ANALYSIS & PDE

Volume 13 No. 4 2020

RAIMUNDO LEITÃO, EDGARD A. PIMENTEL AND MAKSON S. SANTOS

GEOMETRIC REGULARITY FOR ELLIPTIC EQUATIONS IN DOUBLE-DIVERGENCE FORM

msp

ANALYSIS AND PDE
Vol. 13, No. 4, 2020
dx.doi.org/10.2140/apde.2020.13.1129

GEOMETRIC REGULARITY FOR ELLIPTIC EQUATIONS IN DOUBLE-DIVERGENCE FORM

RAIMUNDO LEITÃO, EDGARD A. PIMENTEL AND MAKSON S. SANTOS

We examine the regularity of the solutions to the double-divergence equation. We establish improved Hölder continuity as solutions approach their zero level-sets. In fact, we prove that α-Hölder continuous coefficients lead to solutions of class $\mathcal{C}^{1^-}$, locally. Under the assumption of Sobolev-differentiable coefficients, we establish regularity in the class $\mathcal{C}^{1,1^-}$. Our results unveil improved continuity along a nonphysical free boundary, where the weak formulation of the problem vanishes. We argue through a geometric set of techniques, implemented by approximation methods. Such methods connect our problem of interest with a target profile. An iteration procedure imports information from this limiting configuration to the solutions of the double-divergence equation.

1. Introduction

In the present paper we study the regularity theory for solutions to the double-divergence partial differential equation (PDE)

$$\partial^2_{x_i x_j}(a^{ij}(x)u(x)) = 0 \quad \text{in } B_1, \tag{1}$$

where $(a^{ij})_{i,j=1}^d \in \mathcal{S}(d)$ is uniformly (λ, Λ)-elliptic. We produce new (sharp) regularity results for the solutions to (1). In particular, we are concerned with gains of regularity as solutions approach their zero level-sets. We argue through a genuinely geometric class of methods, inspired by the ideas introduced by L. Caffarelli [1989].

Introduced in [Littman 1959], equations in the double-divergence form have been the object of important advances. See [Sjögren 1973; Bogachev and Shaposhnikov 2017; Hervé 1962; Littman 1963; Fabes and Stroock 1984; Bogachev et al. 2015]. The interest in (1) is due to its own mathematical merits, as well as to its varied set of applications.

The primary motivation for the study of (1) is in the realm of stochastic analysis. In fact, (1) is the Kolmogorov–Fokker–Planck equation associated with the stochastic process whose infinitesimal generator is given by

$$Lv(x) := a^{ij}(x)\,\partial^2_{x_i x_j} v(x).$$

Therefore, one can derive information on the stochastic process through the understanding of (1).

MSC2010: 35B65, 35J15.
Keywords: double-divergence equations, geometric regularity, improved regularity at zero level-sets.

A further instance where double-divergence equations play a role is the fully nonlinear mean-field games theory. The model-problem here is

$$\begin{cases} F(D^2V) = g(u) & \text{in } B_1, \\ \partial^2_{x_i x_j}(F^{ij}(D^2V)u(x)) = 0 & \text{in } B_1, \end{cases} \tag{2}$$

where $F : \mathcal{S}(d) \to \mathbb{R}$ is a (λ, Λ)-elliptic operator, $F^{ij}(M)$ stands for the derivative of F with respect to the entry $m_{i,j}$ of M and $g : \mathbb{R} \to \mathbb{R}$ is a given function. In this case, the first equation in (2) is a Hamilton–Jacobi, associated with an optimal control problem. Its solution V accounts for the value function of the game. On the other hand, the population of players, whose density is denoted by u, solves a double-divergence (Fokker–Planck) equation. The mean-field coupling g encodes the preferences of the players with respect to the density of the entire population. Therefore, the solution u describes the equilibrium distribution of a population of rational players facing a scenario of strategic interaction. Through this framework, double-divergence equations are relevant in the modeling of several phenomena in the life and social sciences. As regards the mean-field games theory, we refer the reader to [Gomes et al. 2016].

A further application of equations in double-divergence form occurs in the theory of Hamiltonian stationary Lagrangian manifolds [Chen and Warren 2019]. Let $\Omega \subset \mathbb{R}^d$ be a domain and consider $u \in C^\infty(\Omega)$. The gradient graph of u is the set

$$\Gamma_u := \{(x, Du(x)) : x \in \Omega\},$$

whereas the volume of Γ_u is given by

$$F_\Omega(u) = \int_\Omega \sqrt{\det(I + (D^2u)^T D^2u)}\, \mathrm{d}x.$$

Given $\Omega \subset \mathbb{R}$, the study of critical points/minimizers for $F_\Omega(u)$ yields the compactly supported first variation

$$\int_\Omega \sqrt{\det g}\, g^{ij} \delta^{kl} u_{x_i x_k} \phi_{x_j x_l}\, \mathrm{d}x = 0 \tag{3}$$

for all $\phi \in C_c^\infty(\Omega)$, where

$$g := I + (D^2u)^T D^2u$$

is the induced metric. It is easy to check that (3) is the weak (distributional) formulation of

$$\partial^2_{x_j x_l}(\sqrt{\det g}\, g^{ij} \delta^{kl} u_{x_i x_k}) = 0 \quad \text{in } \Omega.$$

Hence, given a domain, the minimizers of the volume of the gradient graph relate to the solutions of a PDE in the double-divergence form.

As mentioned before, the study of (1) starts in [Littman 1959]. In that paper, the author considers weak solutions to the inequality

$$\partial^2_{x_i x_j}(a^{ij}(x)u(x)) \geq 0 \quad \text{in } B_1,$$

and establishes a strong maximum principle. In [Hervé 1962], the author develops a potential theory associated with (1). This theory is shown to satisfy the same axioms as the potential theory for the elliptic operator

$$a^{ij}(\,\cdot\,)\,\partial^2_{x_i x_j}.$$

Hence, the study of the former provides information on the latter. An improved maximum principle, as well as a preliminary approximation scheme for (1), are the subject of [Littman 1963].

It was only in [Sjögren 1973] that the regularity for the solutions to (1) was first investigated. In that paper, the author proves that solutions coincide with a continuous function, except in a set of measure zero. Together with its converse — and under further conditions — this is called the *fundamental equivalence*. In addition, a result on the α-Hölder continuity of the solutions is presented. Namely, solutions are proven to be locally α-Hölder continuous provided the coefficients satisfy $a^{ij} \in \mathcal{C}^{\alpha}_{\mathrm{loc}}(B_1)$.

In [Fabes and Stroock 1984], the authors examine properties of the Green's function associated with the operator driving (1). One of the results in that paper regards gains of integrability for the solutions. In fact, it is reported that locally integrable, nonnegative solutions are in $L^{d/(d-1)}_{\mathrm{loc}}(B_1)$.

A distinct approach to (1) regards the study of the *densities* of solutions, that is, their Radon–Nikodym derivatives with respect to the Lebesgue measure. In this realm, several developments have been produced; see [Bogachev et al. 2015]. For example it is widely known that, if $(a^{ij})^d_{i,j=1}$ is nondegenerate in B_1, every solution to (1) has a density; see [loc. cit.].

In [Bogachev et al. 2001] the authors prove that $\det[(a^{ij})^d_{i,j=1}]u$ has a density in $L^{d/(d-1)}_{\mathrm{loc}}(B_1)$, provided $u \geq 0$. If, in addition, $(a^{ij})^d_{i,j=1}$ is Hölder continuous and uniformly elliptic, u is proven to have a density in $L^{d/(d-1)}_{\mathrm{loc}}(B_1)$. Regularity in Sobolev spaces is also studied in [loc. cit.]. Under the assumptions that $(a^{ij})^d_{i,j=1}$ is in $W^{1,p}_{\mathrm{loc}}(B_1)$ and $\det[(a^{ij})^d_{i,j=1}]$ is bounded away from zero, the authors prove that solutions have a density in $W^{1,p}_{\mathrm{loc}}(B_1)$. It is worth noticing that [loc. cit.] addresses differential *inequalities* of the form

$$\int_{B_1} a^{ij}(x)\,u(x)\,\phi_{x_i x_j}(x)\,\mathrm{d}x \leq C\|\phi\|_{W^{1,\infty}(B_1)}$$

for some $C > 0$. The corpus of results reported in [loc. cit.] refines important previous developments; see, for instance [Bogachev et al. 1997; Krylov 1986].

In the recent paper [Bogachev and Shaposhnikov 2017], the authors consider densities of the solutions to (1) and investigate their regularity in Hölder and Lebesgue spaces. In addition, they prove a Harnack inequality for nonnegative solutions; see [loc. cit., Corollary 3.6]. Among other things, this result is relevant as it answers in the positive an open question raised in [Mamedov 1992]. In fact, it is shown that densities are in $L^p_{\mathrm{loc}}(B_1)$, for every $p \geq 1$, if $(a^{ij})^d_{i,j=1} \in \mathrm{VMO}(B_1)$. Moreover, the authors examine the regularity of densities in Hölder spaces, provided the coefficients are in the same class.

A remarkable feature of PDEs in the double-divergence form is the following: the regularity of $(a^{ij})^d_{i,j=1}$ acts as an upper bound for the regularity of the solutions. It means that gains of regularity are not (universally) available for the solutions, vis-a-vis the data of the problem. To see this phenomenon in a (very) simple setting, we detail an example presented in [Bogachev and Shaposhnikov 2017]. Set $d = 1$ and consider the homogeneous problem

$$(a(x)v(x))_{xx} = 0 \quad \text{in }]-1, 1[. \tag{4}$$

Take an arbitrary affine function $\ell : B_1 \to \mathbb{R}$ and let $u(x) := \ell(x)/a(x)$. Notice that

$$\int_{B_1} a(x)\frac{\ell(x)}{a(x)}\phi_{xx}\,\mathrm{d}x = 0$$

for every $\phi \in \mathcal{C}^2_0(]-1,1[)$. Therefore, u is a solution to (4). It is clear that, if $a(x)$ is discontinuous, then u is as well.

Although solutions lack gains of regularity in the entire domain, a natural question regards the conditions under which improvements on the Hölder continuity can be established. Let $S \subset B_1$ be a fixed subset of the domain and suppose that further, natural conditions are placed on $(a^{ij})^d_{i,j=1} \in \mathcal{C}^\beta_{\mathrm{loc}}(B_1)$. The regularity of the solutions along S will be important. Even more relevant in some settings is the regularity of the solutions as they *approach* $S \subset B_1$.

In this paper, we consider the zero level-set of the solutions to (1). That is,

$$S_0[u] := \{x \in B_1 : u(x) = 0\}.$$

We prove that, along S_0, solutions to (1) are of class $\mathcal{C}^\alpha$ for every $\alpha \in (0,1)$, provided $(a^{ij})^d_{i,j=1}$ is Hölder continuous and satisfies a proximity regime of the form

$$\|a^{ij} - a^{ij}(0)\|_{L^\infty(B_1)} \ll \tfrac{1}{2}.$$

The precise statement of our first main result is the following:

Theorem 1. *Let $u \in L^1_{\mathrm{loc}}(B_1)$ be a weak solution to* (1)*. Suppose assumptions A1–A2, to be set forth in Section 2.1, are in force. Let $x_0 \in S_0(u)$. Then u is of class $\mathcal{C}^{1-}$ at x_0 and there exists a constant $C > 0$ such that*

$$\sup_{B_r(x_0)} |u(x_0) - u(x)| \le C r^{\alpha^*}$$

for every $\alpha^* \in (0,1)$.

The contribution of Theorem 1 is to ensure gains of regularity for the solutions to (1) *as they approach the zero level-set*, though estimates in the whole domain are constrained by the regularity of the coefficients a^{ij}. From a heuristic viewpoint, whichever level of ε-Hölder continuity is available for the coefficients — with $0 < \varepsilon \ll \frac{1}{2}$ — suffices to produce $\mathcal{C}^{1^-}$ regularity for the solutions along $S_0[u]$. See Figure 1.

The choice for S_0 is two-fold. Indeed, along this set, the weak formulation of (1) vanishes. Hence, at least intuitively, the weak formulation of the problem fails to provide information on the original equation along $S_0[u]$. A remarkable feature of (1) is related to this apparent lack of information across the zero level-set. As a matter of fact, the structure of the equation is capable of enforcing higher regularity of the solutions along the set where the weak formulation vanishes.

A second instance of motivation for the choice of S_0 falls within the scope of the *nonphysical free boundaries*. Introduced as a technology inspired by free boundary problems in the regularity theory of (nonlinear) partial differential equations, this class of methods has advanced the understanding of fine properties of solutions to a number of important examples. We refer the reader to [Teixeira 2014].

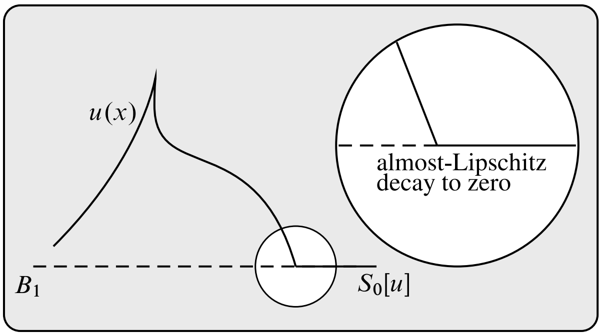

Figure 1. Almost-Lipschitz decay to zero: although the graph of the solutions to (1) admits cusps in the presence of merely Hölder continuous coefficients, they approach their zero level-sets with $\mathcal{C}^\alpha$-regularity for every $\alpha \in (0, 1)$. It means that solutions reach the nonphysical free boundary in an almost-Lipschitz manner.

In addition to the study of (1) in the presence of Hölder continuous coefficients, we also consider the case $(a^{ij})_{i,j=1}^d \in W^{2,p}_{\mathrm{loc}}(B_1)$ for $p > d$. In this setting, (1) becomes

$$\partial_{x_i}\big(a^{ij}(x)\,\partial_{x_j}u(x) + \partial_{x_j}a^{ij}(x)u(x)\big) = 0 \quad \text{in } B_1. \tag{5}$$

Here, two new layers of information are unveiled. First, it is known that solutions to (5) are in $\mathcal{C}^{1,1-d/p}_{\mathrm{loc}}(B_1)$ — see [Ladyzhenskaya and Uraltseva 1968, Chapter 3, Theorem 15.1]; i.e., the gradient of the solutions exists in classical sense. Second, the weak formulation of the problem vanishes at a different subset of the domain, namely

$$S_1[u] := \{x \in B_1 : u(x) = 0 \text{ and } Du(x) = \mathbf{0}\}.$$

Under the assumption $(a^{ij})_{i,j=1}^d \in W^{2,p}_{\mathrm{loc}}(B_1)$, and the appropriate proximity regime, we prove that solutions to (1) are locally of class $\mathcal{C}^{1,1^-}$ along $S_1[u]$. This is the content of our second main result:

Theorem 2 (Hölder regularity of the gradient). *Let $u \in L^1_{\mathrm{loc}}(B_1)$ be a weak solution to* (1). *Suppose A1 and A3, to be introduced in Section 2.1, hold true. Let $x_0 \in S_1[u]$. Then u is of class $\mathcal{C}^{1,1^-}$ at x_0 and there exists a constant $C > 0$ such that*

$$\sup_{x \in B_r(x_0)} |Du(x) - Du(x_0)| \le Cr^{\alpha^*}$$

for every $\alpha^* \in (0, 1)$.

The regularity of the coefficients in Sobolev spaces is pivotal in establishing Theorem 2. Here, Sobolev-differentiable coefficients switch the regularity regime of (1) allowing for an alternative weak formulation of the problem.

We remark that our methods accommodate equations with explicit dependence on lower-order terms; i.e.,

$$\partial^2_{x_i x_j}(a^{ij}(x)u(x)) - \partial_{x_i}(b^i(x)u(x)) + c(x)u(x) = 0 \quad \text{in } B_1,$$

provided the vector field $b : B_1 \to \mathbb{R}^d$ and the function $c : B_1 \to \mathbb{R}$ are well-prepared. See Remarks 13 and 18.

Our arguments are intrinsically geometric. We approximate weak solutions to (1) by solutions to a homogeneous, fixed-coefficient equation of the form

$$a^{ij}(0)\,\partial^2_{x_i x_j} v(x) = 0 \quad \text{in } B_1. \tag{6}$$

Among such solutions, we select v such that $S_0[u] \subset S_0[v]$, and $S_1[v] \subset S_1[u]$, when appropriate. An approximation routine builds upon the regularity theory available for the solutions to (6). This is achieved through a geometric strategy, which produces a preliminary oscillation control. To turn this initial information into an oscillation control in every scale, an iterative method takes place. This line of reasoning is inspired by trail-blazing ideas first introduced in [Caffarelli 1989]. See also [Caffarelli and Cabré 1995].

The remainder of this paper is organized as follows: Section 2.1 details our main assumption, whereas Section 2.2 collects a few elementary facts and notions, together with auxiliary results. In Section 3 we put forward a zero level-set approximation lemma and present the proof of Theorem 1. A finer approximation result appears in Section 4, where we conclude the proof of Theorem 2.

2. Preliminary material and main assumptions

In this section we introduce the main elements used in our arguments throughout the paper. Firstly we discuss our assumptions on the structure of the problem. Then, we collect a few definitions and results.

2.1. ***Main assumptions.*** In what follows, we detail the main hypotheses under which we work in the present paper. We start with an assumption on the uniform ellipticity of the coefficients matrix $(a^{ij})_{i,j=1}^d$.

A1 (uniform ellipticity). *We assume the symmetric matrix* $(a^{ij}(x))_{i,j=1}^d$ *satisfies a* (λ, Λ)*-ellipticity condition of the form*

$$\lambda\,\mathrm{Id} \le (a^{ij}(x))_{i,j=1}^d \le \Lambda\,\mathrm{Id}$$

for some $0 < \lambda \le \Lambda$, *uniformly in* $x \in B_1$.

The next assumption concerns the regularity requirements on the coefficients to ensure Hölder continuity of the solutions to (1). This fact is central in the proof of Theorem 1.

A2 (α-Hölder continuity). *The map* $(a^{ij}(x))_{i,j=1}^d : B_1 \to \mathcal{S}(d)$ *is locally uniformly* α*-Hölder continuous. That is, we have*

$$a^{ij} \in \mathcal{C}^{\alpha}_{\mathrm{loc}}(B_1)$$

for every $1 \le i \le d$ *and* $1 \le j \le d$.

We conclude this section with a further set of conditions on the coefficients a^{ij}. Such an assumption unlocks the study of the gradient-regularity for the solutions to (1), along $S_1[u]$.

A3 (Sobolev differentiability of the coefficients). *Let* $p > d$. *The map*

$$(a^{ij})_{i,j=1}^d : B_1 \to \mathcal{S}(d)$$

is in $W^{2,p}_{\mathrm{loc}}(B_1)$. *That is, we have*

$$a^{ij} \in W^{2,p}_{\mathrm{loc}}(B_1)$$

for every $1 \le i \le d$ *and* $1 \le j \le d$.

In the next section we gather elementary notions and basic facts used further in the paper.

2.2. *Preliminary notions and results.* We start with a result first proven in [Sjögren 1973]. It concerns the existence of a continuous version to the weak solutions to (1).

Proposition 3 (continuous version of weak solutions). *Let $u \in L^1_{\mathrm{loc}}(B_1)$ be a weak solution to* (1). *Then, there exists a null set $\Omega \subset B_1$ and $v \in \mathcal{C}(B_1)$ such that*

$$u \equiv v \quad \text{in } B_1 \setminus \Omega.$$

Proof. For the proof of the proposition, we refer the reader to [Sjögren 1973, Lemma 1]; see also [Sjögren 1975]. □

Remark 4. Hereafter, we suppose that every locally integrable function solving (1) in the weak sense is continuous.

Before proceeding we recall the fundamental solution of the operator

$$a^{ij}(y)\,\partial^2_{x_i x_j};$$

such a function will be denoted by $H(x, y)$. In the case $d > 2$, H is defined as

$$H(x, y) := \frac{[a_{ij}(y)(x_i - y_i)(x_j - y_j)]^{(2-d)/2}}{(d-2)\alpha(d)\sqrt{\det[(a^{ij})^d_{i,j=1}]}}, \tag{7}$$

where $(a_{ij})^d_{i,j=1} := [(a^{ij})^d_{i,j=1}]^{-1}$ and $\alpha(d)$ stands for the volume of the unit ball in dimension d.

A fundamental result in the context of this paper regards initial levels of compactness for the solutions to (1). This is the subject of the next proposition, which we recall here for the sake of completeness.

Proposition 5 (compactness of the solutions). *Let $u \in L^1_{\mathrm{loc}}(B_1)$ be a weak solution to* (1). *Suppose A1–A2 are in force. Then, $u \in \mathcal{C}^\alpha_{\mathrm{loc}}(B_1)$ and there exists a constant $C > 0$ such that*

$$\|u\|_{\mathcal{C}^\alpha(B_{1/2})} \le C, \tag{8}$$

with $C = C(d, \lambda, \Lambda, \|a^{ij}\|_{\mathcal{C}^\alpha(B_1)}, \|u\|_{L^\infty(B_1)})$.

Proof. The inclusion $u \in \mathcal{C}^\alpha_{\mathrm{loc}}(B_1)$ is a well-known result; see, for instance [Sjögren 1973, Theorem 2]. As for the estimate in (8), it follows from considerations on the oscillation of the fundamental solution H, defined in (7), and its derivatives; see the proof of [loc. cit., Theorem 2]. □

We proceed with a proposition on the sequential stability of the solutions to (1). It will be used later to establish two approximation lemmas.

Proposition 6 (sequential stability of weak solutions). *Suppose that*

$$([a^{ij}_n]^d_{i,j=1})_{n\in\mathbb{N}} \subset \mathcal{C}^\alpha_{\mathrm{loc}}(B_1; \mathcal{S}(d))$$

is a sequence of matrices such that

$$\|a^{ij}_n - a^{ij}(x_0)\|_{L^\infty(B_1)} \to 0$$

as $n \to \infty$*. Suppose further that* $(f_n)_{n\in\mathbb{N}} \subset L^p(B_1)$ *is such that*

$$\|f_n\|_{L^p(B_1)} \to 0$$

as $n \to \infty$*. Let* $(u_n)_{n\in\mathbb{N}} \subset L^1_{\text{loc}}(B_1)$ *satisfy*

$$\partial^2_{x_i x_j}(a_n^{ij}(x)u_n(x)) = f_n \quad \text{in } B_1.$$

If there exists $u_\infty \in \mathcal{C}(B_1)$ *such that*

$$\|u_n - u_\infty\|_{L^\infty(B_1)} \to 0$$

as $n \to \infty$, *then* u_∞ *satisfies*

$$\int_{B_1} a^{ij}(x_0)u_\infty(x)\phi_{x_i x_j}(x)\,\mathrm{d}x = 0$$

for every $\phi \in \mathcal{C}^2_c(B_1)$.

Proof. First, notice that we have $a_n^{ij}(x_0) \to a^{ij}(x_0)$ as $n \to \infty$. Now, for every $\phi \in \mathcal{C}^2_c(B_1)$ we have

$$\left|\int_{B_1} \phi_{x_i x_j} a^{ij}(x_0)u_\infty(x)\,\mathrm{d}x\right| \le \int_{B_1} |\phi_{x_i x_j}|\,|a^{ij}(x_0) - a_n^{ij}(x)|\,|u_\infty(x)|\,\mathrm{d}x + \int_{B_1} |\phi_{x_i x_j}|\,|a_n^{ij}(x)|\,|u_n(x) - u_\infty(x)|\,\mathrm{d}x + \int_{B_1} |\phi|\,|f_n|\,\mathrm{d}x.$$

Notice that the right-hand side of this inequality converges to zero as $n \to \infty$. Therefore,

$$\int_{B_1} \phi_{x_i x_j} a^{ij}(x_0)u_\infty(x)\,\mathrm{d}x = 0. \qquad \square$$

In addition to the sequential stability, our arguments require an initial degree of compactness for the solutions to (1). When it comes to the proof of Theorem 1, uniform compactness comes from Proposition 5. In the case of Theorem 2, we turn to a well-known result on the regularity of the (weak) solutions to equations in the divergence form. We start with an observation.

In case A3 is in force, we claim that (1) can be written as

$$\partial_{x_i}\big(a^{ij}(x)\,\partial_{x_j}u(x) + \partial_{x_j}a^{ij}(x)u(x)\big) = 0 \quad \text{in } B_1. \tag{9}$$

Indeed, if a^{ij} is weakly differentiable, we have

$$\int_{B_1} a^{ij}u\,\partial_{x_i x_j}\phi\,\mathrm{d}x = -\int_{B_1} (a^{ij}\,\partial_{x_j}u + \partial x_j a^{ij}u)\,\partial_{x_i}\phi\,\mathrm{d}x$$

for every $\phi \in \mathcal{C}^2_c(B_1)$. Hence, under A3, the homogeneous version of (1) is equivalent to (9). Now we are in position to state the following:

Proposition 7. *Let* $v \in W^{1,p}(B_1)$ *be a weak solution to* (9)*. Suppose A1 and A3 are in force. Then*, $v \in \mathcal{C}^{1,\alpha}_{\text{loc}}(B_1)$, *where*

$$\alpha := \frac{p-d}{p}.$$

Moreover, *there exists a universal constant* $C > 0$ *such that*

$$\|v\|_{\mathcal{C}^{1,\alpha}(B_{1/2})} \le C\|v\|_{L^\infty(B_1)}.$$

For the proof of Proposition 7, we refer the reader to [Ladyzhenskaya and Uraltseva 1968, Chapter 3, Theorem 15.1]. The former proposition is paramount in establishing Theorem 2. Apart from compactness, it produces gradient-continuity for the solutions to (9). This information plays a critical role in the treatment of fine regularity properties of the solutions to the homogeneous version of (1) along $S_1[u]$. In particular, it unlocks a first zero level-set approximation result.

We conclude this section with a comment on the scaling properties of (1). Indeed, we consider weak solutions satisfying $\|u\|_{L^\infty(B_1)} \leq 1$. Let $\bar{u} \in \mathcal{C}(B_1)$ be defined as

$$\bar{u}(x) := \frac{u(x)}{\max\{1, \|u\|_{L^\infty(B_1)}\}},$$

where u is a weak solution to (1). It is clear that $\bar{u}$ is a weak solution to

$$\partial^2_{x_i x_j}(a^{ij}(x)\bar{u}(x)) = 0 \quad \text{in } B_1.$$

Notice that $\|\bar{u}\|_{L^\infty(B_1)} \leq 1$. Then, hereinafter we consider, without loss of generality, normalized solutions to (1). In the sequel, we set forth the proof of Theorem 1.

3. Improved regularity of the solutions

In this section we detail the proof of Theorem 1. As mentioned before, we reason through an approximation/geometric method. At the core of our argument lies a zero level-set approximation lemma. It reads as follows:

Proposition 8 (zero level-set approximation lemma). *Let $u \in L^1_{\text{loc}}(B_1)$ be a weak solution to* (1), *$x_0 \in S_0[u] \cap B_{9/10}$ and suppose A1–A2 are in force. Given $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that, if*

$$\sup_{x \in B_1} |a^{ij}(x) - a^{ij}(0)| < \varepsilon,$$

there exists $h \in \mathcal{C}^{1,1}(B_{9/10})$ satisfying

$$\|u - h\|_{L^\infty(B_{9/10})} < \delta,$$

with

$$h(x_0) = 0.$$

Proof. The proof follows from a contradiction argument. We start by supposing that the statement of the proposition is false. Therefore, there exist $\delta_0 > 0$ and sequences $([a^{ij}_n]^d_{i,j=1})_{n \in \mathbb{N}}$ and $(u_n)_{n \in \mathbb{N}} \subset L^\infty(B_1)$ such that

$$\sup_{x \in B_1} |a^{ij}_n(x) - a^{ij}_n(0)| \sim \frac{1}{n},$$
$$x_0 \in S_0[u_n] \cap B_{9/10},$$
$$\partial^2_{x_i x_j}(a^{ij}_n(x) u_n(x)) = 0 \quad \text{in } B_1,$$

but

$$|u_n(x) - h(x)| > \delta_0 \quad \text{or} \quad h(x_0) \neq 0$$

for every $h \in \mathcal{C}^{1,1}(B_{9/10})$ and every $n \in \mathbb{N}$.

Notice that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $\mathcal{C}^{\alpha}(B_1)$. Therefore, there exists u_∞ such that

$$\|u_n - u_\infty\|_{\mathcal{C}^{\beta}(B_1)} \to 0$$

for every $0 < \beta < \alpha$, through a subsequence, if necessary. On the other hand, we have $a_n^{ij}(0) \to \bar{a}^{ij}(0)$ as $n \to \infty$; hence

$$|a_n^{ij}(x) - \bar{a}^{ij}(0)| \le |a_n^{ij}(x) - a_n^{ij}(0)| + |a_n^{ij}(0) - \bar{a}^{ij}(0)|.$$

Therefore

$$\|a_n^{ij} - \bar{a}^{ij}(0)\|_{L^\infty(B_1)} \to 0$$

as $n \to \infty$. Hence, the sequential stability of weak solutions (Proposition 6) leads to

$$\partial^2_{x_i x_j}(\bar{a}^{ij}(0)u_\infty(x)) = 0 \quad \text{in } B_{9/10}.$$

The regularity theory for constant-coefficient equations implies that $u_\infty \in \mathcal{C}^{1,1}(B_{9/10})$ and, moreover, $u_\infty(x_0) = 0$. Finally, there exists $N \in \mathbb{N}$ such that

$$|u_n(x) - u_\infty(x)| < \delta_0,$$

provided $n > N$. By taking $h \equiv u_\infty$, we produce a contradiction and conclude the proof. □

Remark 9. The proof of Proposition 8 shows that the approximating function h solves the problem

$$\begin{cases} \partial^2_{x_i x_j}(\bar{a}^{ij}(0)h(x)) = 0 & \text{in } B_{9/10}, \\ h = h_0 & \text{on } \partial B_{9/10}, \end{cases} \tag{10}$$

where

$$\|h_0\|_{L^\infty(\partial B_{9/10})} \le \delta + \|u\|_{L^\infty(B_1)}.$$

Therefore, it follows from standard results in elliptic regularity theory that

$$\|h\|_{\mathcal{C}^{1,1}(B_{9/10})} \le C(1 + \|u\|_{L^\infty(B_1)}),$$

where $C > 0$ depends on the dimension d, the ellipticity constants λ and Λ and $\bar{a}^{ij}(0)$. We notice the constant C does not depend on u.

Remark 10. A priori, the parameter $\varepsilon > 0$ depends only on $\delta > 0$. We notice however that (a universal) choice of δ, made further in the paper, implies that ε will depend on the exponent α, the dimension d, λ, Λ and $\|u\|_{L^\infty(B_1)}$. Therefore, we have

$$\varepsilon = \varepsilon(\alpha, d, \lambda, \Lambda, \|u\|_{L^\infty(B_1)}).$$

Next, we control the oscillation of the solutions to (1) within a ball of radius $0 < \rho \ll \frac{1}{2}$, to be determined later.

Proposition 11. *Let $u \in L^1(B_1)$ be a weak solution to* (1). *Suppose A1–A2 are in force. Then, for every $\alpha \in (0, 1)$, there exists $\varepsilon > 0$ such that, if $x_0 \in S_0[u] \cap B_{9/10}$ and*

$$\sup_{x \in B_1} |a^{ij}(x) - a^{ij}(0)| < \varepsilon,$$

we can find $0 < \rho \ll \frac{1}{2}$ *for which*

$$\sup_{B_\rho(x_0)} |u(x)| \le \rho^\alpha.$$

Proof. We start by taking a function $h \in \mathcal{C}^{1,1}_{\mathrm{loc}}(B_{9/10})$ satisfying

$$\|u-h\|_{L^\infty(B_{9/10})} < \delta,$$

with

$$h(x_0) = 0.$$

The existence of such a function is guaranteed by Proposition 8. We have

$$\sup_{x\in B_\rho(x_0)} |h(x)-h(x_0)| \le C\rho$$

for some constant $C > 0$; see Remark 9. Therefore,

$$\sup_{x\in B_\rho(x_0)} |u(x)-h(x_0)| \le \sup_{x\in B_\rho(x_0)} |u(x)-h(x)| + \sup_{x\in B_\rho(x_0)} |h(x)-h(x_0)| \le \delta + C\rho. \tag{11}$$

In the sequel, we make universal choices for ρ and δ; in fact, for a given $\alpha \in (0, 1)$, we set

$$\rho := \left(\frac{1}{2C}\right)^{1/(1-\alpha)} \quad \text{and} \quad \delta := \frac{\rho^\alpha}{2}. \tag{12}$$

Finally, we combine (11) with (12) to obtain

$$\sup_{B_\rho(x_0)} |u(x)| \le \rho^\alpha. \qquad \square$$

Proposition 12. *Let* $u \in L^1_{\mathrm{loc}}(B_1)$ *be a weak solution to* (1)*. Suppose assumptions A1–A2 are in force. Then, there exists* $\varepsilon > 0$ *so that, if* $x_0 \in S_0[u] \cap B_{9/10}$ *and*

$$\sup_{x\in B_1} |a^{ij}(x) - a^{ij}(0)| < \varepsilon,$$

we can find $0 < \rho \ll \frac{1}{2}$ *for which*

$$\sup_{B_{\rho^n}(x_0)} |u(x)| \le \rho^{n\alpha}$$

for every $n \in \mathbb{N}$.

Proof. We resort to an induction argument. First, we make the same choices as in (12); this (universally) determines the parameter ε. The first step of induction — the case $n = 1$ — follows from Proposition 11. The induction hypothesis refers to the case $n = k$; i.e.,

$$\sup_{B_{\rho^k}(x_0)} |u(x)| \le \rho^{k\alpha}$$

for some $k \in \mathbb{N}$.

In the sequel we address the case $n = k+1$. To that end, we introduce an auxiliary function $v_k : B_1 \to \mathbb{R}$, defined as

$$v_k(x) := \frac{u(x_0 + \rho^k x)}{\rho^{k\alpha}}.$$

We observe that $v_k(0)=0$. In addition v_k solves

$$\partial^2_{x_ix_j}(a_k^{ij}(x)v_k(x))=0 \quad \text{in } B_1, \tag{13}$$

where

$$a_k^{ij}(x):=a^{ij}(x_0+\rho^k x).$$

Now, notice that

$$|a_k^{ij}(x)-a^{ij}(0)|=|a^{ij}(x_0+\rho^k x)-a^{ij}(0)|\leq \varepsilon.$$

Finally, the matrix $(a_k^{ij})_{i,j=1}^d$ inherits the Hölder continuity and the (λ,Λ)-ellipticity of $(a^{ij})_{i,j=1}^d$. Therefore, (13) falls within the scope of Proposition 11. Hence,

$$\sup_{B_{\rho^k}}|v_k(x)|\leq \rho^\alpha;$$

by rescaling back to the unitary setting, we get

$$\sup_{B_{\rho^{k+1}}(x_0)}|u(x)|\leq \rho^{(k+1)\alpha}$$

and complete the proof. □

Proof of Theorem 1. Let $0<r\ll \frac{1}{2}$ be fixed and take $x_0\in S_0[u]$. We must verify that

$$\sup_{B_r(x_0)}|u(x)-u(x_0)|\leq Cr^\alpha,$$

where $C>0$ is universal. Fix $n\in\mathbb{N}$ such that $\rho^{n+1}\leq r\leq \rho^n$. Observe that

$$\begin{aligned}\sup_{B_r(x_0)}|u(x)-u(x_0)| &\leq \sup_{B_{\rho^n}(x_0)}|u(x)-u(x_0)|\\ &\leq \rho^{-\alpha}\rho^{(n+1)\alpha}\leq Cr^\alpha.\end{aligned}$$ □

We conclude this section with a remark on double divergence equations with explicit dependence on lower-order terms.

Remark 13. To extend our result to model-problems of the form

$$\partial^2_{x_ix_j}(a^{ij}(x)u(x))+\partial_{x_i}(b^i(x)u(x))+c(x)u(x)=0 \quad \text{in } B_1,$$

it suffices to impose two conditions on $b:B_1\to\mathbb{R}^d$ and $c:B_1\to\mathbb{R}$. Indeed, these maps must be Hölder continuous; such a requirement unlocks the uniform compactness of the solutions. Secondly, a proximity regime must be in force; that is, there must be $\bar{b}\in\mathbb{R}^d$ and $\bar{c}\in\mathbb{R}$ so that

$$\|b^i-\bar{b}^i\|_{L^\infty(B_1)}+\|c-\bar{c}|_{L^\infty(B_1)}\ll \tfrac{1}{2}.$$

In what follows we focus on the proof of Theorem 2.

4. Hölder continuity of the gradient

This section sets forth the proof of Theorem 2. As before, the main ingredient is a first level-set approximation lemma.

Proposition 14 (first level-set approximation lemma). *Let $u \in L^1_{\text{loc}}(B_1)$ be a weak solution to* (1) *and suppose A1 and A3 are in force. Given $\delta > 0$, there exists $\varepsilon > 0$ such that, if $x_0 \in S_1[u] \cap B_{9/10}$ and*

$$\sup_{x\in B_1} |a^{ij}(x) - a^{ij}(x_0)| <, \varepsilon,$$

there exists $h \in C^{1,1}(B_{9/10})$ satisfying

$$\|u - h\|_{\mathcal{C}^{1,\beta}(B_{9/10})} < \delta$$

for some $\beta \in (0, 1)$, with

$$h(x_0) = 0 \quad \textit{and} \quad Dh(x_0) = \mathbf{0}.$$

Proof. We argue by contradiction. Suppose the statement of the proposition is false, in this case there exists $\delta_0 > 0$ and sequences $([a_n^{ij}]_{i,j=1}^d)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$ such that

$$\|a_n^{ij}(x) - a_n^{ij}(x_0)\|_{L^\infty(B_1)} \sim \frac{1}{n},$$
$$x_0 \in S_1[u_n] \cap B_{9/10},$$
$$\partial^2_{x_i x_j}(a_n^{ij}(x)u_n(x)) = 0 \quad \text{in } B_1,$$

with

$$|u_n(x) - h(x)| > \delta_0,$$

and either $h(x_0) \neq 0$ or $Dh(x_0) \neq \mathbf{0}$ for every $h \in C^{1,1}(B_{9/10})$ and $n \in \mathbb{N}$. By Proposition 7 we have that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C^{1,\alpha}(B_1)$. Then, through a subsequence, if necessary, there exists a function u_∞ such that

$$\|u_n - u_\infty\|_{C^{1,\gamma}(B_1)} \to 0$$

for every $0 < \gamma < \beta$. In particular

$$u_n(x_0) \to u_\infty(x_0) \quad \text{and} \quad Du_n(x_0) \to Du_\infty(x_0).$$

Then $u_\infty(x_0) = 0$ and $Du_\infty(x_0) = \mathbf{0}$. Furthermore, $a_n^{ij}(x_0) \to \bar{a}^{ij}(x_0)$ as $n \to \infty$; hence, as before, $a_n^{ij}(x) \to \bar{a}^{ij}(x_0)$ as $n \to \infty$.

Here, we evoke once again the sequential stability of the weak solutions, Proposition 6, to conclude that u_∞ solves

$$\partial^2_{x_i x_j}(\bar{a}^{ij}(x_0)u_\infty(x)) = 0 \quad \text{in } B_{9/10}.$$

The regularity theory for constant coefficients implies that $u_\infty \in C^{1,1}(B_{9/10})$. By taking $h \equiv u_\infty$, we produce a contradiction and establish the result. □

Remark 15. As in Remark 9, we notice that the norm of h in $\mathcal{C}^2$ depends on the solution u only through its L^∞-norm.

Proposition 16. *Let $u \in L^1_{\text{loc}}(B_1)$ be a weak solution to* (1) *and suppose A1 and A3 are in force. Then, for every $\alpha \in (0, 1)$, there exists $\varepsilon > 0$ such that, if $x_0 \in S_1[u] \cap B_{9/10}$ and*

$$\sup_{x\in B_1} |a^{ij}(x) - a^{ij}(x_0)| < \varepsilon,$$

we can find $0 < \rho \ll \frac{1}{2}$ such that

$$\sup_{B_\rho(x_0)} |Du(x) - Du(x_0)| \leq \rho^\alpha.$$

Proof. By Proposition 14, there exists $h \in C^{1,1}(B_1)$ such that

$$\|u - h\|_{C^{1,\beta}(B_{9/10})} < \delta,$$

with $x_0 \in S_1[u] \cap B_{9/10}$. We have

$$\begin{aligned}\sup_{B_\rho(x_0)} |Du(x)-Du(x_0)| &\leq \sup_{B_\rho(x_0)} |Du(x)-Dh(x)| + \sup_{B_\rho(x_0)} |Dh(x)-Dh(x_0)| + \sup_{B_\rho(x_0)} |Dh(x_0)-Du(x_0)| \\ &\leq \delta + C\rho.\end{aligned}$$

Now, by choosing

$$\rho := \left(\frac{1}{2C}\right)^{1/(1-\alpha)} \quad \text{and} \quad \delta := \frac{\rho^\alpha}{2},$$

we obtain

$$\sup_{B_\rho(x_0)} |Du(x) - Du(x_0)| \leq \rho^\alpha$$

and finish the proof. □

Proposition 17. *Let $u \in L^1_{\text{loc}}(B_1)$ be a weak solution to* (1) *and suppose A1 and A3 are in force. Then, there exists $\varepsilon > 0$ such that, if $x_0 \in S_1[u] \cap B_{9/10}$ and*

$$\sup_{x\in B_1} |a^{ij}(x) - a^{ij}(x_0)| < \varepsilon,$$

we can find $0 < \rho \ll \frac{1}{2}$ for which

$$\sup_{B_{\rho^n}(x_0)} |Du(x) - Du(x_0)| \leq \rho^{n\alpha}$$

for every $n \in \mathbb{N}$ and every $\alpha \in (0, 1)$.

Proof. We shall verify the proposition by induction. Notice that Proposition 16 amounts to the first step in the induction argument. Suppose we have verified the statement for $n = k$. It remains to verify it in the case $n = k + 1$. Define the function

$$v_k(x) := \frac{u(x_0 + \rho^k x)}{\rho^{k(1+\alpha)}}.$$

We start by noting that $0 \in S_1[v_k]$. Additionally, v_k solves

$$\partial^2_{x_i x_j}(a^{ij}_k(x) v_k(x)) = 0 \quad \text{in } B_1, \tag{14}$$

where

$$a^{ij}_k(x) := a^{ij}(x_0 + \rho^k x).$$

It is clear that

$$\int_{B_1} |a^{ij}(x_0+\rho^k x)|^p \,\mathrm{d}x = \frac{1}{\rho^{dk}} \int_{B_{\rho^k}(x_0)} |a^{ij}(y)|^p \,\mathrm{d}y < C,$$

where the inequality follows from A1. Also,

$$\int_{B_1} |D(a^{ij}(x_0+\rho^k x))|^p \,\mathrm{d}x = \rho^{k(p-d)} \int_{B_{\rho^k}(x_0)} |Da^{ij}(y)|^p \,\mathrm{d}y < C,$$

since $p > d$, by hypothesis. Similarly

$$\int_{B_1} |D^2(a^{ij}(x_0+\rho^k x))|^p \,\mathrm{d}x = \rho^{k(2p-d)} \int_{B_{\rho^k}(x_0)} |D^2a^{ij}(y)|^p \,\mathrm{d}y < C.$$

Hence, (14) falls within the scope of Proposition 16. Therefore

$$\sup_{B_\rho} |Dv_k(x) - Dv_k(0)| \le \rho^\alpha.$$

Rescaling back to the unit ball, the former inequality implies

$$\sup_{B_{\rho^{k+1}}(x_0)} |Du(x) - Du(x_0)| \le \rho^{(k+1)\alpha}.$$ □

Proof of Theorem 2. The proof follows the general lines of the proof of Theorem 1 and will be omitted. □

Remark 18. As in the previous case, it is possible to extend this result to model-problems of the form

$$\partial^2_{x_i x_j}(a^{ij}(x)u(x)) + \partial_{x_i}(b^i(x)u(x)) + c(x)u(x) = f(x) \quad \text{in } B_1.$$

As before, it suffices to impose two conditions on $b : B_1 \to \mathbb{R}^d$ and $c : B_1 \to \mathbb{R}$. Indeed, the map b must be $W^{1,p}(B_1)$, and the map c must be $L^p(B_1),\ p > d$; such a requirement unlocks the uniform compactness of the solutions. Secondly, a proximity regime must be in force; that is, there must be $\bar{b} \in \mathbb{R}^d$ and $\bar{c} \in \mathbb{R}$ so that

$$\|b^i - \bar{b}^i\|_{W^{1,p}(B_1)} + \|c - \bar{c}\|_{L^\infty(B_1)} \ll \tfrac{1}{2}.$$

Acknowledgements

The authors are very grateful to an anonymous referee for comments and suggestions, which led to a substantial improvement in the material in the paper. Pimentel is partially funded by CNPq-Brazil (grants #433623/2018-7 and # 307500/2017-9) and FAPERJ (grant #E26/200.002/2018). Santos is supported by CAPES-Brazil. Part of this work was developed during a visit of Pimentel and Santos to the International Centre for Theoretical Physics (ICTP, Trieste); the authors are grateful for the Centre's support and hospitality. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brazil (CAPES) - Finance Code 001.

References

[Bogachev and Shaposhnikov 2017] V. I. Bogachev and S. V. Shaposhnikov, "Integrability and continuity of solutions to double divergence form equations", *Ann. Mat. Pura Appl.* (4) **196**:5 (2017), 1609–1635. MR Zbl

[Bogachev et al. 1997] V. I. Bogachev, N. V. Krylov, and M. Röckner, "Elliptic regularity and essential self-adjointness of Dirichlet operators on $\mathbb{R}^n$", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **24**:3 (1997), 451–461. MR Zbl

[Bogachev et al. 2001] V. I. Bogachev, N. V. Krylov, and M. Röckner, "On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions", *Comm. Partial Differential Equations* **26**:11-12 (2001), 2037–2080. MR Zbl

[Bogachev et al. 2015] V. I. Bogachev, N. V. Krylov, M. Röckner, and S. V. Shaposhnikov, *Fokker–Planck–Kolmogorov equations*, Math. Surveys and Monographs **207**, Amer. Math. Soc., Providence, RI, 2015. MR Zbl

[Caffarelli 1989] L. A. Caffarelli, "Interior a priori estimates for solutions of fully nonlinear equations", *Ann. of Math.* (2) **130**:1 (1989), 189–213. MR Zbl

[Caffarelli and Cabré 1995] L. A. Caffarelli and X. Cabré, *Fully nonlinear elliptic equations*, Amer. Math. Soc. Colloquium Publ. **43**, Amer. Math. Soc., Providence, RI, 1995. MR Zbl

[Chen and Warren 2019] J. Chen and M. Warren, "On the regularity of Hamiltonian stationary Lagrangian submanifolds", *Adv. Math.* **343** (2019), 316–352. MR Zbl

[Fabes and Stroock 1984] E. B. Fabes and D. W. Stroock, "The L^p-integrability of Green's functions and fundamental solutions for elliptic and parabolic equations", *Duke Math. J.* **51**:4 (1984), 997–1016. MR Zbl

[Gomes et al. 2016] D. A. Gomes, E. A. Pimentel, and V. Voskanyan, *Regularity theory for mean-field game systems*, Springer, 2016. MR Zbl

[Hervé 1962] R.-M. Hervé, "Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel", *Ann. Inst. Fourier* (*Grenoble*) **12** (1962), 415–571. MR Zbl

[Krylov 1986] N. V. Krylov, "An approach to controlled diffusion processes", *Teor. Veroyatnost. i Primenen.* **31**:4 (1986), 685–709. In Russian; translated in *Theory Probab. Appl.* **31**:4 (1987), 604–626. MR Zbl

[Ladyzhenskaya and Uraltseva 1968] O. A. Ladyzhenskaya and N. N. Uraltseva, *Linear and quasilinear elliptic equations*, Math. Sci. Engineering **46**, Academic Press, New York, 1968. MR Zbl

[Littman 1959] W. Littman, "A strong maximum principle for weakly L-subharmonic functions", *J. Math. Mech.* **8** (1959), 761–770. MR Zbl

[Littman 1963] W. Littman, "Generalized subharmonic functions: monotonic approximations and an improved maximum principle", *Ann. Scuola Norm. Sup. Pisa* (3) **17** (1963), 207–222. MR Zbl

[Mamedov 1992] F. I. Mamedov, "On the Harnack inequality for an equation that is formally conjugate to a linear elliptic differential equation", *Sibirsk. Mat. Zh.* **33**:5 (1992), 100–106. In Russian; translated in *Siberian Math. J.* **33**:5 (1992), 835–841. MR Zbl

[Sjögren 1973] P. Sjögren, "On the adjoint of an elliptic linear differential operator and its potential theory", *Ark. Mat.* **11**:1-2 (1973), 153–165. MR Zbl

[Sjögren 1975] P. Sjögren, "Harmonic spaces associated with adjoints of linear elliptic operators", *Ann. Inst. Fourier* (*Grenoble*) **25**:3-4 (1975), 509–518. MR Zbl

[Teixeira 2014] E. V. Teixeira, "Regularity for quasilinear equations on degenerate singular sets", *Math. Ann.* **358**:1-2 (2014), 241–256. MR Zbl

Received 8 Aug 2018. Revised 18 Mar 2019. Accepted 18 Apr 2019.

RAIMUNDO LEITÃO: rleitao@mat.ufc.br
Department of Mathematics, Universidade Federal do Ceara, Fortaleza, Brazil

EDGARD A. PIMENTEL: pimentel@puc-rio.br
Department of Mathematics, Pontifical Catholic University of Rio de Janeiro, Rio de Janeiro, Brazil

MAKSON S. SANTOS: makson@mat.puc-rio.br
Department of Mathematics, Pontifical Catholic University of Rio de Janeiro, Rio de Janeiro, Brazil

mathematical sciences publishers msp

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2020 is US $340/year for the electronic version, and $550/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 13 No. 4 2020

2157-5045(2020)13:4;1-A