

# ANALYSIS & PDE

Volume 13

No. 3

2020

DORIN BUCUR, ILARIA FRAGALÀ AND ALESSANDRO GIACOMINI

**LOCAL MINIMALITY RESULTS FOR  
THE MUMFORD-SHAH FUNCTIONAL VIA MONOTONICITY**

# LOCAL MINIMALITY RESULTS FOR THE MUMFORD–SHAH FUNCTIONAL VIA MONOTONICITY

DORIN BUCUR, ILARIA FRAGALÀ AND ALESSANDRO GIACOMINI

Let  $\Omega \subseteq \mathbb{R}^2$  be a bounded piecewise  $C^{1,1}$  open set with convex corners, and let

$$MS(u) := \int_{\Omega} |\nabla u|^2 dx + \alpha \mathcal{H}^1(J_u) + \beta \int_{\Omega} |u - g|^2 dx$$

be the Mumford–Shah functional on the space  $SBV(\Omega)$ , where  $g \in L^\infty(\Omega)$  and  $\alpha, \beta > 0$ . We prove that the function  $u \in H^1(\Omega)$  such that

$$\begin{cases} -\Delta u + \beta u = \beta g & \text{in } \Omega, \\ \partial u / \partial \nu = 0 & \text{on } \partial \Omega \end{cases}$$

is a local minimizer of  $MS$  with respect to the  $L^1$ -topology. This is obtained as an application of interior and boundary monotonicity formulas for a weak notion of quasiminimizers of the Mumford–Shah energy. The local minimality result is then extended to more general free discontinuity problems taking into account also boundary conditions.

1. Introduction	865
2. Notation and preliminaries	869
3. Harmonic extension results	870
4. The monotonicity formula up to the boundary	878
5. The local minimality result	882
6. The case of general energies with boundary conditions	890
Acknowledgments	898
References	898

## 1. Introduction

The Mumford–Shah functional was introduced in [Mumford and Shah 1985; 1989] in the context of image segmentation and has found important applications in several other fields, including variational theories in fracture mechanics; see [Francfort and Marigo 1998; Bourdin et al. 2008]. It can be considered the typical example of a *free discontinuity functional*, characterized by the coupling of bulk and surface energies.

The *weak* formulation of the functional due to De Giorgi and Ambrosio [1988] takes the form

$$MS(u) := \int_{\Omega} |\nabla u|^2 dx + \alpha \mathcal{H}^1(J_u) + \beta \int_{\Omega} |u - g|^2 dx, \tag{1-1}$$

---

Bucur is member of the Institut Universitaire de France. Fragalà and Giacomini are members of the Gruppo Nazionale per L'Analisi Matematica, la Probabilità e loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).  
*MSC2010:* 35R35, 35A16, 28A75, 35J25, 35Q74, 49J45, 94A08.

*Keywords:* local minimality, monotonicity formulas, free discontinuity functionals.

where  $\Omega \subset \mathbb{R}^2$  is an open bounded set,  $\alpha, \beta > 0$  and  $g \in L^\infty(\Omega)$ . Here  $\mathcal{H}^1$  denotes the one-dimensional Hausdorff measure,  $u$  belongs to the class of *special functions of bounded variation*  $SBV(\Omega)$  (see Section 2), and  $J_u$  is the jump set of  $u$ . The last term involving  $g$  is usually called the *fidelity term*: in image segmentation,  $g$  is the level of gray of the picture, which has to be approximated by choosing conveniently the jump set  $J_u$  (the *edges*) and the function  $u$  outside it (*regularized image*). When  $\beta = 0$ , we speak of the homogeneous version of MS.

Within this framework, and in general dimension  $N$ , existence of minimizers can be proved easily through the direct method of the calculus of variations, as a direct application of Ambrosio’s compactness and lower semicontinuity theorem. Moreover, thanks to the regularity result of De Giorgi, Carriero and Leaci [De Giorgi et al. 1989], minimizers have a topologically closed jump set and are regular outside, yielding an admissible configuration of the original formulation of [Mumford and Shah 1985; 1989], in which the discontinuity set was considered as an independent variable. The regularity of the discontinuity set was then improved by Ambrosio, Fusco and Pallara, who proved that up to an  $\mathcal{H}^{N-1}$ -negligible set,  $J_u$  is a manifold of class  $C^{1,\delta}$  for any  $\delta < 1$  and of class  $C^{1,1}$  if  $N = 2$ ; see also [Bonnet 1996; David 1996].

The issue of detecting minimizers, or more generally *local minimizers* of the Mumford–Shah functional is very delicate. Since MS is only lower semicontinuous (with respect to the natural  $L^1$ -topology), necessary conditions for minimality cannot be obtained by “standard” differentiation.

First-order necessary conditions are established for jump sets  $\Gamma = J_u$  sufficiently regular by considering *inner variations* (see [Ambrosio et al. 2000, Chapter 7]): they yield that  $u$  satisfies an elliptic PDE outside  $\Gamma$ , while its (mean) curvature  $H_\Gamma$  is involved in a transmission condition coupling the values of  $u$  and  $\nabla u$  on both sides of  $\Gamma$ .

For the homogeneous version of MS, a second-order necessary condition for minimality has been proposed by Cagnetti, Mora and Morini [Cagnetti et al. 2008], involving the positive semidefiniteness of a suitable quadratic form defined on  $H_0^1(\Gamma)$ . Under strict positivity, the authors prove that  $u$  is a “local” minimizer among those functions  $v \in SBV(\Omega)$  such that  $J_v \subseteq \Phi(\Gamma)$  and  $v = u$  on  $\partial\Omega$ , where  $\Phi$  is any diffeomorphism of  $\mathbb{R}^N$  which is sufficiently  $C^2$ -close to the identity and such that  $\text{Id} - \Phi$  is compactly supported in  $\Omega$ . This local minimality was then extended to a full local minimality in the  $L^1$  topology in dimension  $N = 2$  by Bonacini and Morini [2015], employing a penalization/regularization technique together with results from the regularity theory of the area and the Mumford–Shah functional.

The aim of this paper is to show that, in dimension  $N = 2$  and under mild regularity assumptions on the geometry of  $\Omega$ , the Mumford–Shah functional (1-1) admits a natural local minimizer with no jumps. More precisely, the main result of the paper is the following.

**Theorem 1.1.** *Let  $\Omega \subseteq \mathbb{R}^2$  be an open, bounded, piecewise  $C^{1,1}$ -domain with convex corners. Then the function  $u \in H^1(\Omega)$  such that*

$$\begin{cases} -\Delta u + \beta u = \beta g & \text{in } \Omega, \\ \partial u / \partial \nu = 0 & \text{on } \partial\Omega \end{cases} \tag{1-2}$$

*is a local minimizer of MS with respect to the  $L^1$ -topology.*

In other words, the minimizer of MS within the class of Sobolev functions (which satisfies (1-2)) turns out to be a local minimizer for the natural  $L^1$ -topology in the full class of SBV competitors.

A similar result has been obtained by Chambolle, Ponsiglione and the third author [Chambolle et al. 2008] for a generalization of the homogeneous version of the Mumford–Shah functional under boundary conditions, motivated by the study of the issue of *crack initiation* in brittle materials within variational theories of crack propagation; see [Francfort and Marigo 1998; Bourdin et al. 2008]. The technique of [Chambolle et al. 2008] is based essentially on the maximum principle through a truncation argument, and can be used to deal with nonlinear energies for the gradient (with  $p$ -growth for example) and more general geometries for  $\Omega$ , but does not apply to the Mumford–Shah functional since it fails when the fidelity term is present.

Our proof of Theorem 1.1 is based on the use of the monotonicity formula introduced by Luckhaus and the first author [Bucur and Luckhaus 2014] for quasiminimizers of MS, which, in dimension 2, we extend up to boundary points and establish for a quite weak notion of quasiminimizers.

Namely we consider functions  $u \in \text{SBV}(\Omega)$  which are *weak local almost-quasiminimizers* of the Mumford–Shah energy with respect to their own jump sets, i.e., such that

$$\int_{B_\rho(x) \cap \Omega} |\nabla u|^2 dx + \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)) \leq \int_{B_\rho(x) \cap \Omega} |\nabla v|^2 dx + \Lambda \mathcal{H}^1(J_v \cap \bar{B}_\rho(x)) + c_\gamma \rho^{1+\gamma} \tag{1-3}$$

for every  $v \in \text{SBV}(\Omega)$  with  $\{v \neq u\} \subseteq B_\rho(x)$  and  $J_v \subseteq J_u$ , where  $x \in \bar{\Omega}$ ,  $\rho < \rho_0$  are such that

$$\int_{B_\rho(x) \cap \Omega} |\nabla u|^2 dx + \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)) \leq \rho,$$

while  $\Lambda \geq 1$  and  $\gamma, c_\gamma > 0$ . We then prove that if  $x \in \Omega$ , then the quantity

$$E_x(\rho) := \frac{\mathcal{E}_x(\rho)}{\rho} \wedge 1 + \frac{c_\gamma}{\gamma} \rho^\gamma \tag{1-4}$$

is nondecreasing on  $(0, \rho_0 \wedge \text{dist}(x, \partial\Omega))$ , where

$$\mathcal{E}_x(\rho) := \int_{\Omega \cap B_\rho(x)} |\nabla u|^2 dx + \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)),$$

while if  $x \in \partial\Omega$ , monotonicity holds true for the modification

$$\tilde{E}_x(\rho) := \frac{\mathcal{E}_x(\rho)}{\rho} \wedge 1 + \frac{c_\gamma}{\gamma} \rho^\gamma + k_\Omega \int_0^\rho \left( \frac{\mathcal{E}_x(r)}{r} \wedge 1 \right) dr \tag{1-5}$$

on the interval  $(0, \rho_0 \wedge r_\Omega \wedge \text{dist}(x, S \setminus \{x\}))$ , where  $S$  denotes the set of corners of  $\partial\Omega$  and  $r_\Omega, k_\Omega > 0$ .

The proof of these monotonicities follows that of [Bucur and Luckhaus 2014], which is based on a precise harmonic extension estimate on a ball (see Proposition 3.1). The dimension  $N = 2$  gives drastic simplifications in the arguments, and this is the main reason for which we can deal with a weaker class of quasiminimizers (with respect to their own jump set, see also Remark 4.2). The extension of the monotonicity to the boundary requires a generalization of the key harmonic extension estimate to domains of the form  $\Omega \cap B_\rho(x)$  (see Proposition 3.7): this motivates the piecewise  $C^{1,1}$ -regularity assumption for  $\partial\Omega$  and the

restriction to *convex* angles. We also refer the reader to [Chambolle et al. ≥ 2020] by Chambolle, Séré and Zanini, where a boundary monotonicity formula is proved on flat boundaries, under Neumann conditions.

The use of monotonicity in the proof of Theorem 1.1 is roughly as follows. Assuming by contradiction that there exists  $u_n \rightarrow u$  strongly in  $L^1(\Omega)$  such that  $MS(u_n) < MS(u)$ , it is easily seen (through Ambrosio’s theorem) that  $\mathcal{H}^1(J_{u_n}) = \varepsilon_n \rightarrow 0$ . We then consider the *constrained* minimization problem

$$\min_{\substack{w \in \text{SBV}(\Omega) \\ \mathcal{H}^1(J_w) \leq \varepsilon_n}} MS(w)$$

whose minimizers  $w_n$  are such that

$$MS(w_n) \leq MS(u_n) < MS(u). \tag{1-6}$$

Writing  $w_n = u + v_n$ , we can show that  $v_n$  satisfy the weak minimality property (1-3), so that monotonicity is available, and that

$$\int_{\Omega} |\nabla v_n|^2 dx + \mathcal{H}^1(J_{v_n}) \rightarrow 0. \tag{1-7}$$

Since  $J_{w_n} = J_{v_n}$ , inequality (1-6) can hold only for  $J_{v_n} \neq \emptyset$ . If  $x_n \in J_{v_n}$  is a point of density 1 with  $x_n \rightarrow x \in \Omega$ , monotonicity gives a strictly positive uniform lower bound for  $E_{x_n}(\rho)$  defined in (1-4). But (1-7) yields the existence of  $\rho_n$  such that  $E_{x_n}(\rho_n) \rightarrow 0$ , a contradiction. If  $x_n$  approaches  $\partial\Omega$ , boundary monotonicity for (1-5) can be employed to get the same conclusion.

In Section 6 we extend the minimality result to a free discontinuity functional of the type

$$F(u) := \int_{\Omega} A(x) \nabla u \cdot \nabla u dx + \int_{J_u} b(x, u^+, u^-, \nu_u) d\mathcal{H}^1 + \beta \int_{\Omega} |u - g|^2 dx \tag{1-8}$$

under suitable assumptions on the coefficient  $A$  and  $b$  (here  $\nu_u$  denotes the normal vector to  $J_u$ ,  $u^\pm$  are the two traces of  $u$  on the jump set, see Section 2) and considering also boundary condition of Dirichlet type on a part  $\Gamma_D$  of the boundary (here  $\beta = 0$  is allowed). Again, provided that  $\Omega$  satisfies some geometric assumptions (see Theorem 6.7), the minimizer of  $F$  within the class  $H^1(\Omega)$  under boundary conditions is a local minimizer with respect to the  $L^1$  topology in  $\text{SBV}(\Omega)$ . We thus recover for  $\beta = 0$  a particular case of the local minimality result of [Chambolle et al. 2008], and extend it to the case  $\beta > 0$ , i.e., in the presence of a fidelity term: by employing monotonicity instead of a maximum principle argument, we need to restrict to quadratic gradient energies and require additional geometric assumptions on  $\Gamma_D$  (see also Remark 6.6). Finally, as a numerical consequence of our result, it is worth observing that for any iterative local descent method the regularized image given by (1-2) cannot be used as initial point. In particular, the topological derivative of the Mumford–Shah functional will be nonnegative at any point; hence no jump can be naturally detected in this way.

The paper is organized as follows. In Section 2 we fix the notation and recall some basic facts of the space  $\text{SBV}$  used in the main proofs. In Section 3 we prove some preliminary results concerning harmonic extensions in corner domains which are pivotal to extend the monotonicity formula up to boundary points in Section 4. Section 5 is devoted to the proof of the main local minimality result, while the extension to the case with boundary conditions with the general form (1-8) is contained in Section 6.

## 2. Notation and preliminaries

In this section we fix the notation and recall some basic facts concerning the space SBV.

**General notation.** Throughout the paper  $B_\rho(x)$  will denote the open ball of center  $x \in \mathbb{R}^2$  and radius  $\rho > 0$ . We will write  $B_1$  for the ball of center 0 and radius 1, and set  $S^1 := \partial B_1$ . If  $E \subseteq \mathbb{R}^2$  and  $x \in \mathbb{R}^2$ ,  $\text{dist}(x, E)$  will stand for the distance between  $x$  and  $E$ . By  $\mathcal{H}^1$  we will denote the one-dimensional Hausdorff measure, which coincides with the usual length measure on sufficiently regular curves.

If  $\Omega \subseteq \mathbb{R}^2$  is open,  $L^p(\Omega)$  will denote the usual Lebesgue space of measurable functions which are  $p$ -summable, while  $H^1(\Omega)$  will stand for the Sobolev space of functions in  $L^2(\Omega)$  with square integrable gradient.

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is of class  $C^{1,1}$  if it belongs to  $C^1(\mathbb{R})$  and  $f'$  is Lipschitz continuous. We say that  $f$  is piecewise  $C^{1,1}$  if it is continuous and there exists a (locally finite) subdivision of  $\mathbb{R}$  such that on every associated open subinterval  $f$  is of class  $C^{1,1}$ . An open bounded domain  $\Omega \subseteq \mathbb{R}^2$  has a piecewise  $C^{1,1}$ -boundary if for every  $x \in \partial\Omega$  there exists a neighborhood  $U$  and a piecewise  $C^{1,1}$  function  $f$  on  $\mathbb{R}$  such that, up to a rotation,  $\Omega \cap U = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > f(x_1)\} \cap U$ .

Finally, for  $a, b \in \mathbb{R}$  we set

$$a \wedge b := \min\{a, b\} \quad \text{and} \quad a \vee b := \max\{a, b\}.$$

**Special functions of bounded variation.** Let  $\Omega \subseteq \mathbb{R}^N$  be an open set. The space  $\text{SBV}(\Omega)$  of special functions of bounded variation is given by all functions  $u \in L^1(\Omega)$  such that the distributional derivative  $Du$  of  $u$  can be represented as a vector-valued bounded Radon measure of the form

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \nu_u \mathcal{H}^{N-1} \llcorner J_u.$$

Here  $\mathcal{L}^N$  is the Lebesgue measure,  $\nabla u \in L^1(\Omega; \mathbb{R}^N)$  is the approximate gradient of  $u$ , and  $J_u$  is the jump set of  $u$ . The set  $J_u$  turns out to be countably  $\mathcal{H}^{N-1}$ -rectifiable; i.e., it is contained up to a set of  $\mathcal{H}^{N-1}$ -measure zero in the union of  $C^1$ -submanifolds of  $\mathbb{R}^N$ . It is possible to define  $\mathcal{H}^{N-1}$ -a.e. on  $J_u$  an approximate normal denoted by  $\nu_u$ , as well as traces  $u^\pm$ . We refer to [Ambrosio et al. 2000] for a detailed account of this topic.

The following result is fundamental when dealing with the analysis of the Mumford–Shah functional.

**Theorem 2.1** (Ambrosio’s theorem). *Let  $\Omega \subseteq \mathbb{R}^N$  be open and bounded, and let  $(u_n)_{n \in \mathbb{N}}$  be a sequence in  $\text{SBV}(\Omega)$  such that*

$$\|\nabla u_n\|_p + \mathcal{H}^{N-1}(J_{u_n}) + \|u_n\|_\infty \leq C,$$

where  $p > 1$  and  $C \geq 0$ . Then, there exist a subsequence  $(u_{n_k})_{k \in \mathbb{N}}$  and a function  $u \in \text{SBV}(\Omega)$  such that

$$\begin{aligned} \nabla u_{n_k} &\rightharpoonup \nabla u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^N), \\ \mathcal{H}^{N-1}(J_u) &\leq \liminf_{k \rightarrow +\infty} \mathcal{H}^{N-1}(J_{u_{n_k}}), \\ u_{n_k} &\rightarrow u \quad \text{strongly in } L^1(\Omega). \end{aligned}$$

### 3. Harmonic extension results

In this section we collect some harmonic extension estimates on different kinds of domains.

We start by recalling the interior harmonic extension estimate for a disk, which was already exploited in [Bucur and Luckhaus 2014] to obtain the monotonicity formula. As a straightforward extension, we generalize the estimate to the case of circular sectors. Then, through a careful deformation procedure, we handle the case of what we call “corner domains”, namely the epigraphs of functions with a corner-type singularity which obey suitable estimates. Finally, via a localization argument, we cover the case of admissible  $C^{1,1}$  domains, which will be a crucial tool to extend the monotonicity formula up to the boundary. Below and throughout this section, we denote by  $\nabla_\tau$  the tangential gradient.

Let us start with the simple case of the unit disk  $B_1$ .

**Proposition 3.1** (harmonic extension estimate in a disk). *Let  $w \in H^1(\partial B_1)$ , and let  $h_w \in H^1(B_1)$  be its harmonic extension. Then*

$$\int_{B_1} |\nabla h_w|^2 dx \leq \int_{\partial B_1} |\nabla_\tau w|^2 d\mathcal{H}^1. \tag{3-1}$$

*Proof.* Several proofs of this inequality are available in the literature; see for instance [Bucur and Luckhaus 2014] for a proof in  $N$  dimensions. For further needs, we give below a short two-dimensional proof employing Fourier expansions. Let us write

$$w(\vartheta) = \sum_n [a_n \cos(n\vartheta) + b_n \sin(n\vartheta)].$$

The harmonic extension on  $B_1$  is given in polar coordinates by

$$h_w(r, \vartheta) = \sum_n r^n [a_n \cos(n\vartheta) + b_n \sin(n\vartheta)].$$

A direct computation shows that

$$\int_{B_1} |\nabla h_w|^2 dx = \pi \sum_n n(a_n^2 + b_n^2) \quad \text{and} \quad \int_{\partial B_1} |\nabla_\tau w|^2 d\mathcal{H}^1 = \pi \sum_n n^2(a_n^2 + b_n^2),$$

so that the estimate easily follows. □

Let us generalize the estimate (3-1) to the case of circular sectors. Given  $\vartheta_0 \in [0, 2\pi]$ , we consider the circular unit sector and associated arc given in polar coordinates by

$$S_{\vartheta_0} := \left\{ (r, \vartheta) : r \in [0, 1], \vartheta \in \left[ -\frac{\vartheta_0}{2}, \frac{\vartheta_0}{2} \right] \right\} \quad \text{and} \quad \Gamma_{\vartheta_0} := \left\{ (r, \vartheta) : r = 1, \vartheta \in \left[ -\frac{\vartheta_0}{2}, \frac{\vartheta_0}{2} \right] \right\}.$$

The following extension of Proposition 3.1 holds true.

**Proposition 3.2** (harmonic extension estimate in a sector). *Let  $w \in H^1(\Gamma_{\vartheta_0})$ , and let  $h_w \in H^1(S_{\vartheta_0})$  denote a harmonic extension of  $w$ . Then the inequality*

$$\int_{S_{\vartheta_0}} |\nabla h_w|^2 dx \leq \int_{\Gamma_{\vartheta_0}} |\nabla_\tau w|^2 d\mathcal{H}^1$$

*holds true provided that one of the following assumptions is fulfilled:*

- (a)  $\vartheta_0 \in ]0, \pi]$  and  $h_w$  satisfies Neumann conditions or Dirichlet homogeneous conditions on  $\partial S_{\vartheta_0} \setminus \Gamma_{\vartheta_0}$ .
- (b)  $\vartheta_0 \in ]0, \pi/2]$  and  $h_w$  satisfies homogeneous Dirichlet conditions for  $\vartheta = -\vartheta_0/2$  and Neumann conditions for  $\vartheta = \vartheta_0/2$ .

Above, a harmonic function  $h \in H^1(S_{\vartheta_0})$  is said to satisfy Neumann conditions on a portion of the boundary  $\Gamma_N \subset \partial S_{\vartheta_0}$  and Dirichlet conditions  $g$  on the complement  $\Gamma_D = \partial S_{\vartheta_0} \setminus \Gamma_N$  for some function  $g \in H^1(S_{\vartheta_0})$  if  $h$  is a minimizer of

$$\min \left\{ \int_{S_{\vartheta_0}} |\nabla v|^2 dx : v \in H^1(S_{\vartheta_0}), v = g \text{ on } \Gamma_D \right\}.$$

*Proof.* Let us consider the case of homogeneous Dirichlet conditions. We may develop  $w$  as

$$w = \sum_{j=1}^{\infty} c_j \sin \left[ j \frac{\pi}{\vartheta_0} \left( \vartheta + \frac{\vartheta_0}{2} \right) \right]$$

so that the associated harmonic extension takes the form

$$h_w = \sum_{j=1}^{\infty} c_j r^{j\pi/\vartheta_0} \sin \left[ j \frac{\pi}{\vartheta_0} \left( \vartheta + \frac{\vartheta_0}{2} \right) \right].$$

A straightforward computation for the energies of the gradients in polar coordinates shows that

$$\frac{\pi}{\vartheta_0} \int_{S_{\vartheta_0}} |\nabla h_w|^2 dx \leq \int_{\Gamma_{\vartheta_0}} |\nabla_{\tau} w|^2 d\mathcal{H}^1,$$

so that the result follows.

Let us come to the case of Neumann conditions. The case of the semicircle, i.e.,  $\vartheta_0 = \pi$ , is easily obtained through an even extension of  $w$  to  $\partial B_1$  and employing [Proposition 3.1](#). If  $\vartheta_0 \in ]0, \pi[$ , we pass from  $w$  to a function  $\tilde{w}$  by prolonging the constant values of the extremes. Applying the inequality on the semicircle we may write

$$\int_{S_{\vartheta_0}} |\nabla h_w|^2 dx \leq \int_{S_{\vartheta_0}} |\nabla h_{\tilde{w}}|^2 dx \leq \int_{S_{\pi}} |\nabla h_{\tilde{w}}|^2 dx \leq \int_{\Gamma_{\pi}} |\nabla_{\tau} \tilde{w}|^2 d\mathcal{H}^1 = \int_{\Gamma_{\vartheta_0}} |\nabla_{\tau} w|^2 d\mathcal{H}^1,$$

so that the result follows, and point (a) is proved.

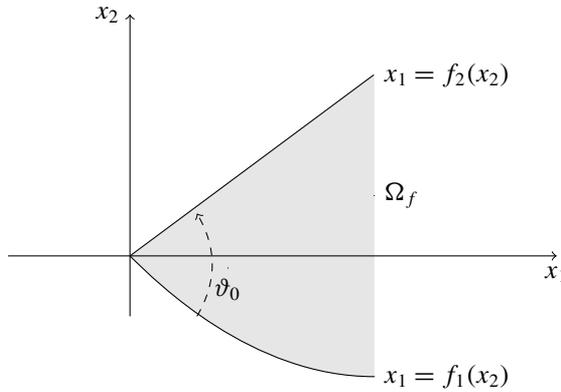
Let us come finally to point (b) which follows by combing the idea used above. If  $\vartheta_0 = \pi/2$ , we can perform an even extension to a semicircle and employ the harmonic estimate with Dirichlet conditions of point (a), and then restrict to  $\Gamma_{\pi/2}$  to get the result. If  $\vartheta_0 \in ]0, \pi/2[$ , we can proceed by extension to  $\Gamma_{\pi/2}$  and using the associated inequality. □

We are now going to consider the case of *corner domains*. By corner domain, we mean the epigraph (see [Figure 1](#))

$$\Omega_f := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > f(x_2)\} \tag{3-2}$$

of a given function  $f$  of the form

$$f(x) := \begin{cases} f_1(x) & \text{if } x < 0, \\ f_2(x) & \text{if } x \geq 0, \end{cases} \tag{3-3}$$



**Figure 1.** The domain  $\Omega_f$ .

with  $f_1 \in C^1(]-\infty, 0])$  and  $f_2 \in C^1([0, +\infty[)$  nonnegative functions satisfying

$$f_1(0) = f_2(0) = 0, \quad f'_2(0) = -f'_1(0). \tag{3-4}$$

We define

$$\vartheta_0(\Omega_f) := \text{angle of } \Omega_f \text{ at the origin}, \tag{3-5}$$

meant as the angle at the origin of the sector  $\{x_1 \geq f'_1(0)x_2\} \cap \{x_1 \geq f'_2(0)x_2\}$ . In particular,  $\vartheta_0(\Omega_f) = \pi$  in the case  $\partial\Omega_f$  is smooth ( $f'_1(0) = f'_2(0) = 0$ ). We also set

$$\partial_1\Omega_f := \{(f_1(x), x) : x < 0\} \cap \bar{B}_1(0) \quad \text{and} \quad \partial_2\Omega_f := \{(f_2(x), x) : x \geq 0\} \cap \bar{B}_1(0). \tag{3-6}$$

The following result holds true.

**Proposition 3.3** (harmonic extension estimate in corner domains). *For  $\rho > 0$ , let  $\Omega_{f_\rho}$  be a family of corner domains as in (3-2), where each function  $f_\rho$  is of the kind (3-3)–(3-4). Assume further that:*

- The left and right derivatives of  $f_\rho$  at the origin are independent of  $\rho$ ; i.e.,

$$f'_{\rho,2}(0) = -f'_{\rho,1}(0) =: \lambda \geq 0. \tag{3-7}$$

- $f'_{\rho,1}, f'_{\rho,2}$  satisfy the following Lipschitz-type estimates for some constant  $c > 0$ :

$$\begin{aligned} \text{for all } x \in [-1, 0], \quad & |f'_{\rho,1}(x) + \lambda| \leq c\rho|x|, \\ \text{for all } x \in [0, 1], \quad & |f'_{\rho,2}(x) - \lambda| \leq c\rho x. \end{aligned} \tag{3-8}$$

Then there exist  $\rho_0 > 0$  and  $c_0 > 0$  such that, for every  $\rho \in (0, \rho_0]$  and  $w \in H^1(\Omega_{f_\rho} \cap \partial B_1(0))$ , denoting by  $h_w$  a harmonic extension of  $w$  to  $\Omega_{f_\rho} \cap B_1(0)$ , it holds that

$$(1 - c_0\rho) \int_{\Omega_{f_\rho} \cap B_1(0)} |\nabla h_w|^2 dx \leq \int_{\Omega_{f_\rho} \cap \partial B_1(0)} |\nabla_\tau w|^2 d\mathcal{H}^1, \tag{3-9}$$

provided that  $\vartheta_0 := \vartheta_0(\Omega_{f_\rho})$  (the inner angle at the origin of  $\Omega_{f_\rho}$ ) and  $h_w$  fulfill one of the following assumptions:

- (a)  $\vartheta_0 \in ]0, \pi]$  and  $h_w$  satisfies Neumann conditions or Dirichlet homogeneous conditions on  $\partial_1 \Omega_{f_\rho} \cup \partial_2 \Omega_{f_\rho}$ .
- (b)  $\vartheta_0 \in ]0, \pi/2]$  and  $h_w$  satisfies homogeneous Dirichlet conditions on  $\partial_1 \Omega_{f_\rho}$  and Neumann conditions on  $\partial_2 \Omega_{f_\rho}$ .

In order to prove the previous proposition, we need some preliminary work. Let us consider the planar domain given in polar coordinates  $(r, \vartheta)$  by

$$C_{\vartheta_0, g} := \left\{ (r, \vartheta) : r \in [0, 1], \vartheta \in \left[ 0, \frac{\vartheta_0}{2} + g(r) \right] \right\}, \tag{3-10}$$

where

$$\vartheta_0 \in ]0, \pi] \quad \text{and} \quad g \in C^1([0, 1]), \quad g(0) = 0, \quad g(r) \in \left[ -\frac{\vartheta_0}{2}, -\frac{\vartheta_0}{2} + \frac{\pi}{2} \right]. \tag{3-11}$$

The domain  $C_{\vartheta_0, g}$  can be mapped to the unit sector

$$S_{\vartheta_0}^+ := \left\{ (r, \vartheta) : r \in [0, 1], \vartheta \in \left[ 0, \frac{\vartheta_0}{2} \right] \right\}$$

by means of the transformation defined by

$$T : C_{\vartheta_0, g} \rightarrow S_{\vartheta_0}^+, \quad T(r, \vartheta) := \left( r, \frac{\vartheta_0}{2} \frac{\vartheta}{\vartheta_0/2 + g(r)} \right), \tag{3-12}$$

with inverse

$$T^{-1}(r', \vartheta') := \left( r', \frac{2}{\vartheta_0} \left( \frac{\vartheta_0}{2} + g(r') \right) \vartheta' \right).$$

Notice that  $T$  maps  $C_{\vartheta_0, g} \cap \partial B_1(0)$  onto  $\Gamma_{\vartheta_0}^+ := \{(r, \vartheta) : r = 1, \vartheta \in [0, \vartheta_0/2]\}$ . Accordingly, with a given function  $v$  defined on  $C_{\vartheta_0, g}$  (resp. on  $C_{\vartheta_0, g} \cap \partial B_1(0)$ ), we can associate the function  $v^\sharp$  defined on  $S_{\vartheta_0}^+$  (resp.  $\Gamma_{\vartheta_0}^+$ ) by

$$v^\sharp := v \circ T^{-1}. \tag{3-13}$$

**Lemma 3.4.** *For  $\rho > 0$ , let  $C_{\vartheta_0, g_\rho}$  be a family of domains as in (3-10), with  $\vartheta_0$  (independent of  $\rho$ ) and  $g_\rho$  as in (3-11). Assume further that  $g'_\rho$  satisfy the following  $L^\infty$  estimate for some constant  $c > 0$ :*

$$|g'_\rho(r)| \leq c\rho \quad \text{for every } r \in [0, 1]. \tag{3-14}$$

Let  $T : C_{\vartheta_0, g_\rho} \rightarrow S_{\vartheta_0}^+$  be defined as in (3-12). There exist  $\rho_0 > 0$  and  $c_0 > 0$  such that for every  $\rho \in (0, \rho_0]$  the following items hold true:

- (a) If  $v^\sharp \in H^1(S_{\vartheta_0}^+)$  is associated with  $v \in H^1(C_{\vartheta_0, g_\rho})$  as in (3-13), then

$$\int_{S_{\vartheta_0}^+} |\nabla v^\sharp|^2 dx' \geq (1 - c_0\rho) \int_{C_{\vartheta_0, g_\rho}} |\nabla v|^2 dx.$$

- (b) If  $w^\sharp \in H^1(\Gamma_{\vartheta_0}^+)$  is associated with  $w \in H^1(C_{\vartheta_0, g} \cap \partial B_1(0))$  as in (3-13), then

$$\int_{\Gamma_{\vartheta_0}^+} |\nabla_\tau w^\sharp|^2 d\mathcal{H}^1 \leq (1 + c_0\rho) \int_{C_{\vartheta_0, g_\rho} \cap \partial B_1(0)} |\nabla_\tau w|^2 d\mathcal{H}^1.$$

*Proof.* Let us prove item (a). We write for simplicity  $g$  in place of  $g_\rho$ . By (3-13), we have

$$v(r, \vartheta) = v^\sharp(T(r, \vartheta)) = v^\sharp\left(r, \frac{\vartheta_0}{2} \frac{\vartheta}{\vartheta_0/2 + g(r)}\right).$$

Computing the partial derivatives of the function  $v$ , we obtain

$$\begin{aligned} \partial_r v(r, \vartheta) &= \partial_{r'} v^\sharp(T(r, \vartheta)) + \partial_{\vartheta'} v^\sharp(T(r, \vartheta)) \frac{\vartheta_0}{2} \frac{-\vartheta g'(r)}{(\vartheta_0/2 + g(r))^2}, \\ \partial_\vartheta v(r, \vartheta) &= \partial_{\vartheta'} v^\sharp(T(r, \vartheta)) \frac{\vartheta_0}{2} \frac{1}{\vartheta_0/2 + g(r)}, \end{aligned}$$

so that

$$\begin{aligned} \partial_{r'} v^\sharp(T(r, \vartheta)) &= \partial_r v(r, \vartheta) + \frac{\vartheta g'(r)}{\vartheta_0/2 + g(r)} \partial_\vartheta v(r, \vartheta), \\ \partial_{\vartheta'} v^\sharp(T(r, \vartheta)) &= \frac{2}{\vartheta_0} \left(\frac{\vartheta_0}{2} + g(r)\right) \partial_\vartheta v(r, \vartheta). \end{aligned} \tag{3-15}$$

Since the jacobian of  $T$  is given by

$$J_T(r, \vartheta) = \frac{\vartheta_0}{2} \frac{1}{\vartheta_0/2 + g(r)},$$

the change of variable formula together with (3-15) yields

$$\int_{S_{\vartheta_0}^+} \left[ (\partial_{r'} v^\sharp)^2 + \left(\frac{\partial_{\vartheta'} v^\sharp}{r'}\right)^2 \right] r' dr' d\vartheta' = \int_{C_{\vartheta_0, g}} \left[ (\partial_r v)^2 + \left(\frac{\partial_\vartheta v}{r}\right)^2 + e(\rho) \right] \frac{\vartheta_0}{2} \frac{1}{\vartheta_0/2 + g(r)} r dr d\vartheta, \tag{3-16}$$

where

$$e(\rho) := 2 \frac{\vartheta g'(r)}{\vartheta_0/2 + g(r)} \partial_r v \partial_\vartheta v + \left(\frac{\vartheta g'(r)}{\vartheta_0/2 + g(r)}\right)^2 (\partial_\vartheta v)^2 + \left[\left(\frac{2}{\vartheta_0} \left(\frac{\vartheta_0}{2} + g(r)\right)\right) - 1\right] \left(\frac{\partial_\vartheta v}{r}\right)^2. \tag{3-17}$$

It is easy to check that there exist  $\rho_0, \tilde{c} > 0$  (depending only on the constant  $c$  in (3-14) and on  $\vartheta_0$ ) such that for every  $\rho \in (0, \rho_0]$

$$-\tilde{c}\rho \left[ (\partial_r v)^2 + \left(\frac{\partial_\vartheta v}{r}\right)^2 \right] \leq e(\rho) \leq \tilde{c}\rho \left[ (\partial_r v)^2 + \left(\frac{\partial_\vartheta v}{r}\right)^2 \right]. \tag{3-18}$$

Indeed, let us show for instance how the first term on the right-hand side of (3-17) can be handled (the other terms being similar). Notice first that the assumption (3-14) gives also  $|g(r)| \leq c\rho$  for every  $r \in [0, 1]$ . Then, taking  $\rho$  sufficiently small we obtain, for a suitable constant  $\tilde{c} > 0$ ,

$$\begin{aligned} \left| 2 \frac{\vartheta g'(r)}{\vartheta_0/2 + g(r)} \partial_r v \partial_\vartheta v \right| &\leq \left| \frac{\vartheta g'(r)}{\vartheta_0/2 + g(r)} \right| \left[ r^2 (\partial_r v)^2 + \left(\frac{\partial_\vartheta v}{r}\right)^2 \right] \\ &\leq \frac{(\vartheta_0/2 + c\rho)c\rho}{\vartheta_0/2 - c\rho} \left[ (\partial_r v)^2 + \left(\frac{\partial_\vartheta v}{r}\right)^2 \right] \leq \tilde{c}\rho \left[ (\partial_r v)^2 + \left(\frac{\partial_\vartheta v}{r}\right)^2 \right]. \end{aligned}$$

In view of (3-18), coming back to (3-16) we get

$$\begin{aligned} \int_{S_{\vartheta_0}^+} \left[ (\partial_r v^\sharp)^2 + \left( \frac{\partial_{\vartheta'} v^\sharp}{r'} \right)^2 \right] r' dr' d\vartheta' &\geq (1 - \tilde{c}\rho) \int_{C_{\vartheta_0}} \left[ (\partial_r v)^2 + \left( \frac{\partial_{\vartheta} v}{r} \right)^2 \right] \frac{\vartheta_0}{2} \frac{1}{\vartheta_0/2 + g(r)} r dr d\vartheta \\ &\geq \frac{1 - \tilde{c}\rho}{1 - (2/\vartheta_0)c\rho} \int_{C_{\vartheta_0,g}} \left[ (\partial_r v)^2 + \left( \frac{\partial_{\vartheta} v}{r} \right)^2 \right] r dr d\vartheta, \end{aligned}$$

from which part (a) of the statement follows.

The proof of part (b) is analogous, by using the lower bound inequality in (3-18) in place of the upper bound one. □

We are now in a position to prove Proposition 3.3.

*Proof of Proposition 3.3.* Throughout the proof we write for simplicity  $f$  in place of  $f_\rho$  and  $g$  in place of  $g_\rho$ . We claim that

$$\Omega_f^+ := \Omega_f \cap B_1(0) \cap \{x_2 > 0\}$$

corresponds to the domain  $C_{\vartheta_0,g}$  defined in (3-10), with  $\vartheta_0$  and  $g$  satisfying (3-11) and (3-14), if we take

$$\vartheta_0 = \vartheta_0(\Omega_f), \quad g(r) := \arccos\left(\frac{f_2(x_2)}{\sqrt{x_2^2 + f_2(x_2)^2}}\right) - \frac{\vartheta_0}{2}, \tag{3-19}$$

where  $\vartheta_0(\Omega_f)$  is the angle of  $\Omega_f$  at the origin defined according to (3-5), and  $x_2 = x_2(r)$  is obtained by inverting the relation

$$r = \sqrt{x_2^2 + f_2(x_2)^2}. \tag{3-20}$$

To prove the claim notice firstly that  $x_2(r)$  is well-defined because (3-20) defines a bijection if  $\rho$  is sufficiently small: if  $f_2'(0) \neq 0$ , this follows immediately from (3-8); if  $f_2'(0) = 0$ , setting  $h(x_2) := x_2^2 + f_2(x_2)^2$  and using again (3-8), we get

$$h'(x_2) = 2x_2 + 2f_2(x_1)f_2'(x_2) \geq 2x_2(1 - c_2^2\rho^2) > 0 \quad \text{for } x_2 > 0 \text{ and } \rho \ll 1.$$

Next observe that, since by assumption  $f_2$  is nonnegative, the domain  $\Omega_f^+$  can be written in polar coordinates  $(r, \vartheta)$  as

$$0 \leq r \leq 1, \quad 0 \leq \vartheta \leq \varphi(r) := \arccos\left(\frac{f_2(x_2)}{\sqrt{x_2^2 + f_2(x_2)^2}}\right) \tag{3-21}$$

(with  $x_2 = x_2(r)$  as above). In view of (3-21), and noticing that the function  $\varphi$  satisfies

$$\varphi(0^+) = \arccos\left(\frac{f_2'(0)}{\sqrt{1 + f_2'(0)^2}}\right) = \frac{\vartheta_0(\Omega_f)}{2},$$

we see that  $\Omega_f^+$  corresponds to  $C_{\vartheta_0,g}$  with  $\vartheta_0$  and  $g$  as in (3-19). To achieve the proof of the claim, it remains to check that  $\vartheta_0$  and  $g$  satisfy (3-11) and (3-14). Since  $\vartheta_0$  is the angle of  $\Omega_f$  at the origin, recalling that  $f_1$  and  $f_2$  are nonnegative by assumption, it is clear that  $\vartheta_0 \in [0, \pi[$ . It is also immediate from the above definition of  $g$  that  $g \in C^1([0, 1])$  and that  $g(0) = \varphi(0) - \vartheta_0/2 = 0$ . Moreover, since

the function  $\varphi$  in (3-21) takes values into  $[0, \pi/2]$ ,  $g$  satisfies the bounds in (3-11). Finally, taking into account (3-8), a straightforward computation shows that there exists  $c > 0$  such that (3-14) is fulfilled.

A similar change of variable can be operated on the set  $\Omega_f^- := \Omega_f \cap B_1(0) \cap \{x_2 < 0\}$ . We conclude that we can map  $\Omega_f \cap B_1(0)$  onto the sector  $S_{\vartheta_0}$ , with an estimate on the  $L^2$ -norm of the gradients according to Lemma 3.4.

We are now ready to prove the estimate (3-9). Let us consider firstly the case of Neumann conditions. Given  $w \in H^1(\Omega_f \cap \partial B_1(0))$ , let  $w^\sharp \in H^1(\Gamma_{\vartheta_0})$  be associated with  $w$  according to (3-13). If  $h_{w^\sharp}$  is the harmonic extension of  $w^\sharp$  to  $S_{\vartheta_0}$  with Neumann conditions on  $\partial S_{\vartheta_0} \setminus \Gamma_{\vartheta_0}$ , thanks to Proposition 3.2(a) we have

$$\int_{S_{\vartheta_0}} |\nabla h_{w^\sharp}|^2 dx \leq \int_{\Gamma_{\vartheta_0}} |\nabla_\tau w^\sharp|^2 d\mathcal{H}^1. \tag{3-22}$$

Coming back to the original domain, if we consider the function  $h \in H^1(\Omega_f \cap B_1(0))$  defined by  $h := h_{w^\sharp} \circ T$  (so that  $h^\sharp = h_{w^\sharp}$ ), by using in the order Lemma 3.4(a), (3-22), and Lemma 3.4(b), we infer

$$\begin{aligned} (1 - c_0\rho) \int_{\Omega_f \cap B_1(0)} |\nabla h|^2 dx &\leq \int_{S_{\vartheta_0}} |\nabla h_{w^\sharp}|^2 dx \\ &\leq \int_{\Gamma_{\vartheta_0}} |\nabla_\tau w^\sharp|^2 d\mathcal{H}^1 \leq (1 + c_0\rho) \int_{\Omega_f \cap \partial B_1(0)} |\nabla_\tau w|^2 d\mathcal{H}^1. \end{aligned}$$

Since the harmonic extension  $h_w$  of  $w$  which satisfies Neumann conditions on  $\partial_1 \Omega_f \cup \partial_2 \Omega_f$  satisfies the inequality

$$\int_{\Omega_f \cap B_1(0)} |\nabla h_w|^2 dx \leq \int_{\Omega_f \cap B_1(0)} |\nabla h|^2 dx,$$

the result follows.

The case of Dirichlet homogeneous conditions or mixed Dirichlet/Neumann conditions can be handled in a similar way, by employing the corresponding inequalities on sectors established in Proposition 3.2.  $\square$

In order to handle more general geometries and also boundary conditions, the following definition will be useful.

**Definition 3.5** (admissible domains). Let  $\Omega \subset \mathbb{R}^2$  be an open bounded set with a piecewise  $C^{1,1}$ -boundary. Let  $S$  denote the set of corners of  $\partial\Omega$ , and let  $\gamma_1, \dots, \gamma_k$  be the open arcs (connected components) of  $\partial\Omega \setminus S$ . We will say that  $\Omega$  is *admissible* if  $\partial\Omega$  can be decomposed as

$$\partial\Omega = \Gamma_D \cup \Gamma_N \cup S, \tag{3-23}$$

where  $\Gamma_D \cup \Gamma_N$  is a partition of  $\{\gamma_1, \dots, \gamma_k\}$  and satisfies the following conditions:

- (i) The angle formed (on the side of  $\Omega$ ) by any pair of consecutive arcs in  $\Gamma_D$  or in  $\Gamma_N$  is less than or equal to  $\pi$ .
- (ii) The angle formed (on the side of  $\Omega$ ) by any arc of  $\Gamma_D$  adjacent to an arc of  $\Gamma_N$  is less than or equal to  $\pi/2$ .

**Remark 3.6.** In our further considerations  $\Gamma_D$  and  $\Gamma_N$  stand for the arcs with Dirichlet and Neumann boundary conditions respectively. For  $\Gamma_D = \partial\Omega$  or  $\Gamma_N = \partial\Omega$ , admissibility according to the above definition reduces to the assumption that  $\Omega$  has convex corners. In particular, smooth  $C^{1,1}$  domains are admissible.

**Proposition 3.7** (harmonic extension estimate in admissible domains). *Let  $\Omega \subset \mathbb{R}^2$  be an open bounded set with a piecewise  $C^{1,1}$ -boundary, which is admissible according to Definition 3.5. Let  $\partial\Omega$  be decomposed as in (3-23). There exist  $\rho_\Omega > 0$  and  $c_\Omega > 0$  such that, for every  $x \in \partial\Omega$  and every  $\rho < \rho_\Omega \wedge \text{dist}(x, S \setminus \{x\})$ , if  $w \in H^1(\Omega \cap \partial B_\rho(x))$  and  $h_w$  denotes a harmonic extension of  $w$  to  $\Omega \cap B_\rho(x)$ , it holds that*

$$\frac{(1 - c_\Omega \rho)}{\rho} \int_{\Omega \cap B_\rho(x)} |\nabla h_w|^2 dx \leq \int_{\Omega \cap \partial B_\rho(x)} |\nabla_\tau w|^2 d\mathcal{H}^1, \tag{3-24}$$

provided one of the following assumptions is fulfilled:

- (a)  $h_w$  satisfies Neumann conditions or homogeneous Dirichlet conditions on  $\partial\Omega \cap B_\rho(x)$ .
- (b)  $h_w$  satisfies homogeneous Dirichlet conditions on  $\Gamma_D \cap B_\rho(x)$  and Neumann conditions on  $\Gamma_N \cap B_\rho(x)$ .

*Proof.* Let us first consider the case when  $h_w$  satisfies Neumann conditions on  $\partial\Omega \cap B_\rho(x)$ . Since  $\Omega$  is compact and with a piecewise  $C^{1,1}$ -boundary, there exists  $\rho_\Omega > 0$  such that, for every  $x \in \partial\Omega$  and  $\rho < \rho_\Omega$ , we have that  $\Omega \cap B_\rho(x)$  is given in a suitable coordinate system with center in  $x$  by the intersection with  $B_\rho(x)$  of the epigraph of a piecewise  $C^{1,1}$ -function  $f_x : \mathbb{R} \rightarrow \mathbb{R}$ . In view of the compactness of  $\partial\Omega$ , it is not restrictive to assume

$$\sup_{x \in \partial\Omega} \|f_x\|_{C^{1,1}_{pw}} = C < +\infty. \tag{3-25}$$

Let  $x \in \partial\Omega$  be a smooth point, and let

$$\rho < \rho_\Omega \wedge \text{dist}(x, S).$$

Up to a translation and a rotation, we may assume that  $x = 0$  and that  $\Omega \cap B_\rho(x)$  is given by

$$\Omega_f \cap B_\rho(0),$$

where

$$\Omega_f := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > f(x_2)\}$$

for a suitable  $f \in C^{1,1}(\mathbb{R})$  with  $f(0) = f'(0) = 0$  and  $\|f\|_{C^{1,1}} \leq C$ . If we rescale to unit size, i.e., we consider the map

$$x \mapsto x/\rho,$$

then  $\Omega_f \cap B_\rho(0)$  is transformed into the set  $\Omega_{f_\rho} \cap B_1(0)$ , where

$$f_\rho(s) := \frac{1}{\rho} f(\rho s).$$

Since  $f'_\rho(s) = f'(\rho s)$  and  $f(0) = f'(0) = 0$ , in view of (3-25) we deduce that for every  $s \in [-1, 1]$

$$|f'_\rho(s)| \leq c\rho|s|.$$

Hence, the family of functions  $f_\rho$  satisfies conditions (3-7) and (3-8). Therefore, up to reducing  $\rho_\Omega$ , we can apply Proposition 3.3 (under assumption (a), the case of Neumann conditions) to the function

$$\tilde{w}(y) := w(\rho y) \in H^1(\Omega_{f_\rho} \cap \partial B_1(0)).$$

Rescaling back to size  $\rho$ , we obtain easily the result.

If  $x$  is a convex corner for  $\partial\Omega$ , the proof is similar, using again Proposition 3.3 (under assumption (a), the case of Neumann conditions).

The cases when  $h_w$  satisfies homogeneous Dirichlet conditions on  $\partial\Omega \cap B_\rho(x)$ , or homogeneous Dirichlet conditions on  $\Gamma_D \cap B_\rho(x)$  and Neumann conditions on  $\Gamma_N \cap B_\rho(x)$ , can be settled in the analogous way, by using the parts of Proposition 3.3 in which the corresponding boundary conditions are considered. □

**Remark 3.8.** An inspection in the proof of Proposition 3.7 shows that the constants  $c_\Omega, \rho_\Omega$  remain bounded if  $\Omega$  is replaced by the domain  $L(\Omega)$  where  $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  varies in a family of linear transformations with bounded norm. This observation will be useful in Section 6.

### 4. The monotonicity formula up to the boundary

In this section we prove a monotonicity formula up to boundary points for a suitable notion of quasiminimizers of the Mumford–Shah functional in dimension 2. We point out that this notion is much weaker than the classical one employed in [Bucur and Luckhaus 2014], since the family of test functions is much smaller. The results of this section are typically two-dimensional and cannot, a priori, be extended in  $N$  dimensions.

Let us start with the following definition.

**Definition 4.1** (weak minimizers). Let  $\Omega \subseteq \mathbb{R}^2$  be an open set, and  $u \in \text{SBV}(\Omega)$ . We say that  $u$  is a weak local almost-quasiminimizer of the Mumford–Shah energy with respect to its own jump set at the point  $x \in \bar{\Omega}$  if there exist  $\rho_x > 0, \gamma > 0, c_\gamma > 0$  and  $\Lambda \geq 1$  such that for every  $\rho < \rho_x$  with

$$\int_{\Omega \cap B_\rho(x)} |\nabla u|^2 dx + \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)) \leq \rho \tag{4-1}$$

and for every  $v \in \text{SBV}(\Omega)$  with  $\{v \neq u\} \subseteq B_\rho(x)$  and  $J_v \subseteq J_u$  we have

$$\int_{\Omega \cap B_\rho(x)} |\nabla u|^2 dx + \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)) \leq \int_{\Omega \cap B_\rho(x)} |\nabla v|^2 dx + \Lambda \mathcal{H}^1(J_v \cap \bar{B}_\rho(x)) + c_\gamma \rho^{1+\gamma}. \tag{4-2}$$

**Remark 4.2.** Notice that local almost-quasiminimizers of the Mumford–Shah functional considered in [Bucur and Luckhaus 2014] (in particular absolute minimizers) are weak local almost-quasiminimizers with respect to their own jump set: indeed they satisfy the minimality property (4-2) on every ball  $B_\rho(x)$  (with  $\rho$  sufficiently small) not necessarily satisfying the energy bound (4-1), and for every competitor  $v$  such that  $\{v \neq u\} \subseteq B_\rho(x)$ , without the restriction  $J_v \subseteq J_u$ .

With a given weak local almost-quasiminimizer  $u$  according to the previous definition, we associate for every  $x \in \bar{\Omega}$  and  $\rho > 0$  the quantity

$$\mathcal{E}_x(\rho) := \int_{\Omega \cap B_\rho(x)} |\nabla u|^2 dx + \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)). \tag{4-3}$$

Our monotonicity formula at interior points reads as follows.

**Theorem 4.3** (interior monotonicity). *Let  $\Omega \subset \mathbb{R}^2$  be open, and let  $u \in \text{SBV}(\Omega)$  be a weak local almost-quasiminimizer for the Mumford–Shah energy with respect to its own jump set at the point  $x \in \Omega$  according to Definition 4.1. Let  $\mathcal{E}_x(\rho)$  be associated with  $u$  as in (4-3).*

Then the quantity

$$E_x(\rho) := \frac{\mathcal{E}_x(\rho)}{\rho} \wedge 1 + \frac{c_\gamma}{\gamma} \rho^\gamma \tag{4-4}$$

is nondecreasing on  $(0, \rho_x \wedge \text{dist}(x, \partial\Omega))$ .

This formula appears to have a similar expression to the one in [Bucur and Luckhaus 2014], but applies to a much weaker notion of minimizer. In particular there is no natural upper bound for  $\mathcal{E}_x(\rho)/\rho$  as in [Bucur and Luckhaus 2014] (such a bound is usually obtained by using as a test for minimality the function  $u 1_{\Omega \setminus B_\rho(x)}$ , which is not a priori an admissible competitor for weak local almost-quasiminimizers with respect to their own jump set).

Coming to boundary points, we are going to state a monotonicity formula for domains which are admissible according to Definition 3.5. First, let us consider the case of  $C^{1,1}$ -domains with convex corners (see Remark 3.6), with no imposition of Dirichlet boundary condition. In the case of a flat boundary, such a monotonicity formula has been obtained in [Chambolle et al.  $\geq$  2020], in which case  $k_\Omega = 0$ .

**Theorem 4.4** (boundary monotonicity). *Let  $\Omega \subset \mathbb{R}^2$  be an open, bounded, piecewise  $C^{1,1}$ -domain with convex corners, and let  $u \in \text{SBV}(\Omega)$  be a weak local almost-quasiminimizer for the Mumford–Shah energy with respect to its own jump set at the point  $x \in \partial\Omega$  according to Definition 4.1. Let  $\mathcal{E}_x(\rho)$  be associated with  $u$  as in (4-3).*

Then there exist  $r_\Omega > 0$  and  $k_\Omega \geq 0$  such that the quantity

$$\tilde{E}_x(\rho) := \frac{\mathcal{E}_x(\rho)}{\rho} \wedge 1 + \frac{c_\gamma}{\gamma} \rho^\gamma + k_\Omega \int_0^\rho \left( \frac{\mathcal{E}_x(r)}{r} \wedge 1 \right) dr \tag{4-5}$$

is nondecreasing on  $(0, \rho_x \wedge r_\Omega \wedge \text{dist}(x, S \setminus \{x\}))$ , where  $S$  denotes the set of corners of  $\partial\Omega$ .

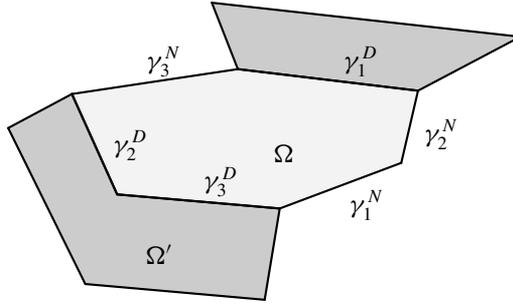
To formulate a monotonicity result which takes into account also boundary conditions, we consider the case when  $\Omega$  is an admissible domain according to Definition 3.5, and homogeneous Dirichlet boundary conditions are imposed on the (nonempty) portion  $\Gamma_D$  of its boundary. To this aim, let us consider an open bounded domain  $\Omega'$  such that  $\Omega \Subset \Omega' \subset \mathbb{R}^2$  and with (see Figure 2)

$$\partial\Omega \cap \Omega' = \Gamma_D. \tag{4-6}$$

Setting

$$\mathcal{A}_0 := \{u \in \text{SBV}(\Omega') : u = 0 \text{ on } \Omega' \setminus \bar{\Omega}\}. \tag{4-7}$$

we adapt the notion of weak local almost-quasiminimizers as follows.



**Figure 2.** The domain  $\Omega'$ .

**Definition 4.5** (weak minimizers in  $\mathcal{A}_0$ ). We say that  $u \in \mathcal{A}_0$  is a weak local almost-quasiminimizer in  $\mathcal{A}_0$  of the Mumford–Shah energy with respect to its own jump set at the point  $x \in \bar{\Omega}$  if there exist  $\rho_x > 0$ ,  $\gamma > 0$ ,  $c_\gamma > 0$  and  $\Lambda \geq 1$  such that for every  $\rho < \rho_x$  with

$$\int_{\Omega \cap B_\rho(x)} |\nabla u|^2 dx + \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)) \leq \rho$$

and for every  $v \in \mathcal{A}_0$  with  $\{v \neq u\} \subseteq B_\rho(x)$  and  $J_v \subseteq J_u$  we have

$$\int_{\Omega \cap B_\rho(x)} |\nabla v|^2 dx + \mathcal{H}^1(J_v \cap \bar{B}_\rho(x)) \leq \int_{\Omega \cap B_\rho(x)} |\nabla u|^2 dx + \Lambda \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)) + c_\gamma \rho^{1+\gamma}.$$

The following variant of [Theorem 4.4](#) holds true.

**Theorem 4.6** (boundary monotonicity in  $\mathcal{A}_0$ ). Let  $\Omega \subset \mathbb{R}^2$  be an admissible domain according to [Definition 3.5](#), and let  $u \in \mathcal{A}_0$  be a weak local almost-quasiminimizer in  $\mathcal{A}_0$  for the Mumford–Shah energy with respect to its own jump set at the point  $x \in \partial\Omega$  according to [Definition 4.5](#). Let  $\mathcal{E}_x(\rho)$  be associated with  $u$  as in (4-3).

Then there exist  $r_\Omega > 0$  and  $k_\Omega > 0$  such that the quantity

$$\tilde{E}_x(\rho) := \frac{\mathcal{E}_x(\rho)}{\rho} \wedge 1 + \frac{c_\gamma}{\gamma} \rho^\gamma + k_\Omega \int_0^\rho \left( \frac{\mathcal{E}_x(r)}{r} \wedge 1 \right) dr \tag{4-8}$$

is nondecreasing on  $(0, \rho_x \wedge r_\Omega \wedge \text{dist}(x, S \setminus \{x\}))$ , where  $S$  denotes the set of corners of  $\partial\Omega$ .

In order to establish these results, we need to revisit the proof of the monotonicity formula given in [\[Bucur and Luckhaus 2014\]](#). The new ingredients here are the jump constraint (for which the two-dimensional setting is crucial) and the fact that the point  $x$  can belong to the boundary. We start with the boundary case, which contains the relevant modifications with respect to the case treated in [\[Bucur and Luckhaus 2014\]](#), and then we go back to the (simpler) interior case.

*Proof of Theorems 4.4 and 4.6.* Let us consider first the case of [Theorem 4.4](#). Following [\[Bucur and Luckhaus 2014\]](#), to prove the result it is enough to show that there exist  $r_\Omega > 0$  and  $k_\Omega > 0$  such that, for  $\rho \in (0, \rho_x \wedge r_\Omega \wedge \text{dist}(x, S \setminus \{x\}))$ , it holds that  $\tilde{E}'_x(\rho) \geq 0$  at almost every differentiability point  $\rho$  of  $\tilde{E}_x$

such that  $\mathcal{E}_x(\rho) < \rho$ . Let  $\rho_\Omega$  and  $c_\Omega$  be as in [Proposition 3.7](#), and let us choose  $k_\Omega = 2c_\Omega$  and  $r_\Omega < \rho_\Omega$ . We argue by contradiction. Assume that

$$\tilde{E}'_x(\rho) = \frac{\mathcal{E}'_x(\rho)}{\rho} - \frac{\mathcal{E}_x(\rho)}{\rho^2} + c_\gamma \rho^{\gamma-1} + k_\Omega \frac{\mathcal{E}_x(\rho)}{\rho} < 0,$$

so that

$$\rho \mathcal{E}'_x(\rho) + c_\gamma \rho^{\gamma+1} + k_\Omega \rho \mathcal{E}_x(\rho) < \mathcal{E}_x(\rho). \tag{4-9}$$

In particular, we infer  $\mathcal{E}'_x(\rho) < 1$  so that from

$$\int_{\Omega \cap \partial B_\rho(x)} |\nabla u|^2 dx + \mathcal{H}^0(J_u \cap \partial B_\rho(x)) \leq \mathcal{E}'_x(\rho) < 1$$

we get  $J_u \cap \partial B_\rho(x) = \emptyset$ . This means that the restriction of the SBV function  $u$  on  $\partial B_\rho(x) \cap \Omega$  is a Sobolev function which we denote by  $w$ . Notice that [\(4-9\)](#) gives

$$\rho \int_{\Omega \cap \partial B_\rho(x)} |\nabla_\tau w|^2 d\mathcal{H}^1 + c_\gamma \rho^{\gamma+1} + k_\Omega \rho \mathcal{E}_x(\rho) < \mathcal{E}_x(\rho). \tag{4-10}$$

Let  $h_w$  denote a harmonic extension  $h_w$  of  $w$  to  $\Omega \cap B_\rho(x)$  which satisfies Neumann conditions on  $\partial\Omega \cap B_\rho(x)$ . By [Proposition 3.7](#), we have

$$\frac{(1 - c_\Omega \rho)}{\rho} \int_{\Omega \cap B_\rho(x)} |\nabla h_w|^2 dx \leq \int_{\Omega \cap \partial B_\rho(x)} |\nabla_\tau w|^2 d\mathcal{H}^1.$$

From [\(4-10\)](#) we deduce that

$$(1 - c_\Omega \rho) \int_{\Omega \cap B_\rho(x)} |\nabla h_w|^2 dx + c_\gamma \rho^{\gamma+1} + k_\Omega \rho \mathcal{E}_x(\rho) < \mathcal{E}_x(\rho).$$

Up to reducing  $r_\Omega$  if necessary, we infer in particular

$$\int_{\Omega \cap B_\rho(x)} |\nabla h_w|^2 dx \leq 2\mathcal{E}_x(\rho)$$

so that for every  $\rho < r_\Omega \wedge \text{dist}(x, S \setminus \{x\})$

$$\int_{\Omega \cap B_\rho(x)} |\nabla h_w|^2 dx + c_\gamma \rho^{\gamma+1} + (k_\Omega - 2c_\Omega) \rho \mathcal{E}_x(\rho) < \mathcal{E}_x(\rho). \tag{4-11}$$

We now consider the admissible competitor for  $u$  given by

$$v(y) := \begin{cases} h_w(y) & \text{if } y \in \Omega \cap B_\rho(x), \\ u(y) & \text{otherwise.} \end{cases} \tag{4-12}$$

Notice that  $J_v \subseteq J_u$ , since  $v$  has no jumps inside  $\Omega \cap B_\rho(x)$  and coincides with  $u$  on  $\Omega \cap \partial B_\rho(x)$ . Then, recalling that we have chosen  $k_\Omega = 2c_\Omega$ , inequality [\(4-11\)](#) for  $\rho < \rho_x \wedge r_\Omega \wedge \text{dist}(x, S \setminus \{x\})$  contradicts the weak local almost-quasiminimality property of  $u$  according to [Definitions 4.1](#) (recall that  $\mathcal{E}_x(\rho) < \rho$ ).

Coming to [Theorem 4.6](#), we can follow the previous arguments by considering the harmonic extension  $h_w$  which satisfies homogeneous Dirichlet conditions on  $\Gamma_D \cap B_\rho(x)$  and Neumann conditions on  $\Gamma_N \cap B_\rho(x)$ , and using again [Proposition 3.7](#). The contradiction then follows by noting that the competitor  $v$  in [\(4-12\)](#) belongs to  $\mathcal{A}_0$ . □

**Remark 4.7.** Notice that the uniformity property of Remark 3.8 holds also for the constants  $r_\Omega$  and  $k_\Omega$ .

*Proof of Theorem 4.3.* The proof reduces essentially to the original case of [Bucur and Luckhaus 2014] by noting that, since we work in dimension 2, the key competitors involved in the arguments turn out to have a jump set contained in  $J_u$ . More precisely, we can follow the proof of Theorem 4.4, by using the estimate of Proposition 3.1 in place of that of Proposition 3.7: in this way we can choose  $k_\Omega = 0$  and obtain the monotonicity for the energy in the simpler form (4-4). □

### 5. The local minimality result

Let  $\Omega \subseteq \mathbb{R}^2$  be a bounded open set, and let

$$MS(u) := \int_\Omega |\nabla u|^2 dx + \alpha \mathcal{H}^1(J_u) + \beta \int_\Omega |u - g|^2 dx$$

denote the Mumford–Shah functional on  $SBV(\Omega)$ , where  $g \in L^\infty(\Omega)$  and  $\alpha, \beta > 0$ .

The following result holds true.

**Theorem 5.1** (small jump sets are not convenient). *Let  $\Omega \subseteq \mathbb{R}^2$  be an open, bounded, piecewise  $C^{1,1}$ -domain with convex corners and let  $g \in L^\infty(\Omega)$ . Let  $U \in H^1(\Omega)$  be the solution to*

$$\min_{v \in H^1(\Omega)} \int_\Omega |\nabla v|^2 dx + \beta \int_\Omega |v - g|^2 dx,$$

*i.e., such that*

$$\begin{cases} -\Delta U + \beta U = \beta g & \text{in } \Omega, \\ \partial U / \partial \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{5-1}$$

*Then there exists  $\varepsilon > 0$  such that for every  $u \in SBV(\Omega)$  with  $\mathcal{H}^1(J_u) < \varepsilon$  we have*

$$MS(U) < MS(u).$$

The main result of the paper, already stated in the Introduction as Theorem 1.1, is a simple consequence of the previous theorem. For convenience, we restate it hereafter:

**Theorem 5.2** (local minimality in  $L^1$ ). *Under the assumptions of Theorem 5.1, the function  $U$  is a local minimizer for the Mumford–Shah functional in  $SBV(\Omega)$  with respect to the  $L^1$ -topology.*

*Proof.* Assume by contradiction that there exists  $v_n \in SBV(\Omega)$  such that

$$v_n \rightarrow U \quad \text{strongly in } L^1(\Omega),$$

with

$$MS(v_n) < MS(U). \tag{5-2}$$

By truncation we may assume that  $\|v_n\|_\infty \leq \|g\|_\infty$ , so that the convergence holds also in  $L^2(\Omega)$ . By (5-2), we may apply Ambrosio’s theorem (see Theorem 2.1) and deduce

$$\int_\Omega |\nabla U|^2 dx \leq \liminf_n \int_\Omega |\nabla v_n|^2 dx,$$

so that, since the fidelity terms are converging, we infer  $\lim_n \mathcal{H}^1(J_{v_n}) = 0$ : but then (5-2) is in contradiction with Theorem 5.1.  $\square$

**Remark 5.3.** Under suitable regularity assumptions on  $g$ , it has been proved in [Alberti et al. 2003, Sections 5.1 and 5.3] by using the *calibration method* that  $U$  is a *global minimizer* for MS if  $\beta$  is sufficiently small or if  $\beta$  is sufficiently large.

The proof of Theorem 5.1 rests on a suitable use of the monotonicity formulas up to the boundary developed in Section 4.

*Proof of Theorem 5.1.* First of all, by considering the change of variable  $x \mapsto \sqrt{\alpha}x$ , it is not restrictive to assume  $\alpha = 1$ .

Let  $u_\varepsilon \in \text{SBV}(\Omega)$  be a minimizer of

$$\min_{\substack{u \in \text{SBV}(\Omega) \\ \mathcal{H}^1(J_u) \leq \varepsilon}} \text{MS}(u).$$

We shall prove by a contradiction argument that  $J_{u_\varepsilon} = \emptyset$  for  $\varepsilon$  small enough, so that  $u_\varepsilon = U$  and the proof follows.

Let  $\varepsilon_n$  be an infinitesimal sequence, and let us denote by  $u_n$  the corresponding functions  $u_{\varepsilon_n}$ . By truncation we may assume that

$$\|u_n\|_\infty \leq \|g\|_\infty \quad \text{and} \quad \|U\|_\infty \leq \|g\|_\infty. \tag{5-3}$$

Comparing  $u_n$  with the zero function we get

$$\int_\Omega |\nabla u_n|^2 dx + \mathcal{H}^1(J_{u_n}) + \beta \int_\Omega |u_n - g|^2 dx \leq \beta \int_\Omega g^2 dx. \tag{5-4}$$

We will concentrate on

$$v_n := u_n - U.$$

We divide the proof into several steps.

Step 1: regularity for  $U$ . In view of the bound (5-3) and of the regularity of  $\partial\Omega$ , we infer from the elliptic problem (5-1) satisfied by  $U$  that  $U \in H^2(\Omega)$ ; see [Grisvard 1985, Theorem 3.2.1.3 and Remark 3.2.4.6]. In particular we deduce that  $\nabla U \in L^p(\Omega)$  for every  $p > 1$ . Then we may write for every  $x \in \bar{\Omega}$ ,  $\rho > 0$  and  $p > 4$

$$\int_{\Omega \cap B_\rho(x)} |\nabla U|^2 dx \leq \left( \int_{\Omega \cap B_\rho(x)} |\nabla U|^p dx \right)^{2/p} |\Omega \cap B_\rho(x)|^{1-2/p}.$$

As a consequence, for every  $\gamma \in (0, 1)$  we get the estimate

$$\int_{\Omega \cap B_\rho(x)} |\nabla U|^2 dx \leq c_1 \rho^{1+\gamma} \tag{5-5}$$

for some  $c_1 > 0$  (here,  $c_1$  depends on  $\gamma, \Omega, \beta, g$ , but not on  $\rho$ ).

Step 2: weak local almost-quasiminimality of  $v_n$  with respect to its own jump set. Let us show that there exist  $\delta > 0$  and  $c_\delta > 0$  such that for every  $x \in \bar{\Omega}$ , for every  $\rho \leq 1$  with

$$\int_{\Omega \cap B_\rho(x)} |\nabla v_n|^2 dx + \mathcal{H}^1(J_{v_n} \cap \bar{B}_\rho(x)) \leq \rho, \tag{5-6}$$

and for every  $v \in \text{SBV}(\Omega)$  with  $\{v \neq v_n\} \subseteq B_\rho(x)$  and  $\mathcal{H}^1(J_v) \leq \varepsilon_n$  we have

$$\int_{\Omega \cap B_\rho(x)} |\nabla v_n|^2 dx + \mathcal{H}^1(J_{v_n} \cap \bar{B}_\rho(x)) \leq \int_{\Omega \cap B_\rho(x)} |\nabla v|^2 dx + \mathcal{H}^1(J_v \cap \bar{B}_\rho(x)) + c_\delta \rho^{1+\delta}. \tag{5-7}$$

In particular we get that  $v_n$  is a weak local almost-quasiminimizer of the Mumford–Shah energy with respect to its own jump set at any point  $x \in \bar{\Omega}$  according to [Definition 4.1](#).

Recalling [\(5-3\)](#) we have  $\|v_n\|_\infty \leq 2\|g\|_\infty$ , so that it is not restrictive to assume that also

$$\|v\|_\infty \leq 2\|g\|_\infty. \tag{5-8}$$

Moreover, we may assume

$$\int_{\Omega \cap B_\rho(x)} |\nabla v|^2 dx < \int_{\Omega \cap B_\rho(x)} |\nabla v_n|^2 dx + \mathcal{H}^1(J_{v_n} \cap \bar{B}_\rho(x)), \tag{5-9}$$

since otherwise [\(5-7\)](#) is immediately satisfied.

Since

$$\{v + U \neq u_n\} \subseteq B_\rho(x)$$

by the minimality of  $u_n$  we get

$$\begin{aligned} \int_{\Omega \cap B_\rho(x)} |\nabla v_n + \nabla U|^2 dx + \mathcal{H}^1(J_{v_n} \cap \bar{B}_\rho(x)) + \beta \int_{\Omega \cap B_\rho(x)} |u_n + U - g|^2 dx \\ \leq \int_{\Omega \cap B_\rho(x)} |\nabla v + \nabla U|^2 dx + \mathcal{H}^1(J_v \cap \bar{B}_\rho(x)) + \beta \int_{\Omega \cap B_\rho(x)} |v + U - g|^2 dx \end{aligned}$$

so that

$$\int_{\Omega \cap B_\rho(x)} |\nabla v_n|^2 dx + \mathcal{H}^1(J_{v_n} \cap \bar{B}_\rho(x)) \leq \int_{\Omega \cap B_\rho(x)} |\nabla v|^2 dx + \mathcal{H}^1(J_v \cap \bar{B}_\rho(x)) + c(\rho),$$

where

$$c(\rho) := 2 \int_{\Omega \cap B_\rho(x)} \nabla U \cdot (\nabla v - \nabla v_n) dx + \beta \int_{\Omega \cap B_\rho(x)} |v + U - g|^2 dx - \beta \int_{\Omega \cap B_\rho(x)} |v_n + U - g|^2 dx.$$

Recalling [\(5-3\)](#) and [\(5-8\)](#), we have

$$\int_{\Omega \cap B_\rho(x)} |v + U - g|^2 dx - \int_{\Omega \cap B_\rho(x)} |v_n + U - g|^2 dx \leq c_2 \rho^2 \tag{5-10}$$

for some  $c_2 > 0$ . Thanks to the estimate [\(5-5\)](#) for  $\nabla U$  obtained in Step 1, and taking into account [\(5-9\)](#) and [\(5-6\)](#), we have

$$\begin{aligned} \left| \int_{\Omega \cap B_\rho(x)} \nabla U \cdot (\nabla v - \nabla v_n) dx \right| &\leq \|\nabla U\|_{L^2(\Omega \cap B_\rho(x); \mathbb{R}^2)} (\|\nabla v\|_{L^2(\Omega \cap B_\rho(x); \mathbb{R}^2)} + \|\nabla v_n\|_{L^2(\Omega \cap B_\rho(x); \mathbb{R}^2)}) \\ &\leq c_3 \rho^{(1+\gamma)/2} \rho^{1/2} = c_3 \rho^{1+\gamma/2} \end{aligned} \tag{5-11}$$

for some  $c_3 > 0$ .

Collecting (5-10) and (5-11), we get for every  $\rho \leq 1$

$$c(\rho) \leq c_\delta \rho^{1+\delta}$$

for  $\delta = \gamma/2$  and  $c_\delta > 0$ , so that inequality (5-7) is proved.

Step 3: vanishing energy. We claim that

$$\int_\Omega |\nabla v_n|^2 dx + \mathcal{H}^1(J_{v_n}) \rightarrow 0. \tag{5-12}$$

Indeed, in view of (5-4) and of (5-3) we may apply Ambrosio’s theorem (see Theorem 2.1) to the sequence  $(u_n)_{n \in \mathbb{N}}$ : there exists  $u \in \text{SBV}(\Omega)$  such that up to a subsequence

$$\begin{aligned} u_n &\rightarrow u \quad \text{strongly in } L^2(\Omega), \\ \nabla u_n &\rightharpoonup \nabla u \quad \text{weakly in } L^2(\Omega; \mathbb{R}^2) \end{aligned} \tag{5-13}$$

and

$$\mathcal{H}^1(J_u) \leq \liminf_n \mathcal{H}^1(J_{u_n}) = 0. \tag{5-14}$$

In particular,  $u \in H^1(\Omega)$ . Moreover, by the minimality of  $u_n$ , we deduce that in the limit

$$\int_\Omega |\nabla u|^2 dx + \beta \int_\Omega |u - g|^2 dx \leq \int_\Omega |\nabla \varphi|^2 dx + \beta \int_\Omega |\varphi - g|^2 dx$$

for every  $\varphi \in H^1(\Omega)$ , which yields  $u = U$ . Passing to the limit in the inequality

$$\int_\Omega |\nabla u_n|^2 dx + \mathcal{H}^1(J_{u_n}) + \beta \int_\Omega |u_n - g|^2 dx \leq \int_\Omega |\nabla U|^2 dx + \beta \int_\Omega |U - g|^2 dx,$$

we deduce

$$\limsup_n \int_\Omega |\nabla u_n|^2 dx \leq \int_\Omega |\nabla U|^2 dx,$$

which together with the weak convergence (5-13) yields

$$\nabla u_n \rightarrow \nabla U \quad \text{strongly in } L^2(\Omega; \mathbb{R}^2).$$

Recalling that  $v_n = u_n - U$  and that  $\mathcal{H}^1(J_{v_n}) = \mathcal{H}^1(J_{u_n}) = \varepsilon_n$ , claim (5-12) follows.

Step 4: conclusion. We can now conclude the proof via a contradiction argument. Assume that  $\mathcal{H}^1(J_{u_n}) > 0$  for every  $n$ . Since  $J_{v_n} = J_{u_n}$ , this implies that for every  $n$

$$\mathcal{H}^1(J_{v_n}) > 0. \tag{5-15}$$

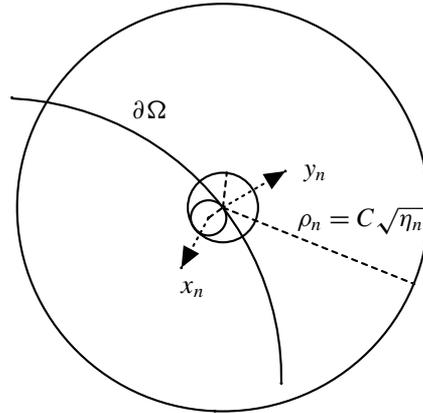
To derive a contradiction from (5-15) when  $n$  is large enough, we consider for every  $n$  a point  $x_n \in J_{v_n}$  of density 1, i.e., such that

$$\lim_{\rho \rightarrow 0^+} \frac{\mathcal{H}^1(J_{v_n} \cap B_\rho(x_n))}{2\rho} = 1. \tag{5-16}$$

Let  $y_n \in \partial\Omega$  be a projection of  $x_n$  on  $\partial\Omega$ ; since  $\Omega$  has convex corners,  $y_n$  is a smooth point of  $\partial\Omega$ . Let us set

$$d_n := \text{dist}(x_n, \partial\Omega) = |x_n - y_n| \quad \text{and} \quad d'_n := \text{dist}(y_n, S),$$

where  $S$  is the set of corners of  $\Omega$ .



**Figure 3.** Illustration of Case 2.

For  $x \in \bar{\Omega}$ , we set

$$\mathcal{E}_x^n(\rho) := \int_{\Omega \cap B_\rho(x)} |\nabla v_n|^2 dx + \mathcal{H}^1(J_{v_n} \cap \bar{B}_\rho(x)).$$

Notice that, in view of Step 3,

$$\eta_n := \int_{\Omega} |\nabla v_n|^2 dx + \mathcal{H}^1(J_{v_n}) \rightarrow 0 \tag{5-17}$$

and

$$\mathcal{E}_x^n(\rho) \leq \eta_n. \tag{5-18}$$

We distinguish four cases.

Case 1: Assume that

$$\limsup_n \frac{d_n}{\sqrt{\eta_n}} > 0. \tag{5-19}$$

Thanks to Step 2 and the interior monotonicity formula of [Theorem 4.3](#) we infer that the map

$$\rho \mapsto \frac{\mathcal{E}_{x_n}^n(\rho)}{\rho} \wedge 1 + c_\gamma \rho^\gamma \tag{5-20}$$

is nondecreasing on  $(0, d_n \wedge 1)$ . Thanks to (5-19), possibly passing to a subsequence, for  $n$  large we may choose as an admissible radius  $\rho_n = C\sqrt{\eta_n}$  for some  $C > 0$ , and write

$$1 \leq \frac{\mathcal{E}_{x_n}^n(\rho_n)}{\rho_n} \wedge 1 + c_\gamma \rho_n^\gamma \leq \frac{\mathcal{E}_{x_n}^n(\rho_n)}{\rho_n} + c_\gamma \rho_n^\gamma \leq \frac{\eta_n}{C\sqrt{\eta_n}} + c_\gamma (C\sqrt{\eta_n})^\gamma,$$

where the first inequality comes from monotonicity at  $x_n$  and (5-16), and the last one by (5-18). In view of (5-17), the above relation gives a contradiction for  $n$  large enough.

Case 2: Assume that

$$\lim_n \frac{d_n}{\sqrt{\eta_n}} = 0 \quad \text{and} \quad \limsup_n \frac{d'_n}{\sqrt{\eta_n}} > 0. \tag{5-21}$$

By Step 2 and the boundary monotonicity formula of [Theorem 4.4](#) we infer that the map

$$\rho \mapsto \frac{\mathcal{E}_{y_n}^n(\rho)}{\rho} \wedge 1 + c_\gamma \rho^\gamma + k_\Omega \int_0^\rho \left( \frac{\mathcal{E}_{y_n}^n(r)}{r} \wedge 1 \right) dr \tag{5-22}$$

is nondecreasing on  $(0, 1 \wedge r_\Omega \wedge d'_n)$ , where  $r_\Omega, k_\Omega > 0$ . Thanks to [\(5-21\)](#), possibly passing to a subsequence, for  $n$  large we may choose as an admissible radius  $\rho_n = C\sqrt{\eta_n}$ , assume that  $\rho_n > 2d_n$ , and write

$$\begin{aligned} \frac{\mathcal{E}_{y_n}^n(\rho_n)}{\rho_n} \wedge 1 + c_\gamma \rho_n^\gamma + k_\Omega \int_0^{\rho_n} \left( \frac{\mathcal{E}_{y_n}^n(r)}{r} \wedge 1 \right) dr &\geq \frac{\mathcal{E}_{y_n}^n(2d_n)}{2d_n} \wedge 1 + c_\gamma (2d_n)^\gamma + k_\Omega \int_0^{2d_n} \left( \frac{\mathcal{E}_{y_n}^n(r)}{r} \wedge 1 \right) dr \\ &\geq \frac{\mathcal{E}_{x_n}^n(d_n)}{2d_n} \wedge 1 + c_\gamma d_n^\gamma \geq \frac{1}{2} \left[ \frac{\mathcal{E}_{x_n}^n(d_n)}{d_n} \wedge 1 + c_\gamma d_n^\gamma \right] \geq \frac{1}{2}, \end{aligned}$$

where we have used in the first inequality monotonicity at  $y_n$ , then the inclusion  $B_{d_n}(x_n) \subseteq B_{2d_n}(y_n)$  (see [Figure 3](#)), and finally monotonicity at  $x_n$  combined with [\(5-16\)](#). We infer

$$\frac{1}{2} \leq \frac{\mathcal{E}_{y_n}^n(\rho_n)}{\rho_n} \wedge 1 + c_\gamma \rho_n^\gamma + k_\Omega \int_0^{\rho_n} \left( \frac{\mathcal{E}_{y_n}^n(r)}{r} \wedge 1 \right) dr \leq \frac{\eta_n}{C\sqrt{\eta_n}} + c_\gamma (C\sqrt{\eta_n})^\gamma + k_\Omega C\sqrt{\eta_n}, \tag{5-23}$$

which is a contradiction for  $n$  large in view of [\(5-17\)](#).

Case 3: Assume that

$$\lim_n \frac{d_n}{\sqrt{\eta_n}} = 0, \quad \lim_n \frac{d'_n}{\sqrt{\eta_n}} = 0 \quad \text{and} \quad \lim_n \frac{d_n}{d'_n} = 0. \tag{5-24}$$

Then there exists a vertex  $z \in S$  such that, possibly passing to a subsequence,  $x_n, y_n \rightarrow z$ . Set

$$d'_n = \text{dist}(y_n, z).$$

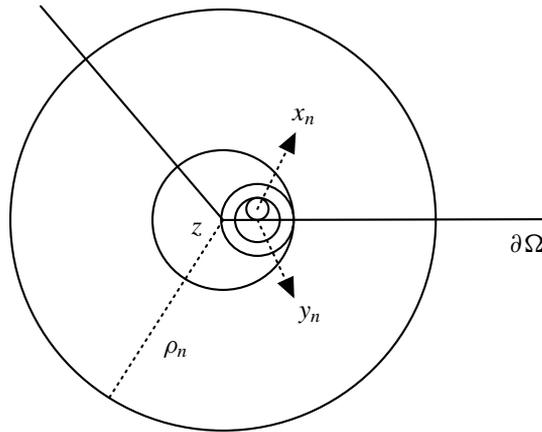
We choose  $\rho_n := \sqrt{\eta_n}$ . Since for  $n$  large we have  $\rho_n \geq 2d'_n$ , we may write using the monotonicity at  $z$  and the inclusion  $B_{d'_n}(y_n) \subseteq B_{2d'_n}(z)$  (see [Figure 4](#))

$$\begin{aligned} \frac{\mathcal{E}_z^n(\rho_n)}{\rho_n} \wedge 1 + c_\gamma \rho_n^\gamma + k_\Omega \int_0^{\rho_n} \left( \frac{\mathcal{E}_z^n(r)}{r} \wedge 1 \right) dr &\geq \frac{\mathcal{E}_z^n(2d'_n)}{2d'_n} \wedge 1 + c_\gamma (2d'_n)^\gamma + k_\Omega \int_0^{2d'_n} \left( \frac{\mathcal{E}_z^n(r)}{r} \wedge 1 \right) dr \\ &\geq \frac{\mathcal{E}_{y_n}^n(d'_n)}{2d'_n} \wedge 1 + c_\gamma (d'_n)^\gamma. \end{aligned}$$

On the other hand, since for  $n$  large we also have  $d'_n \geq 2d_n$ , by monotonicity at  $y_n$  and the inclusion  $B_{d_n}(x_n) \subseteq B_{2d_n}(y_n)$  (see again [Figure 4](#)), we have

$$\begin{aligned} \frac{\mathcal{E}_{y_n}^n(d'_n)}{2d'_n} \wedge 1 + c_\gamma (d'_n)^\gamma + k_\Omega \int_0^{d'_n} \left( \frac{\mathcal{E}_{y_n}^n(r)}{r} \wedge 1 \right) dr &\geq \frac{1}{2} \left[ \frac{\mathcal{E}_{y_n}^n(d'_n)}{d'_n} \wedge 1 + c_\gamma (d'_n)^\gamma + k_\Omega \int_0^{d'_n} \left( \frac{\mathcal{E}_{y_n}^n(r)}{r} \wedge 1 \right) dr \right] \\ &\geq \frac{1}{2} \left[ \frac{\mathcal{E}_{y_n}^n(2d_n)}{2d_n} \wedge 1 + c_\gamma (2d_n)^\gamma + k_\Omega \int_0^{2d_n} \left( \frac{\mathcal{E}_{y_n}^n(r)}{r} \wedge 1 \right) dr \right] \\ &\geq \frac{1}{2} \left[ \frac{\mathcal{E}_{x_n}^n(d_n)}{2d_n} \wedge 1 + c_\gamma d_n^\gamma \right] \geq \frac{1}{4} \left[ \frac{\mathcal{E}_{x_n}^n(d_n)}{d_n} \wedge 1 + c_\gamma d_n^\gamma \right] \geq \frac{1}{4}, \end{aligned}$$

the last inequality coming from monotonicity at  $x_n$  combined with [\(5-16\)](#).



**Figure 4.** Illustration of Case 3.

Collecting the previous inequalities we obtain

$$\begin{aligned} \frac{1}{4} &\leq k_\Omega \int_0^{d'_n} \left( \frac{\mathcal{E}_{y_n}^n(r)}{r} \wedge 1 \right) dr + \frac{\mathcal{E}_z^n(\rho_n)}{\rho_n} \wedge 1 + c_\gamma \rho_n^\gamma + k_\Omega \int_0^{\rho_n} \left( \frac{\mathcal{E}_z^n(r)}{r} \wedge 1 \right) dr \\ &\leq k_\Omega d'_n + \frac{\eta_n}{\sqrt{\eta_n}} + c_\gamma (\sqrt{\eta_n})^\gamma + k_\Omega \sqrt{\eta_n}, \end{aligned}$$

which yields a contradiction for  $n$  large.

Case 4: Assume finally that

$$\lim_n \frac{d_n}{\sqrt{\eta_n}} = 0, \quad \lim_n \frac{d'_n}{\sqrt{\eta_n}} = 0 \quad \text{and} \quad \limsup_n \frac{d_n}{d'_n} > 0. \tag{5-25}$$

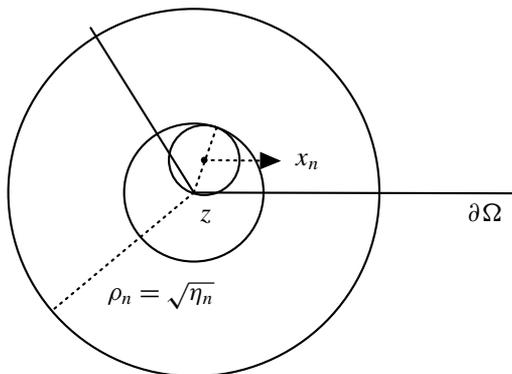
Let  $z \in S$  be a vertex such that, up to a subsequence if necessary,  $x_n, y_n \rightarrow z$ , so that  $d'_n = \text{dist}(y_n, z)$ . Set  $d''_n := \text{dist}(x_n, z)$ . Up to a subsequence we may assume that, for  $n$  large,  $d'_n \leq C d_n$  with  $C > 0$ , so that from the inequalities

$$d_n = \text{dist}(x_n, \partial\Omega) \leq d''_n = \text{dist}(x_n, z) \leq \text{dist}(x_n, y_n) + \text{dist}(y_n, z) = d_n + d'_n \leq (1 + C)d_n \tag{5-26}$$

we infer that  $d_n$  and  $d''_n$  are comparable.

Then, if we choose  $\rho_n := \sqrt{\eta_n}$ , in view of (5-25) and (5-26), for  $n$  large we have  $\rho_n \geq 2d''_n$ . By monotonicity at  $z$  and the inclusion  $B_{d''_n}(x_n) \subseteq B_{2d''_n}(z)$  (see Figure 5), we obtain

$$\begin{aligned} \frac{\mathcal{E}_z^n(\rho_n)}{\rho_n} \wedge 1 + c_\gamma \rho_n^\gamma + k_\Omega \int_0^{\rho_n} \left( \frac{\mathcal{E}_z^n(r)}{r} \wedge 1 \right) dr &\geq \frac{\mathcal{E}_z^n(2d''_n)}{2d''_n} \wedge 1 + c_\gamma (2d''_n)^\gamma + k_\Omega \int_0^{2d''_n} \left( \frac{\mathcal{E}_z^n(r)}{r} \wedge 1 \right) dr \\ &\geq \frac{\mathcal{E}_{x_n}^n(d''_n)}{2d''_n} \wedge 1 + c_\gamma (d''_n)^\gamma \geq \frac{\mathcal{E}_{x_n}^n(d_n)}{2(1 + C)d_n} \wedge 1 + c_\gamma d_n^\gamma \\ &\geq \frac{1}{2(1 + C)} \left[ \frac{\mathcal{E}_{x_n}^n(d_n)}{d_n} \wedge 1 + c_\gamma d_n^\gamma \right] \geq \frac{1}{2(1 + C)}, \end{aligned}$$



**Figure 5.** Illustration of Case 4.

where as above the last inequality comes from monotonicity at  $x_n$  combined with (5-16). Then,

$$\frac{1}{2(1+C)} \leq \frac{\mathcal{E}_z^n(\rho_n)}{\rho_n} \wedge 1 + c_\gamma \rho_n^\gamma + k_\Omega \int_0^{\rho_n} \left( \frac{\mathcal{E}_z^n(r)}{r} \wedge 1 \right) dr \leq \frac{\eta_n}{\sqrt{\eta_n}} + c_\gamma (\sqrt{\eta_n})^\gamma + k_\Omega \sqrt{\eta_n},$$

which yields a contradiction for  $n$  large. □

**Remark 5.4.** For later use, let us notice that in order to carry out Step 4 in the previous proof, the monotonicity properties given by Theorems 4.3 and 4.4 can be replaced respectively by *interior and boundary quasimonotonicity properties* of the following type: if  $u$  is a weak local almost-quasiminimizer for the Mumford–Shah energy with respect to its own jump set at every point of  $\bar{\Omega}$ , and  $E_x$  and  $\tilde{E}_x$  are defined respectively by (4-4) and (4-5), there exist  $c > 1$  and  $\tilde{r}_\Omega > 0$  such that

$$E_x(\rho_2) \geq \frac{1}{c} E_x\left(\frac{\rho_1}{c}\right) \quad \text{if } x \in \Omega \text{ and } \rho_1 \leq \rho_2 \leq \tilde{r}_\Omega \wedge \text{dist}(x, \partial\Omega),$$

$$\tilde{E}_x(\rho_2) \geq \frac{1}{c} \tilde{E}_x\left(\frac{\rho_1}{c}\right) \quad \text{if } x \in \partial\Omega \text{ and } \rho_1 \leq \rho_2 \leq \tilde{r}_\Omega \wedge \text{dist}(x, S \setminus \{x\}).$$

Indeed, considering for example Case 2, we can choose again  $\rho_n := C\sqrt{\eta_n}$  for some  $C > 0$ , and use the *boundary quasimonotonicity* at  $y_n$  for the radii  $\rho_n \geq 2cd_n$  to write

$$\begin{aligned} \frac{\mathcal{E}_{y_n}^n(\rho_n)}{\rho_n} \wedge 1 + c_\gamma \rho_n^\gamma + k_\Omega \int_0^{\rho_n} \left( \frac{\mathcal{E}_{y_n}^n(r)}{r} \wedge 1 \right) dr &\geq \frac{1}{c} \left[ \frac{\mathcal{E}_{y_n}^n(2d_n)}{2d_n} \wedge 1 + c_\gamma (2d_n)^\gamma + k_\Omega \int_0^{2d_n} \left( \frac{\mathcal{E}_{y_n}^n(r)}{r} \wedge 1 \right) dr \right] \\ &\geq \frac{1}{2c} \left[ \frac{\mathcal{E}_{x_n}^n(d_n)}{d_n} \wedge 1 + c_\gamma d_n^\gamma \right]. \end{aligned} \tag{5-27}$$

Since  $x_n$  is a point of density 1 for  $J_{v_n}$ , there exists  $r_n$  such that  $cr_n \leq d_n$  and

$$\frac{\mathcal{H}^1(J_{v_n} \cap \bar{B}_{r_n}(x_n))}{r_n} \geq \frac{3}{2}.$$

Then, by using the *interior quasimonotonicity* at  $x_n$  for the radii  $cr_n \leq d_n$ , we get

$$\frac{1}{2c} \left[ \frac{\mathcal{E}_{x_n}^n(d_n)}{d_n} \wedge 1 + c_\gamma d_n^\gamma \right] \geq \frac{1}{2c^2} \left[ \frac{\mathcal{E}_{x_n}^n(r_n)}{r_n} \wedge 1 + c_\gamma r_n^\gamma \right] \geq \frac{1}{2c^2}.$$

But this lower bound is incompatible with (5-27) since the first line of (5-27) is going to zero as  $n \rightarrow \infty$  (see (5-23)). The other cases can be treated similarly.

### 6. The case of general energies with boundary conditions

In this section we deal with the local minimality of the Sobolev minimizer of a generalization of the Mumford–Shah functional under prescribed boundary conditions.

**6A. Setting of the problem and the local minimality result.** Given an open bounded set  $\Omega \subset \mathbb{R}^2$ , which is admissible according to Definition 3.5, we want to impose Dirichlet boundary conditions on the (nonempty) portion  $\Gamma_D$  of its boundary. To this aim, similarly as in Section 4 we consider an open bounded domain  $\Omega'$  such that  $\Omega \Subset \Omega' \subset \mathbb{R}^2$  and with  $\partial\Omega \cap \Omega' = \Gamma_D$  (see Figure 2). Then, given  $w \in H^1(\Omega') \cap L^\infty(\Omega')$ , we set

$$\mathcal{A}_w := \{u \in \text{SBV}(\Omega') : u = w \text{ on } \Omega' \setminus \bar{\Omega}\}. \tag{6-1}$$

We are interested in the minimization on the class  $\mathcal{A}_w$  of the Mumford–Shah-type functional

$$F(u) := \int_{\Omega'} A(x) \nabla u \cdot \nabla u \, dx + \int_{J_u} b(x, u^+, u^-, \nu_u) \, d\mathcal{H}^1 + \beta \int_{\Omega'} |u - g|^2 \, dx. \tag{6-2}$$

Here  $\nu_u$  denotes a normal along  $J_u$ , and  $u^\pm$  the associated traces of  $u$  (see Section 2), while the assumptions satisfied by the matrix  $A$ , the function  $b$ , and the datum  $g$  are specified below.

**Remark 6.1.** Notice that working on the set  $\mathcal{A}_w$  the boundary condition

$$u = w \quad \text{on } \Gamma_D$$

is taken into account in a relaxed sense: indeed, the parts of  $\Gamma_D$  on which  $u \neq w$  are contained in  $J_u$ , so that they turn out to be penalized by the functional  $F$ . This is usual in variational problems for functions of bounded variation (like for example the graph area problem). Finally, observe that the bulk terms provide a fixed contribution on  $\Omega' \setminus \Omega$ , since  $u = w$  on this set, so that for the minimization it suffices simply to integrate on  $\Omega$ .

**Assumptions on the functional.** Concerning the volume terms in (6-2), we require

$$A \in C^{0,\delta}(\bar{\Omega}'; M_{\text{sym}}^{2 \times 2}), \quad c_1^A |\eta|^2 \leq A(x) \eta \cdot \eta \leq c_2^A |\eta|^2 \quad \text{for every } x \in \Omega' \text{ and } \eta \in \mathbb{R}^2, \tag{6-3}$$

and

$$g \in L^\infty(\Omega'), \quad \beta \geq 0 \tag{6-4}$$

for suitable constants  $c_1^A, c_2^A > 0$  and  $\delta \in (0, 1)$ .

Concerning the surface term

$$\Psi(u) := \int_{J_u} b(x, u^+, u^-, \nu_u) d\mathcal{H}^1,$$

we require for  $b : \Omega' \times \mathbb{R} \times \mathbb{R} \times S^1 \rightarrow [0, +\infty[$  to be such that

$$c_1^b \leq b(x, s_1, s_2, \nu) \leq c_2^b + \Phi(s_1, s_2), \tag{6-5}$$

where  $c_1^b, c_2^b > 0$  and  $\Phi : \mathbb{R}^2 \rightarrow [0, +\infty[$ . In addition we ask that

$$u \mapsto \Psi(u) \text{ is l.s.c with respect to the weak convergence in } SBV(\Omega), \tag{6-6}$$

and the *monotonicity under truncation* satisfies

$$\Psi((u \wedge c_2) \vee c_1) \leq \Psi(u) \quad \text{for every } c_1 \leq c_2. \tag{6-7}$$

**Remark 6.2** (example of admissible surface energies). An admissible surface term could be given for example by

$$b(x, s_1, s_2, \nu) := \varphi(x, \nu) + |s_1 - s_2|,$$

where  $\varphi : \Omega' \times \mathbb{R}^2 \rightarrow ]0, +\infty[$  is such that

$$x \mapsto \varphi(x, \nu) \quad \text{is lower semicontinuous on } \Omega' \text{ for every } \nu \in S^1,$$

$$\nu \mapsto \varphi(x, \nu) \quad \text{is convex, positively one homogeneous on } \mathbb{R}^2 \text{ for every } x \in \Omega',$$

and

$$0 < c_1^b \leq \varphi(x, \nu) \leq c_2^b.$$

Requirements (6-5) and (6-7) are immediately fulfilled. The lower semicontinuity (6-6) is a consequence of the Reshetnyak theorem (see [Ambrosio et al. 2000, Theorem 2.38]) and of the lower semicontinuity result in SBV [Ambrosio et al. 2000, Theorem 5.22].

**A-admissible domains.** To control the interaction between Neumann and Dirichlet conditions in connection with monotonicity, we need to introduce the following property concerning the interplay between the geometry of  $\Omega$  and the bulk energy.

**Definition 6.3** (A-admissible domains). Let  $\Omega \subset \mathbb{R}^2$  be an open bounded set with a piecewise  $C^{1,1}$ -boundary, decomposed as in (3-23), and let  $A$  satisfy (6-3). We say that  $\Omega$  is A-admissible if, for every  $x \in \bar{\Omega}$ , the domain  $\Omega_x := A(x)^{-1/2}\Omega$  is admissible according to Definition 3.5.

**Remark 6.4.** Notice that admissibility according to Definition 3.5 is equivalent to Id-admissibility according to Definition 6.3. In particular, when  $\Gamma_D = \partial\Omega$  or  $\Gamma_N = \partial\Omega$ , A-admissibility reduces to the assumption of convex corners, and smooth  $C^{1,1}$  domains are always A-admissible.

**The local minimality result under boundary conditions.** The analogue of Theorem 5.1 under boundary conditions is the following.

**Theorem 6.5** (small jump sets are not convenient under boundary conditions). *Assume the functional  $F$  satisfies assumptions (6-3)–(6-7). Let  $\Omega \subseteq \mathbb{R}^2$  be  $A$ -admissible according to Definition 6.3, and let  $w \in H^1(\Omega') \cap L^\infty(\Omega')$ . Denoting by  $U_w \in H^1(\Omega')$  the solution to*

$$\min_{v \in \mathcal{A}_w \cap H^1(\Omega')} F(v),$$

*assume that  $U_w$  admits at most uniformly weak singularities in  $\bar{\Omega}$ ; i.e., there exist  $C > 0$  and  $\alpha > 0$  such that for every  $x \in \bar{\Omega}$  and  $\rho \leq 1$*

$$\int_{\Omega \cap B_\rho(x)} |\nabla U_w|^2 dx \leq C\rho^{1+\alpha}. \tag{6-8}$$

*Then there exists  $\varepsilon > 0$  such that for every  $u \in \mathcal{A}_w$  with  $\mathcal{H}^1(J_u) < \varepsilon$  we have*

$$F(U_w) < F(u).$$

**Remark 6.6.** Assumptions on the kind of *singularities* which the function  $U_w$  can exhibit are essential to exclude that small jump sets are not convenient. Indeed, assume for example that

$$\int_{\Omega \cap B_\rho(x)} |\nabla U_w|^2 dx \approx c\rho^\gamma, \tag{6-9}$$

with  $0 < \gamma < 1$  and  $c > 0$  at some point  $x \in \bar{\Omega}$ : this means that for small  $\rho$ , the singularity has an energy content in  $\Omega \cap B_\rho(x)$  which is higher than the length of  $\partial(\Omega \cap B_\rho(x))$ . It turns out that the admissible competitor

$$v := \begin{cases} 0 & \text{in } \Omega \cap B_\rho(x), \\ U_w & \text{otherwise} \end{cases}$$

is such that  $F(v) < F(U_w)$  if  $\rho$  is sufficiently small, so that the minimality property of Theorem 6.5 cannot hold.

Theorem 6.5 states that the energy content given by (6-8) is not enough to destroy the minimality of  $U_w$ , which is preserved even if small jump sets are allowed. In the case  $\Gamma_D = \partial\Omega$ , inequality (6-8) holds for example if  $w \in H^2(\Omega')$  and  $A$  is Lipschitz continuous on  $\bar{\Omega}'$ , since elliptic regularity gives  $U_w \in H^2(\Omega)$ ; see, e.g., [Grisvard 1985, 3.2.1.2].

In the language of fracture mechanics, the situation (6-9) is defined as a *strong singularity*: the “elastic energy” stored in  $\Omega \cap B_\rho(x)$  is much higher than the energy required to create a crack along  $\partial(\Omega \cap B_\rho(x))$  so that the elastic configuration is not in equilibrium, and the creation of a small crack is energetically convenient. “*Weak singularities*” are on the contrary compatible with local equilibrium.

As mentioned in the Introduction, the minimality property of Theorem 6.5 (without fidelity term but with nonlinear bulk energies and general Lipschitz boundaries) was derived in [Chambolle et al. 2008, Theorem 1] under the additional assumption that the competitors have a closed jump set with a preset number of connected components, while the full SBV case was derived under the stronger assumption  $U_w \in C^1(\bar{\Omega})$ ; see [Chambolle et al. 2008, Theorem 6].

As in Section 5, we can draw the following local minimality result with respect to the  $L^1$  topology under boundary conditions.

**Theorem 6.7** (local minimality in  $L^1$  under boundary conditions). *Under the assumptions of Theorem 6.5, the function  $U_w$  is a local minimizer for the Mumford–Shah-type energy  $F$  on  $\mathcal{A}_w$  with respect to the  $L^1$ -topology.*

*Proof.* Assume by contradiction that there exists  $v_n \in \mathcal{A}_w$  such that

$$v_n \rightarrow U_w \quad \text{strongly in } L^1(\Omega'),$$

with

$$F(v_n) < F(U_w). \tag{6-10}$$

It is not restrictive to assume that  $\|v_n\|_\infty \leq \|g\|_\infty + \|w\|_\infty$ , so that the convergence holds also in  $L^2(\Omega')$  and is weak in  $\text{SBV}(\Omega')$  (thanks to the coercivity assumptions on  $A$  and  $b$ ). By Ambrosio’s theorem we deduce

$$\liminf_n F(v_n) \geq F(U_w),$$

so that from (6-10) we infer  $\lim_n \mathcal{H}^1(J_{v_n}) = 0$ ; but then (6-10) is in contrast with Theorem 6.5. □

**6B. Proof of Theorem 6.5.** Let us start by deriving some properties of a solution  $v \in \mathcal{A}_0$  to

$$\min_{\substack{h \in \mathcal{A}_0 \\ \mathcal{H}^1(J_h) \leq \varepsilon}} F(U_w + h). \tag{6-11}$$

They are stated in two separate lemmas below, and concern respectively a uniform almost-quasiminimality property, and a crucial quasimonotonicity property. Some preliminary notation and remarks are in order.

Firstly notice that existence of minimizers to (6-11) is guaranteed by the application of Ambrosio’s theorem in view of the assumptions (6-3)–(6-7) on the terms appearing in  $F$ . Notice that we may assume (thanks in particular to the truncation assumption (6-7) for the surface energy)

$$\|U_w\|_\infty \leq \|w\|_\infty + \|g\|_\infty \quad \text{and} \quad \|v\|_\infty \leq 2\|w\|_\infty + \|g\|_\infty. \tag{6-12}$$

For every  $\xi \in \bar{\Omega}$ , let us consider the matrix

$$L_\xi := A(\xi)^{1/2}$$

and the sets

$$\tilde{\Omega}_\xi := L_\xi^{-1}\Omega \quad \text{and} \quad \tilde{\Omega}'_\xi := L_\xi^{-1}\Omega'.$$

To every function  $u \in \text{SBV}(\Omega')$  let us associate the function  $\tilde{u}_\xi \in \text{SBV}(\tilde{\Omega}'_\xi)$  given by

$$\tilde{u}_\xi(y) := \sqrt{\frac{2 \det L_\xi}{c_1^b \sqrt{c_1^A}}} u(L_\xi y), \tag{6-13}$$

where  $c_1^A$  and  $c_1^b$  are the constants appearing in (6-3) and (6-5). Let us denote by  $\tilde{\mathcal{A}}_0$  the space of functions associated to  $\mathcal{A}_0$  under the previous transformation, and let  $\tilde{\xi}$  be the point corresponding to  $\xi$ .

**Lemma 6.8** (uniform almost-quasiminimality). *Under the assumptions of Theorem 6.5, let  $v \in \mathcal{A}_0$  be a solution to the minimization problem (6-11). There exist  $\rho_0 > 0$ ,  $\Lambda \geq 1$ ,  $\gamma > 0$ ,  $c_\gamma > 0$  such that for every  $\xi \in \bar{\Omega}$ , for every  $\rho < \rho_0$  with*

$$\int_{B_\rho(\tilde{\xi}) \cap \tilde{\Omega}_\xi} |\nabla \tilde{v}_\xi|^2 dx + \mathcal{H}^1(J_{\tilde{v}_\xi} \cap \bar{B}_\rho(\tilde{\xi})) \leq \rho, \tag{6-14}$$

and for every  $\tilde{h} \in \tilde{\mathcal{A}}_0$  with  $\{\tilde{h} \neq \tilde{v}_\xi\} \subseteq B_\rho(\tilde{x})$  and  $J_{\tilde{h}} \subseteq J_{\tilde{v}_\xi}$  we have

$$\int_{B_\rho(\tilde{\xi}) \cap \tilde{\Omega}_\xi} |\nabla \tilde{v}_\xi|^2 dx + \mathcal{H}^1(J_{\tilde{v}_\xi} \cap \bar{B}_\rho(\tilde{\xi})) \leq \int_{B_\rho(\tilde{\xi}) \cap \tilde{\Omega}_\xi} |\nabla \tilde{h}|^2 dx + \Lambda \mathcal{H}^1(J_{\tilde{h}} \cap \bar{B}_\rho(\tilde{\xi})) + c_\gamma \rho^{1+\gamma}.$$

In other words, uniformly in  $\xi \in \bar{\Omega}$ , the function  $\tilde{v}_\xi$  is a weak local almost-quasiminimizer in  $\tilde{\mathcal{A}}_0$  of the Mumford–Shah energy with respect to its own jump set at the point  $\tilde{\xi}$ .

*Proof.* Let us divide the proof in two steps.

Step 1: assume that  $A(\xi) = \text{Id}$ . Let us show that if  $h \in \mathcal{A}_0$  with  $\{h \neq v\} \subseteq B_\rho(\xi)$ ,  $\rho \leq 1$  and  $J_h \subseteq J_v$ , we have

$$\begin{aligned} & \int_{\Omega \cap B_\rho(\xi)} |\nabla v|^2 dx + c_1^b \mathcal{H}^1(J_v \cap \bar{B}_\rho(\xi)) \\ & \leq \int_{\Omega \cap B_\rho(\xi)} |\nabla h|^2 dx + C_1 \mathcal{H}^1(J_h \cap \bar{B}_\rho(\xi)) + 2c_A(1 + c_1^b)\rho^{1+\delta} + C_2 \rho^{(1+\alpha/2)\wedge 2}, \end{aligned} \tag{6-15}$$

where  $c_1^b$  is the constant appearing in the estimates (6-5) for the surface energy  $b$ ,  $c_A$  is the Holder constant of  $A \in C^{0,\delta}(\bar{\Omega}', M_{\text{sym}}^{2 \times 2})$ ,  $\alpha$  is the constant appearing in (6-8), and  $C_1, C_2 > 0$  are suitable constants depending only on the data of the problem.

Indeed, recalling (6-12) and using a truncation argument, it is not restrictive to assume that

$$\|h\|_\infty \leq 2\|w\|_\infty + \|g\|_\infty. \tag{6-16}$$

Moreover, we may assume

$$\int_{\Omega \cap B_\rho(\xi)} |\nabla h|^2 dx < \int_{\Omega \cap B_\rho(\xi)} |\nabla v|^2 dx + c_1^b \mathcal{H}^1(J_v \cap \bar{B}_\rho(\xi)) \tag{6-17}$$

since otherwise (6-15) is immediately satisfied.

By the minimality of  $v$  (since the inclusion  $J_h \subseteq J_v$  ensures that  $\mathcal{H}^1(J_h) \leq \varepsilon$ ), we get

$$\begin{aligned} & \int_{\Omega \cap B_\rho(\xi)} A(x) \nabla v \cdot \nabla v dx + c_1^b \mathcal{H}^1(J_v \cap \bar{B}_\rho(\xi)) \\ & \leq \int_{\Omega \cap B_\rho(\xi)} A(x) \nabla h \cdot \nabla h dx + C_1 \mathcal{H}^1(J_h \cap \bar{B}_\rho(\xi)) + c(\rho), \end{aligned} \tag{6-18}$$

where

$$c(\rho) := 2 \int_{\Omega \cap B_\rho(\xi)} A(x) \nabla U_w \cdot (\nabla h - \nabla v) dx + \beta \int_{\Omega \cap B_\rho(\xi)} (h - v)(2U_w + h + v - 2g) dx$$

and

$$C_1 := c_2^b + \Phi(2\|w\|_\infty + \|g\|_\infty, -(2\|w\|_\infty + \|g\|_\infty)),$$

where  $c_2^b$  and  $\Phi$  appear in (6-5).

In view of the energy bounds (6-8), (6-14), (6-17) and of the  $L^\infty$ -bounds (6-12) and (6-16) we deduce for every  $\rho \leq 1$

$$\begin{aligned} c(\rho) &\leq 2c_2^A \int_{\Omega \cap B_\rho(\xi)} |\nabla U_w| |\nabla h - \nabla v| \, dx + \tilde{C} \rho^2 \\ &\leq 2c_2^A \|\nabla U_w\|_{L^2(\Omega \cap B_\rho(\xi))} (\|\nabla h\|_{L^2(\Omega \cap B_\rho(\xi))} + \|\nabla v\|_{L^2(\Omega \cap B_\rho(\xi))}) + \tilde{C} \rho^2 \\ &\leq 2c_2^A \sqrt{C} \rho^{(1+\alpha)/2} (\sqrt{(1+c_1^b)\rho} + \sqrt{\rho}) + \tilde{C} \rho^2 \leq C_2 \rho^{(1+\alpha/2)\wedge 2} \end{aligned}$$

for a suitable  $C_2 > 0$ .

Since  $A(\xi) = \text{Id}$  and  $A \in C^{0,\delta}(\bar{\Omega}'; M_{\text{sym}}^{2 \times 2})$ , coming back to (6-18) we may write

$$\begin{aligned} \int_{\Omega \cap B_\rho(\xi)} |\nabla v|^2 \, dx - c_A \rho^\delta \int_{\Omega \cap B_\rho(\xi)} |\nabla v|^2 \, dx + c_1^b \mathcal{H}^1(J_v \cap \bar{B}_\rho(x)) \\ \leq \int_{\Omega \cap B_\rho(\xi)} |\nabla h|^2 \, dx + c_A \rho^\delta \int_{\Omega \cap B_\rho(\xi)} |\nabla h|^2 \, dx + C_1 \mathcal{H}^1(J_h \cap \bar{B}_\rho(\xi)) + C_2 \rho^{(1+\alpha/2)\wedge 2}, \end{aligned}$$

where  $c_A > 0$  is the Hölder constant of  $A$ , so that taking into account (6-17) and (6-14) we infer

$$\begin{aligned} \int_{\Omega \cap B_\rho(\xi)} |\nabla v|^2 \, dx + c_1^b \mathcal{H}^1(J_v \cap \bar{B}_\rho(\xi)) \\ \leq \int_{\Omega \cap B_\rho(\xi)} |\nabla h|^2 \, dx + C_1 \mathcal{H}^1(J_h \cap \bar{B}_\rho(\xi)) + 2c_A(1+c_1^b)\rho^{1+\delta} + C_2 \rho^{(1+\alpha/2)\wedge 2}, \end{aligned}$$

so that (6-15) follows.

**Step 2:** let us come to the general case. Under the transformation (6-13), a direct computation shows that the functional  $F$  is transformed up to a multiplicative constant into the functional on  $\text{SBV}(\tilde{\Omega}'_\xi)$  given by

$$\tilde{F}(\tilde{u}) := \int_{\tilde{\Omega}'_\xi} \tilde{A}(y) \nabla \tilde{u} \cdot \nabla \tilde{u} \, dy + \int_{J_{\tilde{u}}} \tilde{b}(y, \tilde{u}^+, \tilde{u}^-, v_{\tilde{u}}) \, d\mathcal{H}^1(y) + \beta \int_{\tilde{\Omega}'_\xi} |\tilde{u} - \tilde{g}_\xi|^2 \, dy,$$

with

$$\tilde{A}(y) := L_\xi^{-1} A(L_\xi y) L_\xi^{-1}$$

and

$$\tilde{b}(y, s_1, s_2, \nu) := \frac{2}{c_1^b \sqrt{c_1^A}} b\left(L_\xi y, \sqrt{\frac{2 \det L_\xi}{c_1^b}} s_1, \sqrt{\frac{2 \det L_\xi}{c_1^b}} s_2, L_\xi \nu^\perp\right) |L_\xi \nu^\perp|,$$

where  $\nu^\perp$  is a unit vector orthogonal to  $\nu$  (so that  $|L_\xi \nu^\perp|$  turns out to be the one-dimensional jacobian of the transformation  $y \mapsto L_\xi y$  involved in the change of variable to pass from an integral on  $J_u$  to that on  $J_{\tilde{u}}$ , see [Ambrosio et al. 2000, Theorem 2.91]). Notice that by construction

$$\tilde{b}(y, s_1, s_2, \nu) \geq 2. \tag{6-19}$$

Clearly the function

$$\tilde{u} := (\widetilde{U_w})_\xi + \tilde{v}_\xi$$

is a minimizer of the functional  $\tilde{F}$  among the functions in  $\text{SBV}(\tilde{\Omega}'_\xi)$  such that  $\tilde{u} = \tilde{w}_\xi$  on  $\tilde{\Omega}'_\xi \setminus \tilde{\Omega}_\xi$  and  $\mathcal{H}^1(L_\xi J_{\tilde{v}}) \leq \varepsilon$ .

Notice that  $\tilde{A}(\tilde{\xi}) = \text{Id}$ . Therefore, taking into account (6-19), inequality (6-15) of Step 1 gives that, for every  $\tilde{h} \in \tilde{\mathcal{A}}_0$  with  $\{\tilde{h} \neq \tilde{v}_\xi\} \subseteq B_\rho(\tilde{\xi})$  and  $J_{\tilde{h}} \subseteq J_{\tilde{v}_\xi}$  (which is an admissible competitor for  $\tilde{v}_\xi$  since automatically  $\mathcal{H}^1(L_\xi J_{\tilde{h}}) \leq \varepsilon$ ),

$$\begin{aligned} & \int_{\tilde{\Omega}_\xi \cap B_\rho(\tilde{\xi})} |\nabla \tilde{v}_\xi|^2 dx + \mathcal{H}^1(J_{\tilde{v}_\xi} \cap \bar{B}_\rho(\tilde{\xi})) \\ & \leq \int_{\tilde{\Omega}_\xi \cap B_\rho(\tilde{\xi})} |\nabla \tilde{h}|^2 dx + \tilde{C}_1 \mathcal{H}^1(J_{\tilde{h}} \cap \bar{B}_\rho(\tilde{\xi})) + 2c_{\tilde{A}}(1 + c_1^b)\rho^{1+\delta} + \tilde{C}_2 \rho^{(1+\alpha/2)\wedge 2} \end{aligned} \quad (6-20)$$

if  $\rho$  is sufficiently small. The statement follows thanks to the uniform estimates on  $L_\xi$  coming from (6-3). □

**Lemma 6.9** (quasimonotonicity). *Under the assumptions of Theorem 6.5, let  $v \in \mathcal{A}_0$  be a solution to the minimization problem (6-11).*

*There exist  $c > 1$ ,  $\tilde{r}_\Omega, \tilde{k}_\Omega > 0$  such that, if  $E_x(\rho), \tilde{E}_x(\rho)$  are defined respectively as in (4-4), (4-5), where  $\gamma, c_\gamma$  are the constants of Lemma 6.8, and*

$$\mathcal{E}_x(\rho) := \int_{\Omega \cap B_\rho(x)} |\nabla v|^2 dx + \mathcal{H}^1(J_v \cap \bar{B}_\rho(x)),$$

*the following properties hold true:*

(a) *For every  $x \in \Omega$  and for every  $\rho_1 < \rho_2 < \tilde{r}_\Omega \wedge \text{dist}(x, \partial\Omega)$*

$$E_x(\rho_2) \geq \frac{1}{c} E_x\left(\frac{\rho_1}{c}\right). \quad (6-21)$$

(b) *For every  $x \in \partial\Omega$  and for every  $\rho_1 < \rho_2 < \tilde{r}_\Omega \wedge \text{dist}(x, S \setminus \{x\})$ , where  $S$  denotes the set of corners of  $\partial\Omega$ ,*

$$\tilde{E}_x(\rho_2) \geq \frac{1}{c} \tilde{E}_x\left(\frac{\rho_1}{c}\right).$$

*Proof.* Let us start with point (a). Thanks to Lemma 6.8, we have that  $\tilde{v}_x \in \tilde{\mathcal{A}}_0$  is a weak local almost-quasiminimizer (in  $\tilde{\mathcal{A}}_0$ ) for the Mumford–Shah energy with respect to its own jump set at the point  $\tilde{x} = L_x^{-1}x$ , so that the quantity

$$\rho \mapsto \frac{\int_{\tilde{\Omega}_x \cap B_\rho(\tilde{x})} |\nabla \tilde{v}_x|^2 dy + \mathcal{H}^1(J_{\tilde{v}_x} \cap B_\rho(\tilde{x}))}{\rho} \wedge 1 + \frac{c_\gamma}{\gamma} \rho^\gamma \quad (6-22)$$

is nondecreasing on  $(0, \rho_0 \wedge r_{\tilde{\Omega}_x} \wedge \text{dist}(\tilde{x}, \partial\tilde{\Omega}_x))$  in view of the interior monotonicity formula of Theorem 4.4. Coming back to the domain  $\Omega$ , thanks to the bounds on  $A$ , we find universal constants  $c_i > 0$  such that

$$c_3 \mathcal{E}_x(c_4 \rho) \leq \int_{\tilde{\Omega}_x \cap B_\rho(\tilde{x})} |\nabla \tilde{v}_x|^2 dy + \mathcal{H}^1(J_{\tilde{v}_x} \cap \bar{B}_\rho(\tilde{x})) \leq c_1 \mathcal{E}_x(c_2 \rho).$$

This estimate, together with the monotonicity of (6-22) and Remarks 3.8 and 4.7, yields easily point (a) of the statement for a suitable  $\tilde{r}_\Omega > 0$ .

The proof of point (b) is similar: we need to invoke, in place of Theorem 4.4, the boundary monotonicity formula of since  $\Omega$  is  $A$ -admissible by hypothesis, together with Remarks 3.8 and 4.7.  $\square$

We are now in a position to prove Theorem 6.5.

*Proof of Theorem 6.5.* It suffices to show that, for  $\varepsilon$  small enough, a minimizer  $v_\varepsilon \in \text{SBV}(\Omega)$  of

$$\min_{\substack{v \in \mathcal{A}_0 \\ \mathcal{H}^1(J_v) \leq \varepsilon}} F(U_w + v)$$

is such that  $J_{v_\varepsilon} = \emptyset$ , which gives  $v_\varepsilon = 0$ , and the proof follows.

We proceed by contradiction, assuming that there exists  $\varepsilon_n \rightarrow 0$  such that, for the corresponding  $v_n := v_{\varepsilon_n}$ , it holds that  $\mathcal{H}^1(J_{v_n}) > 0$ . Let us divide the proof in two steps.

Step 1: vanishing energy. We claim that

$$\int_{\Omega'} |\nabla v_n|^2 dx + \mathcal{H}^1(J_{v_n}) \rightarrow 0. \tag{6-23}$$

Notice indeed that

$$F(U_w + v_n) \leq F(U_w)$$

so that we get easily, taking into account (6-3)–(6-7) and (6-12)

$$\int_{\Omega'} |\nabla v_n|^2 dx + \mathcal{H}^1(J_{v_n}) + \|v_n\|_\infty \leq C.$$

By applying Ambrosio’s theorem to the sequence  $(v_n)_{n \in \mathbb{N}}$ , there exists  $v \in \mathcal{A}_0$  such that, up to a subsequence,

$$\begin{aligned} v_n &\rightarrow v && \text{strongly in } L^2(\Omega'), \\ \nabla v_n &\rightharpoonup \nabla v && \text{weakly in } L^2(\Omega'; \mathbb{R}^2) \end{aligned} \tag{6-24}$$

and

$$\mathcal{H}^1(J_v) \leq \liminf_n \mathcal{H}^1(J_{v_n}) \leq \lim_n \varepsilon_n = 0. \tag{6-25}$$

In particular  $v \in H^1(\Omega')$ . Moreover, by the minimality of  $v_n$ , in the limit we deduce that

$$F(U_w + v) \leq F(U_w + \varphi)$$

for every  $\varphi \in H^1(\Omega')$  with  $\varphi = 0$  on  $\Omega' \setminus \bar{\Omega}$ , which yields  $v = 0$  in view of the definition of  $U_w$ . Passing to the limit in the inequality

$$\begin{aligned} \int_{\Omega'} A(x)[\nabla U_w + \nabla v_n][\nabla U_w + \nabla v_n] dx + \int_{J_{v_n}} b(x, v_n^+, v_n^-, v_{v_n}) d\mathcal{H}^1 + \beta \int_{\Omega'} |U_w + v_n - g|^2 \\ \leq \int_{\Omega'} A(x) \nabla U_w \nabla U_w dx + \beta \int_{\Omega'} |U_w - g|^2 \end{aligned}$$

and taking into account (6-24), (6-25) and the coercivity of  $A$  and  $b$ , we deduce that

$$\nabla v_n \rightarrow 0 \quad \text{strongly in } L^2(\Omega'; \mathbb{R}^2),$$

so that the claim follows.

Step 2: conclusion. By Step 1 we have

$$\eta_n := \int_{\Omega'} |\nabla v_n|^2 dx + \mathcal{H}^1(J_{v_n}) \rightarrow 0.$$

In view of the *quasimonotonicity* properties enjoyed by  $v$  by Lemma 6.9, and taking into account Remark 5.4, we can repeat the arguments of Step 4 in the proof of Theorem 5.1 and get a contradiction, so that the conclusion follows.  $\square$

### Acknowledgments

Bucur was supported by the ANR-15-CE40-0006 COMEDIC project, the “Geometry and Spectral Optimization” research programme LabEx PERSYVAL-Lab GeoSpec ANR-11-LABX-0025-01 and the Institut Universitaire de France.

### References

- [Alberti et al. 2003] G. Alberti, G. Bouchitté, and G. Dal Maso, “The calibration method for the Mumford–Shah functional and free-discontinuity problems”, *Calc. Var. Partial Differential Equations* **16**:3 (2003), 299–333. [MR](#) [Zbl](#)
- [Ambrosio et al. 2000] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Univ. Press, 2000. [MR](#) [Zbl](#)
- [Bonacini and Morini 2015] M. Bonacini and M. Morini, “Stable regular critical points of the Mumford–Shah functional are local minimizers”, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **32**:3 (2015), 533–570. [MR](#) [Zbl](#)
- [Bonnet 1996] A. Bonnet, “On the regularity of edges in image segmentation”, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **13**:4 (1996), 485–528. [MR](#) [Zbl](#)
- [Bourdin et al. 2008] B. Bourdin, G. A. Francfort, and J.-J. Marigo, “The variational approach to fracture”, *J. Elasticity* **91**:1-3 (2008), 5–148. [MR](#) [Zbl](#)
- [Bucur and Luckhaus 2014] D. Bucur and S. Luckhaus, “Monotonicity formula and regularity for general free discontinuity problems”, *Arch. Ration. Mech. Anal.* **211**:2 (2014), 489–511. [MR](#) [Zbl](#)
- [Cagnetti et al. 2008] F. Cagnetti, M. G. Mora, and M. Morini, “A second order minimality condition for the Mumford–Shah functional”, *Calc. Var. Partial Differential Equations* **33**:1 (2008), 37–74. [MR](#) [Zbl](#)
- [Chambolle et al. 2008] A. Chambolle, A. Giacomini, and M. Ponsiglione, “Crack initiation in brittle materials”, *Arch. Ration. Mech. Anal.* **188**:2 (2008), 309–349. [MR](#) [Zbl](#)
- [Chambolle et al.  $\geq$  2020] A. Chambolle, E. Séré, and C. Zanini, “Progressive water-waves: a global variational approach”, in progress.
- [David 1996] G. David, “ $C^1$ -arcs for minimizers of the Mumford–Shah functional”, *SIAM J. Appl. Math.* **56**:3 (1996), 783–888. [MR](#) [Zbl](#)
- [De Giorgi and Ambrosio 1988] E. De Giorgi and L. Ambrosio, “Un nuovo tipo di funzionale del calcolo delle variazioni”, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) **82**:2 (1988), 199–210. [MR](#) [Zbl](#)
- [De Giorgi et al. 1989] E. De Giorgi, M. Carriero, and A. Leaci, “Existence theorem for a minimum problem with free discontinuity set”, *Arch. Ration. Mech. Anal.* **108**:3 (1989), 195–218. [MR](#) [Zbl](#)

- [Francfort and Marigo 1998] G. A. Francfort and J.-J. Marigo, “Revisiting brittle fracture as an energy minimization problem”, *J. Mech. Phys. Solids* **46**:8 (1998), 1319–1342. [MR](#) [Zbl](#)
- [Grisvard 1985] P. Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and Studies in Math. **24**, Pitman, Boston, 1985. [MR](#) [Zbl](#)
- [Mumford and Shah 1985] D. Mumford and J. Shah, “Boundary detection by minimizing functionals, I”, pp. 22–26 in *IEEE Proceedings on Computer Vision and Pattern Recognition* (San Francisco, 1985), IEEE, Washington, DC, 1985.
- [Mumford and Shah 1989] D. Mumford and J. Shah, “Optimal approximations by piecewise smooth functions and associated variational problems”, *Comm. Pure Appl. Math.* **42**:5 (1989), 577–685. [MR](#) [Zbl](#)

Received 16 Oct 2018. Revised 9 Feb 2019. Accepted 3 Apr 2019.

DORIN BUCUR: [dorin.bucur@univ-savoie.fr](mailto:dorin.bucur@univ-savoie.fr)  
*Université Savoie Mont Blanc, CNRS, LAMA, Chambéry, France*

ILARIA FRAGALÀ: [ilaria.fragala@polimi.it](mailto:ilaria.fragala@polimi.it)  
*Dipartimento di Matematica, Politecnico di Milano, Milano, Italy*

ALESSANDRO GIACOMINI: [alessandro.giacomini@unibs.it](mailto:alessandro.giacomini@unibs.it)  
*DICATAM, Sezione di Matematica, Università degli studi di Brescia, Brescia, Italy*

# Analysis & PDE

[msp.org/apde](http://msp.org/apde)

## EDITORS

EDITOR-IN-CHIEF

Patrick Gérard

[patrick.gerard@math.u-psud.fr](mailto:patrick.gerard@math.u-psud.fr)

Université Paris Sud XI

Orsay, France

## BOARD OF EDITORS

Massimiliano Berti	Scuola Intern. Sup. di Studi Avanzati, Italy <a href="mailto:berti@sissa.it">berti@sissa.it</a>	Gilles Pisier	Texas A&M University, and Paris 6 <a href="mailto:pisier@math.tamu.edu">pisier@math.tamu.edu</a>
Michael Christ	University of California, Berkeley, USA <a href="mailto:mchrist@math.berkeley.edu">mchrist@math.berkeley.edu</a>	Tristan Rivière	ETH, Switzerland <a href="mailto:riviere@math.ethz.ch">riviere@math.ethz.ch</a>
Charles Fefferman	Princeton University, USA <a href="mailto:cf@math.princeton.edu">cf@math.princeton.edu</a>	Igor Rodnianski	Princeton University, USA <a href="mailto:irod@math.princeton.edu">irod@math.princeton.edu</a>
Ursula Hamenstaedt	Universität Bonn, Germany <a href="mailto:ursula@math.uni-bonn.de">ursula@math.uni-bonn.de</a>	Yum-Tong Siu	Harvard University, USA <a href="mailto:siu@math.harvard.edu">siu@math.harvard.edu</a>
Vadim Kaloshin	University of Maryland, USA <a href="mailto:vadim.kaloshin@gmail.com">vadim.kaloshin@gmail.com</a>	Terence Tao	University of California, Los Angeles, USA <a href="mailto:tao@math.ucla.edu">tao@math.ucla.edu</a>
Herbert Koch	Universität Bonn, Germany <a href="mailto:koch@math.uni-bonn.de">koch@math.uni-bonn.de</a>	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA <a href="mailto:met@math.unc.edu">met@math.unc.edu</a>
Izabella Laba	University of British Columbia, Canada <a href="mailto:ilaba@math.ubc.ca">ilaba@math.ubc.ca</a>	Gunther Uhlmann	University of Washington, USA <a href="mailto:gunther@math.washington.edu">gunther@math.washington.edu</a>
Richard B. Melrose	Massachusetts Inst. of Tech., USA <a href="mailto:rhm@math.mit.edu">rhm@math.mit.edu</a>	András Vasy	Stanford University, USA <a href="mailto:andras@math.stanford.edu">andras@math.stanford.edu</a>
Frank Merle	Université de Cergy-Pontoise, France <a href="mailto:Frank.Merle@u-cergy.fr">Frank.Merle@u-cergy.fr</a>	Dan Virgil Voiculescu	University of California, Berkeley, USA <a href="mailto:dvv@math.berkeley.edu">dvv@math.berkeley.edu</a>
William Minicozzi II	Johns Hopkins University, USA <a href="mailto:minicozz@math.jhu.edu">minicozz@math.jhu.edu</a>	Steven Zelditch	Northwestern University, USA <a href="mailto:zelditch@math.northwestern.edu">zelditch@math.northwestern.edu</a>
Clément Mouhot	Cambridge University, UK <a href="mailto:c.mouhot@dpms.cam.ac.uk">c.mouhot@dpms.cam.ac.uk</a>	Maciej Zworski	University of California, Berkeley, USA <a href="mailto:zworski@math.berkeley.edu">zworski@math.berkeley.edu</a>
Werner Müller	Universität Bonn, Germany <a href="mailto:mueller@math.uni-bonn.de">mueller@math.uni-bonn.de</a>		

## PRODUCTION

[production@msp.org](mailto:production@msp.org)

Silvio Levy, Scientific Editor

---

See inside back cover or [msp.org/apde](http://msp.org/apde) for submission instructions.

---

The subscription price for 2020 is US \$340/year for the electronic version, and \$550/year (+\$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

---

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

---

APDE peer review and production are managed by EditFlow<sup>®</sup> from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2020 Mathematical Sciences Publishers

# ANALYSIS & PDE

Volume 13 No. 3 2020

---

On the gap between the Gamma-limit and the pointwise limit for a nonlocal approximation of the total variation	627
CLARA ANTONUCCI, MASSIMO GOBBINO and NICOLA PICENNI	
External boundary control of the motion of a rigid body immersed in a perfect two-dimensional fluid	651
OLIVIER GLASS, JÓZSEF J. KOLUMBÁN and FRANCK SUEUR	
Distance graphs and sets of positive upper density in $\mathbb{R}^d$	685
NEIL LYALL and ÁKOS MAGYAR	
Isolated singularities for semilinear elliptic systems with power-law nonlinearity	701
MARIUS GHERGU, SUNGHAN KIM and HENRIK SHAHGHOIAN	
Regularity of the free boundary for the vectorial Bernoulli problem	741
DARIO MAZZOLENI, SUSANNA TERRACINI and BOZHIDAR VELICHKOV	
On the discrete Fuglede and Pompeiu problems	765
GERGELY KISS, ROMANOS DIOGENES MALIKIOSIS, GÁBOR SOMLAI and MÁTÉ VIZER	
Energy conservation for the compressible Euler and Navier–Stokes equations with vacuum	789
IBROKHIMBEK AKRAMOV, TOMASZ DĘBIEC, JACK SKIPPER and EMIL WIEDEMANN	
A higher-dimensional Bourgain–Dyatlov fractal uncertainty principle	813
RUI HAN and WILHELM SCHLAG	
Local minimality results for the Mumford–Shah functional via monotonicity	865
DORIN BUCUR, ILARIA FRAGALÀ and ALESSANDRO GIACOMINI	
The gradient flow of the Möbius energy: $\varepsilon$ -regularity and consequences	901
SIMON BLATT	
Correction to the article The heat kernel on an asymptotically conic manifold	943
DAVID A. SHER	