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C^*-ALGEBRAS ISOMORPHICALLY REPRESENTABLE ON l^p

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Let $p \in (1, \infty)\backslash\{2\}$. We show that every homomorphism from a C^*-algebra $\mathcal{A}$ into $B(l^p(J))$ satisfies a compactness property where J is any set. As a consequence, we show that a C^*-algebra $\mathcal{A}$ is isomorphic to a subalgebra of $B(l^p(J))$, for some set J, if and only if $\mathcal{A}$ is residually finite-dimensional.

1. Introduction

For $1 \leq p < \infty$ and a set J, let $l^p(J)$ be the space

$$\left\{ f : J \to \mathbb{C} : \sum_{j \in J} |f(j)|^p < \infty \right\}$$

with norm

$$\|f\| = \left(\sum_{j \in J} |f(j)|^p \right)^{\frac{1}{p}}.$$

Two Banach algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are *isomorphic* if there exist a bijective homomorphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ and $C > 0$ such that

$$\frac{1}{C}\|a\| \leq \|\phi(a)\| \leq C\|a\|$$

for all $a \in \mathcal{A}_1$. The algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are *isometrically isomorphic* if, moreover, ϕ can be chosen so that $\|\phi(a)\| = \|a\|$ for all $a \in \mathcal{A}_1$.

Gardella and Thiel [2020] showed that for $p \in [1, \infty)\backslash\{2\}$, a C^*-algebra $\mathcal{A}$ is isometrically isomorphic to a subalgebra of $B(l^p(J))$, for some set J, if and only if $\mathcal{A}$ is commutative. So it is natural to consider the question of whether this result holds if we relax the condition of isometrically isomorphic to isomorphic. In this paper, we show that for $p \in (1, \infty)\backslash\{2\}$, a C^*-algebra $\mathcal{A}$ is isomorphic to a subalgebra of $B(l^p(J))$, for some set J, if and only if $\mathcal{A}$ is residually finite-dimensional (Corollary 2.2). We prove this by showing that every homomorphism from a C^*-algebra $\mathcal{A}$ into $B(l^p(J))$ satisfies a compactness property (Theorem 2.1).

The proofs of the main results Theorem 2.1 and Corollary 2.2 in this paper are quite different from the proof of Gardella and Thiel's result. Lamperti's characterization [1958] of isometries on L^p, for $p \neq 2$, plays a crucial role in the proof of Gardella and Thiel's result, while uniform convexity of l^p, for $1 < p < \infty$, and an argument in probability that imitates the proof of Khintchine's inequality [Lindenstrauss and Tzafriri 1977, Theorem 2.b.3], for $p = 1$, are used in the proof of Theorem 2.1.

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2. Main results and proofs

Throughout this paper, the scalar field is $\mathbb{C}$. For algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, a *homomorphism* $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a bounded linear map such that $\phi(a_1a_2) = \phi(a_1)\phi(a_2)$ for all $a_1, a_2 \in \mathcal{A}$. For an element a of a C^*-algebra, $|a| = \sqrt{a^*a}$. The algebra of bounded linear operators on a Banach space $\mathcal{X}$ is denoted by $B(\mathcal{X})$ and the dual of $\mathcal{X}$ is denoted by $\mathcal{X}^*$. For $1 \le p \le \infty$, the l^p direct sum of Banach spaces $\mathcal{X}_\alpha$, for $\alpha \in \Lambda$, is denoted by $\left(\bigoplus_{\alpha\in\Lambda} \mathcal{X}_\alpha\right)_{l^p}$. Two Banach spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ are *isomorphic* if there is an invertible operator $S : \mathcal{X}_1 \to \mathcal{X}_2$. A C^*-algebra $\mathcal{A}$ is *residually finite-dimensional* if for every $a \in \mathcal{A}$, there is a $*$-representation ϕ of $\mathcal{A}$ on a finite-dimensional space such that $\phi(a) \neq 0$.

Theorem 2.1. *Let $p \in (1, \infty)\backslash\{2\}$. Let J be a set. Let $\mathcal{A}$ be a C^*-algebra. Let $\phi : \mathcal{A} \to B(l^p(J))$ be a homomorphism. Then*

(i) *the norm closure of $\{\phi(a)x : a \in \mathcal{A},\ \|a\| \le 1\}$ in $l^p(J)$ is norm compact for every $x \in l^p(J)$, and*

(ii) *$\mathcal{A}/\ker\phi$ is a residually finite-dimensional C^*-algebra.*

Corollary 2.2. *Let $p \in (1, \infty)\backslash\{2\}$. A C^*-algebra $\mathcal{A}$ is isomorphic to a subalgebra of $B(l^p(J))$, for some set J, if and only if $\mathcal{A}$ is residually finite-dimensional.*

Theorem 2.1 and Corollary 2.2 will be proved at the end of this section after a series of lemmas are proved. Theorem 2.1 has an easier proof when ϕ is contractive. Indeed, if $\phi : \mathcal{A} \to B(l^p(J))$ is a contractive homomorphism, then the range of ϕ is in the algebra of diagonal operators on $l^p(J)$ by [Blecher and Phillips 2019, Proposition 2.12] (or by [Gardella and Thiel 2020, Lemma 5.2] when J is countable). Thus, $\{\phi(a)x : a \in \mathcal{A},\ \|a\| \le 1\}$ is norm relatively compact, for every $x \in l^p(J)$, and $\mathcal{A}/\ker\phi$ is commutative.

It is not known if Theorem 2.1 and Corollary 2.2 hold for $p = 1$. However, throughout their proofs, we use, in an essential way, the assumption that p is in the reflexive range. For example, in the proof of Theorem 2.1(i), we use the fact that every bounded sequence in $l^p(J)$ has a weakly convergent subsequence. In the proof of Corollary 2.2, we use a classical result of Pełczyński that the l^p direct sum of finite-dimensional Hilbert spaces is isomorphic to $l^p(J)$ for some set J. This result of Pełczyński holds only when p is in the reflexive range.

The structure of the proof of Theorem 2.1(i) goes as follows: If the closure of $\{\phi(a)x_0 : a \in \mathcal{A},\ \|a\| \le 1\}$ is not compact for some $x_0 \in l^p(J)$, then we can find a bounded sequence in $(b_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that $\phi(b_k)x_0 \to 0$ weakly, as $k \to \infty$, and $\inf_{k\in\mathbb{N}} \|\phi(b_k)x_0\| > 0$. Assume that $p > 2$. In Lemma 2.5, we show that $\phi(b_k) \to 0$ weakly implies that $\omega(b_k^*b_k) \to 0$ for all positive linear functionals $\omega : \mathcal{A} \to \mathbb{C}$ of the form $\omega(a) = y_0^*(\phi(a)x_0)$. This is proved by considering $\sum_{k=1}^n \delta_k b_k$ for random $\delta_1, \ldots, \delta_n$ in $\{-1, 1\}$ and by exploiting $p > 2$. Lemma 2.9 says that when $y_0^* \in (l^p(J))^*$ is suitably chosen, $\omega(b_k^*b_k) \to 0$ implies that $\|\phi(b_k)x_0\| \to 0$, which contradicts $\inf_{k\in\mathbb{N}} \|\phi(b_k)x\| > 0$. This is proved by using the uniform convexity of $l^p(J)$.

Theorem 2.1(ii) follows from Theorem 2.1(i) by using a GNS-type construction and a classical result about compact unitary representations of groups on Hilbert spaces.

The following two lemmas are needed for the proof of Lemma 2.5.

Lemma 2.3. *Let $\mathcal{A}$ be a unital C^*-algebra. Let $a \in \mathcal{A}$. Then there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $\|c_n\| \le 1$ for all $n \in \mathbb{N}$ and $|a| = \lim_{n\to\infty} c_n a$.*

Proof. Without loss of generality, we may assume that $\|a\| \le 1$. For $n \in \mathbb{N}$, define $g_n \in C[0, 1]$ by

$$g_n(x) = \begin{cases} \frac{1}{\sqrt{x}}, & \frac{1}{n} \le x \le 1, \\ n\sqrt{n}x, & 0 \le x \le \frac{1}{n}. \end{cases}$$

Take $c_n = g_n(a^*a)a^*$. Then $c_n c_n^* = g_n(a^*a)a^*ag_n(a^*a)$. Note that

$$xg_n(x)^2 = \begin{cases} 1, & \frac{1}{n} \le x \le 1, \\ n^3x^3, & 0 \le x \le \frac{1}{n}. \end{cases}$$

Thus, $0 \le xg_n(x)^2 \le 1$ for all $x \in [0, 1]$ and so $0 \le c_n c_n^* \le 1$. Hence $\|c_n\| \le 1$.

We have

$$xg_n(x) = \begin{cases} \sqrt{x}, & \frac{1}{n} \le x \le 1, \\ n\sqrt{n}x^2, & 0 \le x \le \frac{1}{n}, \end{cases}$$

and so

$$|xg_n(x) - \sqrt{x}| \le \frac{1}{\sqrt{n}} \quad \text{for all } x \in [0, 1].$$

Since $c_n a = g_n(a^*a)a^*a$, it follows that

$$\|c_n a - \sqrt{a^*a}\| \le \frac{1}{\sqrt{n}}.$$

Thus, the result follows. □

Lemma 2.4. *Let $\mathcal{A}$ be a unital C^*-algebra. Let ω be a positive linear functional on $\mathcal{A}$. Let $a \in \mathcal{A}$. If $a \ge 0$ then*

$$\omega(a^2) \le \omega(a)^{\frac{2}{3}}\omega(a^4)^{\frac{1}{3}}.$$

Proof. There exists a measure μ on $[0, \|a\|]$ such that

$$\omega(f(a)) = \int f(x)\, d\mu(x),$$

for all $f \in C[0, \|a\|]$. So

$$\omega(a^2) = \int x^2\, d\mu(x) \le \left(\int x\, d\mu(x)\right)^{\frac{2}{3}} \left(\int x^4\, d\mu(x)\right)^{\frac{1}{3}} = \omega(a)^{\frac{2}{3}}\omega(a^4)^{\frac{1}{3}}.$$ □

Lemma 2.5. *Let $2 < p < \infty$. Let J be a set. Let $\mathcal{A}$ be a unital C^*-algebra. Let $\phi : \mathcal{A} \to B(l^p(J))$ be a unital homomorphism. Let $x_0 \in l^p(J)$. Let y_0^* be a bounded linear functional on $l^p(J)$. Define $\omega : \mathcal{A} \to \mathbb{C}$ by*

$$\omega(a) = y_0^*(\phi(a)x_0),$$

for $a \in \mathcal{A}$. Assume that ω is a positive linear functional. Let $(b_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ such that $\|b_k\| \le 1$ for all $k \in \mathbb{N}$ and $\phi(b_k)x_0 \to 0$ weakly as $k \to \infty$. Then $\omega(b_k^* b_k) \to 0$ as $k \to \infty$.

Proof. By contradiction, suppose that $\omega(b_k^* b_k)$ does not converge to 0. Passing to a subsequence, we have that there exists $\gamma > 0$ such that $\omega(b_k^* b_k) \geq \gamma$ for all $k \in \mathbb{N}$.

Since $\|\phi(b_k)x_0\| \leq \|\phi\| \, \|x_0\|$ and $\phi(b_k)x_0 \to 0$ weakly, passing to a further subsequence, we may assume that there are $z_1, z_2, \ldots$ in $l^p(J)$ with disjoint supports such that $\|z_k\| \leq \|\phi\| \|x_0\|$ and $\|\phi(b_k)x_0 - z_k\| \leq 1/2^k$ for all $k \in \mathbb{N}$.

Let $n \in \mathbb{N}$. For each $\delta = (\delta_1, \ldots, \delta_n) \in \{-1, 1\}^n$, let

$$a_\delta = \left| \sum_{k=1}^{n} \delta_k b_k \right| \in \mathcal{A}.$$

By Lemma 2.4,

$$\omega(a_\delta^2) \leq \omega(a_\delta)^{\frac{2}{3}} \omega(a_\delta^4)^{\frac{1}{3}}.$$

Thus,

$$\mathbb{E}\omega(a_\delta^2) \leq [\mathbb{E}\omega(a_\delta)]^{\frac{2}{3}} \, [\mathbb{E}\omega(a_\delta^4)]^{\frac{1}{3}},$$

where $\mathbb{E}$ denotes expectation over $\delta = (\delta_1, \ldots, \delta_n)$ uniformly distributed on $\{-1, 1\}^n$.

Note that

$$\begin{aligned}\mathbb{E}\omega(a_\delta^2) &= \mathbb{E}\omega\left(\left(\sum_{k=1}^{n} \delta_k b_k \right)^* \left(\sum_{k=1}^{n} \delta_k b_k \right) \right) \\ &= \mathbb{E}\omega\left(\sum_{1 \leq j,k \leq n} \delta_j \delta_k b_j^* b_k \right) = \sum_{1 \leq j,k \leq n} \mathbb{E}(\delta_j \delta_k) \omega(b_j^* b_k) = \sum_{k=1}^{n} \omega(b_k^* b_k) \geq n\gamma.\end{aligned}$$

Therefore,

$$n\gamma \leq [\mathbb{E}\omega(a_\delta)]^{\frac{2}{3}} \, [\mathbb{E}\omega(a_\delta^4)]^{\frac{1}{3}}. \tag{2-1}$$

We have

$$a_\delta^4 = \left[\left(\sum_{k=1}^{n} \delta_k b_k \right)^* \left(\sum_{k=1}^{n} \delta_k b_k \right) \right]^2 = \sum_{1 \leq k_1, \ldots, k_4 \leq n} \delta_{k_1} \delta_{k_2} \delta_{k_3} \delta_{k_4} b_{k_1}^* b_{k_2} b_{k_3}^* b_{k_4}.$$

Since $\|b_k\| \leq 1$, it follows that

$$\mathbb{E}\omega(a_\delta^4) = \sum_{1 \leq k_1, \ldots, k_4 \leq n} \mathbb{E}(\delta_{k_1} \delta_{k_2} \delta_{k_3} \delta_{k_4}) \omega(b_{k_1}^* b_{k_2} b_{k_3}^* b_{k_4}) \leq \sum_{1 \leq k_1, \ldots, k_4 \leq n} \mathbb{E}(\delta_{k_1} \delta_{k_2} \delta_{k_3} \delta_{k_4}).$$

Note that $\mathbb{E}(\delta_{k_1} \delta_{k_2} \delta_{k_3} \delta_{k_4}) = 0$ unless the following occurs:

$$(k_1 = k_2 \text{ and } k_3 = k_4) \quad \text{or} \quad (k_1 = k_3 \text{ and } k_2 = k_4) \quad \text{or} \quad (k_1 = k_4 \text{ and } k_2 = k_3).$$

Thus, $\mathbb{E}\omega(a_\delta^4) \leq 3n^2$. So by (2-1), we have $n\gamma \leq 3^{\frac{1}{3}} n^{\frac{2}{3}} [\mathbb{E}\omega(a_\delta)]^{\frac{2}{3}}$. Hence,

$$\mathbb{E}\omega(a_\delta) \geq \frac{\gamma^{\frac{3}{2}}}{3^{\frac{1}{2}}} n^{\frac{1}{2}}. \tag{2-2}$$

Fix $\delta \in \{-1, 1\}^n$. By Lemma 2.3,

$$\omega(a_\delta) = \omega\left(\left| \sum_{k=1}^{n} \delta_k b_k \right| \right) \leq \sup_{c \in \mathcal{A},\, \|c\| \leq 1} \left| \omega\left(c \sum_{k=1}^{n} \delta_k b_k \right) \right|.$$

For $c \in \mathcal{A}$ with $\|c\| \le 1$,

$$\begin{aligned}\left|\omega\left(c\sum_{k=1}^{n}\delta_k b_k\right)\right| &= \left|y_0^*\left(\phi(c)\left(\sum_{k=1}^{n}\delta_k\phi(b_k)x_0\right)\right)\right|\\ &\le \|y_0^*\|\,\|\phi\|\left\|\sum_{k=1}^{n}\delta_k\phi(b_k)x_0\right\|\\ &\le \|y_0^*\|\,\|\phi\|\left(\left\|\sum_{k=1}^{n}\delta_k z_k\right\|+\sum_{k=1}^{n}\frac{1}{2^k}\right) \le \|y_0^*\|\,\|\phi\|(\|\phi\|\,\|x_0\|n^{\frac{1}{p}}+1),\end{aligned}$$

where the last two inequalities follow from the fact that $z_1, z_2, \dots$ have disjoint supports, $\|z_k\| \le \|\phi\|\,\|x_0\|$ and $\|\phi(b_k)x_0 - z_k\| \le 1/2^k$. Thus,

$$\omega(a_\delta) \le \|y_0^*\|\,\|\phi\|(\|\phi\|\,\|x_0\|n^{\frac{1}{p}}+1) \quad \text{for all } \delta \in \{-1,1\}^n.$$

So by (2-2),

$$\frac{\gamma^{\frac{3}{2}}}{3^{\frac{1}{2}}}n^{\frac{1}{2}} \le \|y_0^*\|\,\|\phi\|(\|\phi\|\,\|x_0\|n^{\frac{1}{p}}+1).$$

Since n can be chosen to be arbitrarily large and $p > 2$, an absurdity follows. □

For $1 < p < 2$, we have the following result, where the order of b_k^* and b_k is switched, by using the dual l^p space in Lemma 2.5.

Lemma 2.6. *Let $1 < p < 2$. Let J be a set. Let $\mathcal{A}$ be a unital C^*-algebra. Let $\phi : \mathcal{A} \to B(l^p(J))$ be a unital homomorphism. Let $x_0 \in l^p(J)$. Let y_0^* be a bounded linear functional on l^p. Define $\omega : \mathcal{A} \to \mathbb{C}$ by*

$$\omega(a) = y_0^*(\phi(a)x_0),$$

for $a \in \mathcal{A}$. Let $(b_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ such that $\|b_k\| \le 1$ for all $k \in \mathbb{N}$ and such that the sequence $y_0^* \circ \phi(b_k)$ of bounded linear functionals on $l^p(J)$ converges to 0 weakly as $k \to \infty$. Assume that ω is a positive linear functional. Then $\omega(b_k b_k^*) \to 0$ as $k \to \infty$.

Proof. Let $\mathcal{A}_1$ be the unital C^*-algebra consisting of the same elements as $\mathcal{A}$ but with reverse order multiplication

$$a \cdot b = ba.$$

Define a unital homomorphism $\phi_1 : \mathcal{A}_1 \to B((l^p(J))^*)$ by

$$\phi_1(a)y^* = y^* \circ \phi(a),$$

for all $a \in \mathcal{A}_1$, $y^* \in (l^p(J))^*$. Define $\omega_1 : \mathcal{A}_1 \to \mathbb{C}$ by

$$\omega_1(a) = \omega(a) = x_0^{**}(\phi(a)y_0^*),$$

for all $a \in \mathcal{A}_1$, where x_0^{**} is the image of x_0 in the bidual $(l^p)^{**}$. By Lemma 2.5, the result follows. □

The following two lemmas are needed for the proof of Lemma 2.9.

Lemma 2.7 [Clarkson 1936]. *Let $1 < p < \infty$. Let J be a set. For every $\epsilon > 0$, there exists $\gamma > 0$ such that, for all $x, y \in l^p(J)$ satisfying $\|x\|, \|y\| \le 1$ and $\|x + y\| > 2 - \gamma$, we have $\|x - y\| < \epsilon$.*

Lemma 2.8 [Russo and Dye 1966]. *Let $\mathcal{A}$ be a unital C^*-algebra. Then the closed unital ball of $\mathcal{A}$ is the closed convex hull of the set of all unitary elements of $\mathcal{A}$.*

Lemma 2.9. *Let $1 < p < \infty$. Let J be a set. Let $\mathcal{A}$ be a unital C^*-algebra. Let $\phi : \mathcal{A} \to B(l^p(J))$ be a unital homomorphism. Let $x_0 \in l^p(J)$. Then there exists $y_0^* \in (l^p(J))^*$ such that $\omega : \mathcal{A} \to \mathbb{C}$,*

$$\omega(a) = y_0^*(\phi(a)x_0), \quad a \in \mathcal{A},$$

defines a positive linear functional and, for every $\epsilon > 0$, there exists $\gamma > 0$ such that whenever $a \in \mathcal{A}$ satisfies $\|a\| \le 1$ and $\omega(a^*a) < \gamma$, we have $\|\phi(a)x_0\| < \epsilon$.

Proof. Let $\mathcal{U}(\mathcal{A})$ be the set of all unitary elements of $\mathcal{A}$. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{U}(\mathcal{A})$ such that

$$\lim_{n\to\infty} \|\phi(v_n)x_0\| = \sup_{u\in\mathcal{U}(\mathcal{A})} \|\phi(u)x_0\|.$$

For each $n \in \mathbb{N}$, let x_n^* be a bounded linear functional on $l^p(J)$ such that $\|x_n^*\| = 1$ and $x_n^*(\phi(v_n)x_0) = \|\phi(v_n)x_0\|$. Then $x_n^* \circ \phi(v_n)$ is a bounded sequence in $(l^p(J))^*$. Passing to a subsequence, we may assume that $x_n^* \circ \phi(v_n)$ converges weakly to a bounded linear functional $y_0^* \in (l^p(J))^*$ as $n \to \infty$. Thus, $\omega : \mathcal{A} \to \mathbb{C}$,

$$\omega(a) = y_0^*(\phi(a)x_0) = \lim_{n\to\infty} x_n^*(\phi(v_n a)x_0),$$

for $a \in \mathcal{A}$, defines a bounded linear functional on $\mathcal{A}$. Note that

$$\omega(1) = \lim_{n\to\infty} x_n^*(\phi(v_n)x_0) = \lim_{n\to\infty} \|\phi(v_n)x_0\| = \sup_{u\in\mathcal{U}(\mathcal{A})} \|\phi(u)x_0\|,$$

and, for every $u_0 \in \mathcal{U}(\mathcal{A})$,

$$|\omega(u_0)| = \lim_{n\to\infty} |x_n^*(\phi(v_n u_0)x_0)| \le \sup_{u\in\mathcal{U}(\mathcal{A})} \|\phi(u)x_0\|.$$

So by Lemma 2.8, we have $\|\omega\| \le \sup_{u\in\mathcal{U}(\mathcal{A})} \|\phi(u)x_0\|$. Thus, $\omega(1) = \|\omega\|$ and hence ω is a positive linear functional.

By contradiction, suppose that there are $\epsilon > 0$ and a sequence $(a_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that $\|a_k\| \le 1$ and $\|\phi(a_k)x_0\| \ge \epsilon$ for all $k \in \mathbb{N}$ and $\omega(a_k^* a_k) \to 0$ as $k \to \infty$. We have

$$\|a_k\| \ge \frac{\|\phi(a_k)x_0\|}{\|\phi\|\,\|x_0\|} \ge \frac{\epsilon}{\|\phi\|\,\|x_0\|},$$

for all $k \in \mathbb{N}$. For $k \in \mathbb{N}$, let $b_k = a_k/\|a_k\|$. We have $\|b_k\| = 1$ and $\|\phi(b_k)x_0\| \ge \epsilon$ for all $k \in \mathbb{N}$ and $\omega(b_k^* b_k) \to 0$ as $k \to \infty$.

Since $\|x_n^*\| = 1$,

$$\begin{aligned}\liminf_{n\to\infty} \|\phi(v_n)\phi(1 - |b_k|)x_0 + \phi(v_n)x_0\| &\ge \liminf_{n\to\infty} [x_n^*(\phi(v_n)\phi(1 - |b_k|)x_0) + x_n^*(\phi(v_n)x_0)] \\ &= \omega(1 - |b_k|) + \omega(1) = 2\omega(1) - \omega(|b_k|).\end{aligned}$$

Thus,

$$\liminf_{n\to\infty} \|\phi(v_n)\phi(1-|b_k|)x_0+\phi(v_n)x_0\| \geq 2\omega(1)-\omega(|b_k|).$$

But

$$\|\phi(v_n)\phi(1-|b_k|)x_0\| \leq \sup_{b\in\mathcal{A},\,\|b\|\leq 1} \|\phi(b)x_0\|\,\|1-|b_k|\| \leq \sup_{u\in\mathcal{U}(\mathcal{A})} \|\phi(u)x_0\| = \omega(1)$$

and $\|\phi(v_n)x_0\| \leq \omega(1)$ for all $n\in\mathbb{N}$. Take

$$x = \frac{1}{\omega(1)}\phi(v_n)\phi(1-|b_k|)x_0 \quad \text{and} \quad y = \frac{1}{\omega(1)}\phi(v_n)x_0$$

in Lemma 2.7 and note that $\omega(|b_k|) \leq \omega(b_k^*b_k)^{\frac{1}{2}}\omega(1)^{\frac{1}{2}} \to 0$ as $k\to\infty$. We have

$$\lim_{k\to\infty}\limsup_{n\to\infty} \|\phi(v_n)\phi(1-|b_k|)x_0-\phi(v_n)x_0\| = 0.$$

Thus,

$$\lim_{k\to\infty}\limsup_{n\to\infty} \|\phi(v_n)\phi(|b_k|)x_0\| = 0.$$

So $\|\phi(|b_k|)x_0\| \to 0$ as $k\to\infty$. Since

$$b_k = b_k\Big(|b_k|+\frac{1}{k}\Big)^{-1}\Big(|b_k|+\frac{1}{k}\Big) \quad \text{and} \quad \Big\|b_k\Big(|b_k|+\frac{1}{k}\Big)^{-1}\Big\| \leq 1,$$

it follows that $\|\phi(b_k)x_0\| \to 0$ as $k\to\infty$ which contradicts $\|\phi(b_k)x_0\| \geq \epsilon$. □

Proof of Theorem 2.1(i). Without loss generality, we may assume that $\mathcal{A}$ is unital by extending ϕ to a homomorphism from the unitization of $\mathcal{A}$ into $B(l^p(J))$. We may also assume that ϕ is unital since $\phi(1)$ is an idempotent on $l^p(J)$ and the range of every idempotent on $l^p(J)$ is isomorphic to $l^p(J_0)$ for some set J_0 [Pełczyński 1960; Johnson 2012].

Let $x_0 \in l^p$. Let $(a_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ such that $\|a_k\| \leq \frac{1}{2}$ for all $k\in\mathbb{N}$. We need to show that $(\phi(a_k)x_0)_{k\in\mathbb{N}}$ has a norm-convergent subsequence.

Case 1: $p>2$. Passing to a subsequence, we may assume that $(\phi(a_k)x_0)_{k\in\mathbb{N}}$ converges weakly to an element of $l^p(J)$. Thus, $\phi(a_{k_1}-a_{k_2})x_0 \to 0$ weakly as $k_1,k_2\to\infty$.

By Lemma 2.5, we have

$$\lim_{k_1,k_2\to\infty} \omega((a_{k_1}-a_{k_2})^*(a_{k_1}-a_{k_2})) = 0$$

for every positive linear functional $\omega:\mathcal{A}\to\mathbb{C}$ of the form $\omega(a)=y_0^*(\phi(a)x_0)$ for $a\in\mathcal{A}$. By Lemma 2.9, we have $\lim_{k_1,k_2\to\infty}\|\phi(a_{k_1}-a_{k_2})x_0\|=0$. So $(\phi(a_k)x_0)_{k\in\mathbb{N}}$ is norm-convergent.

Case 2: $p<2$. Passing to a subsequence, we may assume that $(y_0^*\circ\phi(a_k^*))_{k\in\mathbb{N}}$ converges weakly to an element of $(l^p(J))^*$. Thus, $y^*\circ\phi(a_{k_1}^*-a_{k_2}^*) \to 0$ weakly as $k_1,k_2\to\infty$ for every $y^*\in(l^p(J))^*$.

By Lemma 2.6, we have

$$\lim_{k\to\infty} \omega((a_{k_1}^*-a_{k_2}^*)(a_{k_1}^*-a_{k_2}^*)^*) = 0$$

for every positive linear functional $\omega:\mathcal{A}\to\mathbb{C}$ of the form $\omega(a)=y_0^*(\phi(a)x_0)$ for $a\in\mathcal{A}$. By Lemma 2.9, we have $\lim_{k_1,k_2\to\infty}\|\phi(a_{k_1}-a_{k_2})x_0\|=0$. So $(\phi(a_k)x_0)_{k\in\mathbb{N}}$ is norm-convergent. □

Lemma 2.10 [Kerr and Li 2016, Theorem 2.24]. *Let G be a group. Let $\mathcal{H}$ be a Hilbert space. Let $\psi : G \to B(\mathcal{H})$ be a unital homomorphism such that $\psi(g)$ is unitary for all $g \in G$. If $\{\psi(g)x : g \in G\}$ is norm precompact in $\mathcal{H}$ for all $x \in \mathcal{H}$, then $\mathcal{H}$ is the direct sum of some finite-dimensional subspaces $\mathcal{H}_\alpha$, for $\alpha \in \Lambda$, such that $\mathcal{H}_\alpha$ is invariant under $\psi(g)$ for all $\alpha \in \Lambda$ and $g \in G$.*

Proof of Theorem 2.1(ii). As in the proof Theorem 2.1(i), we may assume that $\mathcal{A}$ is unital and ϕ is unital. We may also assume that $\ker\phi = \{0\}$. Let $a_0 \neq 0$. There exists $x_0 \in l^p(J)$ such that $\phi(a_0)x_0 \neq 0$. By Lemma 2.9, there exists $y_0^* \in (l^p(J))^*$ such that $\omega : \mathcal{A} \to \mathbb{C}$,

$$\omega(a) = y_0^*(\phi(a)x_0),$$

for $a \in \mathcal{A}$, defines a positive linear functional and $\omega(a_0^* a_0) \neq 0$.

Equip $\mathcal{A}$ with the positive semidefinite sesquilinear form

$$\langle a, b\rangle = \omega(b^* a),$$

for $a, b \in \mathcal{A}$. Consider the ideal $\mathcal{A}_0 = \{a \in \mathcal{A} : \langle a, a\rangle = 0\}$ of $\mathcal{A}$. Let $\mathcal{H}$ be the completion of the quotient space $\mathcal{A}/\mathcal{A}_0$. Then $\mathcal{H}$ is a Hilbert space. For each $a \in \mathcal{A}$, we can define a bounded linear operator on $\mathcal{H}$ by sending $b + \mathcal{A}_0$ to $ab + \mathcal{A}_0$ for $b \in \mathcal{A}$. So $\eta : \mathcal{A} \to B(\mathcal{H})$,

$$\eta(a)(b + \mathcal{A}_0) = ab + \mathcal{A}_0,$$

for $a, b \in \mathcal{A}$, defines a unital $*$-homomorphism. We have

$$\begin{aligned}\|\eta(a_1)(b + \mathcal{A}_0) - \eta(a_2)(b + \mathcal{A}_0)\| &= \omega(b^*(a_1 - a_2)^*(a_1 - a_2)b)\\ &= y_0^*(\phi(b^*(a_1 - a_2)^*(a_1 - a_2)b)x_0)\\ &\le \|y_0^*\|\,\|\phi\|\,\|b^*\|\,\|a_1 - a_2\|\,\|\phi(a_1 - a_2)\phi(b)x_0\|,\end{aligned}$$

for all $a_1, a_2, b \in \mathcal{A}$. By Theorem 2.1(i), we have that $\{\phi(a)x_0 : a \in \mathcal{A},\ \|a\| \le 1\}$ is norm precompact so $\{\eta(a)(b + \mathcal{A}_0) : a \in \mathcal{A},\ \|a\| \le 1\}$ is norm precompact for all $b \in \mathcal{A}$. Let $\mathcal{U}(\mathcal{A})$ be the set of all unitary elements of $\mathcal{A}$. By Lemma 2.10, we have that $\mathcal{H}$ is the direct sum of some finite-dimensional subspaces $\mathcal{H}_\alpha$, for $\alpha \in \Lambda$, such that $\mathcal{H}_\alpha$ is invariant under $\eta(u)$ for all $\alpha \in \Lambda$ and $u \in \mathcal{U}(\mathcal{A})$. Note that $\mathcal{H}_\alpha$ is thus invariant under $\eta(a)$ for all $a \in \mathcal{A}$.

Since $\omega(a_0^* a_0) \neq 0$, we have $\eta(a_0) \neq 0$. So $\eta(a_0) \neq 0$ on $\mathcal{H}_{\alpha_0}$ for some $\alpha_0 \in \Lambda$. Thus, $\mathcal{A}$ is residually finite-dimensional. □

Proof of Corollary 2.2. One direction follows from Theorem 2.1. For the other direction, suppose that $\mathcal{A}$ is a residually finite-dimensional C^*-algebra. Then there is a collection $(\phi_\alpha)_{\alpha\in\Lambda}$ of $*$-representations of $\mathcal{A}$ on finite-dimensional Hilbert spaces $\mathcal{H}_\alpha$ such that $\|a\| = \sup_{\alpha\in\Lambda} \|\phi_\alpha(a)\|$ for all $a \in \mathcal{A}$. Define $\phi : \mathcal{A} \to B\big(\big(\bigoplus_{\alpha\in\Lambda} \mathcal{H}_\alpha\big)_{l^p}\big)$ by $\phi = \bigoplus_{\alpha\in\Lambda} \phi_\alpha$. Thus ϕ is a norm-preserving homomorphism. However, it is a classical result of [Pełczyński 1960] that for $1 < p < \infty$, the l^p direct sum of finite-dimensional Hilbert spaces is isomorphic to $l^p(J)$ for some set J. Therefore, $\mathcal{A}$ is isomorphic to a subalgebra of $B(l^p(J))$, via the map $a \mapsto S\phi(a)S^{-1}$, where $S : \big(\bigoplus_{\alpha\in\Lambda} \mathcal{H}_\alpha\big)_{l^p} \to l^p(J)$ is any invertible operator. □

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