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REGULARITY RESULTS FOR GENERALIZED DOUBLE PHASE FUNCTIONALS

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We consider a wide class of functionals with the property of changing their growth and ellipticity properties according to the modulating coefficients in the framework of Musielak–Orlicz spaces. In particular, we provide an optimal condition on the modulating coefficient to establish the Hölder regularity and Harnack inequality for quasiminimizers of the generalized double phase functional with (G, H)-growth for two Young functions G and H.

1. Introduction

There have been systematic and extensive research activities on the variational problems with nonstandard growth. In particular, functionals whose structure exhibits a phase transition have attracted increasing attention over the last couple of decades. These functionals intervene in the homogenization of strongly anisotropic materials [Zhikov 1986; Zhikov et al. 1994] and in the Lavrentiev phenomenon [Zhikov 1993; 1995]. In this paper, we are concerned with the functionals of the type

$$v \in W^{1,1}(\Omega) \mapsto \mathcal{F}(v, \Omega) := \int_\Omega [G(|Dv|) + a(x)H(|Dv|)]\,dx, \tag{1-1}$$

where $G, H : [0, \infty) \to [0, \infty)$ are Young functions satisfying a suitable gap condition, see (2-24), $a : \Omega \to [0, \infty)$ is a continuous function, and Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 2$.

The main feature of the functional (1-1) is that the energy density changes its growth and ellipticity properties according to the modulating coefficient $a(\cdot)$. The double phase functional (1-1) is a natural generalization of the one with (p, q)-type

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega [|Dv|^p + a(x)|Dv|^q]\,dx, \quad q > p > 1, \tag{1-2}$$

and the one in a borderline case

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega [|Dv|^p + a(x)|Dv|^p \ln(1 + |Dv|)]\,dx, \quad p > 1. \tag{1-3}$$

Zhikov [1986; 1994] first introduced a family of functionals including (1-2) for the purpose of describing a feature of strongly anisotropic materials: the modulating coefficient $a(\cdot)$ presents the geometry of the mixture of two different materials. As shown in [Esposito et al. 2004; Fonseca et al. 2004; Zhikov 1995; 1997], such functionals exhibit Lavrentiev phenomenon whereby minimizers are irregular and even

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discontinuous. On the other hand, the functionals (1-2) and (1-3) belong to the class of functionals having (p, q)-growth condition. These are functionals of the type

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega F(x, Dv)\, dx, \tag{1-4}$$

where the energy density $F(x, \xi)$ satisfies

$$|\xi|^p \lesssim F(x, \xi) \lesssim |\xi|^q + 1, \quad q > p > 1. \tag{1-5}$$

This (p, q)-growth condition was first treated by Marcellini [1986; 1989; 1991] and extensively studied in recent years; see [Breit 2012; Esposito et al. 1999; 2002; 2004; Fonseca et al. 2004; Fusco and Sbordone 1990; Schmidt 2008; 2009].

In the case $p > n$, it is clear from the Sobolev embedding theorem that quasiminimizers of the functionals (1-2) and (1-3) are locally bounded and Hölder continuous. Recently, Baroni, Colombo and Mingione [Baroni et al. 2015a; Colombo and Mingione 2015a; 2015b] found that when $p \le n$, the optimal condition for Hölder continuity of quasiminimizers of the functional (1-2) is $a(\cdot) \in C^{0,\alpha}(\Omega)$, with $\alpha \in (0, 1]$ and $q \le p + \alpha$. For the functional (1-3), the log-Hölder continuity of $a(\cdot)$ is sufficient in order to obtain the Hölder continuity of quasiminimizers; see [Baroni et al. 2015a; 2015b]. These results show that the regularity of the modulating coefficient $a(\cdot)$ is closely related to how to control the size of the associated phase transition. In addition, $C^{1,\beta}$-regularity results for minimizers of the double phase functionals (1-2) and (1-3) have been obtained in [Baroni et al. 2015b; 2018; Colombo and Mingione 2015a; 2015b] and the regularity of the modulating coefficient is directly linked to the gap between two phases. For further regularity results including $C^{0,1}$-regularity for minimizers of functionals with general (p, q)-growth, we refer the reader to [Beck and Mingione 2018; Cupini et al. 2017; 2018; Esposito et al. 2006].

The main object of this paper is to investigate an optimal condition on the modulating coefficient $a(\cdot)$ in the functional (1-1) under which the Hölder regularity result holds for local quasiminimizers. We provide a reasonable condition on the modulus of continuity of $a(\cdot)$, see (4-6), and prove local boundedness, Hölder continuity via De Giorgi's method and the Harnack inequality under this condition. Harjulehto, Hästö and Toivanen [Harjulehto et al. 2017] considered a general setting and developed a set of assumptions on the energy density. Some of the assumptions in [Harjulehto et al. 2017] are the same as ours in the setting of the double phase functionals, see Remark 3.3, but we introduce refined conditions on G and H, and prove that these are sharp conditions for the absence of the Lavrentiev phenomenon, see Theorem 3.1, which also yields the regularity of local quasiminimizers for the generalized double phase functionals. The results in [Harjulehto et al. 2017] and ours complement each other. We also remark that our condition agrees with the known one in the classical case, see Remark 3.2, and serves the natural assumption for the modulating coefficient in a wide variety of double phase functionals such as

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega [|Dv|^p + a(x)|Dv|^p[\ln(1+|Dv|)]^\gamma]\, dx, \quad p > 1,\ \gamma > 0,$$

and

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega [|Dv|^p + a(x)|Dv|^p \ln\ln(e+|Dv|)]\, dx, \quad p > 1;$$

see Remark 4.13.

The method used in this paper is influenced by [Baroni et al. 2015b; 2018; Colombo and Mingione 2015a; 2015b]. For the Hölder continuity of quasiminimizers, we first derive a Caccioppoli-type inequality which is similar to the one that holds for the functional $v \mapsto \int_\Omega G(|Dv|)\,dx$ by using the condition (4-8) on the modulus of continuity of $a(\cdot)$. We then consider a sequence of nested and shrinking balls $\{B_{4^{-i}r_0}\}_{i=0}^{\infty}$ in order to control the oscillation of quasiminimizers along the sequence of balls. Here we should verify for each ball whether the condition (4-8) holds true. If this condition holds true for every ball, then we obtain the Hölder continuity of quasiminimizers. Otherwise, we reduce the oscillation until we reach the exit time for ball $B_{4^{-j}r_0}$, and then we use the existing regularity theory, see Lemma 4.11, for the frozen functional

$$v \in W^{1,1}(B_{4^{-j}r_0}) \mapsto \int_{B_{f^{-j}r_0}} [G(|Dv|) + a_0 H(|Dv|)]\,dx, \quad a_0 = \sup_{B_{4^{-j}r_0}} a(\cdot).$$

For the proof of the Harnack inequality, we first deduce the weak Harnack inequality and the local sup-estimates under the assumption (4-8). Then we apply the exit-time argument as above to obtain the desired inequality.

This paper is organized as follows. In the next section, we introduce some background and investigate the gap conditions. Section 3 deals with the Lavrentiev phenomenon. In Section 4, we establish the local boundedness and the Hölder continuity for (1-1). Section 5 is devoted to proving the Harnack inequality.

2. Preliminaries

Notation. We start this section with introducing notation that will be used in this paper.

Let $B_\rho(y) = \{x \in \mathbb{R}^n : |x-y| < \rho\}$ be the open ball in $\mathbb{R}^n$ centered at $y \in \mathbb{R}^n$ with radius $\rho > 0$. If the center is clear in the context, we shall denote it by $B_\rho \equiv B_\rho(y)$.

For a function v, we write $v_\pm := \max\{\pm v, 0\}$.

For $k \in \mathbb{R}$, $\rho > 0$ and a quasiminimizer u of the functional $\mathcal{F}$, we set

$$A(k,\rho) := \{x \in B_\rho : u(x) > k\} \quad \text{and} \quad A^-(k,\rho) := \{x \in B_\rho : u(x) \le k\}.$$

Hereafter, for the sake of the convenience, we employ the letter c to denote any universal constants which can be explicitly computed in terms of known quantities, and so c might vary from line to line.

Orlicz spaces and Musielak–Orlicz spaces. A Young function $\Phi : [0,\infty) \to [0,\infty)$ is an increasing convex function satisfying

$$\Phi(0) = 0, \quad \lim_{t\to\infty} \Phi(t) = \infty, \quad \lim_{t\to 0+} \frac{\Phi(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty.$$

Definition 2.1. Let Φ be a Young function:

(1) Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if there exists a positive number $\Delta_2(\Phi)$ such that $\Phi(2t) \le \Delta_2(\Phi)\Phi(t)$ for all $t \ge 0$.

(2) Φ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \nabla_2$, if there exists a positive number $\nabla_2(\Phi) > 1$ such that $\Phi(\nabla_2(\Phi)t) \ge 2\nabla_2(\Phi)\Phi(t)$ for all $t \ge 0$.

(3) We write $\Phi \in \Delta_2 \cap \nabla_2$ if $\Phi \in \Delta_2$ and $\Phi \in \nabla_2$.

We note that if $\Phi \in \Delta_2$, then $\Delta_2(\Phi) > 2$. Indeed, by the convexity of Φ, we get

$$\Phi(2t) \le \Delta_2(\Phi)\Phi(t) \le \frac{\Delta_2(\Phi)}{2}\Phi(2t) \quad \text{for all } t \ge 0, \tag{2-1}$$

and hence $\Delta_2(\Phi) \ge 2$. If $\Delta_2(\Phi) = 2$, then it follows from (2-1) that $\Phi(2t) = 2\Phi(t)$ for all $t \ge 0$, and so $\Phi(t) \equiv \Phi(1)t$ is not a Young function. Thus $\Delta_2(\Phi) > 2$.

For a given Young function Φ, we define the complementary Young function Φ^* of Φ by

$$\Phi^*(t) = \sup\{st - \Phi(s) : s \ge 0\}.$$

We remark that Φ^* satisfies all the conditions to be a Young function and that $(\Phi^*)^* = \Phi$. Moreover, $\Phi \in \nabla_2$ if and only if $\Phi^* \in \Delta_2$ with $2\nabla_2(\Phi) = \Delta_2(\Phi^*)$.

We will use the following basic properties of Young functions satisfying Δ_2 and ∇_2 conditions; see for instance [Adams and Fournier 2003; Ok 2016; Rao and Ren 1991].

Lemma 2.2. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$:*

(1) *For any $1 \le \Lambda < \infty$ and $t \ge 0$, we have*

$$\Phi(\Lambda t) \le \Delta_2(\Phi)\Lambda^{\log_2 \Delta_2(\Phi)}\Phi(t). \tag{2-2}$$

(2) *For any $0 < \lambda \le 1$ and $t \ge 0$, we have*

$$\Phi(\lambda t) \le 2\nabla_2(\Phi)\lambda^{1+\log_{\nabla_2(\Phi)} 2}\Phi(t). \tag{2-3}$$

(3) (*Young's inequality*) *For any $\varepsilon \in (0, 1]$, there exists a positive constant c depending only on $\Delta_2(\Phi)$, $\nabla_2(\Phi)$ and ε such that*

$$st \le \varepsilon\Phi(s) + c\Phi^*(t) \quad \textit{for all } s, t \ge 0. \tag{2-4}$$

(4) *If $\Phi \in C^1([0, \infty))$, then for any $t \ge 0$, we have*

$$c_1^{-1}\Phi(t) \le t\Phi'(t) \le c_1\Phi(t) \tag{2-5}$$

and

$$\Phi^*(\Phi'(t)) \le c_2\Phi(t) \tag{2-6}$$

for some constants $c_1, c_2 > 1$ depending only on $\Delta_2(\Phi)$ and $\nabla_2(\Phi)$.

(5) (*a modified form of Young's inequality*) *If $\Phi \in C^1([0, \infty))$, then for any $\varepsilon \in (0, 1]$, there exists a positive constant c depending only on $\Delta_2(\Phi)$, $\nabla_2(\Phi)$ and ε such that*

$$s\Phi'(t) \le \varepsilon\Phi(s) + c\Phi(t) \quad \textit{for all } s, t \ge 0. \tag{2-7}$$

For a Young function Φ, the Orlicz class $K^\Phi(\Omega; \mathbb{R}^N)$, $N \in \mathbb{N}$, consists of all measurable functions $v : \Omega \to \mathbb{R}^N$ satisfying

$$\int_\Omega \Phi(|v(x)|)\, dx < +\infty.$$

The Orlicz space $L^{\Phi}(\Omega;\mathbb{R}^N)$ is the vector space generated by the Orlicz class $K^{\Phi}(\Omega;\mathbb{R}^N)$. If $\Phi\in\Delta_2$, then $K^{\Phi}(\Omega;\mathbb{R}^N)=L^{\Phi}(\Omega;\mathbb{R}^N)$ and this space is a Banach space under the Luxemburg norm

$$\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}=\inf\left\{\sigma>0:\int_{\Omega}\Phi\left(\frac{|v(x)|}{\sigma}\right)dx\le 1\right\}.$$

For $N=1$, we simply write $L^{\Phi}(\Omega):=L^{\Phi}(\Omega;\mathbb{R})$.

We state some relevant inequalities regarding the Luxemburg norm; see [Rao and Ren 1991].

Lemma 2.3. *Let Φ be a Young function with $\Phi\in\Delta_2$:*

(1) $\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}\le 1 \implies \int_{\Omega}\Phi(|v|)\,dx\le\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}$.

(2) $\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}\ge 1 \implies \int_{\Omega}\Phi(|v|)\,dx\ge\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}$.

(3) $\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}\le 1 \iff \int_{\Omega}\Phi(|v|)\,dx\le 1$.

(4) $0<\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}<\infty \implies \int_{\Omega}\Phi\left(\frac{|v|}{\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}}\right)dx=1$.

(5) (*Hölder's inequality*) *For any $v\in L^{\Phi}(\Omega)$ and $w\in L^{\Phi^*}(\Omega)$,*

$$\int_{\Omega}|vw|\,dx\le 2\|v\|_{L^{\Phi}(\Omega)}\|w\|_{L^{\Phi^*}(\Omega)}. \tag{2-8}$$

We now introduce a partial order relation between Young functions, see [Verde 2011], and present a series of lemmas which will be used frequently throughout the paper.

Definition 2.4. Let Φ_1, Φ_2 be Young functions. We shall write

$$\Phi_1\prec\Phi_2$$

if $\Phi_2\circ\Phi_1^{-1}$ is a Young function.

Lemma 2.5. *Let Φ_1, Φ_2 be Young functions with $\Phi_1\prec\Phi_2$. Then*

$$\Phi_1(t)\le\frac{1}{(\Phi_2\circ\Phi_1^{-1})(1)}\Phi_2(t)\quad\text{for all } t\ge\Phi_1^{-1}(1). \tag{2-9}$$

Proof. We first note that for a Young function Φ, there holds

$$\Phi(1)s\le\Phi(s)\quad\text{for all } s\ge 1.$$

Indeed, this follows from the convexity of Φ. Since $\Phi_1\prec\Phi_2$, we have

$$(\Phi_2\circ\Phi_1^{-1})(1)s\le(\Phi_2\circ\Phi_1^{-1})(s)\quad\text{for all } s\ge 1.$$

Setting $t=\Phi_1^{-1}(s)$, we obtain the desired conclusion (2-9). □

Corollary 2.6. *Let Φ_1, Φ_2 be Young functions with $\Phi_1 \prec \Phi_2$. Then*

$$\Phi_1(t) \le c(\Phi_2(t)+1) \quad \text{for all } t \ge 0, \tag{2-10}$$

where c is a positive constant depending only on Φ_1 and Φ_2.

Lemma 2.7. *Let Φ_1, Φ_2 be Young functions with $\Phi_1 \prec \Phi_2$. Then the function*

$$t \mapsto \left(\frac{\Phi_2}{\Phi_1}\right)(t) = \frac{\Phi_2(t)}{\Phi_1(t)}$$

is nondecreasing.

Proof. We first note that the function Φ_2/Φ_1 is nondecreasing if and only if the function $(\Phi_2/\Phi_1)\circ\Phi_1^{-1}$ is nondecreasing, as $t \mapsto \Phi_1(t)$ is increasing and continuous. Since $\Phi_1 \prec \Phi_2$, we see that $\Phi_2 \circ \Phi_1^{-1}$ is a Young function. Hence, it follows from the convexity of $\Phi_2 \circ \Phi_1^{-1}$ that the function

$$t \mapsto \left(\frac{\Phi_2}{\Phi_1}\circ \Phi_1^{-1}\right)(t) = \frac{(\Phi_2\circ\Phi_1^{-1})(t)}{t}$$

is nondecreasing. □

The following lemma and its proof can be found in [Lieberman 1991; Rao and Ren 1991, Chapter II].

Lemma 2.8. *Let $\Phi \in C^1([0,\infty)) \cap C^2((0,\infty))$ be a Young function satisfying*

$$\frac{1}{c_\Phi} \le \frac{t\Phi''(t)}{\Phi'(t)} \le c_\Phi \quad \text{for all } t>0, \tag{2-11}$$

for some $c_\Phi \ge 1$. Then:

(1) $\Phi \in \Delta_2 \cap \nabla_2$, *and the constants $\Delta_2(\Phi)$, $\nabla_2(\Phi)$ depend only on c_Φ.*

(2) *For any $1 \le \Lambda < \infty$ and $t \ge 0$, we have*

$$\Phi(\Lambda t) \le \Lambda^{c_\Phi+1}\Phi(t). \tag{2-12}$$

(3) *For any $0 < \lambda \le 1$ and $t \ge 0$, we have*

$$\Phi(\lambda t) \le \lambda^{(1/c_\Phi)+1}\Phi(t). \tag{2-13}$$

Lemma 2.9. *Let Φ be a Young function with $\Phi \in C^1([0,\infty)) \cap C^2((0,\infty))$. If*

$$\frac{t\Phi''(t)}{\Phi'(t)} \le c_\Phi \quad \text{for all } t>0,$$

for some $c_\Phi \ge 1$, then $t \mapsto \Phi(t^{1/(1+c_\Phi)})$ is a concave function.

Proof. Set $\varphi(t) := \Phi(t^{1/(1+c_\Phi)})$ for $t \ge 0$. Then we have

$$\varphi'(t) = \frac{1}{1+c_\Phi}\Phi'(t^{1/(1+c_\Phi)})t^{-c_\Phi/(1+c_\Phi)},$$

and hence

$$\begin{aligned}\varphi''(t) &= \frac{1}{(1+c_\Phi)^2}\Phi''(t^{1/(1+c_\Phi)})(t^{-c_\Phi/(1+c_\Phi)})^2 - \frac{c_\Phi}{(1+c_\Phi)^2}\Phi'(t^{1/(1+c_\Phi)})t^{-c_\Phi/(1+c_\Phi)-1} \\ &= \frac{1}{(1+c_\Phi)^2}t^{-c_\Phi/(1+c_\Phi)-1}[t^{1/(1+c_\Phi)}\Phi''(t^{1/(1+c_\Phi)}) - c_\Phi\Phi'(t^{1/(1+c_\Phi)})] \le 0\end{aligned}$$

for all $t > 0$. □

We now introduce the Musielak–Orlicz spaces which generalize the Orlicz spaces. Let $\Phi : \Omega \times [0, \infty) \to [0, \infty)$ be a function satisfying the following conditions:

(1) $\Phi(x, \cdot)$ is a Young function for every $x \in \Omega$.

(2) $\Phi(\cdot, t)$ is a measurable function for every $t \ge 0$.

Such a function $\Phi(x, t)$ is called a Musielak–Orlicz function. As before, we present some definitions and properties regarding Musielak–Orlicz functions.

Definition 2.10. Let Φ be a Musielak–Orlicz function:

(1) Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if there exists a positive number $\Delta_2(\Phi)$ such that $\Phi(x, 2t) \le \Delta_2(\Phi)\Phi(x, t)$ for all $x \in \Omega$ and $t \ge 0$.

(2) Φ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \nabla_2$, if there exists a positive number $\nabla_2(\Phi) > 1$ such that $\Phi(x, \nabla_2(\Phi)t) \ge 2\nabla_2(\Phi)\Phi(x, t)$ for all $x \in \Omega$ and $t \ge 0$.

(3) We write $\Phi \in \Delta_2 \cap \nabla_2$ if $\Phi \in \Delta_2$ and $\Phi \in \nabla_2$.

For a given Musielak–Orlicz function Φ, we define the complementary Φ^* of Φ by, for each $x \in \Omega$,

$$\Phi^*(x, t) = \sup\{st - \Phi(x, s) : s \ge 0\}.$$

Then Φ^* satisfies all the conditions to be a Musielak–Orlicz function. Also we note that $(\Phi^*)^* = \Phi$ and that $\Phi \in \nabla_2$ if and only if $\Phi^* \in \Delta_2$ with $2\nabla_2(\Phi) = \Delta_2(\Phi^*)$.

The following lemma can be directly obtained from the definitions of Δ_2-condition, ∇_2-condition and the complementary of Musielak–Orlicz function.

Lemma 2.11. *Let Φ be a Musielak–Orlicz function with $\Phi \in \Delta_2 \cap \nabla_2$:*

(1) *For any $1 \le \Lambda < \infty$, $t \ge 0$ and $x \in \Omega$, we have*

$$\Phi(x, \Lambda t) \le \Delta_2(\Phi)\Lambda^{\log_2 \Delta_2(\Phi)}\Phi(x, t). \tag{2-14}$$

(2) *For any $0 < \lambda \le 1$, $t \ge 0$ and $x \in \Omega$, we have*

$$\Phi(x, \lambda t) \le 2\nabla_2(\Phi)\lambda^{1+\log_{\nabla_2(\Phi)} 2}\Phi(x, t). \tag{2-15}$$

(3) (*Young's inequality*) *For any $\varepsilon \in (0, 1]$, there exists a positive constant c depending only on $\Delta_2(\Phi)$, $\nabla_2(\Phi)$ and ε such that*

$$st \le \varepsilon\Phi(x, s) + c\Phi^*(x, t) \tag{2-16}$$

for all $s, t \ge 0$ and $x \in \Omega$.

For a Musielak–Orlicz function Φ, the Musielak–Orlicz class $K^{\Phi}(\Omega;\mathbb{R}^N)$, $N\in\mathbb{N}$, consists of all measurable functions $v:\Omega\to\mathbb{R}^N$ satisfying

$$\int_\Omega \Phi(x,|v(x)|)\,dx<+\infty.$$

The Musielak–Orlicz space $L^{\Phi}(\Omega;\mathbb{R}^N)$ is the vector space generated by $K^{\Phi}(\Omega;\mathbb{R}^N)$. If $\Phi\in\Delta_2$, then $K^{\Phi}(\Omega;\mathbb{R}^N)=L^{\Phi}(\Omega;\mathbb{R}^N)$ and this space is a Banach space under the Luxemburg norm

$$\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}=\inf\left\{\sigma>0:\int_\Omega \Phi\left(x,\frac{|v(x)|}{\sigma}\right)dx\le 1\right\}.$$

The Musielak–Orlicz–Sobolev space $W^{1,\Phi}(\Omega;\mathbb{R}^N)$ is the function space of all measurable functions $v\in L^{\Phi}(\Omega;\mathbb{R}^N)$ such that its distributional gradient vector Dv belongs to $L^{\Phi}(\Omega;\mathbb{R}^{Nn})$. For $v\in W^{1,\Phi}(\Omega;\mathbb{R}^N)$, we define its norm to be

$$\|v\|_{W^{1,\Phi}(\Omega;\mathbb{R}^N)}=\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}+\|Dv\|_{L^{\Phi}(\Omega;\mathbb{R}^{Nn})}.$$

The space $W_0^{1,\Phi}(\Omega;\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\Omega;\mathbb{R}^N)$ in $W^{1,\Phi}(\Omega;\mathbb{R}^N)$. For $N=1$, we simply write $L^{\Phi}(\Omega):=L^{\Phi}(\Omega;\mathbb{R})$ and $W^{1,\Phi}(\Omega):=W^{1,\Phi}(\Omega;\mathbb{R})$. For a detailed discussion of the Musielak–Orlicz space and the associated Sobolev space, we refer the reader to [Benkirane and Sidi El Vally 2014; Diening 2005; Fan 2012; Fan and Guan 2010; Harjulehto et al. 2016; Musielak 1983; Sidi El Vally 2013].

Gap conditions. We now consider the double phase functional

$$\mathcal{F}(v,\Omega)=\int_\Omega [G(|Dv|)+a(x)H(|Dv|)]\,dx,\qquad v\in W^{1,1}(\Omega),$$

and investigate gap conditions on two Young functions G and H.

In the rest of the paper we shall use the notation

$$\Psi(x,\xi):=G(|\xi|)+a(x)H(|\xi|), \tag{2-17}$$

when $x\in\Omega$ and $\xi\in\mathbb{R}^n$. By abuse of notation, we will continue to write $\Psi(x,\xi)$ also when $x\in\Omega$ and $\xi\in\mathbb{R}$.

Proposition 2.12. *Let $G, H:[0,\infty)\to[0,\infty)$ be Young functions. Suppose that the function $a=a(\cdot):\Omega\to[0,\infty)$ has a modulus of continuity ω satisfying*

$$\limsup_{\rho\to 0+}\omega(\rho)\frac{(H\circ G^{-1})(\rho^{-n})}{\rho^{-n}}<\infty. \tag{2-18}$$

If $H\succ G^\kappa$ for some $\kappa>1+1/n$, then $a(\cdot)$ is a constant function.

Proof. It follows from the condition (2-18) that there exists a constant $L>0$ such that

$$\omega(\rho)\frac{(H\circ G^{-1})(\rho^{-n})}{\rho^{-n}}\le L$$

for all $0 < \rho \le 1$. Since $H \succ G^\kappa$, we have

$$\omega(\rho)\frac{(G^\kappa \circ G^{-1})(\rho^{-n})}{\rho^{-n}} \le c\omega(\rho)\frac{(H \circ G^{-1})(\rho^{-n})}{\rho^{-n}} \le cL \tag{2-19}$$

for all small $\rho > 0$. Here, we see that

$$\omega(\rho)\frac{(G^\kappa \circ G^{-1})(\rho^{-n})}{\rho^{-n}} = \omega(\rho)\frac{[(G \circ G^{-1})(\rho^{-n})]^\kappa}{\rho^{-n}} = \omega(\rho)\rho^{-n(\kappa-1)}. \tag{2-20}$$

Combining (2-19) with (2-20) yields

$$\omega(\rho) \le cL\rho^{n(\kappa-1)} \quad \text{for all } \rho \le \rho_0, \tag{2-21}$$

for some small $\rho_0 > 0$. Then we conclude from the definition of the modulus of continuity that

$$\frac{|a(x)-a(y)|}{|x-y|} \le cL|x-y|^{n(\kappa-1)-1} \tag{2-22}$$

for every $x, y \in \Omega$ with $0 < |x-y| \le \rho_0$. Since $n(\kappa-1)-1 > 0$, it follows immediately that $a(\cdot)$ is a constant function. ☐

Proposition 2.13. *Let $G, H : [0,\infty) \to [0,\infty)$ be Young functions. Suppose that the function $a = a(\cdot) : \Omega \to [0,\infty)$ has a modulus of continuity ω satisfying*

$$\limsup_{\rho\to 0+} \omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} < \infty. \tag{2-23}$$

If $H \succ G^\kappa$ for some $\kappa > 2$, then $a(\cdot)$ is a constant function.

Proof. It follows from the condition (2-23) that there exists a constant $L > 0$ such that

$$\omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} \le L$$

for all $0 < \rho \le 1$. We note from the convexity of G that

$$G(1)s \le G(s) \quad \text{for all } s \ge 1.$$

Since $H \succ G^\kappa$, we get

$$\omega(\rho)\rho^{-(\kappa-1)} \le c\omega(\rho)[G(\rho^{-1})]^{\kappa-1} = c\omega(\rho)\frac{[G(\rho^{-1})]^\kappa}{G(\rho^{-1})} \le c\omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} \le cL$$

for all small $\rho > 0$. As in the previous proof, we conclude that $a(\cdot)$ is a constant function if $\kappa > 2$. ☐

Remark 2.14. If $G(t) \succ t^n$, then it follows from Lemmas 2.5 and 2.7 that

$$\frac{(H \circ G^{-1})(\rho^{-n})}{\rho^{-n}} = \left(\frac{H}{G}\right)(G^{-1}(\rho^{-n})) \le \left(\frac{H}{G}\right)(c\rho^{-1}) \le c\frac{H(\rho^{-1})}{G(\rho^{-1})},$$

and hence the condition (2-23) implies (2-18). On the contrary, if $G(t) \prec t^n$, then

$$\frac{H(\rho^{-1})}{G(\rho^{-1})} = \left(\frac{H}{G}\right)(\rho^{-1}) \le \left(\frac{H}{G}\right)(cG^{-1}(\rho^{-n})) \le c\frac{(H \circ G^{-1})(\rho^{-n})}{\rho^{-n}},$$

and consequently the condition (2-18) implies (2-23). These agree with the known results in the classical case; see Remark 3.2 below.

From this point of view, we shall assume that $G, H : [0,\infty) \to [0,\infty)$ are Young functions with $G, H \in \Delta_2 \cap \nabla_2$ and

$$G \prec H \prec G^{1+1/n}. \tag{2-24}$$

We remark that $\Psi \in \Delta_2 \cap \nabla_2$. To get regularity results, we shall concentrate on nice Young functions, or the N-functions. Thus we further assume that $G, H \in C^1([0,\infty)) \cap C^2((0,\infty))$ and there exist constants $c_G, c_H \geq 1$ such that

$$\frac{1}{c_G} \leq \frac{tG''(t)}{G'(t)} \leq c_G \quad \text{and} \quad \frac{1}{c_H} \leq \frac{tH''(t)}{H'(t)} \leq c_H \tag{2-25}$$

hold for all $t > 0$.

3. Lavrentiev phenomenon

When considering the functionals of the type

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega F(x, Dv)\,dx,$$

with

$$G(|\xi|) \lesssim F(x,\xi) \lesssim H(|\xi|) + 1, \quad G \prec H,$$

the Lavrentiev phenomenon

$$\inf_{v \in W^{1,G}(\Omega)} \int_\Omega F(x, Dv)\,dx < \inf_{v \in W^{1,H}(\Omega)} \int_\Omega F(x, Dv)\,dx$$

may occur. However, for the functional $\mathcal{F}$ defined in (1-1), there is no Lavrentiev phenomenon under a suitable condition on the modulating coefficient $a(\cdot)$.

Theorem 3.1. *Let $\mathcal{F}$ be the functional defined in* (1-1):

(1) *If the modulating coefficient $a(\cdot)$ has a modulus of continuity ω satisfying*

$$\limsup_{\rho \to 0+} \omega(\rho) \frac{(H \circ G^{-1})(\rho^{-n})}{\rho^{-n}} < \infty, \tag{3-1}$$

then for every function $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and balls $B \Subset \widetilde{B} \Subset \Omega$ with $\mathcal{F}(v, \widetilde{B}) < \infty$, there exists a sequence $\{v_k\} \subset W^{1,\infty}(B)$ such that

$$v_k \to v \quad \text{in } W^{1,G}(B) \qquad \text{and} \qquad \mathcal{F}(v_k, B) \to \mathcal{F}(v, B). \tag{3-2}$$

(2) *If the modulating coefficient $a(\cdot)$ has a modulus of continuity ω satisfying*

$$\limsup_{\rho \to 0+} \omega(\rho) \frac{H(\rho^{-1})}{G(\rho^{-1})} < \infty, \tag{3-3}$$

then for every function $v \in W^{1,1}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ *and balls* $B \Subset \widetilde{B} \Subset \Omega$ *with* $\mathcal{F}(v, \widetilde{B}) < \infty$, *there exists a sequence* $\{v_k\} \subset W^{1,\infty}(B)$ *such that*

$$v_k \to v \quad in\ W^{1,G}(B) \qquad and \qquad \mathcal{F}(v_k, B) \to \mathcal{F}(v, B). \tag{3-4}$$

Proof. Let $R > 0$ be the radius of the ball B. Take $\varepsilon_0 \in (0, 1)$ in such a way that $B \equiv B_R \Subset B_{R+\varepsilon_0} \Subset \widetilde{B} \Subset \Omega$. Let $\varphi \in C_0^{\infty}(B_1)$ be a mollifier with $\varphi \ge 0$, $\int_{\mathbb{R}^n} \varphi\, dx = 1$, and set

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)$$

for $x \in B_\varepsilon$ with $\varepsilon > 0$. Then it is obvious that $\varphi_\varepsilon \in C_0^{\infty}(B_\varepsilon)$, $\int_{\mathbb{R}^n} \varphi_\varepsilon\, dx = 1$, $0 \le \varphi_\varepsilon \le c(n)\varepsilon^{-n}$ and $|D\varphi_\varepsilon| \le c(n)\varepsilon^{-(n+1)}$. Now we define, for $0 < \varepsilon < \varepsilon_0$,

$$v_\varepsilon(x) := (v * \varphi_\varepsilon)(x), \quad a_\varepsilon(x) := \inf_{y \in B_\varepsilon(x)} a(y), \quad \Psi_\varepsilon(x, \xi) := G(|\xi|) + a_\varepsilon(x) H(|\xi|)$$

for $x \in B_R$ and $\xi \in \mathbb{R}^n$.

(1) It follows from Jensen's inequality that

$$G(|Dv_\varepsilon(x)|) = G(|Dv * \varphi_\varepsilon(x)|) \le \int_{\mathbb{R}^n} G(|Dv(x-y)|)\varphi_\varepsilon(y)\, dy \le c\varepsilon^{-n}$$

for every $x \in B_R$. By the definitions of $a_\varepsilon(\,\cdot\,)$, we obtain

$$\begin{aligned}\Psi(x, Dv_\varepsilon(x)) &\le |a(x) - a_\varepsilon(x)| H(|Dv_\varepsilon(x)|) + \Psi_\varepsilon(x, Dv_\varepsilon(x)) \\ &\le c\omega(\varepsilon) H(|Dv_\varepsilon(x)|) + \Psi_\varepsilon(x, Dv_\varepsilon(x)).\end{aligned}$$

We now observe from Lemmas 2.2 and 2.7 that

$$\begin{aligned}H(|Dv_\varepsilon(x)|) &= \left(\frac{H}{G}\right)(|Dv_\varepsilon(x)|) G(|Dv_\varepsilon(x)|) \\ &\le \left(\frac{H}{G}\right)(G^{-1}(c\varepsilon^{-n})) G(|Dv_\varepsilon(x)|) = \frac{(H \circ G^{-1})(c\varepsilon^{-n})}{c\varepsilon^{-n}} G(|Dv_\varepsilon(x)|) \\ &\le c\frac{(H \circ G^{-1})(\varepsilon^{-n})}{\varepsilon^{-n}} G(|Dv_\varepsilon(x)|) \le c\frac{(H \circ G^{-1})(\varepsilon^{-n})}{\varepsilon^{-n}} \Psi_\varepsilon(x, Dv_\varepsilon(x)).\end{aligned}$$

Therefore, we see from (3-1) that

$$\begin{aligned}\Psi(x, Dv_\varepsilon(x)) &\le c\omega(\varepsilon)\frac{(H \circ G^{-1})(\varepsilon^{-n})}{\varepsilon^{-n}} \Psi_\varepsilon(x, Dv_\varepsilon(x)) + \Psi_\varepsilon(x, Dv_\varepsilon(x)) \\ &\le c\Psi_\varepsilon(x, Dv_\varepsilon(x)).\end{aligned} \tag{3-5}$$

By Jensen's inequality, we have

$$\begin{aligned}\Psi_\varepsilon(x, Dv_\varepsilon(x)) &\le \int_{B_\varepsilon(x)} \Psi_\varepsilon(x, Dv(y))\varphi_\varepsilon(x-y)\, dy \le \int_{B_\varepsilon(x)} \Psi(y, Dv(y))\varphi_\varepsilon(x-y)\, dy \\ &= [\Psi(\,\cdot\,, Dv(\,\cdot\,)) * \varphi_\varepsilon](x) =: [\Psi(\,\cdot\,, Dv(\,\cdot\,))]_\varepsilon(x).\end{aligned} \tag{3-6}$$

Combining (3-5) and (3-6), we deduce that

$$\Psi(x, Dv_\varepsilon(x)) \le c[\Psi(\,\cdot\,, Dv(\,\cdot\,))]_\varepsilon(x). \tag{3-7}$$

Using the fact that $[\Psi(\,\cdot\,, Dv(\,\cdot\,))]_\varepsilon \to \Psi(\,\cdot\,, Dv(\,\cdot\,))$ strongly in $L^1(B_R)$, we can apply a generalized version of the Lebesgue dominated convergence theorem to obtain a sequence of functions $\{v_k\} := \{v_{\varepsilon_k}\} \subset C_0^\infty(B_R)$ satisfying (3-2) for a suitable sequence $\varepsilon_k \to 0$.

(2) Since v is locally bounded in Ω, we have

$$\begin{aligned}|Dv_\varepsilon(x)| = |v * D\varphi_\varepsilon(x)| &\le \int_{\mathbb{R}^n} |v(x-y)|\,|D\varphi_\varepsilon(y)|\,dy \le \|v\|_{L^\infty(\widetilde{B})} \int_{B_\varepsilon} |D\varphi_\varepsilon(y)|\,dy \\ &\le \|v\|_{L^\infty(\widetilde{B})} c(n)\varepsilon^{-(n+1)}|B_\varepsilon| \le c\varepsilon^{-1}\end{aligned}$$

for every $x \in B_R$. Then we obtain from Lemmas 2.2 and 2.7 that

$$\begin{aligned}H(|Dv_\varepsilon(x)|) &= \left(\frac{H}{G}\right)(|Dv_\varepsilon(x)|)G(|Dv_\varepsilon(x)|) \\ &\le \left(\frac{H}{G}\right)(c\varepsilon^{-1})G(|Dv_\varepsilon(x)|) = \frac{H(c\varepsilon^{-1})}{G(c\varepsilon^{-1})}G(|Dv_\varepsilon(x)|) \\ &\le c\frac{H(\varepsilon^{-1})}{G(\varepsilon^{-1})}G(|Dv_\varepsilon(x)|) \le c\frac{H(\varepsilon^{-1})}{G(\varepsilon^{-1})}\Psi_\varepsilon(x, Dv_\varepsilon(x)).\end{aligned}$$

As in the proof of (1), it follows from (3-3) and (3-6) that

$$\begin{aligned}\Psi(x, Dv_\varepsilon(x)) &\le c\omega(\varepsilon)H(|Dv_\varepsilon(x)|) + \Psi_\varepsilon(x, Dv_\varepsilon(x)) \\ &\le c\omega(\varepsilon)\frac{H(\varepsilon^{-1})}{G(\varepsilon^{-1})}\Psi_\varepsilon(x, Dv_\varepsilon(x)) + \Psi_\varepsilon(x, Dv_\varepsilon(x)) \\ &\le c\Psi_\varepsilon(x, Dv_\varepsilon(x)) \le c[\Psi(\,\cdot\,, Dv(\,\cdot\,))]_\varepsilon(x).\end{aligned}$$

Again, by a generalized version of the Lebesgue dominated convergence theorem, we get a sequence of functions $\{v_k\} := \{v_{\varepsilon_k}\} \subset C_0^\infty(B_R)$ satisfying (3-4) for a suitable sequence $\varepsilon_k \to 0$. $\square$

Remark 3.2. In the special case $(G(t), H(t)) = (t^p, t^q)$ with $1 < p < q$, and $a(\,\cdot\,) \in C^{0,\alpha}(\Omega)$ with $\alpha \in (0, 1]$, a simple computation shows that

$$\text{the condition (3-1)} \iff \frac{q}{p} \le 1 + \frac{\alpha}{n},$$

and

$$\text{the condition (3-3)} \iff q \le p + \alpha.$$

Therefore, Theorem 3.1 generalizes [Colombo and Mingione 2015a, Proposition 3.6; 2015b, Theorem 4.1]. In addition, as in Remark 2.14 and [Colombo and Mingione 2015b], one can check that the condition (3-3) implies the condition (3-1) if $G(t) \succ t^n$, and that the condition (3-1) implies the condition (3-3) if $G(t) \prec t^n$.

Moreover, in the case $(G(t), H(t)) = (t^p, t^p \ln(1+t))$ with $p > 1$, we see that the condition (3-1) and the condition (3-3) are equivalent to

$$\limsup_{\rho\to 0+} \omega(\rho) \ln\left(\frac{1}{\rho}\right) < \infty.$$

This shows that when $a(\cdot)$ is log-Hölder continuous, the functional

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega [|Dv|^p + a(x)|Dv|^p \ln(1+|Dv|)]\, dx, \qquad p > 1,$$

has no Lavrentiev phenomenon.

Remark 3.3. In the setting of the generalized double phase functionals, the conditions (A1) and (A1-n) in [Harjulehto et al. 2017] are same as the conditions (3-1) and (3-3), respectively. From this, it is to be expected that the functionals of the general type (1-4) satisfying the conditions introduced in [Harjulehto et al. 2017] have no Lavrentiev phenomenon.

Remark 3.4. The conditions in Theorem 3.1 are sharp for the absence of the Lavrentiev phenomenon. Indeed, for any ball $B \subset \Omega$, there exist Young functions G, H satisfying (2-24), a nonnegative coefficient $a(\cdot)$ which has a modulus of continuity ω satisfying

$$\lim_{\rho\to 0+} \omega(\rho) \frac{H(\rho^{-1})}{G(\rho^{-1})} = \infty \tag{3-8}$$

and a boundary datum $v_0 \in W^{1,G}(B) \cap L^\infty(B)$ such that

$$\inf_{v\in v_0+W_0^{1,G}(B)} \mathcal{F}(v, B) < \inf_{v\in v_0+W_0^{1,G}(B)\cap W_{\mathrm{loc}}^{1,H}(B)} \mathcal{F}(v, B). \tag{3-9}$$

That is, local minimizers of $\mathcal{F}$ do not belong to $W_{\mathrm{loc}}^{1,H}(B)$ in general. Moreover, they can be discontinuous.

To see this, let us consider the classical case $G(t) = t^p$, $H(t) = t^q$ and $a(\cdot) \in C^{0,\alpha}(\Omega)$ with $1 < p < q$, $\alpha \in (0, 1]$ satisfying

$$1 < p < n < n + \alpha < q. \tag{3-10}$$

Then it follows from [Colombo and Mingione 2015b, Theorem 4.1; Esposito et al. 2004, Section 3] that there exists a coefficient function $a(\cdot) \in C^{0,\alpha}(\Omega)$ and a boundary datum $v_0 \in W^{1,p}(B) \cap L^\infty(B)$ such that the Lavrentiev phenomenon (3-9) occurs. Also we deduce from Remark 3.2 and (3-10) that the coefficient function $a(\cdot)$ has a modulus of continuity ω satisfying (3-8). Furthermore, the modulus of continuity ω does not satisfy the condition (3-1).

4. Local boundedness and Hölder continuity

In the following, we deal with local quasiminimizers of $\mathcal{F}$.

Definition 4.1. We say that $u \in W_{\mathrm{loc}}^{1,1}(\Omega)$ is a local quasiminimizer of $\mathcal{F}$ for $Q \geq 1$, or a local Q-minimizer of $\mathcal{F}$, if for any $v \in W_{\mathrm{loc}}^{1,1}(\Omega)$ with $K := \mathrm{supp}(u - v) \Subset \Omega$, we have $\mathcal{F}(u, K) < +\infty$ and

$$\mathcal{F}(u, K) \leq Q\mathcal{F}(v, K).$$

If $Q = 1$, we say that u is a local minimizer of $\mathcal{F}$.

We remark that if $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local minimizer of the functional

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega F(x, v, Dv)\, dx$$

under the assumption that

$$c_1 \Psi(x,\xi) \le F(x,z,\xi) \le c_2 \Psi(x,\xi)$$

for all $x \in \Omega$, $z \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ with some constants $0 < c_1 \le 1 \le c_2$, then u is also a local quasiminimizer of the functional (1-1) with $Q = c_2/c_1 \ge 1$.

To prove the local boundedness of quasiminimizers of $\mathcal{F}$, we derive the following growth condition on the energy density $\Psi(x,\xi)$ of $\mathcal{F}$.

Lemma 4.2. *Suppose that the gap condition* (2-24) *holds. If $a \in L^\infty(\Omega)$, then*

$$G(|\xi|) \le \Psi(x,\xi) \le c(1 + [G(|\xi|)]^{1+1/n}) \tag{4-1}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, where c is a positive constant depending only on n, G, H and $\|a\|_{L^\infty(\Omega)}$.

Proof. Since $a(\cdot) \ge 0$, it is clear that

$$G(|\xi|) \le G(|\xi|) + a(x)H(|\xi|) = \Psi(x,\xi)$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Moreover, it follows from Corollary 2.6 and (2-24) that

$$\begin{aligned}\Psi(x,\xi) &= G(|\xi|) + a(x)H(|\xi|) \le G(|\xi|) + \|a\|_{L^\infty(\Omega)} H(|\xi|)\\ &\le ([G(|\xi|)]^{1+1/n} + 1) + c\|a\|_{L^\infty(\Omega)}([G(|\xi|)]^{1+1/n} + 1)\\ &\le c([G(|\xi|)]^{1+1/n} + 1)\end{aligned}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. □

We notice that

$$1 + \frac{1}{n} < 1 + \frac{1}{n-1} = 1^*,$$

where 1^* is the Sobolev exponent of 1. The local boundedness of quasiminimizers of $\mathcal{F}$ now follows from the result of [Cupini et al. 2015, Theorem 2.1].

Theorem 4.3 (local boundedness). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local quasiminimizer of $\mathcal{F}$ under the assumption* (2-24), *with $a \in L^\infty_{\mathrm{loc}}(\Omega)$. Then u is locally bounded in Ω.*

Once the local boundedness of quasiminimizers has been obtained, we can prove the Hölder continuity of u without the assumption (2-24). Therefore, we shall consider an a priori bounded quasiminimizer $u \in W^{1,1}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ of $\mathcal{F}$ from Lemma 4.7 on.

Let us start the proof of the Hölder continuity of locally bounded quasiminimizers of $\mathcal{F}$. First, we present some technical lemmas.

Lemma 4.4 [Ladyzhenskaya and Uraltseva 1968]. *Let $\{Y_i\}_{i=0}^{\infty}$ be a sequence of nonnegative numbers satisfying the recursive inequalities*

$$Y_{i+1} \le C b^i Y_i^{1+\delta}, \quad i = 0, 1, 2, \dots, \tag{4-2}$$

where $C, b > 1$ and $\delta > 0$ are given numbers. If

$$Y_0 \le C^{-1/\delta} b^{-1/\delta^2}, \tag{4-3}$$

then $Y_i \to 0$ as $i \to \infty$.

Lemma 4.5 [Ladyzhenskaya and Uraltseva 1968]. *Let $v \in W^{1,1}(B_\rho)$. For any $l > k$, we have*

$$(l-k)|B_\rho \cap \{v > l\}|^{1-1/n} \le \frac{c|B_\rho|}{|B_\rho \setminus \{v > k\}|} \int_{B_\rho \cap \{k < v \le l\}} |Dv|\, dx$$

for some positive constant c depending only on n.

We now state and prove the following Caccioppoli-type inequality.

Lemma 4.6 (Caccioppoli inequality). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}$. Then there exists a constant $c = c(Q, \Delta_2(G), \Delta_2(H)) > 0$ such that for any concentric balls $B_{\rho'} \subset B_\rho \subset \Omega$ with $0 < \rho' < \rho < \infty$, and $k \in \mathbb{R}$, we have*

$$\int_{B_{\rho'}} \Psi(x, D(u-k)_\pm)\, dx \le c \int_{B_\rho} \Psi\left(x, \frac{(u-k)_\pm}{\rho - \rho'}\right) dx. \tag{4-4}$$

Proof. We note that it suffices to prove the version with $(u-k)_+$, as $-u$ is also a Q-minimizer of $\mathcal{F}$. Let $\eta \in C_0^\infty(B_\rho)$ be a cut-off function with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{\rho'}$, and $|D\eta| \le 2/(\rho - \rho')$. We set $v := u - \eta(u-k)_+$, to be used as a competitor. Note that $\operatorname{supp}(u - v) \subset A(k, \rho)$. Then the Q-minimality of u gives

$$\begin{aligned}
\int_{A(k,\rho')} \Psi(x, Du)\, dx &\le Q \int_{A(k,\rho)} \Psi(x, Dv)\, dx \\
&= Q \int_{A(k,\rho)} \Psi(x, (1-\eta)Du - (u-k)_+ D\eta)\, dx \\
&\le c_* \left(\int_{A(k,\rho) \setminus A(k,\rho')} \Psi(x, Du)\, dx + \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{\rho - \rho'}\right) dx \right)
\end{aligned}$$

for some constant $c_* = c_*(Q, \Delta_2(\Psi)) = c_*(Q, \Delta_2(G), \Delta_2(H)) \ge 1$. We now use the "hole-filling" method; that is, we add to both sides the quantity

$$c_* \int_{A(k,\rho')} \Psi(x, Du)\, dx,$$

and divide by $c_* + 1$. Then we discover that

$$\int_{A(k,\rho')} \Psi(x, Du)\, dx \le \vartheta \int_{A(k,\rho)} \Psi(x, Du)\, dx + \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{\rho - \rho'}\right) dx, \tag{4-5}$$

where $\vartheta = c_*/(c_* + 1) < 1$, for any $0 < \rho' < \rho < \infty$ with $B_\rho \subset \Omega$.

Now fix $\rho' < \rho$ and consider a sequence

$$\rho_0 := \rho' \quad \text{and} \quad \rho_{i+1} = (1-\lambda)\lambda^i(\rho-\rho') + \rho_i, \quad i = 0, 1, 2, \dots,$$

where $\lambda \in (0, 1)$ is to be chosen later. Applying (4-5) inductively, we obtain from (2-14) that

$$\begin{aligned}
\int_{A(k,\rho')} \Psi(x, Du)\,dx &\le \vartheta \int_{A(k,\rho_1)} \Psi(x, Du)\,dx + \int_{A(k,\rho_1)} \Psi\Big(x, \frac{u-k}{(1-\lambda)(\rho-\rho')}\Big)\,dx \\
&\le \vartheta^2 \int_{A(k,\rho_2)} \Psi(x, Du)\,dx + \int_{A(k,\rho)} \Psi\Big(x, \frac{u-k}{(1-\lambda)(\rho-\rho')}\Big)\,dx \\
&\qquad\qquad + \vartheta \int_{A(k,\rho_2)} \Psi\Big(x, \frac{u-k}{(1-\lambda)\lambda(\rho-\rho')}\Big)\,dx \\
&\le \vartheta^2 \int_{A(k,\rho_2)} \Psi(x, Du)\,dx + \int_{A(k,\rho)} \Psi\Big(x, \frac{u-k}{(1-\lambda)(\rho-\rho')}\Big)\,dx \\
&\qquad\qquad + \Delta_2(\Psi)\vartheta\lambda^{-\log_2 \Delta_2(\Psi)} \int_{A(k,\rho)} \Psi\Big(x, \frac{u-k}{(1-\lambda)(\rho-\rho')}\Big)\,dx \\
&\le \vartheta^i \int_{A(k,\rho_i)} \Psi(x, Du)\,dx \\
&\qquad\qquad + \Delta_2(\Psi) \sum_{j=0}^{i-1} (\vartheta\lambda^{-\log_2 \Delta_2(\Psi)})^j \int_{A(k,\rho)} \Psi\Big(x, \frac{u-k}{(1-\lambda)(\rho-\rho')}\Big)\,dx \\
&\le \vartheta^i \int_{A(k,\rho_i)} \Psi(x, Du)\,dx \\
&\qquad\qquad + \frac{\Delta_2(\Psi)}{(1-\lambda)^{\log_2 \Delta_2(\Psi)}} \sum_{j=0}^{i-1} (\vartheta\lambda^{-\log_2 \Delta_2(\Psi)})^j \int_{A(k,\rho)} \Psi\Big(x, \frac{u-k}{\rho-\rho'}\Big)\,dx.
\end{aligned}$$

Finally, choosing $\lambda = \lambda(Q, \Delta_2(\Psi)) = \lambda(Q, \Delta_2(G), \Delta_2(H)) \in (0, 1)$ in such a way that $\vartheta\lambda^{-\log_2 \Delta_2(\Psi)} < 1$ and passing to the limit for $i \to \infty$, we get

$$\int_{A(k,\rho')} \Psi(x, Du)\,dx \le \frac{\Delta_2(\Psi)}{(1-\lambda)^{\log_2 \Delta_2(\Psi)}(1-\vartheta\lambda^{-\log_2 \Delta_2(\Psi)})} \int_{A(k,\rho)} \Psi\Big(x, \frac{u-k}{\rho-\rho'}\Big)\,dx,$$

which proves the lemma. □

For the Hölder continuity of local quasiminimizers of $\mathcal{F}$, we assume that the modulating coefficient $a(\cdot)$ has a modulus of continuity ω satisfying

$$\limsup_{\rho\to 0+} \omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} < \infty, \tag{4-6}$$

or, in other words

$$\omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} \le L \quad \text{for every } 0 < \rho \le 1, \tag{4-7}$$

for some $L > 0$.

We remark that when $(G(t), H(t)) = (t^p, t^q)$ with $1 < p < q$, and $a(\cdot) \in C^{0,\alpha}(\Omega)$ with $\alpha \in (0, 1]$, the condition (4-6) is equivalent to $q \le p + \alpha$. In addition, when $(G(t), H(t)) = (t^p, t^p \ln(1+t))$ with

$p > 1$, the condition (4-6) is equivalent to

$$\limsup_{\rho\to 0+} \omega(\rho)\ln\left(\frac{1}{\rho}\right) < \infty.$$

Therefore, the condition (4-6) agrees with the classical ones essentially used in [Baroni et al. 2015a; 2015b; Colombo and Mingione 2015a; 2015b].

In addition, the condition (4-7) ensures that quasiminimizers of $\mathcal{F}$ satisfy the following Caccioppoli-type inequality provided the modulating coefficient $a(\cdot)$ is suitably small in the right scale.

Lemma 4.7 (almost standard Caccioppoli inequality). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_r \Subset \Omega$ be a ball with $r \le 1$. Suppose that*

$$\sup_{x\in B_r} a(x) \le 4\omega(r). \tag{4-8}$$

Then for every $r/2 \le r_1 < r_2 \le r$ and $k \in \mathbb{R}$ with $|k| \le \|u\|_{L^\infty(B_r)}$,

$$⨍_{B_{r_1}} G(|D(u-k)_\pm|)\,dx \le c\left(\frac{r}{r_2-r_1}\right)^{c_G+c_H+2} ⨍_{B_{r_2}} G\left(\frac{(u-k)_\pm}{r}\right)dx \tag{4-9}$$

holds for some constant $c = c(Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}) > 0$.

Proof. If follows from Lemmas 2.2, 2.7 and 4.6, and (4-8) and (2-12) that

$$\begin{aligned}
\int_{B_{r_1}} G(|D(u-k)_\pm|)\,dx &\le \int_{B_{r_1}} \Psi(x, D(u-k)_\pm)\,dx \le c\int_{B_{r_2}} \Psi\left(x, \frac{(u-k)_\pm}{r_2-r_1}\right)dx \\
&= c\int_{B_{r_2}} \left(1 + a(x)\left(\frac{H}{G}\right)\left(\frac{(u-k)_\pm}{r_2-r_1}\right)\right) G\left(\frac{(u-k)_\pm}{r_2-r_1}\right)dx \\
&\le c\int_{B_{r_2}} \left(1 + \omega(r)\left(\frac{H}{G}\right)\left(\frac{2\|u\|_{L^\infty(B_r)}}{r_2-r_1}\right)\right) G\left(\frac{(u-k)_\pm}{r}\frac{r}{r_2-r_1}\right)dx \\
&\le c\left(\frac{r}{r_2-r_1}\right)^{c_G+1}\left(1 + \omega(r)\left(\frac{H}{G}\right)\left(\frac{2\|u\|_{L^\infty(B_r)}}{r_2-r_1}\right)\right)\int_{B_{r_2}} G\left(\frac{(u-k)_\pm}{r}\right)dx.
\end{aligned}$$

We observe from Lemma 2.7, (2-12) and (4-7) that

$$\begin{aligned}
\omega(r)\left(\frac{H}{G}\right)\left(\frac{2\|u\|_{L^\infty(B_r)}}{r_2-r_1}\right) &\le \omega(r)\left(\frac{H}{G}\right)\left(\frac{2(\|u\|_{L^\infty(B_r)}+1)r}{r_2-r_1}\frac{1}{r}\right) \\
&\le \omega(r)\left(\frac{2(\|u\|_{L^\infty(B_r)}+1)r}{r_2-r_1}\right)^{c_H+1}\left(\frac{H}{G}\right)\left(\frac{1}{r}\right) \\
&\le c\left(\frac{r}{r_2-r_1}\right)^{c_H+1}\omega(r)\frac{H(r^{-1})}{G(r^{-1})} \le c\left(\frac{r}{r_2-r_1}\right)^{c_H+1} L,
\end{aligned}$$

which completes the proof. □

Lemma 4.8. *Under the assumptions of Lemma 4.7, we further suppose that the density condition*

$$\left|\left\{x \in B_{r/2} : u(x) > \sup_{B_r} u - \frac{1}{2} \operatorname*{osc}_{B_r} u\right\}\right| \le \frac{1}{2}|B_{r/2}| \tag{4-10}$$

holds. Then for any $\tau \in (0, 1)$, *there exists a large natural number* $m \ge 3$ *depending on* n, Q, c_G, c_H, L, $\|u\|_{L^\infty(B_r)}$ *and* τ *such that*

$$\left|\left\{x \in B_{r/2} : u(x) > \sup_{B_r} u - \frac{1}{2^m} \operatorname*{osc}_{B_r} u\right\}\right| \le \tau|B_{r/2}|.$$

Proof. Let $m \ge 3$ be a large natural number as selected below. Define for $i = 1, 2, \dots, m$,

$$k_i := \sup_{B_r} u - \frac{1}{2^i} \operatorname*{osc}_{B_r} u, \qquad D_i := A\Big(k_i, \frac{r}{2}\Big) \setminus A\Big(k_{i+1}, \frac{r}{2}\Big),$$

and

$$w_i(x) := \begin{cases} k_{i+1} - k_i & \text{if } u(x) > k_{i+1}, \\ u(x) - k_i & \text{if } k_i < u(x) \le k_{i+1}, \\ 0 & \text{if } u(x) \le k_i. \end{cases}$$

We note that $G(w_i) \in W^{1,1}(B_{r/2})$ and $G(w_i) = 0$ in $B_{r/2} \setminus A(k_1, r/2)$ for all $i = 1, 2, \dots, m$, and that $|B_{r/2} \setminus A(k_1, r/2)| \ge \frac{1}{2}|B_{r/2}|$. Using Hölder's inequality, Sobolev's inequality and a modified form of Young's inequality (2-7) with $\varepsilon \in (0, 1)$, we have

$$\begin{aligned} \Big|A\Big(k_{i+1}, \frac{r}{2}\Big)\Big| G\Big(\frac{k_{i+1} - k_i}{r/2}\Big) &\le \int_{A(k_i, r/2)} G\Big(\frac{w_i}{r/2}\Big)\, dx \\ &\le \Big|A\Big(k_i, \frac{r}{2}\Big)\Big|^{1/n} \bigg(\int_{A(k_i, r/2)} \Big[G\Big(\frac{w_i}{r/2}\Big)\Big]^{n/(n-1)} dx\bigg)^{(n-1)/n} \\ &\le cr \bigg(\int_{A(k_i, r/2)} \Big[G\Big(\frac{w_i}{r/2}\Big)\Big]^{n/(n-1)} dx\bigg)^{(n-1)/n} \\ &\le c \int_{D_i} G'\Big(\frac{u - k_i}{r/2}\Big)|Du|\, dx \\ &\le \varepsilon \int_{D_i} G(|Du|)\, dx + c(\varepsilon) \int_{D_i} G\Big(\frac{u - k_i}{r/2}\Big)\, dx. \end{aligned} \tag{4-11}$$

It follows from Lemma 4.7 that

$$\begin{aligned} \int_{D_i} G(|Du|)\, dx \le c \int_{A(k_i, r)} G\Big(\Big|\frac{u - k_i}{r}\Big|\Big)\, dx \le c \int_{A(k_i, r)} G\Big(\frac{1}{2^i r} \operatorname*{osc}_{B_r} u\Big)\, dx \\ = cG\Big(\frac{k_{i+1} - k_i}{r/2}\Big)|A(k_i, r)| \le cG\Big(\frac{k_{i+1} - k_i}{r/2}\Big) r^n. \end{aligned} \tag{4-12}$$

Also, it is clear that

$$\int_{D_i} G\Big(\frac{u - k_i}{r/2}\Big)\, dx \le \int_{D_i} G\Big(\frac{k_{i+1} - k_i}{r/2}\Big)\, dx = G\Big(\frac{k_{i+1} - k_i}{r/2}\Big)|D_i|. \tag{4-13}$$

Combining (4-11) with (4-12) and (4-13), we see that, for $i = 1, 2, \dots, m-1$,

$$\left|A\left(k_{m-1}, \frac{r}{2}\right)\right| \le \left|A\left(k_{i+1}, \frac{r}{2}\right)\right| \le c\varepsilon r^n + c(\varepsilon)|D_i|.$$

Summing over i from 1 to $m-1$ yields that

$$\begin{aligned}(m-1)\left|A\left(k_{m-1}, \frac{r}{2}\right)\right| &\le c(m-1)\varepsilon r^n + c(\varepsilon)\left|A\left(k_1, \frac{r}{2}\right)\right| \\ &\le (c(m-1)\varepsilon + c(\varepsilon))r^n\end{aligned}$$

and hence

$$\left|A\left(k_{m-1}, \frac{r}{2}\right)\right| \le \left(c\varepsilon + \frac{c(\varepsilon)}{m-1}\right)r^n \le \tau|B_{r/2}|$$

by taking sufficiently small $\varepsilon = \varepsilon(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}, \tau) \in (0, 1)$ and sufficiently large $m = m(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}, \tau) \in \mathbb{N}$. □

Lemma 4.9. *Under the assumptions of Lemma 4.8, we further find that there exists a small* $\tau_0 = \tau_0(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}) \in (0, 2^{-(n+1)})$ *such that if*

$$0 < \nu < \frac{1}{2}\operatorname*{osc}_{B_r} u \quad \textit{and} \quad \left|A\left(k_0, \frac{r}{2}\right)\right| \le \tau_0|B_{r/2}|, \tag{4-14}$$

where $k_0 := \sup_{B_r} u - \nu$, *then*

$$\sup_{B_{r/4}} u \le k_0 + \frac{\nu}{2} = \sup_{B_r} u - \frac{\nu}{2}. \tag{4-15}$$

Proof. We first set the sequences

$$\rho_i := \frac{r}{4}\left(1 + \frac{1}{2^i}\right) \quad \text{and} \quad k_i := k_0 + \frac{\nu}{2}\left(1 - \frac{1}{2^i}\right), \quad i = 0, 1, 2, \dots,$$

and define

$$D_{i+1} := A(k_i, \rho_{i+1}) \setminus A(k_{i+1}, \rho_{i+1}) \quad \text{and} \quad Y_i := \frac{|A(k_i, \rho_i)|}{|B_{r/2}|}.$$

We note from the definitions of k_i that $(u - k_i)_+ \le \nu \le \|u\|_{L^\infty(B_r)}$. Then we discover from (4-9) and (4-14) that

$$\begin{aligned}\int_{A(k_i,\rho_{i+1})} G(|Du|)\,dx &\le c2^{(i+3)(c_G+c_H+2)} \int_{A(k_i,\rho_i)} G\left(\frac{(u-k_i)_+}{r}\right)dx \\ &\le c2^{i(c_G+c_H+2)} G\left(\frac{\nu}{r}\right)|A(k_i, \rho_i)|.\end{aligned}$$

It follows from the convexity of G that

$$\begin{aligned}G\left(\fint_{D_{i+1}} |Du|\,dx\right) &\le \fint_{D_{i+1}} G(|Du|)\,dx \le c2^{i(c_G+c_H+2)} \frac{|A(k_i, \rho_i)|}{|D_{i+1}|} G\left(\frac{\nu}{r}\right) \\ &\le G\left(c2^{i(c_G+c_H+2)} \frac{|A(k_i, \rho_i)|}{|D_{i+1}|} \frac{\nu}{r}\right).\end{aligned}$$

Therefore, we obtain

$$\fint_{D_{i+1}} |Du|\,dx \le c2^{i(c_G+c_H+2)} \frac{|A(k_i,\rho_i)|}{|D_{i+1}|}\frac{\nu}{r}.$$

On the other hand, using Lemma 4.5 and the fact that $\tau_0 \in (0, 2^{-(n+1)})$, we have

$$\begin{aligned}\int_{D_{i+1}} |Du|\,dx &\ge c(k_{i+1}-k_i)|A(k_{i+1},\rho_{i+1})|^{1-1/n}|B_{\rho_{i+1}} \setminus A(k_i,\rho_{i+1})|\rho_{i+1}^{-n}\\ &\ge c2^{-i}\nu|A(k_{i+1},\rho_{i+1})|^{1-1/n}(|B_{r/4}|-\tau_0|B_{r/2}|)r^{-n}\\ &\ge c2^{-i}\nu|A(k_{i+1},\rho_{i+1})|^{1-1/n}\\ &\ge c2^{-i}\nu r^{n-1}Y_{i+1}^{1-1/n}.\end{aligned}$$

Combining these inequalities gives

$$Y_{i+1}^{1-1/n} \le c2^{i(c_G+c_H+3)}r^{-n}|A(k_i,\rho_i)| \le c2^{i(c_G+c_H+3)}Y_i,$$

and hence

$$Y_{i+1} \le c_* 2^{n(c_G+c_H+3)/(n-1)i}\, Y_i^{1+1/(n-1)}$$

for some constant $c_* = c_*(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}) > 1$.

Consequently, Lemma 4.4 implies that $Y_i \to 0$ as $i \to \infty$, provided

$$Y_0 = \frac{|A(k_0, r/2)|}{|B_{r/2}|} \le \tau_0 \le c_*^{-(n-1)}2^{-n(n-1)(c_G+c_H+3)}.$$

Then we obtain

$$\left|A\left(k_0+\frac{\nu}{2}, \frac{r}{4}\right)\right| = 0,$$

which implies (4-15). □

The following proposition follows from the above lemma in a standard way by taking $\nu = (1/2^m)\operatorname{osc}_{B_r} u$; see for instance [Baroni et al. 2015b; DiBenedetto 1995].

Proposition 4.10. *Under the assumptions of Lemma 4.8, let $m \ge 3$ be the natural number determined in Lemma 4.8 with $\tau = \tau_0 \in (0, 2^{-(n+1)})$ which is given in Lemma 4.9. Then we see that $m \in \mathbb{N}$ depends only on n, Q, c_G, c_H, L, $\|u\|_{L^\infty(B_r)}$, and we have*

$$\operatorname*{osc}_{B_{r/4}} u \le \left(1-\frac{1}{2^{m+1}}\right)\operatorname*{osc}_{B_r} u. \tag{4-16}$$

The following lemma provides the Hölder continuity of quasiminimizers of the functional

$$v \in W^{1,1}(\Omega) \mapsto \mathcal{F}_0(v,\Omega) := \int_\Omega [G(|Dv|)+a_0H(|Dv|)]\,dx, \tag{4-17}$$

where $0 \le a_0 \le \|a\|_{L^\infty(\Omega)}$ is a fixed constant. For simplicity, we set

$$\Psi_0(t) := G(t)+a_0H(t) \tag{4-18}$$

for $t \ge 0$.

Lemma 4.11. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}_0$ under the assumption* (2-25)*. Then there exist $\beta_0 \in (0,1)$ and $c > 0$, both depending on n, Q, c_G, c_H, but independent of a_0 and u, such that for any fixed ball $B_{r_0} \Subset \Omega$*

$$\operatorname*{osc}_{B_r} u \le c\left(\frac{r}{r_0}\right)^{\beta_0} \operatorname*{osc}_{B_{r_0}} u \tag{4-19}$$

holds for every $0 < r \le r_0$.

Proof. We first observe from [Baroni et al. 2015b, Remark 3.1] that

$$\frac{1}{2\max\{c_G, c_H\}} \le \frac{t\Psi_0''(t)}{\Psi_0'(t)} \le 2\max\{c_G, c_H\} \quad \text{for all } t > 0.$$

We deduce from Theorem 4.3 that u is locally bounded in Ω. Therefore, the result (4-19) follows from [Lieberman 1991, Section 6]. □

We are now ready to prove the Hölder continuity of locally bounded quasiminimizers of $\mathcal{F}$.

Theorem 4.12 (Hölder continuity). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7)*. Then for every open subset $\Omega' \Subset \Omega$ there exists $\beta \in (0,1)$, depending on n, Q, c_G, c_H, L and $\|u\|_{L^\infty(\Omega')}$, such that*

$$u \in C^{0,\beta}_{\mathrm{loc}}(\Omega').$$

Proof. Since the proof is analogous to that of [Baroni et al. 2015b, Theorem 4.1], we only sketch the proof. We shall show that for a fixed ball $B_{8r_0} \subset \Omega'$ with $8r_0 \le 1$, there holds

$$\operatorname*{osc}_{B_r} u \le c\left(\frac{r}{r_0}\right)^{\beta} \operatorname*{osc}_{B_{r_0}} u \quad \text{for all } r \in (0, r_0], \tag{4-20}$$

for some positive constant c depending only on n, Q, c_G, c_H, L and $\|u\|_{L^\infty(\Omega')}$.

Let us define

$$\mathcal{J} := \left\{ i \in \mathbb{N}_0 : \text{(4-8) does not hold for } r = \frac{r_0}{4^i} \right\},$$

and

$$j := \begin{cases} \min \mathcal{J} & \text{if } \mathcal{J} \ne \varnothing, \\ \infty & \text{if } \mathcal{J} = \varnothing. \end{cases}$$

If $j \ge 1$, then we obtain from Proposition 4.10 that for each $r = 4^{-i} r_0$ with $i = 0, \dots, j-1$,

$$\operatorname*{osc}_{B_{r/4}} u \le \left(1 - \frac{1}{2^{m+1}}\right) \operatorname*{osc}_{B_r} u,$$

which yields

$$\operatorname*{osc}_{B_r} u \le 4\left(\frac{r}{r_0}\right)^{\beta_1} \operatorname*{osc}_{B_{r_0}} u \quad \text{for all } r \in (4^{-(j+1)} r_0, r_0], \tag{4-21}$$

for some $\beta_1 \in (0,1)$. If $j = \infty$, then (4-21) holds for every $r \in (0, r_0]$, which is the desired conclusion (4-20) with $\beta = \beta_1$.

In the case $1 \le j < \infty$, one can check that u is a $(2Q)$-minimizer of the functional

$$v \mapsto \int_{B_{4^{-j}r_0}} [G(|Dv|) + a_0 H(|Dv|)]\,dx, \quad a_0 = \sup_{B_{4^{-j}r_0}} a(\,\cdot\,).$$

Now, Lemma 4.11 gives

$$\operatorname*{osc}_{B_r} u \le c\left(\frac{r}{4^{-j}r_0}\right)^{\beta_0} \operatorname*{osc}_{B_{4^{-j}r_0}} u \tag{4-22}$$

for every $r \in (0, 4^{-j}r_0]$. Here, $\beta_0 \in (0,1)$ and $c > 0$ both depend only on n, Q, c_G, c_H. Combining (4-21) and (4-22), we conclude that (4-20) holds for $\beta = \min\{\beta_0, \beta_1\}$. Finally, if $j = 0$, then u is a $(2Q)$-minimizer of the functional

$$v \mapsto \int_{B_{r_0}} [G(|Dv|) + a_0 H(|Dv|)]\,dx, \quad a_0 = \sup_{B_{r_0}} a(\,\cdot\,),$$

and hence we have the desired conclusion (4-20) with $\beta = \beta_0$. □

Remark 4.13. Our condition (4-6) provides a characterization of the modulating coefficient $a(\,\cdot\,)$. More precisely, a modulus of continuity of $a(\,\cdot\,)$ is exactly calibrated to the size of the phase transition. For example, it is evident that the natural assumption for the modulating coefficient in the functional

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega [|Dv|^p + a(x)|Dv|^p[\ln(1+|Dv|)]^\gamma]\,dx,$$

with $p > 1$ and $\gamma > 0$, is

$$\limsup_{\rho\to 0+} \omega(\rho)\left[\ln\left(\frac{1}{\rho}\right)\right]^\gamma < \infty.$$

Similarly, for the functional

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega [|Dv|^p + a(x)|Dv|^p \ln\ln(e+|Dv|)]\,dx,$$

with $p > 1$, the natural assumption for the modulating coefficient is

$$\limsup_{\rho\to 0+} \omega(\rho)\ln\ln\left(\frac{1}{\rho}\right) < \infty.$$

5. The Harnack inequality

In this section, we prove the Harnack inequality for locally bounded quasiminimizers of $\mathcal{F}$. We first present some technical tools.

Lemma 5.1 [Ladyzhenskaya and Uraltseva 1968]. *Let $v \in W^{1,1}(B_\rho)$. For any $l > k$, we have*

$$(l-k)|B_\rho \cap \{v < k\}|^{1-1/n} \le \frac{c|B_\rho|}{|B_\rho \setminus \{v < l\}|} \int_{B_\rho\cap\{k<v\le l\}} |Dv|\,dx$$

for some positive constant c depending only on n.

Lemma 5.2 [Giusti 2003]. *Let ψ be a bounded nonnegative function in the interval $[\rho, r]$ such that*

$$\psi(t) \le \vartheta\psi(s) + \frac{A}{(s-t)^{\kappa}} \quad \text{for every } \rho \le t < s \le r,$$

with $A \ge 0$, $\kappa > 0$ and $0 \le \vartheta < 1$. Then we have

$$\psi(\rho) \le c(\kappa, \vartheta)\frac{A}{(r-\rho)^{\kappa}}.$$

The following lemma provides the weak Harnack inequality of quasiminimizers of the functional $\mathcal{F}_0$ in (4-17); see [Lieberman 1991].

Lemma 5.3. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}_0$ under the assumption* (2-25), *and let $B \Subset \Omega$ be a ball. Then for any exponent $q_+ > 0$ and every $0 < t < s < 1$, we have*

$$\sup_{tB} |u| \le c^*\left(\fint_{sB} |u|^{q_+}\,dx\right)^{1/q_+} \tag{5-1}$$

for some constant $c^* = c^*(n, Q, c_G, c_H, s-t, q_+) > 1$. Moreover, if u is nonnegative, then there exists an exponent $q_- = q_-(n, Q, c_G, c_H) \in (0,1)$ such that for every $t, s \in (0,1)$

$$\inf_{tB} u \ge \frac{1}{c_*}\left(\fint_{sB} u^{q_-}\,dx\right)^{1/q_-} \tag{5-2}$$

holds for some constant $c_* = c_*(n, Q, c_G, c_H, t, s) > 1$.

Analysis similar to that in the proof of Lemma 4.8 gives the following lemma.

Lemma 5.4. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a nonnegative and locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{3r} \Subset \Omega$ be a ball with $3r \le 1$. Suppose that*

$$\sup_{x\in B_{3r}} a(x) \le 12\omega(r). \tag{5-3}$$

For any $\tau_1, \tau_2 \in (0,1)$, there exists a large natural number m depending on $n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}, \tau_1$ and τ_2 such that for any $0 < \lambda \le \|u\|_{L^\infty(B_{3r})}$ if

$$|\{x \in B_r : u(x) \ge \lambda\}| \ge \tau_1|B_r| \tag{5-4}$$

holds, then

$$|\{x \in B_{2r} : u(x) \le 2^{-m}\lambda\}| \le \tau_2|B_{2r}|. \tag{5-5}$$

Now we can obtain a lower bound of u under some density condition as follows.

Proposition 5.5. *Let the assumptions in Lemma 5.4 hold. For any $\tau \in (0,1)$, there exists a small $\delta_1 = \delta_1(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}, \tau) > 0$ such that for any $0 < \lambda \le \|u\|_{L^\infty(B_{3r})}$, if*

$$|\{x \in B_r : u(x) \ge \lambda\}| \ge \tau|B_r| \tag{5-6}$$

holds, then

$$\inf_{B_r} u \ge \delta_1\lambda. \tag{5-7}$$

Proof. We first note that it suffices to prove the proposition for $\tau \in (0, 2^{-(n+1)})$. We fix $m_0 \in \mathbb{N}$, and set the sequences

$$\rho_i := r\left(1 + \frac{1}{2^i}\right) \quad \text{and} \quad k_i := \left(\frac{1}{2} + \frac{1}{2^i}\right)2^{-m_0}\lambda, \quad i = 0, 1, 2, \dots.$$

We also define

$$D_{i+1}^- := A^-(k_i, \rho_{i+1}) \setminus A^-(k_{i+1}, \rho_{i+1}) \quad \text{and} \quad Y_i := \frac{|A^-(k_i, \rho_i)|}{|B_{\rho_i}|}.$$

Since u is nonnegative, we have $(u - k_i)_- \le 2^{-m_0}\lambda$. By (4-9), we get

$$\begin{aligned}\int_{A^-(k_i,\rho_{i+1})} G(|Du|)\,dx &\le c2^{(i+3)(c_G+c_H+2)} \int_{A^-(k_i,\rho_i)} G\left(\frac{(u-k_i)_-}{2r}\right)dx \\ &\le c2^{i(c_G+c_H+2)} G\left(\frac{2^{-m_0}\lambda}{r}\right)|A^-(k_i, \rho_i)|.\end{aligned}$$

We deduce from the convexity of G that

$$\begin{aligned}G\left(\fint_{D_{i+1}^-} |Du|\,dx\right) \le \fint_{D_{i+1}^-} G(|Du|)\,dx &\le c2^{i(c_G+c_H+2)} \frac{|A^-(k_i, \rho_i)|}{|D_{i+1}^-|} G\left(\frac{2^{-m_0}\lambda}{r}\right) \\ &\le G\left(c2^{i(c_G+c_H+2)} \frac{|A^-(k_i, \rho_i)|}{|D_{i+1}^-|} \frac{2^{-m_0}\lambda}{r}\right).\end{aligned}$$

Therefore, we obtain

$$\fint_{D_{i+1}^-} |Du|\,dx \le c2^{i(c_G+c_H+2)} \frac{|A^-(k_i, \rho_i)|}{|D_{i+1}^-|} \frac{2^{-m_0}\lambda}{r}.$$

On the other hand, using Lemma 5.1 and the fact that $\tau \in (0, 2^{-(n+1)})$, we have

$$\begin{aligned}\int_{D_{i+1}^-} |Du|\,dx &\ge c(k_i - k_{i+1})|A^-(k_{i+1}, \rho_{i+1})|^{1-1/n} |B_{\rho_{i+1}} \setminus A^-(k_i, \rho_{i+1})| \rho_{i+1}^{-n} \\ &\ge c2^{-i} \cdot 2^{-m_0}\lambda |A^-(k_{i+1}, \rho_{i+1})|^{1-1/n} (|B_{2r}| - \tau|B_r|) r^{-n} \\ &\ge c2^{-i} \cdot 2^{-m_0}\lambda |A^-(k_{i+1}, \rho_{i+1})|^{1-1/n} \\ &\ge c2^{-i} \cdot 2^{-m_0}\lambda r^{n-1} Y_{i+1}^{1-1/n}.\end{aligned}$$

Combining these inequalities gives

$$Y_{i+1}^{1-1/n} \le c2^{i(c_G+c_H+3)} r^{-n} |A^-(k_i, \rho_i)| \le c2^{i(c_G+c_H+3)} Y_i,$$

and hence

$$Y_{i+1} \le c_0 2^{in(c_G+c_H+3)/(n-1)} Y_i^{1+1/(n-1)}$$

for some constant $c_0 = c_0(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}) > 1$. Here we note from Lemma 5.4 that there exists a large natural number m_0 depending only on $n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}$ such that

$$|\{x \in B_{2r} : u(x) \le 2^{-m_0}\lambda\}| \le c_0^{-(n-1)} 2^{-n(n-1)(c_G+c_H+3)} |B_{2r}|.$$

Then it is clear that

$$Y_0 = \frac{|A^-(k_0, 2r)|}{|B_{2r}|} = \frac{|\{x \in B_{2r} : u(x) \le 2^{-m_0}\lambda\}|}{|B_{2r}|} \le c_0^{-(n-1)} 2^{-n(n-1)(c_G+c_H+3)},$$

and hence $Y_i \to 0$ as $i \to \infty$ by Lemma 4.4. Consequently, we obtain

$$|A^-(2^{-(m_0+1)}\lambda, r)| = 0,$$

which implies (5-7) with $\delta_1 = 2^{-(m_0+1)}$. □

Proposition 5.6. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a nonnegative and locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{3r} \Subset \Omega$ be a ball with $3r \le 1$. Suppose that*

$$\sup_{x \in B_{3r}} a(x) > 12\omega(r). \tag{5-8}$$

For any $\tau \in (0, 1)$, there exists a small $\delta_2 = \delta_2(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}, \tau) > 0$ such that if

$$|\{x \in B_r : u(x) \ge \lambda\}| \ge \tau |B_r| \tag{5-9}$$

for $0 < \lambda \le \|u\|_{L^\infty(B_{3r})}$, then

$$\inf_{B_r} u \ge \delta_2 \lambda. \tag{5-10}$$

Proof. By (5-8), there exists $x_M \in \bar{B}_{3r}$ such that $a(x_M) = a_0 > 12\omega(r)$. Then for every $x \in B_{3r}$

$$a(x_M) - a(x) \le \omega(6r) \le 6\omega(r),$$

and hence

$$a_0 \le 2a_0 - 12\omega(r) \le 2a(x) \le 2a_0.$$

Since $\Psi(x, Du) \in L^1(B_{3r})$, it follows that

$$G(|Dv|) + a_0 H(|Dv|) \in L^1(B_{3r}).$$

Furthermore, one can see that u is a $(2Q)$-minimizer of the functional

$$v \mapsto \int_{B_{3r}} [G(|Dv|) + a_0 H(|Dv|)]\, dx, \quad a_0 = \sup_{B_{3r}} a(\cdot).$$

Now, using (5-2) in Lemma 5.3 with $B \equiv B_{3r}$ and $t = s = \frac{1}{3}$, we see from (5-9) that

$$\inf_{B_r} u \ge \frac{\tau^{1/q_-}\lambda}{c_*},$$

which implies (5-10) with $\delta_2 := \tau^{1/q_-} c_*^{-1}$. □

An immediate consequence of Propositions 5.5 and 5.6 is the following.

Corollary 5.7. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a nonnegative and locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{3r} \Subset \Omega$ be a ball with $3r \le 1$. For any $\tau \in (0, 1)$, there exists a small $\delta = \delta(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}, \tau) > 0$ such that if*

$$|\{x \in B_r : u(x) \ge \lambda\}| \ge \tau |B_r|$$

for $0 < \lambda \le \|u\|_{L^\infty(B_{3r})}$, then

$$\inf_{B_r} u \ge \delta \lambda.$$

From Corollary 5.7 and the covering arguments in [Kinnunen and Shanmugalingam 2001, Section 7], we obtain the following weak Harnack inequality for quasiminimizers of $\mathcal{F}$. For the proof we refer the reader to [Baroni et al. 2015a, Theorem 3.5; Harjulehto et al. 2008, Theorem 5.7].

Theorem 5.8 (the weak Harnack inequality). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a nonnegative and locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{9r} \equiv B_{9r}(x_0) \Subset \Omega$ with $9r \le 1$. Then there exists an exponent $q_- > 0$ and a constant $c > 1$, depending on n, Q, c_G, c_H, L and $\|u\|_{L^\infty(B_{9r})}$, such that*

$$\inf_{B_r} u \ge \frac{1}{c}\left(\fint_{B_{2r}} u^{q_-}\,dx\right)^{1/q_-}. \tag{5-11}$$

To prove the sup-estimate for quasiminimizers of $\mathcal{F}$, we now introduce the scaled functions and the corresponding functional. Let us define, for $R \in (0, 1]$ and $r > 0$ with $B_r \Subset \Omega$,

$$u_R(x) := \frac{u(Rx)}{R}, \quad a_R(x) := a(Rx), \quad x \in B_r,$$

and

$$\mathcal{F}_R(v, K) := \int_K [G(|Dv|) + a_R(x)H(|Dv|)]\,dx, \quad K \Subset B_r.$$

Lemma 5.9. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}$. Let $R \in (0, 1]$ and suppose that $B_r \Subset \Omega$. Then u_R is a Q-minimizer of $\mathcal{F}_R$ in B_r.*

Proof. We first observe that $Du_R(x) = Du(Rx)$. Since $B_r \Subset \Omega$, we see that $\mathcal{F}(u, B_r) < +\infty$, and hence

$$\begin{aligned}\mathcal{F}_R(u_R, B_r) &= \int_{B_r} [G(|Du(Rx)|) + a(Rx)H(|Du(Rx)|)]\,dx \\ &= \frac{1}{R^n}\int_{B_{Rr}} [G(|Du(y)|) + a(y)H(|Du(y)|)]\,dy \\ &\le \frac{1}{R^n}\int_{B_r} [G(|Du(y)|) + a(y)H(|Du(y)|)]\,dy \\ &= \frac{1}{R^n}\mathcal{F}(u, B_r) < +\infty.\end{aligned}$$

Furthermore, for any $v_R \in W^{1,1}_{\mathrm{loc}}(B_r)$ with $K := \operatorname{supp}(u_R - v_R) \Subset B_r$, we have

$$\operatorname{supp}(u - v) = \{Rx : x \in K\} =: RK,$$

and

$$\begin{aligned}\mathcal{F}_R(u_R, K) &= \int_K [G(|Du(Rx)|) + a(Rx)H(|Du(Rx)|)]\,dx \\ &= \frac{1}{R^n}\int_{RK} [G(|Du(y)|) + a(y)H(|Du(y)|)]\,dy \\ &\le \frac{Q}{R^n}\int_{RK} [G(|Dv(y)|) + a(y)H(|Dv(y)|)]\,dy \\ &= Q\int_K [G(|Dv(Rx)|) + a(Rx)H(|Dv(Rx)|)]\,dx = Q\mathcal{F}_R(v_R, K).\end{aligned}$$

Therefore, u_R is a Q-minimizer of $\mathcal{F}_R$ in B_r. □

From the definition of the scaled function $a_R(\cdot)$, one can directly obtain the following lemma.

Lemma 5.10. *Let $R \in (0, 1]$ and suppose that $B_{4r} \subset B_1 \subset \Omega$. Then the function $a_R : B_{1/R} \to [0, \infty)$ has a modulus of continuity ω_R satisfying*

$$\omega_R(\rho) = \omega(R\rho) \quad \text{for all } 0 < \rho \le \frac{1}{R}.$$

Moreover, we have

$$\sup_{x \in B_{3r}} a(x) \le 12\omega(r) \quad \Longleftrightarrow \quad \sup_{x \in B_{3r/R}} a_R(x) \le 12\omega_R\left(\frac{r}{R}\right).$$

We now prove the sup-estimate for quasiminimizers of $\mathcal{F}$. For this, we consider two cases separately, as in the proof of the weak Harnack inequality.

Proposition 5.11. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{4r} \Subset \Omega$ be a ball with $4r \le 1$. Suppose that*

$$\sup_{x \in B_{3r}} a(x) \le 12\omega(r).$$

Then for any exponent $q_+ > 0$, we have the estimate

$$\sup_{B_r} |u| \le c\left(\fint_{B_{2r}} |u|^{q_+}\, dx\right)^{1/q_+} \tag{5-12}$$

for some constant $c > 1$ depending on n, Q, c_G, c_H, L, $\|u\|_{L^\infty(B_{4r})}$ and q_+.

Proof. Let us consider the scaled functions

$$u_r(x) = \frac{u(rx)}{r}, \quad a_r(x) = a(rx), \quad x \in B_4.$$

Then by Lemmas 5.9 and 5.10, we see that the Caccioppoli inequality (4-9) holds for u_r. For $1 \le t < s \le 2$, we now set the sequences

$$\rho_i := t + \frac{s-t}{2^i} \quad \text{and} \quad k_i := 2d\left(1 - \frac{1}{2^{i+1}}\right), \quad i = 0, 1, 2, \dots,$$

where $d > 0$ is to be chosen later. We further define

$$\tilde{\rho}_i := \frac{\rho_i + \rho_{i+1}}{2} \quad \text{and} \quad Y_i := \frac{1}{G(d)} \int_{A_r(k_i, \rho_i)} G(u_r - k_i)\, dx,$$

where

$$A_r(k, \rho) := \{x \in B_\rho : u_r > k\}.$$

Let $\eta_i \in C_0^\infty(B_{\tilde{\rho}_i})$ be a cut-off function with $0 \le \eta_i \le 1$, $\eta_i \equiv 1$ on $B_{\rho_{i+1}}$, and

$$|D\eta_i| \le \frac{4}{\rho_i - \rho_{i+1}}.$$

Using Hölder's inequality, Sobolev's inequality and a modified form of Young's inequality (2-7) with $\varepsilon = 1$, we have

$$\begin{aligned}
G(d)Y_{i+1} &\le \int_{B_{\tilde\rho_i}} G((u_r-k_{i+1})_+\eta_i)\,dx \\
&\le |A_r(k_{i+1},\rho_i)|^{1/n}\biggl(\int_{B_{\tilde\rho_i}} [G((u_r-k_{i+1})_+\eta_i)]^{n/(n-1)}\,dx\biggr)^{(n-1)/n} \\
&\le c|A_r(k_{i+1},\rho_i)|^{1/n}\int_{B_{\tilde\rho_i}} G'((u_r-k_{i+1})_+\eta_i)[|D(u_r-k_{i+1})_+|\eta_i+(u_r-k_{i+1})_+|D\eta_i|]\,dx \\
&\le c|A_r(k_{i+1},\rho_i)|^{1/n}\int_{B_{\tilde\rho_i}} G'((u_r-k_{i+1})_+)|D(u_r-k_{i+1})_+|\,dx \\
&\qquad\qquad + c|A_r(k_{i+1},\rho_i)|^{1/n}\frac{2^{i+3}}{s-t}\int_{B_{\tilde\rho_i}} G'((u_r-k_{i+1})_+)(u_r-k_{i+1})_+\,dx \\
&\le c|A_r(k_{i+1},\rho_i)|^{1/n}\biggl[\int_{B_{\tilde\rho_i}} G(|D(u_r-k_{i+1})_+|)\,dx+\int_{B_{\tilde\rho_i}} G((u_r-k_{i+1})_+)\,dx\biggr] \\
&\qquad\qquad + c|A_r(k_{i+1},\rho_i)|^{1/n}\frac{2^{i+3}}{s-t}\int_{B_{\tilde\rho_i}} G((u_r-k_{i+1})_+)\,dx \\
&\le c|A_r(k_{i+1},\rho_i)|^{1/n}\biggl[\int_{B_{\tilde\rho_i}} G(|D(u_r-k_{i+1})_+|)\,dx+\frac{2^{i+3}}{s-t}\int_{B_{\tilde\rho_i}} G((u_r-k_{i+1})_+)\,dx\biggr] \\
&\le c|A_r(k_{i+1},\rho_i)|^{1/n}\biggl(\frac{2^{i+3}}{s-t}\biggr)^{c_G+c_H+2}\int_{B_{\rho_i}} G((u_r-k_{i+1})_+)\,dx.
\end{aligned}$$

Here we observe from (2-12) that

$$\begin{aligned}
|A_r(k_{i+1},\rho_i)| &\le \frac{1}{G(k_{i+1}-k_i)}\int_{A_r(k_{i+1},\rho_i)} G(u_r-k_i)\,dx \\
&= \frac{1}{G(d/2^{i+1})}\int_{A_r(k_{i+1},\rho_i)} G(u_r-k_i)\,dx \\
&\le \frac{G(d)}{G(d/2^{i+1})}Y_i \le 2^{(i+1)(c_G+1)}Y_i \le c\biggl(\frac{2^{i+3}}{s-t}\biggr)^{c_G+c_H+2}Y_i
\end{aligned}$$

and

$$\int_{B_{\rho_i}} G((u_r-k_{i+1})_+)\,dx = \int_{A_r(k_{i+1},\rho_i)} G(u_r-k_{i+1})\,dx \le \int_{A_r(k_i,\rho_i)} G(u_r-k_i)\,dx = G(d)Y_i.$$

Combining these inequalities yields

$$Y_{i+1} \le \frac{c_0}{(s-t)^\kappa}2^{i\kappa}Y_i^{1+1/n}$$

for some constant $c_0 > 1$ depending only on n, Q, c_G, c_H, L and $\|u\|_{L^\infty(B_{4r})}$, where

$$\kappa = \Bigl(1+\frac{1}{n}\Bigr)(c_G+c_H+2) > 1.$$

Applying Lemma 4.4, we have $Y_i \to 0$ as $i \to \infty$, provided

$$Y_0 = \frac{1}{G(d)} \int_{A_r(d,s)} G(u_r - d)\,dx \le \left[\frac{c_0}{(s-t)^\kappa}\right]^{-n} 2^{-n^2\kappa}. \tag{5-13}$$

It is clear that (5-13) is satisfied if we choose $d > 0$ such that

$$G(d) = \frac{2^{n^2\kappa} c_0^n}{(s-t)^{n\kappa}} \int_{B_s} G((u_r)_+)\,dx. \tag{5-14}$$

Then we obtain $u_r \le 2d$ in B_t, which together with (5-14) implies

$$G\big(\sup_{B_t}(u_r)_+\big) \le \frac{c}{(s-t)^{n\kappa}} \fint_{B_s} G((u_r)_+)\,dx. \tag{5-15}$$

We note from Lemma 2.9 that there exists $\gamma = \gamma(c_G) > 1$ such that $t \mapsto G(t^{1/\gamma})$ is a concave function. Then it follows from (5-15) and Jensen's inequality that

$$\begin{aligned} G\big(\sup_{B_t}(u_r)_+\big) &\le \frac{c}{(s-t)^{n\kappa}} \fint_{B_s} G((u_r)_+)\,dx = \frac{c}{(s-t)^{n\kappa}} \fint_{B_s} G(((u_r)_+^\gamma)^{1/\gamma})\,dx \\ &\le \frac{c}{(s-t)^{n\kappa}} G\bigg(\bigg(\fint_{B_s} (u_r)_+^\gamma\,dx\bigg)^{1/\gamma}\bigg) \le G\bigg(\frac{c}{(s-t)^{n\kappa}} \bigg(\fint_{B_s} (u_r)_+^\gamma\,dx\bigg)^{1/\gamma}\bigg), \end{aligned}$$

and hence

$$\sup_{B_t}(u_r)_+ \le \frac{c}{(s-t)^{n\kappa}} \bigg(\fint_{B_s} (u_r)_+^\gamma\,dx\bigg)^{1/\gamma}.$$

Since $-u$ is also a Q-minimizer of $\mathcal{F}$, we get

$$\sup_{B_t}|u_r| \le \frac{c}{(s-t)^{n\kappa}} \bigg(\fint_{B_s} |u_r|^\gamma\,dx\bigg)^{1/\gamma}.$$

Moreover, for $0 < q_+ < \gamma$, we obtain from Young's inequality that

$$\begin{aligned} \sup_{B_t}|u_r| &\le \frac{c}{(s-t)^{n\kappa}} \big[\sup_{B_s}|u_r|\big]^{1-q_+/\gamma} \bigg(\fint_{B_s} |u_r|^{q_+}\,dx\bigg)^{1/\gamma} \\ &\le \frac{1}{2}\sup_{B_s}|u_r| + \frac{c}{(s-t)^{n\kappa\gamma/q_+}} \bigg(\fint_{B_2} |u_r|^{q_+}\,dx\bigg)^{1/q_+} \end{aligned}$$

as $1 \le t < s \le 2$. Then Lemma 5.2 with $\psi(t) := \sup_{B_t}|u_r|$ yields

$$\sup_{B_1}|u_r| \le c\bigg(\fint_{B_2} |u_r|^{q_+}\,dx\bigg)^{1/q_+}, \tag{5-16}$$

where c is a positive constant depending on n, Q, c_G, c_H, L, $\|u\|_{L^\infty(B_{4r})}$ and q_+.

On the other hand, the inequality (5-16) also holds for $q_+ \ge \gamma$ by Hölder's inequality. Finally, from the definition of u_r, we obtain the desired conclusion (5-12). □

Proposition 5.12. *Let $u \in W^{1,1}_{\text{loc}}(\Omega)$ be a locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{4r} \Subset \Omega$ be a ball with $4r \le 1$. Suppose that*

$$\sup_{x \in B_{3r}} a(x) > 12\omega(r).$$

Then for any exponent $q_+ > 0$, we have the estimate

$$\sup_{B_r} |u| \le c \left(\fint_{B_{2r}} |u|^{q_+} \, dx \right)^{1/q_+} \tag{5-17}$$

for some constant $c > 1$ depending on n, Q, c_G, c_H, L, $\|u\|_{L^\infty(B_{4r})}$ and q_+.

Proof. As in the proof of Proposition 5.6, we see that u is a $(2Q)$-minimizer of the functional

$$v \mapsto \int_{B_{3r}} [G(|Dv|) + a_0 H(|Dv|)] \, dx, \quad a_0 = \sup_{B_{3r}} a(\cdot) > 0.$$

Therefore, (5-1) in Lemma 5.3 with $B \equiv B_{3r}$, $t = \frac{1}{3}$ and $s = \frac{2}{3}$ directly gives (5-17). ☐

Combining Propositions 5.11 and 5.12 yields the following sup-estimate.

Corollary 5.13. *Let $u \in W^{1,1}_{\text{loc}}(\Omega)$ be a locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{4r} \Subset \Omega$ be a ball with $4r \le 1$ Then for any exponent $q_+ > 0$, we have the estimate*

$$\sup_{B_r} |u| \le c \left(\fint_{B_{2r}} |u|^{q_+} \, dx \right)^{1/q_+} \tag{5-18}$$

for some constant $c > 1$ depending on n, Q, c_G, c_H, L, $\|u\|_{L^\infty(B_{4r})}$ and q_+.

Finally, from Theorem 5.8 and Corollary 5.13 with $q_+ = q_-$, we obtain the Harnack inequality of quasiminimizers of $\mathcal{F}$. We remark that the following theorem has no extra term in (5-19), so it can be regarded as a refined version of the result in [Harjulehto et al. 2017] for the generalized double phase case.

Theorem 5.14 (the Harnack inequality). *Let $u \in W^{1,1}_{\text{loc}}(\Omega)$ be a nonnegative and locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{9r} \Subset \Omega$ be a ball with $9r \le 1$. Then there exists a constant $c > 1$, depending on n, Q, c_G, c_H, L and $\|u\|_{L^\infty(B_{9r})}$, such that*

$$\sup_{B_r} u \le c \inf_{B_r} u. \tag{5-19}$$

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