^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t.$$

For the integral over $\mathscr{A}^c$ we see that

$$\int_{\mathscr{A}^c} |\varphi(\rho^\varepsilon-\rho)(u^\varepsilon-u)\nabla P'(\rho^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathscr{A}^c} |\varphi(\rho^\varepsilon-\rho)(u^\varepsilon-u)P''(\rho^\varepsilon)\cdot\nabla\rho^\varepsilon|\,\mathrm{d}x\,\mathrm{d}t$$

and we observe that by the definition of P we have $\rho^\varepsilon P''(\rho^\varepsilon) = p'(\rho^\varepsilon)$, and by assumption p' is bounded. Therefore we have the bound

$$\int_{\mathscr{A}^c} |\varphi(\rho^\varepsilon-\rho)(u^\varepsilon-u)P''(\rho^\varepsilon)\nabla\rho^\varepsilon|\,\mathrm{d}x\,\mathrm{d}t \le \int_{\mathscr{A}^c}\left|\varphi(\rho^\varepsilon-\rho)(u^\varepsilon-u)p'(\rho^\varepsilon)\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon}\right|\mathrm{d}x\,\mathrm{d}t.$$

We have assumed that $p'(0) = 0$ and $p' \in C^{0,\gamma-1}$ and so take any ρ_1, ρ_2 such that $p'(\rho_2) = 0$ and we obtain

$$|p'(\rho_1)| = |p'(\rho_1) - p'(\rho_2)| \leq C|\rho_1 - \rho_2|^{\gamma-1} \leq C|\rho_1|^{\gamma-1}$$

using the definition of Hölder continuity. Thus letting $\rho_1 = \rho^\varepsilon(x)$ for each x we see that $|p'(\rho^\varepsilon)(x)| \leq C|\rho^\varepsilon|^{\gamma-1}(x)$ and so we obtain

$$\int_{\mathscr{A}^c} \left| \varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u) p'(\rho^\varepsilon) \frac{\nabla \rho^\varepsilon}{\rho^\varepsilon} \right| \mathrm{d}x\, \mathrm{d}t \leq C \int_{\mathscr{A}^c} \left| \varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)(\rho^\varepsilon)^{\gamma-1} \frac{\nabla \rho^\varepsilon}{\rho^\varepsilon} \right| \mathrm{d}x\, \mathrm{d}t.$$

We will split the integral over $\mathscr{A}^c$ further into different disjoint domains, $\mathscr{B}_{\varepsilon^\beta} := \{y : 0 < \rho^\varepsilon(y) < \varepsilon^\beta\}$ and $\mathscr{C}_{\varepsilon^\beta} := \{y : \rho^\varepsilon(y) \geq \varepsilon^\beta\}$. For the integral over $\mathscr{B}_{\varepsilon^\beta}$ we see that

$$\begin{aligned}
&\left| \int_{\mathscr{B}_{\varepsilon^\beta}} \varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)(\rho^\varepsilon)^{\gamma-1} \frac{\nabla \rho^\varepsilon}{\rho^\varepsilon} \,\mathrm{d}x\, \mathrm{d}t \right| \\
&\qquad \leq C\|\varphi\|_{C^0} \varepsilon^{\beta-1+\alpha} \|\rho\|_{B_q^{\beta,\infty}} \|u\|_{B_p^{\alpha,\infty}} \|(\rho^\varepsilon)^{\gamma-1}\|_{L^\infty(\mathscr{B}_{\varepsilon^\beta})} \left\| \frac{\rho^\varepsilon - \rho}{\rho^\varepsilon} \right\|_{L^q(\mathscr{B}_{\varepsilon^\beta})} \\
&\qquad \leq C\|\varphi\|_{C^0} \varepsilon^{\gamma\beta-1+\alpha} \|\rho\|_{B_q^{\beta,\infty}} \|u\|_{B_p^{\alpha,\infty}} \left\| \frac{\rho^\varepsilon - \rho}{\rho^\varepsilon} \right\|_{L^q(\mathscr{B}_{\varepsilon^\beta})},
\end{aligned}$$

where for the last line, as $\rho^\varepsilon(y) \leq \varepsilon^\beta$, we have $(\rho^\varepsilon(y))^{\gamma-1} \leq \varepsilon^{\beta(\gamma-1)}$ as $\gamma - 1 > 0$. We also have the assumption that $\|(\rho^\varepsilon - \rho)/\rho^\varepsilon\|_{L^q(\mathscr{B}_{\varepsilon^\beta})} \leq C$ and so we have the bound $C\varepsilon^{\gamma\beta-1+\alpha}$ as wanted. We are left with the integral over $\mathscr{C}_{\varepsilon^\beta}$ and see that

$$\left| \int_{\mathscr{C}_{\varepsilon^\beta}} \varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)(\rho^\varepsilon)^{\gamma-1} \frac{\nabla \rho^\varepsilon}{\rho^\varepsilon} \,\mathrm{d}x\, \mathrm{d}t \right| \leq C\|\varphi\|_{C^0} \varepsilon^{\beta-1+\alpha} \|\rho\|_{B_q^{\beta,\infty}} \|u\|_{B_p^{\alpha,\infty}} \left\| \frac{\rho^\varepsilon - \rho}{(\rho^\varepsilon)^{2-\gamma}} \right\|_{L^q(\mathscr{C}_{\varepsilon^\beta})}.$$

As $\rho^\varepsilon \geq \varepsilon^\beta$, we have $(\rho^\varepsilon)^{-1} \leq \varepsilon^{-\beta}$, and so $(\rho^\varepsilon)^{\gamma-2} \leq \varepsilon^{\beta(\gamma-2)}$, and we obtain

$$\left\| \frac{\rho^\varepsilon - \rho}{(\rho^\varepsilon)^{2-\gamma}} \right\|_{L^q(\mathscr{C}_{\varepsilon^\beta})} \leq \|\rho^\varepsilon - \rho\|_{L^q(\mathscr{C}_{\varepsilon^\beta})} \varepsilon^{\beta(\gamma-2)} \leq C\varepsilon^\beta \|\rho\|_{B_q^{\beta,\infty}} \varepsilon^{\beta(\gamma-2)} \leq C\varepsilon^{\beta(\gamma-1)}.$$

We are thus done as we have obtained convergence to zero as long as $\gamma\beta + \alpha > 1$.

We have thus shown that, under the assumptions of the theorem, we have $R^\varepsilon, S^\varepsilon \to 0$. The result follows. □

We have written Theorem 4.1 in the most general from but observe that the condition

$$\left\| \frac{\rho^\varepsilon - \rho}{\rho^\varepsilon} \right\|_{L^q(\mathscr{B}_{\varepsilon^\beta})} \leq C$$

feels rather artificial and is not in the $p \in C^2$ result from [Feireisl et al. 2017]. We will now focus on finding conditions on ρ for different L^q norms that will control this term.

Our first result will show that when we assume that $q = 1$ and so u, ρ are Hölder continuous, not just Besov functions, we can control this term directly as expected and do not have to ask for any special extra conditions.

Lemma 4.3. *Let $w \in L^1(\Omega)$ be nonnegative, where $\Omega \subset (0,T)\times\mathbb{T}^d$ satisfies $|\Omega| \neq 0$ and $w^\varepsilon|_\Omega > 0$. Then $\|(w^\varepsilon - w)/w^\varepsilon\|_{L^1(\Omega)} \le C$, where C does not depend on ε but may depend on w and Ω.*

Proof. It suffices to show that $\|w/w^\varepsilon\|_{L^1(\Omega)} \le C$. Indeed, since $|\Omega| \le C$,

$$\left\|\frac{w^\varepsilon - w}{w^\varepsilon}\right\|_{L^1(\Omega)} \le \|1\|_{L^1(\Omega)} + \left\|\frac{w}{w^\varepsilon}\right\|_{L^1(\Omega)} = C + \left\|\frac{w}{w^\varepsilon}\right\|_{L^1(\Omega)}.$$

Fix $\varepsilon > 0$, $N = d+1$, and let $\{Q_j\}_{j=1}^n$ be a partition of $(0,T)\times\mathbb{T}^d$ into disjoint cubes with side length ε/C_N, where C_N is a constant depending only on the dimension, and select the cubes such that $|\Omega \cap Q_j| \neq 0$. Decomposing w as $w = \sum_{j=1}^n w\chi_{Q_j}$, we see that

$$\left\|\frac{w}{w^\varepsilon}\right\|_{L^1(\Omega)} = \left\|\frac{\sum_{j=1}^n w\chi_{Q_j}}{\sum_{k=1}^n (w\chi_{Q_k})^\varepsilon}\right\|_{L^1(\Omega)} \le \sum_{j=1}^n \left\|\frac{w\chi_{Q_j}}{(w\chi_{Q_j})^\varepsilon}\right\|_{L^1(\Omega)}.$$

We now want to bound $(w\chi_{Q_j})^\varepsilon$ from below. Recalling from Section 2 that $\eta = 1$ for $|x| < \frac{1}{3}$, we have, for $x \in Q_j$, that

$$\begin{aligned}(w\chi_{Q_j})^\varepsilon(x) &\ge \frac{1}{\varepsilon^N}\int_{\{|(x-y)/\varepsilon|\le 1/3\}} \eta\left(\frac{x-y}{\varepsilon}\right)(w\chi_{Q_j})(y)\,\mathrm{d}y = \frac{1}{\varepsilon^N}\int_{B_{\varepsilon/3}(x)} (w\chi_{Q_j})(y)\,\mathrm{d}y \\ &= \frac{\omega_N}{|B_\varepsilon|}\int_{B_{\varepsilon/3}(x)} (w\chi_{Q_j})(y)\,\mathrm{d}y \ge \frac{\omega_N}{|B_\varepsilon|}\int_{Q_j} w(y)\,\mathrm{d}y,\end{aligned}$$

where we obtain the last inequality provided C_N is large enough so that $B_{\varepsilon/3}(x) \supset Q_j$ for all $x \in Q_j$. Thus we obtain

$$\left\|\frac{w}{w^\varepsilon}\right\|_{L^1(\Omega)} \le \sum_{j=1}^n \left\|\frac{w\chi_{Q_j}}{(w\chi_{Q_j})^\varepsilon}\right\|_{L^1(\Omega)} \le \sum_{j=1}^n \frac{|B_\varepsilon|}{\omega_N}\frac{\int_{Q_j} w\,\mathrm{d}x}{\int_{Q_j} w\,\mathrm{d}x} \le C\sum_{j=1}^n |Q_j| \le C, \tag{4-8}$$

where we have used a dimensional constant to relate the measure of the balls to the associated cubes. □

As a consequence we obtain the following corollary, where by assuming Hölder continuity of u and ρ we obtain a natural extension of Theorem 1.1 to the case where $p \in C^{1,\gamma-1}$.

Corollary 4.4. *Let ρ, u be a solution of* (1-1) *in the sense of distributions. Assume*

$$u \in C^\alpha((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in C^\beta((0,T)\times\mathbb{T}^d), \quad 0 \le \underline{\rho} \le \rho \le \bar{\rho} \ \text{ a.e. in } (0,T)\times\mathbb{T}^d$$

for some constants $\underline{\rho}$, $\bar{\rho}$ and $0 \le \alpha, \beta \le 1$ such that

$$\alpha + \gamma\beta > 1 \quad \text{and} \quad 2\alpha + \beta > 1.$$

Assume further that $p \in C^{1,(\gamma-1)}([\underline{\rho}, \bar{\rho}])$, and, in addition

$$p'(0) = 0 \quad \text{as soon as } \underline{\rho} = 0.$$

Then the energy is locally conserved; *i.e.*, (1-2) *holds in the sense of distributions on* $(0,T)\times\mathbb{T}^d$.

Proof. For the integral over $\mathscr{B}_{\varepsilon^\beta}$, in the proof of Theorem 4.1, we see that

$$\left|\int_{\mathscr{B}_{\varepsilon^\beta}} \varphi(\rho^\varepsilon-\rho)(u^\varepsilon-u)(\rho^\varepsilon)^{\gamma-1}\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon}\,\mathrm{d}x\,\mathrm{d}t\right|$$
$$\leq C\|\varphi\|_{C^0}\varepsilon^{\beta-1+\alpha}\|\rho\|_{\mathscr{C}^\beta}\|u\|_{\mathscr{C}^\alpha}\|(\rho^\varepsilon)^{\gamma-1}\|_{L^\infty(\mathscr{B}_{\varepsilon^\beta})}\left\|\frac{\rho^\varepsilon-\rho}{\rho^\varepsilon}\right\|_{L^1(\mathscr{B}_{\varepsilon^\beta})}$$
$$\leq C\|\varphi\|_{C^0}\varepsilon^{\gamma\beta-1+\alpha}\|\rho\|_{\mathscr{C}^\beta}\|u\|_{\mathscr{C}^\alpha}.$$

For the other bounds, as we are on a domain with finite measure, we can bound the Besov norms by the Hölder norms. □

Remark 4.5. Notice that the conditions $u\in C^\alpha((0,T)\times\mathbb{T}^d)$ and $\rho\in C^\beta((0,T)\times\mathbb{T}^d)$ imply that $\rho u\in C^{\min(\alpha,\beta)}((0,T)\times\mathbb{T}^d)$. Therefore, if one has $\alpha\geq\beta$, then the requirement that ρu be in $C^\beta((0,T)\times\mathbb{T}^d)$ can be dropped. See also Remark 3.2(2) in [Feireisl et al. 2017].

When we still want to consider Besov spaces for ρ and u we have to consider extra conditions on ρ in order to control the term $\|(\rho^\varepsilon-\rho)/\rho^\varepsilon\|_{L^q(\mathscr{B}_{\varepsilon^\beta})}$. Our first method will be to ask for an integrability condition on $1/\rho$.

Lemma 4.6. *Assume that* $1/w\in L^p$ *and* $w\in L^q$. *Then*

$$\left\|\frac{w^\varepsilon-w}{w^\varepsilon}\right\|_{L^r}\leq C\quad \textit{for } \frac{1}{p}+\frac{1}{q}\leq\frac{1}{r},$$

and in fact if $r<\infty$,

$$\lim_{\varepsilon\to 0}\left\|\frac{w^\varepsilon-w}{w^\varepsilon}\right\|_{L^r}=0\quad \textit{for } \frac{1}{p}+\frac{1}{q}\leq\frac{1}{r}.$$

Proof. Using Hölder's inequality and then Jensen's inequality, as the integral of the mollifier is 1 and $1/x$ is a convex function, we get that $\|1/w^\varepsilon\|\leq\|(1/w)^\varepsilon\|\leq\|1/w\|$ and so

$$\left\|\frac{w^\varepsilon-w}{w^\varepsilon}\right\|_{L^r}\leq\|w^\varepsilon-w\|_{L^q}\left\|\frac{1}{w^\varepsilon}\right\|_{L^p}\leq\|w^\varepsilon-w\|_{L^q}\left\|\frac{1}{w}\right\|_{L^p}\leq C.$$

As long as $q<\infty$ we see that this, in fact, converges to zero. □

We now obtain the following corollary adding this condition into Theorem 4.1. We note that when $p=q=3$, we obtain the best result with the weakest integrability assumption in the Besov norms.

Corollary 4.7. *Let* ρ, u *be a solution of* (1-1) *in the sense of distributions. Assume*

$$u\in B_p^{\alpha,\infty}((0,T)\times\mathbb{T}^d),\quad \rho,\rho u\in B_q^{\beta,\infty}((0,T)\times\mathbb{T}^d),\quad 0\leq\underline{\rho}\leq\rho\leq\bar{\rho}\ \textit{a.e. in } (0,T)\times\mathbb{T}^d$$

for some constants $\underline{\rho}$, $\bar{\rho}$ *and* $0\leq\alpha,\beta\leq 1$ *such that*

$$\frac{1}{p}+\frac{2}{q}\leq 1,\quad \frac{2}{p}+\frac{1}{q}\leq 1,\quad p,q\geq 2,\quad \alpha+\gamma\beta>1,\quad \textit{and}\quad 2\alpha+\beta>1.$$

Define $\mathscr{E}:=\{x:\rho\neq 0\}$ *and assume that*

$$\frac{1}{\rho}\in L^q(\mathscr{E}).$$

Assume further that $p \in C^{1,(\gamma-1)}([\underline{\rho}, \bar{\rho}])$, *and, in addition*

$$p'(0) = 0 \quad \textit{as soon as } \underline{\rho} = 0.$$

Then the energy is locally conserved; *i.e.*, (1-2) *holds in the sense of distributions on* $(0, T) \times \mathbb{T}^d$.

Proof. For the integral over $\mathscr{B}_{\varepsilon^\beta}$, in the proof of Theorem 4.1, we see that as $\rho \in L^\infty$ and $\varepsilon^\beta \geq \rho^\varepsilon$ then

$$\left| \int_0^T \int_{\mathscr{B}_{\varepsilon^\beta}} \varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)(\rho^\varepsilon)^{\gamma-1} \frac{\nabla \rho^\varepsilon}{\rho^\varepsilon} \, dx \, dt \right|$$
$$\leq C \|\varphi\|_{C^0} \varepsilon^{\beta-1+\alpha} \|\rho\|_{B_q^{\beta,\infty}} \|u\|_{B_p^{\alpha,\infty}} \|(\rho^\varepsilon)^{\gamma-1}\|_{L^\infty(\mathscr{B}_{\varepsilon^\beta})} \left\| \frac{\rho^\varepsilon - \rho}{\rho^\varepsilon} \right\|_{L^q(\mathscr{B}_{\varepsilon^\beta})}$$
$$\leq C \|\varphi\|_{C^0} \varepsilon^{\gamma\beta-1+\alpha} \|\rho\|_{B_q^{\beta,\infty}} \|u\|_{B_p^{\alpha,\infty}} \left\| \frac{1}{\rho} \right\|_{L^q(\mathscr{E})},$$

and so we are done, using Lemma 4.6 for the final step. □

Remark 4.8. Even though we have written $1/\rho \in L^q(\mathscr{E})$, we can fix some $\delta > 0$ and only need this condition on some $\mathscr{B}_\delta$, as for $\varepsilon^1 > \varepsilon^2$ we have $\mathscr{B}_{\varepsilon^2} \subset \mathscr{B}_{\varepsilon^1}$, and so when $\varepsilon^\beta < \delta$ we have $\mathscr{B}_{\varepsilon^\beta} \subset \mathscr{B}_\delta$.

One can see that the condition $1/\rho \in L^q(\mathscr{B}_\delta)$ is quite a strong assumption and requires a quick approach of the function to the null set. Above we used conventional bounds to obtain a general integral result but do not consider the local structure of the function. We notice that a pointwise estimate $\rho \leq C\rho^\varepsilon$ would allow us to control the L^q norm of $(\rho^\varepsilon - \rho)/\rho^\varepsilon$ and, though convexity of ρ would do, we will now show a nice link between this and quasinearly subharmonic functions which are much more general functions than subharmonic, quasisubharmonic and nearly subharmonic functions [Pavlović and Riihentaus 2011]. The main motivation for the study of this notion in this paper is that it happens, as will be shown below, to be equivalent to the L^∞-boundedness of our problem term $(\rho^\varepsilon - \rho)/\rho^\varepsilon$.

Definition 4.9. Let $X \subset \mathbb{R}^d$ be a set and $u : X \to [0, +\infty)$ be Borel measurable. Then u is quasinearly subharmonic on X, that is $u \in \mathrm{QNS}(X)$, if there is a constant $\varepsilon_0 = \varepsilon_0(u)$, $0 < \varepsilon_0 < 1$, such that for each open set $O \subset X$, $O \neq X$, for each $x \in O$ and each r, $0 < r \leq \varepsilon_0 \delta^O(x)$, one has $u \in L^1(B_r(x))$ and

$$u(x) \leq \frac{C}{|B_r(x)|} \int_{B_r(x)} u(y) \, dy \quad \text{for some constant } C \geq 1, \tag{4-9}$$

where C is independent of r, $|B_r(x)| = \omega_d r^d$ is the volume of the ball and

$$\delta^O(x) = \mathrm{dist}(x, O^c) \quad \text{for the complement } O^c \text{ of } O \text{ in } X.$$

Lemma 4.10. *Let* $u : X \to [0, +\infty)$ *be a Borel measurable function. Then* u *is quasinearly subharmonic if and only if for every* $O \Subset X$ *there exist* M, ε_0 *such that for any* $0 < \varepsilon < \varepsilon_0$

$$u(x) \leq M u^\varepsilon(x) \quad \textit{for any } x \in O.$$

Proof. Let $u : X \to [0, +\infty)$ be a quasinearly subharmonic function. Then for any $\varepsilon < \mathrm{dist}(O, \partial X)$, u^ε is a well-defined smooth function on O. Suppose that $O \Subset X$ is a precompact set. Then $\delta_0 = \mathrm{dist}(O, \partial X)$

is a positive number and for $\varepsilon < \delta_0$

$$O \subset \{x : \mathrm{dist}(x, \partial X) > \varepsilon\}$$

and u^ε is well-defined on O. We prove that there exist M and ε_0 such that

$$u(x) \le M u^\varepsilon(x) \quad \text{for any } x \in O,\ 0 < \varepsilon < \varepsilon_0.$$

Indeed, we have

$$u^\varepsilon(x) = \frac{1}{\varepsilon^d} \int_X \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\,\mathrm{d}y.$$

Note that $y \in X$ for $x \in O$ and $|x - y| < \varepsilon$. Since $u \ge 0$ and recalling that from the definition of η we know that $\eta = 1$ for $|x| < \frac{1}{3}$, we have

$$\begin{aligned} u^\varepsilon(x) &\ge \frac{1}{\varepsilon^d} \int_{\{|(x-y)/\varepsilon| \le 1/3\}} \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\,\mathrm{d}y \\ &= \frac{1}{\varepsilon^d} \int_{\{|(x-y)/\varepsilon| \le 1/3\}} u(y)\,\mathrm{d}y = \frac{\omega_d}{3^d |B_{\varepsilon/3}(x)|} \int_{B_{\varepsilon/3}(x)} u(y)\,\mathrm{d}y \ge \frac{\omega_d u(x)}{3^d C} \end{aligned}$$

for sufficiently small ε. Therefore, we obtain

$$u(x) \le \frac{3^d C u^\varepsilon(x)}{\omega_d} \quad \text{for sufficiently small } \varepsilon \le \varepsilon_0 \delta^O(x).$$

On the other hand, if $u(x) \le M u^\varepsilon(x)$, then we have

$$u(x) \le \frac{M}{\varepsilon^d} \int_X \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\,\mathrm{d}y = \frac{M\omega_d}{\omega_d \varepsilon^d} \int_{|x-y|\le\varepsilon} \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\,\mathrm{d}y \le \frac{M\omega_d}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y)\,\mathrm{d}y. \tag{4-10}$$

Hence we deduce

$$u(x) \le \frac{C}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y)\,\mathrm{d}y. \qquad \square$$

From this pointwise control showing that $\rho(x) \le M\rho^\varepsilon(x)$ we obtain another corollary to our main result.

Corollary 4.11. *Let ρ, u be a solution of* (1-1) *in the sense of distributions. Assume*

$$u \in B_p^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B_q^{\beta,\infty}((0,T)\times\mathbb{T}^d), \quad 0 \le \underline{\rho} \le \rho \le \bar{\rho} \ \text{ a.e. in } (0,T)\times\mathbb{T}^d$$

for some constants $\underline{\rho}$, $\bar{\rho}$ and $0 \le \alpha, \beta \le 1$ such that

$$\frac{1}{p} + \frac{2}{q} \le 1, \quad \frac{2}{p} + \frac{1}{q} \le 1, \quad p, q \ge 2, \quad \alpha + \gamma\beta > 1, \quad \text{and} \quad 2\alpha + \beta > 1.$$

Assume that $\rho \in \mathrm{QNS}(\mathscr{B}_\delta)$ for some $\delta > 0$ and $p \in \mathscr{C}^{1,(\gamma-1)}([\underline{\rho}, \bar{\rho}])$ with

$$p'(0) = 0 \quad \text{as soon as } \underline{\rho} = 0.$$

Then the energy is locally conserved; *i.e.*, (1-2) *holds in the sense of distributions on* $(0,T)\times\mathbb{T}^d$.

Proof. For the integral over $\mathscr{B}_{\varepsilon^\beta}$, in the proof of Theorem 4.1, we see that

$$\left\| \frac{\rho^\varepsilon - \rho}{\rho^\varepsilon} \right\|_{L^q(\mathscr{B}_{\varepsilon^\beta})} \le \left\| \frac{\rho^\varepsilon + C\rho^\varepsilon}{\rho^\varepsilon} \right\|_{L^q(\mathscr{B}_{\varepsilon^\beta})} \le C \quad \text{for } \varepsilon^\beta < \delta$$

and so we are done. □

Remark 4.12. (1) The condition $\rho \in \mathrm{QNS}(B_\delta)$ deals with ρ, $\rho_\varepsilon = 0$ without splitting into cases and so using this condition the proof is simplified.

(2) We note that this condition is weaker than local convexity of ρ on $\mathscr{B}_\delta$, which would also give the same result.

(3) In view of Lemma 4.10, it is essentially a matter of taste if one prefers to formulate Corollary 4.11 in terms of quasinearly subharmonicity or directly under the assumption $\rho \le C\rho^\varepsilon$.

4A. *Counterexample for the L^p case.* We indicate in this subsection why Lemma 4.3 is no longer true when the L^1-norm is replaced with the L^p-norm for $p > 1$. This shows that the Hölder assumption of Corollary 4.4 cannot easily be relaxed.

We can see $\rho^\varepsilon(x)$ is like a weighted average of ρ over the ball $B_\varepsilon(x)$ and so heuristically we can see

$$\frac{\rho - \rho^\varepsilon}{\rho^\varepsilon} \simeq \frac{\rho(x) - (1/|B_\varepsilon|)\int_{B_\varepsilon(x)} \rho(y)\,\mathrm{d}y}{(1/|B_\varepsilon|)\int_{B_\varepsilon(x)} \rho(y)\,\mathrm{d}y}$$

(which is rigorous for $\eta_\varepsilon = (1/|B_\varepsilon|)\chi_{B_\varepsilon(0)}(x)$), and assuming the right-hand side is bounded and rearranging gives the condition (4-9). We see that a condition of the form

$$\left\| \frac{\rho(\,\cdot\,) - (1/|B_\varepsilon|)\int_{B_\varepsilon(\,\cdot\,)} \rho(y)\,\mathrm{d}y}{(1/|B_\varepsilon|)\int_{B_\varepsilon(\,\cdot\,)} \rho(y)\,\mathrm{d}y} \right\|_{L^p} < C,$$

in a sense a "relatively weighted L^p mean oscillation condition", could potentially be the weakest condition to control (4-1).

We notice that for the L^1 norm we obtain perfect cancellation in the fraction when calculating (4-8), as a mollifier acts like a local weighted average. However, when we perform the calculation in (4-8), but in L^p, then instead we obtain

$$\sum_{j=1}^n \frac{|B_\varepsilon|}{\omega_N} \frac{\|w\chi_{Q_j}\|_{L^p}}{\int_{Q_j} w\,\mathrm{d}x} = \sum_{j=1}^n \frac{|B_\varepsilon|}{\omega_N} \frac{\left(\int_{Q_j} w^p\,\mathrm{d}x\right)^{1/p}}{\int_{Q_j} w\,\mathrm{d}x}$$

and if we assume that $w = 1$ then we get $\sum_{j=1}^n (|B_\varepsilon|/\omega_N)|Q_j|^{1/p-1}$. As $1/p - 1 < 0$, for certain functions this term could blow up.

In fact if one chooses a function made of separated spikes where the supports get smaller and smaller then we can show this blow-up. We will formulate a simple counterexample so that it is in one dimension, discontinuous and nonnegative, though more regular counterexamples can be constructed in higher dimensions that are, for instance, even smooth and strictly positive.

Firstly, note that if we show that $\|f/f^\varepsilon\|_{L^p}$ blows up as $\varepsilon \to 0$ then $\|f/f^\varepsilon - f^\varepsilon/f^\varepsilon\|_{L^p}$ will also blow up. We can take $x \in \mathbb{T}$ and define our counterexample

$$f(x) := \sum_{i=1}^{\infty} \chi_{[1/i,\,1/i+1/2^i]}(x).$$

It is easy to see that $f \in B_p^{\alpha,\infty}(\mathbb{T})$ for $p > 1$ and any $0 < \alpha < 1-1/p$ by regularizing and using Lemma 2.49 from [Bahouri et al. 2011]. Thus we have the sum of separated spikes so they are further than $1/i^2$ apart yet have supports of size $1/2^i$. Let $\varepsilon = 1/(2i^2)$ and see that as f is nonnegative we can bound the sum below by just the i-th spike and see that as mollification only acts locally, the value on the denominator is only dependent on the i-th spike; thus we obtain

$$\left\|\frac{f}{f^\varepsilon}\right\|_{L^p(\mathbb{T})} \geq \left\|\frac{1}{f^\varepsilon}\right\|_{L^p(1/i,\,1/i+1/2^i)} = \|((\chi_{[1/i,\,1/i+1/2^i]})^{1/(2i^2)})^{-1}\|_{L^p(1/i,\,1/i+1/2^i)}. \tag{4-11}$$

We can then the bound mollification of $\chi_{[1/i,\,1/i+1/2^i]}$ in a similar method to (4-10) but in one dimension and so we can bound (4-11) below by

$$\left\|\frac{f}{f^\varepsilon}\right\|_{L^p(\mathbb{T})} \geq C\frac{2^i}{2i^2}\|1\|_{L^p(1/i,\,1/i+1/2^i)} = C\frac{2^i}{2i^2}2^{-i/p} = C\frac{2^{i(1-1/p)}}{2i^2}.$$

As f is the sum of infinitely many spikes there will exist an appropriate spike for any ε_i and thus we can send $i \to \infty$ and, as $1-1/p > 0$, we have $C2^{i(1-1/p)}/(2i^2) \to \infty$, which implies $\|f/f^\varepsilon\|_{L^p(\mathbb{T})} \to \infty$.

5. Energy conservation on domains with boundary

We have derived the local energy conservation equations on $(0,T)\times\mathbb{T}^d$ and so for an $\varphi \in C_c^\infty((0,T)\times\mathbb{T}^d)$ we have

$$\int_0^T\!\!\int_{\mathbb{T}^d} \partial_t\varphi\cdot\big(\tfrac{1}{2}\rho|u|^2 + P(\rho)\big) + \nabla\varphi\cdot\big[\big(\tfrac{1}{2}\rho|u|^2 + p(\rho) + P(\rho)\big)u\big]\,\mathrm{d}t\,\mathrm{d}x = 0. \tag{5-1}$$

The local energy equation is derived by taking momentum balance equations and testing with $(\varphi u^\varepsilon)^\varepsilon$ and using that mollification is symmetric to regularize the equation. For the continuity equation we just use φ^ε to test the equation and again move the mollification onto the equation. Once this is done, all the calculations are done locally on $\operatorname{supp}(\varphi)$.

When studying the isentropic Euler equations on a bounded domain with Lipschitz boundary Ω we have

$$\begin{aligned} \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla p(\rho) &= 0 \quad \text{in } [0,T]\times\Omega,\\ \partial_t\rho + \operatorname{div}(\rho u) &= 0 \quad \text{in } [0,T]\times\Omega,\\ u\cdot n &= 0 \quad \text{on } [0,T]\times\partial\Omega, \end{aligned} \tag{5-2}$$

where n denotes the outward normal vector field for $\partial\Omega$. For any $\varphi \in C_c^\infty((0,T)\times\Omega)$ we can find an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ we have φ^ε, $(\varphi u^\varepsilon)^\varepsilon \in C_c^\infty((0,T)\times\Omega)$, and so we can apply the same method as above to obtain a local energy equation on $(0,T)\times\Omega$ of the form (5-1). Here we are assuming the same conditions on u, ρ and p as in the previous theorems and in the corollaries in Sections 3 and 4, yet making the appropriate changes so that u and ρ are defined on the domain $(0,T)\times\Omega$ rather than $(0,T)\times\mathbb{T}^d$.

The following theorem and its proof follow ideas from [Bardos et al. 2018]:

Theorem 5.1. *Let ρ, u be a solution of* (5-2) *in the sense of distributions. Assume that ρ, u, and p satisfy the conditions necessary to derive the local energy equality* (5-1). *Assume further that $\rho\in L^\infty((0,T)\times\partial\Omega)$, $\partial\Omega$ is C^2, and $u\cdot n$ is continuous at the boundary. Then we have energy conservation on Ω; that is, for $\Theta(t)\in C_c^\infty(0,T)$*

$$\int_0^T\int_\Omega \partial_t\Theta(t)\cdot\big(\tfrac12\rho|u|^2+P(\rho)\big)\,\mathrm{d}t\,\mathrm{d}x=0 \tag{5-3}$$

and further if u, ρ are weakly continuous in time then

$$\int_\Omega \tfrac12\rho|u|^2(t_1,x)+P(\rho)(t_1,x)\,\mathrm{d}x=\int_\Omega \tfrac12\rho|u|^2(t_2,x)+P(\rho)(t_2,x)\,\mathrm{d}x \tag{5-4}$$

for any $t_1,t_2\in[0,T]$.

Proof. For any $\varphi\in C_c^\infty((0,T)\times\Omega)$ we can find an $\varepsilon_0>0$ such that for all $0<\varepsilon<\varepsilon_0$ we have φ^ε, $(\varphi u^\varepsilon)^\varepsilon\in C_c^\infty((0,T)\times\Omega)$ and so assuming sufficient regularity of ρ, u and p we obtain

$$\int_0^T\int_\Omega \partial_t\varphi\cdot\big(\tfrac12\rho|u|^2+P(\rho)\big)+\nabla\varphi\cdot\big[\big(\tfrac12\rho|u|^2+p(\rho)+P(\rho)\big)u\big]\,\mathrm{d}t\,\mathrm{d}x=0. \tag{5-5}$$

Let $\chi:\mathbb{R}^+\to\mathbb{R}$ be a nonnegative, smooth function such that

$$\chi(s):=\begin{cases}0 & \text{if } s<1,\\ 1 & \text{if } s>2,\end{cases}$$

and define for $x\in\overline{\Omega}$ the function $d_{\partial\Omega}(x)$ as the euclidean distance from x to the closest point on the boundary. We can then define for any $\delta>0$ the composition $\chi(d_{\partial\Omega}(x)/\delta)$ and see that as $\delta\to0$ so does $\chi(d_{\partial\Omega}(x)/\delta)\to\mathbb{1}_\Omega$. Further, let $\Theta(t)\in C_c^\infty(0,T)$.

We can for any $\delta>0$ let $\varphi(x,t)=\chi(d_{\partial\Omega}(x)/\delta)\Theta(t)$ in (5-5) and we obtain

$$\int_\Omega\chi\Big(\frac{d_{\partial\Omega}(x)}{\delta}\Big)\int_0^T\partial_t\Theta(t)\cdot\big(\tfrac12\rho|u|^2+P(\rho)\big)\,\mathrm{d}x\,\mathrm{d}t \\ +\int_0^T\Theta(t)\int_\Omega\nabla\chi\Big(\frac{d_{\partial\Omega}(x)}{\delta}\Big)\cdot\big[\big(\tfrac12\rho|u|^2+p(\rho)+P(\rho)\big)u\big]\,\mathrm{d}t\,\mathrm{d}x=0,$$

and by the chain rule we see that

$$\nabla\chi\Big(\frac{d_{\partial\Omega}(x)}{\delta}\Big)=\frac1\delta\chi'\Big(\frac{d_{\partial\Omega}(x)}{\delta}\Big)\nabla d_{\partial\Omega}(x),$$

and so

$$0=\int_\Omega\chi\Big(\frac{d_{\partial\Omega}(x)}{\delta}\Big)\int_0^T\partial_t\Theta(t)\cdot\big(\tfrac12\rho|u|^2+P(\rho)\big)\,\mathrm{d}x\,\mathrm{d}t \\ +\int_0^T\Theta(t)\int_\Omega\frac1\delta\chi'\Big(\frac{d_{\partial\Omega}(x)}{\delta}\Big)\nabla d_{\partial\Omega}(x)\cdot\big[\big(\tfrac12\rho|u|^2+p(\rho)+P(\rho)\big)u\big]\,\mathrm{d}t\,\mathrm{d}x. \tag{5-6}$$

As $\chi(d_{\partial\Omega}(x)/\delta) \to \mathbb{1}_\Omega$ strongly, the first integral on the right-hand side of (5-6) will converge to

$$\int_0^T\int_\Omega \partial_t\Theta(t)\cdot\left(\tfrac{1}{2}\rho|u|^2+P(\rho)\right)\mathrm{d}x\,\mathrm{d}t$$

as we wanted. All that is left is to show that the other term on the right-hand side of (5-6) vanishes in the limit.

As $\partial\Omega$ is C^2 we can use [Gilbarg and Trudinger 1977], specifically Lemma 14.16, to see that there exists an $a>0$ such that $d_{\partial\Omega}(x)\in C^2(\Gamma_a)$, where $\Gamma_a := \{x\in\overline{\Omega} : d_{\partial\Omega}(x)<a\}$. Further, in a similar argument to [Bardos et al. 2014, Section 7], when $x\in\Omega$ is sufficiently close to $\partial\Omega$, there exists a unique point $\hat{x}\in\partial\Omega$ such that $x=\hat{x}+n(\hat{x})\,d_{\partial\Omega}(x)$, where $n(\hat{x})$ is the unit outward normal to the boundary at x. We see that we can bound the modulus for the second term on the right-hand side of (5-6) by

$$\begin{aligned}\left\|\chi'\left(\frac{d_{\partial\Omega}}{\delta}\right)\right\|_{L^\infty}\int_0^T\Theta(t)\frac{1}{\delta}\int_{\Gamma_{2\delta}}|\nabla d_{\partial\Omega}(x)\cdot u|\left|\left(\tfrac{1}{2}\rho|u|^2+p(\rho)+P(\rho)\right)\right|\mathrm{d}t\,\mathrm{d}x&\\ \le C\int_0^T\Theta(t)\frac{1}{\delta}\int_{\Gamma_{2\delta}}|\nabla d_{\partial\Omega}(x)\cdot u|\,\mathrm{d}t\,\mathrm{d}x&\quad(5\text{-}7)\end{aligned}$$

as we know that $\|\chi'(d_{\partial\Omega}/\delta)\|_{L^\infty}\le C$ and by our assumptions $\left\|\left(\tfrac{1}{2}\rho|u|^2+p(\rho)+P(\rho)\right)\right\|_{L^\infty}\le C$ as well. For $2\delta<a$ we know that $d_{\partial\Omega}\in C^2$ and furthermore as $\nabla d_{\partial\Omega}\in C^1$, in the region $\Gamma_{2\delta}$, $|\nabla d_{\partial\Omega}(x)\cdot u|\to C|n(\hat{x})\cdot u(\hat{x})|$ as long as $u(x)\to u(\hat{x})$ as $x\to\hat{x}$, and for this the assumption that $u\cdot n$ is continuous at the boundary will suffice. Thus as $\partial\Omega$ is at least Lipschitz so $|\Gamma_{2\delta}|\le C\delta|\partial\Omega|$ and so we can apply the Lebesgue differentiation theorem to (5-7) and see that as $\delta\to 0$,

$$C\int_0^T\Theta(t)\frac{1}{\delta}\int_{\Gamma_{2\delta}}|\nabla d_{\partial\Omega}(x)\cdot u|\,\mathrm{d}t\,\mathrm{d}x\to C\int_0^T\Theta(t)\int_{\partial\Omega}|n(\hat{x})\cdot u(\hat{x})|\,\mathrm{d}t\,\mathrm{d}\hat{x}=0$$

as $n(\hat{x})\cdot u(\hat{x})=0$ and so we have shown (5-3).

We now want to show (5-4) with the extra assumptions of weak continuity in time of both u and ρ. To do this we define the sequence of functions $\Theta_\nu : [0,T]\to\mathbb{R}$ which are nonnegative and smooth, where for any point $t_1, t_2\in[0,T]$ with $t_1<t_2$ we have

$$\Theta_\nu(\tau) := \begin{cases}0 & \text{if } \tau<t_1+\nu \text{ or } \tau>t_2-\nu,\\ 1 & \text{if } \tau>t_1+2\nu \text{ or } \tau<t_2-2\nu,\end{cases}$$

and see similarly that as $\nu\to 0$ we have $\Theta_\nu(t)\to\mathbb{1}_{[t_1,t_2]}$. We see that $\Theta_\nu\in C_c^\infty(0,T)$ for every $\nu>0$ and so substituting this function into (5-6) we obtain

$$\int_0^T\int_\Omega \partial_t\Theta_\nu(t)\cdot\left(\tfrac{1}{2}\rho|u|^2+P(\rho)\right)\mathrm{d}t\,\mathrm{d}x=0$$

for every ν. From our choice of Θ_ν we see that

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d}\partial_t\Theta_\nu(t)\cdot\left(\tfrac{1}{2}\rho|u|^2+P(\rho)\right)\mathrm{d}t\,\mathrm{d}x=\int_{t_1}^{t_1+2\nu}\partial_t\Theta_\nu(t)\cdot\int_\Omega\left(\tfrac{1}{2}\rho|u|^2+P(\rho)\right)\mathrm{d}t\,\mathrm{d}x&\\ +\int_{t_2-2\nu}^{t_2}\partial_t\Theta_\nu(t)\cdot\int_\Omega\left(\tfrac{1}{2}\rho|u|^2+P(\rho)\right)\mathrm{d}t\,\mathrm{d}x.&\end{aligned}$$

We know that $\int_{t_1}^{t_1+2\nu} \partial_t \Theta_\nu(t)\,\mathrm{d}t = 1$ and $\int_{t_2-2\nu}^{t_2} \partial_t \Theta_\nu(t)\,\mathrm{d}t = -1$ by the fundamental theorem of calculus and as $\nu \to 0$ these terms approximate the identity at t_1 and t_2, and thus these terms converge to

$$\int_\Omega \tfrac{1}{2}\rho|u|^2(t_1,x) + P(\rho)(t_1,x)\,\mathrm{d}x \quad \text{and} \quad -\int_\Omega \tfrac{1}{2}\rho|u|^2(t_2,x) + P(\rho)(t_2,x)\,\mathrm{d}x$$

respectively, assuming weak continuity of ρ and u in time. Thus we are done. □

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