ANALYSIS & PDE

Volume 13 No. 7 2020

FABIO CAVALLETTI AND ANDREA MONDINO

NEW FORMULAS FOR THE LAPLACIAN OF DISTANCE FUNCTIONS AND APPLICATIONS

msp

ANALYSIS AND PDE
Vol. 13, No. 7, 2020
https://doi.org/10.2140/apde.2020.13.2091

NEW FORMULAS FOR THE LAPLACIAN OF DISTANCE FUNCTIONS AND APPLICATIONS

FABIO CAVALLETTI AND ANDREA MONDINO

The goal of the paper is to prove an exact representation formula for the Laplacian of the distance (and more generally for an arbitrary 1-Lipschitz function) in the framework of metric measure spaces satisfying Ricci curvature lower bounds in a synthetic sense (more precisely in essentially nonbranching $\mathsf{MCP}(K, N)$-spaces). Such a representation formula makes apparent the classical upper bounds together with lower bounds and a precise description of the singular part. The exact representation formula for the Laplacian of a general 1-Lipschitz function holds also (and seems new) in a general complete Riemannian manifold.

We apply these results to prove the equivalence of $\mathsf{CD}(K, N)$ and a dimensional Bochner inequality on signed distance functions. Moreover we obtain a measure-theoretic splitting theorem for infinitesimally Hilbertian, essentially nonbranching spaces satisfying $\mathsf{MCP}(0, N)$.

1. Introduction

The Laplacian comparison theorem for the distance function from a point in a manifold with Ricci curvature bounded from below is one of the most fundamental results in Riemannian geometry. The local version states that if (M, g) is a smooth Riemannian manifold of dimension $N \geq 2$ satisfying

MSC2010: 49J52, 53C23.
Keywords: Ricci curvature, optimal transport, Laplacian comparison, distance function.

$\mathrm{Ric}_g \geq (N-1)g$ then, calling $\mathsf{d}_p(\cdot) := \mathsf{d}(p, \cdot)$ the distance from a point $p \in M$, until the distance function is smooth the following upper bound holds:

$$\Delta \mathsf{d}_p \leq (N-1) \cot \mathsf{d}_p. \tag{1-1}$$

Of course here Δ denotes the Laplacian (also called Laplace–Beltrami operator) of the Riemannian manifold (M, g) and cot is the cotangent (for a general lower bound $\mathrm{Ric}_g \geq Kg$, an analogous upper bound holds by replacing the right-hand side of (1-1) with the suitable (hyperbolic-)trigonometric function). The result is very classical and can be proved either via the Bochner inequality (see for instance [Cheeger 2001, Section 2]) or by Jacobi fields computations (see for instance [Petersen 1998, Chapter 7]).

It was Calabi [1958] who first extended the upper bound (1-1) to the whole manifold in the weak sense of barriers. Cheeger and Gromoll [1971], in their celebrated proof of the splitting theorem, then proved that the upper bound (1-1) also holds globally on M in a distributional sense (see also [Cheeger 2001, Section 4]). Since those classical works, the Laplacian comparison theorem has become a fundamental technical tool in the investigation of Riemannian manifolds satisfying Ricci curvature lower bounds (see for instance [Cheeger 2001; Cheeger and Colding 1996; 1997; 2000a; 2000b; Colding 1996a; 1996b; 1997; Colding and Naber 2012; Li and Yau 1986; Petersen 1998]).

We finally mention that recently Mantegazza, Mascellani, and Uraltsev [Mantegazza et al. 2014] obtained an exact representation formula for the distributional Hessian (and Laplacian) of the distance function from a point and that Gigli [2015] extended to the nonsmooth setting the upper bound (1-1).

The goal of this paper is to sharpen the Laplacian comparison theorem in several ways. First of all we will give an *exact* representation formula for the Laplacian of a general distance function (and for a general 1-Lipschitz function on its transport set; see later for the details) which describes exactly also the singular part concentrated on the cut locus. Such a representation formula will hold on *every complete* Riemannian manifold, without any curvature assumption. When specialised to Riemannian manifolds with Ricci curvature bounded below, such an exact representation formula will make apparent not only the celebrated *global upper bound* (1-1) but also *a lower bound on the regular part of the Laplacian*. The results will be proved in the much higher generality of (not necessarily smooth) metric measure spaces satisfying Ricci curvature lower bounds in a synthetic sense (more precisely, essentially nonbranching $\mathsf{MCP}(K, N)$-spaces); see the final part of the introduction.

In order to fix the ideas, we start the introduction discussing the smooth setting of Riemannian manifolds.

Let us introduce some notation in order to state the results. Given a point $p \in M$, denote by $\mathcal{C}_p$ the cut locus of p. The negative gradient flow $g_t : M \to M$ of the distance function d_p induces a partition $\{X_\alpha\}_{\alpha \in Q}$ of $M \setminus (\{p\} \cup \mathcal{C}_p)$ into minimising geodesics; each X_α is called a (transport) ray and Q is a suitable set of indices. We will denote the initial and final points of the ray X_α as $a(X_\alpha)$ and $b(X_\alpha)$ respectively; it is not hard to see that $a(X_\alpha) \in \mathcal{C}_p$ and $b(X_\alpha) = p$ for every $\alpha \in Q$. Let us stress that in this case the endpoints $a(X_\alpha), b(X_\alpha)$ are not elements of the ray X_α (in general, endpoints may or may not be elements of the ray, depending on the specific case; see also Remark 3.1). Such a partition induces a disintegration (the nonexpert reader can think of a kind of "nonstraight Fubini theorem") of the

Riemannian volume measure $\mathfrak{m}$ into measures $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha}$ concentrated on X_α:

$$\mathfrak{m} = \int_Q h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha} \, \mathfrak{q}(d\alpha), \tag{1-2}$$

where $\mathfrak{q}$ is a suitable probability measure on the set of indices Q. We refer to Section 3A for all the details on the disintegration formula. Here we only mention that once the probability $\mathfrak{q}$ is fixed within a suitable family of probability measures, then the functions h_α are uniquely determined.

The fact that (M, g) satisfies $\mathrm{Ric}_g \geq (N-1)g$ is inherited by the disintegration as concavity properties of the densities h_α; for the details see Section 3. For simplicity of notation, we will define

$$(\log h_\alpha)'(x) := \frac{d}{dt}\Big|_{t=0} \log h_\alpha(g_t(x));$$

thanks to the disintegration (1-2) and the (semi-)concavity of h_α along X_α, the quantity $(\log h_\alpha)'$ is well-defined $\mathfrak{m}$-a.e.

The first main result of the paper is an exact representation formula for the Laplacian of the distance function in nonsmooth spaces satisfying synthetic lower bounds on the Ricci curvature (see later in the introduction). In order to fix the ideas, we state it here for smooth Riemannian manifolds. We denote by $C_c(M)$ the space of real-valued continuous functions with compact support in M endowed with the final topology and by $(C_c(M))'$ its dual space made of real-valued continuous linear functionals on $C_c(M)$.

Theorem 1.1. *Let (M, g) be a smooth complete N-dimensional Riemannian manifold, where $N \geq 2$. Fix $p \in M$, and consider $\mathsf{d}_p := \mathsf{d}(p, \cdot)$ and an associated disintegration $\mathfrak{m} = \int_Q h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha} \mathfrak{q}(d\alpha)$.*

Then $\Delta \mathsf{d}_p$ is an element of $(C_c(M))'$ with the representation formula

$$\Delta \mathsf{d}_p = -(\log h_\alpha)' \mathfrak{m} - \int_Q h_\alpha \delta_{a(X_\alpha)} \, \mathfrak{q}(d\alpha). \tag{1-3}$$

It can be written as the sum of three Radon measures

$$\Delta \mathsf{d}_p = [\Delta \mathsf{d}_p]^+_{\mathrm{reg}} - [\Delta \mathsf{d}_p]^-_{\mathrm{reg}} + [\Delta \mathsf{d}_p]_{\mathrm{sing}},$$

with

$$[\Delta \mathsf{d}_p]^\pm_{\mathrm{reg}} = -[(\log h_\alpha)']^\pm \mathfrak{m}, \quad [\Delta \mathsf{d}_p]_{\mathrm{sing}} = -\int_Q h_\alpha \delta_{a(X_\alpha)} \, \mathfrak{q}(d\alpha) \leq 0,$$

where $\pm$ stands for the positive and negative parts. Here, $[\Delta \mathsf{d}_p]_{\mathrm{reg}} := [\Delta \mathsf{d}_p]^+_{\mathrm{reg}} - [\Delta \mathsf{d}_p]^-_{\mathrm{reg}}$ is the regular part of $\Delta \mathsf{d}_p$ (i.e., absolutely continuous with respect to $\mathfrak{m}$), and $[\Delta \mathsf{d}_p]_{\mathrm{sing}}$ is the singular part.

In particular, if (M, g) is compact $\Delta \mathsf{d}_p$ is a finite signed Borel (and in particular Radon) measure.

Moreover, if $\mathrm{Ric}_g \geq Kg$ for some $K \in \mathbb{R}$, the following comparison results hold true (for simplicity here we assume $K = N-1$ for the bounds corresponding to a general $K \in \mathbb{R}$; see (4-15)):

$$\Delta \mathsf{d}_p \leq (N-1) \cot \mathsf{d}_p \mathfrak{m}, \tag{1-4}$$

$$[\Delta \mathsf{d}_p]_{\mathrm{reg}} = -(\log h_\alpha)' \mathfrak{m} \geq -(N-1) \cot \mathsf{d}_{a(X_\alpha)} \mathfrak{m}. \tag{1-5}$$

Remark 1.2 (on the lower bound (1-5)). Denote by $\mathcal{C}_p := \{a(X_\alpha)\}_{\alpha \in Q}$ the cut locus of p and by g_t the negative gradient flow of d_p at time t. More precisely, g_t is defined ray by ray as the translation by t in the direction of the negative gradient of d_p for $t \in (0, |X_\alpha|)$, where $|X_\alpha|$ denotes the length of the transport ray X_α, i.e., $|X_\alpha| = \mathsf{d}(a(X_\alpha), b(X_\alpha)) = \mathsf{d}(a(X_\alpha), p)$. Then for every $\varepsilon > 0$ there exists $C_{K,N,\varepsilon} > 0$ so that

$$[\Delta \mathsf{d}_p]_{\mathrm{reg}} \geq -C_{K,N,\varepsilon}\mathfrak{m} \quad \text{on } \{x = g_t(a_\alpha) : t \geq \varepsilon\} \supset \{x \in X : \mathsf{d}(x, \mathcal{C}_p) \geq \varepsilon\}.$$

Let us stress that such a lower bound depends just on the dimension N, on the lower bound $K \in \mathbb{R}$ over the Ricci tensor, and on the distance $\varepsilon > 0$ from the cut locus $\mathcal{C}_p$, but is independent of the specific manifold (M, g).

We will prove the next more general statement for any signed distance function. Let us first give some definitions: Given a continuous function $v : M \to \mathbb{R}$ such that $\{v = 0\} \neq \varnothing$, the function

$$d_v : M \to \mathbb{R}, \quad d_v(x) := \mathsf{d}(x, \{v = 0\}) \operatorname{sgn}(v), \tag{1-6}$$

is called the *signed distance function* (from the zero-level set of v). With a slight abuse of notation, we denote by d both the distance between points and the induced distance between sets; more precisely

$$\mathsf{d}(x, \{v = 0\}) := \inf\{\mathsf{d}(x, y) : y \in \{v = 0\}\}.$$

Analogously to d_p, a signed distance function d_v induces a partition of M (up to a set of measure zero) into rays $\{X_\alpha\}_{\alpha \in Q}$ and a corresponding disintegration of the Riemannian volume measure $\mathfrak{m}$. The orientation of the rays is analogous. More precisely, if X_α is a transport ray associated with d_v and $a(X_\alpha), b(X_\alpha)$ are its initial and final points, then $d_v(b(X_\alpha)) \leq 0,\ d_v(a(X_\alpha)) \geq 0$, so that transport rays are oriented from $\{v \geq 0\}$ towards $\{v \leq 0\}$.

Theorem 1.3. *Let (M, g) be a smooth complete N-dimensional Riemannian manifold, where $N \geq 2$.*

Consider the signed distance function d_v for some continuous function $v : X \to \mathbb{R}$ and an associated disintegration

$$\mathfrak{m} = \int_Q h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha} \mathfrak{q}(d\alpha).$$

Then Δd_v^2 is an element of $(C_c(M))'$ with the representation formula

$$\Delta d_v^2 = 2(1 - d_v(\log h_\alpha)')\mathfrak{m} - 2\int_Q (h_\alpha d_v)[\delta_{a(X_\alpha)} - \delta_{b(X_\alpha)}]\, \mathfrak{q}(d\alpha). \tag{1-7}$$

It can be written as the sum of three Radon measures

$$\Delta d_v^2 = [\Delta d_v^2]^+_{\mathrm{reg}} - [\Delta d_v^2]^-_{\mathrm{reg}} + [\Delta d_v^2]_{\mathrm{sing}},$$

with

$$[\Delta d_v^2]^\pm_{\mathrm{reg}} := 2(1 - d_v(\log h_\alpha)')^\pm \mathfrak{m}, \quad [\Delta d_v^2]_{\mathrm{sing}} := -2\int_Q (h_\alpha d_v)[\delta_{a(X_\alpha)} - \delta_{b(X_\alpha)}]\, \mathfrak{q}(d\alpha) \leq 0,$$

where $\pm$ stands for the positive and negative parts; in particular if (M, g) is compact, Δd_v^2 is a finite signed Borel (and in particular Radon) measure.

Moreover, if $\mathrm{Ric}_g \geq Kg$ *for some* $K \in \mathbb{R}$, *the following comparison results hold true* (*for simplicity here we assume* $K = N-1$ *for the bounds corresponding to a general* $K \in \mathbb{R}$; *see* (4-23), (4-24)):

$$[\Delta d_v^2]^+_{\mathrm{reg}} \leq 2\mathfrak{m} + 2(N-1)\mathsf{d}(\{v=0\}, x)(\cot \mathsf{d}_{b(X_\alpha)} \mathfrak{m}\llcorner_{\{v \geq 0\}} + \cot \mathsf{d}_{a(X_\alpha)} \mathfrak{m}\llcorner_{\{v<0\}}), \tag{1-8}$$

$$[\Delta d_v^2]^-_{\mathrm{reg}} \leq 2\mathfrak{m} - 2(N-1)\mathsf{d}(\{v=0\}, \cdot\,)(\cot \mathsf{d}_{a(X_\alpha)} \mathfrak{m}\llcorner_{\{v \geq 0\}} + \cot \mathsf{d}_{b(X_\alpha)} \mathfrak{m}\llcorner_{\{v<0\}}). \tag{1-9}$$

We will also present a general statement (Corollary 4.10) valid for any 1-Lipschitz function $u : M \to \mathbb{R}$, provided the rays of the induced disintegration satisfy a suitable integrability condition (roughly, they should not be too short), obtaining the same representation formula together with the two-sided estimate we mentioned before.

An interesting feature of Corollary 4.10 is that it will hold *for every* 1-Lipschitz function $u : X \to \mathbb{R}$. Let us stress that the 1-Lipschitz assumption is clearly a *first-order* condition, with no information on second-order derivatives. Nevertheless, Corollary 4.10 will imply that in a general complete Riemannian manifold it is possible to deduce some information on the *second derivatives once restricted to a suitable subset*. More precisely, if one considers only the set of points “saturating the 1-Lipschitz assumption” then the Laplacian of u is a continuous linear functional on C_c. We stress that we will obtain an *exact representation formula* of Δu (restricted to such a set) which, in the case the Ricci curvature of the ambient N-manifold is bounded below by $K \in \mathbb{R}$, will give a *two-sided bound* on the regular part in terms of K, N. We refer to Corollary 4.10 for the details.

Up to now we focussed the introduction on the setting of complete Riemannian manifolds (satisfying Ricci curvature lower bounds). However, everything will be proved in the much higher generality of (possibly nonsmooth) essentially nonbranching, metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$ satisfying the measure contraction property $\mathsf{MCP}(K, N)$ for some $K \in \mathbb{R}$, $N \in (1, \infty)$. We refer to Section 2A for the detailed definitions; here let us just recall that $\mathsf{MCP}(K, N)$, introduced independently in [Ohta 2007a] and [Sturm 2006b], is the weakest among the synthetic conditions of Ricci curvature bounded below by K and dimension bounded above by N for metric measure spaces. In particular it is strictly weaker than the celebrated curvature dimension condition $\mathsf{CD}(K, N)$ pioneered in [Lott and Villani 2009; Sturm 2006a; 2006b] and than the (weaker) reduced curvature dimension condition $\mathsf{CD}^*(K, N)$ [Bacher and Sturm 2010]. The essential nonbranching condition, introduced by T. Rajala and Sturm [2014], roughly amounts to asking that W_2-geodesics are concentrated on nonbranching geodesics.

Remark 1.4 (notable examples of spaces fitting in the framework of the paper). The class of essentially nonbranching $\mathsf{MCP}(K, N)$ spaces include many remarkable families of spaces, among them:

- Smooth Finsler manifolds where the norm on the tangent spaces is strongly convex, and which satisfy lower Ricci curvature bounds. More precisely we consider a C^∞-manifold M, endowed with a function $F : TM \to [0, \infty]$ such that $F|_{TM \setminus \{0\}}$ is C^∞ and for each $p \in M$ it holds that $F_p := T_pM \to [0, \infty]$ is a strongly convex norm; i.e.,

$$g^p_{ij}(v) := \frac{\partial^2 (F_p^2)}{\partial v^i \partial v^j}(v) \quad \text{is a positive definite matrix at every } v \in T_pM \setminus \{0\}.$$

Under these conditions, it is known that one can write the geodesic equations and geodesics do not branch; in other words these spaces are nonbranching. We also assume (M, F) to be geodesically complete and endowed with a C^∞ measure $\mathfrak{m}$ in such a way that the associated metric measure space $(X, F, \mathfrak{m})$ satisfies the $\mathsf{MCP}(K, N)$ condition; see [Ohta 2007b; Ohta and Sturm 2014].

• Sub-Riemannian manifolds. The following are all examples of essentially nonbranching $\mathsf{MCP}(K, N)$-spaces: the $(2n+1)$-dimensional Heisenberg group [Juillet 2009], any corank-1 Carnot group [Rizzi 2016], any ideal Carnot group [Rifford 2013], any generalised H-type Carnot group of rank k and dimension n [Barilari and Rizzi 2018].

• Strong $\mathsf{CD}^*(K, N)$ spaces, and in particular $\mathsf{RCD}^*(K, N)$ spaces (see below). The class of $\mathsf{RCD}^*(K, N)$ spaces includes the following remarkable subclasses:

 – Measured Gromov Hausdorff limits of Riemannian N-dimensional manifolds satisfying Ricci $\geq K$; see [Ambrosio et al. 2014b; Gigli et al. 2015b].

 – Finite-dimensional Alexandrov spaces with curvature bounded from below; see [Petrunin 2011].

In the context of metric measure spaces satisfying Ricci curvature lower bounds in a synthetic form, the Laplacian comparison theorem in its classical form (1-1) was established in [Gigli 2015]. More precisely, that work developed a notion of a possibly multivalued Laplacian holding on a general metric measure space $(X, \mathsf{d}, \mathfrak{m})$; it also introduces a property of the space called *infinitesimal strict convexity*, which grants, among other things, uniqueness of the Laplacian. Finally, assuming infinitesimal strict convexity and $\mathsf{CD}^*(K, N)$, a sharp upper bound for the Laplacian of a general Kantorovich potential for the W_2 distance is obtained, in particular, for d_p^2. The comparison in [Gigli 2015] is stated for $\mathsf{CD}^*(K, N)$ but the same proof, in the case of d_p^2, works assuming the weaker $\mathsf{MCP}(K, N)$.

Our results therefore extend the ones in [Gigli 2015] removing the assumption of infinitesimal strict convexity (hence including the possibility of a multivalued Laplacian, see Definition 2.12); moreover we precisely describe the Laplacian of a general signed distance function or a 1-Lipschitz function with sufficiently long transport rays, obtaining also a lower bound on the regular part and a representation formula for the singular part. We stress the fundamental role of the exact representation formulas: it will be the key in our application to the Bochner inequality (signed distance functions) and for the splitting theorem (general 1-Lipschitz function); see the discussions below.

We conclude this part on the related results in the literature mentioning that the Laplacian comparison results [Gigli 2015, Theorem 5.14, Corollary 5.15] seem to claim the stronger conclusion that $\Delta\mathsf{d}_p^2$ is a Radon measure in the classical sense (see Definition 2.11 and comments shortly afterwards). This however seems to not follow from the proof, when (X, d) is not compact: $\Delta\mathsf{d}_p^2$ is proved to be an element of $(C_c(X))'$ so, by the Riesz theorem, it is a difference of positive Radon measures but it may fail to be a Borel measure (see [Gigli 2015, Proposition 4.13] and the application of the Riesz theorem in the last part of its proof). We will therefore adapt the definition of Laplacian (see Definition 2.12), weakening [Gigli 2015, Definition 4.4]. With this new definition also [Gigli 2015, Proposition 4.13] together with its applications seem to work.

The second part of the paper is devoted to applications.

In Section 6 we will use the representation formula for the Laplacian to show that, under essential nonbranching, the $\mathsf{CD}(K, N)$ condition is equivalent to a dimensional Bochner inequality on signed distance functions. The Bochner inequality corresponds to an *Eulerian* formulation of Ricci curvature lower bounds, while the $\mathsf{CD}(K, N)$ condition, based on convexity of entropies along W_2-geodesics of probability measures, corresponds to a *Lagrangian* approach.

It has been a long-standing open problem, see for instance the celebrated book [Villani 2009, Open Problem 17.38, Conclusions and Open Problems, p. 923], to show that the Eulerian and the Lagrangian formulations of Ricci curvature lower bounds are equivalent. Such an equivalence has already been proved to hold true under the additional assumption that the heat flow $\mathrm{H}_t : L^2(X, \mathfrak{m}) \to L^2(X, \mathfrak{m})$ is linear for every $t \geq 0$ (or, equivalently, the Cheeger energy $\mathrm{Ch}(f) := \int_X |\nabla f|_w^2 \, \mathfrak{m}$ satisfies the parallelogram identity). The class of $\mathsf{CD}(K, N)$ spaces satisfying such a linearity condition is called $\mathsf{RCD}(K, N)$. After its birth in [Ambrosio et al. 2014b] (see also [Ambrosio et al. 2015a]) for $N = \infty$ and further developments for $N < \infty$ (see [Ambrosio et al. 2019; Erbar et al. 2015; Gigli 2015] and the subsequent [Cavalletti and Milman 2016]), the theory of metric measure spaces satisfying $\mathsf{RCD}(K, N)$ (called $\mathsf{RCD}(K, N)$-spaces for short) has been flourishing in the last years (for a survey of results, see [Villani 2019; Ambrosio 2018]).

The equivalence between $\mathsf{RCD}(K, N)$ and the Bochner inequality (properly written in a weak form, called Bakry-Émery condition $\mathrm{BE}(K, N)$) was proved for $N = \infty$ by Ambrosio, Gigli, and Savaré [Ambrosio et al. 2014b; 2015b], and in the finite-dimensional case by Erbar, Kuwada, and Sturm [Erbar et al. 2015] and Ambrosio, Mondino, and Savaré [Ambrosio et al. 2019].

Let us stress that the linearity of the heat flow was a crucial assumption in all of the aforementioned works.

The equivalence between the Bochner inequality and $\mathsf{CD}(K, N)$ was proved also in *smooth* Finsler manifolds by Ohta and Sturm. In [Ohta and Sturm 2014] no linearity of the heat flow is assumed, on the other hand the smoothness of the Finsler structure is heavily used in the computations. In the present paper, in contrast to the aforementioned works, we assume *neither that the heat flow is linear nor that the space is smooth*, thus showing that the equivalence between *Lagrangian* and *Eulerian* approaches to Ricci curvature lower bounds holds in the higher generality of nonsmooth "possibly Finslerian" spaces.

The proof of the equivalence seems also to follow rather easily once the representation formula for the Laplacian of signed distance functions is at our disposal. Here we also crucially use [Cavalletti and Milman 2016], where it is shown that a control on the behaviour of signed distance functions is sufficient to control the geometry of the space (see the statement: $\mathsf{CD}^1(K, N)$ implies $\mathsf{CD}(K, N)$). This also motivates our interest in the Laplacian of this family of functions (Theorem 4.14).

A second application is a measure-theoretic splitting theorem stating, roughly, that an infinitesimally Hilbertian (i.e., the Cheeger energy satisfies the parallelogram identity), essentially nonbranching $\mathsf{MCP}(0, N)$ space containing a line is isomorphic as a *measure space* to a splitting (for the precise statement see Theorem 7.1).

For smooth Riemannian manifolds [Cheeger and Gromoll 1971], as well as for Ricci-limits [Cheeger and Colding 1996] and $\mathsf{RCD}(0, N)$ spaces [Gigli 2013], the splitting theorem has a stronger statement giving an *isometric splitting*. However under the assumptions of Theorem 7.1 it is not conceivable to expect also a splitting of the metric. Indeed the Heisenberg group $\mathbb{H}^n$ is an example of a nonbranching infinitesimally Hilbertian $\mathsf{MCP}(0, N)$ space [Juillet 2009] containing a line, which is homeomorphic and isomorphic as a measure space to a splitting (indeed it is homeomorphic to $\mathbb{R}^n$ and the measure is exactly the n-dimensional Lebesgue measure) but it is not isometric to a splitting.

2. Prerequisites

In this section we review the basic material needed throughout the paper. The standing assumptions are that (X, d) is a complete, proper and separable metric space endowed with a positive Radon measure $\mathfrak{m}$ satisfying $\mathrm{supp}(\mathfrak{m}) = X$. The triple $(X, \mathsf{d}, \mathfrak{m})$ is said to be a metric measure space, m.m.s. for short.

The properness assumption is motivated by the synthetic Ricci curvature lower bounds we will assume to hold.

2A. ***Essentially nonbranching,*** $\mathsf{MCP}(K, N)$ ***and*** $\mathsf{CD}(K, N)$ ***metric measure spaces.*** We denote by

$$\mathrm{Geo}(X) := \{\gamma \in C([0, 1], X) : \mathsf{d}(\gamma_s, \gamma_t) = |s-t|\mathsf{d}(\gamma_0, \gamma_1) \text{ for every } s, t \in [0, 1]\}$$

the space of constant-speed geodesics. The metric space (X, d) is a *geodesic space* if and only if for each $x, y \in X$ there exists $\gamma \in \mathrm{Geo}(X)$ so that $\gamma_0 = x,\ \gamma_1 = y$.

Recall that, for complete geodesic spaces, local compactness is equivalent to properness (a metric space is proper if every closed ball is compact).

We denote by $\mathcal{P}(X)$ the space of all Borel probability measures over X and by $\mathcal{P}_2(X)$ the space of probability measures with finite second moment. $\mathcal{P}_2(X)$ can be endowed with the L^2-Kantorovich–Wasserstein distance W_2 defined as follows: for $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, set

$$W_2^2(\mu_0, \mu_1) := \inf_{\pi} \int_{X\times X} \mathsf{d}^2(x, y)\, \pi(dx\, dy), \tag{2-1}$$

where the infimum is taken over all $\pi \in \mathcal{P}(X \times X)$ with μ_0 and μ_1 as the first and the second marginals. The space (X, d) is geodesic if and only if the space $(\mathcal{P}_2(X), W_2)$ is geodesic.

For any $t \in [0, 1]$, let e_t denote the evaluation map

$$\mathrm{e}_t : \mathrm{Geo}(X) \to X, \quad \mathrm{e}_t(\gamma) := \gamma_t.$$

Any geodesic $(\mu_t)_{t\in[0,1]}$ in $(\mathcal{P}_2(X), W_2)$ can be lifted to a measure $\nu \in \mathcal{P}(\mathrm{Geo}(X))$, so that $(\mathrm{e}_t)_\sharp \nu = \mu_t$ for all $t \in [0, 1]$.

Given $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, we denote by $\mathrm{OptGeo}(\mu_0, \mu_1)$ the space of all $\nu \in \mathcal{P}(\mathrm{Geo}(X))$ for which $(\mathrm{e}_0, \mathrm{e}_1)_\sharp \nu$ realises the minimum in (2-1). Such a ν will be called *dynamical optimal plan*. If (X, d) is geodesic, then the set $\mathrm{OptGeo}(\mu_0, \mu_1)$ is nonempty for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$.

We will also consider the subspace $\mathcal{P}_2(X, \mathsf{d}, \mathfrak{m}) \subset \mathcal{P}_2(X)$ formed by all those measures absolutely continuous with respect to $\mathfrak{m}$.

A set $G \subset \mathrm{Geo}(X)$ is a set of nonbranching geodesics if and only if for any $\gamma^1, \gamma^2 \in G$, it holds

$$\text{there exists } \bar{t} \in (0, 1) \text{ such that, for all } t \in [0, \bar{t}], \quad \gamma_t^1 = \gamma_t^2 \quad \Longrightarrow \quad \gamma_s^1 = \gamma_s^2 \quad \text{for all } s \in [0, 1].$$

In this paper we will only consider essentially nonbranching spaces; let us recall their definition (introduced in [Rajala and Sturm 2014]).

Definition 2.1. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is *essentially nonbranching* (e.n.b. for short) if and only if for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, with μ_0, μ_1 absolutely continuous with respect to $\mathfrak{m}$, any element of $\mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of nonbranching geodesics.

It is clear that if (X, d) is a smooth Riemannian manifold then any subset $G \subset \mathrm{Geo}(X)$ is a set of nonbranching geodesics, in particular any smooth Riemannian manifold is essentially nonbranching.

In order to formulate curvature properties for $(X, \mathsf{d}, \mathfrak{m})$ we recall the definition of the distortion coefficients: for $K \in \mathbb{R}$, $N \in [1, \infty)$, $\theta \in (0, \infty)$, $t \in [0, 1]$, set

$$\tau_{K,N}^{(t)}(\theta) := t^{1/N} \sigma_{K,N-1}^{(t)}(\theta)^{(N-1)/N}, \tag{2-2}$$

where the σ-coefficients are defined as follows: given two numbers $K, N \in \mathbb{R}$ with $N \geq 0$, we set, for $(t, \theta) \in [0, 1] \times \mathbb{R}_+$,

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} \infty & \text{if } K\theta^2 \geq N\pi^2, \\ \dfrac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0 < K\theta^2 < N\pi^2, \\ t & \text{if } K\theta^2 < 0 \text{ and } N = 0, \text{ or if } K\theta^2 = 0, \\ \dfrac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2 \leq 0 \text{ and } N > 0. \end{cases} \tag{2-3}$$

Let us also recall the definition of the Rényi entropy functional $\mathcal{E}_N : \mathcal{P}(X) \to [0, \infty]$,

$$\mathcal{E}_N(\mu) := \int_X \rho^{1-1/N}(x)\, \mathfrak{m}(dx), \tag{2-4}$$

where $\mu = \rho\mathfrak{m} + \mu^s$ with $\mu^s \perp \mathfrak{m}$.

Next we recall the definition of $\mathsf{MCP}(K, N)$ given independently in [Ohta 2007a] and [Sturm 2006b]. On general metric measure spaces the two definitions slightly differ, but on essentially nonbranching spaces they coincide (see for instance Appendix A in [Cavalletti and Mondino 2017a] or Proposition 9.1 in [Cavalletti and Milman 2016]). We report the one given in [Ohta 2007a].

Definition 2.2 (MCP condition). Let $K \in \mathbb{R}$ and $N \in [1, \infty)$. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies $\mathsf{MCP}(K, N)$ if for any $\mu_0 \in \mathcal{P}_2(X)$ of the form

$$\mu_0 = \frac{1}{\mathfrak{m}(A)} \mathfrak{m}\llcorner_A$$

for some Borel set $A \subset X$ with $\mathfrak{m}(A) \in (0, \infty)$, and any $o \in X$ there exists $\nu \in \mathrm{OptGeo}(\mu_0, \delta_o)$ such that

$$\frac{1}{\mathfrak{m}(A)} \mathfrak{m} \geq (\mathrm{e}_t)_\sharp \big(\tau_{K,N}^{(1-t)}(\mathsf{d}(\gamma_0, \gamma_1))\nu(d\gamma)\big) \quad \text{for all } t \in [0, 1]. \tag{2-5}$$

From [Cavalletti and Milman 2016, Proposition 9.1], in the setting of essentially nonbranching spaces Definition 2.2 is equivalent to the following condition: for all $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with $\mu_0 \ll \mathfrak{m}$ and $\mathrm{supp}(\mu_1) \subset \mathrm{supp}(\mathfrak{m})$, there exists a unique $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$, ν is induced by a map (i.e., $\nu = S_\sharp(\mu_0)$

for some map $S : X \to \mathrm{Geo}(X)$), $\mu_t := (\mathrm{e}_t)_\# \nu \ll \mathfrak{m}$ for all $t \in [0,1)$, and writing $\mu_t = \rho_t \mathfrak{m}$, we have for all $t \in [0,1)$

$$\rho_t^{-1/N}(\gamma_t) \geq \tau_{K,N}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))\rho_0^{-1/N}(\gamma_0) \quad \text{for } \nu\text{-a.e. } \gamma \in \mathrm{Geo}(X). \tag{2-6}$$

The curvature-dimension condition was introduced independently in [Lott and Villani 2009] and [Sturm 2006a; 2006b]; let us recall its definition.

Definition 2.3 (CD condition). Let $K \in \mathbb{R}$ and $N \in [1,\infty)$. A metric measure space $(X,\mathsf{d},\mathfrak{m})$ satisfies $\mathsf{CD}(K,N)$ if for any two $\mu_0,\mu_1 \in \mathcal{P}_2(X,\mathsf{d},\mathfrak{m})$ with bounded support there exist $\nu \in \mathrm{OptGeo}(\mu_0,\mu_1)$ and $\pi \in \mathcal{P}(X\times X)$ a W_2-optimal plan such that $\mu_t := (\mathrm{e}_t)_\sharp \nu \ll \mathfrak{m}$ and for any $N' \geq N$, $t \in [0,1]$,

$$\mathcal{E}_{N'}(\mu_t) \geq \int \tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))\rho_0^{-1/N'} + \tau_{K,N'}^{(t)}(\mathsf{d}(x,y))\rho_1^{-1/N'}\,\pi(dx\,dy). \tag{2-7}$$

Throughout this paper, we will always assume the proper metric measure space $(X,\mathsf{d},\mathfrak{m})$ satisfies $\mathsf{MCP}(K,N)$ for some $K,N \in \mathbb{R}$, and is essentially nonbranching. This will imply in particular that (X,d) is geodesic.

It is not difficult to see that if $(X,\mathsf{d},\mathfrak{m})$ satisfies $\mathsf{CD}(K,N)$ then it also satisfies $\mathsf{MCP}(K,N)$, but the converse implication is false in general (for example the sub-Riemannian Heisenberg group satisfies $\mathsf{MCP}(K,N)$ for some suitable K,N, but does not satisfy $\mathsf{CD}(K',N')$ for any choice of K',N').

It is worth recalling that if (M,g) is a Riemannian manifold of dimension n and $h \in C^2(M)$ with $h > 0$, then the m.m.s. $(M,\mathsf{d}_g,h\,\mathrm{Vol}_g)$ (where d_g and Vol_g denote the Riemannian distance and volume induced by g) satisfies $\mathsf{CD}(K,N)$ with $N \geq n$ if and only if (see [Sturm 2006b, Theorem 1.7])

$$\mathrm{Ric}_{g,h,N} \geq Kg, \quad \mathrm{Ric}_{g,h,N} := \mathrm{Ric}_g - (N-n)\frac{\nabla_g^2 h^{1/(N-n)}}{h^{1/(N-n)}}.$$

In particular if $N = n$, the generalised Ricci tensor $\mathrm{Ric}_{g,h,N} = \mathrm{Ric}_g$ makes sense only if h is constant.

A variant of the CD condition, called reduced curvature dimension condition and denoted by $\mathsf{CD}^*(K,N)$ [Bacher and Sturm 2010], asks for the same inequality (2-7) as $\mathsf{CD}(K,N)$ but the coefficients $\tau_{K,N}^{(t)}(\mathsf{d}(\gamma_0,\gamma_1))$ and $\tau_{K,N}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))$ are replaced by $\sigma_{K,N}^{(t)}(\mathsf{d}(\gamma_0,\gamma_1))$ and $\sigma_{K,N}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))$, respectively. For both definitions there is a local version and it was recently proved in [Cavalletti and Milman 2016] that on an essentially nonbranching m.m.s. with $\mathfrak{m}(X) < \infty$, the $\mathsf{CD}^*_{\mathrm{loc}}(K,N)$, $\mathsf{CD}^*(K,N)$, $\mathsf{CD}_{\mathrm{loc}}(K,N)$, $\mathsf{CD}(K,N)$ conditions are all equivalent for all $K \in \mathbb{R}$, $N \in (1,\infty)$, via the $\mathsf{CD}^1(K,N)$ condition defined in terms of the L^1-optimal transport problem. For more details we refer to [Cavalletti and Milman 2016].

2B. *Lipschitz functions and Laplacians in metric measure spaces.* We recall some facts about calculus in metric measure spaces following the approach of [Ambrosio et al. 2014a; 2014b; Gigli 2015] with the slight difference that here we confine the presentation to the (easier) setting of Lipschitz functions (instead of Sobolev), as in the paper we will work in such a framework. For this subsection it is enough to assume the metric space (X,d) is complete and separable and $\mathfrak{m}$ is a nonnegative locally finite measure.

A function $f : X \to \mathbb{R}$ is Lipschitz (or more precisely L-Lipschitz) if there exists a constant $L \geq 0$ such that

$$|f(x) - f(y)| \leq L\,\mathsf{d}(x, y) \quad \text{for all } x, y \in X.$$

The minimal constant $L \geq 0$ satisfying the last inequality is called *global Lipschitz constant* of f and is denoted by $\mathrm{Lip}(f)$.

We denote by $\mathrm{LIP}(X)$ the space of real-valued Lipschitz functions on (X, d) and by $\mathrm{LIP}_c(\Omega) \subset \mathrm{LIP}(X)$ the subspace of Lipschitz functions of X with compact support contained in the open subset $\Omega \subset X$.

Given $f \in \mathrm{LIP}(X)$, the *local Lipschitz constant* $|Df|(x_0)$ of f at $x_0 \in X$ is defined as

$$|Df|(x_0) := \limsup_{x \to x_0} \frac{|f(x) - f(x_0)|}{\mathsf{d}(x, x_0)} \quad \text{if } x_0 \text{ is not isolated,} \qquad |Df|(x_0) = 0 \quad \text{otherwise.}$$

It is clear that $|Df| \leq \mathrm{Lip}(f)$ on all X.

Definition 2.4. Let $f, u \in \mathrm{LIP}(X)$. Define the functions $D^{\pm} f(\nabla u) : X \to \mathbb{R}$ by

$$D^+ f(\nabla u) := \inf_{\varepsilon > 0} \frac{|D(u + \varepsilon f)|^2 - |Du|^2}{2\varepsilon},$$

while $D^- f(\nabla u)$ is obtained by replacing $\inf_{\varepsilon>0}$ with $\sup_{\varepsilon<0}$.

If $D^+ f(\nabla u) = D^- f(\nabla u)$ $\mathfrak{m}$-a.e. for all $f, u \in \mathrm{LIP}(X)$, then $(X, \mathsf{d}, \mathfrak{m})$ is said to be (Lipschitz-) infinitesimally strictly convex and we set $Df(\nabla u) := D^+ f(\nabla u)$; if moreover $Df(\nabla u) = Du(\nabla f)$ $\mathfrak{m}$-a.e. for all $f, u \in \mathrm{LIP}(X)$, then $(X, \mathsf{d}, \mathfrak{m})$ is said to be (Lipschitz)-infinitesimally Hilbertian.

Remark 2.5. Given $f, u \in \mathrm{LIP}(X)$, it is easily seen the map $\varepsilon \mapsto |D(u + \varepsilon f)|^2$ is convex and real-valued. Thus

$$\inf_{\varepsilon > 0} \frac{|D(u + \varepsilon f)|^2 - |Du|^2}{2\varepsilon} = \liminf_{\varepsilon \downarrow 0} \frac{|D(u + \varepsilon f)|^2 - |Du|^2}{2\varepsilon},$$
$$\sup_{\varepsilon < 0} \frac{|D(u + \varepsilon f)|^2 - |Du|^2}{2\varepsilon} = \limsup_{\varepsilon \uparrow 0} \frac{|D(u + \varepsilon f)|^2 - |Du|^2}{2\varepsilon}.$$

Remark 2.6. The local doubling and Poincaré conditions will be satisfied throughout the paper as we will work in essentially nonbranching $\mathsf{MCP}(K, N)$-spaces, with $K \in \mathbb{R}$, $N \in (1, \infty)$ thanks to [von Renesse 2008, Corollary p. 28]. The standing assumptions in that paper are $\mathsf{MCP}(K, N)$ and that the set

$$C_x := \{y \in X : \text{there exists } \gamma^1 \neq \gamma^2 \in \mathrm{Geo}(X) \text{ such that } x = \gamma^1_0 = \gamma^2_0,\ y = \gamma^1_1 = \gamma^2_1\}$$

has $\mathfrak{m}$-measure zero for $\mathfrak{m}$-a.e. $x \in X$.

In an essentially nonbranching $\mathsf{MCP}(K, N)$ space the previous property can be obtained as follows: for any $r > 0$ invoke [Cavalletti and Mondino 2017a, Theorem 5.2] with $\mu_0 := \mathfrak{m}\llcorner_{B_r(x)}/\mathfrak{m}(B_r(x))$ and $\mu_1 := \delta_x$; existence of a map pushing μ_0 to the unique element of $\mathrm{OptGeo}(\mu_0, \mu_1)$ yields that $\mathfrak{m}(C_x \cap B_r(x)) = 0$, actually for any $x \in X$.

Remark 2.7. The notions of infinitesimally strictly convex and infinitesimally Hilbertian have been introduced in [Ambrosio et al. 2014b; Gigli 2015] in the setting of Sobolev spaces, with the local Lipschitz constant replaced by the minimal weak upper gradient. The corresponding Lipschitz counterparts that we defined above have been already considered in [Mondino 2015] and coincide with the ones of [Gigli 2015] provided the space satisfies doubling and Poincaré locally, thanks to a deep result of [Cheeger 1999]. Thanks to Remark 2.6 we will avoid therefore the prefix "Lipschitz" in the corresponding notions, for simplicity of notation.

Definition 2.8 (test plans, [Ambrosio et al. 2014a]). Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space as above and $\pi \in \mathcal{P}(C([0,1], X))$. We say that π is a test plan provided it has bounded compression; i.e., there exists $C > 0$ such that

$$(\mathrm{e}_t)_\sharp \pi = \mu_t \le C\mathfrak{m} \quad \text{for all } t \in [0,1],$$

and

$$\iint_0^1 |\dot\gamma_t|^2 \, dt \, \pi(d\gamma) < \infty.$$

Definition 2.9 (plans representing gradients). Let $(X, \mathsf{d}, \mathfrak{m})$ be an m.m.s., $g \in \mathrm{LIP}(X)$ and π a test plan. We say that π represents the gradient of g provided it is a test plan and we have

$$\liminf_{t\to 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t} \, \pi(d\gamma) \ge \frac{1}{2} \int |Dg|^2(\gamma_0) \, \pi(d\gamma) + \frac{1}{2} \limsup_{t\to 0} \frac{1}{t} \iint_0^t |\dot\gamma_s|^2 \, ds \, \pi(d\gamma)$$

Theorem 2.10 [Ambrosio et al. 2014b, Lemma 4.5; Gigli 2015, Theorem 3.10]. *Let $f, u \in \mathrm{LIP}(X)$ and π be any plan representing the gradient of u; then*

$$\begin{aligned}\int D^+ f(\nabla u)(\mathrm{e}_0)_\sharp \pi &\ge \limsup_{t\to 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \pi(d\gamma) \\ &\ge \liminf_{t\to 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \pi(d\gamma) \\ &\ge \int D^- f(\nabla u)(\mathrm{e}_0)_\sharp \pi.\end{aligned}$$

In particular, if $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally strictly convex then

$$\int_X Df(\nabla u)(\mathrm{e}_0)_\sharp \pi = \lim_{t\to 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \pi(d\gamma).$$

In order to define the Laplacian, let us recall the definition of Radon functional. For simplicity, from now on, we will assume (X, d) to be locally compact (this will be satisfied throughout the paper as we will work in the setting of $\mathsf{MCP}(K, N)$ spaces which are, even more strongly, locally doubling).

Definition 2.11. • A *Radon functional over an open set* $\Omega \subset X$ is a linear functional $T : \mathrm{LIP}_c(\Omega) \to \mathbb{R}$ such that for every compact subset $W \subset \Omega$ there exists a constant $C_W \ge 0$ so that

$$|T(f)| \le C_W \max_W |f| \quad \text{for all } f \in \mathrm{LIP}_c(\Omega) \text{ with } \operatorname{supp}(f) \subset W.$$

• A *nonnegative Radon measure over an open set* $\Omega \subset X$ is a Borel, nonnegative measure $\mu : \mathcal{B}(\Omega) \to [0, +\infty]$ that is locally finite; i.e., for any $x \in \Omega$ there exists a neighbourhood U_x of finite μ-measure: $\mu(U_x) < +\infty$. A nonnegative Radon measure is said to be *finite* if $\mu(X) < \infty$.

• A *signed Radon measure over an open set* $\Omega \subset X$ is a Borel measure $\mu : \mathcal{B}(\Omega) \to \mathbb{R} \cup \{\pm\infty\}$ that can be written as $\mu = \mu^+ - \mu^-$ with μ^+, μ^- nonnegative Radon measures, where at least one of the two is finite.

A signed Radon measure is said to be *finite* if, denoting by $\|\mu\| := \mu^+ + \mu^-$ the total variation measure, it holds $\|\mu\|(X) < \infty$.

Note that, by the classical Riesz–Markov–Kakutani representation theorem, for every *nonnegative* Radon functional T over X there exists a nonnegative Radon measure μ_T representing T via integration, i.e.,

$$T(f) = \int_X f(x)\, \mu_T(dx) \quad \text{for all } f \in \mathrm{LIP}_c(X).$$

In particular, every Radon functional can be written as the sum of two Radon measures (i.e., the positive and negative parts, respectively).

Let us stress that the nonnegativity assumption is crucial. Indeed a general Radon functional may not be representable by a measure; for example consider $X = \mathbb{R}$, $\Omega = \mathbb{R} \setminus \{0\}$ and $T : \mathrm{LIP}_c(\Omega) \to \mathbb{R}$ defined by

$$T : \mathrm{LIP}_c(\Omega) \to \mathbb{R}, \qquad T(f) := \int_\Omega \frac{f(x)}{x}\, dx.$$

It is straightforward to see that T is a real-valued Radon functional over Ω but cannot be represented by a signed Radon measure over Ω, the point being that $(-\infty, 0)$ would have "measure" $-\infty$ and $(0, +\infty)$ would have "measure" $+\infty$, thus failing the additivity axiom. An expert reader may recognise that $T(f)$ is (up to a multiplicative constant) the Hilbert transform of f evaluated at 0.

Definition 2.12. Let $\Omega \subset X$ be an open subset and let $u \in \mathrm{LIP}(X)$. We say that u is in the domain of the Laplacian of Ω, and write $u \in D(\boldsymbol{\Delta}, \Omega)$, provided there exists a Radon functional T over Ω such that for any $f \in \mathrm{LIP}_c(\Omega)$ it holds

$$\int_X D^- f(\nabla u)\, \mathfrak{m} \le -T(f) \le \int_X D^+ f(\nabla u)\, \mathfrak{m}. \tag{2-8}$$

In this case we write $T \in \boldsymbol{\Delta} u_{\llcorner\Omega}$. In the case T can be represented by a signed measure μ over Ω, with a slight abuse of notation we will identify T with μ and write $\mu \in \boldsymbol{\Delta} u_{\llcorner\Omega}$.

Let us stress that in general there is not a unique operator T satisfying (2-8); in other words the Laplacian can be multivalued.

2C. *Synthetic Ricci lower bounds over the real line.* Given $K \in \mathbb{R}$ and $N \in (1, \infty)$, a nonnegative Borel function h defined on an interval $I \subset \mathbb{R}$ is called an $\mathsf{MCP}(K, N)$ density on I if for all $x_0, x_1 \in I$ and $t \in [0, 1]$

$$h(tx_1 + (1-t)x_0) \ge \sigma^{(1-t)}_{K,N-1}(|x_1 - x_0|)^{N-1} h(x_0). \tag{2-9}$$

Even though it is a folklore result, we will include a proof of the following fact:

Lemma 2.13. *A one-dimensional metric measure space, that for simplicity we directly identify with* $(I, |\cdot|, h\mathcal{L}^1)$, *satisfies* $\mathsf{MCP}(K, N)$ *if and only there exists* $\tilde{h}$, *an* $\mathsf{MCP}(K, N)$ *density, such that* $h = \tilde{h}$ $\mathcal{L}^1$*-a.e. on* I.

Proof. Assume h is an $\mathsf{MCP}(K, N)$ density on I. From [Cavalletti and Milman 2016, Proposition 9.1(iv)], it will be enough to prove (2-6) under the additional assumption that

$$\mu_0 = \frac{1}{\mathfrak{m}(A)} \chi_A \mathfrak{m}$$

for some $A \subset I$ such that $0 < \mathfrak{m}(A) < \infty$, with $\mathfrak{m} = h\mathcal{L}^1$.

Without any loss in generality we assume $o = 0 \in I$. Given then any $A \subset I$ as above, the unique W_2 geodesic $(\mu_t)_{t\in[0,1]}$ connecting μ_0 to δ_0 is

$$\mu_t = (f_t)_\sharp \mu_0, \qquad f_t(x) = (1-t)x.$$

Then using the change of variable formula,

$$\mu_t = \rho_t \mathfrak{m}, \qquad \rho_t(x) = \frac{h(x/(1-t))}{h(x)} \frac{\chi_A(x/(1-t))}{(1-t)\mathfrak{m}(A)},$$

implying that

$$\left(\frac{\rho_t(f_t(x))}{\rho_0(x)}\right)^{-1/N} = \left(\frac{(1-t)h((1-t)x)}{h(x)}\right)^{1/N} \geq (1-t)^{1/N} \sigma_{K,N-1}^{(1-t)}(|x|)^{(N-1)/N} = \tau_{K,N}^{(1-t)}(|x|),$$

proving (2-6). In order to prove the converse implication, we fix $x_1 = 0 = o$ and take

$$\mu_0 := \frac{1}{\mathcal{L}^1(A)} \mathcal{L}^1 \llcorner_A, \qquad A \subset I,\ 0 < \mathcal{L}^1(A) < \infty.$$

Then

$$\mu_t := \frac{1}{\mathcal{L}^1((1-t)A)} \mathcal{L}^1 \llcorner_{(1-t)A}$$

is the unique W_2-geodesic connecting μ_0 to δ_o. Hence (2-9) can be applied to

$$\mu_t = \rho_t \mathfrak{m}, \qquad \rho_t(x) = \frac{1}{(1-t)\mathcal{L}^1(A)} \frac{\chi_{(1-t)A}(x)}{h(x)}.$$

Then (2-9) along (μ_t) implies the claim. □

The estimate (2-9) implies several known properties that we collect in what follows. To write them in a unified way we define for $\kappa \in \mathbb{R}$ the function $s_\kappa : [0, +\infty) \to \mathbb{R}$ (on $[0, \pi/\sqrt{\kappa})$ if $\kappa > 0$),

$$s_\kappa(\theta) := \begin{cases} (1/\sqrt{\kappa}) \sin(\sqrt{\kappa}\theta) & \text{if } \kappa > 0, \\ \theta & \text{if } \kappa = 0, \\ (1/\sqrt{-\kappa}) \sinh(\sqrt{-\kappa}\theta) & \text{if } \kappa < 0. \end{cases} \tag{2-10}$$

For the moment we confine ourselves to the case $I = (a, b)$ with $a, b \in \mathbb{R}$; hence (2-9) implies

$$\left(\frac{s_{K/(N-1)}(b-x_1)}{s_{K/(N-1)}(b-x_0)}\right)^{N-1} \leq \frac{h(x_1)}{h(x_0)} \leq \left(\frac{s_{K/(N-1)}(x_1-a)}{s_{K/(N-1)}(x_0-a)}\right)^{N-1} \tag{2-11}$$

for $x_0 \le x_1$ (see the proof of Lemma 2.17 for the easier estimate in the case $K = 0$). Hence denoting by $D = b - a$ the length of I, for any $\varepsilon > 0$ it follows that

$$\sup\left\{\frac{h(x_1)}{h(x_0)} : x_0, x_1 \in [a+\varepsilon, b-\varepsilon]\right\} \le C_\varepsilon, \tag{2-12}$$

where C_ε only depends on K, N, provided $2\varepsilon \le D \le 1/\varepsilon$.

Moreover (2-11) implies that h is locally Lipschitz in the interior of I and an easy manipulation of it (see [Cavalletti and Milman 2016, Lemma A.9]) yields the following bound on the derivative of h:

$$-(N-1)\frac{s'_{K/(N-1)}(b-x)}{s_{K/(N-1)}(b-x)} \le (\log h)'(x) \le (N-1)\frac{s'_{K/(N-1)}(x-a)}{s_{K/(N-1)}(x-a)} \tag{2-13}$$

if $x \in (a, b)$ is a point of differentiability of h. Finally if $K > 0$, then $b - a \le \pi\sqrt{(N-1)/K}$.

Remark 2.14. The estimate (2-11) also implies that an $\mathsf{MCP}(K, N)$ density $h : (a, b) \to (0, \infty)$, $a, b \in \mathbb{R}$, can always be extended to a continuous function on the closed interval $[a, b]$. Notice indeed that the map

$$(a, b) \ni x \mapsto \frac{h(x)}{(s_{K/(N-1)}(b-x))^{N-1}}$$

is nondecreasing and strictly positive. Hence the following limit exists and is a real number:

$$\lim_{x\to a} \frac{h(x)}{(s_{K/(N-1)}(b-x))^{N-1}}.$$

Since $b - a > 0$, we obtain that also the limit $\lim_{x\to a} h(x)$ exists, for every $K \le 0$ and for $K > 0$ provided $b - a \ne \pi\sqrt{(N-1)/K}$. The case $K > 0$ and $b - a = \pi\sqrt{(N-1)/K}$ follows by rigidity: (2-11) implies

$$\frac{\sin(\pi - x_1\sqrt{K/(N-1)})}{\sin(\pi - x_0\sqrt{K/(N-1)})} \le \frac{h(x_1)}{h(x_0)} \le \frac{\sin(x_1\sqrt{K/(N-1)})}{\sin(x_0\sqrt{K/(N-1)})},$$

showing that $h(x)$, up to a renormalisation constant, coincides with $\sin(x\sqrt{K/(N-1)})$. To show that h can also be extended to a continuous function at b, one can argue as above starting from the nonincreasing property of the function

$$(a, b) \ni x \mapsto \frac{h(x)}{(s_{K/(N-1)}(x-a))^{N-1}},$$

following again from (2-11).

The next lemma was stated and proved in [Cavalletti and Milman 2016, Lemma A.8] under the CD condition; as the proof only uses $\mathsf{MCP}(K, N)$ we report it in this more general version.

Lemma 2.15. *Let h denote an $\mathsf{MCP}(K, N)$ density on a finite interval (a, b), $N \in (1, \infty)$, which integrates to* 1. *Then*

$$\sup_{x\in(a,b)} h(x) \le \frac{1}{b-a}\begin{cases} N, & K \ge 0, \\ \left(\int_0^1 (\sigma^{(t)}_{K,N-1}(b-a))^{N-1}\, dt\right)^{-1}, & K < 0. \end{cases} \tag{2-14}$$

In particular, for fixed K and N, h is uniformly bounded from above as long as $b - a$ is uniformly bounded away from 0 *(and from above if $K < 0$).*

From the previous auxiliary results we obtain the following lemma that will be used throughout the paper.

Lemma 2.16. *Let h denote an $\mathsf{MCP}(K,N)$ density on a finite interval (a,b), $N\in(1,\infty)$, which integrates to* 1. *Then*

$$\int_{(a,b)} |h'(x)|\,dx \le \frac{1}{b-a} C^{(K,N)}_{(b-a)} \tag{2-15}$$

for some $C^{(K,N)}_{(b-a)}>0$ with the property that, for fixed $K\in\mathbb{R}$ and $N\in(1,\infty)$, it holds

$$\sup_{r\in(0,R)} C^{(K,N)}_r<\infty \quad \textit{for every } R>0, \qquad \lim_{r\uparrow\infty} C^{(K,N)}_r=\infty. \tag{2-16}$$

Proof. Case 1: $K\le 0$. The two inequalities in (2-13) give for each point $x\in(a,b)$ of differentiability of h

$$\begin{aligned} w_1 &:= h'(x)+(N-1)\frac{s'_{K/(N-1)}(b-x)}{s_{K/(N-1)}(b-x)}h(x)\ge 0,\\ w_2 &:= h'(x)-(N-1)\frac{s'_{K/(N-1)}(x-a)}{s_{K/(N-1)}(x-a)}h(x)\le 0. \end{aligned} \tag{2-17}$$

Thus, we can write

$$\begin{aligned} \int_{[a,b]}|h'|\,dx \le \int_{[a,a+b-a/(2)]} w_1\,dx &+ \int_{[a,a+(b-a)/2]} |w_1-h'|\,dx \\ -\int_{[a+(b-a)/2,b]} w_2\,dx &+ \int_{[a+(b-a)/2,b]} |w_2-h'|\,dx. \end{aligned} \tag{2-18}$$

First of all, observing that for $K\le 0$ one has

$$\frac{s'_{K/(N-1)}(t)}{s_{K/(N-1)}(t)}\ge 0$$

for all $t\ge 0$, we get

$$\begin{aligned} \int_{[a,a+(b-a)/2]} w_1\,dx &\le h\Big(a+\frac{b-a}{2}\Big)-h(a)+(N-1)\|h\|_{L^\infty(a,b)}\log\Big(\frac{s_{K/(N-1)}(b-a)}{s_{K/(N-1)}((b-a)/2)}\Big)\\ &\le C^{(K,N)}_{(b-a)}\|h\|_{L^\infty(a,b)}, \end{aligned}$$

$$\int_{[a,a+(b-a)/2]} |w_1-h'|\,dx=\int_{[a,a+(b-a)/2]}(N-1)\frac{s'_{K/(N-1)}(b-x)}{s_{K/(N-1)}(b-x)}\,h(x)\le C^{(K,N)}_{(b-a)}\|h\|_{L^\infty(a,b)},$$

where $r\mapsto C^{(K,N)}_r$ satisfies (2-16). The bounds for the second line of (2-18) are analogous. Thus we conclude

$$\int_{[a,b]}|h'|\,dx\le C^{(K,N)}_{(b-a)}\|h\|_{L^\infty(a,b)},$$

which, recalling (2-14), gives the claim (2-15).

Case 2: $K>0$. In order to simplify the notation, we assume $K=N-1>0$ (so that $b-a\le\pi$), $a=0$ and $b-a=D\le\pi$. The discussion for general $K>0$, $a<b\in[0,\pi]$ is analogous.

We first consider the case $D \le \pi/2$. Using (2-11), notice that

$$\frac{h(x)}{\sin(D-x)} \le \frac{h(D/2)}{\sin(D/2)} \quad \text{for all } x \in [0, D/2],$$
$$\frac{h(x)}{\sin(x)} \le \frac{h(D/2)}{\sin(D/2)} \quad \text{for all } x \in [D/2, D].$$

For $x \in [0, D/2]$, these yield (recall that $\cos(x) \ge 0$)

$$\omega_0'(x) := h'(x) + \frac{\cos(D-x)}{\sin(D/2)} h(D/2) \ge h'(x) + \frac{\cos(D-x)}{\sin(D-x)} h(x) \ge 0,$$

and for $x \in [D/2, D]$ (recall that $\cos(x) \ge 0$)

$$\omega_1'(x) := h'(x) - \frac{\cos(x)}{\sin(D/2)} h(D/2) \le h'(x) - \frac{\cos(x)}{\sin(x)} h(x) \le 0.$$

Then we can collect all the estimates together:

$$\begin{aligned} \int_{[0,D]} |h'(x)| &\le \int_{[0,D/2]} \omega_0'(x)\,dx + \int_{[0,D/2]} |\omega_0'(x) - h'(x)|\,dx \\ &\quad - \int_{[D/2,D]} \omega_1'(x)\,dx + \int_{[D/2,D]} |\omega_1'(x) - h'(x)|\,dx \\ &\le C\|h\|_{L^\infty(0,D)}. \end{aligned} \tag{2-19}$$

The claim (2-15) then follows applying Lemma 2.15.

If $D > \pi/2$, like in the case $K \le 0$, the two inequalities in (2-13) give for each point $x \in (0, D)$ of differentiability of h

$$h'(x) + (N-1)\frac{\cos(D-x)}{\sin(D-x)} h(x) \ge 0,$$
$$h'(x) - (N-1)\frac{\cos(x)}{\sin(x)} h(x) \le 0.$$

Hence for $x \in (0, D-\pi/2)$ we have $h'(x) \ge 0$ and for $x \in [\pi/2, D]$ we have $h'(x) \le 0$. Then Lemma 2.15 and the bound $D \le \pi$ imply that

$$\begin{aligned} \int_{[0,D-\pi/2]\cup[\pi/2,D]} |h'(x)|\,dx &= \int_{[0,D-\pi/2]} h'(x)\,dx - \int_{[\pi/2,D]} h'(x)\,dx \\ &\le 4 \sup_{[0,D]} |h| \le \frac{4N}{D}. \end{aligned}$$

In order to complete the proof it is then enough to bound $\int_{[D-\pi/2,\pi/2]} |h'(x)|\,dx$. Since (2-19) was obtained for any h MCP-density on $[0, D]$ with $D \le \pi/2$ without using the assumption of $\int h = 1$, it implies

$$\int_{[0,\pi/2]} |h'(x)|\,dx \le C\|h\|_{L^\infty[0,\pi/2]}$$

for any MCP-density on $[0, D]$ with $D \ge \pi/2$. Lemma 2.15 gives the claim. □

In the proof of the splitting theorem for $\mathsf{MCP}(0, N)$ spaces we will use the next lemma.

Lemma 2.17. *Let h be a $\mathsf{MCP}(0, N)$ measure on the whole real line $\mathbb{R}$. Then h is identically equal to a real constant.*

Proof. We show that $h(x_0) = h(x_1)$ for all $x_0, x_1 \in \mathbb{R}$. The $\mathsf{MCP}(0, N)$ condition reads as

$$h(tx_1 + (1-t)x_0) \geq (1-t)^{N-1} h(x_0).$$

For $a < z < b$ apply the previous estimate for $z = x_0$ and $x_1 = b$. It implies

$$\frac{h(tb + (1-t)z)}{h(z)} \geq (1-t)^{N-1};$$

if $w \in (z, b)$ and $w = tb + (1-t)z$ for some $t \in (0, 1)$, then $1 - t = (b - w)/(b - z)$. Plugging in the previous inequality the explicit expression of $(1 - t)$ and repeating the argument taking now $x_0 = a$ and $x_1 = z$, we obtain the next two-sided estimate

$$\left(\frac{b - x_1}{b - x_0}\right)^{N-1} \leq \frac{h(x_1)}{h(x_0)} \leq \left(\frac{x_1 - a}{x_0 - a}\right)^{N-1}, \tag{2-20}$$

valid for all $a \leq x_0 \leq x_1 \leq b$. Since

$$\lim_{b\to+\infty} \left(\frac{b - x_1}{b - x_0}\right)^{N-1} = 1 = \lim_{a\to-\infty} \left(\frac{x_1 - a}{x_0 - a}\right)^{N-1},$$

and since (2-20) holds for all $a \in (-\infty, x_0)$ and all $b \in (x_1, +\infty)$, the thesis follows. $\square$

We now review a few facts about $\mathsf{CD}(K, N)$ densities of the real line (see [Cavalletti and Milman 2016, Appendix]). Given $K \in \mathbb{R}$ and $N \in (1, \infty)$, a nonnegative Borel function h defined on an interval $I \subset \mathbb{R}$ is called a $\mathsf{CD}(K, N)$ density on I if for all $x_0, x_1 \in I$ and $t \in [0, 1]$

$$h^{1/(N-1)}((1-t)x_0 + tx_1) \geq h^{1/(N-1)}(x_0)\sigma^{(1-t)}_{K,N}(|x_1 - x_0|) + h^{1/(N-1)}(x_1)\sigma^{(t)}_{K,N-1}(|x_1 - x_0|). \tag{2-21}$$

A one-dimensional metric measure space, say $(I, |\cdot|, h\mathcal{L}^1)$, satisfies $\mathsf{CD}(K, N)$ if and only h has a continuous representative $\tilde{h}$ that is a $\mathsf{CD}(K, N)$ density.

We will make use of the fact that a $\mathsf{CD}(K, N)$ density $h : I \to [0, \infty)$ is locally semiconcave in the interior; i.e., for all x_0 in the interior of I, there exists $C_{x_0} \in \mathbb{R}$ so that $h(x) - C_{x_0} x^2$ is concave in a neighbourhood of x_0.

Recall moreover that if $f : I \to \mathbb{R}$ denotes a convex function on an open interval $I \subset \mathbb{R}$, it is well known that the left and right derivatives $f^{\prime,-}$ and $f^{\prime,+}$ exist at every point in I and that f is locally Lipschitz; in particular, f is differentiable at a given point if and only if the left and right derivatives coincide. Denoting by $D \subset I$ the differentiability points of f in I, it is also well known that $I \setminus D$ is at most countable. Clearly, all of these results extend to locally semiconvex and locally semiconcave functions as well. We finally recall the next regularisation property for $\mathsf{CD}(K, N)$ densities obtained in [Cavalletti and Milman 2016, Proposition A.10]

Proposition 2.18. *Let h be a $\mathsf{CD}(K, N)$ density on an interval (a, b). Let ψ_ε denote a nonnegative C^2 function supported on $[-\varepsilon, \varepsilon]$ with $\int \psi_\varepsilon = 1$. For any $\varepsilon \in (0, (b-a)/2)$, define the function h^ε on $(a+\varepsilon, b-\varepsilon)$ by*

$$\log h^\varepsilon := \log h * \psi_\varepsilon := \int \log h(y)\psi_\varepsilon(x-y)\, dy.$$

Then h^ε is a C^2-smooth $\mathsf{CD}(K, N)$ density on $(a+\varepsilon, b-\varepsilon)$.

Part I. A representation formula for the Laplacian

3. Transport set and disintegration

Throughout this section we assume $(X, \mathsf{d}, \mathfrak{m})$ to be a metric measure space with $\mathrm{supp}(\mathfrak{m}) = X$ and (X, d) geodesic and proper (and hence complete).

3A. ***Disintegration of σ-finite measures.*** To any 1-Lipschitz function $u : X \to \mathbb{R}$ there is a naturally associated d-cyclically monotone set

$$\Gamma_u := \{(x, y) \in X \times X : u(x) - u(y) = \mathsf{d}(x, y)\}. \tag{3-1}$$

Its transpose is given by $\Gamma_u^{-1} = \{(x, y) \in X \times X : (y, x) \in \Gamma_u\}$. We define the *transport relation* R_u and the *transport set* $\mathcal{T}_u$ as

$$R_u := \Gamma_u \cup \Gamma_u^{-1}, \quad \mathcal{T}_u := P_1(R_u \setminus \{x = y\}), \tag{3-2}$$

where $\{x = y\}$ denotes the diagonal $\{(x, y) \in X^2 : x = y\}$ and P_i is the projection onto the i-th component. Recall that $\Gamma_u(x) = \{y \in X : (x, y) \in \Gamma_u\}$ denotes the section of Γ_u through x in the first coordinate, and similarly for $R_u(x)$ (through either of the coordinates by symmetry). Since u is 1-Lipschitz, Γ_u, Γ_u^{-1} and R_u are closed sets, and so are $\Gamma_u(x)$ and $R_u(x)$.

Also recall the following definitions, introduced in [Cavalletti 2014]:

$$A_+ := \{x \in \mathcal{T}_u : \text{there exists } z, w \in \Gamma_u(x) \text{ such that } (z, w) \notin R_u\},$$
$$A_- := \{x \in \mathcal{T}_u : \text{there exists } z, w \in \Gamma_u^{-1}(x) \text{ such that } (z, w) \notin R_u\}.$$

$A_\pm$ are called the *sets of forward and backward branching points*, respectively. If $x \in A_+$ and $(y, x) \in \Gamma_u$ then necessarily also $y \in A_+$ (as $\Gamma_u(y) \supset \Gamma_u(x)$ by the triangle inequality); similarly, if $x \in A_-$ and $(x, y) \in \Gamma_u$ then necessarily $y \in A_-$.

Consider the *nonbranched transport set*

$$\mathcal{T}_u^{\mathrm{nb}} := \mathcal{T}_u \setminus (A_+ \cup A_-), \tag{3-3}$$

and define the *nonbranched transport relation*

$$R_u^{\mathrm{nb}} := R_u \cap (\mathcal{T}_u^{\mathrm{nb}} \times \mathcal{T}_u^{\mathrm{nb}}).$$

In was shown in [Cavalletti 2014] (see also [Bianchini and Cavalletti 2013]) that R_u^{nb} is an equivalence relation over $\mathcal{T}_u^{\mathrm{nb}}$ and that for any $x \in \mathcal{T}_u^{\mathrm{nb}}$, $R_u(x) \subset (X, \mathsf{d})$ is isometric to a closed interval in $(\mathbb{R}, |\cdot|)$, and $R_u^{\mathrm{nb}}(x) \subset (X, \mathsf{d})$ is isometric to either a closed, semiclosed or open interval in $(\mathbb{R}, |\cdot|)$.

Therefore, from the nonbranched transport relation R_u^{nb}, one obtains a partition of the nonbranched transport set $\mathcal{T}_u^{\mathrm{nb}}$ into a disjoint family (of equivalence classes) $\{X_\alpha\}_{\alpha \in Q}$, each of them isometric to a real interval (depending on the situation, the interval can be bounded or unbounded, closed, semiclosed or open). Here Q is any set of indices. Concerning the measurability, as the space (X, d) is proper, $\mathcal{T}_u$ and $A_\pm$ are σ-compact sets and, consequently, $\mathcal{T}_u^{\mathrm{nb}}$ and R_u^{nb} are Borel.

Remark 3.1 (initial and final points). It will be useful to isolate two families of distinguished points of the transport set, the *sets of initial and final points*, respectively:

$$a := \{x \in \mathcal{T}_u : \text{there does not exist } y \in \mathcal{T}_u,\ y \neq x, \text{ such that } (y, x) \in R_u\},$$
$$b := \{x \in \mathcal{T}_u : \text{there does not exist } y \in \mathcal{T}_u,\ y \neq x, \text{ such that } (x, y) \in R_u\}.$$

Notice that no inclusion of the form $a \subset A_+,\ b \subset A_-$ is valid. For instance consider

$$X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$$

endowed with the Euclidean distance and

$$u(x) := \mathrm{dist}(x, \{x_1 = 0\});$$

then $a = \{x_1 = 0\}$ and $A_\pm = \varnothing$. In particular, sets a and b may or may not be subsets of $\mathcal{T}_u^{\mathrm{nb}}$. See also the discussion right above (1-2). Curvature assumptions will, however, imply that a and b have measure zero. We will also use the notation $a(X_\alpha), b(X_\alpha)$ to denote the starting and final points, respectively, of the transport set X_α, whenever they exist.

Once a partition of the nonbranched transport set $\mathcal{T}_u^{\mathrm{nb}}$ is at our disposal, a decomposition of the reference measure $\mathfrak{m}\llcorner_{\mathcal{T}_u^{\mathrm{nb}}}$ can be obtained using the disintegration theorem. In the recent literature of optimal transportation, the disintegration formulas have always been obtained under the additional assumption of finiteness of the measure $\mathfrak{m}(X) < \infty$. We will therefore spend few words on how to use disintegration theorem to obtain a disintegration associated to the family of transport rays without assuming $\mathfrak{m}(X) < \infty$.

We first introduce the quotient map $\mathfrak{Q} : \mathcal{T}_u^{\mathrm{nb}} \to Q$ induced by the partition

$$\alpha = \mathfrak{Q}(x) \quad \Longleftrightarrow \quad x \in X_\alpha. \tag{3-4}$$

The set of indices (or quotient set) Q can be endowed with the quotient σ-algebra $\mathscr{Q}$ (of the σ-algebra $\mathscr{X}$ over X of $\mathfrak{m}$-measurable subsets),

$$C \in \mathscr{Q} \quad \Longleftrightarrow \quad \mathfrak{Q}^{-1}(C) \in \mathscr{X},$$

i.e., the finest σ-algebra on Q such that $\mathfrak{Q}$ is measurable.

The set of indices Q can be identified with any subset of $\overline{Q} \subset X$ satisfying the following two properties:

- For all $x \in \mathcal{T}_u^{\mathrm{nb}}$ there exists a unique $\bar{x} \in \overline{Q}$ such that $(x, \bar{x}) \in R_u^{\mathrm{nb}}$.
- If $x, y \in \mathcal{T}_u^{\mathrm{nb}}$ and $(x, y) \in R_u^{\mathrm{nb}}$, then $\bar{x} = \bar{y}$.

In particular $\overline{Q}$ has to contain a single element for each equivalence class X_α.

Another way to obtain a quotient set is to look instead first for an explicit quotient map: in particular, any map $\overline{\mathfrak{Q}} : \mathcal{T}_u^{\mathrm{nb}} \to \mathcal{T}_u^{\mathrm{nb}}$ satisfying the properties

- $(x, \bar{\mathfrak{Q}}(x)) \in R_u^{\mathrm{nb}}$,
- if $(x, y) \in R_u^{\mathrm{nb}}$, then $\bar{\mathfrak{Q}}(x) = \bar{\mathfrak{Q}}(y)$,

will be a quotient map for the equivalence relation R_u^{nb} over $\mathcal{T}_u^{\mathrm{nb}}$; then the quotient set associated to $\bar{\mathfrak{Q}}$ will be the set $\{x \in R_u^{\mathrm{nb}} : x = \bar{\mathfrak{Q}}(x)\}$.

Existence of $\bar{Q}$ or of $\bar{\mathfrak{Q}}$ can be always deduced by the axiom of choice. However, in order to apply the disintegration theorem, measurability properties are needed.

A rather explicit construction of the quotient map has been already obtained under the additional assumption of $\mathfrak{m}(X) < \infty$ (see [Cavalletti and Mondino 2017b; 2018, Lemma 3.8]); however, $\mathfrak{m}(X) < \infty$ did not play any role in the proof and we therefore simply report the next statement.

We will denote by $\mathcal{A}$ the σ-algebra generated by the analytic sets of X.

Lemma 3.2 (Q is locally contained in level sets of u). *There exists an $\mathcal{A}$-measurable quotient map $\mathfrak{Q} : \mathcal{T}_u^{\mathrm{nb}} \to Q$ such that the quotient set $Q \subset X$ is $\mathcal{A}$-measurable and can be written locally as a level set of u in the following sense*:

$$Q = \bigcup_{n \in \mathbb{N}} Q_n, \qquad Q_n \subset u^{-1}(l_n),$$

where $l_n \in \mathbb{Q}$ and $Q_i \cap Q_j = \varnothing$ for $i \neq j$.

Lemma 3.2 allows us to apply the disintegration theorem (see [Cavalletti and Milman 2016, Section 6.3]), provided the ambient measure $\mathfrak{m}$ is suitably modified into a finite measure. To this aim, the next elementary lemma will be useful.

Lemma 3.3. *Let $\mathfrak{m}$ be a σ-finite measure over the proper metric space (X, d) with $\operatorname{supp}(\mathfrak{m}) = X$. Then there exists a Borel function $f : X \to (0, \infty)$ satisfying*

$$\inf_{\mathcal{K}} f > 0 \quad \textit{for any compact subset } \mathcal{K} \subset X, \qquad \int_{\mathcal{T}_u^{\mathrm{nb}}} f \, \mathfrak{m} = 1. \tag{3-5}$$

Proof. Since by assumption (X, d) is proper, for every $x_0 \in X$ and $R > 0$ the closed metric ball $\bar{B}_R(x_0)$ is compact. Thus, using that $\mathfrak{m}$ is σ-finite and $\operatorname{supp}(\mathfrak{m}) = X$, we get

$$0 < \mathfrak{m}(B_n(x_0) \setminus B_{n-1}(x_0)) < \infty \quad \text{for all } n \in \mathbb{N}_{\geq 1}.$$

It is then readily checked that $f : X \to (0, \infty)$ defined by

$$f := \frac{1}{2^n \mathfrak{m}(B_n(x_0) \setminus B_{n-1}(x_0))}$$

on $B_{n+1}(x_0) \setminus B_n(x_0)$ for all $n \in \mathbb{N}_{\geq 1}$ satisfies (3-5). □

Under the assumption that $\mathfrak{m}$ is σ-finite, let $f : X \to (0, \infty)$ satisfy (3-5), set

$$\mu := f \mathfrak{m}\llcorner_{\mathcal{T}_u^{\mathrm{nb}}}, \tag{3-6}$$

and define the normalised quotient measure

$$\mathfrak{q} := \mathfrak{Q}_\sharp \mu. \tag{3-7}$$

Notice that $\mathfrak{q}$ is a Borel probability measure over X. It is straightforward to check that

$$\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}_u^{\mathrm{nb}}}) \ll \mathfrak{q}.$$

Take indeed $E \subset Q$ with $\mathfrak{q}(E) = 0$; then by definition $\int_{\mathfrak{Q}^{-1}(E)} f(x)\,\mathfrak{m}(dx) = 0$, implying $\mathfrak{m}(\mathfrak{Q}^{-1}(E)) = 0$, since $f > 0$.

From the disintegration theorem [Fremlin 2003, Section 452], we deduce the existence of a map

$$Q \ni \alpha \mapsto \mu_\alpha \in \mathcal{P}(X)$$

satisfying the following properties:

(1) For any μ-measurable set $B \subset X$, the map $\alpha \mapsto \mu_\alpha(B)$ is $\mathfrak{q}$-measurable.

(2) For $\mathfrak{q}$-a.e. $\alpha \in Q$, μ_α is concentrated on $\mathfrak{Q}^{-1}(\alpha)$.

(3) For any μ-measurable set $B \subset X$ and $\mathfrak{q}$-measurable set $C \subset Q$, the following disintegration formula holds:
$$\mu(B \cap \mathfrak{Q}^{-1}(C)) = \int_C \mu_\alpha(B)\,\mathfrak{q}(d\alpha).$$

Finally the disintegration is $\mathfrak{q}$-essentially unique; i.e., if any other map $Q \ni \alpha \mapsto \bar{\mu}_\alpha \in \mathcal{P}(X)$ satisfies the previous three points, then

$$\bar{\mu}_\alpha = \mu_\alpha \quad \mathfrak{q}\text{-a.e. } \alpha \in Q.$$

Hence once $\mathfrak{q}$ is given (recall that $\mathfrak{q}$ depends on f from Lemma 3.3), the disintegration is unique up to a set of $\mathfrak{q}$-measure zero. In the case $\mathfrak{m}(X) < \infty$, the natural choice, that we tacitly assume, is to take as f the characteristic function of $\mathcal{T}_u^{\mathrm{nb}}$ divided by $\mathfrak{m}(\mathcal{T}_u^{\mathrm{nb}})$ so that $\mathfrak{q} := \mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}_u^{\mathrm{nb}}}/\mathfrak{m}(\mathcal{T}_u^{\mathrm{nb}}))$.

All the previous properties will be summarised saying that $Q \ni \alpha \mapsto \mu_\alpha$ is a *disintegration of* μ *strongly consistent with respect to* $\mathfrak{Q}$.

It follows from [Fremlin 2003, Proposition 452F] that

$$\int_X g(x)\,\mu(dx) = \int_Q \int g(x)\,\mu_\alpha(dx)\,\mathfrak{q}(d\alpha)$$

for every $g : X \to \mathbb{R} \cup \{\pm\infty\}$ such that $\int g\,\mu$ is well-defined in $\mathbb{R} \cup \{\pm\infty\}$. Hence picking $g = 1/f$ (where f is the one used to define μ), we get

$$\mathfrak{m}\llcorner_{\mathcal{T}_u^{\mathrm{nb}}} = \int_Q \frac{\mu_\alpha}{f}\,\mathfrak{q}(d\alpha); \tag{3-8}$$

the previous identity has to be understood with test functions as the previous formula.

Defining $\mathfrak{m}_\alpha := \mu_\alpha/f$, we obtain that $\mathfrak{m}_\alpha$ is a nonnegative Radon measure over X satisfying all the measurability properties (with respect to $\alpha \in Q$) of μ_α and giving a disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}_u^{\mathrm{nb}}}$ strongly consistent with respect to $\mathfrak{Q}$. Moreover, for every compact subset $\mathcal{K} \subset X$, it holds

$$\frac{1}{\sup_{\mathcal{K}} f}\mu_\alpha(\mathcal{K}) \le \mathfrak{m}_\alpha(\mathcal{K}) = \frac{\mu_\alpha}{f}(\mathcal{K}) \le \frac{1}{\inf_{\mathcal{K}} f} \quad \text{for } \mathfrak{q}\text{-a.e. } \alpha \in Q. \tag{3-9}$$

In the next statement, we summarise what we have obtained so far concerning the disintegration of a σ-finite reference measure $\mathfrak{m}$ with respect to the nonbranched transport relation induced by any 1-Lipschitz function $u : X \to \mathbb{R}$.

We denote by $\mathcal{M}_+(X)$ the space of nonnegative Radon measures over X.

Theorem 3.4. *Let $(X, \mathsf{d}, \mathfrak{m})$ be any geodesic and proper (hence complete) m.m.s. with $\operatorname{supp}(\mathfrak{m}) = X$ and $\mathfrak{m}$ σ-finite. Then for any 1-Lipschitz function $u : X \to \mathbb{R}$, the measure $\mathfrak{m}$ restricted to the nonbranched transport set $\mathcal{T}_u^{\mathrm{nb}}$ admits the disintegration formula*

$$\mathfrak{m}\llcorner_{\mathcal{T}_u^{\mathrm{nb}}} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha),$$

where $\mathfrak{q}$ is a Borel probability measure over $Q \subset X$ such that $\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}_u^{\mathrm{nb}}}) \ll \mathfrak{q}$ and the map $Q \ni \alpha \mapsto \mathfrak{m}_\alpha \in \mathcal{M}_+(X)$ satisfies the following properties:

(1) *For any $\mathfrak{m}$-measurable set B, the map $\alpha \mapsto \mathfrak{m}_\alpha(B)$ is $\mathfrak{q}$-measurable.*

(2) *For $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha$ is concentrated on $\mathfrak{Q}^{-1}(\alpha) = R_u^{\mathrm{nb}}(\alpha)$ (strong consistency).*

(3) *For any $\mathfrak{m}$-measurable set B and $\mathfrak{q}$-measurable set C, the following disintegration formula holds:*

$$\mathfrak{m}(B \cap \mathfrak{Q}^{-1}(C)) = \int_C \mathfrak{m}_\alpha(B) \, \mathfrak{q}(d\alpha).$$

(4) *For every compact subset $\mathcal{K} \subset X$ there exists a constant $C_{\mathcal{K}} \in (0, \infty)$ such that*

$$\mathfrak{m}_\alpha(\mathcal{K}) \le C_{\mathcal{K}} \quad \textit{for } \mathfrak{q}\textit{-a.e. } \alpha \in Q.$$

Moreover, for any $\mathfrak{q}$ as above such that $\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}_u^{\mathrm{nb}}}) \ll \mathfrak{q}$, the disintegration is $\mathfrak{q}$-essentially unique (see above).

3B. *Localisation of Ricci bounds.* Under the additional assumption of a synthetic lower bound on the Ricci curvature, one can obtain regularity properties both on $\mathcal{T}_u^{\mathrm{nb}}$ and on the conditional measures $\mathfrak{m}_\alpha$. As some of these results were obtained assuming $\mathfrak{m}(X) < \infty$, in what follows we review how to obtain the same regularity with no finiteness assumption on $\mathfrak{m}$. First of all recall that, for any $K \in \mathbb{R}$ and $N \in (1, \infty)$, $\mathsf{CD}(K, N)$ implies $\mathsf{MCP}(K, N)$, which in turn implies that $\mathfrak{m}$ is σ-finite. Thus Theorem 3.4 can be applied.

Lemma 3.5. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially nonbranching m.m.s. with $\operatorname{supp}(\mathfrak{m}) = X$ and satisfying $\mathsf{MCP}(K, N)$ for some $K \in \mathbb{R}$, $N \in (1, \infty)$. Then for any 1-Lipschitz function $u : X \to \mathbb{R}$, it holds $\mathfrak{m}(\mathcal{T}_u \setminus \mathcal{T}_u^{\mathrm{nb}}) = 0$.*

Lemma 3.5 has been proved in [Cavalletti 2014] for metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$ satisfying $\mathsf{RCD}(K, N)$ with $N < \infty$ and $\operatorname{supp}(\mathfrak{m}) = X$. The $\mathsf{RCD}(K, N)$ assumption was used in that proof only to have at our disposal the following property: given $\mu_0, \mu_1 \in \mathcal{P}(X)$ with $\mu_0 \ll \mathfrak{m}$, there exists a unique optimal dynamical plan for the W_2-distance and it is induced by a map. In [Cavalletti and Mondino 2017a, Theorem 1.1] this property is also satisfied by an e.n.b. metric measure space satisfying $\mathsf{MCP}(K, N)$

with $\mathrm{supp}(\mathfrak{m}) = X$, without any finiteness assumption on $\mathfrak{m}$. Hence Lemma 3.5 can be proved following verbatim [Cavalletti 2014].

Building on [Cavalletti and Mondino 2017a], in [Cavalletti and Milman 2016, Theorem 7.10] additional information on the transport rays was proved: for $\mathfrak{q}$-a.e. $\alpha \in Q$ it holds

$$R_u(\alpha) = \overline{R_u^{\mathrm{nb}}(\alpha)} \supset R_u^{\mathrm{nb}}(\alpha) \supset \mathring{R}_u(\alpha), \tag{3-10}$$

with the latter to be interpreted as the relative interior. The additional assumption of $\mathfrak{m}(X) < \infty$ was used in the proof only to obtain the existence of a disintegration of $\mathfrak{m}$ strongly consistent with the nonbranched equivalence relation. Hence from Theorem 3.4 also (3-10) is valid in the present framework.

To conclude, we assert that the localisation results for the synthetic Ricci curvature lower bounds $\mathsf{MCP}(K, N)$ and $\mathsf{CD}(K, N)$, with $K, N \in \mathbb{R}$ and $N > 1$, are valid also in our framework.

• *Localisation of* $\mathsf{MCP}(K, N)$. In [Bianchini and Cavalletti 2013, Theorem 9.5], assuming nonbranching and the $\mathsf{MCP}(K, N)$ condition, it is proved (adopting slightly different notation) that for $\mathfrak{q}$-a.e. $\alpha \in Q$ it holds

$$\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha},$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. Moreover, the one-dimensional metric measure space $(\bar{X}_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$, isomorphic to $([0, D_\alpha], |\cdot|, h_\alpha \mathcal{L}^1)$, is proved to satisfy $\mathsf{MCP}(K, N)$; here $\bar{X}_\alpha$ stands for the closure of the transport ray X_α with respect to d. Note that $\bar{X}_\alpha$ might not be a subset of $\mathcal{T}_u^{\mathrm{nb}}$ because of its endpoints but this will not affect any argument as $\mathfrak{m}_\alpha(\bar{X}_\alpha \setminus X_\alpha) = 0$. No finiteness assumption was assumed in [Bianchini and Cavalletti 2013, Theorem 9.5] and, since here we restrict to the nonbranched transport set, the arguments can be carried over to give the same statement.

• *Localisation of* $\mathsf{CD}(K, N)$. The localisation of $\mathsf{CD}(K, N)$ was proved in [Cavalletti and Mondino 2017a, Theorem 5.1] under the assumption $\mathfrak{m}(X) = 1$. Nevertheless, in that work the $\mathsf{CD}(K, N)$ condition was assumed to be valid only locally; i.e., the space was assumed to satisfy $\mathsf{CD}_{\mathrm{loc}}(K, N)$. In particular the proof first shows that the one-dimensional metric measure space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ satisfies $\mathsf{CD}_{\mathrm{loc}}(K, N)$ for $\mathfrak{q}$-a.e. $\alpha \in Q$ and then, thanks to the local-to-global property of one-dimensional $\mathsf{CD}(K, N)$ condition, concludes with the full claim. Hence, if $(X, \mathsf{d}, \mathfrak{m})$ is e.n.b. and satisfies $\mathsf{CD}(K, N)$, since by Theorem 3.4 a disintegration formula is at our disposal and the reference measure $\mathfrak{m}$ is locally finite, one can repeat the arguments in [Cavalletti and Mondino 2017a, Theorem 5.1] and obtain that the one-dimensional metric measure space $(\bar{X}_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$, isomorphic to $([0, D_\alpha], |\cdot|, h_\alpha \mathcal{L}^1)$, satisfies $\mathsf{CD}(K, N)$.

We summarise the above discussion in the next statement.

Theorem 3.6. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an essentially nonbranching m.m.s. with* $\mathrm{supp}(\mathfrak{m}) = X$ *and satisfying* $\mathsf{MCP}(K, N)$ *for some* $K \in \mathbb{R}$, $N \in (1, \infty)$.

Then, for any 1*-Lipschitz function* $u : X \to \mathbb{R}$, *there exists a disintegration of* $\mathfrak{m}$ *strongly consistent with* R_u^{nb} *satisfying*

$$\mathfrak{m}\llcorner_{\mathcal{T}_u^{\mathrm{nb}}} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha), \qquad \mathfrak{q}(Q) = 1.$$

Moreover, for $\mathfrak{q}$*-a.e.* α*,* $\mathfrak{m}_\alpha$ *is a Radon measure with* $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha} \ll \mathcal{H}^1 \llcorner_{X_\alpha}$ *and* $(\bar{X}_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ *satisfies* $\mathsf{MCP}(K, N)$.

If, additionally, $(X, \mathsf{d}, \mathfrak{m})$ *satisfies* $\mathsf{CD}_{\mathrm{loc}}(K, N)$*, then* h_α *is a* $\mathsf{CD}(K, N)$ *density on* X_α *for* $\mathfrak{q}$*-a.e.* α*.*

It is worth recalling that, once we know that $(\bar{X}_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ satisfies $\mathsf{MCP}(K, N)$, it is straightforward to get that $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha}$ for some density h_α. We refer to Section 2C for all the properties satisfied by one-dimensional metric measure spaces satisfying lower Ricci curvature bounds.

We conclude the section by specialising the results to the smooth framework of Riemannian manifolds (cf. [Klartag 2017]).

Corollary 3.7. *Let* (M, g) *be a complete* N*-dimensional Riemannian manifold, where* $N \geq 2$*, and let* $\mathfrak{m}$ *denote its Riemannian volume measure.*

Then, for any 1*-Lipschitz function* $u : M \to \mathbb{R}$*, there exists a disintegration of* $\mathfrak{m}$ *strongly consistent with* R_u^{nb} *satisfying*

$$\mathfrak{m} \llcorner_{\mathcal{T}_u^{\mathrm{nb}}} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha), \qquad \mathfrak{q}(Q) = 1.$$

Moreover:

(1) *For* $\mathfrak{q}$*-a.e.* α*,* $\mathfrak{m}_\alpha$ *is a Radon measure with* $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha} \ll \mathcal{H}^1 \llcorner_{X_\alpha}$*.*

(2) *For every* $x \in M$ *there exist a (compact, geodesically convex) neighbourhood* U *of* x *and* $\bar{K} \in \mathbb{R}$ *such that* $h_\alpha \llcorner_U$ *is a* $\mathsf{CD}(\bar{K}, N)$ *density on* $X_\alpha \cap U$ *for* $\mathfrak{q}$*-a.e.* α*.*

(3) *If, additionally,* $\mathrm{Ric}_g \geq K g$ *for some* $K \in \mathbb{R}$*, then* h_α *is a* $\mathsf{CD}(K, N)$ *density on* X_α *for* $\mathfrak{q}$*-a.e.* α*.*

Proof. The corollary follows directly from Theorems 3.4 and 3.6, reasoning as follows. A complete Riemannian manifold is geodesic and proper; hence Theorem 3.4 implies the first part of the claim,

$$\mathfrak{m} \llcorner_{\mathcal{T}_u^{\mathrm{nb}}} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha), \qquad \mathfrak{q}(Q) = 1,$$

and for $\mathfrak{q}$-a.e. α, $\mathfrak{m}_\alpha$ is a Radon measure with $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha} \ll \mathcal{H}^1 \llcorner_{X_\alpha}$.

Moreover every point $x \in M$ admits a geodesically convex compact neighbourhood U where, by compactness, the Ricci tensor is bounded below by some $\bar{K} \in \mathbb{R}$. In particular $(U, \mathsf{d}, \mathfrak{m} \llcorner_U)$ is an essentially nonbranching $\mathsf{CD}(\bar{K}, N)$ space and thus we can apply Theorem 3.6 to $(U, \mathsf{d}, \mathfrak{m} \llcorner_U)$. Since the partition associated to $u : U \to \mathbb{R}$ is given by the restriction of transport rays, the quotient measure of $\mathfrak{m}$ restricted to $U \cap \mathcal{T}_u^{\mathrm{nb}}$ will be absolutely continuous with respect to $\mathfrak{q}$; hence by $\mathfrak{q}$-essential uniqueness of disintegration we deduce that $h_\alpha \llcorner_U$ is a $\mathsf{CD}(\bar{K}, N)$ density on $X_\alpha \cap U$ for $\mathfrak{q}$-a.e. α. The third claim is already contained in Theorem 3.6. □

4. Representation formula for the Laplacian

From now on we will assume $(X, \mathsf{d}, \mathfrak{m})$ to be an e.n.b. metric measure space satisfying $\mathsf{MCP}(K, N)$ for some $K \in \mathbb{R}$ and $N \in (1, \infty)$. In particular $(X, \mathsf{d}, \mathfrak{m})$ is a locally doubling and Poincaré space (recall Remark 2.6).

We will obtain an explicit representation formula for the Laplacian for a general 1-Lipschitz function

$$u : X \to \mathbb{R}, \quad |u(x) - u(y)| \le \mathsf{d}(x, y),$$

assuming a mild regularity property on $\mathcal{T}_u$, the associated transport set defined in Section 3.

A distinguished role will be played by a particular family of 1-Lipschitz functions, namely the so-called signed distance functions. Such a class played a key role in the recent proof [Cavalletti and Milman 2016] of the local-to-global property of $\mathsf{CD}(K, N)$ under the e.n.b. assumption.

Definition 4.1 (signed distance function). Given a continuous function $v : (X, \mathsf{d}) \to \mathbb{R}$ so that $\{v = 0\} \neq \varnothing$, the function

$$d_v : X \to \mathbb{R}, \quad d_v(x) := \mathsf{d}(x, \{v = 0\}) \operatorname{sgn}(v), \tag{4-1}$$

is called the signed distance function (from the zero-level set of v).

With a slight abuse of notation, we denote by d both the distance between points and the induced distance between sets; more precisely

$$\mathsf{d}(x, \{v = 0\}) := \inf\{\mathsf{d}(x, y) : y \in \{v = 0\}\}.$$

Lemma 4.2. *The signed distance function d_v is 1-Lipschitz on $\{v \ge 0\}$ and $\{v \le 0\}$. If (X, d) is a length space, then d_v is 1-Lipschitz on the entire X.*

For the proof we refer to [Cavalletti and Milman 2016, Lemma 8.4].

We now fix once and for all a 1-Lipschitz function $u : X \to \mathbb{R}$. In order not to have empty statements, throughout the section we will assume that $\mathfrak{m}(\mathcal{T}_u) > 0$.

4A. ***Representing the gradient of $-u$.*** The translation along $\mathcal{T}_u^{\mathrm{nb}}$ is defined as

$$g : \mathbb{R} \times \mathcal{T}_u^{\mathrm{nb}} \to \mathcal{T}_u^{\mathrm{nb}} \subset X, \quad \operatorname{graph}(g) = \{(t, x, y) \in \mathbb{R} \times R_u^{\mathrm{nb}} : u(x) - u(y) = t\}.$$

Since R_u^{nb} is Borel, the same applies to g, while $\operatorname{Dom}(g) = P_{12}(\operatorname{graph}(g))$ is analytic. We will write g_t for $g(t, \cdot)$. Notice that

$$\operatorname{graph}(g_t) = \{(x, y) \in R_u^{\mathrm{nb}} : u(x) - u(y) = t\}$$

is Borel as well and thus for $t \in \mathbb{R}$

$$\operatorname{Dom}(g_t) = \mathcal{T}_u^{\mathrm{nb}}(t) := \{x \in \mathcal{T}_u^{\mathrm{nb}} : \text{there exists } y \in R_u^{\mathrm{nb}}(x) \text{ with } u(x) - u(y) = t\}$$

is an analytic set. The rough intuitive picture is of course that g_t plays the role of negative gradient flow of u, restricted to the points of maximal slope 1. In order to handle the case when $\mathfrak{m}(\mathcal{T}_u^{\mathrm{nb}}) = +\infty$, it is useful to introduce the following definition.

Definition 4.3. A measurable subset $E \subset X$ is said to be R_u^{nb}*-convex* if for any $x \in \mathcal{T}_u^{\mathrm{nb}}$ the set $E \cap R_u^{\mathrm{nb}}(x)$ is isometric to an interval.

For every bounded R_u^{nb}-convex subset $E \subset \mathcal{T}_u^{\mathrm{nb}}(2\varepsilon)$ with $\mathfrak{m}(E) > 0$, consider the function $\Lambda : E \to C([0,1]; X)$ defined by

$$[0,1] \ni \tau \mapsto \Lambda(x)_\tau := \begin{cases} g_\tau(x), & \tau \in [0,\varepsilon], \\ g_\varepsilon(x), & \tau \in [\varepsilon, 1], \end{cases}$$

and set

$$\pi_E := \frac{1}{\mathfrak{m}(E)} \Lambda_\sharp \mathfrak{m}\llcorner_E. \tag{4-2}$$

Note that

$$\mathfrak{m}(E)(\mathrm{e}_\tau)_\sharp \pi_E = (\mathrm{e}_\tau \circ \Lambda)_\sharp \mathfrak{m}\llcorner_E = \begin{cases} (g_\tau)_\sharp \mathfrak{m}\llcorner_E =: \mathfrak{m}_E^\tau, & \tau \in [0,\varepsilon], \\ (g_\varepsilon)_\sharp \mathfrak{m}\llcorner_E =: \mathfrak{m}_E^\varepsilon, & \tau \in [\varepsilon, 1]. \end{cases} \tag{4-3}$$

The rough intuitive idea is of course that $\mathfrak{m}_E^\tau$ is the push forward of $\mathfrak{m}\llcorner_E$ via the negative gradient flow of u at time τ.

Proposition 4.4. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an e.n.b. metric measure space satisfying $\mathsf{MCP}(K,N)$ and u be as before. For every bounded R_u^{nb}-convex subset $E \subset \mathcal{T}_u^{\mathrm{nb}}(2\varepsilon)$ with $\mathfrak{m}(E) > 0$, the measure π_E defined in (4-2) is a test plan representing the gradient of $-u$ (see Definition 2.9).*

Proof. Fix $t \in [0,\varepsilon]$. First of all write

$$\mathfrak{m}(E)(\mathrm{e}_t)_\sharp \pi_E = \mathfrak{m}_E^t = \int_Q (g_t)_\sharp \mathfrak{m}_\alpha\llcorner_E \, \mathfrak{q}(d\alpha). \tag{4-4}$$

Since $\mathfrak{m}_\alpha\llcorner_E = h_\alpha \mathcal{H}^1\llcorner_{X_\alpha \cap E}$, we have

$$(g_t)_\sharp \mathfrak{m}_\alpha\llcorner_E = \frac{h_\alpha \circ g_{-t}}{h_\alpha} \mathfrak{m}_\alpha\llcorner_{g_t(E)}. \tag{4-5}$$

Identifying $X_\alpha \cap \left(\bigcup_{t\in[0,\varepsilon]} g_t(E)\right)$ with an interval $[a_\alpha, b_\alpha] \subset \mathbb{R}$ (for the sake of the argument we assume the interval to be closed, but all the other cases are completely analogous), from (2-11), for $x \in [a_\alpha + t, b_\alpha - 2\varepsilon + t]$ and $t \le \varepsilon$ it holds

$$\frac{h_\alpha(x-t)}{h_\alpha(x)} \le \left[\frac{s_{K/(N-1)}(b_\alpha - x + t)}{s_{K/(N-1)}(b_\alpha - x)} \right]^{N-1} \le C_\varepsilon \quad \text{for all } x \in [a_\alpha + t, b_\alpha - 2\varepsilon + t] \text{ and } t \le \varepsilon, \tag{4-6}$$

where the last inequality follows from the fact that $b_\alpha - x \ge 2\varepsilon - t \ge \varepsilon > 0$. We stress that $C_\varepsilon > 0$ is independent of $\alpha \in Q$. The combination of (4-4), (4-5) and (4-6) gives that

$$(\mathrm{e}_t)_\sharp \pi \le \frac{C_\varepsilon}{\mathfrak{m}(E)} \mathfrak{m}$$

for all $t \in [0,1]$; i.e., π_E has bounded compression. Moreover since $\mathfrak{m}(E) < \infty$, and $|\dot\gamma| = 1$ for π-a.e. γ, it follows that π_E is a test plan (Definition 2.8).

We now prove that π_E represents the gradient of $-u$. Since by construction $u(x) - u(g_\tau(x)) = \tau$ for $\mathfrak{m}\llcorner E$-a.e. x, we have

$$\liminf_{\tau\to 0} \int \frac{u(\gamma_0) - u(\gamma_\tau)}{\tau} \pi_E(d\gamma) = \frac{1}{\mathfrak{m}(E)} \liminf_{\tau \to 0} \int_E \frac{u(x) - u(g_\tau(x))}{\tau} \mathfrak{m}(dx) = 1.$$

Hence the claim (recall Definition 2.9) follows by the fact that the 1-Lipschitz regularity of u implies $|Du| \le 1$ $\mathfrak{m}$-a.e. and thus

$$1 \ge \frac{1}{2\mathfrak{m}(E)} \int_{\mathcal{T}_u^{\rm nb}(2\varepsilon)} |Du|^2(x)\, \mathfrak{m}(dx) + \tfrac{1}{2}\pi_E(C([0,1];X)). \qquad \square$$

In the next statement and in the rest of the paper, we will often consider the restriction of a Lipschitz function f to some transport ray $R_u^{\rm nb}(\alpha)$ giving a real variable Lipschitz function: $[a_\alpha, b_\alpha] \ni t \mapsto f(g(t, a_\alpha))$. It will make sense then to compute the t-derivative of the previous map: whenever it exists, we will use the notation

$$f'(x) := \lim_{t\to 0} \frac{f(g(t,x)) - f(x)}{t}. \tag{4-7}$$

Note that f' is roughly the directional derivative of f “in the direction of $-\nabla u$”. Observe that, if $(X, \mathsf{d}, \mathfrak{m})$ is $\mathsf{MCP}(K,N)$ e.n.b., for every $f \in \mathrm{LIP}(X)$ the quantity f' is well-defined $\mathfrak{m}$-a.e. on $\mathcal{T}_u$.

Theorem 4.5. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an e.n.b. metric measure space satisfying $\mathsf{MCP}(K,N)$ and u be as before. Then for any Lipschitz function $f : X \to \mathbb{R}$ it holds*

$$D^- f(-\nabla u) \le f' \le D^+ f(-\nabla u) \quad \mathfrak{m}\text{-a.e. on } \mathcal{T}_u. \tag{4-8}$$

Proof. Given $f \in \mathrm{LIP}(X)$, fix $\varepsilon > 0$ and let $E \subset \mathcal{T}_u^{\rm nb}(2\varepsilon)$ be any bounded $R_u^{\rm nb}$-convex subset with $\mathfrak{m}(E) > 0$. Theorem 2.10 together with Proposition 4.4 and (4-3) implies

$$\begin{aligned}\int_E D^- f(-\nabla u)\,\mathfrak{m} &\le \liminf_{\tau\to 0} \int_E \frac{f(g_\tau(x)) - f(x)}{\tau}\,\mathfrak{m}(dx)\\ &\le \limsup_{\tau\to 0} \int_E \frac{f(g_\tau(x)) - f(x)}{\tau}\,\mathfrak{m}(dx) \le \int_E D^+ f(-\nabla u)\,\mathfrak{m}.\end{aligned}$$

To conclude it is enough to observe that

$$\int_E \frac{f(g_\tau(x)) - f(x)}{\tau}\,\mathfrak{m}(dx) = \int_Q \int_{E\cap X_\alpha} \frac{f(g_\tau(x)) - f(x)}{\tau}\,\mathfrak{m}_\alpha(dx)\,\mathfrak{q}(d\alpha),$$

and notice that for each $\alpha \in Q$ the incremental ratio $(f(g_\tau(x)) - f(x))/\tau$ converges to $f'(x)$ for $\mathfrak{m}_\alpha$-a.e. $x \in X_\alpha$ and is dominated by the Lipschitz constant of f. Therefore, by the dominated convergence theorem, for each E as above it holds

$$\int_E D^- f(-\nabla u)\,\mathfrak{m} \le \int_E f'\,\mathfrak{m} \le \int_E D^+ f(-\nabla u)\,\mathfrak{m}.$$

The claim follows by the arbitrariness of $\varepsilon > 0$ and $E \subset \mathcal{T}_u^{\rm nb}(2\varepsilon)$. $\square$

The chain rule [Gigli 2015, Proposition 3.15] combined with Theorem 4.5 allows us to obtain:

Corollary 4.6. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an e.n.b. metric measure space satisfying $\mathsf{MCP}(K,N)$ and u be as before. Then for any Lipschitz function $f : X \to \mathbb{R}$*

$$D^- f(-\nabla u^2) \le 2uf' \le D^+ f(-\nabla u^2),$$

where the inequalities hold true $\mathfrak{m}$-a.e. over $\mathcal{T}_u$.

Proof. We show that $D^- f(-\nabla u^2) \le 2uf'$; the argument for proving $2uf' \le D^+ f(-\nabla u^2)$ is completely analogous.

By the chain rule [Gigli 2015, Proposition 3.15], we know that

$$D^- f(-\nabla u^2) = 2u D^{-\operatorname{sgn} u} f(-\nabla u).$$

Combining the last identity with Theorem 4.5 yields

$$D^- f(-\nabla u^2) = \begin{cases} 2u D^+ f(-\nabla u) \le 2uf' & \mathfrak{m}\text{-a.e. on } \{u \le 0\}, \\ 2u D^- f(-\nabla u) \le 2uf' & \mathfrak{m}\text{-a.e. on } \{u \ge 0\}, \end{cases}$$

giving the claim. □

4B. ***A formula for the Laplacian of a general 1-Lipschitz function.*** The next proposition, which is key to showing that Δu is a Radon functional, follows from Lemmas 2.15 and 2.16. We use the notation that $a(X_\alpha)$ and $b(X_\alpha)$ denote the initial and final points respectively of the transport ray X_α. Recall also that h_α is positive and differentiable a.e. on X_α; in particular $\log h_\alpha$ is well-defined and differentiable a.e. along X_α.

Proposition 4.7. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an e.n.b. metric measure space satisfying $\mathsf{MCP}(K, N)$ for some $K \in \mathbb{R}$, $N \in (1, \infty)$. Let $u : X \to \mathbb{R}$ be a 1-Lipschitz function with associated disintegration $\mathfrak{m}\llcorner_{\mathcal{T}_u^{\mathrm{nb}}} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha)$, with $\mathfrak{q}(Q) = 1$, $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1\llcorner_{X_\alpha}$, $h_\alpha \in L^1(\mathcal{H}^1\llcorner_{X_\alpha})$ for $\mathfrak{q}$-a.e. $\alpha \in Q$. Assume that*

$$\int_Q \frac{1}{\mathsf{d}(a(X_\alpha), b(X_\alpha))} \, \mathfrak{q}(d\alpha) < \infty.$$

Then $T_u : \mathrm{LIP}_c(X) \to \mathbb{R}$

$$T_u(f) := \int_Q \int_{X_\alpha} (\log h_\alpha)' f \, \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha) + \int_Q (h_\alpha f)(a(X_\alpha)) - (h_\alpha f)(b(X_\alpha)) \, \mathfrak{q}(d\alpha) \tag{4-9}$$

is a Radon functional over X.

Proof. Fix any bounded open subset $W \subset X$ and observe that we can find a bounded R_u^{nb}-convex measurable subset $E \subset \mathcal{T}_u^{\mathrm{nb}}$ such that $W \cap \mathcal{T}_u^{\mathrm{nb}} \subset E$ (take for instance on each X_α the convex-hull of $W \cap X_\alpha$) and

$$\mathsf{d}(a(X_\alpha \cap E), b(X_\alpha \cap E)) \ge \min\{1, \mathsf{d}(a(X_\alpha), b(X_\alpha))\} \quad \text{for all } \alpha \in Q. \tag{4-10}$$

Note that E depends just on W and the ray relation R_u^{nb}. For any $f \in \mathrm{LIP}_c(X)$ with $\mathrm{supp}(f) \subset W$, it is clear that

$$\begin{aligned} &\int_Q \int_{X_\alpha} (\log h_\alpha)' f \, \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha) + \int_Q (h_\alpha f)(a(X_\alpha)) - (h_\alpha f)(b(X_\alpha)) \, \mathfrak{q}(d\alpha) \\ &\qquad = \int_Q \int_{X_\alpha \cap E} (\log h_\alpha)' f \, \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha) + \int_Q (h_\alpha f)(a(X_\alpha \cap E)) - (h_\alpha f)(b(X_\alpha \cap E)) \, \mathfrak{q}(d\alpha). \end{aligned}$$

Since E is bounded, we have $\sup_{\alpha \in Q} \mathsf{d}(a(X_\alpha \cap E), b(X_\alpha \cap E)) \le C_W$ for some $C_W \in (0, \infty)$ depending only on $W \subset X$. Moreover, Theorem 3.4(4) implies $\sup_{\alpha \in Q} \int_{X_\alpha \cap E} h_\alpha \, d\mathcal{H}^1 \le C_W$. Therefore, applying

Lemmas 2.15 and 2.16 to the renormalised densities

$$\tilde{h}_\alpha := \frac{1}{\int_{X_\alpha \cap E} h_\alpha \, d\mathcal{H}^1} h_\alpha$$

and rescaling back to get h_α, recalling also (4-10) we infer

$$\sup_{X_\alpha \cap E} h_\alpha(x) + \int_{X_\alpha \cap E} |h'_\alpha| \, d\mathcal{H}^1 \le C_W \frac{1}{\mathsf{d}(a(X_\alpha), b(X_\alpha))} \quad \text{for } \mathfrak{q}\text{-a.e. } \alpha \in \mathfrak{Q}(E) \subset Q.$$

We can thus estimate

$$\left| \int_Q \int_{X_\alpha \cap E} (\log h_\alpha)' f \, \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha) + \int_Q (h_\alpha f)(a(X_\alpha \cap E)) - (h_\alpha f)(b(X_\alpha \cap E)) \, \mathfrak{q}(d\alpha) \right| \\ \le \left(C_W \int_Q \frac{1}{\mathsf{d}(a(X_\alpha), b(X_\alpha))} \, \mathfrak{q}(d\alpha) \right) \max |f|.$$

We can therefore conclude that (4-9) defines a Radon functional. □

The first main result follows by combining Theorem 4.5 and Proposition 4.7.

Theorem 4.8. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an e.n.b. metric measure space with $\mathrm{supp}(\mathfrak{m}) = X$ and satisfying $\mathsf{MCP}(K, N)$ for some $K \in \mathbb{R}$, $N \in (1, \infty)$. Let $u : X \to \mathbb{R}$ be a 1-Lipschitz function with associated disintegration $\mathfrak{m} \llcorner_{\mathcal{T}_u^{\mathrm{nb}}} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha)$, with $\mathfrak{q}(Q) = 1$, $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha}$, $h_\alpha \in L^1(\mathcal{H}^1 \llcorner_{X_\alpha})$ for $\mathfrak{q}$-a.e. $\alpha \in Q$. Assume that*

$$\int_Q \frac{1}{\mathsf{d}(a(X_\alpha), b(X_\alpha))} \, \mathfrak{q}(d\alpha) < \infty.$$

Then, for any open subset $U \subset X$ such that $\mathfrak{m}(U \setminus \mathcal{T}_u) = 0$, it holds $u \in D(\mathbf{\Delta}, U)$. More precisely, $T_U : \mathrm{LIP}_c(U) \to \mathbb{R}$, defined by

$$T_U(f) := -\int_Q f h'_\alpha \mathcal{H}^1 \llcorner_{X_\alpha \cap U} \, \mathfrak{q}(d\alpha) + \int_Q (f h_\alpha)(b(X_\alpha)) - (f h_\alpha)(a(X_\alpha)) \, \mathfrak{q}(d\alpha),$$

is a Radon functional with $T_U \in \mathbf{\Delta} u \llcorner_U$. Moreover, writing $T_U = T_U^{\mathrm{reg}} + T_U^{\mathrm{sing}}$, with

$$T_U^{\mathrm{reg}}(f) := -\int_Q f h'_\alpha \mathcal{H}^1 \llcorner_{X_\alpha \cap U} \, \mathfrak{q}(d\alpha), \quad T_U^{\mathrm{sing}}(f) := \int_Q (f h_\alpha)(b(X_\alpha)) - (f h_\alpha)(a(X_\alpha)) \, \mathfrak{q}(d\alpha),$$

it holds that T_U^{reg} can be represented by $T_U^{\mathrm{reg}} = -(\log h_\alpha)' \mathfrak{m} \llcorner_U$ and satisfies the bounds

$$-(N-1) \frac{s'_{K/(N-1)}(\mathsf{d}(b(X_\alpha), x))}{s_{K/(N-1)}(\mathsf{d}(b(X_\alpha), x))} \le (\log h_\alpha)'(x) \le (N-1) \frac{s'_{K/(N-1)}(\mathsf{d}(x, a(X_\alpha)))}{s_{K/(N-1)}(\mathsf{d}(x, a(X_\alpha)))}. \tag{4-11}$$

Remark 4.9 (interpretation in the case X_α is unbounded). Let us explicitly note that, in the case the ray X_α is isometric to $(-\infty, b)$ (respectively $(a, +\infty)$), then by definition $(f h_\alpha)(a(X_\alpha)) = 0$ (resp. $(f h_\alpha)(b(X_\alpha)) = 0$). Let us discuss the case $K = -(N-1)$, the other cases being analogous. In the case the ray X_α is isometric to $(-\infty, b)$ (respectively $(a, +\infty)$), then the upper bound (resp. the lower bound) in (4-11) should be interpreted as $(\log h_\alpha)' \le N - 1$ (resp. $(\log h_\alpha)' \ge -(N-1)$). In particular, if for

$\mathfrak{q}$-a.e. $\alpha \in Q$ the ray X_α is isometric to $(-\infty, +\infty)$, then for any open subset $U \subset X$ with $\mathfrak{m}(U \setminus \mathcal{T}_u) = 0$ the singular part T_U^{sing} vanishes and it holds $-(N-1)\mathfrak{m}\llcorner_U \leq T_U^{\mathrm{reg}} \leq (N-1)\mathfrak{m}\llcorner_U$.

Proof of Theorem 4.8. Fix an arbitrary open subset $U \subset X$ such that $\mathfrak{m}(U \setminus \mathcal{T}_u) = 0$. Let $f : X \to \mathbb{R}$ be any Lipschitz function compactly supported in U and let f' be defined $\mathfrak{m}$-a.e. by (4-7). Recall that the closure of the transport ray $(\overline{X}_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ is isomorphic to a (possibly unbounded, possibly not open) real interval $[a(X_\alpha), b(X_\alpha)]$ endowed with the weighted measure $h_\alpha \mathcal{L}^1$, so we can integrate by parts Lipschitz functions on X_α analogously as on a weighted real interval.

Via an integration by parts, we thus obtain

$$\int_{X_\alpha} h_\alpha(x) f'(x)\, \mathcal{H}^1(dx) = -\int_{X_\alpha} h'_\alpha(x) f(x)\, \mathcal{H}^1(dx) + (h_\alpha f)(b(X_\alpha)) - (h_\alpha f)(a(X_\alpha)) \quad \mathfrak{q}\text{-a.e. } \alpha,$$

which, together with Theorem 3.6, gives

$$\int_U f'(x)\, \mathfrak{m}(dx) = -\int_Q \int_{X_\alpha} h'_\alpha(x) f(x)\, \mathcal{H}^1(dx) + (h_\alpha f)(b(X_\alpha)) - (h_\alpha f)(a(X_\alpha))\, \mathfrak{q}(d\alpha). \tag{4-12}$$

Proposition 4.7 ensures that, under the assumptions of Theorem 4.8, the expression

$$T_{\Delta u}(f) := -\int_Q f h'_\alpha \mathcal{H}^1\llcorner_{X_\alpha}\, \mathfrak{q}(d\alpha) + \int_Q (h_\alpha f)(b(X_\alpha)) - (h_\alpha f)(a(X_\alpha))\, \mathfrak{q}(d\alpha)$$

defines a Radon functional on U.

The combination of (4-12) with Theorem 4.5 gives

$$\int_U D^- f(-\nabla u)\, \mathfrak{m} \leq T_{\Delta u}(f) \leq \int_U D^+ f(-\nabla u)\, \mathfrak{m}.$$

Noting that (see [Gigli 2015, Proposition 3.15])

$$D^- f(-\nabla u) = -D^+ f(\nabla u), \quad D^+ f(-\nabla u) = -D^- f(\nabla u) \quad \mathfrak{m}\text{-a.e.},$$

the previous inequalities imply

$$\int_U D^- f(\nabla u)\, \mathfrak{m} \leq -T_{\Delta u}(f) \leq -\int_U D^+ f(\nabla u)\, \mathfrak{m}.$$

Recalling (2-13), the proof of all the claims is complete. □

The next result, dealing with smooth Riemannian manifolds, can be proved using Corollary 3.7 in the proof of Theorem 4.8 and following verbatim the arguments. Let us just mention that the Laplacian here is single-valued, i.e., $\{T_U\} = \mathbf{\Delta} u\llcorner_U$, since on a smooth Riemannian manifold (M, g) it holds $D^+ f(\nabla u) = D^- f(\nabla u) = g(\nabla f, \nabla u)$.

Corollary 4.10. *Let (M, g) be an N-dimensional complete Riemannian manifold, where $N \geq 2$. Let $u : M \to \mathbb{R}$ be a 1-Lipschitz function with associated disintegration $\mathfrak{m}\llcorner_{\mathcal{T}_u} = \int_Q \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha)$, with $\mathfrak{q}(Q) = 1$, $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1\llcorner_{X_\alpha}$, $h_\alpha \in L^1(\mathcal{H}^1\llcorner_{X_\alpha})$ for $\mathfrak{q}$-a.e. $\alpha \in Q$. Assume that*

$$\int_Q \frac{1}{\mathsf{d}(a(X_\alpha), b(X_\alpha))}\, \mathfrak{q}(d\alpha) < \infty.$$

Then, for any open subset $U \subset M$ such that $\mathfrak{m}(U \setminus \mathcal{T}_u) = 0$, it holds $u \in D(\mathbf{\Delta}, U)$. More precisely, $T_U : \mathrm{LIP}_c(U) \to \mathbb{R}$, defined by

$$T_U(f) := -\int_Q f h'_\alpha \mathcal{H}^1 \llcorner_{X_\alpha \cap U} \, \mathfrak{q}(d\alpha) + \int_Q (f h_\alpha)(b(X_\alpha)) - (f h_\alpha)(a(X_\alpha)) \, \mathfrak{q}(d\alpha),$$

is a Radon functional with $\{T_U\} = \mathbf{\Delta} u \llcorner_U$. Moreover, writing $T_U = T_U^{\mathrm{reg}} + T_U^{\mathrm{sing}}$, with

$$T_U^{\mathrm{reg}}(f) := -\int_Q f h'_\alpha \mathcal{H}^1 \llcorner_{X_\alpha \cap U} \, \mathfrak{q}(d\alpha), \quad T_U^{\mathrm{sing}}(f) := \int_Q (f h_\alpha)(b(X_\alpha)) - (f h_\alpha)(a(X_\alpha)) \, \mathfrak{q}(d\alpha),$$

it holds that T_U^{reg} can be represented by $T_U^{\mathrm{reg}} = -(\log h_\alpha)' \mathfrak{m} \llcorner_U$.

In addition, if $\mathrm{Ric}_g \geq K g$ for some $K \in \mathbb{R}$, then the following bounds hold:

$$-(N-1) \frac{s'_{K/(N-1)}(\mathsf{d}(b(X_\alpha), x))}{s_{K/(N-1)}(\mathsf{d}(b(X_\alpha), x))} \leq (\log h_\alpha)'(x) \leq (N-1) \frac{s'_{K/(N-1)}(\mathsf{d}(x, a(X_\alpha)))}{s_{K/(N-1)}(\mathsf{d}(x, a(X_\alpha)))}. \tag{4-13}$$

Specialising Corollary 4.10 to the distance function gives Theorem 1.1; we briefly discuss the details below.

Proof of Theorem 1.1. Fix $p \in M$.

<u>Step 1</u>: $u := \mathsf{d}_p := \mathsf{d}(p, \cdot)$ satisfies the assumptions of Corollary 4.10.

Since by hypothesis (M, g) is complete, any point $x \in M$ can be joined to p with a length-minimising geodesic. Thus $\mathcal{T}_{\mathsf{d}_p} = M$, $b(X_\alpha) = p$ and $a(X_\alpha) \in \mathcal{C}_p$ for every $\alpha \in Q$. Moreover, there exists $\varepsilon = \varepsilon(p) > 0$ such that all the minimising geodesics X_α emanating from p have length $\mathsf{d}(a(X_\alpha), b(X_\alpha)) = \mathsf{d}(a(X_\alpha), p) \geq \varepsilon$. Since by construction $\mathfrak{q}(Q) = 1$, we conclude that the assumptions of Corollary 4.10 are satisfied.

<u>Step 2</u>: The representation formula (1-3) holds. We are left to show that

$$\int_Q h_\alpha \delta_{b(X_\alpha)} \, \mathfrak{q}(d\alpha) = 0.$$

Clearly, it is enough to show that

$$h_\alpha(p) = 0 \quad \text{for } \mathfrak{q}\text{-a.e. } \alpha \in Q. \tag{4-14}$$

Suppose by contradiction that there exists $\bar{Q} \subset Q$, where $h_\alpha(p) \geq c > 0$, with $\mathfrak{q}(\bar{Q}) > 0$. For simplicity of notation, we identify the minimising geodesic X_α with the real interval $[a_\alpha, b_\alpha]$, where p corresponds to b_α. Then by Fatou's lemma it holds

$$\begin{aligned} \infty > \omega_N = \liminf_{r \downarrow 0} \frac{\mathfrak{m}(B_r(p))}{r^N} &\geq \liminf_{r \downarrow 0} \int_{\bar{Q}} \frac{1}{r^N} \int_{[b_\alpha - r, b_\alpha]} h_\alpha(t) \, dt \, \mathfrak{q}(d\alpha) \\ &\geq \int_{\bar{Q}} \liminf_{r \downarrow 0} \frac{1}{r} \int_{[b_\alpha - r, b_\alpha]} \frac{h_\alpha(t)}{r^{N-1}} \, dt \, \mathfrak{q}(d\alpha) = \infty, \end{aligned}$$

giving a contradiction and thus proving the claim (4-14).

Step 3: We define three nonnegative Radon measures $[\Delta \mathsf{d}_p]^{\pm}_{\mathrm{reg}} := -[(\log h_\alpha)']^{\pm}\mathfrak{m}$ and $[\Delta \mathsf{d}_p]_{\mathrm{sing}} := -\int_Q h_\alpha \delta_{a(X_\alpha)}\,\mathfrak{q}(d\alpha)$, and let $\Delta \mathsf{d}_p = [\Delta \mathsf{d}_p]^{+}_{\mathrm{reg}} - [\Delta \mathsf{d}_p]^{-}_{\mathrm{reg}} + [\Delta \mathsf{d}_p]_{\mathrm{sing}}$.

Combining Corollary 3.7(2) with (2-13), it follows that $[\Delta \mathsf{d}_p]_{\mathrm{reg}} := -(\log h_\alpha)'\mathfrak{m}$ defines a Radon functional; by the Riesz theorem, its positive and negative parts are thus Radon measures. Also $[\Delta \mathsf{d}_p]_{\mathrm{sing}} := -\int_Q h_\alpha \delta_{a(X_\alpha)}\,\mathfrak{q}(d\alpha) = \Delta \mathsf{d}_p - [\Delta \mathsf{d}_p]_{\mathrm{reg}}$ is a nonpositive Radon functional (as a difference of Radon functionals) and thus, by the Riesz theorem, it defines a Radon measure.

Step 4: Upper and lower bounds in the case $\mathrm{Ric}_g \geq Kg$ for some $K \in \mathbb{R}$.

If $\mathrm{Ric}_g \geq Kg$, by Corollary 3.7(3) we know that h_α is a $\mathsf{CD}(K, N)$ (and in particular $\mathsf{MCP}(K, N)$) density over X_α for $\mathfrak{q}$-a.e. α. Thus (2-13) gives the bounds

$$-(N-1)\frac{s'_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)})}{s_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)})}\mathfrak{m} \leq [\Delta \mathsf{d}_p]_{\mathrm{reg}} \leq (N-1)\frac{s'_{K/(N-1)}(\mathsf{d}_p)}{s_{K/(N-1)}(\mathsf{d}_p)}\mathfrak{m}, \tag{4-15}$$

completing the proof. □

Remark 4.11 (on the bounds under the assumption $\mathrm{Ric}_g \geq Kg$). A few comments are in order:

- *The upper bound* in (4-15) is the celebrated Laplacian comparison theorem. Note that a similar upper bound is proved above to hold more generally for the (regular part of the) Laplacian of a (rather) general 1-Lipschitz function (4-13) in the high generality of e.n.b. $\mathsf{MCP}(K, N)$-spaces (4-11).

- *The case of the round sphere*. Let $p, q \in \mathbb{S}^N$ be a couple of antipodal points; clearly the cut locus of p coincides with q. In this case, choosing $u = \mathsf{d}_p$ in the construction above gives the partition of $\mathbb{S}^N \setminus \{p, q\}$ into meridians, and each ray is a meridian without its endpoints p, q, oriented from q to p. Theorem 1.1 thus yields
$$-(N-1)\cot \mathsf{d}_q \leq \Delta \mathsf{d}_p \leq (N-1)\cot \mathsf{d}_p \quad \text{on } \mathbb{S}^N.$$
Note that (for the round sphere) the same conclusion could be achieved by applying the Laplacian comparison theorem to d_p and to d_q and using that $\mathsf{d}_p = \pi - \mathsf{d}_q$.

- *The lower bound for a smooth Riemannian manifold.* Arguing analogously to the spherical case, one can achieve the lower bound along a (minimising) geodesic $\gamma : [0, 1] \to M$ with (M, g) satisfying $\mathrm{Ric}_g \geq (N-1)g$ (see [Colding and Naber 2012, Lemma 3.2]). In this case, the function $x \mapsto \mathsf{d}_{\gamma_0}(x) + \mathsf{d}_{\gamma_1}(x)$ achieves its minimum $\mathsf{d}(\gamma_0, \gamma_1)$ along $\gamma([0, 1])$; thus $\Delta(\mathsf{d}_{\gamma_0} + \mathsf{d}_{\gamma_1}) \geq 0$ along $\gamma((0, 1))$ and, applying the upper bound (1-1) to $\mathsf{d}_{\gamma_0}, \mathsf{d}_{\gamma_1}$ and exploiting the linearity of the Laplacian we get
$$-(N-1)\cot \mathsf{d}_{\gamma_1} \leq \Delta \mathsf{d}_{\gamma_0} \leq (N-1)\cot \mathsf{d}_{\gamma_0} \quad \text{along } \gamma((0, 1)). \tag{4-16}$$
"Gluing" together all the inequalities (4-16) corresponding to all the (minimising) geodesics emanating from p gives (4-15). Clearly this argument holds for smooth Riemannian manifolds, but in situations where the space is a priori not smooth and the Laplacian is a priori not linear (as for e.n.b. $\mathsf{MCP}(K, N)$-spaces), one has to argue differently. As the reader could already appreciate (see, e.g., the proof of Theorem 1.1), we attacked the problem by using techniques from L^1-optimal transport.

A crucial fact in order to apply Theorem 4.8 to the distance function in the smooth case was that the cut locus of a point p is at strictly positive distance from p. This fact is clearly not at our disposal in the general setting of an e.n.b. $\mathsf{MCP}(K, N)$ space (e.g., the boundary of a convex body in $\mathbb{R}^3$ whose cut locus is dense). In Section 4C we will thus argue differently, showing first the result for the distance squared, and then getting the claim for the distance via the chain rule.

4C. *A formula for the Laplacian of a signed distance function.* The goal of the subsection is to prove the existence of the Laplacian of d_v and d_v^2 as Radon measures and to show upper and lower bounds; let us stress that, contrary to the previous subsection, here there will be no integrability assumption on the reciprocal of the length of the transport rays.

Recall that given a continuous function $v : (X, \mathsf{d}) \to \mathbb{R}$ so that $\{v = 0\} \neq \varnothing$, the signed distance function

$$d_v : X \to \mathbb{R}, \quad d_v(x) := \mathsf{d}(x, \{v = 0\}) \operatorname{sgn}(v),$$

is 1-Lipschitz.

Notice also that since (X, d) is proper, $\mathcal{T}_{d_v} \supset X \setminus \{v = 0\}$. Indeed, given $x \in X \setminus \{v = 0\}$, consider the distance minimising $z \in \{v = 0\}$ (whose existence is guaranteed by the compactness of closed bounded sets). Then $(x, z) \in R_{d_v}$ and thus $x \in \mathcal{T}_{d_v}$, as $x \neq z$. The next remark follows.

Remark 4.12. Let X_α be any transport ray associated with d_v and let $a(X_\alpha), b(X_\alpha)$ be its starting and final points, respectively. Then

$$d_v(b(X_\alpha)) \leq 0, \quad d_v(a(X_\alpha)) \geq 0,$$

whenever $b(X_\alpha)$ and $a(X_\alpha)$ exist.

The next lemma will be key to showing the existence of the Laplacian of d_v^2 as a Radon measure.

Lemma 4.13. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an e.n.b. metric measure space satisfying $\mathsf{MCP}(K, N)$ for some $K \in \mathbb{R}$, $N \in (1, \infty)$.*

The expression

$$\begin{aligned} \nu := {} & 2\left(1 + \mathsf{d}(\{v = 0\}, x)(N-1)\frac{s'_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}\right)\mathfrak{m}\llcorner_{\{v \geq 0\}} \\ & + 2\left(1 + \mathsf{d}(\{v = 0\}, x)(N-1)\frac{s'_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}\right)\mathfrak{m}\llcorner_{\{v < 0\}} \end{aligned} \tag{4-17}$$

defines a signed Radon measure over X. More precisely:

<u>Case 1</u>: $K > 0$. *In this case ν is a signed finite measure on X satisfying $\nu \leq C_{K,N}\mathfrak{m}$. Then*:

- *If*

$$\sup_{x \in \{v \geq 0\}} \mathsf{d}(x, b(X_\alpha)) < \pi\sqrt{(N-1)/K}, \qquad \sup_{x \in \{v < 0\}} \mathsf{d}(x, a(X_\alpha)) < \pi\sqrt{(N-1)/K}$$

  then ν has density bounded in $L^\infty(X, \mathfrak{m})$.

- *If*

$$\sup_{x\in\{v\geq 0\}} \mathsf{d}(x, b(X_\alpha)) = \pi\sqrt{(N-1)/K} \quad \textit{or} \quad \sup_{x\in\{v<0\}} \mathsf{d}(x, a(X_\alpha)) = \pi\sqrt{(N-1)/K}$$

then there exist exactly two points $\bar{a}, \bar{b} \in X$ *with* $\mathsf{d}(\bar{a}, \bar{b}) = \pi\sqrt{(N-1)/K}$ *such that for* $\mathfrak{q}$*-a.e.* α

$$a(X_\alpha) = \bar{a}, \quad b(X_\alpha) = \bar{b},$$

and ν *has density bounded in*

$$L^\infty_{\mathrm{loc}}(\{v \geq 0\} \setminus \{\bar{a}\}, \mathfrak{m}) \cap L^\infty_{\mathrm{loc}}(\{v \leq 0\} \setminus \{\bar{b}\}, \mathfrak{m}) \cap L^1(X, \mathfrak{m}).$$

Moreover, in this case $(X, \mathfrak{m})$ *is isomorphic to a spherical suspension as a measure space. If in addition* $(X, \mathsf{d}, \mathfrak{m})$ *is an* $\mathsf{RCD}(K, N)$ *space, then* $(X, \mathsf{d}, \mathfrak{m})$ *is isomorphic to a spherical suspension as a metric measure space.*

Case 2: $K = 0$. *In this case* $\nu = 2\mathfrak{m}$ *is a nonnegative Radon measure; if* $b(X_\alpha)$ *or* $a(X_\alpha)$ *does not exist, the two ratios in* (4-17) *are posed by definition equal to* 0, *respectively.*

Case 3: $K < 0$. *In this case* ν *is a nonnegative Radon measure. If* $b(X_\alpha)$ *or* $a(X_\alpha)$ *does not exist, the two ratios in* (4-17) *are posed by definition equal to* 1, *respectively.*

Proof. Case 2: For $K = 0$ the bounds are a straightforward consequence of the definition of the coefficients $s_{K/(N-1)}$ given in (2-10).

Case 3: For $K < 0$ observe that, since $(0, \infty) \ni t \mapsto \coth t \in (0, \infty)$ is decreasing and $\mathsf{d}(\{v = 0\}, x) \leq \mathsf{d}_{b(X_\alpha)}(x)$ for all $x \in \{v \leq 0\}$, it holds

$$\begin{aligned} 0 \leq \mathsf{d}(\{v=0\}, x)\frac{s'_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))} &\leq \mathsf{d}(\{v=0\}, x)\frac{s'_{K/(N-1)}(\mathsf{d}(\{v=0\}, x))}{s_{K/(N-1)}(\mathsf{d}(\{v=0\}, x))} \\ &= \mathsf{d}(\{v=0\}, x)\sqrt{\frac{-K}{N-1}}\coth\left(\sqrt{\frac{-K}{N-1}}\mathsf{d}(\{v=0\}, x)\right) \quad \text{for all } x \in \{v \leq 0\}. \end{aligned}$$

Since the function

$$[0, \infty) \ni t \mapsto t\coth\left(\sqrt{\frac{-K}{N-1}}t\right)$$

is locally bounded and the discussion for the second line of (4-17) is completely analogous, the claim follows.

Case 1: For $K > 0$, recall that an $\mathsf{MCP}(K, N)$-space has diameter at most $\pi\sqrt{(N-1)/K}$. Since $(0, \pi) \ni t \mapsto \cot t$ is decreasing and $\mathsf{d}(\{v = 0\}, x) \leq \mathsf{d}_{b(X_\alpha)}(x)$ for all $x \in \{v \leq 0\}$, it holds

$$\begin{aligned} \mathsf{d}(\{v=0\}, x)\frac{s'_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))} &\leq \mathsf{d}(\{v=0\}, x)\frac{s'_{K/(N-1)}(\mathsf{d}(\{v=0\}, x))}{s_{K/(N-1)}(\mathsf{d}(\{v=0\}, x))} \\ &= \mathsf{d}(\{v=0\}, x)\sqrt{\frac{K}{N-1}}\cot\left(\sqrt{\frac{K}{N-1}}\mathsf{d}(\{v=0\}, x)\right) \quad \text{for all } x \in \{v \leq 0\}. \end{aligned}$$

It is easily checked that

$$\sup_{t\in[0,\pi\sqrt{(N-1)/K}]}\left(1+t\sqrt{K(N-1)}\cot\left(t\sqrt{\frac{K}{N-1}}\right)\right)\le C'_{K,N};$$

thus the bound $\nu\le C_{K,N}\mathfrak{m}$ follows.

Since

$$\inf_{t\in[0,\pi\sqrt{(N-1)/K-\varepsilon}]}\cot\left(t\sqrt{\frac{K}{N-1}}\right)>-\infty\quad\text{for every } \varepsilon\in\left(0,\pi\sqrt{\frac{N-1}{K}}\right],$$

we have that ν is a measure with L^∞-bounded density unless the second bullet of case 1 holds.

To discuss the second bullet of case 1 we assume $K=N-1$ in order to simplify the notation, the case for general $K>0$ being completely analogous.

Using the maximal diameter theorem (proved in [Ohta 2007b] in the nonbranched $\mathsf{MCP}(N-1,N)$-setting and easily extendable to the present e.n.b situation) one can show that all the rays X_α are of length π; for the reader's convenience we give a self-contained argument. Let $X_{\bar\alpha}$ be a ray of length π and X_α be any other ray; then

$$\begin{aligned}\mathsf{d}(a(X_\alpha),b(X_\alpha))+\pi&=\mathsf{d}(a(X_\alpha),b(X_\alpha))+\mathsf{d}(a(X_{\bar\alpha}),b(X_\alpha))+\mathsf{d}(b(X_\alpha),b(X_{\bar\alpha}))\\&\ge\mathsf{d}(a(X_{\bar\alpha}),b(X_\alpha))+\mathsf{d}(a(X_\alpha),b(X_{\bar\alpha})),\end{aligned}\tag{4-18}$$

where the first equality follows from [Ohta 2007b, Lemma 5.2] (since $|X_{\bar\alpha}|=\pi$, for each $x\in X$, $\mathsf{d}(x,a(X_{\bar\alpha}))+\mathsf{d}(x,b(X_{\bar\alpha}))=\pi$). By d-cyclical monotonicity also the reverse inequality is valid giving

$$\mathsf{d}(a(X_\alpha),b(X_\alpha))+\pi=\mathsf{d}(a(X_{\bar\alpha}),b(X_\alpha))+\mathsf{d}(a(X_\alpha),b(X_{\bar\alpha})).\tag{4-19}$$

In particular, $a(X_{\bar\alpha})\neq b(X_\alpha)$; indeed otherwise (4-19) would give $\mathsf{d}(a(X_\alpha),b(X_\alpha))+\pi=\mathsf{d}(a(X_\alpha),b(X_{\bar\alpha}))$, which, by virtue of the Myers diameter bound, would imply $a(X_\alpha)=b(X_\alpha)$. Contradicting the fact that the rays have strictly positive length.

Adding $\mathsf{d}(b(X_\alpha),b(X_{\bar\alpha}))-\pi$ (resp. $\mathsf{d}(a(X_\alpha),a(X_{\bar\alpha}))-\pi$) to both sides of (4-19) and using again [Ohta 2007b, Lemma 5.2], we get

$$\begin{aligned}\mathsf{d}(a(X_\alpha),b(X_\alpha))+\mathsf{d}(b(X_\alpha),b(X_{\bar\alpha}))&=\mathsf{d}(a(X_\alpha),b(X_{\bar\alpha})),\\\mathsf{d}(a(X_\alpha),b(X_\alpha))+\mathsf{d}(a(X_\alpha),a(X_{\bar\alpha}))&=\mathsf{d}(a(X_{\bar\alpha}),b(X_\alpha)).\end{aligned}$$

Summing up the last two identities, together with (4-19), yields

$$\mathsf{d}(a(X_\alpha),b(X_\alpha))+\mathsf{d}(a(X_\alpha),a(X_{\bar\alpha}))+\mathsf{d}(b(X_{\bar\alpha}),b(X_\alpha))=\pi.$$

Since $\mathsf{d}(a(X_{\bar\alpha}),b(X_{\bar\alpha}))=\pi$, the last identity forces the four points $a(X_{\bar\alpha}),a(X_\alpha),b(X_\alpha),b(X_{\bar\alpha})$ to lie on the same geodesic γ. If $a(X_\alpha)\neq a(X_{\bar\alpha})$ then $a(X_\alpha)$ would be an internal point of γ, contradicting that $a(X_\alpha)$ is the initial point of the nonextendible ray X_α, and if $b(X_\alpha)\neq b(X_{\bar\alpha})$ then $b(X_\alpha)$ would be an internal point of γ, contradicting that $b(X_\alpha)$ is the final point of the nonextendible ray X_α.

Moreover $(X, \mathfrak{m})$ is isomorphic as a measure space to a spherical suspension over any transport ray of length π [Ohta 2007b, p. 235].

We are left to show that the density of ν is in $L^1(X, \mathfrak{m})$. By symmetry it is enough to show that

$$\int_{\{v\geq 0\}} \left|1+(N-1)\mathsf{d}(\{v=0\}, x)\cot(\mathsf{d}(\bar{b}, x))\right| \mathfrak{m}(dx) < \infty. \tag{4-20}$$

Notice that, for every fixed $\varepsilon \in [0, \pi/2]$, the integrand is bounded for $\mathsf{d}(\bar{b}, x) \in [\varepsilon, \pi-\varepsilon]$.

Since $\bar{b} \in \{v \leq 0\}$, if $\bar{b}$ is an accumulation point for $\{v \geq 0\}$, then $v(\bar{b}) = 0$. As v is strictly decreasing on the rays, which cover a dense subset, it follows that $\{v=0\} = \{\bar{b}\}$. Thus, in this case, the integrand becomes $1+\mathsf{d}(\bar{b}, x)\cot(\mathsf{d}(\bar{b}, x))$, which is bounded for $\mathsf{d}(\bar{b}, x) \in [0, \varepsilon]$.

We now show that the integral is finite also on

$$\{x : \mathsf{d}(\bar{b}, x) \in [\pi-\varepsilon, \pi]\} \cap \{v \geq 0\}.$$

Since

$$\mathsf{d}(\{v=0\}, x) \leq \mathsf{d}(\bar{b}, x),$$

it is enough to show that

$$\int_{\{v\geq 0\}\cap \mathsf{d}(\bar{b},x)\in[\pi-\varepsilon,\pi]} |1+(N-1)\mathsf{d}(\bar{b}, x)\cot(\mathsf{d}(\bar{b}, x))|\, \mathfrak{m}(dx) < \infty. \tag{4-21}$$

Recalling that $(X, \mathfrak{m})$ is isomorphic as a measure space to a spherical suspension over any transport ray of length π, the integral in (4-21) is bounded by

$$\begin{aligned}\int_{[\pi-\varepsilon,\pi]} |1+(N-1)t\cot t|\sin^{N-1}(t)\,dt &= \int_{[0,\varepsilon]} [(N-1)(\pi-s)\cot s - 1]\sin^{N-1}(s)\,ds \\ &= (N-1)\pi \int_{[0,\varepsilon]} s^{N-2}\,ds + O(\varepsilon) < \infty,\end{aligned}$$

since $N > 1$. This concludes the proof that the density of ν is in $L^1(X, \mathfrak{m})$. The stronger rigidity statement under the stronger $\mathsf{RCD}(K, N)$ assumption is a direct consequence of the maximal diameter theorem proved in [Ketterer 2015] in the $\mathsf{RCD}(K, N)$-setting. □

Corollary 4.6 and Lemma 4.13 have far-reaching consequences.

Theorem 4.14. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an e.n.b. metric measure space satisfying* $\mathsf{MCP}(K, N)$ *for some* $K \in \mathbb{R}$, $N \in (1, \infty)$.

Consider the signed distance function d_v *for some continuous function* $v : X \to \mathbb{R}$ *and the associated disintegration*

$$\mathfrak{m}\llcorner_{X\setminus\{v=0\}} = \int_Q h_\alpha \mathcal{H}^1\llcorner_{X_\alpha}\, \mathfrak{q}(d\alpha).$$

Then $d_v^2 \in D(\mathbf{\Delta})$ *and one element of* $\mathbf{\Delta}(d_v^2)$, *which we denote by* Δd_v^2, *has the representation formula*

$$\Delta d_v^2 = 2(1-d_v(\log h_\alpha)')\mathfrak{m} - 2\int_Q (h_\alpha d_v)[\delta_{a(X_\alpha)} - \delta_{b(X_\alpha)}]\, \mathfrak{q}(d\alpha). \tag{4-22}$$

Moreover Δd_v^2 *is a sum of two signed Radon measures and the following comparison results hold true*:

$$\begin{aligned}\Delta d_v^2 \le \nu := 2\mathfrak{m} &+ 2(N-1)\mathsf{d}(\{v=0\},x)\frac{s'_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}\mathfrak{m}\llcorner_{\{v\ge 0\}}\\ &+2(N-1)\mathsf{d}(\{v=0\},x)\frac{s'_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}\mathfrak{m}\llcorner_{\{v<0\}},\end{aligned}\tag{4-23}$$

$$\begin{aligned}[\Delta d_v^2]^{\mathrm{reg}} &:= 2(1-d_v(\log h_\alpha)')\mathfrak{m}\\ &\ge 2\mathfrak{m} - 2(N-1)\mathsf{d}(\{v=0\},x)\frac{s'_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}\mathfrak{m}\llcorner_{\{v\ge 0\}}\\ &\qquad -2(N-1)\mathsf{d}(\{v=0\},x)\frac{s'_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}\mathfrak{m}\llcorner_{\{v<0\}},\end{aligned}\tag{4-24}$$

where $[\Delta d_v^2]^{\mathrm{reg}}$ *is the regular part of* Δd_v^2 (*i.e., absolutely continuous with respect to* $\mathfrak{m}$).

Proof. Fix any compactly supported Lipschitz function $f : X \to \mathbb{R}$ and integrate by parts on each ray X_α to obtain

$$\begin{aligned}&\int_{X_\alpha} d_v(x)f'(x)h_\alpha(x)\,\mathcal{H}^1(dx)\\ &= -\int_{X_\alpha} f(x)d_v'(x)h_\alpha(x)\,\mathcal{H}^1(dx) - \int_{X_\alpha} f(x)d_v(x)h_\alpha'(x)\,\mathcal{H}^1(dx) + (f d_v h_\alpha)(b(X_\alpha)) - (f d_v h_\alpha)(a(X_\alpha))\\ &= \int_{X_\alpha} f(x)h_\alpha(x)\,\mathcal{H}^1(dx) - \int_{X_\alpha} f(x)d_v(x)h_\alpha'(x)\,\mathcal{H}^1(dx) + (f d_v h_\alpha)(b(X_\alpha)) - (f d_v h_\alpha)(a(X_\alpha))\\ &= \int_{X_\alpha} f(x)\big(1-d_v(x)(\log h_\alpha)'(x)\big)h_\alpha(x)\,\mathcal{H}^1(dx) + (f d_v h_\alpha)(b(X_\alpha)) - (f d_v h_\alpha)(a(X_\alpha)).\end{aligned}\tag{4-25}$$

Then considering along each ray X_α the two regions $\{v \ge 0\}$ and $\{v < 0\}$, we notice that (2-13) gives

$$\begin{aligned}-d_v(x)(\log h_\alpha)'(x) \le \mathsf{d}(\{v=0\},x)(N-1)&\frac{s'_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}\chi_{\{v\ge 0\}}(x)\\ &+\mathsf{d}(\{v=0\},x)(N-1)\frac{s'_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}\chi_{\{v<0\}}(x) =: V_\alpha(x).\end{aligned}$$

Hence we can collect the estimates, using Remark 4.12, and obtain

$$\int_{X_\alpha} d_v(x)f'(x)h_\alpha(x)\,\mathcal{H}^1(dx) \le \int_{X_\alpha} (1+V_\alpha(x))f(x)h_\alpha(x)\,\mathcal{H}^1(dx),$$

provided f is nonnegative. Thanks to Lemma 4.13,

$$\nu = 2\int_Q (1+V_\alpha(x))\mathfrak{m}_\alpha\,\mathfrak{q}(d\alpha) = 2(1+V)\mathfrak{m}$$

is a well-defined Radon (possibly signed) measure.

Hence, continuing from (4-25), the expression

$$\Delta d_v^2 := 2\int_Q (h_\alpha - d_v h'_\alpha)\mathcal{H}^1 \llcorner_{X_\alpha}\, \mathfrak{q}(d\alpha) + 2\int_Q (h_\alpha d_v)[\delta_{b(X_\alpha)} - \delta_{a(X_\alpha)}]\, \mathfrak{q}(d\alpha), \tag{4-26}$$

once restricted to bounded subsets, defines a Borel measure with values in $\mathbb{R}\cup\{-\infty\}$ which satisfies $\Delta d_v^2 \le \nu$. Now, combining Theorem 3.6 with (4-25) and (4-26), we get

$$\int_X f\, \Delta d_v^2(dx) = 2\int_Q\int_{X_\alpha} d_v(x) f'(x) h_\alpha(x)\, \mathcal{H}^1(dx)\, \mathfrak{q}(d\alpha) = 2\int_{\mathcal{T}_{d_v}} d_v(x) f'(x)\, \mathfrak{m}(dx)$$

for any compactly supported Lipschitz function $f : X \to \mathbb{R}$. Therefore, Corollary 4.6 yields

$$\int_{\mathcal{T}_{d_v}} D^- f(-\nabla d_v^2)\, \mathfrak{m} \le \int_X f\, \Delta d_v^2(dx) \le \int_{\mathcal{T}_{d_v}} D^+ f(-\nabla d_v^2)\, \mathfrak{m}$$

for any compactly supported Lipschitz function $f : X \to \mathbb{R}$. Since $X\setminus\mathcal{T}_{d_v} \subset \{v=0\} = \{d_v = 0\}$, from the locality properties of differentials (see [Gigli 2015, equation (3.7)]) we can turn the previous inequalities into

$$\int_X D^- f(-\nabla d_v^2)\, \mathfrak{m} \le \int_X f\, \Delta d_v^2(dx) \le \int_X D^+ f(-\nabla d_v^2)\, \mathfrak{m}, \tag{4-27}$$

valid for any compactly supported Lipschitz function $f : X \to \mathbb{R}$. In order to show that $d_v^2 \in D(\mathbf{\Delta})$ with $\Delta d_v^2 \in \mathbf{\Delta}(d_v^2)$, we are thus left to prove that Δd_v^2 is a signed Radon measure.

We now claim that Δd_v^2 is a sum of two Radon measures over X. Since $\Delta d_v^2 \le \nu$ with ν a signed Radon measure, thanks to the Riesz–Markov–Kakutani representation theorem it is enough to show that Δd_v^2 defines a Radon functional.

To this aim, fix a compact subset $W \subset X$ and fix a compactly supported Lipschitz cutoff function $\chi_W : X \to [0,1]$ satisfying $\chi_W \equiv 1$ on W. First observe that, using (4-27), for any Lipschitz function $f : X \to \mathbb{R}$ with $\mathrm{supp}(f) \subset W$ we have

$$\begin{aligned}\left|\int_X \chi_W\, \Delta d_v^2(dx)\right| &\le 2\big(\max_{x\in\mathrm{supp}(\chi_W)} d_v(x)\big)\, \mathrm{Lip}(\chi_W)\mathfrak{m}(\mathrm{supp}(\chi_W)) \in (0,\infty),\\ \left|\int_X (f\chi_W)\, \Delta d_v^2(dx)\right| &\le 2\big(\max_{x\in\mathrm{supp}(\chi_W)} d_v(x)\big)\, \mathrm{Lip}(f\chi_W)\mathfrak{m}(\mathrm{supp}(\chi_W)) \in (0,\infty).\end{aligned}$$

Thus for any Lipschitz function $f : X \to \mathbb{R}$ with $\mathrm{supp}(f) \subset W$, using that $\Delta d_v^2 \le \nu \le \nu^+$, on one hand we have

$$\begin{aligned}\int_X f\, \Delta d_v^2 &= -\int_X (\max f - f)\chi_W\, \Delta d_v^2 + \int_X (\max f)\chi_W\, \Delta d_v^2\\ &\ge -\int_X (\max f - f)\chi_W \nu^+ - C_W(\max f),\end{aligned} \tag{4-28}$$

where $C_W := 2(\mathrm{Lip}\, \chi_W) \max_{x\in\mathrm{supp}(\chi_W)} \mathsf{d}_p(x)\mathfrak{m}(\mathrm{supp}(\chi_W)) \in (0,\infty)$ depends only on χ_W.

On the other hand,

$$\begin{aligned}\int_X f\,\Delta d_v^2 &= \int_X f^+\,\Delta d_v^2 - \int_X f^-\,\Delta d_v^2\\ &\le \int_X f^+ v^+ + \int_X (\max f^- - f^-)\chi_W v^+ + C_W(\max f^-)\\ &\le \max|f|\big(v^+(W) + v^+(\mathrm{supp}(\chi_W)) + C_W\big).\end{aligned} \tag{4-29}$$

The combination of (4-28) and (4-29) gives that, for every compact subset $W \subset X$, there exists a constant $C'_W = 2(v^+(\mathrm{supp}(\chi_W)) + \mathrm{Lip}\,\chi_W \max_{x\in\mathrm{supp}(\chi_W)} d_v(x)\mathfrak{m}(\mathrm{supp}(\chi_W))) \in (0,\infty)$ such that

$$\left|\int_X f\,\Delta d_v^2\right| \le C'_W \max|f|$$

for every Lipschitz function $f : X \to \mathbb{R}$ with $\mathrm{supp}(f) \subset W$, showing that Δd_v^2 is a Radon functional and thus $d_v^2 \in D(\mathbf{\Delta})$ with $\Delta d_v^2 \in \mathbf{\Delta}(d_v^2)$.

In order to complete the proof we are left with showing (4-24): again from (2-13)

$$\begin{aligned}-d_v(x)(\log h_\alpha)'(x) \ge -(N-1)\mathsf{d}(\{v=0\},x)\frac{s'_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}\chi_{\{v\ge 0\}}(x)\\ -(N-1)\mathsf{d}(\{v=0\},x)\frac{s'_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}\chi_{\{v<0\}}(x),\end{aligned}$$

and the claim is proved. □

Remark 4.15. • In the case X is bounded, in the proof of Theorem 4.14 one can pick $W = X$ and $\chi_W \equiv 1$, giving that the total variation of Δd_v^2 is bounded by $\|\Delta d_v^2\| \le 2v^+(X)$.

• Theorem 1.3 can be proved using Corollary 3.7 in the proof of Theorem 4.14 and following verbatim the arguments. Uniqueness of the representation of the Laplacian follows then from infinitesimal Hilbertianity of smooth manifolds.

The representation formula for the Laplacian of the signed distance function on $X \setminus \{v=0\}$ follows from Theorem 4.14 by the chain rule [Gigli 2015, Proposition 4.11].

Corollary 4.16. *Let $(X,\mathsf{d},\mathfrak{m})$ be an e.n.b. metric measure space satisfying $\mathsf{MCP}(K,N)$ for some $K\in\mathbb{R}$, $N\in(1,\infty)$.*

Consider the signed distance function d_v for some continuous function $v : X \to \mathbb{R}$ and the associated disintegration

$$\mathfrak{m}\llcorner_{X\setminus\{v=0\}} = \int_Q h_\alpha \mathcal{H}^1\llcorner_{X_\alpha}\,\mathfrak{q}(d\alpha).$$

Then

(1) $|d_v| \in D(\mathbf{\Delta}, X\setminus\{v=0\})$ *and one element of* $\mathbf{\Delta}(|d_v|)\llcorner_{X\setminus\{v=0\}}$, *which we denote by* $\Delta|d_v|\llcorner_{X\setminus\{v=0\}}$, *is the Radon functional on* $X\setminus\{v=0\}$ *with the representation formula*

$$\Delta|d_v|\llcorner_{X\setminus\{v=0\}} = -\,\mathrm{sgn}(v)(\log h_\alpha)'\mathfrak{m}\llcorner_{X\setminus\{v=0\}} - \int_Q (h_\alpha[\delta_{a(X_\alpha)\cap\{v>0\}} + \delta_{b(X_\alpha)\cap\{v<0\}}]\,\mathfrak{q}(d\alpha). \tag{4-30}$$

Moreover the following comparison results hold true:

$$\begin{aligned}\Delta|d_v|\llcorner_{X\setminus\{v=0\}} \le (N-1)&\frac{s'_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}\mathfrak{m}\llcorner_{\{v>0\}}\\ &+(N-1)\frac{s'_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}\mathfrak{m}\llcorner_{\{v<0\}},\end{aligned} \tag{4-31}$$

$$\begin{aligned}[\Delta|d_v|\llcorner_{X\setminus\{v=0\}}]^{\mathrm{reg}} :=& -\mathrm{sgn}(v)(\log h_\alpha)'\mathfrak{m}\llcorner_{X\setminus\{v=0\}}\\ \ge& -(N-1)\frac{s'_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}\mathfrak{m}\llcorner_{\{v>0\}}\\ &-(N-1)\frac{s'_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}\mathfrak{m}\llcorner_{\{v<0\}},\end{aligned} \tag{4-32}$$

where $[\Delta|d_v|\llcorner_{X\setminus\{v=0\}}]^{\mathrm{reg}}$ *is the regular part of* $\Delta|d_v|\llcorner_{X\setminus\{v=0\}}$ (*i.e., absolutely continuous with respect to* $\mathfrak{m}$).

(2) $d_v \in D(\boldsymbol{\Delta}, X\setminus\{v=0\})$ *and one element of* $\boldsymbol{\Delta}(d_v)\llcorner_{X\setminus\{v=0\}}$, *which we denote by* $\Delta d_v\llcorner_{X\setminus\{v=0\}}$, *is the Radon functional on* $X\setminus\{v=0\}$ *with the representation formula*

$$\Delta d_v\llcorner_{X\setminus\{v=0\}} = -(\log h_\alpha)'\mathfrak{m}\llcorner_{X\setminus\{v=0\}} - \int_Q (h_\alpha[\delta_{a(X_\alpha)\cap\{v>0\}} - \delta_{b(X_\alpha)\cap\{v<0\}}]\,\mathfrak{q}(d\alpha). \tag{4-33}$$

Moreover the following comparison results hold true:

$$\Delta d_v\llcorner_{X\setminus\{v=0\}} \le (N-1)\frac{s'_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{b(X_\alpha)}(x))}\mathfrak{m}\llcorner_{X\setminus\{v=0\}} + \int_Q h_\alpha\delta_{b(X_\alpha)\cap\{v<0\}}\,\mathfrak{q}(d\alpha), \tag{4-34}$$

$$\Delta d_v\llcorner_{X\setminus\{v=0\}} \ge -(N-1)\frac{s'_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}\mathfrak{m}\llcorner_{X\setminus\{v=0\}} - \int_Q (h_\alpha[\delta_{a(X_\alpha)\cap\{v>0\}}]\,\mathfrak{q}(d\alpha). \tag{4-35}$$

Proof. Writing $\mathrm{sgn}(v)d_v = \sqrt{d_v^2}$, a direct application of the chain rule [Gigli 2015, Proposition 4.11] combined with Theorem 4.14 gives that $|d_v| \in D(\boldsymbol{\Delta}, X\setminus\{v=0\})$ and that $\Delta|d_v|$ defined in (4-30) is an element of $\boldsymbol{\Delta}|d_v|\llcorner_{X\setminus\{v=0\}}$. The comparison results (4-31), (4-32) follow from the definition (4-30) together with (2-13).

Since $d_v = \mathrm{sgn}(v)\,|d_v|$, it is clear that $d_v \in D(\boldsymbol{\Delta}, X\setminus\{v=0\})$, with

$$\boldsymbol{\Delta}(d_v)\llcorner_{X\setminus\{v=0\}} = \mathrm{sgn}(v)\,\boldsymbol{\Delta}(|d_v|)\llcorner_{X\setminus\{v=0\}};$$

thus $\Delta d_v\llcorner_{X\setminus\{v=0\}}$defined in (4-33) is an element of $\boldsymbol{\Delta}(d_v)\llcorner_{X\setminus\{v=0\}}$ and the comparison results (4-34), (4-35) follow again from (2-13). □

We now specialise the above results to the distance function from a point $p \in X$; i.e., we pick $v = \mathsf{d}_p$ so that $\{v=0\} = p$ and $v \ge 0$ everywhere. Note that, in this case, $b(X_\alpha) = p$ for $\mathfrak{q}$-a.e. $\alpha \in Q$.

Corollary 4.17. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an e.n.b. metric measure space satisfying* $\mathsf{MCP}(K, N)$ *for some* $K \in \mathbb{R}$, $N \in (1, \infty)$. *Fix* $p \in X$, *consider* $\mathsf{d}_p := \mathsf{d}(p, \cdot)$ *and the associated disintegration*

$$\mathfrak{m} = \int_Q h_\alpha \mathcal{H}^1 \llcorner X_\alpha \, \mathfrak{q}(d\alpha).$$

Then $\mathsf{d}_p^2 \in D(\mathbf{\Delta})$ *and one element of* $\mathbf{\Delta}(\mathsf{d}_p^2)$, *which we denote by* $\Delta \mathsf{d}_p^2$, *is a sum of two signed Radon measures and satisfies the representation formula*

$$\Delta \mathsf{d}_p^2 = 2(1 - \mathsf{d}_p (\log h_\alpha)')\mathfrak{m} - 2 \int_Q h_\alpha \mathsf{d}_p \delta_{a(X_\alpha)} \, \mathfrak{q}(d\alpha). \tag{4-36}$$

Moreover, the following comparison results hold true:

$$\Delta \mathsf{d}_p^2 \le \nu := 2\left(1 + (N-1)\mathsf{d}_p(x) \frac{s'_{K/(N-1)}(\mathsf{d}_p(x))}{s_{K/(N-1)}(\mathsf{d}_p(x))}\right)\mathfrak{m}, \tag{4-37}$$

$$[\Delta \mathsf{d}_p^2]^{\mathrm{reg}} := 2(1 - \mathsf{d}_p(\log h_\alpha)')\mathfrak{m} \ge 2\left(1 - (N-1)\mathsf{d}_p \frac{s'_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}\right)\mathfrak{m}, \tag{4-38}$$

where $[\Delta \mathsf{d}_p^2]^{\mathrm{reg}}$ *is the regular part of* $\Delta \mathsf{d}_p^2$ (*i.e.*, *absolutely continuous with respect to* $\mathfrak{m}$).

Remark 4.18 (on the lower bound (4-38)). Denote by $\mathcal{C}_p := \{a(X_\alpha)\}_{\alpha \in Q}$ the cut locus of p. Then for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ so that for every bounded subset $W \subset X$ it holds

$$\begin{aligned} [\Delta \mathsf{d}_p^2]^{\mathrm{reg}}_{\llcorner W} &\ge 2\left(1 - (N-1)\mathsf{d}_p \frac{s'_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}\right)\mathfrak{m}\llcorner_W \\ &\ge -C_{\varepsilon, W}\mathfrak{m}\llcorner_W \quad \text{on } W \cap \{x = g_t(a_\alpha) : t \ge \varepsilon\} \supset W \cap \{x \in X : \mathsf{d}(x, \mathcal{C}_p) \ge \varepsilon\}. \end{aligned}$$

The representation formula for the Laplacian of the distance function follows from Corollary 4.17 by the chain rule [Gigli 2015, Proposition 4.11], writing $\mathrm{sgn}(v) d_v = \sqrt{d_v^2}$.

Corollary 4.19. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an e.n.b. metric measure space satisfying* $\mathsf{MCP}(K, N)$ *for some* $K \in \mathbb{R}$, $N \in (1, \infty)$. *Fix* $p \in X$, *consider* $\mathsf{d}_p := \mathsf{d}(p, \cdot)$ *and the associated disintegration*

$$\mathfrak{m} = \int_Q h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha} \, \mathfrak{q}(d\alpha).$$

Then $\mathsf{d}_p \in D(\mathbf{\Delta}, X \setminus \{p\})$ *and one element of* $\mathbf{\Delta}\mathsf{d}_{p \llcorner X \setminus \{p\}}$, *which we denote by* $\Delta \mathsf{d}_{p \llcorner X \setminus \{p\}}$, *is a Radon functional with the representation formula*

$$\Delta \mathsf{d}_{p \llcorner X \setminus \{p\}} = -(\log h_\alpha)'\mathfrak{m} - \int_Q h_\alpha \delta_{a(X_\alpha)} \, \mathfrak{q}(d\alpha). \tag{4-39}$$

Moreover, the following comparison results hold true:

$$\Delta \mathsf{d}_{p \llcorner X \setminus \{p\}} \le (N-1) \frac{s'_{K/(N-1)}(\mathsf{d}_p(x))}{s_{K/(N-1)}(\mathsf{d}_p(x))}\mathfrak{m}, \tag{4-40}$$

$$[\Delta \mathsf{d}_{p \llcorner X \setminus \{p\}}]^{\mathrm{reg}} := -(\log h_\alpha)'\mathfrak{m} \ge -(N-1) \frac{s'_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}{s_{K/(N-1)}(\mathsf{d}_{a(X_\alpha)}(x))}\mathfrak{m}, \tag{4-41}$$

where $[\Delta \mathsf{d}_{p \llcorner X \setminus \{p\}}]^{\mathrm{reg}}$ *is the regular part of* $\Delta \mathsf{d}_{p \llcorner X \setminus \{p\}}$ (*i.e.*, *absolutely continuous with respect to* $\mathfrak{m}$).

Remark 4.20. Corollary 4.19 should be compared with [Gigli 2015, Corollary 5.15, Remark 5.16], where it was proved that $\mathsf{d}_p \in D(\mathbf{\Delta}, X \setminus \{p\})$ together with the upper bound (4-40) under the assumption that $(X, \mathsf{d}, \mathfrak{m})$ is an infinitesimally strictly convex $\mathsf{MCP}(K, N)$-space.

Let us stress that, by the very definition, the Laplacian in the infinitesimally strictly convex setting is single-valued, simplifying the treatment.

One novelty of Corollary 4.19 is that the infinitesimal strict convexity is replaced by the essentially nonbranching property which, a priori, does not exclude a multivalued Laplacian. In addition to that, the geometrically new content of Corollary 4.19 when compared with [Gigli 2015] is that it contains an *exact representation formula* (4-39) which also gives the new lower bound (4-41).

Part II. Applications

In Part II of the paper we collect all the main applications of the results obtained in Part I.

5. The singular part of the Laplacian

In order to state the next corollary recall that from essentially nonbranching and $\mathsf{MCP}(K, N)$ it follows that for every fixed $p \in X$ and $\mathfrak{m}$-a.e. $x \in X$ (precisely on $\mathcal{T}^{\mathrm{nb}}_{\mathsf{d}_p}$) there exists a unique geodesic γ^x starting from x and arriving at p, i.e., $\gamma^x_0 = x$ and $\gamma^x_1 = p$. For each $t \in [0, 1]$, define the map

$$T_t : \mathcal{T}^{\mathrm{nb}}_{\mathsf{d}_p} \to \mathcal{T}^{\mathrm{nb}}_{\mathsf{d}_p}, \quad T_t(x) := \gamma^x_t. \tag{5-1}$$

It is worth noting that T_t is also the W_2-optimal transport map from the (renormalised) ambient measure $\mathfrak{m}$ to δ_p, provided $\mathfrak{m}(X) < \infty$.

The goal of the next proposition is to get some refined information on the cut locus $\mathcal{C}_p$ of p; more precisely, we infer an upper bound on an optimal transport-type Minkowski content of $\mathcal{C}_p$.

Proposition 5.1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an e.n.b. metric measure space satisfying $\mathsf{MCP}(K, N)$ for some $K \in \mathbb{R}$, $N \in (1, \infty)$. Fix any point $p \in X$ and consider for each $t \in [0, 1]$ the map T_t defined by (5-1).*

Then, for every bounded open subset $W \subset X$ it holds

$$\limsup_{\varepsilon \downarrow 0} \frac{\mathfrak{m}((X \setminus T_\varepsilon(X)) \cap W)}{\varepsilon} \le \|[\Delta \mathsf{d}_p^2]_{\mathrm{sing}}\|(W) < \infty. \tag{5-2}$$

Remark 5.2 (geometric meaning of Proposition 5.1). Fix $p \in X$, and consider $\mathsf{d}_p := \mathsf{d}(p, \cdot)$ and the associated disintegration $\mathfrak{m} = \int_Q h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha} \mathfrak{q}(d\alpha)$. Then the cut locus $\mathcal{C}_p$ of p coincides with the set of initial points $\{a(X_\alpha)\}_{\alpha \in Q}$ of the transport rays. The set $X \setminus T_\varepsilon(X)$ thus can be seen as an "optimal transport neighbourhood" of the cut locus $\mathcal{C}_p$ and therefore (5-2) gives an optimal transport-type estimate on a weak version of the codimension-1 Minkowski content of $\mathcal{C}_p$.

Since the cut locus of a point in an e.n.b. $\mathsf{MCP}(K, N)$ space can be dense (this can be the case already for the boundary of a convex body in $\mathbb{R}^3$), one cannot expect an upper bound on the classical codimension-1 Minkowski content of $\mathcal{C}_p$. The bound (5-2) looks interesting already in the classical setting of a smooth Riemannian manifold. Indeed it is well known that $\mathcal{C}_p$ is rectifiable with locally finite codimension-1

Hausdorff measure (see for instance [Mantegazza and Mennucci 2003]), but in the literature it seems not to be present any (local) bound on its codimension-1 Minkowski content.

Proof. If X is bounded, one can choose $W = X$ and the proof is easier (there is no need to introduce an intermediate set U in the arguments below); we thus discuss directly the case when X is not bounded.

Let $U \supset W$ be a bounded open subset such that W is compactly contained in U, in particular $\mathsf{d}(W, X \setminus U) > 0$.

With a slight abuse of notation, for ease of writing, in the next computations we identify the ray $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ with the real interval $((a_\alpha, b_\alpha), |\cdot|, h_\alpha \mathcal{L}^1)$ isomorphic to it as an m.m.s.

Recalling from Remark 2.14 that $h_\alpha : X_\alpha \simeq (a_\alpha, b_\alpha) \to \mathbb{R}^+$ is continuous up to the initial point a_α, it is clear that

$$h_\alpha(a(X_\alpha))\, \mathsf{d}_p(a(X_\alpha)) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{[a_\alpha, a_\alpha + \varepsilon |X_\alpha|]} h_\alpha(s)\, ds,$$

where $|X_\alpha|$ denotes the length of the transport ray X_α, i.e., $|X_\alpha| = \mathsf{d}(a(X_\alpha), b(X_\alpha)) = \mathsf{d}(a(X_\alpha), p)$. Hence, for any bounded open subset $U \subset X$ it holds

$$\begin{aligned} \|[\Delta \mathsf{d}_p^2]^{\text{sing}}\|(U) &= \int_{\{\alpha \in Q : a(X_\alpha) \in U\}} (h_\alpha \mathsf{d}_p)(a(X_\alpha))\, \mathfrak{q}(d\alpha) \\ &= \int_{\{\alpha \in Q : a(X_\alpha) \in U\}} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{[a_\alpha, a_\alpha + \varepsilon |X_\alpha|]} h_\alpha(s)\, ds\, \mathfrak{q}(d\alpha), \end{aligned}$$

where $\|[\Delta \mathsf{d}_p^2]^{\text{sing}}\|(U)$ denotes the total variation measure of U. Since by Corollary 4.17 we know that $\|[\Delta \mathsf{d}_p^2]^{\text{sing}}\|(U) < \infty$, by Fatou's lemma we infer

$$\limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{\{\alpha \in Q : a(X_\alpha) \in U\}} \int_{[a_\alpha, a_\alpha + \varepsilon |X_\alpha|]} h_\alpha(s)\, ds\, \mathfrak{q}(d\alpha) \le \|[\Delta \mathsf{d}_p^2]^{\text{sing}}\|(U) < \infty. \tag{5-3}$$

We then look for a more convenient expression of the left-hand side of the previous inequality. First, note that for ε sufficiently small such that

$$\varepsilon/(1-\varepsilon) < \frac{\mathsf{d}(W, X \setminus U)}{\mathsf{d}_p(W)}$$

it holds

$$\int_Q \int_{[a_\alpha, a_\alpha + \varepsilon |X_\alpha|] \cap W} h_\alpha(s)\, ds\, \mathfrak{q}(d\alpha) \le \int_{\{\alpha \in Q : a(X_\alpha) \in U\}} \int_{[a_\alpha, a_\alpha + \varepsilon |X_\alpha|]} h_\alpha(s)\, ds\, \mathfrak{q}(d\alpha). \tag{5-4}$$

Recalling the definition of the map T_t given in (5-1), we now claim that

$$\mathfrak{m}((X \setminus T_\varepsilon(X)) \cap W) = \int_Q \int_{[a_\alpha, a_\alpha + \varepsilon |X_\alpha|] \cap W} h_\alpha(s)\, ds\, \mathfrak{q}(d\alpha). \tag{5-5}$$

Indeed, on the one hand, by the disintegration theorem, Theorem 3.6, we know that

$$\mathfrak{m}((X \setminus T_\varepsilon(X)) \cap W) = \int_Q \int_{X_\alpha \cap (X \setminus T_\varepsilon(X)) \cap W} h_\alpha(s)\, ds\, \mathfrak{q}(d\alpha).$$

On the other hand, since trivially

$$X_\alpha \cap (X \setminus T_t(X)) \cap W = X_\alpha \setminus T_t(X) \cap W,$$

and since, as T_t is translating along $\mathcal{T}_{\mathsf{d}_p}^{\mathrm{nb}}$, one has $X_\alpha \setminus T_t(X) = X_\alpha \setminus T_t(X_\alpha)$, we obtain

$$X_\alpha \cap (X \setminus T_t(X)) \cap W = X_\alpha \setminus T_t(X_\alpha) \cap W.$$

The claim (5-5) follows. The combination of (5-3), (5-4) and (5-5) gives

$$\limsup_{\varepsilon \downarrow 0} \frac{\mathfrak{m}((X \setminus T_\varepsilon(X)) \cap W)}{\varepsilon} \leq \|[\Delta \mathsf{d}_p^2]^{\mathrm{sing}}\|(U) < \infty$$

for every U bounded open subset compactly containing the open set W. Since from Theorem 4.14 we know that $\Delta \mathsf{d}_p^2$ is a Radon measure, the thesis (5-2) follows. $\square$

We next give some sufficient condition implying that the densities h_α, given by the disintegration theorem, Theorem 3.6, are null at the final points.

Lemma 5.3. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an e.n.b. $\mathsf{MCP}(K, N)$ space for some $K \in \mathbb{R}$, $N \in (1, \infty)$.*

Let $u = \mathsf{d}_p = \mathsf{d}(p, \cdot)$ for some $p \in X$ and consider the disintegration associated to d_p:

$$\mathfrak{m} = \int_Q h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha} \mathfrak{q}(d\alpha).$$

Assume there exists $s > 1$ such that

$$\liminf_{r \downarrow 0} \frac{\mathfrak{m}(B_r(p))}{r^s} < \infty; \tag{5-6}$$

then $h_\alpha(p) = 0$ for $\mathfrak{q}$-a.e. $\alpha \in Q$.

More generally, for any 1-Lipschitz function u, denoting by

$$\mathfrak{m}\llcorner_{\mathcal{T}_u^{\mathrm{nb}}} = \int_Q h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha} \mathfrak{q}(d\alpha)$$

the associated disintegration, it holds that

$$\left\| \int_Q h_\alpha(a(X_\alpha)) \delta_{a(X_\alpha)} \, \mathfrak{q}(d\alpha) \right\| \leq \liminf_{r \downarrow 0} \frac{\mathfrak{m}\big(\bigcup_\alpha [a(X_\alpha), a(X_\alpha) + r]\big)}{r} = \beta \in [0, +\infty], \tag{5-7}$$

where the leftmost term is the total variation of the corresponding measure.

Proof. Suppose by contradiction the claim was false, i.e., there exists $\bar{Q} \subset Q$ where $h_\alpha(p) \geq c > 0$, with $\mathfrak{q}(\bar{Q}) > 0$. Observe that a.e. transport ray X_α ends in p, i.e., $b(X_\alpha) = p$ for $\mathfrak{q}$-a.e. $\alpha \in Q$. As usual, we identify the transport ray X_α with the real interval $[a_\alpha, b_\alpha]$ (the cases of semiclosed and open intervals are analogous). Then by Fatou's lemma it holds

$$\begin{aligned}
\liminf_{r \downarrow 0} \frac{\mathfrak{m}(B_r(p))}{r^s} &\geq \liminf_{r \downarrow 0} \int_{\bar{Q}} \frac{1}{r^s} \int_{[b_\alpha - r, b_\alpha]} h_\alpha(t) \, dt \, \mathfrak{q}(d\alpha) \\
&\geq \int_{\bar{Q}} \liminf_{r \downarrow 0} \frac{1}{r} \int_{[b_\alpha - r, b_\alpha]} \frac{h_\alpha(t)}{r^{s-1}} \, dt \, \mathfrak{q}(d\alpha) = \infty,
\end{aligned}$$

giving a contradiction and proving the claim.

The second part of the lemma follows along analogous arguments. $\square$

Remark 5.4. If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}(K, N)$ space not isometric to a circle or to a (possibly unbounded) real interval then (5-6) is satisfied for $\mathfrak{m}$-a.e. $p \in X$.

Indeed if $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}(K, N)$ space, using the rectifiability result [Mondino and Naber 2019, Theorem 1.1] (see also [Gigli et al. 2015a] and compare with [Cheeger and Colding 1997; 2000a; 2000b]) together with the absolute continuity of the reference measure $\mathfrak{m}$ with the respect to the Hausdorff measure of the bi-Lipschitz charts obtained independently in [Kell and Mondino 2018, Theorem 1.2] and [Gigli and Pasqualetto 2016, Theorem 3.5], it follows that for $\mathfrak{m}$-a.e. $p \in X$ there exists $n = n(p) \in \mathbb{N} \cap [1, \infty)$ such that

$$\liminf_{r \downarrow 0} \frac{\mathfrak{m}(B_r(p))}{r^n} < \infty.$$

If moreover we assume (X, d) not to be isometric to a circle or to a (possibly unbounded) real interval, then by [Kitabeppu and Lakzian 2016] it follows that $n(p) > 1$ for $\mathfrak{m}$-a.e. $p \in X$.

If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{MCP}(K, N)$ space then the validity of (5-6) is not known.

6. $\mathsf{CD}(K, N)$ is equivalent to a (K, N)-Bochner-type inequality

The Bochner inequality is one of the most fundamental estimates in geometric analysis. For a smooth N-dimensional Riemannian manifold (M, g) with $\mathrm{Ricci}_g \geq K g$, for some $K \in \mathbb{R}$, it states that for any smooth function $u \in C^3(M)$ it holds

$$\frac{1}{2}\Delta|\nabla u|^2 - \langle \nabla u, \nabla \Delta u \rangle \geq K|\nabla u|^2 + |\nabla^2 u|^2 \geq K|\nabla u|^2 + \frac{1}{N}(\Delta u)^2,$$

where $|\nabla^2 u|^2$ is the Hilbert–Schmidt norm of the Hessian matrix $\nabla^2 u$ and the rightmost inequality follows directly by the Cauchy–Schwarz inequality. Note in particular that if u is a distance function, then on an open dense set of full measure, $|\nabla u|^2 = 1$ and the Hessian is a block matrix with vanishing slot in the direction of the "gradient of the distance"; in particular, for a distance function the inequality can be improved to

$$-\langle \nabla u, \nabla \Delta u \rangle \geq K + \frac{1}{N-1}(\Delta u)^2 \quad \text{a.e.} \tag{6-1}$$

Finally, note that the term $-\langle \nabla u, \nabla \Delta u \rangle$ corresponds to "the derivative of Δu in the direction of $-\nabla u$"; thus, if we consider the transport set associated to u, such a term would correspond to what we denoted as $(\Delta u)'$. Since in a general m.m.s. it is not clear there is enough regularity to write $(\Delta u)'$, it is natural to consider the following version of (6-1) "integrated along transport rays":

$$\Delta u(g_t(x)) - \Delta u(x) \geq Kt + \frac{1}{N-1}\int_{(0,t)} (\Delta u)^2(g_s(x))\, ds. \quad \text{a.e. } x, t. \tag{6-2}$$

This is the (K, N)-Bochner inequality that will be proved to be equivalent to the $\mathsf{CD}(K, N)$ condition.

In order to state the results, it is useful to recall that given a 1-Lipschitz function u on an e.n.b. $\mathsf{CD}(K, N)$ space there is a natural disintegration of $\mathfrak{m}$ restricted to the transport set $\mathcal{T}_u^{\mathrm{nb}}$ (see Theorem 3.6):

$$\mathfrak{m}\llcorner_{\mathcal{T}_u^{\mathrm{nb}}} = \int_Q h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha} \mathfrak{q}(d\alpha). \tag{6-3}$$

We will define $\mathrm{int}(\mathcal{T}_u^{\mathrm{nb}}) := \bigcup_{\alpha\in Q} \mathring{X}_\alpha$, where $\mathring{X}_\alpha$ stands for the relative interior of X_α; it can also be identified by isometry with the open interval (a_α, b_α).

The function h_α in (6-3) is a $\mathsf{CD}(K, N)$ density on (a_α, b_α), so in particular it is semiconcave; thus if D_α is the set of differentiability points of h_α, then $(a_\alpha, b_\alpha) \setminus D_\alpha$ is countable.

Our next result roughly states that the (K, N)-Bochner-type inequality (6-2) holds for those 1-Lipschitz functions for which we have found an explicit representation formula for the Laplacian, namely those 1-Lipschitz functions satisfying the hypothesis of Theorem 4.8 and for any distance function with sign d_v. Recall that, for any u belonging to these classes of functions, Δu outside of the initial and final points of transport rays forming $\mathcal{T}_u^{\mathrm{nb}}$ is absolutely continuous with respect to $\mathfrak{m}$.

Theorem 6.1 ($\mathsf{CD}_{\mathrm{loc}}(K, N)$+e.n.b.$\Longrightarrow$$(K, N)$-Bochner-type inequality). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an e.n.b. metric measure space satisfying $\mathsf{CD}_{\mathrm{loc}}(K, N)$. Then the following hold*:

(1) *Let $u : X \to \mathbb{R}$ be any 1-Lipschitz function such that $\int_Q |X_\alpha|^{-1}\, \mathfrak{q}(d\alpha) < \infty$. Then for $\mathfrak{q}$-a.e. $\alpha \in Q$, for each $x \in X_\alpha$ it holds*

$$\Delta u(g_t(x)) - \Delta u(x) \geq Kt + \frac{1}{N-1}\int_{(0,t)} (\Delta u)^2(g_s(x))\, ds \tag{6-4}$$

for all $t \in \mathbb{R}$ such that $g_t(x) \in \mathcal{T}_u$, up to a countable set depending only on α.

(2) *Let $u = d_v$ be a signed distance function. Then for $\mathfrak{q}$-a.e. $\alpha \in Q$, for each $x \in \mathring{X}_\alpha \setminus \{v = 0\}$ the (K, N)-Bochner-type inequality (6-4) holds for all $t \in \mathbb{R}$ such that $g_t(x) \in \mathring{X}_\alpha \setminus \{v = 0\}$ and $\mathrm{sgn}(d_v(x)) = \mathrm{sgn}(d_v(g_t(x)))$, provided the densities $\Delta d_v(x)$ and $\Delta d_v(g_t(x))$ exist.*

Proof. We prove just (1), the proof of (2) being completely analogous (using Corollary 4.16 in place of Theorem 4.8).

Fix $\alpha \in Q$ and $x \in \mathrm{int}(R_u^{\mathrm{nb}}(\alpha)) = (a_\alpha, b_\alpha)$ for which the representation of Δu given by Theorem 4.8 is valid:

$$\Delta u(x) = -(\log h_\alpha)'(x).$$

In particular h_α is differentiable at x. As observed above, for each α, $\Delta u(x)$ is defined on $D_\alpha \subset (a_\alpha, b_\alpha)$, with $(a_\alpha, b_\alpha) \setminus D_\alpha$ countable. Therefore the claim reduces to showing for $\mathfrak{q}$-a.e. $\alpha \in Q$ that

$$(\log h_\alpha)'(x) - (\log h_\alpha)'(g_t(x)) \geq Kt + \frac{1}{N-1}\int_{(0,t)} ((\log h_\alpha)'(g_s(x)))^2\, ds, \tag{6-5}$$

whenever $x, g_t(x) \in D_\alpha$. To prove (6-5), consider a nonnegative C^2 function ψ supported on $[-1, 1]$ with $\int \psi = 1$. Let $\psi_\varepsilon(x) := \psi(x/\varepsilon)$; of course ψ_ε is supported on $[-\varepsilon, \varepsilon]$ with $\int \psi_\varepsilon = 1$. Define the function h_α^ε on $(a_\alpha + \varepsilon, b_\alpha - \varepsilon)$ by

$$\log h_\alpha^\varepsilon := \log h_\alpha * \psi_\varepsilon. \tag{6-6}$$

Since by Theorem 3.6 we know that h_α is a $\mathsf{CD}(K, N)$ density, and h_α^ε is a C^2-smooth $\mathsf{CD}(K, N)$ density on $(a_\alpha + \varepsilon, b_\alpha - \varepsilon)$ by Proposition 2.18; in particular (6-5) is satisfied by h_α^ε. Taking the limit as $\varepsilon \to 0$, we obtain that $(\log h_\alpha^\varepsilon)' \to (\log h_\alpha)'$ pointwise on D_α and in $L^1((a_\alpha, b_\alpha))$. Thus we can pass into the limit as $\varepsilon \to 0$ in (6-5) and get that it is also satisfied by h_α. ☐

Also the converse implication holds, giving a complete equivalence between the (K, N)-Bochner-type inequality (6-4) on signed distance functions and the $\mathsf{CD}(K, N)$ condition.

Theorem 6.2 ($\mathsf{MCP}(K', N')$ + e.n.b. + (K, N)-Bochner-type inequality $\Longrightarrow \mathsf{CD}(K, N)$). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an e.n.b. metric measure space satisfying $\mathsf{MCP}(K', N')$ for some $K' \in \mathbb{R}$, $N' \in (1, \infty)$, with $\mathfrak{m}(X) < \infty$. Assume that, for every signed distance function $d_v : X \to \mathbb{R}$, for $\mathfrak{q}$-a.e. $\alpha \in Q$, for each $x \in \mathring{X}_\alpha \setminus \{v = 0\}$ it holds*

$$\Delta d_v(g_t(x)) - \Delta d_v(x) \geq Kt + \frac{1}{N-1} \int_{(0,t)} (\Delta d_v)^2(g_s(x))\, ds \tag{6-7}$$

for all $t \in \mathbb{R}$ such that $g_t(x) \in \mathring{X}_\alpha \setminus \{v = 0\}$ and $\operatorname{sgn}(d_v(x)) = \operatorname{sgn}(d_v(g_t(x)))$, provided the densities $\Delta d_v(x)$ and $\Delta d_v(g_t(x))$ exist.

Then $(X, \mathsf{d}, \mathfrak{m})$ satisfies $\mathsf{CD}(K, N)$.

Remark 6.3. We briefly comment on the statement of Theorem 6.2. Using the assumption of e.n.b. and of $\mathsf{MCP}(K', N')$, we deduce from Corollary 4.16 that any $d_v \in D(\mathbf{\Delta}, X \setminus \{v = 0\})$. Therefore, in the assumption (6-7), we consider $\Delta d_v(g_t(x))$ only for those $g_t(x)$ belonging to $\{v > 0\}$ or to $\{v < 0\}$, provided $x \in \{v > 0\}$ or $x \in \{v < 0\}$ respectively.

Let us also comment on the assumptions $\mathsf{CD}_{\mathrm{loc}}(K, N)$ vs. $\mathsf{CD}(K, N)$ and $\mathfrak{m}(X) < \infty$ in the last two results. It was proved in [Cavalletti and Milman 2016] that, under the assumption $\mathfrak{m}(X) < \infty$, an e.n.b. $\mathsf{CD}_{\mathrm{loc}}(K, N)$ space satisfies $\mathsf{CD}(K, N)$ globally; on the other hand the implication is open without the assumption $\mathfrak{m}(X) < \infty$. We thus assumed $\mathsf{CD}_{\mathrm{loc}}(K, N)$ in Theorem 6.1 as, a priori, it is more general and still gives that all the conditional densities h_α are $\mathsf{CD}(K, N)$ densities (see Theorem 3.6).

Proof. We show that, given any 1-Lipschitz function $\varphi : X \to \mathbb{R}$, the conditional probabilities associated to the transport set $\mathcal{T}_\varphi^{\mathrm{nb}}$ of φ satisfy $\mathsf{CD}(K, N)$. From [Cavalletti and Milman 2016] it will then follow that $(X, \mathsf{d}, \mathfrak{m})$ satisfy $\mathsf{CD}(K, N)$.

<u>Step 1</u>: Let us fix $\varphi : X \to \mathbb{R}$ a 1-Lipschitz function and the associated nonbranched transport set $\mathcal{T}_\varphi^{\mathrm{nb}}$. Fix also $c \in \mathbb{R}$, let $\varphi_c := \varphi - c$ and consider the associated signed distance function d_{φ_c} from the level set $\{\varphi = c\}$.

Note that the function d_{φ_c} coincides with φ_c along $(R_\varphi^{\mathrm{nb}})^{-1}(\{\varphi = c\})$, i.e., along each transport ray of φ having nonempty intersection with $\{\varphi = c\}$.

Indeed, fix any $x \in \mathcal{T}_{\varphi_c}^{\mathrm{nb}}$ with $\varphi(x) \geq c$ (the argument for $\varphi(x) \leq c$ is analogous) such that there exists $y \in R_\varphi^{\mathrm{nb}}(x)$ with $\varphi(y) = c$ (i.e., $x \in (R_\varphi^{\mathrm{nb}})^{-1}(\{\varphi = c\})$); then for any other $z \in \{\varphi = c\}$ it holds

$$\mathsf{d}(x, y) = \varphi(x) - \varphi(y) = \varphi(x) - \varphi(z) \leq \mathsf{d}(x, z),$$

showing that $\mathsf{d}(x, y) = d_{\varphi_c}(x)$ and that $d_{\varphi_c}(x) = \varphi(x) - \varphi(y) = \varphi(x) - c = \varphi_c(x)$. Hence if $x \in (R_\varphi^{\mathrm{nb}})^{-1}(\{\varphi = c\})$, then

$$|d_{\varphi_c}(x) - d_c(y)| = \mathsf{d}(x, y)$$

for some $(x, y) \in (R_\varphi^{\mathrm{nb}})$ implying $(x, y) \in R_{d_{\varphi_c}}$. Since a branching structure for d_{φ_c} inside $(R_\varphi^{\mathrm{nb}})^{-1}(\{\varphi = c\})$ will imply a branching structure for φ_c, this implies that on $(R_\varphi^{\mathrm{nb}})^{-1}(\{\varphi = c\})$ the equivalence relation R_φ^{nb} implies $R_{d_{\varphi_c}}^{\mathrm{nb}}$. In particular it follows that $\mathcal{T}_\varphi^{\mathrm{nb}} \cap (R_\varphi^{\mathrm{nb}})^{-1}(\{\varphi = c\}) \subset \mathcal{T}_{d_{\varphi_c}}^{\mathrm{nb}}$.

Step 2: Consider the disintegrations associated to $\mathcal{T}_\varphi^{\mathrm{nb}}$ and to $\mathcal{T}_{d_{\varphi c}}^{\mathrm{nb}}$ via Theorem 3.6:

$$\mathfrak{m}\llcorner_{\mathcal{T}_\varphi^{\mathrm{nb}}} = \int_{Q_\varphi} \mathfrak{m}_{\alpha,\varphi}\, \mathfrak{q}_\varphi(d\alpha), \qquad \mathfrak{m}\llcorner_{\mathcal{T}_{d_{\varphi c}}^{\mathrm{nb}}} = \int_{Q_{d_{\varphi c}}} \mathfrak{m}_{\alpha,d_{\varphi c}}\, \mathfrak{q}_{d_{\varphi c}}(d\alpha),$$

with $\mathfrak{m}_{\alpha,\varphi} = h_{\alpha,\varphi}\mathcal{H}^1\llcorner_{X_{\alpha,\varphi}}$ and $\mathfrak{m}_{\alpha,d_{\varphi c}} = h_{\alpha,d_{\varphi c}}\mathcal{H}^1\llcorner_{X_{\alpha,d_{\varphi c}}}$.

From Step 1 and the uniqueness of the disintegration, it follows that up to a constant factor

$$h_{\alpha,\varphi} = h_{\alpha,d_{\varphi c}} \quad \text{on } X_{\alpha,\varphi},$$

for all those α such that $X_{\alpha,\varphi} \cap \{\varphi = c\} \neq \varnothing$. Moreover from Corollary 4.16 we deduce that

$$\Delta d_{\varphi c}\llcorner_{\mathring{X}_{\alpha,d_{\varphi c}}\cap\{\varphi\neq c\}} = -(\log h_{\alpha,d_{\varphi c}})'.$$

The last two identities together with the assumption (6-7) applied to d_{φ_c} imply that for all those α such that $X_{\alpha,\varphi} \cap \{\varphi = c\} \neq \varnothing$, for each $x \in \mathring{X}_{\alpha,\varphi} \cap \{\varphi \neq c\}$ it holds

$$-[(\log h_{\alpha,\varphi})'(g_t(x)) - (\log h_{\alpha,\varphi})'(x)] \geq Kt + \frac{1}{N-1}\int_{(0,t)} [(\log h_{\alpha,\varphi})']^2(g_s(x))\, ds \tag{6-8}$$

for all those t such that $\varphi(g_t(x)) > c$ provided $\varphi(x) > c$ (and appropriate modifications if $\varphi(x) < c$). Identifying $\mathring{X}_\alpha$ with the isometric real interval (a_α, b_α) and denoting with c_α the unique point corresponding to $\mathring{X}_\alpha \cap \{\varphi = c\}$, (6-8) becomes

$$-[(\log h_{\alpha,\varphi})'(x+t) - (\log h_{\alpha,\varphi})'(x)] \geq Kt + \frac{1}{N-1}\int_{(0,t)} [(\log h_{\alpha,\varphi})']^2(x+s)\, ds \tag{6-9}$$

for each $x \in (a_\alpha, c_\alpha)$ and t such that $x + t \leq c_\alpha$. We again regularise by logarithmic convolution, i.e., as in (6-6). In order to simplify the notation, we will omit the subscript φ. We have

$$(\log h_\alpha^\varepsilon)'(x) = \int (\log h_\alpha)'(y)\psi_\varepsilon(x-y)\, dx,$$
$$(\log h_\alpha^\varepsilon)'(y) - (\log h_\alpha^\varepsilon)'(y+t) = \int [(\log h_\alpha)'(x) - (\log h_\alpha)'(x+t)]\psi_\varepsilon(x-y)\, dx.$$

Moreover

$$\begin{aligned}\iint_{(0,t)} ((\log h_\alpha)'(x+s))^2\psi_\varepsilon(x-y)\, ds\, dx &= \int_{(0,t)}\int ((\log h_\alpha)'(x+s))^2\psi_\varepsilon(x-y)\, dx\, ds \\ &\geq \int_{(0,t)}\left(\int (\log h_\alpha)'(x+s)\psi_\varepsilon(x-y)\, dx\right)^2 ds \\ &= \int_{(0,t)} \log(h_\alpha^\varepsilon)'(y+s)^2\, ds.\end{aligned}$$

Hence (6-9) is valid for $\log h_{\alpha,\varphi}^\varepsilon$ for each $\varepsilon > 0$ implying (just differentiate in t) that $h_{\alpha,\varphi}^\varepsilon$ is a $\mathsf{CD}(K, N)$ density on (a_α, c_α). Letting $\varepsilon \downarrow 0$ we obtain that $h_{\alpha,\varphi}$ is a $\mathsf{CD}(K, N)$ density on (a_α, c_α). From the arbitrariness of c, we conclude that $h_{\alpha,\varphi}$ is a $\mathsf{CD}(K, N)$ density. Hence $(X, \mathsf{d}, \mathfrak{m})$ satisfies $\mathsf{CD}^1_{\mathrm{Lip}}(K, N)$

(see [Cavalletti and Milman 2016] for the definition of $\mathsf{CD}^1_{\mathrm{Lip}}(K, N)$). Then we can conclude using [Cavalletti and Milman 2016] that $(X, \mathsf{d}, \mathfrak{m})$ satisfies $\mathsf{CD}(K, N)$. ☐

7. Splitting theorem under MCP(0, N)

Before stating the main result of the section, let us introduce some notation.

Given a metric space (X, d), a curve $\bar\gamma : \mathbb{R} \to X$ is called *line* if it is an isometric immersion, i.e.,

$$\bar\gamma : \mathbb{R} \to X, \qquad \mathsf{d}(\bar\gamma_t, \bar\gamma_s) = |t - s| \quad \text{for all } s, t \in \mathbb{R}.$$

To a line $\bar\gamma : \mathbb{R} \to X$ we associate the Busemann functions

$$\mathsf{b}^+(x) := \lim_{t\to+\infty} \mathsf{d}(x, \bar\gamma_t) - t.$$

Straightforwardly from the triangle inequality, one can check that the Busemann functions are well-defined maps $\mathsf{b}^\pm : X \to \mathbb{R}$ and

$$|\mathsf{b}^\pm(x) - \mathsf{b}^\pm(y)| \le \mathsf{d}(x, y).$$

Since $\mathsf{b}^\pm$ are 1-Lipschitz functions, we can consider the associated nonbranching transport set $\mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^\pm}$ defined in (3-3).

Theorem 7.1 (splitting theorem). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an e.n.b. infinitesimally Hilbertian $\mathsf{MCP}(0, N)$ space containing a line. Then $(X, \mathfrak{m})$ is isomorphic as a measure space to a splitting $Q \times \mathbb{R}$.*

More precisely the following holds. Denoting by $\mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+} = \bigcup_{\alpha\in Q} X_\alpha$ the nonbranching transport set induced by b^+ with the associated (disjoint) decomposition in transport rays, it holds that $\mathfrak{m}(X \setminus \mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+}) = 0$ and the map

$$\Phi : \mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+} \to Q \times \mathbb{R}, \quad x \mapsto \Phi(x) := (\alpha(x), \mathsf{b}^+(x)), \tag{7-1}$$

is an isomorphism of measures spaces, i.e.,

- *Φ is a bijection,*
- *Φ induces an isomorphism between the σ-algebra of $\mathfrak{m}$-measurable subsets of $\mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+}$ and the σ-algebra of $\mathfrak{q} \otimes \mathcal{L}^1$-measurable subsets of $Q \times \mathbb{R}$, where $\mathfrak{q}$ is quotient measure in the disintegration $\mathfrak{m}\llcorner_{\mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+}} = \int_Q \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha)$ given by Theorem 3.6.*
- *$\Phi_\sharp \mathfrak{m}\llcorner_{\mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+}} = \mathfrak{q}' \otimes \mathcal{L}^1$. Here $\mathfrak{q}'$ is a nonnegative measure over Q equivalent to $\mathfrak{q}$, i.e., $\mathfrak{q}' \ll \mathfrak{q}$ and $\mathfrak{q} \ll \mathfrak{q}'$.*

Moreover, for every $\alpha \in Q$, the map $\mathsf{b}^+ : X_\alpha \to \mathbb{R}$ is an isometry.

If in addition (X, d) is nonbranching, then X is homeomorphic to a splitting $Q \times \mathbb{R}$. More precisely, $X = \mathcal{T}_{\mathsf{b}^+} = \mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+}$ and the map $\Phi : X \to Q \times \mathbb{R}$ defined in (7-1) is an homeomorphism. Here the set of rays Q is induced with the compact-open topology as a subset of $C(\mathbb{R}, X)$, where each ray is parametrised by $(\mathsf{b}^+)^{-1}$; i.e.,

given $\beta \in Q$, $\{\alpha_n\}_{n\in\mathbb{N}} \subset Q$, it holds $\beta = \lim_{n\to\infty} \alpha_n$ if and only if

$$0 = \lim_{n\to\infty} \sup_{t\in I} \mathsf{d}\big(X_{\alpha_n}((\mathsf{b}^+)^{-1}(t)), X_\beta((\mathsf{b}^+)^{-1}(t))\big) \quad \text{for every compact interval } I \subset \mathbb{R}. \tag{7-2}$$

Remark 7.2. For smooth Riemannian manifolds [Cheeger and Gromoll 1971], as well as for Ricci-limits [Cheeger and Colding 1996] and $\mathsf{RCD}(0, N)$ spaces [Gigli 2013], the splitting theorem has a stronger statement giving an *isometric splitting*. However under the assumptions of Theorem 7.1 it is not conceivable to expect also a splitting of the metric. Indeed the Heisenberg group $\mathbb{H}^n$ is an example of a nonbranching infinitesimally Hilbertian $\mathsf{MCP}(0, N)$ space [Juillet 2009] containing a line, which is homeomorphic and isomorphic as measure space to a splitting (indeed it is homeomorphic to $\mathbb{R}^n$ and the measure is exactly the n-dimensional Lebesgue measure) but it is not isometric to a splitting.

We start by establishing some preliminary lemmas on the properties of Busemann functions.

Lemma 7.3. *For any proper geodesic space*, $\mathcal{T}_{\mathsf{b}^{\pm}} = X$.

Proof. Fix any $x \in X$ and $s > 0$. For each $t \in \mathbb{R}$, consider a unit speed geodesic

$$\gamma^t : [0, \mathsf{d}(x, \bar{\gamma}_t)] \to X \quad \text{such that} \quad \gamma^t_0 = x \quad \text{and} \quad \gamma^t_{\mathsf{d}(x,\bar{\gamma}_t)} = \bar{\gamma}_t.$$

From the triangle inequality, $\lim_{t\to\pm\infty} \mathsf{d}(x, \bar{\gamma}_t) = \infty$. Hence any fixed $s > 0$, for $|t|$ sufficiently large, belongs to the domain of γ^t. Consider then the trivial identities

$$\mathsf{d}(x, \bar{\gamma}_t) - t - \mathsf{d}(\gamma^t_s, \bar{\gamma}_t) + t = \mathsf{d}(x, \gamma^t_s) = s > 0.$$

Taking the limit as $t \to +\infty$ and using uniform convergence gives

$$\mathsf{b}^+(x) - \mathsf{b}^+(y) = \mathsf{d}(x, y) = s > 0,$$

where y is any accumulation point of $\{\gamma^t_s\}_{t\geq 0}$. In particular this shows that each point $x \in X$ can be moved forwardly with respect to b^+ (into y) proving in particular that $x \in \mathcal{T}_{\mathsf{b}^+}$. The proof for b^- can be achieved along the same lines. □

The proof of Lemma 7.3 also proves the following corollary.

Corollary 7.4. *Let* (X, d) *be proper and geodesic. Then* $\mathfrak{b}_{\mathsf{b}^{\pm}} = \emptyset$; *i.e., the set of final points associated to* b^+ *and to* b^- *are both empty.*

Applying results from Part I we easily obtain the following result.

Proposition 7.5. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an e.n.b. metric measure space satisfying* $\mathsf{MCP}(0, N)$ *containing a line. Then* $\mathsf{b}^{\pm} \in D(\boldsymbol{\Delta}, X)$ *and there exists a Radon measure* $\Delta\mathsf{b}^{\pm} \in \boldsymbol{\Delta}\mathsf{b}^{\pm}$ *satisfying*

$$\Delta\mathsf{b}^{\pm} \leq 0. \tag{7-3}$$

Proof. We only prove the claim for b^+, the proof for b^- being analogous. First of all from Theorem 3.6, we have the disintegration

$$\mathfrak{m} = \int_Q h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha} \mathfrak{q}(d\alpha).$$

Thanks to Corollary 7.4 we deduce that each ray X_α is isomorphic to a right half-line (or to a full line), in particular it has infinite length.

The combination of Theorem 4.8 with Lemma 7.3 thus gives that $\mathsf{b}^+ \in D(\mathbf{\Delta}, X)$ and that

$$\Delta \mathsf{b}^+ := -\int_Q h'_\alpha \mathcal{H}^1 \llcorner_{X_\alpha} \mathfrak{q}(d\alpha) - \int_Q h_\alpha \delta_{a(X_\alpha)\cap U} \mathfrak{q}(d\alpha)$$

defines a Radon measure $\Delta \mathsf{b}^+ \in \mathbf{\Delta} \mathsf{b}^+$. We are left to show that $\Delta \mathsf{b}^+ \leq 0$.

As above, we identify X_α with the right half-line $[a_\alpha, \infty)$ endowed with the $\mathsf{MCP}(0, N)$ density h_α. Using (2-11), we deduce that for $a_\alpha < x_0 \leq x_1 < b < \infty$ it holds

$$\left(\frac{b - x_1}{b - x_0}\right)^{N-1} \leq \frac{h_\alpha(x_1)}{h_\alpha(x_0)}.$$

Letting $b \to \infty$, it follows that $h_\alpha(x_0) \leq h_\alpha(x_1)$, showing that $h'_\alpha \geq 0$ whenever h'_α exists.

Thus $\Delta \mathsf{b}^+ \leq 0$ and the proposition follows. □

Observe also that, by triangle inequality, one has

$$\mathsf{d}(x, \bar{\gamma}_t) - t + \mathsf{d}(x, \bar{\gamma}_{-s}) - s \geq 0.$$

Setting $\mathsf{b} := \mathsf{b}^+ + \mathsf{b}^-$ and letting $t, s \to \infty$, it gives

$$\mathsf{b} \geq 0 \quad \text{on } X \qquad \text{and} \qquad \mathsf{b} \equiv 0 \quad \text{on } \bar{\gamma}. \tag{7-4}$$

From now on we assume $(X, \mathsf{d}, \mathfrak{m})$ to be infinitesimally Hilbertian, which is equivalent to assuming that the Laplacian Δ is single-valued (on its domain) and linear. Proposition 7.5 then implies

$$\mathsf{b} := \mathsf{b}^+ + \mathsf{b}^- \in D(\mathbf{\Delta}, X), \qquad \Delta \mathsf{b} \leq 0. \tag{7-5}$$

It is worth noting that (7-5) will be the only implication of the paper where infinitesimal Hilbertianity plays a role. We now want to combine (7-5) and (7-4) with the strong maximum principle in order to infer that $\mathsf{b} \equiv 0$. The next statement was proved in [Björn and Björn 2011, Theorem 9.13] (actually we report a slightly weaker statement which will suffice for our purposes).

Theorem 7.6 (strong maximum principle). *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space supporting a local weak $(1, 2)$-Poincaré inequality with $\mathfrak{m}$ locally doubling. Let $u \in \mathrm{LIP}(X)$ and $\Omega \subset X$ be a connected bounded open subset.*

If u attains its maximum in an interior point of Ω and

$$\int_\Omega |\nabla u|^2 \,\mathfrak{m} \leq \int_\Omega |\nabla(u + f)|^2 \,\mathfrak{m} \quad \textit{for all } f \in \mathrm{LIP}(X),\ \mathrm{supp}(f) \subset \Omega,\ f \leq 0, \tag{7-6}$$

then u is constant on Ω.

Let us discuss the validity of the strong maximum principle in our setting. Clearly, from Bishop–Gromov inequality it follows that an $\mathsf{MCP}(0, N)$ space is doubling. Moreover, essentially nonbranching $\mathsf{MCP}(0, N)$ spaces satisfy a local weak $(1, 1)$-Poincaré inequality [von Renesse 2008] (that work assumes negligibility of cut-locus from $\mathfrak{m}$-a.e. point that is satisfied whenever the space is essentially nonbranching, see Remark 2.6), which in turns implies that the space supports a local weak $(1, 2)$-Poincaré inequality. In conclusion if $(X, \mathsf{d}, \mathfrak{m})$ is an essentially nonbranching $\mathsf{MCP}(0, N)$ space, then the strong maximum principle holds. The simple link between (7-6) and the measure-valued Laplacian was established in [Gigli and Mondino 2013, Theorem 4.3]; for completeness, below we report the argument together with the desired conclusion $\mathsf{b} \equiv 0$.

Lemma 7.7. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an infinitesimally Hilbertian*, *essentially nonbranching*, *metric measure space satisfying* $\mathsf{MCP}(0, N)$. *Assume* (X, d) *contains a line and let* $\mathsf{b} := \mathsf{b}^+ + \mathsf{b}^-$.

Then $\mathsf{b} \equiv 0$ *on* X.

Proof. It is enough to prove that (7-5) implies (7-6) for $u := -\mathsf{b}$; then the claim will follow by the combination of (7-4) with Theorem 7.6.

Let $\Omega \subset X$ be a connected bounded open subset and $f \in \mathrm{LIP}(X)$ be nonpositive with $\mathrm{supp}(f) \subset \Omega$. Since the map $\varepsilon \mapsto \int_\Omega |\nabla(-\mathsf{b} + \varepsilon f)|^2\, \mathfrak{m}$ is convex and $\Delta \mathsf{b} \le 0$, we have

$$\begin{aligned}\int_\Omega |\nabla(-\mathsf{b} + f)|^2\, \mathfrak{m} - \int_\Omega |\nabla(-\mathsf{b})|^2\, \mathfrak{m} &\ge \lim_{\varepsilon \downarrow 0} \int_\Omega \frac{|\nabla(-\mathsf{b} + \varepsilon f)|^2 - |\nabla(-\mathsf{b})|^2}{\varepsilon}\, \mathfrak{m} \\ &= -2 \int_\Omega \langle \nabla \mathsf{b}, \nabla f \rangle\, \mathfrak{m} = 2 \int_\Omega f\, \Delta \mathsf{b} \ge 0,\end{aligned}$$

proving (7-6) for $u := -\mathsf{b}$. ☐

Lemma 7.8. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an infinitesimally Hilbertian*, *essentially nonbranching*, *metric measure space satisfying* $\mathsf{MCP}(0, N)$. *Assume* (X, d) *contains a line. Let* $\mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+} = \bigcup_{\alpha \in Q} X_\alpha$ *be the ray decomposition of the nonbranching transport set* $\mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+}$ *associated to* b^+.

Then for each $\alpha \in Q$, *the ray* X_α *is isometric to* $\mathbb{R}$; *in other words* $a(X_\alpha) = \varnothing = b(X_\alpha)$.

Proof. From Lemma 7.7 we know that $\mathsf{b}^+ = -\mathsf{b}^-$ on all X. It follows that

$$\{(x, y) \in R^{\mathrm{nb}}_{\mathsf{b}^+}\} = \{(y, x) \in R^{\mathrm{nb}}_{\mathsf{b}^-}\}.$$

Thus $\mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+} = \mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^-}$ with the same ray decomposition (from the support sense); clearly, on each ray, the orientation induced by b^+ is the opposite from the one induced by b^-. In particular, the set of initial points for b^+ coincides with the set of final points for b^-:

$$\mathfrak{a}_{\mathsf{b}^+} := \{x \in \mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+} : (y, x) \in R^{\mathrm{nb}}_{\mathsf{b}^+} \iff y = x\} = \{x \in \mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^-} : (x, y) \in R^{\mathrm{nb}}_{\mathsf{b}^-} \iff x = y\} =: \mathfrak{b}_{\mathsf{b}^-}.$$

Since from Corollary 7.4 the set of final points for b^- is empty, i.e., $\mathfrak{b}_{\mathsf{b}^-} = \varnothing$, it follows that both the sets of initial and final points for b^+ are empty; in other words, each ray X_α is isometric to $\mathbb{R}$. ☐

Proof of the splitting theorem, Theorem 7.1. By combining the lemmas above we can quickly get the first part of Theorem 7.1. Indeed, from Lemma 7.3 we already know that $X = \mathcal{T}_{\mathsf{b}^+}$ and, from Lemma 3.5 we know that $\mathfrak{m}(\mathcal{T}_{\mathsf{b}^+} \setminus \mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+}) = 0$; thus the claim $\mathfrak{m}(X \setminus \mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+}) = 0$ is proved.

Moreover, Theorem 3.6 ensures that there exists a disintegration of $\mathfrak{m}$ satisfying

$$\mathfrak{m}\llcorner_{\mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+}} = \int_Q \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha), \qquad \mathfrak{q}(Q) = 1,$$

where, for $\mathfrak{q}$-a.e. α, $\mathfrak{m}_\alpha$ is a Radon measure $\mathfrak{m}_\alpha \ll \mathcal{H}^1\llcorner_{X_\alpha}$ and $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ satisfies $\mathsf{MCP}(0, N)$.

From Lemma 7.8 we know that (X_α, d) is isometric to the real line (note that the isometry is simply $\mathsf{b}^+ : X_\alpha \to \mathbb{R}$), and thus Lemma 2.17 implies that $\mathfrak{m}_\alpha = c_\alpha \mathcal{H}^1 \llcorner_{X_\alpha}$ for some constant $c_\alpha > 0$, for $\mathfrak{q}$-a.e. $\alpha \in Q$.

Define the measure $\mathfrak{q}'$ on Q as

$$\mathfrak{q}'(B) = \int_B c_\alpha \, \mathfrak{q}(d\alpha) \quad \text{for any } \mathfrak{q}\text{-measurable subset } B \subset Q.$$

It is clear that $\mathfrak{q}' \ll \mathfrak{q}$ and that $\mathfrak{q} \ll \mathfrak{q}'$, i.e., they are equivalent measures, and that

$$\mathfrak{m}\llcorner_{\mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+}} = \int_Q \mathcal{H}^1 \llcorner_{X_\alpha} \, \mathfrak{q}'(d\alpha).$$

The last disintegration formula is equivalent to claiming that the map

$$\Phi : \mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+} \to Q \times \mathbb{R}, \qquad x \mapsto \Phi(x) := (\alpha(x), \mathsf{b}^+(x)),$$

is an isomorphism of measures spaces, i.e., Φ induces an isomorphism between the σ-algebra of $\mathfrak{m}$-measurable subsets of $\mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+}$ and the σ-algebra of $\mathfrak{q}\otimes\mathcal{L}^1$-measurable subsets of $Q\times\mathbb{R}$, and $\Phi_\sharp \mathfrak{m}\llcorner_{\mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+}} = \mathfrak{q}'\otimes\mathcal{L}^1$. It is also clear that $\Phi : \mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+} \to Q \times \mathbb{R}$ is bijective, as $\mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+} = \bigcup_{\alpha\in Q} X_\alpha$ is a partition, and $\mathsf{b}^+ : X_\alpha \to \mathbb{R}$ is an isometry for every $\alpha \in Q$.

We now prove the second part of Theorem 7.1. From the very definition (3-3) of the nonbranched transport set $\mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+}$, if (X, d) is nonbranching then $\mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+} = \mathcal{T}_{\mathsf{b}^+}$. Thus, Lemma 7.3 gives $X = \mathcal{T}_{\mathsf{b}^+} = \mathcal{T}^{\mathrm{nb}}_{\mathsf{b}^+}$.

From the first part, we already know that $\Phi : X \to Q \times \mathbb{R}$ is bijective. Since convergence in Q (see (7-2)) is equivalent to the local uniform convergence of the rays, it is clear that Φ^{-1} is continuous.

It is then enough to show that Φ is continuous. We argue by contradiction. Assume that there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ with $x_n \to x$ in X such that $\{(\alpha(x_n), \mathsf{b}^+(x_n))\}_{n\in\mathbb{N}}$ does not converge to $(\alpha(x), \mathsf{b}^+(x))$. Since $\mathsf{b}^+ : X \to \mathbb{R}$ is continuous (actually it is even 1-Lipschitz), it is clear that $\mathsf{b}^+(x_n) \to \mathsf{b}^+(x)$ and thus it must be that $\{\alpha(x_n)\}_{n\in\mathbb{N}}$ does not converge to $\alpha(x)$. By the definition (7-2) of convergence in Q, it follows that, up to subsequences, it holds

$$0 < \varepsilon = \lim_{n\to\infty} \sup_{t\in I} \mathsf{d}\big(X_{\alpha(x_n)}((\mathsf{b}^+)^{-1}(t)), X_{\alpha(x)}((\mathsf{b}^+)^{-1}(t))\big) \quad \text{for some compact interval } I \subset \mathbb{R}. \tag{7-7}$$

As already observed, $\mathsf{b}^+ : X_\beta \to \mathbb{R}$ is an isometry for every $\beta \in Q$ and thus it can be used to parametrise each ray; in the formula above as well as in the following we fix such a parametrisation.

Since by assumption $x_n \to x$, for every closed interval $I \subset \mathbb{R}$ containing $\mathsf{b}^+(x)$, it is clear that the union of the images of the rays $X_{\alpha(x_n)}$ restricted to I are all contained in a compact subset of X. Thus, the by Arzelà–Ascoli theorem, such restrictions converge uniformly to a geodesic γ of X passing through x. By a standard diagonal argument, γ can be extended to a geodesic defined on the whole $\mathbb{R}$ and

$$X_{\alpha(x_n)} \to \gamma \quad \text{uniformly on compact intervals.} \tag{7-8}$$

Recalling that the relation R_{b^+} is closed (see (3-1) and (3-2)) we get that γ is a ray passing through x, i.e., $\gamma = X_\beta$ for some $\beta \in Q$. Since the rays are pairwise disjoint, it follows that $\beta = \alpha(x)$.

Therefore (7-8) contradicts (7-7). $\square$

Acknowledgements

Mondino acknowledges the support of the EPSRC First Grant EP/R004730/1 and of the ERC Starting Grant 802689. The authors wish to thank the anonymous reviewers for the careful reading and for their comments, which led to an improvement in the exposition.

References

[Ambrosio 2018] L. Ambrosio, "Calculus, heat flow and curvature-dimension bounds in metric measure spaces", pp. 301–340 in *Proc. Int. Congress Math., I* (Rio de Janiero, 2018), World Sci., Singapore, 2018.

[Ambrosio et al. 2014a] L. Ambrosio, N. Gigli, and G. Savaré, "Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below", *Invent. Math.* **195**:2 (2014), 289–391. MR Zbl

[Ambrosio et al. 2014b] L. Ambrosio, N. Gigli, and G. Savaré, "Metric measure spaces with Riemannian Ricci curvature bounded from below", *Duke Math. J.* **163**:7 (2014), 1405–1490. MR Zbl

[Ambrosio et al. 2015a] L. Ambrosio, N. Gigli, A. Mondino, and T. Rajala, "Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure", *Trans. Amer. Math. Soc.* **367**:7 (2015), 4661–4701. MR Zbl

[Ambrosio et al. 2015b] L. Ambrosio, N. Gigli, and G. Savaré, "Bakry–Émery curvature-dimension condition and Riemannian Ricci curvature bounds", *Ann. Probab.* **43**:1 (2015), 339–404. MR Zbl

[Ambrosio et al. 2019] L. Ambrosio, A. Mondino, and G. Savaré, *Nonlinear diffusion equations and curvature conditions in metric measure spaces*, Mem. Amer. Math. Soc. **1270**, Amer. Math. Soc., Providence, RI, 2019. MR Zbl

[Bacher and Sturm 2010] K. Bacher and K.-T. Sturm, "Localization and tensorization properties of the curvature-dimension condition for metric measure spaces", *J. Funct. Anal.* **259**:1 (2010), 28–56. MR Zbl

[Barilari and Rizzi 2018] D. Barilari and L. Rizzi, "Sharp measure contraction property for generalized H-type Carnot groups", *Commun. Contemp. Math.* **20**:6 (2018), art. id. 1750081. MR Zbl

[Bianchini and Cavalletti 2013] S. Bianchini and F. Cavalletti, "The Monge problem for distance cost in geodesic spaces", *Comm. Math. Phys.* **318**:3 (2013), 615–673. MR Zbl

[Björn and Björn 2011] A. Björn and J. Björn, *Nonlinear potential theory on metric spaces*, EMS Tracts in Math. **17**, Eur. Math. Soc., Zürich, 2011. MR Zbl

[Calabi 1958] E. Calabi, "An extension of E. Hopf's maximum principle with an application to Riemannian geometry", *Duke Math. J.* **25**:1 (1958), 45–56. MR Zbl

[Cavalletti 2014] F. Cavalletti, "Monge problem in metric measure spaces with Riemannian curvature-dimension condition", *Nonlinear Anal.* **99** (2014), 136–151. MR Zbl

[Cavalletti and Milman 2016] F. Cavalletti and E. Milman, "The globalization theorem for the curvature dimension condition", preprint, 2016. arXiv

[Cavalletti and Mondino 2017a] F. Cavalletti and A. Mondino, "Optimal maps in essentially non-branching spaces", *Commun. Contemp. Math.* **19**:6 (2017), art. id. 1750007. MR Zbl

[Cavalletti and Mondino 2017b] F. Cavalletti and A. Mondino, "Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds", *Invent. Math.* **208**:3 (2017), 803–849. MR Zbl

[Cavalletti and Mondino 2018] F. Cavalletti and A. Mondino, "Almost Euclidean isoperimetric inequalities in spaces satisfying local Ricci curvature lower bounds", *Int. Math. Res. Not.* (online publication April 2018).

[Cheeger 1999] J. Cheeger, "Differentiability of Lipschitz functions on metric measure spaces", *Geom. Funct. Anal.* **9**:3 (1999), 428–517. MR Zbl

[Cheeger 2001] J. Cheeger, *Degeneration of Riemannian metrics under Ricci curvature bounds*, Scuola Normale Superiore, Pisa, 2001. MR Zbl

[Cheeger and Colding 1996] J. Cheeger and T. H. Colding, "Lower bounds on Ricci curvature and the almost rigidity of warped products", *Ann. of Math.* (2) **144**:1 (1996), 189–237. MR Zbl

[Cheeger and Colding 1997] J. Cheeger and T. H. Colding, "On the structure of spaces with Ricci curvature bounded below, I", *J. Differential Geom.* **45**:3 (1997), 406–480. MR Zbl

[Cheeger and Colding 2000a] J. Cheeger and T. H. Colding, "On the structure of spaces with Ricci curvature bounded below, II", *J. Differential Geom.* **54**:1 (2000), 13–35. MR Zbl

[Cheeger and Colding 2000b] J. Cheeger and T. H. Colding, "On the structure of spaces with Ricci curvature bounded below, III", *J. Differential Geom.* **54**:1 (2000), 37–74. MR Zbl

[Cheeger and Gromoll 1971] J. Cheeger and D. Gromoll, "The splitting theorem for manifolds of nonnegative Ricci curvature", *J. Differential Geom.* **6** (1971), 119–128. MR Zbl

[Colding 1996a] T. H. Colding, "Large manifolds with positive Ricci curvature", *Invent. Math.* **124**:1-3 (1996), 193–214. MR Zbl

[Colding 1996b] T. H. Colding, "Shape of manifolds with positive Ricci curvature", *Invent. Math.* **124**:1-3 (1996), 175–191. MR Zbl

[Colding 1997] T. H. Colding, "Ricci curvature and volume convergence", *Ann. of Math.* (2) **145**:3 (1997), 477–501. MR Zbl

[Colding and Naber 2012] T. H. Colding and A. Naber, "Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications", *Ann. of Math.* (2) **176**:2 (2012), 1173–1229. MR Zbl

[Erbar et al. 2015] M. Erbar, K. Kuwada, and K.-T. Sturm, "On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces", *Invent. Math.* **201**:3 (2015), 993–1071. MR Zbl

[Fremlin 2003] D. H. Fremlin, *Measure theory, IV: Topological measure spaces, I*, Torres Fremlin, Colchester, UK, 2003.

[Gigli 2013] N. Gigli, "The splitting theorem in non-smooth context", preprint, 2013. arXiv

[Gigli 2015] N. Gigli, *On the differential structure of metric measure spaces and applications*, Mem. Amer. Math. Soc. **1113**, Amer. Math. Soc., Providence, RI, 2015. MR Zbl

[Gigli and Mondino 2013] N. Gigli and A. Mondino, "A PDE approach to nonlinear potential theory in metric measure spaces", *J. Math. Pures Appl.* (9) **100**:4 (2013), 505–534. MR Zbl

[Gigli and Pasqualetto 2016] N. Gigli and E. Pasqualetto, "Behaviour of the reference measure on RCD spaces under charts", preprint, 2016. arXiv

[Gigli et al. 2015a] N. Gigli, A. Mondino, and T. Rajala, "Euclidean spaces as weak tangents of infinitesimally Hilbertian metric measure spaces with Ricci curvature bounded below", *J. Reine Angew. Math.* **705** (2015), 233–244. MR Zbl

[Gigli et al. 2015b] N. Gigli, A. Mondino, and G. Savaré, "Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows", *Proc. Lond. Math. Soc.* (3) **111**:5 (2015), 1071–1129. MR Zbl

[Juillet 2009] N. Juillet, "Geometric inequalities and generalized Ricci bounds in the Heisenberg group", *Int. Math. Res. Not.* **2009**:13 (2009), 2347–2373. MR Zbl

[Kell and Mondino 2018] M. Kell and A. Mondino, "On the volume measure of non-smooth spaces with Ricci curvature bounded below", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **18**:2 (2018), 593–610. MR Zbl

[Ketterer 2015] C. Ketterer, "Cones over metric measure spaces and the maximal diameter theorem", *J. Math. Pures Appl.* (9) **103**:5 (2015), 1228–1275. MR Zbl

[Kitabeppu and Lakzian 2016] Y. Kitabeppu and S. Lakzian, "Characterization of low dimensional $\mathrm{RCD}^*(K, N)$ spaces", *Anal. Geom. Metr. Spaces* **4**:1 (2016), 187–215. MR Zbl

[Klartag 2017] B. Klartag, *Needle decompositions in Riemannian geometry*, Mem. Amer. Math. Soc. **1180**, Amer. Math. Soc., Providence, RI, 2017. MR Zbl

[Li and Yau 1986] P. Li and S.-T. Yau, "On the parabolic kernel of the Schrödinger operator", *Acta Math.* **156**:3-4 (1986), 153–201. MR Zbl

[Lott and Villani 2009] J. Lott and C. Villani, "Ricci curvature for metric-measure spaces via optimal transport", *Ann. of Math.* (2) **169**:3 (2009), 903–991. MR Zbl

[Mantegazza and Mennucci 2003] C. Mantegazza and A. C. Mennucci, "Hamilton–Jacobi equations and distance functions on Riemannian manifolds", *Appl. Math. Optim.* **47**:1 (2003), 1–25. MR Zbl

[Mantegazza et al. 2014] C. Mantegazza, G. Mascellani, and G. Uraltsev, "On the distributional Hessian of the distance function", *Pacific J. Math.* **270**:1 (2014), 151–166. MR Zbl

[Mondino 2015] A. Mondino, "A new notion of angle between three points in a metric space", *J. Reine Angew. Math.* **706** (2015), 103–121. MR Zbl

[Mondino and Naber 2019] A. Mondino and A. Naber, "Structure theory of metric measure spaces with lower Ricci curvature bounds", *J. Eur. Math. Soc.* **21**:6 (2019), 1809–1854. MR Zbl

[Ohta 2007a] S.-i. Ohta, "On the measure contraction property of metric measure spaces", *Comment. Math. Helv.* **82**:4 (2007), 805–828. MR Zbl

[Ohta 2007b] S.-i. Ohta, "Products, cones, and suspensions of spaces with the measure contraction property", *J. Lond. Math. Soc.* (2) **76**:1 (2007), 225–236. MR Zbl

[Ohta and Sturm 2014] S.-i. Ohta and K.-T. Sturm, "Bochner–Weitzenböck formula and Li–Yau estimates on Finsler manifolds", *Adv. Math.* **252** (2014), 429–448. MR Zbl

[Petersen 1998] P. Petersen, *Riemannian geometry*, Graduate Texts in Math. **171**, Springer, 1998. MR Zbl

[Petrunin 2011] A. Petrunin, "Alexandrov meets Lott–Villani–Sturm", *Münster J. Math.* **4** (2011), 53–64. MR Zbl

[Rajala and Sturm 2014] T. Rajala and K.-T. Sturm, "Non-branching geodesics and optimal maps in strong CD(K, ∞)-spaces", *Calc. Var. Partial Differential Equations* **50**:3-4 (2014), 831–846. MR Zbl

[von Renesse 2008] M.-K. von Renesse, "On local Poincaré via transportation", *Math. Z.* **259**:1 (2008), 21–31. MR Zbl

[Rifford 2013] L. Rifford, "Ricci curvatures in Carnot groups", *Math. Control Relat. Fields* **3**:4 (2013), 467–487. MR Zbl

[Rizzi 2016] L. Rizzi, "Measure contraction properties of Carnot groups", *Calc. Var. Partial Differential Equations* **55**:3 (2016), art. id. 60. MR Zbl

[Sturm 2006a] K.-T. Sturm, "On the geometry of metric measure spaces, I", *Acta Math.* **196**:1 (2006), 65–131. MR Zbl

[Sturm 2006b] K.-T. Sturm, "On the geometry of metric measure spaces, II", *Acta Math.* **196**:1 (2006), 133–177. MR Zbl

[Villani 2009] C. Villani, *Optimal transport: old and new*, Grundlehren der Math. Wissenschaften **338**, Springer, 2009. MR Zbl

[Villani 2019] C. Villani, "Inégalités isopérimétriques dans les espaces métriques mesurés (d'après F. Cavalletti & A. Mondino)", exposé 1127, pp. 213–265 in *Séminaire Bourbaki,* 2016/2017, Astérisque **407**, Soc. Math. France, Paris, 2019. MR

Received 20 Nov 2018. Revised 21 Jul 2019. Accepted 6 Sep 2019.

FABIO CAVALLETTI: cavallet@sissa.it
Mathematics Area, SISSA, Trieste, Italy

ANDREA MONDINO: andrea.mondino@maths.ox.ac.uk
Current address: *Mathematical Institute, University of Oxford, Oxford, United Kingdom*
Mathematics Institute, University of Warwick, Coventry, United Kingdom

Analysis & PDE
msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2020 is US \$340/year for the electronic version, and \$550/year (+\$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

ANALYSIS & PDE

Volume 13 No. 7 2020