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WELL-POSEDNESS OF THE HYDROSTATIC NAVIER–STOKES EQUATIONS

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We address the local well-posedness of the *hydrostatic Navier–Stokes* equations. These equations, sometimes called *reduced Navier–Stokes/Prandtl*, appear as a formal limit of the Navier–Stokes system in thin domains, under certain constraints on the aspect ratio and the Reynolds number. It is known that without any structural assumption on the initial data, real-analyticity is both necessary and sufficient for the local well-posedness of the system. In this paper we prove that for convex initial data, local well-posedness holds under simple Gevrey regularity.

1. Introduction

The present paper is devoted to the study of the two-dimensional system

$$\partial_t u + u\,\partial_x u + v\,\partial_y u + \partial_x p - \eta\,\partial_y^2 u = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-1a}$$

$$\partial_y p = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-1b}$$

$$\partial_x u + \partial_y v = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-1c}$$

$$u|_{y=0,1} = v|_{y=0,1} = 0, \quad x \in \mathbb{T}, \tag{1-1d}$$

where $\eta > 0$. The unknowns of this system are $(u, v) = (u, v)(x, y, t)$ and $p = p(x, y, t)$, which model respectively the velocity field and pressure of a fluid flow. The boundary condition (1-1d) corresponds to a no-slip condition at the walls $y = 0, 1$. With respect to the tangential variable x we impose $\mathbb{T}$-periodic (lateral) boundary conditions.

Note that upon integrating in y the incompressibility equation (1-1c), using the boundary condition for v (1-1d) we obtain the compatibility condition

$$\partial_x \int_0^1 u(x, y, t)\, dy = 0 \tag{1-2}$$

for all $x \in \mathbb{T}$ and $t \geq 0$, so that the vertical mean of u is just a function of time. Condition (1-2) allows us to compute the pressure gradient, see (2-4) below, and to obtain the boundary condition for the vorticity, see (2-6b) below.

System (1-1) is formally obtained [Lagrée and Lorthois 2005; Renardy 2009] when considering the asymptotics of the two-dimensional Navier–Stokes equation in a thin domain: $\Omega = (0, L) \times (0, l)$ with

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$\delta = l/L \ll 1$. After a proper rescaling

$$t := \frac{Ut}{L}, \quad x := \frac{x}{L}, \quad y := \frac{y}{l}, \quad u := \frac{u}{U}, \quad v := \frac{v}{\delta U},$$

the Navier–Stokes equation becomes

$$\partial_t u + u\,\partial_x u + v\,\partial_y u + \partial_x p - \eta\delta^2\,\partial_x^2 - \eta\,\partial_y^2 u = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-3a}$$

$$\delta^2(\partial_t v + u\,\partial_x v + v\,\partial_y v) + \partial_y p - \eta\delta^4\,\partial_x^2 v - \eta\delta^2\,\partial_y^2 v = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-3b}$$

$$\partial_x u + \partial_y v = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-3c}$$

where

$$\eta = \frac{1}{\delta^2\,\mathrm{Re}},$$

with $\mathrm{Re} = UL/\nu$ the Reynolds number. If we assume $\eta \sim 1$ and keep the leading-order terms as $\delta \to 0$, or if we assume $\eta \ll 1$ and keep both the leading-order and next-order terms in (1-3), we end up with (1-1).

Our concern here will be the local-in-time well-posedness of (1-1). Besides its mathematical relevance, this problem is meaningful from the point of view of hydrodynamic stability, notably with regards to the properties of the so-called *primitive equations*

$$\partial_t u + u\,\partial_x u + v\,\partial_y u + \partial_x p - \eta'\,\partial_x^2 - \eta\,\partial_y^2 u = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-4a}$$

$$\partial_y p = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-4b}$$

$$\partial_x u + \partial_y v = 0, \quad (x, y) \in \mathbb{T} \times (0, 1). \tag{1-4c}$$

This model and its three-dimensional counterpart are very important in atmospheric sciences, after accounting for gravity and many other features [Lions et al. 1992a; 1992b; Temam and Ziane 2004; Petcu et al. 2009]. For positive values of tangential and transverse viscosity coefficients, they are known to be globally well-posed in the Sobolev setting in both the two- and the three-dimensional case [Ziane 1997; Bresch et al. 2003; 2005; Temam and Ziane 2004; Cao and Titi 2007; Kobelkov 2007; Kukavica and Ziane 2007; 2008], and the vanishing viscosity limit $\eta, \eta' \to 0$ can be characterized in the real-analytic category [Kukavica et al. 2016]. Yet, in the absence of additional turbulent viscosity, the dimensional analysis of (1-3) shows that the tangential diffusion coefficient η' is expected to be very small. This allows to relate the well-/ill-posedness of (1-1) and the stability/instability properties of (1-4). For instance, assume that (1-1) is linearly ill-posed without analyticity in x: a result in this direction was shown in [Renardy 2009], and will be discussed later on. It roughly means that, at least in the early stages of the evolution, there are perturbations with wave number $k \gg 1$ in x that grow like $e^{|k|t}$. From there, if η' is small enough so that $\eta'|k|^2 \ll 1$, one can expect the tangential diffusion $-\eta'\,\partial_x^2$ to stay negligible, and the perturbation to be an approximate solution of (1-4) (with Dirichlet conditions). This can result in a growth almost as strong as $e^{t/\sqrt{\eta'}}$, showing the strong instability of (1-4). We note that if one keeps $\eta' > 0$ in (1-4) while setting $\eta = 0$, the local well-posedness can be established for Sobolev initial datum [Cao et al. 2016; 2017], confirming that the horizontal dissipation dominated equation is much more stable than the hydrostatic Navier–Stokes system (1-1) considered in this paper.

From a mathematical perspective, system (1-3) is reminiscent of the two-dimensional Prandtl system, describing boundary layer flows. The latter is set in a half-plane, say $\mathbb{T} \times \mathbb{R}_+$, and reads

$$\partial_t u + u\,\partial_x u + v\,\partial_y u + \partial_x p - \eta\,\partial_y^2 u = 0, \quad (x, y) \in \mathbb{T} \times \mathbb{R}_+, \tag{1-5a}$$
$$\partial_y p = 0, \quad (x, y) \in \mathbb{T} \times \mathbb{R}_+, \tag{1-5b}$$
$$\partial_x u + \partial_y v = 0, \quad (x, y) \in \mathbb{T} \times \mathbb{R}_+, \tag{1-5c}$$
$$u|_{y=0} = v|_{y=0} = 0, \tag{1-5d}$$
$$\lim_{y\to+\infty} u = u^\infty, \tag{1-5e}$$
$$\lim_{y\to+\infty} p = p^\infty. \tag{1-5f}$$

Hence, the only difference with (1-1) lies in the domain and in the boundary conditions. Here, u^∞ and p^∞ are given data, related to the Euler flow above the boundary layer. In particular, as p does not depend on y, it is no longer an unknown of the system. This is a major difference with (1-1), where p can be seen as a Lagrange multiplier, associated to the constraint that $v = -\int_0^y \partial_x u$ vanishes at $y = 1$ (see (2-4) below).

The well-posedness properties of (1-5) are now well-understood, and depend on the monotonicity properties of the initial data. Roughly, if the data have Sobolev regularity, and if furthermore the initial data are monotonic in y, (1-5) has local-in-time Sobolev solutions [Oleinik 1966; Masmoudi and Wong 2015]. On the other hand, without monotonicity, system (1-5) is ill-posed in Sobolev spaces [Gérard-Varet and Dormy 2010; Gérard-Varet and Nguyen 2012]. Local-in-time well-posedness can be achieved when the initial datum is real analytic [Sammartino and Caflisch 1998; Kukavica and Vicol 2013], and even under the milder condition of Gevrey regularity in x [Gérard-Varet and Masmoudi 2015]. We refer to [E and Engquist 1997; Xin and Zhang 2004; Gérard-Varet et al. 2018; Ignatova and Vicol 2016; Kukavica et al. 2017; Dalibard and Masmoudi 2018] for more results on the Prandtl system such as singularities, long-time behavior, and Gevrey-class stability. Interestingly, the instability mechanism that yields ill-posedness in the Sobolev setting involves in a crucial manner the lack of monotonicity and the diffusion term $-\eta\,\partial_y^2 u$. Indeed, the inviscid version of Prandtl, that is,

$$\partial_t u + u\,\partial_x u + v\,\partial_y u + \partial_x p = 0, \quad (x, y) \in \mathbb{T} \times \mathbb{R}_+, \tag{1-6a}$$
$$\partial_y p = 0, \quad (x, y) \in \mathbb{T} \times \mathbb{R}_+, \tag{1-6b}$$
$$\partial_x u + \partial_y v = 0, \quad (x, y) \in \mathbb{T} \times \mathbb{R}_+, \tag{1-6c}$$
$$v|_{y=0} = 0, \tag{1-6d}$$
$$\lim_{y\to+\infty} p = p^\infty \tag{1-6e}$$

has local smooth solutions for smooth data, as can be shown by the method of characteristics [Hong and Hunter 2003].

With regards to this recent understanding of the Prandtl system, it is very natural to ask about the local well-posedness of (1-1), and to start from the consideration of the inviscid case $\eta = 0$, namely

$$\partial_t u + u\,\partial_x u + v\,\partial_y u + \partial_x p = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-7a}$$

$$\partial_y p = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-7b}$$

$$\partial_x u + \partial_y v = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-7c}$$

$$v|_{y=0,1} = 0. \tag{1-7d}$$

This *hydrostatic Euler system* has been the matter of many studies [Brenier 1999; 2003; Grenier 1999; Renardy 2009; Kukavica et al. 2011; 2014; Masmoudi and Wong 2012; Cao et al. 2015; Wong 2015]. Contrary to (1-6), existence of local strong solutions requires a structural assumption, namely the uniform convexity (or concavity) in the variable y of the initial data. A contrario, the presence of an inflection point may trigger high-frequency instability. This point was established in [Renardy 2009], where the author considered the linearization of (1-7) around shear flows $u = U_s(y),\ v = 0$. More precisely, he showed that if the equation $\int_0^1 (U_s(y) - c)^{-2}\,dy = 0$ has complex roots, then the linearized hydrostatic Euler system admits perturbations which have wave number k in x and grow like $e^{\delta k t},\ \delta > 0$, for all $k \gg 1$. Returning to the nonlinear problem (1-7), one can only expect to show short-time stability for data whose Fourier transform in x behaves like $e^{-\delta|k|}$ for large k. This corresponds to analytic data in x. Local well-posedness in the analytic setting was established in [Kukavica et al. 2011]. Moreover, it is mentioned in [Renardy 2009] that this high-frequency instability persists in the case of the viscous system (1-1), at least for small enough η.

Considering all these results, the remaining task is to analyze the viscous system (1-1) for convex (or concave) initial data. This is the purpose of this paper. It raises strong mathematical issues, related to the control of x derivatives of the solution. In particular, we find

$$\partial_t(\partial_x u) + (u\,\partial_x + v\,\partial_y)(\partial_x u) + (\partial_x u)^2 + (\partial_x v)\,\partial_y u + \partial_x(\partial_x p) - \eta\,\partial_y^2(\partial_x u) = 0.$$

One of the main problems in controlling $\partial_x u$ is the term $\partial_x v\,\partial_y u$. Indeed, $\partial_x v = -\int_0^y \partial_x^2 u$ is recovered from the divergence-free condition, so that it can be seen as a first-order operator in x applied to $\partial_x u$. As this first-order term has no skew-symmetry, it does not disappear from energy estimates, so that standard energy arguments can only be conclusive with the help of analyticity. In the case of the hydrostatic Euler system, the way out of this difficulty consists in considering the (approximate) vorticity $\omega = \partial_y u$. Its tangential derivative is seen to satisfy

$$\partial_t(\partial_x \omega) + (u\,\partial_x + v\,\partial_y)(\partial_x \omega) + (\partial_x u)(\partial_x \omega) + (\partial_x v)\,\partial_y \omega = 0.$$

Under a uniform convexity or concavity assumption $|\partial_y \omega| \geq \alpha$, the idea is to test the equation against $\partial_x \omega / \partial_y \omega$ rather than $\partial_x \omega$, to take advantage of the cancellation

$$\int \partial_x v\,\partial_x \omega = -\int \partial_y \partial_x v\,\partial_x u = \int \partial_x^2 u\,\partial_x u = 0.$$

This allows one to get rid of the bad term and is the starting point of the local well-posedness argument. Such an idea was used previously in [Grenier 2000; Masmoudi and Wong 2012].

Unfortunately, this manipulation, which we will call *the hydrostatic trick*, is not fully appropriate for the viscous system (1-1). The reason is that in the estimate for $\partial_x \omega$ the viscous term generates extra boundary integrals such as

$$I^\flat = \eta \int_{\mathbb{T}\times\{0\}} \partial_y \partial_x \omega \, \frac{\partial_x \omega}{\partial_y \omega} \, dx, \qquad I^\sharp = \eta \int_{\mathbb{T}\times\{1\}} \partial_y \partial_x \omega \, \frac{\partial_x \omega}{\partial_y \omega} \, dx.$$

The value of $\partial_y \partial_x \omega$ at the boundary can be obtained from the equation on $\partial_x u$, and yields for instance (the computation will be detailed later)

$$\partial_y \partial_x \omega|_{y=0} = \partial_x^2 p = -2\partial_x \int_0^1 u \, \partial_x u \, dy + \partial_x \omega|_{y=1} - \partial_x \omega|_{y=0}.$$

The issue comes from the first term on the right hand-side, which is again a first-order term in $\partial_x u$ without any skew-symmetric structure. In other words, *there is an additional loss of derivative compared to the Prandtl equation*, so that obtaining well-posedness below analytic regularity is challenging. This is our goal in what follows, and we prove in Theorem 2.1 below the local well-posedness under Gevrey regularity of class $\frac{9}{8}$ in the x-variable, under an extra convexity assumption in y.

2. Main result and strategy

For notational simplicity, from now one we will set $\eta = 1$ in (1-1). Let $\Omega = \mathbb{T} \times (0, 1)$. For $\tau > 0$, $\gamma \geq 1$, we define the Gevrey norm

$$\|f\|_{\gamma,\tau}^2 = \sum_{j=0}^{\infty} \tau^{2j} (j!)^{-2\gamma} \|\partial_x^j f\|_{L^2(\Omega)}^2.$$

Functions f satisfying $\|f\|_{\gamma,\tau} < +\infty$ are in Gevrey class γ with respect to x, measured in L^2 in the y-variable. Our main result is the following:

Theorem 2.1 (well-posedness for convex Gevrey-class initial datum). *Let* $\tau^0 > \tau_1 > 0$, $\gamma \leq \frac{9}{8}$. *Let* u_0 *be a function satisfying the regularity condition*

$$\|\partial_y u_0\|_{\gamma,\tau^0} + \|\partial_y^3 u_0\|_{\gamma,\tau^0} < +\infty, \tag{2-1}$$

the convexity condition

$$\inf_{\Omega} \partial_y^2 u_0 > 0, \tag{2-2}$$

and the compatibility conditions $\partial_x \int_0^1 u_0 \, dy = 0$, $u_0|_{y=0,1} = 0$,

$$\partial_y^2 u_0|_{y=0,1} = \int_0^1 (-\partial_x u_0^2 + \partial_y^2 u_0) \, dy - \int_\Omega \partial_y^2 u_0.$$

Then there exists $T > 0$, *and a unique solution* u *of* (1-1) *with initial data* u_0 *that satisfies*

$$\sup_{t\in[0,T]} (\|\partial_y u(t)\|_{\gamma,\tau_1} + \|\partial_y^3 u(t)\|_{\gamma,\tau_1}) < +\infty.$$

and

$$\inf_{t\in[0,T]\times\Omega} \partial_y^2 u > 0. \tag{2-3}$$

A few remarks are in order:

• The main point in our result is that we prove local well-posedness without analyticity, reaching exponents $\gamma > 1$. The value $\gamma = \frac{9}{8}$ is due to technical limitations, and could certainly be improved. The optimal value that can be expected for γ, or even the possibility of well-posedness in the Sobolev setting are interesting open questions. Our conjecture — based on a formal parallel with Tollmien–Schlichting instabilities for Navier–Stokes [Grenier et al. 2016] — is that the best exponent possible should be $\gamma = \frac{3}{2}$, but such result is for the time being out of reach. If confirmed, it would emphasize the destabilizing role of viscosity.

• We lose on the radius τ of Gevrey regularity, going from τ^0 to τ_1 in positive time. This loss is very standard [Sammartino and Caflisch 1998; Kukavica et al. 2011; Kukavica and Vicol 2013; Gérard-Varet and Masmoudi 2015].

• Besides the Gevrey regularity assumption (2-1), the key assumption is $\inf_\Omega \partial_y^2 u_0 > 0$, which corresponds to a strictly convex initial data. The strict concavity condition $\sup_\Omega \partial_y^2 u_0 < 0$ would work as well. On the opposite, as discussed before, we do not expect such well-posedness to hold for data with inflection points [Renardy 2009].

• The first compatibility condition $\partial_x \int_0^1 u_0 = 0$ is here to ensure that (1-2) holds for all time. Note that we can use (1-2) to determine $\partial_x p$: applying ∂_x to (1-1a), taking the mean over $y \in (0,1)$, integrating by parts in the term $\int_0^1 v\, \partial_y u\, dy$, and using the periodic lateral boundary conditions, we find

$$\partial_x p = \tilde{\omega}|_{y=1} - \tilde{\omega}|_{y=0} - \partial_x \int_0^1 u^2\, dy, \quad x \in \mathbb{T}, \tag{2-4}$$

where $\omega = \partial_y u$ is the vorticity, and we have denoted by

$$\tilde{\omega}(x,y,t) = \omega(x,y,t) - \int_{\mathbb{T}} \omega(x,y,t)\, dx, \quad y \in \{0,1\}, \tag{2-5}$$

the zero mean (in x) boundary vorticity. We will use the notation (2-5) throughout the paper. Note that for $y \in \{0,1\}$, the functions ω and $\tilde{\omega}$ only differ by a function of time.

• The second and third compatibility conditions can be explained as follows. Most of our analysis relies on the control of the vorticity $\omega = \partial_y u$. We notably need some bound on $\sup_{t\in[0,T]} \|\omega\|_{\gamma,\tau}$ for $\tau \in [\tau_1, \tau^0)$. If we leave aside the Gevrey regularity in x, this corresponds to an $L_t^\infty H_y^1$ bound on u. As u satisfies a heat-type equation with Dirichlet condition, it is well known that such an $L_t^\infty H_y^1$ bound requires the compatibility condition $u|_{t=0}|_{y=0,1} = u|_{y=0,1}|_{t=0}$. In view of (1-1c), this amounts to the second compatibility condition of the theorem: $u_0|_{y=0,1} = 0$.

Similarly, the last compatibility condition is related to the fact that we need a bound for $\sup_{t\in[0,T]} \|\partial_t \omega\|_{\gamma,\tau}$ for $\tau \in [\tau_1, \tau^0)$. More precisely, this condition can be derived from the system obeyed by $\omega = \partial_y u$, which is

$$\partial_t \omega + u\, \partial_x \omega + v\, \partial_y \omega - \partial_y^2 \omega = 0, \quad (x,y) \in \mathbb{T} \times (0,1), \tag{2-6a}$$

$$\partial_y \omega|_{y=0,1} = \tilde{\omega}|_{y=1} - \tilde{\omega}|_{y=0} - \partial_x \int_0^1 u^2\, dy. \tag{2-6b}$$

Indeed, (2-6a) follows from differentiating (1-1a) in y, while the boundary condition (2-6b) is obtained by evaluating (1-1a) at $y=0, 1$, using the Dirichlet boundary conditions for u and v in (1-1d), and the formula for the pressure gradient (2-4). Now, from (2-6a), it appears that an $L^\infty_t L^2_y$ control of $\partial_t\omega$ is similar to an $L^\infty_t L^2_y$ control of $\partial^2_y\omega$, meaning an $L^\infty_t H^1_y$ control of $\partial_y\omega$. By differentiating (2-6a), one sees that $\partial_y\omega$ satisfies a heat-like equation, and by (2-6a), it also satisfies a Dirichlet-type condition. Again, an $L^\infty_t H^1_y$ control requires $\partial_y\omega|_{t=0}|_{y=0,1} = \partial_y\omega|_{y=0,1}|_{t=0}$, which by (2-6b) amounts to the third compatibility condition.

General strategy of the proof. Our analysis is based on the vorticity evolution (2-6). We want to benefit from the so-called hydrostatic trick, which consists in establishing L^2 estimates for the weighted derivatives $\partial^j_x\omega/\sqrt{\partial_y\omega}$. The difficulty is that these estimates are not compatible with the diffusion $-\partial^2_y\omega$, which creates boundary terms involving ∂^j_x, $\partial_y\omega|_{y=0}$. Because of the extra x-derivative at the right-hand side of (2-6b), one cannot close an estimate at the Sobolev level.

To overcome this difficulty, our first idea is to write $\omega = \omega^{\text{in}} + \omega^{\text{bl}}$, where ω^{bl} is a boundary corrector which solves (approximately)

$$\partial_t\omega^{\text{bl}} - \partial^2_y\omega^{\text{bl}} = 0, \quad \partial_y\omega^{\text{bl}}|_{y=0,1} = -\partial_x\int_0^1 u^2\,dy,$$

where the right side of the Neumann boundary condition is seen as a given data. With this splitting, the bad term is removed from the Neumann condition on ω^{in}, so that we may apply the hydrostatic trick to this quantity. Still, this approach is obviously not enough: the equation for ω^{in} still involves ω, either directly or through ω^{bl}, so that no closed estimate is available on ω^{in}.

This is where we shall take advantage of Gevrey regularity. To explain this point, it is simpler to consider the linearization of (2-6) around a shear flow $(u_s(y), 0)$:

$$\partial_t\omega + u_s\,\partial_x\omega + u_s''v - \partial^2_y\omega = 0, \quad \partial_x u + \partial_y v = 0, \quad \partial_y\omega|_{y=0,1} = \tilde{\omega}|_{y=1} - \tilde{\omega}|_{y=0} - 2\partial_x\int_0^1 u_s u\,dy.$$

As this system has x-independent coefficients, one can Fourier transform in x. More precisely, looking for local well-posedness in Gevrey class γ, it is natural to look for solutions in the form $\omega = e^{k^{1/\gamma}t}e^{ikx}\hat{\omega}_k(t, y)$. We end up with the following system for the boundary layer corrector:

$$(k^{1/\gamma} + \partial_t)\hat{\omega}^{\text{bl}}_k - \partial^2_y\hat{\omega}^{\text{bl}}_k = 0, \quad \partial_y\hat{\omega}^{\text{bl}}_k|_{y=0,1} = -2ik\int_0^1 u_s\hat{u}_k\,dy.$$

Note that, when taking the boundary layer corrector as a solution of this heat-type system, we implicitly assume that the other terms in the equation, notably the convection term $u_s\,\partial_x\omega \sim iky\hat{\omega}^{\text{bl}}_k$, are negligible in the boundary layer. A formal analysis shows that this should hold as long as $\gamma > \frac{3}{2}$, which is the range considered here. In the limit case $\gamma = \frac{3}{2}$, conjectured to be optimal for well-posedness (see remark above), one should probably replace the heat operator by an Airy-type one, as in [Grenier et al. 2016].

Explicit calculations on the boundary layer system reveal that Gevrey regularity in x is converted into spatial localization in y: for $k \gg 1$, $\hat{\omega}^{\text{bl}}_k$ has a boundary layer behavior, with concentration near $y=0, 1$

at scale $k^{-1/(2\gamma)}$. Roughly, neglecting the upper boundary, one can think of

$$\hat{\omega}_k^{\mathrm{bl}} \approx k^{1-1/(2\gamma)} W(t, k^{1/(2\gamma)} y) \int_0^1 u_s \hat{u}_k \, dy,$$
$$\hat{u}_k^{\mathrm{bl}} \approx k^{1-1/\gamma} U(t, k^{1/(2\gamma)} y) \int_0^1 u_s \hat{u}_k \, dy.$$

Now, the idea is to write

$$\int_0^1 u_s \hat{u}_k \, dy = \int_0^1 u_s \hat{u}_k^{\mathrm{bl}} + \int_0^1 u_s \hat{u}_k^{\mathrm{in}} = \left(k^{1-1/\gamma} \int_0^1 u_s(y) U(t, k^{1/(2\gamma)} y) \, dy \right) \int_0^1 u_s \hat{u}_k \, dy + \int_0^1 u_s \hat{u}_k^{\mathrm{in}}.$$

In short, one can check that for $\gamma \le 2$, we have $k^{1-1/\gamma} \int_0^1 u_s(y) U(t, k^{1/(2\gamma)} y) \, dy = o(1)$ in the limit of large k, so that the first term on the right-hand side can be absorbed in the left-hand side. This leads to a control of $\int_0^1 u_s u$, and thus of ω^{bl}, in terms of ω^{in}. From there, one can get closed estimates on ω^{in}.

Of course, this strategy is made more difficult when dealing with the x-dependent and nonlinear system (2-6). In particular, the Fourier approach is no longer convenient, and we must use the characterization of Gevrey regularity in the physical space, through the family $\{\partial_x^j \omega\}_{j \in \mathbb{N}}$. In order to take advantage of the boundary layer phenomenon, we shall introduce Gevrey norms with extra weight $(j+1)^r$; see (3-1). The boundary layer phenomenon will be reflected by the fact that multiplication by y or integration in y will generate a gain in the exponent r; see Lemma 3.1. Such gain will make possible the control of boundary layer quantities by ω^{in}; see Lemma 3.4.

From there, the analysis will focus on weighted estimates for ω^{in}, using the hydrostatic trick. As usual in nonlinear problems, these estimates will be obtained conditional to certain bounds (notably a lower bound on $\partial_y \omega$, to benefit from convexity). We will show that such bounds are preserved in small time, which will require estimates on the time derivative $\partial_t \omega$, as well as maximum principle arguments for $\partial_y \omega$.

3. Preliminaries

As usual in this kind of analysis, we will focus on a priori estimates. This means that from Section 3 to Section 6, we will assume implicitly that we already have a solution of (1-1) on $[0, T]$ with all necessary smoothness, and we will collect properties and estimates about this solution. Only in Section 7 will we describe the way of constructing solutions.

Norms and notation. Let $\gamma \ge 1$, $r \in \mathbb{R}$, $\tau > 0$. We introduce a refined two-dimensional Gevrey norm

$$\|f\|_{\gamma,r,\tau}^2 = \sum_{j \ge 0} M_j^2 \|\partial_x^j f\|_{L^2_{x,y}(\mathbb{T}\times[0,1])}^2, \quad \text{where} \quad M_j = \frac{(j+1)^r \tau^{j+1}}{(j!)^\gamma}. \tag{3-1}$$

Note that the L^2 norm in space is only used on $\Omega = \mathbb{T} \times [0, 1]$, although the functions may be defined on the half-space $\mathbb{T} \times [0, \infty)$. We note that if $r' \ge r$ then $\|\cdot\|_{\gamma,r',\tau} \ge \|\cdot\|_{\gamma,r,\tau}$.

For functions which are independent of the y-variable, we use the one-dimensional counterpart

$$|f|_{\gamma,r,\tau}^2 = \sum_{j \ge 0} M_j^2 \|\partial_x^j f\|_{L^2_x(\mathbb{T})}^2,$$

where M_j is defined as before. Similarly, if $r' \ge r$ then $|\cdot|_{\gamma,r',\tau} \ge |\cdot|_{\gamma,r,\tau}$.

Let τ^0, τ_1 be as in Theorem 2.1, and let τ_0 such that $\tau^0 > \tau_0 > \tau_1$. Throughout the paper, the Gevrey-class radius τ will be defined by

$$\tau(t) = \tau_0 \exp(-\beta t), \tag{3-2}$$

where $\beta \geq 1$, $t \in [0, T]$, and T is always small enough so that $\tau(t) \geq \tau_1$. In particular $\dot{\tau}(t) = -\beta\tau(t)$.

We will use $a \lesssim b$ to denote the existence of a constant $C > 0$, which may depend only on γ, τ_0, τ_1, and r, such that $a \leq Cb$. Similarly, will use $a \ll b$ to denote the existence of a sufficiently large constant $C > 0$, which may depend only on γ, τ_0, τ_1, and r, such that $Ca \leq b$.

For any function f we use the notation

$$f_j = M_j \, \partial_x^j f, \tag{3-3}$$

where M_j is defined in (3-1) and depends on r, γ, and τ. With this notation we have

$$\|f\|^2_{\gamma,r,\tau} = \sum_{j\geq 0} \|f_j\|^2_{L^2_{x,y}} \quad \text{and} \quad |f|^2_{\gamma,r,\tau} = \sum_{j\geq 0} \|f_j\|^2_{L^2_x}.$$

A boundary layer lift. The boundary condition (2-6b) in the vorticity evolution (2-6) motivates the introduction of a boundary layer lift for the vorticity, which we describe next. Throughout the paper we appeal to Gevrey estimates for the system

$$(\partial_t - \partial_y^2)\omega^\flat = 0, \tag{3-4a}$$

$$(\partial_y \omega^\flat + 2\omega^\flat)|_{y=0} = \partial_x h|_{y=0}, \tag{3-4b}$$

$$\omega^\flat|_{t=0} = 0, \tag{3-4c}$$

posed for $t \in [0, T]$, $x \in \mathbb{T}$, and $y \in \mathbb{R}_+$. Here h is a placeholder for $-\left(\int_0^1 u^2\,dy - \int_{\mathbb{T}}\int_0^1 u^2\,dy\,dx\right)$. Since the boundary datum for $\omega^\flat$ is a pure x-derivative (and this is the only nontrivial datum), we note that (3-4) immediately implies $\int_{\mathbb{T}} \omega^\flat(x, y, t)\,dx = 0$ for any $y \geq 0$. We also define

$$u^\flat(x, y) = \int_{+\infty}^{y} \omega^\flat(x, z)\,dz, \tag{3-5}$$

$$v^\flat(x, y) = \int_{y}^{+\infty} \partial_x u^\flat(x, z)\,dz. \tag{3-6}$$

Lemma 3.1. *Let $r \in \mathbb{R}$, $\beta \geq 1$ and $T > 0$ such that $\tau(t) \geq \tau_1$ for $t \in [0, T]$. The boundary layer vorticity $\omega^\flat$ obeys*

$$\int_0^t \|\omega^\flat(s)\|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{3/2}} \int_0^t |h(s)|^2_{\gamma,r+\gamma-3/4,\tau(s)}\,ds, \tag{3-7a}$$

$$\int_0^t \|y\,\omega^\flat(s)\|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{5/2}} \int_0^t |h(s)|^2_{\gamma,r+\gamma-5/4,\tau(s)}\,ds, \tag{3-7b}$$

$$\int_0^t \|\partial_y\omega^\flat(s)\|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{1/2}} \int_0^t |h(s)|^2_{\gamma,r+\gamma-1/4,\tau(s)}\,ds, \tag{3-7c}$$

$$\int_0^t \|y\,\partial_y \omega^\flat(s)\|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{3/2}} \int_0^t |h(s)|^2_{\gamma,r+\gamma-3/4,\tau(s)}\,ds, \tag{3-7d}$$

$$\int_0^t |\omega^\flat(s)|_{y=1}|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{20}} \int_0^t |h(s)|^2_{\gamma,r+\gamma-10,\tau(s)}\,ds, \tag{3-7e}$$

$$\int_0^t |\partial_y \omega^\flat(s)|_{y=1}|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{20}} \int_0^t |h(s)|^2_{\gamma,r+\gamma-10,\tau(s)}\,ds, \tag{3-7f}$$

the boundary layer velocity $u^\flat$ obeys

$$\int_0^t \|u^\flat(s)\|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{5/2}} \int_0^t |h(s)|^2_{\gamma,r+\gamma-5/4,\tau(s)}\,ds, \tag{3-8a}$$

$$\int_0^t \|y u^\flat(s)\|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{7/2}} \int_0^t |h(s)|^2_{\gamma,r+\gamma-7/4,\tau(s)}\,ds, \tag{3-8b}$$

$$\int_0^t |u^\flat(s)|_{y=1/2}|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{20}} \int_0^t |h(s)|^2_{\gamma,r+\gamma-10,\tau(s)}\,ds, \tag{3-8c}$$

and the boundary layer velocity $v^\flat$ satisfies

$$\int_0^t \|v^\flat(s)\|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{7/2}} \int_0^t |h(s)|^2_{\gamma,r+2\gamma-7/4,\tau(s)}\,ds, \tag{3-9a}$$

$$\int_0^t |v^\flat|_{y=0}(s)|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{3}} \int_0^t |h(s)|^2_{\gamma,r+2\gamma-3/2,\tau(s)}\,ds, \tag{3-9b}$$

$$\int_0^t |v^\flat|_{y=1}(s)|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{20}} \int_0^t |h(s)|^2_{\gamma,r+\gamma-10,\tau(s)}\,ds \tag{3-9c}$$

for all $t \in [0, T]$.

Proof of Lemma 3.1. In view of (3-2), (3-3), and (3-4), the function $\omega_j^\flat = M_j\,\partial_x^j \omega^\flat$ obeys equations

$$(\partial_t + \beta(j+1) - \partial_y^2)\omega_j^\flat = 0, \tag{3-10a}$$

$$(\partial_y \omega_j^\flat + 2\omega_j^\flat)|_{y=0} = \partial_x h_j|_{y=0} = \frac{M_j}{M_{j+1}} h_{j+1}, \tag{3-10b}$$

$$\omega_j^\flat|_{t=0} = 0. \tag{3-10c}$$

For fixed $x \in \mathbb{T}$ we define $f_j(x,t) = (M_j/M_{j+1}) h_{j+1}(x,t)$ for $t \in [0,T]$, and $f_j(x,t) = 0$ for $t \in \mathbb{R} \setminus [0,T]$. Pointwise in x and y we take a Fourier transform in time and solve in $L^2(\mathbb{R}_t \times \mathbb{T}_x \times \mathbb{R}_y^+)$ the system

$$(\partial_t + \beta(j+1) - \partial_y^2)\bar{\omega}_j^\flat = 0,$$
$$(\partial_y \bar{\omega}_j^\flat + 2\bar{\omega}_j^\flat)|_{y=0} = f_j.$$

The solution is obtained by taking the inverse Fourier transform in time (we let ζ denote the dual Fourier variable to t) of the function

$$\hat{\bar{\omega}}_j^\flat(\zeta, x, y) = \frac{\hat{f}_j(\zeta, x)}{2 - \sqrt{\beta(j+1) + i\zeta}} e^{-y\sqrt{\beta(j+1)+i\zeta}}. \tag{3-12}$$

We implicitly assume here that $\beta > 4$ so that for all $j \in \mathbb{N}$, for all ζ with $\mathscr{I}m\,\zeta \le 0$,

$$|2-\sqrt{\beta(j+1)+i\zeta}| \ge |\sqrt{\beta(j+1)+i\zeta}|-2 \ge \sqrt{\beta(j+1)-\mathscr{I}m\,\zeta}-2 \ge \sqrt{\beta}-2 > 0. \tag{3-13}$$

We will make a crucial use of:

Lemma 3.2. *The following two properties hold*:

- $\bar{\omega}_j^\flat \equiv 0$ *for* $t < 0$.
- $\bar{\omega}_j^\flat \equiv \omega_j^\flat$ *for* $t \in [0, T]$.

The proof is postponed to the Appendix. This lemma will allow us to use the explicit formula (3-12) to obtain estimates on $\omega_j^\flat$, starting with (3-7a)–(3-7f).

Let us detail the derivation of (3-7a). A simple calculation based on (3-12) yields

$$\|\hat{\bar{\omega}}_j^\flat\|^2_{L^2_{\zeta,x,y}} \le \frac{C}{(\beta(j+1))^{3/2}}\|\hat{f}_j\|^2_{L^2_{\zeta,x}}$$

for a constant C independent of j (and obviously independent of T, which is only involved in the definition of f_j). By the Plancherel formula in time

$$\|\bar{\omega}_j^\flat\|^2_{L^2_{t,x,y}} \le \frac{C}{(\beta(j+1))^{3/2}}\|f_j\|^2_{L^2_{t,x}} = \frac{C}{\beta(j+1)^{3/2}}\left(\frac{M_j}{M_{j+1}}\right)^2\int_0^T \|h_{j+1}(s)\|^2_{L^2_x}\,ds. \tag{3-14}$$

This implies (by the second item of Lemma 3.2)

$$\int_0^T \|\omega_j^\flat(s)\|^2_{L^2_{x,y}}\,ds \le \frac{C'}{\beta^{3/2}}(j+1)^{2\gamma-3/2}\int_0^T \|h_{j+1}(s)\|^2_{L^2_x}\,ds.$$

Multiplying by $(j+1)^{2r}$ and summing over j, we obtain the inequality (3-7a) *in the special case* $t=T$. For the general case $t \in (0,T)$, the idea is to slightly modify $\bar{\omega}_j^\flat$. Namely, instead of extending $(M_j/M_{j+1})h_{j+1}$ by zero outside $(0,T)$ and then solving the heat equation with the extension f_j as a boundary data, we extend $(M_j/M_{j+1})h_{j+1}|_{(0,t)}$ by zero outside $(0,t)$. We then solve the heat equation with this modified boundary data f_j^t, which is zero outside $(0,t)$, resulting in a new $\bar{\omega}_j^{\flat,t}$. Obviously, Lemma 3.2 and the previous calculation remain true with T replaced by t and $\bar{\omega}_j^\flat$ replaced by $\bar{\omega}_j^{\flat,t}$. This yields (3-7a). Inequalities (3-7b)–(3-8b) follow very similar arguments, which we skip for brevity.

In the case of (3-9a), we need to take into account one more x-derivative. A simple calculation yields (with obvious notation)

$$\|\hat{\bar{v}}_j^\flat\|^2_{L^2_{\zeta,x,y}} \le \frac{C}{(\beta(j+1))^{7/2}}\|\partial_x\hat{f}_j\|^2_{L^2_{\zeta,x}}.$$

The extra factor of $(\beta(j+1))^2$ in the denominator compared to (3-14) comes from taking two antiderivatives in y, while $\hat{f}_j$ is replaced by $\partial_x\hat{f}_j$ due to the extra x-derivative in (3-6). It follows that

$$\int_0^T \|v_j^\flat(s)\|^2_{L^2_{x,y}}\,ds \le \frac{C}{\beta^{7/2}}(j+1)^{2\gamma-7/2}\int_0^T \|\partial_x h_{j+1}(s)\|^2_{L^2_x}\,ds$$

and using that $|\partial_x h_{j+1}| \lesssim (M_{j+1}/M_{j+2})|h_{j+2}| \lesssim (j+2)^\gamma |h_{j+2}|$, we get

$$\int_0^T \|v_j^\flat(s)\|_{L^2_{x,y}}^2 \, ds \le \frac{C}{\beta^{7/2}} (j+1)^{4\gamma-7/2} \int_0^T \|h_{j+2}(s)\|_{L^2_x}^2 \, ds.$$

Multiplying by $(j+1)^{2r}$ and summing over j yields (3-9a) for $t = T$, while the case of an arbitrary time t is treated with the modification explained above. The pointwise estimate (3-9b), taken at $y = 0$, follows from the inequality

$$\|\hat{\bar{v}}_j^\flat|_{y=0}\|_{L^2_{\zeta,x}}^2 \le \frac{C}{(\beta(j+1))^3} \|\partial_x \hat{f}_j\|_{L^2_{\zeta,x}}^2 .$$

The pointwise estimates (3-7f), (3-8c), and (3-9c), taken at $y = 1$ or $y = \frac{1}{2}$ are much better: all boundary layer terms taken at $y = 1$ contain an exponential factor $e^{-\sqrt{\beta(j+1)+i\xi}}$ which allows us to gain an arbitrary number of powers of βj (which explains the arbitrary factor $1/\beta^{20}$ and the index $r - \gamma - 10$). □

Lemma 3.3. *Let $r \in \mathbb{R}$, $\beta \ge 1$ and $T > 0$ such that $\tau(t) \ge \tau_1$ for $t \in [0, T]$. We have*

$$\sup_{[0,t]} \|\omega^\flat(s)\|_{\gamma,r,\tau(s)}^2 \lesssim \frac{1}{\beta^{1/2}} \int_0^t |h(s)|_{\gamma,r+\gamma-1/4,\tau(s)}^2 \, ds \tag{3-15a}$$

for all $t \in [0, T]$.

Proof of Lemma 3.3. In order to establish the estimate (3-15a), we rely on the explicit formula (3-12), which gives an L^1 control of the Fourier transform:

$$\begin{aligned}
\|\hat{\bar{\omega}}_j^\flat\|_{L^1_\zeta(L^2_{x,y})} &\lesssim \int_{\mathbb{R}} \frac{1}{|\sqrt{\beta(j+1)+i\zeta} - 2|} \left(\int_{\mathbb{R}_+} \int_{\mathbb{T}} |e^{-2y\sqrt{\beta(j+1)+i\zeta}}| \, |\hat{f}_j(\zeta, x)|^2 \, dx \, dy \right)^{1/2} d\zeta \\
&\lesssim \int_{\mathbb{R}} \frac{1}{|\sqrt{\beta(j+1)+i\zeta}|^{3/4}} \left(\int_{\mathbb{T}} |\hat{f}_j(\zeta, x)|^2 \, dx \right)^{1/2} d\zeta \\
&\lesssim \left(\int_{\mathbb{R}} \frac{1}{|\sqrt{\beta(j+1)+i\zeta}|^{3/2}} \, d\zeta \right)^{1/2} \left(\int_{\mathbb{R}} \int_{\mathbb{T}} |\hat{f}_j(\zeta, x)|^2 \, dx \, d\zeta \right)^{1/2} \\
&\lesssim \frac{1}{(\beta(j+1))^{1/4}} \left(\int_{\mathbb{R}} \int_{\mathbb{T}} |\hat{f}_j(\zeta, x)|^2 \, dx \, d\zeta \right)^{1/2}.
\end{aligned}$$

This implies

$$\sup_{t \in \mathbb{R}} \|\bar{\omega}_j^\flat(t)\|_{L^2_{x,y}} \lesssim \frac{1}{(\beta(j+1))^{1/4}} \left(\int_{\mathbb{R}} \int_{\mathbb{T}} |f_{j+1}(t, x)|^2 \, dt \right)^{1/2}.$$

Restricting the left-hand side to the supremum over $(0, T)$, we get

$$\sup_{t \in (0,T)} \|\omega_j^\flat(t)\|_{L^2_{x,y}}^2 \lesssim \frac{1}{(\beta(j+1))^{-2\gamma+1/2}} \int_0^T \int_{\mathbb{T}} \|h_{j+1}(t, x)|^2 \, dt.$$

Multiplying by $(j+1)^{2r}$ and summing over j, we get (3-15a) for $t = T$. The general case of $t \in (0, T)$ is treated as in the proof of Lemma 3.1. □

The interior vorticity controls the boundary layer lift. So far, we have only focused on the lower boundary layer lift, which is very small near $y = 0$. We introduce the notation

$$\omega^{\mathrm{bl}}(x, y, t) = \omega^{\flat}(x, y, t) - \omega^{\flat}(x, 1 - y, t), \tag{3-16a}$$

$$u^{\mathrm{bl}}(x, y, t) = u^{\flat}(x, y, t) + u^{\flat}(x, 1 - y, t), \tag{3-16b}$$

$$v^{\mathrm{bl}}(x, y, t) = -\int_0^y \partial_x u^{\mathrm{bl}}(x, z, t)\, dz \tag{3-16c}$$

to denote the cumulative boundary layer profile, and

$$\omega^{\mathrm{in}}(x, y, t) = \omega(x, y, t) - \omega^{\mathrm{bl}}(x, y, t), \tag{3-17a}$$

$$u^{\mathrm{in}}(x, y, t) = u(x, y, t) - u^{\mathrm{bl}}(x, y, t), \tag{3-17b}$$

$$v^{\mathrm{in}}(x, y, t) = v(x, y, t) - v^{\mathrm{bl}}(x, y, t) \tag{3-17c}$$

to denote the interior vorticity, horizontal velocity component, and vertical velocity component. In view of (3-3), (3-16) and (3-17) we also define the objects ω_j^{bl}, u_j^{bl}, v_j^{bl} in terms of the function h, and ω_j^{in}, u_j^{in}, v_j^{in} in terms of h and ω.

Lemma 3.4. *Let $\gamma \in \left[1, \frac{5}{4}\right]$, $r > 2\gamma + 2$, $M > 0$. Assume $\omega = \partial_y u$ is such that*

$$\sup_{[0,T]} \|\omega(t)\|_{\gamma, r/4, \tau(t)} \le M \tag{3-18}$$

and define

$$h(x, t) = -\int_0^1 (u(x, y, t))^2\, dy + \int_{\mathbb{T}} \int_0^1 (u(x, y, t))^2\, dy\, dx.$$

With h as above, let $\omega^{\flat}$ be defined via (3-4), *and let ω^{in} be as defined in* (3-17). *Then there exists $\beta_* = \beta_*(\tau_0, \tau_1, \gamma, r, M)$ such that if $\beta \ge \beta_*$ and if T is such that $\tau(t) \ge \tau_1$ for $t \in [0, T]$, then*

$$\int_0^t |h(s)|^2_{\gamma, r, \tau(s)} ds \lesssim M^2 \int_0^t \|\omega^{\mathrm{in}}(s)\|^2_{\gamma, r, \tau(s)}\, ds$$

for any $t \in [0, T]$.

Note that with h defined as above we have

$$\partial_x h = -\partial_x \int_0^1 u^2\, dy,$$

so that the additional kinetic energy term in h is not seen by ω^{bl}. Combining Lemmas 3.1, 3.3 and 3.4, we see that condition (3-18) implies a sharp control of the Gevrey norm of the boundary layer profiles ω^{bl}, u^{bl}, and v^{bl}, solely in terms of the Gevrey norm of the interior vorticity ω^{in} and of the constants M and β.

Proof of Lemma 3.4. For $j = 0$ we have $h_0 = M_0 h = \tau h$, and since

$$\int_{\mathbb{T}} h(x, t)\, dx = 0,$$

we may apply the Poincaré inequality in the x-variable:

$$\|h_0\|_{L^2_x} \lesssim \|\partial_x h_0\|_{L^2_x} \lesssim \|h_1\|_{L^2_x}. \tag{3-19}$$

Hence, it is enough to estimate h_j for $j \geq 1$. By the Leibniz rule we have

$$-h_j(x,t) = \sum_{\ell=0}^{j} \binom{j}{\ell} \frac{M_j}{M_{j-\ell} M_\ell} \int_0^1 u_\ell(x,y,t) u_{j-\ell}(x,y,t)\, dy. \tag{3-20}$$

We can without loss of generality estimate only the half-sum $\sum_{0 \leq \ell \leq j/2}$, as the other half-sum can be put in the same form through the change of index $\ell' = j - \ell$.

First let us treat the case $\ell \geq 1$. The compatibility condition (1-2) yields $\int_0^1 u_\ell(x,y)\, dy = 0$, which directly implies

$$\int_0^1 u_\ell(x,y) u^{\text{in}}_{j-\ell}(x,y)\, dy = \int_0^1 u_\ell(x,y) \left(u^{\text{in}}_{j-\ell}(x,y) - \int_0^1 u^{\text{in}}_{j-\ell}(x,z)\, dz \right) dy.$$

Using the one-dimensional Gagliardo–Nirenberg inequality, the one-dimensional Hardy inequality, the one-dimensional Poincaré inequality, and the fact that $u_\ell|_{y=0} = u_\ell|_{y=1} = 0$, we have, for $\ell \geq 1$,

$$\begin{aligned}
\left\| \int_0^1 u_\ell(x,y) u_{j-\ell}(x,y)\, dy \right\|_{L^2_x} &\leq \left\| \int_0^1 u_\ell(x,y) u^{\text{in}}_{j-\ell}(x,y)\, dy \right\|_{L^2_x} + \left\| \int_0^1 u_\ell(x,y) u^{\text{bl}}_{j-\ell}(x,y)\, dy \right\|_{L^2_x} \\
&\leq \|u_\ell\|_{L^\infty_x L^2_y} \left\| u^{\text{in}}_{j-\ell} - \int_0^1 u^{\text{in}}_{j-\ell}\, dz \right\|_{L^2_{x,y}} + \left\| \frac{u_\ell}{y(1-y)} \right\|_{L^\infty_x L^2_y} \|y(1-y) u^{\text{bl}}_{j-\ell}\|_{L^2_{x,y}} \\
&\lesssim \|u_\ell\|^{1/2}_{L^2_{x,y}} \|\partial_x u_\ell\|^{1/2}_{L^2_{x,y}} \|\omega^{\text{in}}_{j-\ell}\|_{L^2_{x,y}} + \|\omega_\ell\|^{1/2}_{L^2_{x,y}} \|\partial_x \omega_\ell\|^{1/2}_{L^2_{x,y}} \|y(1-y) u^{\text{bl}}_{j-\ell}\|_{L^2_{x,y}} \\
&\lesssim \frac{M_\ell^{1/2}}{M_{\ell+1}^{1/2}} \|\omega_\ell\|^{1/2}_{L^2_{x,y}} \|\omega_{\ell+1}\|^{1/2}_{L^2_{x,y}} \left(\|\omega^{\text{in}}_{j-\ell}\|_{L^2_{x,y}} + \|y(1-y) u^{\text{bl}}_{j-\ell}\|_{L^2_{x,y}} \right).
\end{aligned}$$

For $\ell = 0$, we estimate the L^2_x norm of $\int_0^1 u_0\, u^{\text{bl}}_j\, dy$ precisely as in the case $\ell \geq 1$. For the interior piece, since $j \geq 1$ we may use (1-2) and the Poincaré inequality in y to estimate

$$\begin{aligned}
\left\| \int_0^1 u_0(x,y) u^{\text{in}}_j(x,y)\, dy \right\|_{L^2_x} &\lesssim \|u_0\|_{L^\infty_x L^2_y} \left(\left\| u^{\text{in}}_j(x,y) - \int_0^1 u^{\text{in}}_j(x,z)\, dz \right\|_{L^2_{x,y}} + \left\| \int_0^1 u^{\text{bl}}_j(x,z)\, dz \right\|_{L^2_{x,y}} \right) \\
&\lesssim M \left(\|\omega^{\text{in}}_j\|_{L^2_{x,y}} + \left\| \int_0^1 u^{\text{bl}}_j(x,z)\, dz \right\|_{L^2_x} \right)
\end{aligned}$$

since

$$\|u_0\|_{L^\infty_x L^2_y} \lesssim \|\omega_0\|_{L^\infty_x L^2_y} \lesssim \|\omega_0\|_{L^2_{x,y}} + \|\omega_1\|_{L^2_{x,y}} \lesssim M.$$

At this point we note that

$$\int_0^1 u^{\text{bl}}_j(x,y)\, dy = -\int_0^{1/2} y \omega^{\text{bl}}_j(x,y)\, dy + u^{\text{bl}}_j\left(x, \tfrac{1}{2}\right) + \int_{1/2}^1 (1-y) \omega^{\text{bl}}_j(x,y)\, dz$$

so that

$$\left\| \int_0^1 u_j^{\mathrm{bl}}(x,y)\,dy \right\|_{L_x^2} \lesssim \|y\omega_j^\flat\|_{L^2_{x,y}} + \left\|u_j^\flat\left(x,\tfrac{1}{2}\right)\right\|_{L_x^2}.$$

Returning to (3-20), and using that in this range of ℓ, namely less than $\frac{1}{2}j$, we have

$$\binom{j}{\ell}\frac{M_j}{M_{j-\ell}M_\ell^{1/2}M_{\ell+1}^{1/2}} \lesssim \frac{1}{\tau^{1/2}}\binom{j}{\ell}^{1-\gamma}\frac{1}{(\ell+1)^{r-\gamma/2}} \lesssim \frac{1}{(\ell+1)^{r-\gamma/2}},$$

for $j \geq 1$ we obtain

$$\begin{aligned}\|h_j\|_{L_x^2} &\lesssim \sum_{\ell=1}^{\lceil j/2\rceil}\binom{j}{\ell}\frac{M_j}{M_{j-\ell}M_\ell^{1/2}M_{\ell+1}^{1/2}}\|\omega_\ell\|_{L^2_{x,y}}^{1/2}\|\omega_{\ell+1}\|_{L^2_{x,y}}^{1/2}(\|\omega_{j-\ell}^{\mathrm{in}}\|_{L^2_{x,y}} + \|y(1-y)u_{j-\ell}^{\mathrm{bl}}\|_{L^2_{x,y}}) \\ &\qquad + M\left(\|\omega_j^{\mathrm{in}}\|_{L^2_{x,y}} + \|yu_j^\flat\|_{L^2_{x,y}} + \|y\omega_j^\flat\|_{L^2_{x,y}} + \left\|u_j^\flat\left(x,\tfrac{1}{2}\right)\right\|_{L_x^2}\right) \\ &\lesssim \sum_{\ell=1}^{\lceil j/2\rceil}\frac{(l+1)^{-3r/4}\|\omega_\ell\|_{L^2_{x,y}}^{1/2}\|\omega_{\ell+1}\|_{L^2_{x,y}}^{1/2}}{(\ell+1)^{r/4-\gamma/2}}(\|\omega_{j-\ell}^{\mathrm{in}}\|_{L^2_{x,y}} + \|yu_{j-\ell}^\flat\|_{L^2_{x,y}}) \\ &\qquad + M\left(\|\omega_j^{\mathrm{in}}\|_{L^2_{x,y}} + \|yu_j^\flat\|_{L^2_{x,y}} + \|y\omega_j^\flat\|_{L^2_{x,y}} + \left\|u_j^\flat\left(x,\tfrac{1}{2}\right)\right\|_{L_x^2}\right). \end{aligned} \tag{3-21}$$

From (3-19) and (3-21), using the discrete Hölder and Young inequalities, inequalities (3-8b), (3-8c), (3-7b) and assumption (3-18) we obtain from the above that

$$\begin{aligned}\int_0^t |h(s)|^2_{\gamma,r,\tau(s)}\,ds &= \int_0^t \sum_{j\geq 0}\|h_j(s)\|^2_{L_x^2}\,ds \\ &\lesssim \sup_{[0,t]}\left(\sum_{j\geq 0}\frac{(j+1)^{-3r/4}(\|\omega_j\|_{L^2_{x,y}}+\|\omega_{j+1}\|_{L^2_{x,y}})}{(j+1)^{r/4-\gamma/2}}\right)^2\int_0^t\left(\sum_{j\geq 0}\|\omega_j^{\mathrm{in}}\|^2_{L^2_{x,y}}+\sum_{j\geq 0}\|yu_j^\flat\|^2_{L^2_{x,y}}\right)ds \\ &\quad + M^2\int_0^t\left(\|\omega^{\mathrm{in}}(s)\|^2_{\gamma,r,\tau(s)}+\|yu^\flat(s)\|^2_{\gamma,r,\tau(s)}+\|y\omega^\flat(s)\|^2_{\gamma,r,\tau(s)}+|u^\flat(s)|_{y=1/2}|^2_{\gamma,r,\tau(s)}\right)ds \\ &\lesssim M^2\left(\int_0^t\|\omega^{\mathrm{in}}(s)\|^2_{\gamma,r,\tau(s)}\,ds+\int_0^t\|yu^\flat(s)\|^2_{\gamma,r,\tau(s)}+\|y\omega^\flat\|^2_{\gamma,r,\tau(s)}+|u^\flat|_{y=1/2}|^2_{\gamma,r,\tau(s)}\,ds\right) \\ &\lesssim M^2\left(\int_0^t\|\omega^{\mathrm{in}}(s)\|^2_{\gamma,r,\tau(s)}\,ds+\frac{1}{\beta^{5/2}}\int_0^t|h(s)|^2_{\gamma,r+\gamma-5/4,\tau(s)}\,ds\right).\end{aligned}$$

Here we have used that $\frac{1}{4}r - \frac{1}{2}\gamma > \frac{1}{2}$. The proof is completed using that $M^2\beta^{-5/2} \ll 1$, which follows once β_* is taken sufficiently large, and the fact that $\gamma \leq \frac{5}{4}$, which allows us to absorb the second term in the right side of the above into the left side. □

4. Estimates involving ω^{in}

From the vorticity evolution (2-6) and the definition of ω^{bl} (3-16) (which obeys $\int_{\mathbb{T}}\omega^{\mathrm{bl}}(x,y,t)\,dx = 0$ for any $y \geq 0$), we obtain that the equation obeyed by the interior vorticity is

$$\partial_t \omega^{\text{in}} - \partial_y^2 \omega^{\text{in}} + u\,\partial_x \omega^{\text{in}} + v\,\partial_y \omega^{\text{in}} = -u\,\partial_x \omega^{\text{bl}} - v\,\partial_y \omega^{\text{bl}}, \tag{4-1a}$$

$$\partial_y \omega^{\text{in}}|_{y=0,1} = \tilde{\omega}^{\text{in}}|_{y=1} - \tilde{\omega}^{\text{in}}|_{y=0} + 2\omega^\flat|_{y=1} - \partial_y \omega^\flat|_{y=1}, \tag{4-1b}$$

$$\omega^{\text{in}}(0) = \omega_0. \tag{4-1c}$$

The initial condition for ω^{in} is obtained from the fact that $\omega^{\text{bl}}(0) = 0$, which holds in view of (3-4c). The main a priori estimate for ω^{in} is provided by the following proposition.

Proposition 4.1. *Let M, δ_0, $\gamma \in \left[1, \frac{9}{8}\right]$ be given, and let β_* be as in Lemma 3.4. There exists $r_0 = r_0(\gamma)$ such that for all $r \geq r_0$, one can find $\beta_0 = \beta_0(M, \delta_0, \tau_0, \tau_1, r, \gamma) > \max(\beta_*, 4)$ satisfying: if $\beta \geq \beta_0$ and $T \leq 1$ is small enough so that $\tau(t) \geq \tau_1$ for all $t \in [0, T]$, under the assumptions*

$$\sup_{t\in[0,T]} \|\omega(t)\|_{\gamma,3r/4,\tau(t)} + \sup_{t\in[0,T]} \|\partial_y \omega(t)\|_{\gamma,r/2,\tau(t)} \leq M, \tag{4-2}$$

$$\delta_0 \leq \partial_y \omega \leq \frac{1}{\delta_0}, \tag{4-3}$$

$$\sup_{t\in[0,T]} \|\partial_y^2 \omega(t)\|_{L_x^\infty L_y^2} \leq M, \tag{4-4}$$

we have

$$\sup_{s\in[0,t]} \|\omega^{\text{in}}(s)\|^2_{\gamma,r,\tau(s)} + \int_0^t \|\partial_y \omega^{\text{in}}(s)\|^2_{\gamma,r,\tau(s)}\, ds + \beta \int_0^t \|\omega^{\text{in}}(s)\|^2_{\gamma,r+1/2,\tau(s)}\, ds \leq \frac{1}{\delta_0^2} \|\omega(0)\|^2_{\gamma,r,\tau_0} \tag{4-5}$$

holds for all $t \in [0, T]$. Moreover, as a consequence we obtain

$$\sup_{s\in[0,t]} \|\omega(s)\|^2_{\gamma,r-\gamma+3/4,\tau(s)} + \int_0^t \|\partial_y \omega(s)\|^2_{\gamma,r-\gamma+3/4,\tau(s)}\, ds + \beta \int_0^t \|\omega(s)\|^2_{\gamma,r-\gamma+5/4,\tau(s)}\, ds \leq \frac{4}{\delta_0^2} \|\omega(0)\|^2_{\gamma,r,\tau_0} \tag{4-6}$$

for all $t \in [0, T]$.

Proof of Proposition 4.1. Using the convention (3-3), from (4-1) we obtain

$$(\partial_t + \beta(j+1) - \partial_y^2)\omega_j^{\text{in}} + (u\,\partial_x + v\,\partial_y)\omega_j^{\text{in}} + v_j^{\text{in}}\,\partial_y \omega = -(u\,\partial_x + v\,\partial_y)\omega_j^{\text{bl}} - v_j^{\text{bl}}\,\partial_y \omega - M_j[\partial_x^j, u\,\partial_x + v\,\partial_y]\omega + v_j\,\partial_y \omega, \tag{4-7a}$$

$$\partial_y \omega_j^{\text{in}}|_{y=0,1} = \tilde{\omega}_j^{\text{in}}|_{y=1} - \tilde{\omega}_j^{\text{in}}|_{y=0} + 2\omega_j^\flat|_{y=1} - \partial_y \omega_j^\flat|_{y=1}. \tag{4-7b}$$

Note that as soon as $j \geq 1$, we may replace $\tilde{\omega}_j^{\text{in}}|_{y=0,1} = \omega_j^{\text{in}}|_{y=0,1}$ in (4-7b). We perform a "hydrostatic energy estimate" on (4-7), which is permissible in view of (4-3). That is, we multiply (4-7a) by $\omega_j^{\text{in}}/\partial_y \omega$ and integrate over $\Omega = \mathbb{T} \times [0, 1]$. We notably use the "hydrostatic trick", which in this case gives

$$\begin{aligned}
\int_\Omega v_j^{\text{in}} \omega_j^{\text{in}}\, dx\, dy &= -\int_\Omega \left(\int_0^y \partial_x u_j^{\text{in}} \right) \partial_y u_j^{\text{in}}\, dx\, dy \\
&= \int_\Omega \partial_x u_j^{\text{in}} u_j^{\text{in}}\, dx\, dy - \int_\mathbb{T} \left(\int_0^1 \partial_x u_j^{\text{in}} \right) u_j^{\text{in}}|_{y=1}\, dx \\
&= -\int_\mathbb{T} \left(\int_0^1 \partial_x u_j^{\text{bl}}(x, y)\, dy \right) u_j^{\text{bl}}(x, 1)\, dx,
\end{aligned}$$

taking into account that $\int_0^1 \partial_x u_j(x,y)\,dy = 0$ and that $u_j|_{y=1}=0$. Thus, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\frac{\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\right\|_{L^2}^2+\beta(j+1)\left\|\frac{\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\right\|_{L^2}^2+\left\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\right\|_{L^2}^2\\
&\quad=\int_{\mathbb{T}}\left(\frac{\partial_y\omega_j^{\mathrm{in}}\omega_j^{\mathrm{in}}}{\partial_y\omega}\bigg|_{y=1}-\frac{\partial_y\omega_j^{\mathrm{in}}\omega_j^{\mathrm{in}}}{\partial_y\omega}\bigg|_{y=0}\right)dx+\int_{\mathbb{T}}\left(\int_0^1\partial_x u_j^{\mathrm{bl}}(x,y)\,dy\right)u_j^{\mathrm{bl}}(x,1)\,dx\\
&\qquad+\int_\Omega\frac{\partial_y\omega_j^{\mathrm{in}}\omega_j^{\mathrm{in}}}{\partial_y\omega}\frac{\partial_y^2\omega}{\partial_y\omega}\,dx\,dy-\frac{1}{2}\int_\Omega\frac{(\omega_j^{\mathrm{in}})^2}{\partial_y\omega}\frac{(u\,\partial_x+v\,\partial_y)\,\partial_y\omega}{\partial_y\omega}\,dx\,dy\\
&\qquad-\int_\Omega u\,\partial_x\omega_j^{\mathrm{bl}}\frac{\omega_j^{\mathrm{in}}}{\partial_y\omega}\,dx\,dy-\int_\Omega v\,\partial_y\omega_j^{\mathrm{bl}}\frac{\omega_j^{\mathrm{in}}}{\partial_y\omega}dxdy-\int_\Omega v_j^{\mathrm{bl}}\omega_j^{\mathrm{in}}\,dx\,dy\\
&\qquad-\sum_{k=1}^{j}\frac{M_j}{M_kM_{j-k+1}}\binom{j}{k}\int_\Omega u_k\omega_{j-k+1}\frac{\omega_j^{\mathrm{in}}}{\partial_y\omega}\,dx\,dy-\sum_{k=1}^{j-1}\frac{M_j}{M_kM_{j-k}}\binom{j}{k}\int_\Omega v_k\,\partial_y\omega_{j-k}\frac{\omega_j^{\mathrm{in}}}{\partial_y\omega}\,dx\,dy\\
&\quad=:T_{1j}+T_{2j}+T_{3j}-T_{4j}-T_{5j}-T_{6j}-T_{7j}-T_{8j}-T_{9j}.
\end{aligned}\tag{4-8}
$$

Summing over j, and integrating on $[0,t)$, with $t\le T$, we obtain

$$
\begin{aligned}
&\|\omega^{\mathrm{in}}(t)\|_{\gamma,r,\tau(t)}^2+2\beta\int_0^t\|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau}^2+\int_0^t\|\partial_y\omega^{\mathrm{in}}\|_{\gamma,r,\tau}^2\\
&\quad\le\frac{1}{\delta_0^2}\|\omega_0^{\mathrm{in}}\|_{\gamma,r,\tau_0}^2+\frac{1}{\delta_0}\int_0^t\sum_{j\ge0}\left(|T_{1j}|-\frac{1}{2}\left\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\right\|_{L^2}^2\right)+|T_{2j}|+\left(|T_{3j}|+|T_{4j}|-\frac{1}{2}\left\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\right\|_{L^2}^2\right)ds\\
&\qquad+\frac{1}{\delta_0}\int_0^t\sum_{j\ge0}|T_{5j}|+|T_{6j}|+|T_{7j}|+|T_{8j}|+|T_{9j}|\,ds.
\end{aligned}\tag{4-9}
$$

The rest of the proof is dedicated to estimating the nine terms on the right side of (4-9).

The T_{1j} bound: From (2-6b) and (4-7b) we obtain

$$
\begin{aligned}
T_{1j}&=\int_{\mathbb{T}}\frac{\partial_y\omega_j^{\mathrm{in}}|_{y=0,1}(\omega_j^{\mathrm{in}}|_{y=1}-\omega_j^{\mathrm{in}}|_{y=0})}{\partial_y\omega|_{y=0,1}}\,dx\\
&=\int_{\mathbb{T}}\frac{(\tilde\omega_j^{\mathrm{in}}|_{y=1}-\tilde\omega_j^{\mathrm{in}}|_{y=0})(\omega_j^{\mathrm{in}}|_{y=1}-\omega_j^{\mathrm{in}}|_{y=0})}{\partial_y\omega|_{y=0,1}}\,dx+\int_{\mathbb{T}}\frac{(2\omega_j^{\flat}|_{y=1}-\partial_y\omega_j^{\flat}|_{y=1})(\omega_j^{\mathrm{in}}|_{y=1}-\omega_j^{\mathrm{in}}|_{y=0})}{\partial_y\omega|_{y=0,1}}\,dx\\
&=T_{11j}+T_{12j}.
\end{aligned}
$$

From the Gagliardo–Nirenberg inequality $\|f\|_{L^\infty(0,1)}\le\|f\|_{L^2(0,1)}+2\|f\|_{L^2(0,1)}^{1/2}\|\partial_y f\|_{L^2(0,1)}^{1/2}$, we have

$$|T_{11j}|\lesssim\frac{1}{\delta_0}(\|\omega_j^{\mathrm{in}}\|_{L^2_{x,y}}^2+\|\omega_j^{\mathrm{in}}\|_{L^2_{x,y}}\|\partial_y\omega_j^{\mathrm{in}}\|_{L^2_{x,y}}).$$

Using Cauchy–Schwarz, we similarly obtain

$$|T_{12j}|\lesssim|T_{11j}|+\frac{1}{\delta_0}(\|\omega_j^{\flat}|_{y=1}\|_{L^2_x}^2+\|\partial_y\omega_j^{\flat}|_{y=1}\|_{L^2_x}^2).$$

Summing up the above two estimates, and summing over $j \geq 0$ we obtain

$$\sum_{j\geq 0}\left(|T_{1j}| - \frac{1}{2}\left\| \frac{\partial_y \omega_j^{\text{in}}}{\sqrt{\partial_y \omega}}\right\|_{L^2}^2\right) \lesssim \frac{1}{\delta_0^2}\|\omega^{\text{in}}\|_{\gamma,r,\tau}^2 + \frac{1}{\delta_0}(|\omega_j^\flat|_{y=1}|_{\gamma,r,\tau}^2 + |\partial_y \omega_j^\flat|_{y=1}|_{\gamma,r,\tau}^2).$$

Using (3-7e)–(3-7f), and combining the resulting bound with Lemma 3.4 (which may be used due to assumption (4-2)), we arrive at

$$\int_0^t \sum_{j\geq 0}\left(|T_{1j}| - \frac{1}{2}\left\| \frac{\partial_y \omega_j^{\text{in}}}{\sqrt{\partial_y \omega}}\right\|_{L^2}^2\right) \lesssim \frac{1}{\delta_0^2}\int_0^t \|\omega^{\text{in}}\|_{\gamma,r,\tau}^2 + \frac{1}{\delta_0 \beta^{20}}\int_0^t |h|_{\gamma,r+\gamma-10,\tau}^2 \lesssim \frac{1}{\delta_0^2}\int_0^t \|\omega^{\text{in}}\|_{\gamma,r,\tau}^2, \quad (4\text{-}10)$$

where we have used that $\delta_0 M^2 \leq \beta^{20}$.

The T_{2j} bound: From (3-16) we obtain

$$\begin{aligned} T_{2j} &= 2\int_{\mathbb{T}}\left(\int_0^1 \partial_x u_j^\flat(x,y)\,dy\right)(u_j^\flat(x,0) + u_j^\flat(x,1))\,dx \\ &= 2\int_{\mathbb{T}} (v_j^\flat(x,0) - v_j^\flat(x,1))(u_j^\flat(x,0) + u_j^\flat(x,1))\,dx, \end{aligned}$$

and thus, also appealing to Gagliardo–Nirenberg, we obtain

$$\begin{aligned} |T_{2j}| &\leq 2(\|v_j^\flat|_{y=0}\|_{L_x^2} + \|v_j^\flat|_{y=1}\|_{L_x^2})(\|u_j^\flat|_{y=0}\|_{L_x^2} + \|u_j^\flat|_{y=1}\|_{L_x^2}) \\ &\lesssim \frac{\|v_j^\flat|_{y=0}\|_{L_x^2} + \|v_j^\flat|_{y=1}\|_{L_x^2}}{(j+1)^{3/2-\gamma}}\big((j+1)^{3/2-\gamma}\|u_j^\flat\|_{L^2_{x,y}} + (j+1)^{7/8-\gamma/2}\|u_j^\flat\|_{L^2_{x,y}}^{1/2}(j+1)^{5/8-\gamma/2}\|\omega_j^\flat\|_{L^2_{x,y}}^{1/2}\big), \end{aligned}$$

and summing over j we arrive at

$$\sum_{j\geq 0}|T_{2j}| \lesssim (|v^\flat|_{y=0}|_{\gamma,r+\gamma-3/2,\tau} + |v_j^\flat|_{y=1}|_{\gamma,r+\gamma-3/2,\tau})(\|u^\flat\|_{\gamma,r+3/2-\gamma,\tau} + \|u^\flat\|_{\gamma,r+7/4-\gamma,\tau}^{1/2}\|\omega^\flat\|_{\gamma,r+5/4-\gamma,\tau}^{1/2}).$$

Upon integrating on $[0,t)$, the above terms are bounded using (3-7a), (3-8a), (3-9b), and (3-9c), after which Lemma 3.4 is used to yield

$$\begin{aligned} \int_0^t \sum_{j\geq 0}|T_{2j}| &\lesssim \frac{1}{\beta^{5/2}}\left(\int_0^t |h|_{\gamma,r+3\gamma-3,\tau}^2\right)^{1/2}\left(\left(\int_0^t |h|_{\gamma,r+1/4,\tau}^2\right)^{1/2} + \left(\int_0^t |h|_{\gamma,r+1/2,\tau}^2\right)^{1/2}\right) \\ &\lesssim \frac{M^2}{\beta^{5/2}}\left(\int_0^t \|\omega^{\text{in}}\|_{\gamma,r+3\gamma-3,\tau}^2\right)^{1/2}\left(\int_0^t \|\omega^{\text{in}}\|_{\gamma,r+1/2,\tau}^2\right)^{1/2}. \end{aligned}$$

For the last inequality, we have applied Lemma 3.4 to both factors on the right-hand side, which is legitimate under the assumptions

$$r + \min\{3\gamma - 3, \tfrac{1}{2}\} \geq 2\gamma + 2, \qquad \sup_{[0,T]}\|\omega(t)\|_{\gamma,(1/4)(r+\max\{3\gamma-3,1/2\}),\tau(t)} \leq M.$$

Both assumptions are satisfied for $r > r(\gamma)$ large enough, the second one being deduced from (4-2). Thus we have proven

$$\int_0^t \sum_{j\geq 0} |T_{2j}| \lesssim \frac{M^2}{\beta^{5/2}} \int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}. \tag{4-11}$$

The T_{3j} and T_{4j} bounds: These are the only terms for which assumption (4-4) is used. In view of (4-3)–(4-4) and the Gagliardo–Nirenberg inequality in y, we immediately obtain

$$\begin{aligned}\sum_{j\geq 0}\biggl(|T_{3j}|-\frac{1}{4}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^2_{L^2}\biggr) &\lesssim \sum_{j\geq 0}\biggl(\frac{M}{\delta_0^{3/2}}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|_{L^2_xL^2_y}\|\omega_j^{\mathrm{in}}\|_{L^2_xL^\infty_y}-\frac{1}{8}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^2_{L^2}\biggr)\\ &\lesssim \sum_{j\geq 0}\biggl(\frac{M}{\delta_0^{3/2}}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|_{L^2}\biggl(\|\omega_j^{\mathrm{in}}\|_{L^2}+\frac{1}{\delta_0^{1/4}}\|\omega_j^{\mathrm{in}}\|^{1/2}_{L^2}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^{1/2}_{L^2}\biggr)-\frac{1}{8}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^2_{L^2}\biggr)\\ &\lesssim \frac{M^4}{\delta_0^7}\|\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau},\end{aligned}$$

and using (4-2) combined with (4-3)–(4-4) we also obtain

$$\begin{aligned}\sum_{j\geq 0}\biggl(|T_{4j}|-\frac{1}{4}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^2_{L^2}\biggr) &\lesssim \sum_{j\geq 0}\biggl(\frac{M^2}{\delta_0^2}\|\omega_j^{\mathrm{in}}\|_{L^2_xL^2_y}\|\omega_j^{\mathrm{in}}\|_{L^2_xL^\infty_y}-\frac{1}{4}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^2_{L^2}\biggr)\\ &\lesssim \sum_{j\geq 0}\biggl(\frac{M^2}{\delta_0^2}\|\omega_j^{\mathrm{in}}\|_{L^2}\biggl(\|\omega_j^{\mathrm{in}}\|_{L^2}+\frac{1}{\delta_0^{1/4}}\|\omega_j^{\mathrm{in}}\|^{1/2}_{L^2}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^{1/2}_{L^2}\biggr)-\frac{1}{4}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^2_{L^2}\biggr)\\ &\lesssim \frac{M^{8/3}}{\delta_0^3}\|\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau}.\end{aligned}$$

Here we have also used the second term on the left side of (4-2), in order to estimate $\|\partial_x\partial_y\omega\|_{L^\infty_xL^2_y}$. Thus,

$$\int_0^t \sum_{j\geq 0}\biggl(|T_{3j}|+|T_{4j}|-\frac{1}{4}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^2_{L^2}\biggr) \lesssim \frac{M^4}{\delta_0^7}\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau}. \tag{4-12}$$

The T_{5j} bound: As it turns out, this term creates the most stringent assumption on γ, namely that $\gamma \leq \frac{9}{8}$. Since $u|_{y=0,1} = 0$, using (4-2) and (4-4), we have

$$\begin{aligned}|T_{5j}| &\leq \frac{1}{\delta_0}\biggl\|\frac{u}{y(1-y)}\biggr\|_{L^\infty}\|y(1-y)\partial_x\omega_j^{\mathrm{bl}}\|_{L^2}\|\omega_j^{\mathrm{in}}\|_{L^2}\\ &\lesssim \frac{\|\omega\|_{L^\infty}}{\delta_0}\frac{M_j}{M_{j+1}(j+1)^{1/2}}\|y\omega^\flat_{j+1}\|_{L^2}(j+1)^{1/2}\|\omega_j^{\mathrm{in}}\|_{L^2},\end{aligned}$$

and thus, upon summing over j and integrating on $[0,t]$ we arrive at

$$\int_0^t \sum_{j\geq 0}|T_{5j}| \lesssim \frac{M}{\delta_0}\biggl(\int_0^t \|y\omega^\flat\|^2_{\gamma,r+\gamma-1/2,\tau}\biggr)^{1/2}\biggl(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\biggr)^{1/2}.$$

We now appeal to (3-7b) and to Lemma 3.4, which is again legitimate for $r > r(\gamma)$ large enough. We obtain

$$\begin{aligned}\int_0^t \sum_{j\geq 0} |T_{5j}| &\lesssim \frac{M}{\delta_0 \beta^{5/4}} \left(\int_0^t |h|^2_{\gamma,r+2\gamma-7/4,\tau}\right)^{1/2} \left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\right)^{1/2} \\ &\lesssim \frac{M^2}{\delta_0 \beta^{5/4}} \left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+2\gamma-7/4,\tau}\right)^{1/2} \left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\right)^{1/2} \\ &\lesssim \frac{M^2}{\delta_0 \beta^{5/4}} \int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}. \end{aligned} \tag{4-13}$$

In the last inequality we have used that $2\gamma - \frac{7}{4} \leq \frac{1}{2}$, which holds since $\gamma \leq \frac{9}{8}$.

The T_{6j} bound: Similarly, using that $v|_{y=0,1} = 0$, we obtain

$$\begin{aligned}|T_{6j}| &\leq \frac{1}{\delta_0} \left\| \frac{v}{y(1-y)} \right\|_{L^\infty} \|y(1-y)\partial_y \omega_j^{\mathrm{bl}}\|_{L^2} \|\omega_j^{\mathrm{in}}\|_{L^2} \\ &\lesssim \frac{\|\partial_x u\|_{L^\infty}}{\delta_0} \|y\, \partial_y \omega_j^\flat\|_{L^2} \|\omega_j^{\mathrm{in}}\|_{L^2} \\ &\lesssim \frac{M}{\delta_0} \frac{\|y\, \partial_y \omega_j^\flat\|_{L^2}}{(j+1)^{1/2}} ((j+1)^{1/2} \|\omega_j^{\mathrm{in}}\|_{L^2}),\end{aligned}$$

so that

$$\int_0^t \sum_{j\geq 0} |T_{6j}| \lesssim \frac{M}{\delta_0} \left(\int_0^t \|y\, \partial_y \omega^\flat\|^2_{\gamma,r-1/2,\tau}\right)^{1/2} \left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\right)^{1/2}.$$

Using (3-7d), and then Lemma 3.4 (applicable for $r > r(\gamma)$ large enough, by (4-2)), we obtain

$$\begin{aligned}\int_0^t \sum_{j\geq 0} |T_{6j}| &\lesssim \frac{M}{\delta_0 \beta^{3/4}} \left(\int_0^t |h|^2_{\gamma,r+\gamma-7/4,\tau}\right)^{1/2} \left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\right)^{1/2} \\ &\lesssim \frac{M^2}{\delta_0 \beta^{3/4}} \left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau}\right)^{1/2} \left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\right)^{1/2}\end{aligned} \tag{4-14}$$

since $\gamma \leq \frac{7}{4}$.

The T_{7j} bound: For T_{7j} we directly estimate

$$\sum_{j\geq 0} |T_{7j}| \leq \sum_{j\geq 0} \frac{1}{\delta_0} (j+1)^{-1/2} \|v_j^\flat\|_{L^2} (j+1)^{1/2} \|\omega_j^{\mathrm{in}}\|_{L^2} \lesssim \frac{1}{\delta_0} \|v^\flat\|_{\gamma,r-1/2,\tau} \|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau}.$$

Integrating in time, appealing to (3-9a), and still using Lemma 3.4 we obtain

$$\begin{aligned}\int_0^t \sum_{j\geq 0} |T_{7j}| &\lesssim \frac{1}{\delta_0 \beta^{7/4}} \left(\int_0^t |h|^2_{\gamma,r+2\gamma-9/4,\tau}\right)^{1/2} \left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\right)^{1/2} \\ &\lesssim \frac{M}{\delta_0 \beta^{7/4}} \left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+2\gamma-9/4,\tau}\right)^{1/2} \left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\right)^{1/2} \\ &\lesssim \frac{M}{\delta_0 \beta^{7/4}} \int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\end{aligned} \tag{4-15}$$

as $2\gamma - \frac{9}{4} \leq \frac{1}{2}$.

The T_{8j} bound: We note that

$$\frac{M_j}{M_k M_{j-k+1}}\binom{j}{k} \lesssim \binom{j}{k}^{1-\gamma}\frac{(j+1)^r}{(k+1)^r(j-k+1)^{r-\gamma}},$$

and for $1 \le k \le [j/2]$ it is convenient to use $\binom{j}{k} \ge (j-k+1)/k$. We obtain

$$\begin{aligned}|T_{8j}| &\lesssim \sum_{k=1}^{[j/2]} \frac{j^{1/2}(j-k+1)^{1/2}}{(k+1)^{r-\gamma+1}}\left|\int_\Omega u_k\omega_{j-k+1}\frac{\omega_j^{\text{in}}}{\partial_y\omega}\right| + \sum_{k=[j/2]+1}^{j}\frac{1}{(j-k+1)^{r-\gamma}}\left|\int_\Omega u_k\omega_{j-k+1}\frac{\omega_j^{\text{in}}}{\partial_y\omega}\right| \\ &=: T_{8j,\text{low}} + T_{8j,\text{high}}.\end{aligned}$$

In order to estimate $T_{8j,\text{low}}$, we split ω_{j-k+1} into $\omega_{j-k+1} = \omega_{j-k+1}^{\text{in}} + \omega_{j-k+1}^{\text{bl}}$. First, using the Gagliardo–Nirenberg inequality on Ω and the Poincaré inequality in x (since $k \ge 1$) we may bound

$$\begin{aligned}\|\omega_k\|_{L^\infty} &\lesssim \|\omega_k\|_{L^2} + \|\partial_x\omega_k\|_{L^2} + (\|\omega_k\|_{L^2}^{1/2} + \|\partial_x\omega_k\|_{L^2}^{1/2})(\|\partial_y\omega_k\|_{L^2}^{1/2} + \|\partial_x\,\partial_y\omega_k\|_{L^2}^{1/2}) \\ &\lesssim \|\partial_x\omega_k\|_{L^2} + \|\partial_x\omega_k\|_{L^2}^{1/2}\|\partial_x\,\partial_y\omega_k\|_{L^2}^{1/2} \\ &\lesssim k^\gamma(\|\omega_{k+1}\|_{L^2} + \|\partial_y\omega_{k+1}\|_{L^2}),\end{aligned} \tag{4-16}$$

from which we conclude that we estimate

$$\begin{aligned}\left|\int_\Omega u_k\omega_{j-k+1}^{\text{bl}}\frac{\omega_j^{\text{in}}}{\partial_y\omega}\,dx\,dy\right| &\lesssim \frac{1}{\delta_0}\left\|\frac{u_k}{y(1-y)}\right\|_{L^\infty}\|y(1-y)\omega_{j-k+1}^{\text{bl}}\|_{L^2}\|\omega_j^{\text{in}}\|_{L^2} \\ &\lesssim \frac{k^\gamma}{\delta_0}(\|\omega_{k+1}\|_{L^2} + \|\partial_y\omega_{k+1}\|_{L^2})\|y\omega_{j-k+1}^\flat\|_{L^2}\|\omega_j^{\text{in}}\|_{L^2} \\ &\lesssim \frac{k^{\gamma+r/2}}{\delta_0}\frac{\|\omega_{k+1}\|_{L^2} + \|\partial_y\omega_{k+1}\|_{L^2}}{k^{r/2}}\|y\omega_{j-k+1}^\flat\|_{L^2}\|\omega_j^{\text{in}}\|_{L^2}.\end{aligned}$$

Similarly,

$$\left|\int_\Omega u_k\omega_{j-k+1}^{\text{in}}\frac{\omega_j^{\text{in}}}{\partial_y\omega}\,dx\,dy\right| \lesssim \frac{k^{\gamma+r/2}}{\delta_0}\frac{\|\omega_{k+1}\|_{L^2} + \|\partial_y\omega_{k+1}\|_{L^2}}{k^{r/2}}\|\omega_{j-k+1}^{\text{in}}\|_{L^2}\|\omega_j^{\text{in}}\|_{L^2}$$

so that from the discrete Young and Hölder inequalities, we obtain

$$\begin{aligned}&\sum_{j\ge 0} T_{8j,\text{low}} \\ &\quad\lesssim \frac{1}{\delta_0}\left(\sum_{j\ne 0}\frac{j^{\gamma+r/2}}{(j+1)^{r-\gamma+1}}\frac{\|\omega_{j+1}\|_{L^2} + \|\partial_y\omega_{j+1}\|_{L^2}}{j^{r/2}}\right)(\|y\omega^\flat\|_{\gamma,r+1/2,\tau} + \|\omega^{\text{in}}\|_{\gamma,r+1/2,\tau})\|\omega^{\text{in}}\|_{\gamma,r+1/2,\tau} \\ &\quad\lesssim \frac{1}{\delta_0}(\|\omega\|_{\gamma,r/2} + \|\partial_y\omega\|_{\gamma,r/2})(\|y\omega^\flat\|_{\gamma,r+1/2,\tau} + \|\omega^{\text{in}}\|_{\gamma,r+1/2,\tau})\|\omega^{\text{in}}\|_{\gamma,r+1/2,\tau} \\ &\quad\lesssim \frac{M}{\delta_0}(\|y\,\omega^\flat\|_{\gamma,r+1/2,\tau} + \|\omega^{\text{in}}\|_{\gamma,r+1/2,\tau})\|\omega^{\text{in}}\|_{\gamma,r+1/2,\tau}.\end{aligned} \tag{4-17}$$

For the second inequality, we have assumed that $\frac{1}{2}r - 2\gamma + 1 > \frac{1}{2}$ (so that $j^{\gamma+r/2}/(j+1)^{r-\gamma+1}$ is square summable), and for the third inequality we have appealed to (4-2).

In order to bound $T_{8j,\mathrm{high}}$, we use that $u_k|_{y=0,1} = 0$, and the one-dimensional Poincaré inequality to obtain

$$\begin{aligned}\left|\int_\Omega u_k \omega_{j-k+1} \frac{\omega_j^{\mathrm{in}}}{\partial_y \omega}\,dx\,dy\right| &\lesssim \frac{1}{\delta_0} \|u_k\|_{L^2_x L^\infty_y} \|\omega_{j-k+1}\|_{L^\infty_x L^2_y} \|\omega_j^{\mathrm{in}}\|_{L^2} \\ &\lesssim \frac{(j-k+1)^\gamma}{\delta_0} \|\omega_k\|_{L^2} \|\omega_{j-k+2}\|_{L^2} \|\omega_j^{\mathrm{in}}\|_{L^2} \\ &\lesssim \frac{(j-k+1)^\gamma}{\delta_0} \frac{\|\omega_k^{\mathrm{in}}\|_{L^2} + \|\omega_k^{\mathrm{bl}}\|_{L^2}}{(k+1)^{1/2}} \|\omega_{j-k+2}\|_{L^2} (j+1)^{1/2} \|\omega_j^{\mathrm{in}}\|_{L^2}.\end{aligned}$$

We again rely on discrete Young and Hölder inequalities, assume that $r > \frac{8}{3}\gamma + \frac{2}{3}$ (so that $(j+1)^{2\gamma - 3r/4}$ is square summable), and use (4-2) to arrive at

$$\begin{aligned}\sum_{j\ge 0} T_{8j,\mathrm{high}} &\lesssim \frac{1}{\delta_0}\left(\sum_j (j+1)^{2\gamma-3r/4} \frac{\|\omega_j\|_{L^2}}{(j+1)^{r/4}}\right) \|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau} (\|\omega^{\mathrm{in}}\|_{\gamma,r,\tau} + \|\omega^\flat\|_{\gamma,r-1/2,\tau}) \\ &\lesssim \frac{M}{\delta_0} \|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau} (\|\omega^{\mathrm{in}}\|_{\gamma,r-1/2,\tau} + \|\omega^\flat\|_{\gamma,r-1/2,\tau}).\end{aligned} \tag{4-18}$$

Combining (4-17), (4-18), integrating in time, using (3-7a), (3-7b), and Lemma 3.4 (which is applicable by assumption (4-2)), we arrive at

$$\begin{aligned}\int_0^t \sum_{j\ge 0} T_{8j} &\lesssim \frac{M}{\delta_0}\left(\left(\int_0^t \|y\,\omega^\flat\|^2_{\gamma,r+1/2,\tau}\right)^{1/2} + \left(\int_0^t \|\omega^\flat\|^2_{\gamma,r-1/2,\tau}\right)^{1/2}\right)\left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\right)^{1/2} \\ &\qquad + \frac{M}{\delta_0}\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau} \\ &\lesssim \frac{M}{\delta_0 \beta^{3/4}}\left(\left(\int_0^t |h|^2_{\gamma,r+\gamma-3/4,\tau}\right)^{1/2} + \left(\int_0^t |h|^2_{\gamma,r+\gamma-5/4,\tau}\right)^{1/2}\right)\left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\right)^{1/2} \\ &\qquad + \frac{M}{\delta_0}\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau} \\ &\lesssim \frac{M^2}{\delta_0 \beta^{3/4}}\left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+\gamma-3/4,\tau}\right)^{1/2}\left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\right)^{1/2} + \frac{M}{\delta_0}\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau} \\ &\lesssim \frac{M^2}{\delta_0}\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\end{aligned} \tag{4-19}$$

since $\gamma \le \frac{5}{4}$.

The T_{9j} bound: In order to estimate T_{9j} we note that for $1 \le k \le j-1$ we have

$$\frac{M_j}{M_k M_{j-k}} \binom{j}{k} \lesssim \binom{j}{k}^{1-\gamma} \frac{(j+1)^r}{(k+1)^r (j-k+1)^r} \lesssim \left(\frac{j}{\min\{k, j-k\}}\right)^{1-\gamma} \frac{1}{(\min\{k, j-k\})^r}$$

and similarly to T_{8j} we take the decomposition

$$\begin{aligned}T_{9j} &\lesssim \sum_{k=1}^{[j/2]} \frac{1}{k^r}\left|\int_\Omega v_k\, \partial_y \omega_{j-k} \frac{\omega_j^{\mathrm{in}}}{\partial_y \omega}\right| + \sum_{k=[j/2]+1}^{j-1} \frac{1}{(j-k)^{r-\gamma+1} j^{\gamma-1}}\left|\int_\Omega v_k\, \partial_y \omega_{j-k} \frac{\omega_j^{\mathrm{in}}}{\partial_y \omega}\right| \\ &=: T_{9j,\mathrm{low}} + T_{9j,\mathrm{high}}.\end{aligned} \tag{4-20}$$

First we treat the case $k \le j/2$. Using the Poincaré inequality in y (which is allowed since $u_{k+1}|_{y=0,1} = 0$) we obtain

$$\begin{aligned}\left|\int_\Omega v_k\, \partial_y \omega_{j-k} \frac{\omega_j^{\text{in}}}{\partial_y \omega}\, dx\, dy\right| &\lesssim \frac{1}{\delta_0}\left\|\frac{v_k}{y(1-y)}\right\|_{L^\infty} \|y(1-y)\, \partial_y \omega_{j-k}\|_{L^2} \|\omega_j^{\text{in}}\|_{L^2}\\ &\lesssim \frac{1}{\delta_0}\|\partial_x u_k\|_{L^\infty}(\|\partial_y \omega_{j-k}^{\text{in}}\|_{L^2} + \|y\, \partial_y \omega_{j-k}^\flat\|_{L^2})\|\omega_j^{\text{in}}\|_{L^2}\\ &\lesssim \frac{k^\gamma}{\delta_0}\|\omega_{k+1}\|_{L_x^\infty L_y^2}(\|\partial_y \omega_{j-k}^{\text{in}}\|_{L^2} + \|y\, \partial_y \omega_{j-k}^\flat\|_{L^2})\|\omega_j^{\text{in}}\|_{L^2}.\end{aligned}$$

Furthermore, using the one-dimensional Gagliardo–Nirenberg and Poincaré inequalities in x, for $1 \le k \le [j/2]$ we arrive at

$$\left|\int_\Omega v_k\, \partial_y \omega_{j-k} \frac{\omega_j^{\text{in}}}{\partial_y \omega}\, dx\, dy\right| \lesssim \frac{k^{2\gamma + r/4}}{\delta_0} \frac{\|\omega_{k+2}\|_{L^2}}{k^{r/4}}(\|\partial_y \omega_{j-k}^{\text{in}}\|_{L^2} + \|y\, \partial_y \omega_{j-k}^\flat\|_{L^2})\|\omega_j^{\text{in}}\|_{L^2}.$$

Summing over j, assuming that $r > \frac{8}{3}\gamma + \frac{2}{3}$, and appealing to (4-2) we obtain

$$\begin{aligned}\sum_{j\ge 0} |T_{9j,\text{low}}| &\lesssim \frac{\|\omega\|_{\gamma,3r/4,\tau}}{\delta_0}(\|\partial_y \omega^{\text{in}}\|_{\gamma,r,\tau} + \|y\, \partial_y \omega^\flat\|_{\gamma,r,\tau})\|\omega^{\text{in}}\|_{\gamma,r,\tau}\\ &\lesssim \frac{M}{\delta_0}(\|\partial_y \omega^{\text{in}}\|_{\gamma,r,\tau} + \|y\, \partial_y \omega^\flat\|_{\gamma,r,\tau})\|\omega^{\text{in}}\|_{\gamma,r,\tau}. \qquad (4\text{-}21)\end{aligned}$$

For the case $k \ge j/2$, we first note that the compatibility condition (1-2) allows us to write

$$\int_{\mathbb{T}}\int_0^1 u_{k+1}^2\, dy\, dx = \int_{\mathbb{T}}\int_0^1 u_{k+1} u_{k+1}^{\text{bl}}\, dy\, dx + \int_{\mathbb{T}}\int_0^1 u_{k+1}\left(u_{k+1}^{\text{in}} - \int_0^1 u_{k+1}^{\text{in}}\, dz\right) dy\, dx.$$

By Cauchy–Schwarz and the Poincaré inequality in y (for zero-mean functions) we conclude

$$\|u_{k+1}\|_{L^2}^2 \lesssim \|u_{k+1}^{\text{bl}}\|_{L^2}^2 + \|\omega_{k+1}^{\text{in}}\|_{L^2}^2.$$

Then we similarly estimate

$$\begin{aligned}&\left|\int_\Omega v_k\, \partial_y \omega_{j-k} \frac{\omega_j^{\text{in}}}{\partial_y \omega}\, dx\, dy\right|\\ &\quad\lesssim \frac{1}{\delta_0}\|v_k\|_{L_x^2 L_y^\infty}\|\partial_y \omega_{j-k}\|_{L_x^\infty L_y^2}\|\omega_j^{\text{in}}\|_{L^2}\\ &\quad\lesssim \frac{1}{\delta_0}\|\partial_x u_k\|_{L^2}\|\partial_x \partial_y \omega_{j-k}\|_{L^2}\|\omega_j^{\text{in}}\|_{L^2}\\ &\quad\lesssim \frac{(j-k)^\gamma j^{\gamma-1}}{\delta_0} k^{1/2}\|u_{k+1}\|_{L^2}\|\partial_y \omega_{j-k+1}\|_{L^2}(j^{1/2}\|\omega_j^{\text{in}}\|_{L^2})\\ &\quad\lesssim \frac{(j-k)^{\gamma+r/2} j^{\gamma-1}}{\delta_0}(k^{1/2}\|\omega_{k+1}^{\text{in}}\|_{L^2} + k^{1/2}\|u_{k+1}^\flat\|_{L^2})\frac{\|\partial_y \omega_{j-k+1}\|_{L^2}}{(j-k)^{r/2}}(j^{1/2}\|\omega_j^{\text{in}}\|_{L^2}).\end{aligned}$$

Summing over j, noting that the powers of j precisely cancel, we find for $r > r(\gamma)$ large enough

$$\begin{aligned}\sum_{j\geq 0}|T_{9j,\mathrm{high}}| &\lesssim \frac{\|\partial_y\omega\|_{\gamma,r/2}}{\delta_0}(\|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau}+\|u^\flat\|_{\gamma,r+1/2,\tau})\|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau}\\ &\lesssim \frac{M}{\delta_0}(\|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau}+\|u^\flat\|_{\gamma,r+1/2,\tau})\|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau}. \qquad (4\text{-}22)\end{aligned}$$

Integrating in time the sum of (4-21) and (4-22), appealing to (3-7a) and (3-7d), and using Lemma 3.4 (which is applicable for $r > r(\gamma)$ large enough, by assumption (4-2)), we obtain

$$\begin{aligned}\int_0^t\sum_{j\geq 0}|T_{9j}| - \frac{1}{2}\int_0^t\|\partial_y\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau} &\lesssim \int_0^t(\|y\,\partial_y\omega^\flat\|^2_{\gamma,r,\tau}+\|u^\flat\|^2_{\gamma,r+1/2,\tau})+\frac{M^2}{\delta_0^2}\int_0^t\|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\\ &\lesssim \frac{1}{\beta^{3/2}}\int_0^t|h|^2_{\gamma,r+\gamma-3/4,\tau}+\frac{M^2}{\delta_0^2}\int_0^t\|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\\ &\lesssim \left(\frac{M^2}{\beta^{3/2}}+\frac{M^2}{\delta_0^2}\right)\int_0^t\|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau} \qquad (4\text{-}23)\end{aligned}$$

since $\gamma - \frac{3}{4} \leq \frac{1}{2}$.

Conclusion of the proof: Inserting the bounds (4-10)–(4-15), (4-19), and (4-23) into estimate (4-9), we obtain

$$\begin{aligned}&\|\omega^{\mathrm{in}}(t)\|^2_{\gamma,r,\tau(t)}+2\beta\int_0^t\|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\,ds+\int_0^t\|\partial_y\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau}\,ds-\frac{1}{\delta_0^2}\|\omega_0^{\mathrm{in}}\|^2_{\gamma,r,\tau_0}\\ &\quad\lesssim\left(\frac{1}{\delta_0^3}+\frac{M^4}{\delta_0^8}+\frac{M}{\delta_0\beta^{3/2}}\right)\int_0^t\|\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau}\,ds\\ &\qquad+\left(\frac{M^2}{\delta_0\beta^{5/2}}+\frac{M^2}{\delta_0^2\beta^{5/4}}+\frac{M^2}{\delta_0\beta^{3/2}}+\frac{M}{\delta_0^2\beta^{7/4}}+\frac{M^2}{\delta_0^2\beta^{3/4}}+\frac{M^2}{\delta_0^3}\right)\int_0^t\|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\,ds. \qquad (4\text{-}24)\end{aligned}$$

Note that $\|\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau} \leq \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}$, so that we may combine the last two terms on the right side of (4-24). Choosing β_0 large enough, depending on $M \geq 1$, $\delta_0 \leq 1$, and the implicit constant in (4-24), for any $\beta \geq \beta_0$ we obtain

$$\|\omega^{\mathrm{in}}(t)\|^2_{\gamma,r,\tau(t)}+\beta\int_0^t\|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\,ds+\int_0^t\|\partial_y\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau}\,ds\leq\frac{1}{\delta_0^2}\|\omega_0^{\mathrm{in}}\|^2_{\gamma,r,\tau_0}.$$

The estimate (4-5) now follows directly from the above estimate.

Finally, in order to prove (4-6), we appeal to (3-15a), Lemma 3.4, and estimate (4-5), to obtain

$$\begin{aligned}\sup_{[0,t]}\|\omega^\flat\|^2_{\gamma,r-\gamma+3/4,\tau(s)} &\lesssim \frac{1}{\beta^{1/2}}\int_0^t|h(s)|^2_{\gamma,r+1/2,\tau(s)}\,ds\\ &\lesssim \frac{M^2}{\beta^{1/2}}\int_0^t\|\omega^{\mathrm{in}}(s)\|^2_{\gamma,r+1/2,\tau(s)}\,ds\leq\frac{1}{2\delta_0^2}\|\omega^{\mathrm{in}}(0)\|^2_{\gamma,r,\tau_0} \qquad (4\text{-}25)\end{aligned}$$

upon ensuring that β is sufficiently large, depending on M, δ_0. Moreover, from (3-7c) and (3-7a) we similarly obtain

$$\int_0^t \|\partial_y \omega^\flat(s)\|^2_{\gamma,r-\gamma+3/4,\tau(s)}\,ds + \beta \int_0^t \|\omega^\flat(s)\|^2_{\gamma,r-\gamma+5/4,\tau(s)}\,ds \lesssim \frac{1}{\beta^{1/2}} \int_0^t |h(s)|^2_{\gamma,r+1/2,\tau(s)}\,ds \le \frac{1}{2\delta_0^2}\|\omega^{\text{in}}(0)\|^2_{\gamma,r,\tau_0} \tag{4-26}$$

as above. Summing (4-25)–(4-26) with (4-5) (and using $(a+b)^2 \le 2a^2 + 2b^2$) we obtain

$$\sup_{s\in[0,t]} \|\omega(s)\|^2_{\gamma,r-\gamma+3/4,\tau(s)} + \int_0^t \|\partial_y \omega(s)\|^2_{\gamma,r-\gamma+3/4,\tau(s)}\,ds + \beta \int_0^t \|\omega(s)\|^2_{\gamma,r-\gamma+5/4,\tau(s)}\,ds \le \frac{4}{\delta_0^2}\|\omega^{\text{in}}(0)\|^2_{\gamma,r,\tau_0}$$

by using $\gamma \le \frac{5}{4}$. This concludes the proof of (4-6). □

As an easy consequence of the estimate (4-6), we state:

Corollary 4.2. *Let M, δ_0 and $\gamma \in \left[1, \frac{9}{8}\right]$ be given. For $r \ge r_0(\gamma)$, $\beta \ge \beta_0$ and T such that $\tau(t) \ge \tau_1$ for all $t \in [0, T]$, if*

$$\frac{4}{\delta_0^2}\|\omega_0\|_{\gamma,r,\tau_0} \le \frac{M}{2} \tag{4-27}$$

then

$$\sup_{t\in[0,T]} \|\omega(t)\|_{\gamma,3r/4,\tau(t)} \le \frac{M}{2}.$$

5. Estimates for $\partial_t \omega$

In order to emphasize the linear nature of the estimates in this section we write $\partial_t \omega = \dot{\omega}$. The equation obeyed by $\dot{\omega}$ is

$$\partial_t \dot{\omega} - \partial_y^2 \dot{\omega} + (u\,\partial_x + v\,\partial_y)\dot{\omega} + (\dot{u}\,\partial_x + \dot{v}\,\partial_y)\omega = 0, \tag{5-1a}$$

$$\partial_y \dot{\omega}|_{y=0,1} = (\tilde{\dot{\omega}}|_{y=1} - \tilde{\dot{\omega}}|_{y=0}) - \partial_x\left(2\int_0^1 u\dot{u}\,dy\right). \tag{5-1b}$$

Proposition 5.1. *Let M, δ_0 and $\gamma \in \left[1, \frac{9}{8}\right]$ be given. There exists $r_1 = r_1(\gamma) \ge r_0$ such that: for all r, r' satisfying $r' \ge r_1$, $\frac{3}{4}r - r' \ge r_1$, one can find $\beta_1 = \beta_1(M, \delta_0, \tau_0, \tau_1, r, r', \gamma) \ge \beta_0$ satisfying: if $\beta \ge \beta_0$, if $T \le 1$ small enough so that $\tau(t) \ge \tau_1$ for all $t \in [0, T]$, and if (4-2)–(4-4) hold, we have*

$$\sup_{s\in[0,t]} \|\dot{\omega}(s)\|^2_{\gamma,r'-\gamma+3/4,\tau(s)} + \int_0^t \|\partial_y \dot{\omega}(s)\|^2_{\gamma,r'-\gamma+3/4,\tau(s)}\,ds + \beta\int_0^t \|\dot{\omega}(s)\|^2_{\gamma,r'-\gamma+5/4,\tau(s)}\,ds \le \frac{4}{\delta_0^2}\|\dot{\omega}(0)\|^2_{\gamma,r',\tau_0}. \tag{5-2}$$

Proof of Proposition 5.1. The proof is very similar to that of Proposition 4.1, since one may view (5-1) as a linearization of (2-6) about ω itself (respectively u for the boundary condition). In order to avoid redundancy, we only emphasize the essential differences.

Estimate (5-2) follows directly from estimates for $\dot{\omega}^{\mathrm{in}}$ which are analogous to (4-5). In order to define $\dot{\omega}^{\mathrm{in}}$, we define $\dot{\omega}^{\flat}$ as the solution of system (3-4) with boundary datum given by

$$\partial_x \dot{h} = -2\partial_x \int_0^1 u\dot{u}\,dy,$$

which is consistent with (5-1b). The function $\dot{\omega}^{\flat}$ obeys all the estimates claimed in Lemma 3.1, except that on the right side we need to replace h with $\dot{h}$. As in (3-16) we define the boundary layer functions corresponding to $\dot{\omega}$, and according to (3-17) we define the interior functions corresponding to $\dot{\omega}$. Note that as before we impose $\dot{\omega}^{\mathrm{bl}}(0) = 0$, and thus $\dot{\omega}^{\mathrm{in}}(0) = \dot{\omega}_0$, where by (2-6a)

$$\dot{\omega}_0 = -u_0\,\partial_x\omega_0 - v_0\,\partial_y\omega_0 - \partial_y^2\omega_0.$$

At this stage, we can prove an analogous statement to the one provided by Lemma 3.4, with h being replaced by

$$\dot{h} = 2\int_0^1 u\dot{u}\,dy - 2\int_{\mathbb{T}}\int_0^1 u\dot{u}\,dy\,dx.$$

Namely, we can show that for any r as in Proposition 4.1 and any r' such that

$$\tfrac{3}{4}r - \tfrac{1}{2}\gamma - 1 \geq r' > 2\gamma + 2,$$

we have

$$\int_0^t |\dot{h}(s)|^2_{\gamma,r',\tau(s)}\,ds \lesssim M^2 \int_0^t \|\dot{\omega}^{\mathrm{in}}(s)\|^2_{\gamma,r',\tau(s)}\,ds. \tag{5-3}$$

Indeed, defining for all f

$$f'_j = (j+1)^{r'-r} f_j = M'_j\,\partial_x^j f, \quad \text{where } M'_j = \frac{(j+1)^{r'}\tau^{j+1}}{(j!)^{\gamma}},$$

similarly to (3-19) we obtain $\|\dot{h}_0\|_{L^2_x} \lesssim \|\dot{h}_1\|_{L^2_x}$, while for $j \geq 1$, as a substitute to (3-21) we obtain the inequality

$$\|\dot{h}'_j\|_{L^2_x} \lesssim \sum_{\ell=1}^{j}\binom{j}{\ell}\frac{M'_j}{M'_{j-\ell}M'^{1/2}_{\ell}M'^{1/2}_{\ell+1}}\|\omega'_\ell\|^{1/2}_{L^2_{x,y}}\|\omega'_{\ell+1}\|^{1/2}_{L^2_{x,y}}(\|\dot{\omega}^{\mathrm{in}'}_{j-\ell}\|_{L^2_{x,y}} + \|y(1-y)\dot{u}^{\mathrm{bl}'}_{j-\ell}\|_{L^2_{x,y}}) \\ + M\big(\|\dot{\omega}^{\mathrm{in}}_j\|_{L^2_{x,y}} + \|y\dot{u}^{\flat}_j\|_{L^2_{x,y}} + \|y\dot{\omega}^{\flat}_j\|_{L^2_{x,y}} + \big\|\dot{u}^{\flat}_j\big(x,\tfrac{1}{2}\big)\big\|_{L^2_x}\big).$$

The half sum $\sum_{\ell=1}^{\lceil j/2\rceil}$ and the last term on the right-hand side can be treated as before, resulting in

$$\int_0^t \bigg(M\big(\|\dot{\omega}^{\mathrm{in}}_j\|_{L^2_{x,y}} + \dots + \big\|\dot{u}^{\flat}_j\big(x,\tfrac{1}{2}\big)\big\|_{L^2_x}\big) + \sum_{\ell=1}^{j/2}\dots\bigg)^2 \\ \lesssim M^2\bigg(\int_0^t \|\dot{\omega}^{\mathrm{in}}(s)\|^2_{\gamma,r',\tau(s)}\,ds + \frac{1}{\beta^{5/2}}\int_0^t |\dot{h}(s)|^2_{\gamma,r'+\gamma-5/4,\tau(s)}\,ds\bigg)$$

if

$$\sup_{t\in[0,T]} \|\omega(t)\|_{\gamma,r'/4,\tau(t)} \leq M,$$

which is satisfied by assumption (4-2) as soon as $r' \leq 3r$.

For the half-sum $\sum_{\ell=\lceil j/2\rceil+1}^{j}$, we cannot proceed symmetrically as in the proof of Lemma 3.4: as we want an L^2-in-time control by $\dot{\omega}$, the bound

$$\binom{j}{\ell}\frac{M'_j}{M'_{j-\ell}M'^{1/2}_{\ell}M'^{1/2}_{\ell+1}} \lesssim (l+1)^{\gamma/2}$$

yields by a discrete convolution inequality

$$\int_0^t\biggl(\sum_{\ell=\lceil j/2\rceil+1}^{j}\cdots\biggr)^2 \lesssim \biggl(\sup_{[0,t]}\sum_{\ell\ge 1}(\ell+1)^{\gamma/2}\|\omega'_\ell\|_{L^2}\biggr)^2\int_0^t (\|\dot{\omega}^{\text{in}}(s)\|^2_{\gamma,r',\tau(s)}+\|y\dot{u}^{\flat}(s)\|^2_{\gamma,r',\tau(s)})\,ds.$$

Writing

$$\sum_{\ell}(\ell+1)^{\gamma/2}\|\omega'_\ell\|_{L^2} = \sum_{\ell}\frac{1}{\ell+1}((\ell+1)^{\gamma/2+1}\|\omega'_\ell\|_{L^2})$$

and using Cauchy–Schwarz, we find

$$\begin{aligned}\int_0^t\biggl(\sum_{\ell=\lceil j/2\rceil+1}^{j}\cdots\biggr)^2 &\lesssim \sup_{[0,t]}\|\omega(s)\|^2_{\gamma,r'+\gamma/2+1,\tau(s)}\biggl(\int_0^t\|\dot{\omega}^{\text{in}}(s)\|^2_{\gamma,r',\tau(s)}\,ds+\frac{1}{\beta^{7/2}}\int_0^t|\dot{h}(s)|^2_{\gamma,r'+\gamma-7/4,\tau(s)}\,ds\biggr)\\ &\lesssim M^2\biggl(\int_0^t\|\dot{\omega}^{\text{in}}(s)\|^2_{\gamma,r',\tau(s)}\,ds+\frac{1}{\beta^{7/2}}\int_0^t|\dot{h}(s)|^2_{\gamma,r'+\gamma-7/4,\tau(s)}\,ds\biggr),\end{aligned}$$

where the last inequality comes from (4-2), under the assumption that $r'+\frac{1}{2}\gamma+1\le\frac{3}{4}r$. Gathering the two previous inequalities yields (5-3) for β sufficiently large.

Now, similarly to (4-7), we have

$$\begin{aligned}&(\partial_t+\beta(j+1)-\partial_y^2)\dot{\omega}_j^{\text{in}'}+(u\,\partial_x+v\,\partial_y)\dot{\omega}_j^{\text{in}'}+\dot{v}_j^{\text{in}'}\,\partial_y\omega\\ &\qquad=-(u\,\partial_x+v\,\partial_y)\dot{\omega}_j^{\text{bl}'}-\dot{v}_j^{\text{bl}'}\,\partial_y\omega-M'_j[\partial_x^j,u\,\partial_x+v\,\partial_y]\dot{\omega}-M'_j\,\partial_x^j(\dot{u}\,\partial_x\omega)-M'_j[\partial_x^j,\partial_y\omega]\dot{v},\end{aligned}\tag{5-4a}$$

$$\partial_y\dot{\omega}_j^{\text{in}}|_{y=0,1}=\tilde{\dot{\omega}}_j^{\text{in}'}|_{y=1}-\tilde{\dot{\omega}}_j^{\text{in}'}|_{y=0}+2\dot{\omega}_j^{\flat'}|_{y=1}-\partial_y\dot{\omega}_j^{\flat'}|_{y=1}.\tag{5-4b}$$

Note that (5-4b) is the same as (4-7b), the left side of (5-4a) is the same as the left side of (4-7a), and the first two terms on the right side of (5-4a) are the same as the first two terms on the right side of (4-7a). The difference comes from the last three terms on the right-side of (4-7a), namely the quadratic terms. The main point is that they now lack symmetry: they involve not only $(\dot{\omega}^{\text{in}'},\dot{\omega}^{\text{bl}'})$ but also ω. In particular, all terms containing ω must be controlled uniformly in time, to allow for the L^2_t control of $\dot{\omega}^{\text{in}'}$ on the left-hand side. This is why we take r' less than $\frac{3}{4}r$: with such a margin we can still use (4-2) to control uniformly in time the terms where most derivatives fall on ω.

More precisely, proceeding as in the proof of (5-3) to handle the linear terms (see the estimates of $T_{1j},\dots,T_{7j}$), we can show that for β large enough

$$\begin{aligned}&\|\dot{\omega}^{\text{in}}(t)\|^2_{\gamma,r',\tau(t)}+2\beta\int_0^t\|\dot{\omega}^{\text{in}}\|^2_{\gamma,r'+1/2,\tau}\,ds+\frac{3}{2}\int_0^t\|\partial_y\dot{\omega}^{\text{in}}\|^2_{\gamma,r',\tau}\,ds-\frac{1}{\delta_0^2}\|\dot{\omega}_0\|^2_{\gamma,r',\tau_0}\\ &\lesssim\frac{M^4}{\delta_0^7}\int_0^t\|\dot{\omega}^{\text{in}}\|^2_{\gamma,r',\tau}\,ds+\frac{M^2}{\delta_0\beta^{3/4}}\int_0^t\|\dot{\omega}^{\text{in}}\|^2_{\gamma,r'+1/2,\tau}\,ds+\sum_{j\ge0}\int_0^t(S_{1j}+S_{2j}+S_{3j}+S_{4j})(s)\,ds,\end{aligned}\tag{5-5}$$

where

$$S_{1j} = -\int_\Omega M'_j[\partial_x^j, u\,\partial_x]\dot\omega \frac{\dot\omega_j^{\mathrm{in}'}}{\partial_y\omega}, \quad S_{2j} = -\int_\Omega M'_j[\partial_x^j, v\,\partial_y]\dot\omega \frac{\dot\omega_j^{\mathrm{in}'}}{\partial_y\omega},$$

$$S_{3j} = -\int_\Omega M'_j\,\partial_x^j(\dot u\,\partial_x\omega) \frac{\dot\omega_j^{\mathrm{in}'}}{\partial_y\omega}, \quad S_{4j} = -\int_\Omega M'_j[\partial_x^j, \partial_y\omega]\dot v \frac{\dot\omega_j^{\mathrm{in}'}}{\partial_y\omega}.$$

The first term is analogous to T_{8j}. One can write

$$S_{1j} = -\Biggl(\sum_{k=1}^{\lceil j/2\rceil} + \sum_{k=\lceil j/2\rceil+1}^{j}\Biggr)\binom{j}{k}\frac{M'_j}{M'_k M'_{j-k+1}}\int_\Omega u'_k\dot\omega'_{j-k+1}\frac{\dot\omega_j^{\mathrm{in}'}}{\partial_y\omega} = S_{1j,\mathrm{low}} + S_{1j,\mathrm{high}}.$$

The treatment of $S_{1j,\mathrm{low}}$ is exactly the same as the one of $T_{8j,\mathrm{low}}$. Similarly to (4-17), (4-19), we get

$$\sum\int_0^t S_{1j,\mathrm{low}}(s)\,ds \lesssim \frac{M^2}{\delta_0}\int_0^t \|\dot\omega^{\mathrm{in}}(s)\|^2_{\gamma,r'+1/2,\tau(s)}\,ds.$$

To treat $S_{1j,\mathrm{high}}$, we use the inequality

$$\binom{j}{k}\frac{M'_j}{M'_k M'_{j-k+1}} \lesssim (j-k+1)^{\gamma-r'}$$

for $k \geq \lceil j/2\rceil + 1$, so that

$$\begin{aligned} S_{1j,\mathrm{high}} &\lesssim \sum_{k=\lceil j/2\rceil+1}^{j} \frac{1}{\delta_0}\|u'_k\|_{L^\infty}(j-k+1)^{\gamma-r'}\|\dot\omega'_{j-k+1}\|_{L^2}\|\dot\omega_j^{\mathrm{in}'}\|_{L^2} \\ &\lesssim \sum_{k=\lceil j/2\rceil+1}^{j} \frac{k^\gamma}{\delta_0}\|\omega'_{k+1}\|_{L^2}(j-k+1)^{\gamma-r'}\|\dot\omega'_{j-k+1}\|_{L^2}\|\dot\omega_j^{\mathrm{in}'}\|_{L^2}, \end{aligned}$$

so that by the discrete Young's inequality

$$\begin{aligned} \sum\int_0^t S_{1j,\mathrm{high}}(s)\,ds &\lesssim \frac{1}{\delta_0}\sup_{s\in[0,t]}\sum_k k^\gamma\|\omega'_k(s)\|_{L^2}\int_0^t \|\dot\omega(s)\|_{\gamma,\gamma,\tau(s)}\|\dot\omega^{\mathrm{in}}\|_{\gamma,r',\tau(s)} \\ &\lesssim \frac{1}{\delta_0}\sup_{s\in[0,t]}\|\omega(s)\|_{\gamma,r'+\gamma+1,\tau(s)}\int_0^t \|\dot\omega(s)\|_{\gamma,\gamma,\tau(s)}\|\dot\omega^{\mathrm{in}}\|_{\gamma,r',\tau(s)}. \end{aligned}$$

The sup in time is controlled as usual by assumption (4-2), under the constraint $r'+\gamma+1 \leq \frac{3}{4}r$. As regards the second factor, one can split $\|\dot\omega(s)\|_{\gamma,\gamma,\tau(s)}$ into

$$\|\dot\omega(s)\|_{\gamma,\gamma,\tau(s)} \leq \|\dot\omega^{\mathrm{in}}(s)\|_{\gamma,\gamma,\tau(s)} + \|\dot\omega^{\mathrm{bl}}(s)\|_{\gamma,\gamma,\tau(s)}$$

and control the second term by the analogue of Lemma 3.1, followed by (5-3). For $r' \geq \gamma + \bigl(\gamma+\frac{3}{4}\bigr)$ we find that

$$\sum\int_0^t S_{1j,\mathrm{high}}(s)\,ds \lesssim \frac{M^2}{\delta_0}\int_0^t \|\dot\omega^{\mathrm{in}}(s)\|^2_{\gamma,r',\tau(s)}\,ds.$$

Estimates on S_{2j} (which is analogous to T_{9j}) and S_{3j} can be established in the same way. We find for r' and $\frac{3}{4}r - r'$ large enough (with thresholds depending on γ)

$$\sum_j \int_0^t S_{2j} \leq \eta \int_0^t \|\partial_y \dot{\omega}^{\text{in}}(s)\|^2_{\gamma,\gamma+r',\tau(s)}\, ds + \frac{C}{\eta}\frac{M^4}{\delta_0^2} \int_0^t \|\dot{\omega}^{\text{in}}(s)\|^2_{\gamma,\gamma+r'+1/2,\tau(s)}\, ds,$$

with $C > 0$, η arbitrarily small, and

$$\sum_j \int_0^t S_{3j} \leq \frac{M^2}{\delta_0} \int_0^t \|\dot{\omega}^{\text{in}}(s)^2\|_{\gamma,\gamma+r',\tau(s)}\, ds.$$

To handle S_{4j}, we proceed slightly differently. We start with the decomposition

$$\begin{aligned} S_{4j} &= -\Bigg(\sum_{k=0}^{\lceil j/2\rceil} + \sum_{k=\lceil j/2\rceil+1}^{j-1}\Bigg)\binom{j}{k}\frac{M'_j}{M'_k M'_{j-k}} \int_\Omega \partial_y \omega'_{j-k}\, \dot{v}'_k \frac{\dot{\omega}_j^{\text{in}'}}{\partial_y \omega} \\ &= S_{4j,\text{low}} + S_{4j,\text{high}}. \end{aligned}$$

$S_{4j,\text{high}}$ can be treated similarly to $T_{9j,\text{high}}$. We obtain, see (4-22),

$$\begin{aligned} \sum_j \int_0^t S_{4j,\text{high}} &\lesssim \frac{1}{\delta_0} \sup_{[0,t]} \|\partial_y \omega\|_{\gamma,r'/2} \int_0^t (\|\dot{\omega}^{\text{in}}(s)\|_{\gamma,r'+1/2,\tau(s)} + \|\dot{u}^\flat\|_{\gamma,r'+1/2,\tau(s)})\|\dot{\omega}^{\text{in}}(s)\|_{\gamma,r'+1/2,\tau(s)}\, ds \\ &\lesssim \frac{M^2}{\delta_0} \int_0^t \|\dot{\omega}^{\text{in}}(s)\|^2_{\gamma,r'+1/2,\tau(s)}\, ds. \end{aligned}$$

Here, we have used the Gevrey control of $\partial_y \omega$ given by (4-2) to bound the first factor, and the analogue of Lemma 3.1 followed by (5-3) to control the boundary layer term in the second factor. As for $S_{4j,\text{low}}$, we integrate by parts in y. As $\dot{v}$ vanishes at the boundary, no boundary term appears, and we get

$$\begin{aligned} S_{4j,\text{low}} &= \sum_{k=0}^{\lceil j/2\rceil}\binom{j}{k}\frac{M'_j}{M'_k M'_{j-k}} \int_\Omega \Bigg(\omega'_{j-k}\, \partial_y \dot{v}'_k \frac{\dot{\omega}_j^{\text{in}'}}{\partial_y \omega} - \omega'_{j-k} \dot{v}'_k \frac{\partial_y^2 \omega}{(\partial_y \omega)^2}\dot{\omega}_j^{\text{in}'} + \omega'_{j-k}\dot{v}'_k \frac{\partial_y \dot{\omega}_j^{\text{in}'}}{\partial_y \omega}\Bigg) \\ &= S_{4j,\text{low},1} + S_{4j,\text{low},2} + S_{4j,\text{low},3}. \end{aligned}$$

We can bound $S_{4j,\text{low},1}$ with the same ideas as before. For r' and $\frac{3}{4}r - r'$ large enough we have

$$\int_0^t \sum_j S_{4j,\text{low},1} \lesssim \frac{M^2}{\delta_0} \int_0^t \|\dot{\omega}^{\text{in}}(s)\|^2_{\gamma,r'+1/2,\tau(s)}\, ds.$$

As for $S_{4j,\text{low},2}$, we start from the bound

$$\begin{aligned} S_{4j,\text{low},2} &\lesssim \frac{1}{\delta_0^2} \sum_{k=0}^{\lceil j/2\rceil} \|\omega'_{j-k}\|_{L^\infty_x L^2_y} (k+1)^{-r'} \|\dot{v}'_k\|_{L^\infty} \|\partial_y^2 \omega\|_{L^\infty_x L^2_y} \|\dot{\omega}_j^{\text{in}'}\|_{L^2_x L^\infty_y} \\ &\lesssim \frac{M}{\delta_0^2} \sum_{k=0}^{\lceil j/2\rceil} \|\omega'_{j-k}\|_{L^\infty_x L^2_y} (k+1)^{-r'} \|\dot{v}'_k\|_{L^\infty} \|\dot{\omega}_j^{\text{in}'}\|_{L^2_x L^\infty_y}, \end{aligned}$$

where the last inequality comes from (4-4) to control $\partial_y^2\omega$. It follows that

$$S_{4j,\mathrm{low},2} \lesssim \frac{M}{\delta_0^2} \sum_{k=0}^{\lceil j/2 \rceil} (j-k+1)^\gamma \|\omega'_{j-k+1}\|_{L^2} (k+1)^{-r'+2\gamma} \|\dot{u}'_{k+2}\|_{L^2} (\|\dot{\omega}_j^{\mathrm{in}'}\|_{L^2} + \|\partial_y \dot{\omega}_j^{\mathrm{in}'}\|_{L^2}).$$

From there, for r' and $\frac{3}{4}r - r'$ large enough (with thresholds depending on γ),

$$\int_0^t \sum_j S_{4j,\mathrm{low},2} \leq \eta \int_0^t \|\partial_y \dot{\omega}^{\mathrm{in}}(s)\|^2_{\gamma,\gamma+r',\tau(s)}\, ds + \frac{C}{\eta} \frac{M^6}{\delta_0^4} \int_0^t \|\dot{\omega}^{\mathrm{in}}(s)\|^2_{\gamma,\gamma+r',\tau(s)}\, ds.$$

With similar manipulations, we get the bound

$$\int_0^t \sum_j S_{4j,\mathrm{low},3} \leq \eta \int_0^t \|\partial_y \dot{\omega}^{\mathrm{in}}(s)\|^2_{\gamma,\gamma+r',\tau(s)}\, ds + \frac{C}{\eta} \frac{M^4}{\delta_0^2} \int_0^t \|\dot{\omega}^{\mathrm{in}}(s)\|^2_{\gamma,\gamma+r',\tau(s)}\, ds.$$

Injecting the previous estimates in (5-5), we get for large enough β

$$\|\dot{\omega}^{\mathrm{in}}(t)\|^2_{\gamma,r',\tau(t)} + \beta \int_0^t \|\dot{\omega}^{\mathrm{in}}\|^2_{\gamma,r'+1/2,\tau}\, ds + \int_0^t \|\partial_y \dot{\omega}^{\mathrm{in}}\|^2_{\gamma,r',\tau}\, ds \leq \frac{1}{\delta_0^2} \|\dot{\omega}_0\|^2_{\gamma,r',\tau_0}.$$

Estimate (5-2) follows from this inequality, in the same way as (4-6) is deduced from (4-5). □

Corollary 5.2. *Let M, δ_0 and $\gamma \in \left[1, \frac{9}{8}\right]$ be given. There exists $r_2 = r_2(\gamma) \geq r_1$ such that for $r \geq r_2(\gamma)$, one can find $\beta_2 = \beta_2(M, \delta_0, \tau_0, \tau_1, \gamma, r) \geq \beta_1$ and*

$$T_0 = T_0(M, \delta_0, \beta, \tau_0, \tau_1, \gamma, r, \|\dot{\omega}_0\|_{\gamma,r/2+\gamma-3/4,\tau_0}) > 0$$

satisfying: *if $\beta \geq \beta_0$, if $T \leq T_0$, if* (4-2)–(4-4) *hold, and if*

$$\|\partial_y \omega_0\|_{\gamma,r/2,\tau_0} \leq \frac{M}{4}, \tag{5-6}$$

then

$$\sup_{t\in[0,T]} \|\partial_y \omega(t)\|_{\gamma,r/2,\tau(t)} \leq \frac{M}{2}. \tag{5-7}$$

Proof of Corollary 5.2. We write $\partial_y\omega(t) = \partial_y\omega_0 + \int_0^t \partial_y\dot{\omega}(s)\, ds$, so that for all $t \in [0, T]$

$$\begin{aligned}
\|\partial_y \omega(t)\|_{\gamma,r/2,\tau(t)} &\leq \|\partial_y \omega_0\|_{\gamma,r/2,\tau(t)} + \int_0^t \|\partial_y \dot{\omega}(s)\|_{\gamma,r/2,\tau(t)}\, ds \\
&\leq \|\partial_y \omega_0\|_{\gamma,r/2,\tau(0)} + \int_0^t \|\partial_y \dot{\omega}(s)\|_{\gamma,r/2,\tau(s)}\, ds \\
&\leq \|\partial_y \omega_0\|_{\gamma,r/2,\tau(0)} + \sqrt{t} \left(\int_0^t \|\partial_y \dot{\omega}(s)\|^2_{\gamma,r/2,\tau(s)}\, ds \right)^{1/2}.
\end{aligned}$$

Taking for instance $r_2 = 4r_1 + 4\gamma + 3$, where r_1 was introduced in Proposition 5.1, and $r \geq r_2$, we ensure that $r' := \frac{1}{2}r + \gamma - \frac{3}{4}$ satisfies $r' \geq r_1$ and $\frac{3}{4}r - r' \geq r_1$. By Proposition 5.1, for $\beta \geq \beta_0$ large enough, and

T such that $\tau(t) \in [\tau_1, \tau_0]$ for all $t \in [0, T]$, we get

$$\sup_{t\in[0,T]} \|\partial_y \omega(t)\|_{\gamma,r/2,\tau(t)} \le \|\partial_y \omega_0\|_{\gamma,r/2,\tau(0)} + 2\sqrt{T}/\delta_0 \|\dot{\omega}(0)\|_{\gamma,r/2+\gamma-3/4,\tau_0}. \tag{5-8}$$

The result follows from the assumption on $\partial_y \omega_0$, once T_0 is taken small enough to ensure that

$$\frac{2\sqrt{T_0}}{\delta_0} \|\dot{\omega(0)}\|_{\gamma,r/2+\gamma-3/4,\tau_0} \le \frac{M}{4}$$

holds. □

Corollary 5.3. *Let M, δ_0 and $\gamma \in \left[1, \frac{9}{8}\right]$ be given. There exists $r_3 = r_3(\gamma) \ge r_2$ such that for $r \ge r_3(\gamma)$, one can find $\beta_3 = \beta_3(M, \delta_0, \tau_0, \tau_1, \gamma, r) \ge \beta_2$, $c_0 = c_0(\tau_0, \tau_1, \gamma, r) > 0$ and*

$$T_0 = T_0(M, \delta_0, \beta, \tau_0, \tau_1, \gamma, r, \|\omega(0)\|_{\gamma,r,\tau_0}, \|\dot{\omega}(0)\|_{\gamma,r/2+\gamma-3/4,\tau_0}) > 0 \tag{5-9}$$

satisfying: *if $\beta \ge \beta_0$, if $T \le T_0$, if* (4-2)–(4-4) *hold, and if*

$$\frac{1}{\delta_0}\|\dot{\omega}_0\|_{\gamma,r/2+\gamma-3/4,\tau_0} + \frac{1}{\delta_0^2}\|\omega_0\|^2_{\gamma,r,\tau_0} + \frac{1}{\delta_0}\|\omega_0\|_{\gamma,r,\tau_0}\|\partial_y\omega_0\|_{\gamma,r/2,\tau_0} \le \frac{c_0 M}{4}, \tag{5-10}$$

then

$$\sup_{t\in[0,T]} \|\partial_y^2 \omega(t)\|_{L^\infty_x L^2_y} \le \frac{M}{2}.$$

Proof of Corollary 5.3. We write the vorticity equation in the form

$$\partial_y^2 \omega = \dot{\omega} + u\,\partial_x \omega + v\,\partial_y \omega.$$

Hence, for all $t \in [0, T]$,

$$\|\partial_y^2\omega(t)\|_{L^\infty_x L^2_y} \le \|\dot{\omega}(t)\|_{L^\infty_x L^2_y} + \|u(t)\|_{L^\infty_{x,y}}\|\partial_x\omega(t)\|_{L^\infty_x L^2_y} + \|v(t)\|_{L^\infty_{x,y}}\|\partial_y\omega(t)\|_{L^\infty_x L^2_y}.$$

For r large enough, we obtain

$$\|\partial_y^2\omega(t)\|_{L^\infty_x L^2_y} \lesssim \|\dot{\omega}(t)\|_{\gamma,r/2,\tau(t)} + \|\omega(t)\|^2_{\gamma,r-\gamma+3/4,\tau(t)} + \|\omega(t)\|_{\gamma,r-\gamma+3/4,\tau(t)}\|\partial_y\omega(t)\|_{\gamma,r/2,\tau(t)}.$$

By Propositions 4.1 and 5.1 applied respectively with r and $r' = \frac{1}{2}r + \gamma - \frac{3}{4}$, and by inequality (5-8), we find

$$\sup_{t\in[0,T]} \|\partial_y^2\omega(t)\|_{L^\infty_x L^2_y} \lesssim \frac{1}{\delta_0}\|\dot{\omega}_0\|_{\gamma,r/2+\gamma-3/4,\tau_0} + \frac{1}{\delta_0^2}\|\omega_0\|^2_{\gamma,r,\tau_0} + \frac{1}{\delta_0}\|\omega_0\|_{\gamma,r,\tau_0}\left(\|\partial_y\omega_0\|_{\gamma,r/2,\tau_0} + \frac{\sqrt{T}}{\delta_0}\|\dot{\omega}_0\|_{\gamma,r/2+\gamma-3/4,\tau_0}\right).$$

Upon taking T sufficiently small, this concludes the proof of the corollary. □

6. Minimum and maximum principle for $\partial_y \omega$

The quantity $\partial_y \omega$ obeys a (degenerate) parabolic equation with Dirichlet boundary conditions

$$\partial_t(\partial_y\omega) - \partial_y^2(\partial_y\omega) + (u\,\partial_x + v\,\partial_y)(\partial_y\omega) + (\partial_x u)(\partial_y\omega) = \omega\,\partial_x\omega, \tag{6-1a}$$

$$\partial_y\omega|_{y=0,1} = (\tilde\omega|_{y=1} - \tilde\omega|_{y=0}) - \partial_x \int_0^1 u^2\,dy. \tag{6-1b}$$

Our goal is to combine this fact with $L_t^2 L_{x,y}^\infty$ estimates on $\omega\,\partial_x\omega$ and the Dirichlet datum, to deduce that the convexity of u is conserved for small time.

Proposition 6.1. *Let $M, \delta_0 > 0$ and $\gamma \in \left[1, \frac{9}{8}\right]$ be given. There exists $r_4 = r_4(\gamma) \geq r_3$ such that for $r \geq r_4(\gamma)$, one can find $\beta_4 = \beta_4(M, \delta_0, \tau_0, \tau_1, \gamma, r) \geq \beta_3$ and T_0 as in* (5-9) *satisfying*: *if $\beta \geq \beta_0$, if $T \leq T_0$, if* (4-2)–(4-4) *hold*, *and if*

$$4\delta_0 \leq \partial_y\omega_0 \leq \frac{1}{4\delta_0}, \tag{6-2}$$

then

$$2\delta_0 \leq \partial_y\omega(t) \leq \frac{1}{2\delta_0} \quad \textit{for all } t \in [0, T]. \tag{6-3}$$

Proof of Proposition 6.1. We wish to apply a version of the parabolic minimum/maximum principle for the following degenerate parabolic problem posed in $\Omega \times (0, T)$, with Ω being the periodic-in-x strip $(x, y) \in \mathbb{T} \times (0, 1)$:

$$(\partial_t - \partial_y^2 + b(x,y,t)\cdot\nabla_{x,y} + c(x,y,t))\psi = d(x,y,t) \quad \text{in } \Omega\times(0,T), \tag{6-4a}$$

$$\psi = a(x,t) \quad \text{on } \partial\Omega\times[0,T), \tag{6-4b}$$

$$\psi|_{t=0} = \psi_0(x,y) \quad \text{in } \Omega. \tag{6-4c}$$

Here $\psi = \partial_y\omega,\ b = (u, v)$ is incompressible and vanishes on the boundary $\mathbb{T}\times\{0,1\},\ c = \partial_x u$ vanishes at the boundary $\mathbb{T}\times\{0,1\},\ d = \omega\,\partial_x\omega$, and the boundary data is $a = (\tilde\omega|_{y=1} - \tilde\omega|_{y=0}) - \partial_x\int_0^1 u^2\,dy$. As emphasized after Theorem 2.1, the third compatibility condition of the theorem corresponds to the relation $a(x,0) = \psi_0(x,0)$.

By (6-2), the initial datum ψ_0 is taken to obey $0 < 4\delta_0 \leq \psi_0(x,y) \leq 1/(4\delta_0)$, for some $\delta_0 \in \left(0, \frac{1}{4}\right)$, uniformly on Ω. Thus, by the compatibility of the initial datum and of the boundary condition, we have $0 < 4\delta_0 \leq a(x,0) \leq 1/(4\delta_0)$, uniformly on $\mathbb{T}$. Thanks to the Gagliardo–Nirenberg inequality

$$\|f\|_{L_y^\infty} \leq C\|f\|_{L_y^2}^{1/2}(\|f\|_{L_y^2}^{1/2} + \|\partial_y f\|_{L_y^2}^{1/2})$$

and the estimate (5-2), we have

$$\begin{aligned}\|\partial_t a(x,t)\|_{L^2(0,T;L_x^\infty)} &\leq 4\|\dot\omega\|_{L^2(0,T;L^\infty)} + 2\left\|\partial_x\int_0^1 u\,\dot u\,dy\right\|_{L^2(0,T;L_x^\infty)} \\ &\lesssim \frac{1}{\delta_0^2}\left(\frac{1}{\beta^{1/4}} + \frac{M}{\beta^{1/2}}\right)\|\dot\omega_0\|_{\gamma,r/2+\gamma-3/4,\tau_0} \leq \|\dot\omega_0\|_{\gamma,r/2+\gamma-3/4,\tau_0}\end{aligned}$$

for β sufficiently large. By the fundamental theorem of calculus in time, and the Cauchy–Schwarz inequality we thus obtain

$$3\delta_0 \le 4\delta_0 - \sqrt{T}\|\dot{\omega}_0\|_{\gamma,r/2+\gamma-3/4,\tau_0} \le a(x,t) \le \frac{1}{4\delta_0} + \sqrt{T}\|\dot{\omega}_0\|_{\gamma,r/2+\gamma-3/4,\tau_0} \le \frac{1}{3\delta_0}$$

uniformly on $\mathbb{T}\times(0,T)$, upon taking T sufficiently small. Thus, on the parabolic boundary $\Omega\times\{0\}\cup\partial\Omega\times(0,T)$, we have $\psi \ge 3\delta_0$.

By the same Gagliardo–Nirenberg inequality, the Poincaré inequality in y, and estimate (4-6), we have

$$\sup_{t\in[0,T]}\|c(t)\|_{L^\infty_x L^\infty_y} = \sup_{t\in[0,T]}\|\partial_x u(t)\|_{L^\infty_x L^\infty_y} \le \frac{C_1}{\delta_0}\|\omega_0\|_{\gamma,r,\tau_0},$$

where $C_1 = C_1(\tau_0,\tau_1,\gamma,r)$. Setting

$$C_* = 1 + \frac{C_1}{\delta_0}\|\omega_0\|_{\gamma,r,\tau_0}, \tag{6-5}$$

the above estimate implies

$$c(x,y,t) + C_* \ge 1.$$

Lastly, we note that by the Gagliardo–Nirenberg inequality and (4-6) we have

$$\int_0^t \|d(s)\|_{L^\infty_x L^\infty_y}\,ds = \int_0^t \|\omega(s)\|_{L^\infty_x L^\infty_y}\|\partial_x\omega(s)\|_{L^\infty_x L^\infty_y}\,ds \lesssim \frac{\sqrt{t}}{\delta_0^2}\|\omega_0\|^2_{\gamma,r,\tau_0}$$

so that for $T \le 1$ we have that

$$\begin{aligned} e(t) &:= t + \int_0^t e^{-C_* s}\|d(s) - 3\delta_0 c(s)\|_{L^\infty_x L^\infty_y}\,ds \\ &\lesssim t + \sqrt{t}\|\omega_0\|^2_{\gamma,r,\tau_0} + tC_1\|\omega_0\|_{\gamma,r,\tau_0} \\ &\le C_2\sqrt{t}(1 + \|\omega_0\|^2_{\gamma,r,\tau_0} + \|\omega_0\|_{\gamma,r,\tau_0}) = \sqrt{t}D_* \end{aligned} \tag{6-6}$$

hold for all $t \in [0,T]$, where C_2 is a constant that only depends on γ, r, τ_0, and τ_1, and we have set

$$D_* = C_2(1 + \|\omega_0\|^2_{\gamma,r,\tau_0} + \|\omega_0\|_{\gamma,r,\tau_0}).$$

With this notation, we make the following change of unknowns:

$$\bar{\psi} = e^{-C_* t}(\psi(x,y,t) - 3\delta_0) + e(t), \tag{6-7a}$$

$$\bar{a} = e^{-C_* t}(a(x,t) - 3\delta_0) + e(t), \tag{6-7b}$$

$$\bar{d} = e^{-C_* t}(d(x,y,t) - 3\delta_0 c(x,y,t)), \tag{6-7c}$$

$$\bar{c} = c(x,y,t) + C_*, \tag{6-7d}$$

$$\bar{\psi}_0 = \psi_0(x,y) - 3\delta_0. \tag{6-7e}$$

The quantity $e(t)$ was chosen so that $\dot{e}(t) = 1 + \|\bar{d}(t)\|_{L^\infty}$. One may then verify directly that

$$(\partial_t - \partial_y^2 + b \cdot \nabla_{x,y} + \bar{c})\bar{\psi} = (\bar{d} + \|\bar{d}\|_{L^\infty}) + 1 + \bar{c}e \geq 1 > 0, \tag{6-8a}$$

$$\bar{\psi}|_{y\in\{0,1\}} = \bar{a} \geq t \geq 0, \tag{6-8b}$$

$$\bar{\psi}|_{t=0} = \bar{\psi}_0 \geq \delta_0 > 0. \tag{6-8c}$$

The parabolic minimum principle then guarantees that

$$\bar{\psi}(x, y, t) \geq 0 \quad \text{on } \Omega \times [0, T]. \tag{6-9}$$

Indeed, if a strictly negative minimum were attained by $\bar{\psi}$, then this point minimum could not lie on the parabolic boundary (since $\bar{a} \geq 0$ and $\bar{\psi}_0 > 0$). If this point lay in the interior, at this point we would need to have $\nabla_{t,x,y}\bar{\psi} = 0$, whereas $(-\partial_y^2 + \bar{c})\bar{\psi} < 0$ since $\bar{c} > 0$. This contradicts $(\bar{d} + \|\bar{d}\|_{L^\infty}) + 1 + \bar{c}e > 0$, which thus proves (6-9).

Working backwards from the definition of $\bar{\psi}$, we see that (6-5), (6-6), and (6-9) imply

$$\psi(x, y, t) \geq 3\delta_0 - e^{C_* t} e(t) \geq 3\delta_0 - \sqrt{T} e^{C_* T} D_* \geq 2\delta_0$$

as long as T is chosen sufficiently small in terms of C_*, D_*, and δ_0, consistent with the dependence given in (5-9). This proves the lower bound in (6-3).

The proof of the upper bound in (6-3) follows from very similar arguments, reducing the problem to a maximum principle for a parabolic equation. To avoid redundancy, we omit these details. □

7. Proof of Theorem 2.1

The proof of the main theorem proceeds as follows. Let $\gamma \leq \frac{9}{8}$ and $r \geq r_4(\gamma)$. For any $\tau_0 < \tau^0$ assumption (2-1) implies that $\omega_0 = \partial_y u_0$ satisfies

$$\|\omega_0\|_{\gamma,r,\tau_0} + \|\partial_y^2 \omega_0\|_{\gamma,r,\tau_0} < +\infty.$$

We fix $\tau_0 \in (\tau_1, \tau^0)$. We then fix δ_0 small enough and M large enough, so that the initial constraints (4-27), (5-6), (5-10) and (6-2) hold. Let $\beta \geq \beta_4$ and $\varepsilon > 0$. We consider the approximate system

$$\partial_t u + u\, \partial_x u + v\, \partial_y u + \partial_x p - \partial_y^2 u - \varepsilon \partial_x^2 u = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{7-1a}$$

$$\partial_y p = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{7-1b}$$

$$\partial_x u + \partial_y v = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{7-1c}$$

$$u|_{y=0,1} = v|_{y=0,1} = 0, \tag{7-1d}$$

with the same initial condition $u|_{t=0} = u_0$. System (7-1) is called the two-dimensional primitive equations, and has been widely studied, in various geometries and under various boundary conditions [Bresch et al. 2003; 2005; Temam and Ziane 2004]. In particular, Gevrey or analytic regularity results were obtained in both periodic and bounded geometries [Petcu 2004; Petcu et al. 2004; Kukavica et al. 2016]. In the context of system (7-1), the well-posedness result stated in Theorem 2.1 can be proved without much difficulty. In fact, the presence of $-\varepsilon\partial_x^2 u$ allows for a classical treatment, and the existence of solutions at fixed $\varepsilon > 0$ follows, e.g., from a Galerkin approximation procedure (which is compatible with the hydrostatic trick [Masmoudi and Wong 2012]). Moreover, the compatibility conditions are the same for (1-1) and (7-1). We find in this way a unique local solution u^ε with the regularity requirements stated in Theorem 2.1. We can then consider $T_{\varepsilon,*}$ the maximal time on which $\|\omega_\varepsilon\|_{\gamma,0,\tau_1} < +\infty$. In particular, if $T_{\varepsilon,*}$ is small enough so that $\tau(T_{\varepsilon,*}) \geq \tau_1$, one has

$$\sup_{t\in[0,T_{\varepsilon,*})} \|\omega_\varepsilon(t)\|_{\gamma,3r/4,\tau(t)} = +\infty. \tag{7-2}$$

By the initial constraint (4-27), the fact that $\tau_0 < \tau^0$, and the continuity of the solution, there exists a maximal time $0 < T_\varepsilon \leq T_{\varepsilon,*}$ on which the conditions (4-2)–(4-4) are satisfied with u replaced by u_ε and T replaced by T_ε. Note that all the estimates that we established for a solution u of (1-1) adapt straightforwardly to a solution u^ε of (7-1). The only notable change is the inclusion of the $-\varepsilon\partial_x^2$ term in (3-4) for defining the boundary layer lift $\omega^{\flat,\varepsilon}$. However, since all estimates for $\omega^{\flat,\varepsilon}$ are obtained by performing a Fourier transform in x and using Plancherel to obtain the desired L^2_x bound, this modification is routine (see also [Ignatova and Vicol 2016] for ε-independent bounds for solutions of the ε-regularization of the Prandtl system that are analytic in x and Sobolev in y). Applying Corollaries 4.2, 5.2, and 5.3, and Proposition 6.1 at positive ε, we see that there exists $T > 0$ independent of ε, such that for all $t \in [0, \min(T_\varepsilon, T)]$, the conditions (4-2)–(4-4) still hold with M replaced by $\frac{1}{2}M$, and δ_0 replaced by $2\delta_0$. If $T_\varepsilon < T$, then one has necessarily $T_\varepsilon = T_{\varepsilon,*}$, otherwise by continuity the inequalities (4-2)–(4-4) would be satisfied beyond T_ε. But then there is a contradiction between (7-2) and the first half of (4-2). Hence, $T_\varepsilon \geq T$, and so $T_{\varepsilon,*} \geq T$.

We have just shown that the approximations u_ε are all defined on a time interval independent of ε, and satisfy uniform Gevrey bounds on it. This allows us to let ε go to zero, and conclude by standard compactness arguments to the existence of a solution.

For the uniqueness of solutions, the equation obeyed by the difference is basically a linearized version of the equation, very similar to the equation obeyed by $\dot{\omega}$. Then an estimate similar to the one from Proposition 5.1, gives the good estimate for the difference of two solutions, implying uniqueness.

Appendix: Proof of Lemma 3.2

To prove the first item, we adapt arguments of [Fernandez et al. 2016, pages 1805–1807]. We fix $x \in \mathbb{T}$, $y > 0$, and drop them from the notation. We write

$$\hat{\tilde{\omega}}^\flat_j(\eta) = \hat{f}_j(\zeta)\, g_j(\zeta), \quad g_j(\zeta) = \frac{1}{2-\sqrt{\beta(j+1)+i\zeta}}\, e^{-y\sqrt{\beta(j+1)+i\zeta}}.$$

Clearly, as f_j is equal to 0 for $t < 0$ and belongs to $L^1(\mathbb{R})$,

$$\hat{f}_j(\zeta) = \int_{\mathbb{R}_+} f_j(t) e^{-i\zeta t}\, dt$$

is holomorphic for $\mathscr{I}m\,\zeta < 0$, and continuous for $\mathscr{I}m\,\zeta \leq 0$. Moreover,

$$\lim_{\mathscr{I}m\,\zeta\to+\infty} \hat{f}_j(\zeta) = 0 \text{ uniformly for } \mathscr{R}e\,\zeta \in \mathbb{R}, \qquad \lim_{\mathscr{R}e\,\zeta\to\pm\infty} \hat{f}_j(\zeta) = 0 \text{ uniformly for } \mathscr{I}m\,\zeta \leq 0. \tag{A-1}$$

The first limit follows directly from the inequality

$$|\hat{f}_j(\zeta)| \le \int_{\mathbb{R}_+} |f_j(t)| e^{-\mathcal{I}m\,\zeta t}\,dt$$

and the dominated convergence theorem. The second limit follows from a close look at the Riemann–Lebesgue lemma: given $\varepsilon > 0$, and some $f_j^\varepsilon \in C_c^1(\mathbb{R}_+)$ with $\int_{\mathbb{R}_+} |f_j - f_j^\varepsilon| \le \varepsilon$, we get

$$|\hat{f}_j(\zeta)| \le \int_{\mathbb{R}+} |f_j - f_j^\varepsilon| + \Big|\int_{\mathbb{R}+} f_j^\varepsilon(t) e^{-i\zeta t}\,dt\Big| \le \varepsilon + \frac{M_\varepsilon}{|\mathcal{R}e\,\zeta|},$$

where the second bound follows from an integration by parts of the second integral.

Obviously, g_j is also holomorphic in $\mathcal{I}m\,\zeta < 0$, continuous over $\mathcal{I}m\,\zeta \le 0$, with bound

$$|g_j(\zeta)| \le \frac{1}{\beta - 2} e^{-\sqrt{|\zeta|}y}; \tag{A-2}$$

see (3-13). We finally apply the Cauchy formula: for any $t < 0$, for any $\mu > 0$,

$$\begin{aligned}\bar{\omega}_j^\flat(t) &= \lim_{s\to+\infty} \frac{1}{2\pi} \int_{-s}^{s} \hat{f}_j(\zeta)\, g_j(\zeta) e^{i\zeta t}\,d\zeta \\ &= -\lim_{s\to+\infty} \frac{1}{2\pi} \bigg(\int_{[-s,s]-i\mu} \hat{f}_j(\zeta)\, g_j(\zeta) e^{i\zeta t}\,d\zeta + \int_{[s,s-i\mu]} \hat{f}_j(\zeta)\, g_j(\zeta) e^{i\zeta t}\,d\zeta \\ &\qquad\qquad + \int_{[-s-i\mu,-s]} \hat{f}_j(\zeta)\, g_j(\zeta) e^{i\zeta t}\,d\zeta \bigg).\end{aligned}$$

As $t < 0$, taking into account the first limit in (A-1), the first integral at the right-hand side goes to zero when $\mu \to +\infty$, while the two other integrals over the vertical segments converge to the integrals over the vertical half-lines:

$$\begin{aligned}\bar{\omega}_j^\flat(t) &= \lim_{s\to+\infty} \frac{1}{2\pi} \bigg(\int_{[s,s-i\infty]} \hat{f}_j(\zeta)\, g_j(\zeta) e^{i\zeta t}\,d\zeta + \int_{[-s-i\infty,-s]} \hat{f}_j(\zeta)\, g_j(\zeta) e^{i\zeta t}\,d\zeta \bigg) \\ &= \lim_{s\to+\infty} \frac{1}{2\pi} \bigg(\int_{[0,-i\infty]} \hat{f}_j(s+\zeta)\, g_j(s+\zeta) e^{i(s+\zeta)t}\,d\zeta + \int_{[-i\infty,0]} \hat{f}_j(-s+\zeta)\, g_j(-s+\zeta) e^{i(-s+\zeta)t}\,d\zeta \bigg).\end{aligned}$$

Using the second limit in (A-1) and the bound (A-2), we can conclude that the limit on the right-hand side is zero thanks to the dominated convergence theorem.

To prove the second item of the lemma, we remark from formula (3-12) that

$$(1+|\zeta|)^{3/4}\hat{\omega}_j^\flat \in L^2_\zeta(\mathbb{R}, L^2_y(\mathbb{R}_+, H^k_x(\mathbb{T}))), \quad (1+|\zeta|)^{1/4}\hat{\omega}_j^\flat \in L^2_\zeta(\mathbb{R}, H^1_y(\mathbb{R}_+, H^k_x(\mathbb{T}))) \quad \text{for all } k$$

using the smoothness of $\hat{f}_j$ with respect to x. We deduce that

$$\bar{\omega}_j^\flat \in H_t^{3/4}(\mathbb{R}, L^2_y(\mathbb{R}_+, H^k_x(\mathbb{T}))), \quad \bar{\omega}_j^\flat \in H_t^{1/4}(\mathbb{R}, H^1_y(\mathbb{R}_+, H^k_x(\mathbb{T})) \quad \text{for all } k. \tag{A-3}$$

Moreover, using again (3-12) and Plancherel in time, we get that, for any $\varphi = \varphi(t, x, y)$ smooth and quickly decreasing as $t \to \pm\infty$ and $y \to +\infty$,

$$\int_{\mathbb{R}\times\mathbb{R}_+\times\mathbb{T}} \bar{\omega}_j^\flat(\beta(j+1) - \partial_t)\varphi + \int_{\mathbb{R}\times\mathbb{R}_+\times\mathbb{T}} \partial_y\bar{\omega}_j^\flat\, \partial_y\varphi - \int_{\mathbb{R}\times\mathbb{T}} (2\bar{\omega}_j^\flat|_{y=0} + f_j)\varphi|_{y=0} = 0.$$

If we take φ with support in time included in $(-\infty, T)$, taking into account that $\bar{\omega}_j^\flat$ is zero for negative times, we end up with

$$\int_{(0,T)\times\mathbb{R}_+\times\mathbb{T}} \bar{\omega}_j^\flat(\beta(j+1) - \partial_t)\varphi + \int_{(0,T)\times\mathbb{R}_+\times\mathbb{T}} \partial_y\bar{\omega}_j^\flat\, \partial_y\varphi - \int_{(0,T)\times\mathbb{T}} \left(2\bar{\omega}_j^\flat|_{y=0} + \frac{M_j}{M_{j+1}} h_{j+1}\right)\varphi|_{y=0} = 0.$$

We recognize the weak formulation of system (3-10). The identity $\bar{\omega}_j^\flat = \omega_j^\flat$ over $(0, T)$ follows from the uniqueness of solutions to this system (for example in the regularity class given by (A-3)).

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