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EXPONENTIAL CONVERGENCE OF PARABOLIC OPTIMAL TRANSPORT ON BOUNDED DOMAINS

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We study the asymptotic behavior of solutions to the second boundary value problem for a parabolic PDE of Monge–Ampère type arising from optimal mass transport. Our main result is an exponential rate of convergence for solutions of this evolution equation to the stationary solution of the optimal transport problem. We derive a differential Harnack inequality for a special class of functions that solve the linearized problem. Using this Harnack inequality and certain techniques specific to mass transport, we control the oscillation in time of solutions to the parabolic equation, and obtain exponential convergence. Additionally, in the course of the proof, we present a connection with the pseudo-Riemannian framework introduced by Kim and McCann in the context of optimal transport, which is interesting in its own right.

1. Introduction

Given two smooth domains Ω, $\Omega^* \subset \mathbb{R}^n$, two probability measures μ, η defined respectively on Ω and Ω^*, and a Borel measurable *cost function* $c : \overline{\Omega} \times \overline{\Omega}^* \to \mathbb{R}$, the optimal transport problem is to find a μ-measurable map $T : \Omega \to \Omega^*$ satisfying $T_\# \mu = \eta$ (where $T_\# \mu(E) := \mu(T^{-1}(E))$ for all measurable $E \subset \Omega^*$) such that

$$\int_\Omega c(x, T(x))\, d\mu(x) = \max_{S_\# \mu = \eta} \int_\Omega c(x, S(x))\, d\mu(x). \tag{1}$$

Under mild assumptions on the cost function and the measures, it can be shown that the solution T to (1) exists; see, for example, [Brenier 1991; Gangbo and McCann 1996]. If the measures μ and η are absolutely continuous with respect to Lebesgue measure, and c satisfies the bitwist condition (6) below, the map T is μ-a.e. single-valued and can be determined by the implicit relation

$$\nabla_x c(x, T(x)) = \nabla u(x),$$

where the scalar-valued potential u is a *c-convex function* (see Definition 2.1) satisfying the Monge–Ampère-type equation

$$\begin{cases} \det[D^2 u(x) - A(x, \nabla u(x))] = B(x, \nabla u(x)), & x \in \Omega, \\ T(\Omega) = \Omega^*, \end{cases} \tag{2}$$

where A is a matrix-valued function and B is scalar-valued, defined in terms of the cost function c and the densities of the measures μ, η. The issue of existence and regularity of solutions to the PDE (2) has

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been an active area of research for many years. For higher-order regularity results, we refer the reader to [Ma et al. 2005; Trudinger and Wang 2009; Urbas 1997].

One possible approach to finding a solution to the PDE above is to solve the parabolic PDE

$$\begin{cases} \partial_t u(x,t) = \log\det[D^2 u(x,t) - A(x, \nabla u(x,t))] - \log B(x, \nabla u(x,t)), & x \in \Omega,\ t > 0, \\ G(x, \nabla u(x,t)) = 0, & x \in \partial\Omega, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{3}$$

for appropriate initial and boundary conditions u_0 and G (see Section 2), and view a stationary solution as $t \to \infty$ as a solution to (2). The study of existence, regularity, and asymptotic behavior of solutions to the parabolic problem (3) was initiated only recently through the works [Kitagawa 2012; Kim et al. 2012].

The main result of this paper is the following theorem on an exponential convergence rate of solutions to the parabolic equation (3). The notation $C_x^{k_1} C_t^{k_2}$ will denote functions on a space-time domain which are C^{k_1} in the space variable and C^{k_2} in the time variable, with corresponding norms finite. Our main result is as follows:

Theorem 1.1. *Suppose $u \in C_x^4 C_t^3(\overline{\Omega} \times [0,\infty))$ is a solution on $\overline{\Omega} \times [0,\infty)$ to the parabolic equation* (3) *converging uniformly on $\overline{\Omega}$ to a stationary solution u^∞ as $t \to \infty$, and $\mathcal{K}$ is a constant such that*

$$\|u\|_{C_x^4 C_t^2(\overline{\Omega}\times[0,\infty))} + \|c\|_{C^4(\overline{\Omega}\times\overline{\Omega}^*)} \le \mathcal{K}. \tag{4}$$

If the cost function c satisfies the bitwist condition (6), *and Ω and Ω^* satisfy the c-convexity conditions* (8) *and* (9), *then*

$$\|u(\,\cdot\,,t) - u^\infty\|_{L^\infty(\Omega)} \le C_1 e^{-C_2 t} \quad \text{for all } t \ge 0,$$

for some constants $C_1, C_2 > 0$ depending only on $\mathcal{K}$ and the dimension n.

Previous work in [Kitagawa 2012] establishes the existence of a function $u \in C_x^2 C_t^1(\overline{\Omega} \times [0,\infty))$ that solves (3) for all times $t \ge 0$ and converges in $C^2(\overline{\Omega})$ to a function $u^\infty(\,\cdot\,)$ as $t \to \infty$, where $u^\infty(\,\cdot\,)$ satisfies the elliptic optimal transport equation (2). Using this result and a bootstrapping argument, we obtain the following corollary.

Corollary 1.2. *Suppose the cost function c satisfies the bitwist condition* (6) *and the Ma–Trudinger–Wang condition* (10), *and suppose Ω and Ω^* satisfy the c-convexity conditions* (8) *and* (9) *with $\delta, \delta^* > 0$. Suppose the source and target measures μ and η are absolutely continuous with smooth densities that are bounded away from zero and infinity on $\overline{\Omega}$ and $\overline{\Omega}^*$ respectively. Finally, suppose the initial condition $u_0 \in C^{4,\alpha}(\Omega)$ for some $\alpha \in (0,1]$ is locally, uniformly c-convex* (*as in Definition 2.1*) *and satisfies the boundary compatibility conditions* (12). *Then u satisfies the hypotheses of Theorem 1.1 above.*

Proof. Under the Ma–Trudinger–Wang condition (10) and the uniform c- and c^*-convexity of the domains (i.e., (8) and (9) with $\delta, \delta^* > 0$), global $C_x^{2,\alpha} C_t^{1,\alpha}$ estimates of the solution $u(x,t)$ to (3) were obtained in [Kitagawa 2012, Theorems 10.1 and 11.2, and Section 12]. Thus, by applying boundary Schauder estimates for linear uniformly parabolic equations in nondivergence form with uniformly oblique boundary conditions (see [Lieberman 1996, Theorems 4.23 and 4.31]) to the linearized equation (18), we obtain the desired higher regularity of u. □

Remark 1.3. One particular motivation for this exponential convergence result comes from numerics for optimal transport. Since the stationary state of (3) gives rise to the solution of the optimal transport problem between the measures μ and η, one could attempt to implement an algorithm that is initiated with some c-convex potential function and flows toward the desired solution via (3). Establishing quantitative rates of convergence for such an algorithm is consequently of paramount importance. One difficulty that should be noted here is that in the case with nonempty boundary, due to compatibility requirements with the boundary condition, there are some restrictions on what can be taken as an initial condition (compare to the case of no boundary, where one can simply take a constant function), and it is not always clear how to generate initial data that will still provide global existence. We plan to explore this issue of finding appropriate initial conditions in future work.

1A. ***Prior results and the contributions of this paper.*** The parabolic flow (3) on Riemannian manifolds with no boundary was considered by Kim, Streets, and Warren [Kim et al. 2012], under a strong form of the Ma–Trudinger–Wang condition (10); their methods strongly use that the boundary is empty. There, the authors prove exponential convergence of the solution u of (3) to the solution u^∞ of the elliptic equation (2); see [Kim et al. 2012, Theorem 1.1]. Their proof relies on establishing a Li–Yau-type Harnack inequality for solutions to the linearization of (3), coupled with the observation that this linearization is actually a heat equation where the elliptic part is a conformal factor times the Laplace–Beltrami operator of a conformal change of a metric defined from the solution of the parabolic evolution itself; see [Kim et al. 2012, Proposition 5.1] and the discussion preceding Proposition 2.7 below.

However, presence of a boundary turns out to be a major obstruction to applying the methods of [Kim et al. 2012]. First, their method of introducing a conformal change of metric cannot be used in two dimensions: when there is no boundary, it is possible to convert the two-dimensional problem to a three-dimensional one, but such a technique simply does not work when the boundary is nonempty and is required to satisfy certain convexity properties. Second, the linearization of (3) is a Neumann boundary-value problem with respect to a time-varying Riemannian metric, for which there is no general known Harnack inequality. Existing results require that the metric itself satisfy some specific evolution, such as Ricci flow [Bailesteanu et al. 2010] or Gauss curvature flow [Chow 1991]. Thus while there is a sizable body of work on differential Harnack inequalities, none of them are directly applicable to the linearization of (3). We also mention the result [Schnürer and Smoczyk 2003], which treats a nonlinear evolution equation arising from Gauss curvature flow that resembles (3) in the case where the cost function is $c(x, y) = \langle x, y\rangle$, with nonempty boundary. The authors of [Schnürer and Smoczyk 2003] also obtain an exponential convergence result, but assume certain structural assumptions on the function B in (3) that are not satisfied in the optimal transport case, and impose additional constraints on the initial data u_0.

The contributions of this paper are as follows. First we show it is possible to obtain a Harnack inequality for a certain subclass of solutions to the linearized equation. In the interior, this can be shown by a series of estimates similar to that of [Kim et al. 2012] with no boundary, but as mentioned above, a different method must be employed to settle the two-dimensional case. In dealing with the boundary, we must carefully exploit the curvature conditions imposed on the boundaries of both the source and target domains in order to choose the correct class of solutions for which we can obtain the Harnack inequality. Once we have a Harnack inequality for such special solutions, we use the fact that solutions of the parabolic flow come from the optimal transport problem, and hence satisfy a mass-preservation condition (see Lemma 2.3 below), to finish the proof of exponential convergence. We heavily stress here that our approach diverges from the traditional proof of exponential convergence via the Harnack inequality, and crucially uses the fact that there is an underlying optimal transport problem. Additionally, we show this analysis of the boundary behavior can also be done by exploiting the pseudo-Riemannian structure introduced in [Kim and McCann 2010] for optimal transport. More specifically, we prove a relation between the second fundamental form with respect to the time-varying Riemannian metric on the source domain, with the Euclidean second fundamental forms of the source and target domains under c-exponential coordinates, which has not previously been explored.

1B. ***Outline and strategy of proof.*** The outline of the remainder of the paper and the strategy behind our proof are as follows. In Section 2 we give the necessary background for the optimal transport problem. We also recall the method of [Kim et al. 2012] for the proof of exponential convergence on manifolds with no boundary, and prove here the important parabolic estimate Proposition 2.7, although with a slightly different proof from that of Kim, Streets, and Warren. In Section 3 we obtain expressions for and estimates on the boundary condition acting on the relevant auxiliary function. For the benefit of the reader, we divide the proof of these estimates into the inner product case and the general cost function case. In Section 4 we finally obtain the exponential convergence result from the estimates derived in the previous sections; the proof we present relies on the underlying optimal transport structure of the problem. The final Section 5 provides the aforementioned alternative, geometric approach to the boundary estimates from Section 3.

2. Preliminaries

2A. ***Basic notions from optimal transport.*** We denote by D^2, ∇, and D_β the Hessian matrix, the gradient vector, and the directional derivative in the direction β of a given function with respect to the space variable x. Spatial partial derivatives will be denoted by subscript indices, with the actual variable specified when necessary, while D_x and D_p will be used for the derivative matrix of a mapping with respect to the variable in the subscript. We will also follow the convention of summing over repeated indices. Time derivatives will be denoted by ∂_t.

When considering a Riemannian manifold (M, g), we will denote the inner product and norm with respect to the metric g by $\langle \cdot, \cdot \rangle_g$ and $|\cdot|_g$ respectively. The notation ∇^g, Hess_g, Δ_g, and Ric_g will be used for the gradient, Hessian, Laplacian, and Ricci tensor with respect to g.

Regarding the cost function $c(x, y)$, derivatives in the x-variable will be denoted by subscripts preceding a comma, while derivatives in the y-variable will be denoted by subscripts following a comma. The notation $c^{i,j}$ denotes the entries of the inverse of the matrix $c_{i,j}$.

We will assume from here onward that Ω, Ω^* are open, smooth, bounded domains in $\mathbb{R}^n$. The outward-pointing unit normals to $\partial\Omega$ and $\partial\Omega^*$ will be denoted by ν and ν^* respectively. The function h^* will be a normalized defining function for Ω^*; i.e., $h^* = 0$ on $\partial\Omega$, $h^* < 0$ on Ω, and $\nabla h^* = \nu^*$ on $\partial\Omega^*$. The measures μ, η are assumed to be absolutely continuous with respect to n-dimensional Lebesgue measure, with densities ρ, ρ^* respectively satisfying the bounds $0 < \lambda \le \rho, \rho^* \le \Lambda < \infty$ and the mass balance condition

$$\int_\Omega \rho = \int_{\Omega^*} \rho^*. \tag{5}$$

We will also assume $c \in C^{4,\alpha}(\overline{\Omega} \times \overline{\Omega}^*)$ for some $\alpha \in (0, 1]$, and

$$\begin{aligned} &y \mapsto \nabla_x c(x, y) \text{ is a diffeomorphism for all } x \in \overline{\Omega}, \\ &x \mapsto \nabla_y c(x, y) \text{ is a diffeomorphism for all } y \in \overline{\Omega}^*. \end{aligned} \tag{6}$$

For any $p \in \nabla_x c(x, \Omega^*)$ and $x \in \Omega$ (resp. $q \in \nabla_y c(\Omega, y)$ and $y \in \Omega^*$), we denote by $Y(x, p)$ (resp. $X(q, y)$) the unique element of Ω^* (resp. Ω) such that

$$(\nabla_x c)(x, Y(x, p)) = p \quad (\text{resp. } (\nabla_y c)(X(q, y), y) = q). \tag{7}$$

We say Ω *is c-convex with respect to* Ω^* if the set $\nabla_y c(\Omega, y)$ is a convex set for each $y \in \Omega^*$. Similarly, Ω^* *is c^*-convex with respect to* Ω if the set $\nabla_x c(x, \Omega^*)$ is a convex set for each $x \in \Omega$. Analytically, these conditions are satisfied if we have

$$[\nu_i^j(x) - c^{\ell,k} c_{ij,\ell}(x, y)\nu^k(x)]\tau^i\tau^j \ge \delta|\tau|^2 \qquad \text{for all } x \in \partial\Omega,\ y \in \overline{\Omega}^*,\ \tau \in T_x(\partial\Omega), \tag{8}$$

$$[(\nu^*)_i^j(y) - c^{k,\ell} c_{\ell,ij}(x, y)(\nu^*)^k(x)](\tau^*)^i(\tau^*)^j \ge \delta^*|\tau^*|^2 \quad \text{for all } y \in \partial\Omega^*,\ x \in \overline{\Omega},\ \tau^* \in T_y(\partial\Omega^*) \tag{9}$$

for some constants $\delta, \delta^* \ge 0$ respectively, where we will always sum over repeated indices. If δ (resp. δ^*) is strictly positive, we say that Ω *is uniformly c-convex with respect to* Ω^* (resp. Ω^* *is uniformly c^*-convex with respect to* Ω).

Define the matrix-valued function A by $A(x, p) := (D_x^2 c)(x, Y(x, p))$. Since $Y(x, p)$ satisfies the equation $(\nabla_x c)(x, Y(x, p)) = p$, we can differentiate implicitly in p to get

$$(D_{x,y}^2 c)(x, Y(x, p)) D_p Y(x, p) = \mathbb{I}_n.$$

Similarly, differentiating the equation $(\nabla_x c)(x, Y(x, p)) = p$ in x gives

$$(D_x^2 c)(x, Y(x, p)) + (D_{x,y}^2 c)(x, Y(x, p)) D_x Y(x, p) = 0.$$

We have chosen the convention $(DY)_{\ell m} = Y_m^\ell$ for differentiation either in the x- or p-variables. It follows that

$$A(x, p) = (D_x^2 c)(x, Y(x, p)) = -(D_p Y)^{-1}(x, p) D_x Y(x, p).$$

Definition 2.1. A function $\varphi : \Omega \to \mathbb{R}$ is said to be *c-convex* if for any point $x_0 \in \Omega$, there exists a $y_0 \in \Omega^*$ and $\lambda_0 \in \mathbb{R}$ such that

$$\begin{aligned} \varphi(x_0) &= c(x_0, y_0) + \lambda_0, \\ \varphi(x) &\ge c(x, y_0) + \lambda_0 \quad \text{for all } x \in \Omega. \end{aligned}$$

A function $\varphi \in C^2(\Omega)$ is said to be *locally, uniformly c-convex* if $D^2\varphi(x) - A(x, \nabla\varphi(x)) > 0$ as a matrix for every $x \in \overline{\Omega}$.

Although we will not use it explicitly in this paper, we also mention the, by now well-known, Ma–Trudinger–Wang condition. This condition (or rather a stronger version of it) was first used to obtain interior $C^{2,\alpha}$ regularity of solutions to the elliptic optimal transport equation (2) in [Ma et al. 2005]. It was proven to be a necessary condition for regularity theory in [Loeper 2009], and it was shown that classical solutions for the parabolic equation (3) exist under the same condition in [Kitagawa 2012].

Definition 2.2. The cost function $c(x, y)$ satisfies the Ma–Trudinger–Wang (MTW) condition if

$$D_{p_i p_j} A_{k\ell}(x, p)\xi^i \xi^j \eta^k \eta^\ell \geq 0 \quad \text{for all } x \in \overline{\Omega},\ p \in \nabla_x c(x, \Omega^*),\ \xi \perp \eta. \tag{10}$$

2B. ***The parabolic optimal transport problem.*** For a function $u \in C_x^4 C_t^2(\overline{\Omega} \times [0, \infty))$ (which, in the sequel, will be the solution to the parabolic optimal transportation problem), we will employ the following notation:

$$\begin{aligned} T(x, t) &= Y(x, \nabla u(x, t)), \\ B(x, p) &= |\det (D^2_{x,y} c)(x, Y(x, p))| \cdot \frac{\rho(x)}{\rho^*(Y(x, p))}, \\ G(x, p) &= h^*(Y(x, p)), \\ \beta(x, t) &= \nabla_p G(x, p)|_{p=\nabla u}, \\ W(x, t) &= D^2 u(x, t) - A(x, \nabla u(x, t)). \end{aligned}$$

The components of the matrix $W(x, t)$ will be denoted by w_{ij}, while the components of the inverse matrix will be denoted by w^{ij}.

Using the above notation, we can now precisely state the parabolic optimal transportation problem. We seek to find a function $u \in C_x^4 C_t^2(\overline{\Omega} \times [0, \infty))$ satisfying the evolution equation

$$\begin{cases} \partial_t u(x, t) = \log\det[D^2 u(x, t) - A(x, \nabla u(x, t))] - \log B(x, \nabla u(x, t)), & x \in \Omega,\ t > 0, \\ G(x, \nabla u(x, t)) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases} \tag{11}$$

We require the function $u_0 \in C^{4,\alpha}(\Omega)$ for some $\alpha \in (0, 1]$ to be locally, uniformly c-convex as in Definition 2.1 and satisfy

$$\begin{cases} h^*(Y(x, \nabla u_0(x))) = 0 & \text{on } \partial\Omega, \\ T_0(\Omega) = \Omega^*, \end{cases} \tag{12}$$

where $T_0(x) := Y(x, \nabla u_0(x))$.

Let us establish some basic facts which will be needed throughout.

Lemma 2.3. *The function $\theta(x, t) := \partial_t u(x, t)$ satisfies*

$$\int_\Omega e^{\theta(x,t)} \rho(x)\, dx = \int_{\Omega^*} \rho^*(y)\, dy \quad \textit{for all } t \geq 0. \tag{13}$$

Proof. Differentiating the identity $T(x, t) = Y(x, \nabla u(x, t))$, we obtain

$$T^k_{x_\ell}(x, t) = Y^k_{x_\ell}(x, \nabla u(x, t)) + Y^k_{p_j}(x, \nabla u(x, t)) u_{x_j x_\ell}, \quad k, \ell = 1, \ldots, n.$$

In matrix notation,

$$\begin{aligned} D_x T(x,t) &= D_x Y(x, \nabla u(x,t)) + D_p Y(x, \nabla u(x,t)) D^2 u(x,t) \\ &= D_p Y(x, \nabla u(x,t))(D^2 u(x,t) - A(x, \nabla u(x,t)) \\ &= D_p Y(x, \nabla u(x,t)) W(x,t) \\ &= (D^2_{x,y} c)^{-1}(x, Y(x, \nabla u(x,t))) W(x,t). \end{aligned} \tag{14}$$

Consequently,

$$|\det D_x T(x,t)| = \frac{\det W(x,t)}{|\det(D^2_{x,y} c)(x, T(x,t))|}. \tag{15}$$

From (11), it follows that

$$e^{\partial_t u(x,t)} \rho(x) = |\det D_x T(x,t)| \rho^*(x, T(x,t)). \tag{16}$$

Integrating over Ω and using the change of variables formula yields the desired identity. □

Observe that, by (13) and the mass balance condition (5), θ must satisfy

$$\sup_{\Omega} \theta(\,\cdot\,, t) \geq 0 \quad \text{and} \quad \inf_{\Omega} \theta(\,\cdot\,, t) \leq 0 \quad \text{for all } t \geq 0. \tag{17}$$

Lemma 2.4. *Let ν denote the outward-pointing unit normal to Ω, and let W and β be defined as above. Then*

$$\nu(x) = \frac{W(x,t)\beta(x,t)}{|W(x,t)\beta(x,t)|} \quad \textit{for all } (x,t) \in \partial\Omega \times [0, \infty).$$

Proof. Fix $t \geq 0$. The boundary condition $G(x, \nabla u(x,t)) = 0$ on $\partial\Omega$ is equivalent to saying $h^*(T(x,t)) = 0$ on $\partial\Omega$. Therefore, by differentiating in any direction τ tangential to $\partial\Omega$, we get

$$h^*_k(T(x,t)) T^k_{x_i}(x,t) \tau^i = 0.$$

In matrix notation,

$$\langle W(x,t)(D_p Y)^T(x, \nabla u(x,t)) \nabla h^*(T(x,t)), \tau \rangle = 0.$$

By definition,

$$\beta(x,t) = (D_p Y)^T(x, \nabla u(x,t)) \nabla h^*(Y(x, \nabla u(x,t))).$$

Therefore,

$$\langle W(x,t)\beta(x,t), \tau \rangle = 0.$$

It follows that $W\beta$ is parallel to the unit outward-pointing normal vector field ν on $\partial\Omega$. Since $h^* < 0$ on Ω, we can write $W\beta = \chi\nu$, where $\chi \geq 0$. Notice that by (15) and (16), W is positive definite. By bitwist (6), and the fact that $\nabla h^* = \nu^*$, we also know β is nonzero. Consequently, $\chi = |W\beta|$ is nonzero. □

2C. *The linearized equation.* Differentiating (11) in t gives the following linear equation for θ:

$$\begin{cases} \mathcal{L}\theta := w^{ij}(\theta_{ij} - D_{p_k} A_{ij} \theta_k) + D_{p_k}(\log B)\theta_k - \partial_t \theta = 0 & \text{on } \mathcal{C}_T := \Omega \times [0, T], \\ D_\beta \theta = 0 & \text{on } \partial\Omega \times [0, T), \end{cases} \tag{18}$$

where $D_\beta\theta := \langle\beta, \nabla\theta\rangle$, and where, in the coefficients, $p = \nabla u(x,t)$. By the global C^2 estimates established in [Kitagawa 2012], the operator $\mathcal{L}$ is uniformly parabolic and, by Theorems 7.1 and 9.2 of that paper, the boundary condition $D_\beta\theta = 0$ is uniformly oblique for all time. Hence, there exist positive constants $c_1, c_2 > 0$ depending only on Ω, Ω^*, B, c and u_0, but independent of t, such that $w^{ij}\xi_i\xi_j \geq c_1|\xi|^2$ for all $(x,t) \in \Omega$ and $\xi \in \mathbb{R}^n$, and $\langle\beta, \nu\rangle \geq c_2 > 0$ for all $x \in \partial\Omega,\ t > 0$.

Solutions to the linearized equation (18) satisfy the following maximum principle; see also [Kitagawa 2012, Theorem 8.1].

Proposition 2.5. *Suppose v is a solution to the linearized equation* (18). *Then*

$$\max_{(x,t)\in\mathcal{C}_T} v(x,t) = \max_{x\in\Omega} v(x,0), \qquad \min_{(x,t)\in\mathcal{C}_T} v(x,t) = \min_{x\in\Omega} v(x,0).$$

Proof. By the parabolic maximum principle, the maximum of v occurs on the parabolic boundary $\partial_P\mathcal{C}_T := (\bar{\Omega}\times\{0\}) \cup (\partial\Omega\times(0,T))$. Suppose there exists $(x_0,t_0) \in \partial\Omega\times(0,T)$ such that $v(x_0,t_0) = \max_{(x,t)\in\mathcal{C}_T} v(x,t)$. It then follows from Hopf's lemma, see [Lieberman 1996, Lemma 2.8 and following paragraph], that $D_\beta v(x_0,t_0) > 0$. However, this violates the boundary condition $D_\beta v = 0$, and so the maximum cannot occur on $\partial\Omega\times(0,T)$. The argument for the minimum follows in similar fashion. □

2D. ***Exponential convergence on manifolds with no boundary.*** In this section we recall the proof for exponential convergence in the case of no boundary as done in [Kim et al. 2012]. At the end of the section, we reprove the parabolic estimate Proposition 2.7 for the linearized operator, but we note our method differs slightly from that of [Kim et al. 2012].

The authors of [Kim et al. 2012] consider the parabolic flow (11) on a Riemannian manifold with no boundary and show exponential convergence of the solution u of (11) to the solution u^∞ of the elliptic equation (2). A key ingredient in their proof of exponential convergence is a Li–Yau-type Harnack inequality for positive solutions v of the linearized equation $\mathcal{L}v = 0$; see [Kim et al. 2012, Theorem 5.2]. This strategy is motivated by the observation that the operator $\mathcal{L}$ is a heat-type equation with respect to the time-varying Riemannian metric g with components $g_{ij} = w_{ij}$ (see the discussion preceding Proposition 2.7 below).

Suppose v is a positive solution to the linearized equation $\mathcal{L}v = 0$ on $\mathcal{C}_T$, where $T > 0$ is chosen to be sufficiently large. Let $f = \log v$ and consider the quantity

$$F = t(|\nabla^g f|_g^2 - \alpha\,\partial_t f) = t(w^{ij} f_i f_j - \alpha\,\partial_t f), \tag{19}$$

where $\alpha > 0$ is a constant to be determined and ∇^g denotes the gradient of a function with respect to the metric g. It is shown in [Kim et al. 2012, Theorem 5.2] that F is sublinear in t everywhere in $\mathcal{C}_T$; that is, there exist constants $C_1, C_2 > 0$ (independent of T) such that $F(x,t) \leq C_1 + C_2 t$ for all $x \in \Omega,\ t \in [0,T]$. The sublinearity in t of F implies the differential Harnack inequality

$$w^{ij} f_i f_j - \alpha\,\partial_t f \leq \frac{C_1}{t} + C_2 \tag{20}$$

for some possibly different constants C_1 and $C_2 > 0$. A standard argument applying the fundamental theorem of calculus to f along an appropriate space-time curve, and then using (20) to estimate the term involving $\partial_t f$ (see, for instance, [Kim et al. 2012, p. 4345, Proof of Theorem 5.2]), yields the parabolic Harnack inequality

$$\sup_{\Omega} v(\,\cdot\,, t) \le C \inf_{\Omega} v(\,\cdot\,, t+1) \quad \text{for all } t \ge 1, \tag{21}$$

where $C > 0$ is a constant independent of t. One then applies (21) to the nonnegative solutions

$$v_k^+(x,t) := \sup_{\Omega} v(\,\cdot\,, k) - v(x, k+t) \quad \text{and} \quad v_k^-(x,t) := v(x, k+t) - \inf_{\Omega} v(\,\cdot\,, k), \quad k = 0, 1, 2, \dots,$$

to obtain decay of oscillation of θ in time; see [Kim et al. 2012, Section 7.1]. This shows that θ converges exponentially fast to a constant function on Ω as $t \to \infty$. Invoking (17), we conclude that $\lim_{t\to\infty} \theta \equiv 0$, and so $u(\,\cdot\,, t)$ converges exponentially fast as $t \to \infty$ to a function $u^{\infty}(\,\cdot\,)$ solving (2).

Below we show the sublinearity of F, which is a standard argument provided here for completeness. The proof relies on an important parabolic inequality satisfied by F, (24), which we will prove in Proposition 2.7 below.

Proposition 2.6. *If F does not attain a positive maximum on $\partial\Omega \times (0, T)$, then there exist constants C_1' and $C_2' > 0$ independent of T such that*

$$F(x,t) \le C_1' + C_2' t \quad \textit{for all } (x,t) \in \mathcal{C}_T. \tag{22}$$

Proof. First note that $F(\,\cdot\,, 0) \equiv 0$ because $\inf_{\Omega} v(\,\cdot\,, 0) > 0$, and so the bound holds at $t = 0$. Suppose there exists a first time $\tau \in (0, T)$ such that $F(y, \tau) \ge C_1' + C_2'\tau$ for some $y \in \Omega$. By going further in time if necessary, we may assume there exists a point $(x_0, t_0) \in \overline{\Omega} \times (0, T]$ such that $F(x_0, t_0) > C_1' + C_2' t_0$ and F attains a local maximum at (x_0, t_0). If (x_0, t_0) is an interior point of $\mathcal{C}_T$, it follows from (24) that

$$C_1 F(x_0, t_0)^2 - F(x_0, t_0) - C_2 t_0^2 \le 0,$$

from which we conclude

$$F(x_0, t_0) \le \frac{1 + \sqrt{1 + 4C_1C_2t_0^2}}{2C_1} \le \widetilde{C}_1 + \widetilde{C}_2 t_0 \tag{23}$$

for a different set of constants $\widetilde{C}_1, \widetilde{C}_2 > 0$ and for $t_0 > 0$ sufficiently large. If C_1', C_2' were chosen at the beginning to satisfy $C_1' > \widetilde{C}_1$ and $C_2' > \widetilde{C}_2$, then we reach a contradiction based on (23). □

Thus it is clear that on a manifold with no boundary, Proposition 2.6 combined with the discussion above yields exponential convergence, as is shown in [Kim et al. 2012].

We finish this section by establishing the parabolic inequality (24) satisfied by F. It is shown in [Kim et al. 2012, Proposition 5.1] that if $n \ge 3$ and

$$\psi(x,t) := \left(\frac{\rho^*(T(x,t))^2 \,\det D_x T(x,t)}{|\det D^2_{x,y} c(x, T(x,t))|} \right)^{1/(n-2)},$$

then

$$\mathcal{L}v = \psi \Delta_{\psi g} v - \partial_t v,$$

where $\Delta_{\psi g}$ is the Laplace–Beltrami operator with respect to the time-varying metric ψg with $g_{ij} := w_{ij}$. By adapting the proof of the differential Harnack inequality for the heat equation established in [Li and Yau 1986], the authors of [Kim et al. 2012] establish a parabolic inequality for F similar to (24) in the case of manifolds with no boundary of dimension $n \geq 3$. The case $n = 2$ is treated in [Kim et al. 2012] through the introduction of a third dummy dimension in a manner giving the solution u of (11) a product structure; see [Kim et al. 2012, Section 7.1.2] for details. In the presence of a boundary, such an argument for dealing with the two-dimensional case is almost certain to fail due to the requirement of uniform c- and c^*-convexity of the domains involved.

We elect to take a different approach which considers the weighted Laplacian $\Delta_\phi := \Delta_g - \langle \nabla^g \phi, \nabla^g \cdot \rangle_g$ for the manifold with density $(\Omega, g, e^{-\phi} d\,\mathrm{Vol}_g)$, where

$$\phi(x,t) := \log\left(\frac{|\det D^2_{x,y} c(x, T(x,t))|}{\rho^*(T(x,t))^2 \det D_x T(x,t)}\right)^{1/2}.$$

It was first noted in [Warren 2014, Section 3] that for such a choice of weighted manifold, $\mathcal{L} = \Delta_\phi - \partial_t$. The advantage of using this representation of $\mathcal{L}$ is that the case of dimension $n = 2$ does not need to be treated separately. As mentioned above, in the case of nonempty boundary, the conversion of the two-dimensional problem to a three-dimensional one as in [Kim et al. 2012] cannot be carried out. To summarize, the following proof follows the spirit of [Kim et al. 2012, Section 6] (which in turn is based on [Li and Yau 1986, Theorem 1.2]), but the details differ as we use the representation of the linearized operator as a weighted Laplacian from [Warren 2014], in contrast with the conformal factor approach used in [Kim et al. 2012].

Proposition 2.7. *Under the same hypotheses as Theorem 1.1, there exist constants C_1, C_2, and $C_3 > 0$, depending only on the constant $\mathcal{K}$ defined in* (4) *and the dimension n, such that whenever v satisfies $\mathcal{L}v = 0$,*

$$\mathcal{L}F + 2\langle \nabla^g f, \nabla^g F\rangle_g \geq \frac{1}{t}(C_1 F^2 - F - C_2 t^2 + C_3 t |\nabla^g f|_g^2 F). \tag{24}$$

Proof. We recall the well-known weighted Bochner formula

$$\Delta_\phi(|\nabla^g f|_g^2) = 2\|\mathrm{Hess}_g f\|^2 + 2\langle \nabla^g f, \nabla^g(\Delta_\phi f)\rangle_g + 2\,\mathrm{Ric}_\phi(\nabla^g f, \nabla^g f), \tag{25}$$

where $\mathrm{Ric}_\phi := \mathrm{Ric}_g + \mathrm{Hess}_g\, \phi$. Clearly, $\mathrm{Ric}_\phi \geq -\mathcal{K}$, where $\mathcal{K}$ is defined in (4). Since $\mathcal{L}v = 0$, the function $f := \log v$ solves the equation

$$\partial_t f = \Delta_\phi f + |\nabla^g f|_g^2. \tag{26}$$

Consider the auxiliary function

$$F := t(|\nabla^g f|_g^2 - \alpha\, \partial_t f), \quad \alpha > 0.$$

By using (25), we obtain

$$\begin{aligned}\Delta_\phi F &= t(\Delta_\phi(|\nabla^g f|_g^2) - \alpha \Delta_\phi(\partial_t f)) \\ &= t\big(2\|\mathrm{Hess}_g f\|^2 + 2\langle \nabla^g f, \nabla^g(\Delta_\phi f)\rangle_g + 2\,\mathrm{Ric}_\phi(\nabla^g f, \nabla^g f) - \alpha \Delta_\phi(\partial_t f)\big).\end{aligned}$$

Direct computation shows that

$$\Delta_\phi(\partial_t f) \le \partial_t(\Delta_\phi f) + C(\|\mathrm{Hess}_g f\| + |\nabla^g f|_g),$$

where $C = C(\partial_t g, \partial_t \nabla g, \partial_t \nabla\phi) \ge 0$ depends only on $\mathcal{K}$. Therefore,

$$\begin{aligned}\Delta_\phi F &\ge t\big(2\|\mathrm{Hess}_g f\|^2 + 2\langle \nabla^g f, \nabla^g(\Delta_\phi f)\rangle_g + 2\,\mathrm{Ric}_\phi(\nabla^g f, \nabla^g f) - \alpha\,\partial_t(\Delta_\phi f) - \alpha C(\|\mathrm{Hess}_g f\| + |\nabla^g f|_g)\big)\\ &\ge t\big(\|\mathrm{Hess}_g f\|^2 + 2\langle \nabla^g f, \nabla^g(\Delta_\phi f)\rangle_g - \alpha\,\partial_t(\Delta_\phi f) - C_1|\nabla^g f|_g^2 - C_2\big),\end{aligned}$$

where we have used Cauchy's inequality and the lower bound for Ric_ϕ. From (26) and the definition of F, it follows that

$$\Delta_\phi f = -\left(\frac{F}{t} + (\alpha-1)\,\partial_t f\right).$$

Therefore,

$$\begin{aligned}2\langle \nabla^g f, \nabla^g(\Delta_\phi f)\rangle_g &= -2\left\langle \nabla^g f, \nabla^g\left(\frac{F}{t} + (\alpha-1)\,\partial_t f\right)\right\rangle_g\\ &= -\frac{2}{t}\langle \nabla^g f, \nabla^g F\rangle_g - 2(\alpha-1)\langle \nabla^g f, \nabla^g(\partial_t f)\rangle_g.\end{aligned}$$

Furthermore,

$$\partial_t F = \frac{F}{t} + t(\partial_t|\nabla^g f|_g^2 - \alpha\,\partial_t^2 f).$$

Therefore,

$$\begin{aligned}-\alpha\,\partial_t(\Delta_\phi f) &= \alpha\,\partial_t\left(\frac{F}{t} + (\alpha-1)\,\partial_t f\right)\\ &= \alpha\left(\frac{\partial_t F}{t} - \frac{F}{t^2} + (\alpha-1)\,\partial_t^2 f\right)\\ &= \alpha\left(\frac{\partial_t F}{t} - \frac{F}{t^2}\right) + (\alpha-1)\alpha\,\partial_t^2 f\\ &= \alpha\left(\frac{\partial_t F}{t} - \frac{F}{t^2}\right) + (\alpha-1)\left(\frac{F}{t^2} - \frac{\partial_t F}{t} + \partial_t|\nabla^g f|_g^2\right)\\ &= \frac{\partial_t F}{t} - \frac{F}{t^2} + (\alpha-1)\,\partial_t|\nabla^g f|_g^2.\end{aligned}$$

It follows that

$$\begin{aligned}&2\langle \nabla^g f, \nabla^g(\Delta_\phi f)\rangle_g - \alpha\,\partial_t(\Delta_\phi f)\\ &\qquad= \frac{1}{t}\left(\partial_t F - 2\langle \nabla^g f, \nabla^g F\rangle_g - \frac{F}{t}\right) + (\alpha-1)(\partial_t|\nabla^g f|_g^2 - 2\langle \nabla^g f, \nabla^g(\partial_t f)\rangle_g)\\ &\qquad\ge \frac{1}{t}\left(\partial_t F - 2\langle \nabla^g f, \nabla^g F\rangle_g - \frac{F}{t}\right) - C_3|\nabla^g f|_g^2,\end{aligned}$$

where we have used the fact

$$\partial_t|\nabla^g f|_g^2 \le 2\langle \nabla^g f, \nabla^g(\partial_t f)\rangle_g + \gamma|\nabla^g f|_g^2$$

for some constant $\gamma = \gamma(\partial_t g) \geq 0$ depending only on $\mathcal{K}$. Inserting the above inequality into the lower bound for $\Delta_\phi F$ yields

$$\Delta_\phi F \geq t\left(\|\mathrm{Hess}_g f\|^2 + \frac{1}{t}\left(\partial_t F - 2\langle \nabla^g f, \nabla^g F\rangle_g - \frac{F}{t}\right) - C_4|\nabla^g f|_g^2 - C_2\right).$$

Now since $\Delta_\phi f = \Delta_g f - \langle \nabla^g \phi, \nabla^g f\rangle_g$, we have

$$\begin{aligned}(\Delta_g f)^2 = (\Delta_\phi f + \langle \nabla^g\phi, \nabla^g f\rangle_g)^2 &= (\Delta_\phi f)^2 + \langle \nabla^g\phi, \nabla^g f\rangle_g^2 + 2(\Delta_\phi f)\langle \nabla^g\phi, \nabla^g f\rangle_g \\ &\geq (\Delta_\phi f)^2 + \langle \nabla^g\phi, \nabla^g f\rangle_g^2 - \frac{(\Delta_\phi f)^2}{2} - 2\langle \nabla^g\phi, \nabla^g f\rangle_g^2 \\ &= \frac{(\Delta_\phi f)^2}{2} - \langle \nabla^g\phi, \nabla^g f\rangle_g^2 \\ &\geq \frac{(\Delta_\phi f)^2}{2} - |\nabla^g\phi|_g^2|\nabla^g f|_g^2.\end{aligned}$$

Therefore, by the arithmetic-geometric mean inequality, we have

$$\|\mathrm{Hess}_g f\|^2 \geq \frac{1}{n}(\Delta_g f)^2 \geq \frac{(\Delta_\phi f)^2}{2n} - \frac{1}{n}|\nabla^g\phi|_g^2|\nabla^g f|_g^2.$$

Since $|\nabla^g\phi|_g \leq \mathcal{K}$, we obtain

$$\Delta_\phi F \geq \frac{t}{2n}(\Delta_\phi f)^2 + \partial_t F - 2\langle \nabla^g f, \nabla^g F\rangle_g - \frac{F}{t} - C_5 t|\nabla^g f|_g^2 - C_2 t.$$

Finally, by (26), and after relabeling constants, we conclude that

$$\Delta_\phi F + 2\langle \nabla^g f, \nabla^g F\rangle_g - \partial_t F \geq \frac{1}{t}[C_1 t^2(|\nabla^g f|_g^2 - \partial_t f)^2 - F - C_2 t^2|\nabla^g f|_g^2 - C_3 t^2].$$

Here the constants $C_1, C_2, C_3 > 0$ depend only on up to fourth-order derivatives of the cost function (through $\mathrm{Hess}_g \phi$) and the $C_x^4 C_t^1$ norm of the solution u to (11) (through the time derivative of g and bounds on the Ricci curvature of g), and hence only on $\mathcal{K}$ and on the dimension n.

Let $y = |\nabla^g f|_g^2$ and $z = \partial_t f$. Then for any $\alpha, \epsilon, \delta > 0$, we have the identity

$$(y-z)^2 = \left(\frac{1}{\alpha} - \frac{\epsilon}{2}\right)(y-\alpha z)^2 + \left(1 - \frac{\epsilon}{2} - \delta - \frac{1}{\alpha}\right)y^2 + \left(1 - \alpha + \frac{\epsilon}{2}\alpha^2\right)z^2 + \epsilon y(y-\alpha z) + \delta y^2.$$

We now choose $\alpha, \epsilon > 0$ such that

$$1 - \frac{\epsilon}{2} - \frac{1}{\alpha} > 0, \quad 1 - \alpha + \frac{\epsilon}{2}\alpha^2 \geq 0, \quad \frac{1}{\alpha} - \frac{\epsilon}{2} > 0.$$

Note that these conditions impose the restriction $\alpha > 1$. A direct verification shows that $\alpha = 2$ and $\epsilon = \frac{1}{2}$ satisfy the above inequalities. We then choose $\delta = \frac{1}{8} \in \left(0, 1 - \frac{\epsilon}{2} - \frac{1}{\alpha}\right) = \left(0, \frac{1}{4}\right)$. With these choices of α, ϵ, δ, we obtain (discarding the second and third terms in the expansion, and using that $F = t(y - \alpha z)$)

$$\Delta_\phi F + 2\langle \nabla^g f, \nabla^g F\rangle_g - \partial_t F \geq \frac{1}{t}\left[C_1 t^2\left\{\frac{F^2}{4t^2} + y\frac{F}{2t} + \frac{y^2}{8}\right\} - F - C_2 t^2 y - C_3 t^2\right].$$

Using Cauchy's inequality, we may eliminate the $-C_2t^2y$ and $C_1t^2y^2/8$ terms to get

$$\Delta_\phi F + 2\langle \nabla^g f, \nabla^g F\rangle_g - \partial_t F \geq \frac{1}{t}\left[C_1 t^2\left\{\frac{F^2}{4t^2} + y\frac{F}{2t}\right\} - F - C_4 t^2\right].$$

Relabeling constants, we have thus established an inequality of the form (24). □

3. Sublinearity of F on domains with boundary

On a domain with boundary, one must deal with the possibility that F attains a maximum at a point $(x_0, t_0) \in \partial_P \mathcal{C}_T = (\bar{\Omega} \times \{0\}) \cup (\partial\Omega \times (0, T))$, the parabolic boundary of the cylinder $\mathcal{C}_T$. Since $F \equiv 0$ on $\bar{\Omega} \times \{0\}$, it suffices to assume $(x_0, t_0) \in \partial\Omega \times (0, T)$. The original argument of [Li and Yau 1986, proof of Theorem 1.1] in the case of the heat equation eliminates the possibility of F attaining a nonnegative maximum on $\partial\Omega \times (0, T)$ by means of a contradiction to Hopf's lemma. For this, they require two additional hypotheses: namely, the solution to the heat equation also satisfies a Neumann boundary condition, and the boundary is mean-convex.

We will obtain a similar contradiction to Hopf's lemma only for the particular nonnegative solution $\Theta(x, t) := \sup_\Omega \theta(\cdot, 0) - \theta(x, t)$ of the linearized equation (18) (as well as for translations of Θ in time) by exploiting the boundary condition $D_\beta\Theta = 0$ on $\partial\Omega \times [0, T]$, and using the assumption that the domains Ω, Ω^* are respectively c-convex and c^*-convex. This gives the desired sublinearity at the boundary of the corresponding function F defined in (19) and establishes the Harnack inequality (21) for Θ, which turns out to be sufficient to prove the exponential convergence of $u(\cdot, t)$ to the steady state solution $u^\infty(\cdot)$ as $t \to \infty$ (see Section 4). As mentioned in the Introduction, it is unclear if such a sublinearity estimate at the boundary holds for an arbitrary nonnegative solution v of the linearized equation (18).

Let us carry on with the proof of the sublinearity of F outlined in Proposition 2.6, now assuming there exists $(x_0, t_0) \in \partial\Omega \times (0, T)$ such that $F(x_0, t_0) > C_1' + C_2' t_0$ and F attains a local maximum at (x_0, t_0). It follows from (24) that, in a spherical cap near (x_0, t_0), we have

$$\mathcal{L}F + 2\langle \nabla^g f, \nabla^g F\rangle_g \geq 0.$$

By the uniform obliqueness of β and Hopf's lemma, it follows that $D_\beta F(x_0, t_0) > 0$. Anticipating a contradiction, we proceed to explicitly compute $D_\beta F(x_0, t_0)$. We first make a rotation centered at x_0 so the directions $e_1, \ldots, e_{n-1}$ form an orthonormal basis for the tangent space to $\partial\Omega$ at x_0, and the direction e_n is the outward-pointing unit normal direction to $\partial\Omega$ at x_0. Differentiating F in these coordinates, we find that

$$\begin{aligned} D_\beta F(x_0, t_0) &= D_\beta|_{(x_0,t_0)} t(w^{ij} f_i f_j - \alpha\,\partial_t f) \\ &= t_0[(D_\beta w^{ij}) f_i f_j + 2w^{ij}(D_\beta f_i) f_j - \alpha D_\beta(\partial_t f)]\big|_{(x_0,t_0)} \\ &= t_0[-w^{i\ell}w^{jk}(D_\beta w_{\ell k}) f_i f_j + 2w^{ij}((D_\beta f)_i - \beta_i^k f_k) f_j - \alpha(\partial_t(D_\beta f) - (\partial_t\beta^k) f_k)]\big|_{(x_0,t_0)}. \end{aligned}$$

Now since $D_\beta f = D_\beta v/v = 0$ on $\partial\Omega$, we have $\partial_t(D_\beta f) = 0$ and $(D_\beta f)_i = 0$ for $i = 1, \ldots, n-1$. Therefore,

$$D_\beta F(x_0, t_0) = t_0[-w^{i\ell}w^{jk}(D_\beta w_{\ell k}) f_i f_j - 2w^{ij}\beta_i^k f_k f_j + 2w^{nj} f_j (D_\beta f)_n + \alpha(\partial_t\beta^k) f_k]\big|_{(x_0,t_0)}.$$

We claim $w^{nj} f_j = 0$ at (x_0, t_0). By Lemma 2.4, $W\beta$ is parallel to the outward-pointing unit normal vector ν on $\partial\Omega$, so $\nu = (1/\chi)W\beta$, where $\chi := |W\beta|$. Again since $D_\beta f = 0$ on $\partial\Omega$,

$$0 = \langle \beta, \nabla f\rangle = \langle W^{-1}W\beta, \nabla f\rangle = \langle W\beta, W^{-1}\nabla f\rangle.$$

Hence,

$$\tau := W^{-1}\nabla f \tag{27}$$

is tangent to $\partial\Omega$. In the coordinate system defined above, we have $\nu(x_0, t_0) = e_n$, and so $\tau^n(x_0, t_0) = 0$. Since $\tau^n = w^{nj} f_j$, the claim is proved. It follows that

$$D_\beta F(x_0, t_0) = t_0[-(D_\beta w_{k\ell})\tau^k\tau^\ell - 2\beta_i^k f_k\tau^i + \alpha(\partial_t\beta^k) f_k]\big|_{(x_0,t_0)}. \tag{28}$$

Note that since $\tau_n = 0$ at (x_0, t_0), it suffices to sum the indices in the first term over $k, \ell = 1, \dots, n-1$.

3A. ***Inner product cost.*** We first show how to explicitly compute $D_\beta F(x_0, t_0)$ in the case when the cost function is given by the Euclidean inner product on $\mathbb{R}^n$ (which is known to be equivalent to taking the cost function to be the Euclidean distance squared). There are a number of simplifications in this case, as $Y(x, p) = p$, $W(x, t) = D^2u(x, t)$, and c- and c^*-convexity of sets and functions reduce to the usual notions of convexity of the domains Ω and Ω^*.

Proposition 3.1. *If* $c(x, y) = \langle x, y\rangle$,

$$D_\beta F(x_0, t_0) = t_0[-\chi\langle (D\nu)\tau, \tau\rangle - \langle D^2h^*(\nabla u)\nabla f, \nabla f\rangle + \alpha\langle D^2h^*(\nabla u)\nabla\theta, \nabla f\rangle]\big|_{(x_0,t_0)}. \tag{29}$$

Proof. We have

$$W = D^2u, \quad \beta = \nabla h^*(\nabla u).$$

Consequently, $\nu = (1/\chi)(D^2u)\beta$. Differentiating ν^k in the e_ℓ-direction for $k, \ell = 1, \dots, n-1$, we find

$$\begin{aligned}\nu_\ell^k = \left(\frac{1}{\chi}u_{kr}\beta^r\right)_\ell &= \frac{1}{\chi}(u_{\ell kr}\beta^r + u_{kr}\beta_\ell^r) - \frac{\chi_\ell}{\chi^2}(u_{kr}\beta^r)\\ &= \frac{1}{\chi}(D_\beta u_{\ell k} + u_{kr}\beta_\ell^r) - (\log\chi)_\ell\nu^k.\end{aligned}$$

Solving for $D_\beta u_{\ell k}$, we obtain

$$D_\beta u_{\ell k} = \chi\nu_\ell^k - u_{kr}\beta_\ell^r + \chi(\log\chi)_\ell\nu^k.$$

Therefore at (x_0, t_0), we have (recall (27))

$$\begin{aligned}-(D_\beta u_{\ell k})\tau^\ell\tau^k &= -(\chi\nu_\ell^k - u_{kr}\beta_\ell^r + \chi(\log\chi)_\ell\nu^k)\tau^\ell\tau^k\\ &= -\chi\nu_\ell^k\tau^\ell\tau^k + u_{kr}\tau^k\beta_\ell^r\tau^\ell\\ &= -\chi\nu_\ell^k\tau^\ell\tau^k + f_r\beta_\ell^r\tau^\ell,\end{aligned}$$

where we sum the indices k, ℓ from 1 to $n-1$. Substituting this into (28) gives

$$D_\beta F(x_0, t_0) = t_0[-\chi\nu_\ell^k\tau^\ell\tau^k - \beta_i^k f_k\tau^i + \alpha(\partial_t\beta^k) f_k]\big|_{(x_0,t_0)}.$$

Since $\beta(x,t) = \nabla h^*(\nabla u(x,t))$, we find that

$$\beta_i^k f_k \tau^i = h^*_{k\ell}(\nabla u) u_{\ell i} f_k \tau^i = h^*_{k\ell}(\nabla u) f_k u_{\ell i} \tau^i = h^*_{k\ell}(\nabla u) f_k f_\ell,$$

and

$$(\partial_t \beta^k) f_k = h^*_{k\ell}(\nabla u)(\partial_t u_\ell) f_k = h^*_{k\ell}(\nabla u)\theta_\ell f_k;$$

hence (29) follows. □

3B. ***General cost.*** We now show how to explicitly compute $D_\beta F(x_0, t_0)$ in the case of a general cost.

Proposition 3.2. *We have*

$$D_\beta F(x_0,t_0) = t_0[-\chi(\nu_i^j - c^{r,\ell} c_{ij,r} \nu^\ell)\tau^i\tau^j - G_{p_k p_s}(x,\nabla u) f_k f_s + \alpha\, G_{p_k p_s}(x,\nabla u) f_k \theta_s]\big|_{(x_0,t_0)}. \tag{30}$$

Proof. We have

$$w_{jk}(x,t) = u_{jk}(x,t) - c_{jk}(x, T(x,t)), \qquad \beta^k(x,t) = h^*_\ell(Y(x,\nabla u(x,t))) Y^\ell_{p_k}(x,\nabla u(x,t)).$$

Recall that $\nu = (1/\chi) W\beta$. As in the case of the inner product cost, we differentiate ν^j in the e_i-direction for $i, j = 1, \dots, n-1$ to get

$$\begin{aligned}\nu_i^j = \left(\frac{1}{\chi} w_{jk}\beta^k\right)_i &= \frac{1}{\chi}((w_{jk})_i \beta^k + w_{jk}\beta_i^k) - \frac{\chi_i}{\chi^2}(w_{jk}\beta^k)\\ &= \frac{1}{\chi}((w_{jk})_i \beta^k + w_{jk}\beta_i^k) - (\log\chi)_i \nu^j.\end{aligned}$$

Differentiating w_{jk} gives

$$\begin{aligned}(w_{jk})_i &= u_{jki} - c_{jki} - c_{jk,r} T_i^r\\ &= (w_{ij})_k + c_{ij,r} T_k^r - c_{jk,r} T_i^r\\ &= (w_{ij})_k + c_{ij,r} c^{r,\ell} w_{\ell k} - c_{jk,r} c^{r,\ell} w_{\ell i},\end{aligned}$$

where we have used (14) in the final line. Therefore,

$$\begin{aligned}\nu_i^j &= \frac{1}{\chi}((w_{jk})_i \beta^k + w_{jk}\beta_i^k) - (\log\chi)_i \nu^j\\ &= \frac{1}{\chi}\big([(w_{ij})_k + c_{ij,r} c^{r,\ell} w_{\ell k} - c_{jk,r} c^{r,\ell} w_{\ell i}]\beta^k + w_{jk}\beta_i^k\big) - (\log\chi)_i \nu^j\\ &= \frac{1}{\chi}\big(D_\beta w_{ij} + [c_{ij,r} c^{r,\ell} w_{\ell k} - c_{jk,r} c^{r,\ell} w_{\ell i}]\beta^k + w_{jk}\beta_i^k\big) - (\log\chi)_i \nu^j.\end{aligned}$$

Solving for $D_\beta w_{ij}$, we obtain

$$D_\beta w_{ij} = \chi \nu_i^j - [c_{ij,r} c^{r,\ell} w_{\ell k} - c_{jk,r} c^{r,\ell} w_{\ell i}]\beta^k - w_{jk}\beta_i^k + \chi(\log\chi)_i \nu^j.$$

Therefore, at (x_0, t_0), we have (again using (27))

$$\begin{aligned}-(D_\beta w_{ij})\tau^i\tau^j &= -\big(\chi\nu_i^j - [c_{ij,r} c^{r,\ell} w_{\ell k} - c_{jk,r} c^{r,\ell} w_{\ell i}]\beta^k - w_{jk}\beta_i^k + \chi(\log\chi)_i \nu^j\big)\tau^i\tau^j\\ &= -(\chi\nu_i^j - c_{ij,r} c^{r,\ell} w_{\ell k}\beta^k + c_{jk,r} c^{r,\ell} w_{\ell i}\beta^k)\tau^i\tau^j + w_{jk}\beta_i^k\tau^i\tau^j\\ &= -\chi(\nu_i^j - c^{r,\ell} c_{ij,r}\nu^\ell)\tau^i\tau^j - c^{r,\ell} c_{jk,r} f_\ell \beta^k \tau^j + f_k \beta_i^k \tau^i\\ &= -\chi(\nu_i^j - c^{r,\ell} c_{ij,r}\nu^\ell)\tau^i\tau^j - c_{jk,r} h_s^* Y^s_{p_k} Y^r_{p_\ell} f_\ell \tau^j + f_k \beta_i^k \tau^i\end{aligned}$$

where we sum the indices i, j from 1 to $n-1$. It follows from (28) that

$$D_\beta F(x_0,t_0)=t_0[-\chi(\nu_i^j-c^{r,\ell}c_{ij,r}\nu^\ell)\tau^i\tau^j-c_{jk,r}h_s^*Y^s_{p_k}Y^r_{p_\ell}f_\ell\tau^j-f_k\beta_i^k\tau^i+\alpha\,\partial_t\beta^k f_k]\big|_{(x_0,t_0)}. \tag{31}$$

We compute

$$\beta_i^k=h^*_{\ell r}(Y^r_{x_i}+Y^r_{p_s}u_{si})Y^\ell_{p_k}+h^*_\ell(Y^\ell_{p_kx_i}+Y^\ell_{p_kp_s}u_{si}). \tag{32}$$

To simplify the first term, recall the identity (see (14))

$$Y^r_{x_i}+Y^r_{p_s}u_{si}=Y^r_{p_s}w_{si}.$$

For the second term in (32), we differentiate the equation $c_{i,\ell}Y^\ell_{p_k}=\delta_{ik}$ with respect to p_s and x_i to obtain

$$Y^\ell_{p_kp_s}=-c^{\ell,j}c_{j,rq}Y^r_{p_k}Y^q_{p_s}$$

and

$$Y^\ell_{p_kx_i}=-c^{\ell,j}c_{ij,r}Y^r_{p_k}+c^{\ell,j}c_{j,rq}Y^r_{p_k}Y^q_{p_s}c_{si}=-c_{ij,r}Y^\ell_{p_j}Y^r_{p_k}-Y^\ell_{p_kp_s}c_{si}.$$

Therefore,

$$Y^\ell_{p_kx_i}+Y^\ell_{p_kp_s}u_{si}=-c_{ij,r}Y^\ell_{p_j}Y^r_{p_k}+w_{si}Y^\ell_{p_kp_s}.$$

Substituting these into the expression (32) gives

$$\beta_i^k=h^*_{\ell r}Y^\ell_{p_k}Y^r_{p_s}w_{si}+h^*_\ell(-c_{ij,r}Y^\ell_{p_j}Y^r_{p_k}+w_{si}Y^\ell_{p_kp_s}).$$

Therefore,

$$\begin{aligned}f_k\beta_i^k\tau^i&=h^*_{\ell r}Y^\ell_{p_k}Y^r_{p_s}f_kw_{si}\tau^i-c_{ij,r}h^*_\ell Y^\ell_{p_j}Y^r_{p_k}f_k\tau^i+h^*_\ell Y^\ell_{p_kp_s}f_kw_{si}\tau^i\\&=h^*_{\ell r}Y^\ell_{p_k}Y^r_{p_s}f_kf_s-c_{ij,r}h^*_\ell Y^\ell_{p_j}Y^r_{p_k}f_k\tau^i+h^*_\ell Y^\ell_{p_kp_s}f_kf_s.\end{aligned}$$

Substituting into (31) and observing that the second term in the above expression cancels the term $-c_{jk,r}h_s^*Y^r_{p_\ell}Y^s_{p_k}f_\ell\tau^j$ in (31), we obtain

$$D_\beta F(x_0,t_0)=t_0[-\chi(\nu_i^j-c^{r,\ell}c_{ij,r}\nu^\ell)\tau^i\tau^j-(h^*_{\ell r}Y^\ell_{p_k}Y^r_{p_s}+h^*_\ell Y^\ell_{p_kp_s})f_kf_s+\alpha\beta_t^kf_k]\big|_{(x_0,t_0)}.$$

Next, we compute

$$\partial_t\beta^k=(h^*_{\ell r}Y^\ell_{p_k}Y^r_{p_s}+h^*_\ell Y^\ell_{p_kp_s})\theta_s.$$

Finally, noticing that $h^*_{\ell r}Y^\ell_{p_k}Y^r_{p_s}+h^*_\ell Y^\ell_{p_kp_s}=G_{p_kp_s}$, we obtain the claimed expression (30). □

4. Proof of exponential convergence

With Propositions 3.1 and 3.2 in hand, we may now prove our main result. We note that the proof presented here is different from the standard proof of exponential convergence via parabolic Harnack inequality outlined in Section 2D, and explicitly uses special properties of the underlying optimal transport problem. In particular, we must take into consideration the c- and c^*-convexity of both domains Ω and Ω^*, and judiciously choose special solutions of the linearized problem that will allow us to utilize the expressions for $D_\beta F$ obtained in Propositions 3.1 and 3.2.

Proof of Theorem 1.1. Consider the function

$$\Theta(x,t) = \sup_{\Omega} \theta(\,\cdot\,,0) - \theta(x,t),$$

which satisfies (18), and is nonnegative by Proposition 2.5. We claim $D_\beta F(x_0,t_0) \le 0$ when $v = \Theta$, which will contradict Hopf's lemma, thus proving F cannot attain a positive maximum on $\partial\Omega \times (0,T)$.

Let us first deal with the case of the inner product cost. Since the domain Ω is convex, we have $\langle (D\nu)\tau, \tau\rangle \ge 0$. Therefore, since $\chi \ge 0$, we obtain using Proposition 3.1

$$D_\beta F(x_0,t_0) \le t_0[-\langle D^2h^*(\nabla u)\nabla f, \nabla f\rangle + \alpha\langle D^2h^*(\nabla u)\nabla\theta, \nabla f\rangle]\big|_{(x_0,t_0)}. \tag{33}$$

Next, the convexity of Ω^* implies D^2h^* is nonnegative, so by substituting for $f = \log\Theta$ in (33), we find

$$D_\beta F(x_0,t_0) \le t_0\left[-\frac{1}{\Theta^2}\langle D^2h^*(\nabla u)\nabla\theta, \nabla\theta\rangle - \frac{\alpha}{\Theta}\langle D^2h^*(\nabla u)\nabla\theta, \nabla\theta\rangle\right]\bigg|_{(x_0,t_0)} \le 0.$$

This is the desired contradiction to Hopf's lemma. For general costs, we use Proposition 3.2, noticing that c-convexity of Ω with respect to Ω^*, given in (8), implies $(\nu_i^j - c^{r,\ell}c_{ij,r}\nu^\ell)\tau^i\tau^j \ge 0$, while the c^*-convexity of Ω^* with respect to Ω, given in (9), implies $G_{p_k p_s}$ is a nonnegative matrix.

It follows from Proposition 2.6 that with the choice $v = \Theta$, the corresponding function F defined in (19) is sublinear in time, and consequently the Harnack inequality (21) holds for Θ. Using this Harnack inequality, we now prove exponential convergence of $\theta(\,\cdot\,,t)$. The argument is similar to [Kim et al. 2012, Section 7], but differs in an essential manner. For each integer $k \ge 1$, consider the function

$$\Theta_k(x,t) := \sup_{\Omega} \theta(\,\cdot\,,k-1) - \theta(x,(k-1)+t).$$

The functions Θ_k are nonnegative by Proposition 2.5 and solve (18). Arguing as above, the corresponding functions F for $v = \Theta_k$ are also sublinear in t (with constants independent of k) and thus the Harnack inequality (21) holds for Θ_k. Applying (21) to Θ_k at $t = 1$ yields

$$\sup_{\Omega} \theta(\,\cdot\,,k-1) - \inf_{\Omega} \theta(\,\cdot\,,k) \le C\big(\sup_{\Omega} \theta(\,\cdot\,,k-1) - \sup_{\Omega} \theta(\,\cdot\,,k+1)\big). \tag{34}$$

Now by (17), we know $\inf_\Omega \theta(\,\cdot\,,k) \le 0$ for each k. Therefore, defining $\epsilon := (C-1)/C < 1$, we find

$$\sup_{\Omega} \theta(\,\cdot\,,k+1) \le \epsilon \sup_{\Omega} \theta(\,\cdot\,,k-1).$$

Iterating this inequality gives the exponential decay of the supremum

$$\sup_{\Omega} \theta(\,\cdot\,,t) \le \sup_{\Omega} \theta(\,\cdot\,,0)e^{-\sigma t}, \quad \text{where } e^{-\sigma} = \epsilon. \tag{35}$$

On the other hand, (34) implies

$$\inf_{\Omega} \theta(\,\cdot\,,k) \ge -(C-1)\sup_{\Omega} \theta(\,\cdot\,,k-1) + C\sup_{\Omega} \theta(\,\cdot\,,k+1) \ge -(C-1)\sup_{\Omega} \theta(\,\cdot\,,k-1),$$

where we have used (17) again to throw away the term $\sup_\Omega \theta(\,\cdot\,,k+1)$. Therefore, by (35), we obtain

$$\inf_{\Omega} \theta(\,\cdot\,,k) \ge -(C-1)\sup_{\Omega} \theta(\,\cdot\,,0)e^{-\sigma(k-1)}. \tag{36}$$

This implies the exponential convergence of $\inf_\Omega \theta(\cdot\,, t)$, which combined with (35) gives the desired exponential convergence of $\theta(\cdot\,, t)$ to zero. □

5. A geometric approach to sublinearity at the boundary

In this section, we present an alternative approach to the computation of $D_\beta F(x_0, t_0)$ arising in the boundary sublinearity above. We will accomplish this using geometric language, exploiting the pseudo-Riemannian framework for optimal transport developed in [Kim and McCann 2010]. All material in this section is new, to the best knowledge of the authors, and constitutes the first treatment of the boundary geometry of domains in the context of the Kim–McCann metric.

In order to stay in line with established conventions, in this section we will mostly follow the notation used in [Kim and McCann 2010]. Thus in this section only, we will refer to the source and target domains as Ω and $\bar{\Omega}$ respectively (in particular, $\bar{\Omega}$ does not denote the closure of a set), which we assume are subsets of some fixed Riemannian manifolds. Points with a bar above will belong to $\bar{\Omega}$, while those without will belong to Ω. We also adopt the Einstein summation convention with the caveat that any indices given by Greek letters will run from 1 to $2n$, while lower case Roman indices run between 1 and n with the convention that an index with a bar above will be that value with n added to it: in other words, $1 \le \gamma \le 2n,\ 1 \le i \le n$ and $\bar{i} := i + n$.

Additionally, we will switch sign conventions at this point to stay in line with the definitions of [Kim and McCann 2010]. This means that c will be replaced by $-c$ everywhere, and the optimal transport problem (1) that is considered will be a minimization instead of a maximization problem.

We also split the tangent and cotangent spaces of $\Omega \times \bar{\Omega}$ in the canonical way according to the product structure, which gives the splitting $dc = Dc \oplus \bar{D}c$ of the one form dc on $\Omega \times \bar{\Omega}$, and given any local coordinate system on $\Omega \times \bar{\Omega}$ we will use the notation X to denote the full $2n$-dimensional coordinate variable: thus given a point $X = (x, \bar{x}) \in \Omega \times \bar{\Omega}$, X^i will indicate the i-th coordinate of x with $1 \le i \le n$, and $X^{\bar{i}}$ will indicate the i-th coordinate of $\bar{x}$. We will also suppress the time variable in this section, as everything considered will be for a fixed time t (in fact, the time dependency of the potential u will be completely irrelevant in the results of this section). Finally, we use the notation

$$[\Omega]_{\bar{x}} := -\bar{D}c(\Omega, \bar{x}) \subset T^*_{\bar{x}}\bar{\Omega}, \quad [\bar{\Omega}]_x := -Dc(x, \bar{\Omega}) \subset T^*_x\Omega \quad \text{for any } (x, \bar{x}) \in \Omega \times \bar{\Omega}.$$

Equip Ω with the pullback metric $w := (\mathrm{Id} \times T)^*h$, where

$$h := \frac{1}{2}\begin{pmatrix} 0 & -\bar{D}Dc \\ -D\bar{D}c & 0 \end{pmatrix}$$

is the Kim–McCann (pseudo-Riemannian) metric on $\Omega \times \bar{\Omega}$ defined as in [Kim and McCann 2010, (2.1)]. By [Kim et al. 2010, Section 3.2], in Euclidean coordinates the coefficients of w at x are exactly $w_{ij}(x) = u_{ij}(x) + c_{ij}(x, T(x))$, and w is a Riemannian metric. We will write ∇^w and ∇^h for the Levi-Civita connections of w and h respectively, Γ for the Christoffel symbols of h, and $|\cdot|_w$ for the length of a vector in w. We will also metrically identify various cotangent spaces naturally with $\mathbb{R}^n$ through the underlying Riemannian metrics on Ω or $\bar{\Omega}$. The inner products and norms in these underlying metrics will be denoted by $\langle\,\cdot\,,\,\cdot\,\rangle$ and $|\cdot|$ respectively. Our main result of the section is the following.

Theorem 5.1. *Let* II^w *be the second fundamental form of* $\partial\Omega$ *defined with respect to the metric* w, *and fix a point* $x_0 \in \partial\Omega$. *If* $\mathrm{II}^{\partial[\Omega]_{T(x_0)}}$, $\mathrm{II}^{\partial[\bar{\Omega}]_{x_0}}$ *are the* (*Euclidean*) *second fundamental forms of* $\partial[\Omega]_{T(x_0)}$ *and* $\partial[\bar{\Omega}]_{x_0}$ *respectively, then for any* $\tau_1, \tau_2 \in T_{x_0}\partial\Omega$ *we have*

$$2|\beta(x_0)|_w \mathrm{II}^w_{x_0}(\tau_1, \tau_2) = |DT(x_0)\beta(x_0)| \mathrm{II}^{\partial[\Omega]_{T(x_0)}}_{-\bar{D}c(x_0,T(x_0))}(\hat{\tau}_1, \hat{\tau}_2) + |\beta(x_0)| \mathrm{II}^{\partial[\bar{\Omega}]_{x_0}}_{-Dc(x_0,T(x_0))}(\hat{\bar{\tau}}_1, \hat{\bar{\tau}}_2), \tag{37}$$

where

$$\hat{\tau}_i := -D\bar{D}c(x_0, T(x_0))\tau_i \in T^*_{T(x_0)}\bar{\Omega},$$
$$\hat{\bar{\tau}}_i := -\bar{D}Dc(x_0, T(x_0))DT(x_0)\tau_i \in T^*_{x_0}\Omega.$$

Proof. Fix any point $x_0 \in \partial\Omega$. Note by Lemma 2.4 that $\beta(x_0)$ is an (outward) normal to $\partial\Omega$ at x_0 with respect to the metric w. Then since $\mathrm{Id} \times T$ is an embedding of Ω into $\Omega \times \bar{\Omega}$, if ∇^h is the Levi-Civita connection of h, we have (using that τ_2 is tangent to $\partial\Omega$ in the second line)

$$\begin{aligned}
\mathrm{II}^w_{x_0}(\tau_1, \tau_2) &= w\left(\nabla^w_{\tau_1}\frac{\beta}{|\beta|_w}, \tau_2\right) = |\beta|_w^{-1} w(\nabla^w_{\tau_1}\beta, \tau_2) + D_{\tau_1}(|\beta|_w^{-1}) w(\beta, \tau_2)\\
&= |\beta|_w^{-1} w(\nabla^w_{\tau_1}\beta, \tau_2) = -|\beta|_w^{-1} w(\beta, \nabla^w_{\tau_1}\tau_2)\\
&= -|\beta|_w^{-1} h\big((\beta \oplus DT(x_0)\beta), \nabla^h_{(\tau_1 \oplus DT(x_0)\tau_1)}(\tau_2 \oplus DT(x_0)\tau_2)\big)\\
&= -|\beta|_w^{-1} (\beta \oplus DT(x_0)\beta)^\flat [\nabla^h_{(\tau_1 \oplus DT(x_0)\tau_1)}(\tau_2 \oplus DT(x_0)\tau_2)],
\end{aligned} \tag{38}$$

where $\flat$ is the operation of lowering the indices of a tangent vector to $\Omega \times \bar{\Omega}$ by the metric h. Next consider the mapping $\Phi(x, \bar{x}) := -Dc(x_0, \bar{x}) \oplus (-\bar{D}c(x, T(x_0)))$. By the bitwist condition (6), Φ is a diffeomorphism on $\Omega \times \bar{\Omega}$; hence Φ^{-1} gives a global coordinate chart on the set. We will use hats to denote quantities related to h written in the coordinates given by Φ^{-1}, while quantities without hats will be in Euclidean coordinates. A quick calculation yields that

$$\frac{\partial \Phi^\delta}{\partial X^\gamma}(x_0, T(x_0)) = 2h_{\delta\gamma}(x_0, T(x_0)). \tag{39}$$

We will now calculate the Christoffel symbols $\widehat{\Gamma}^\delta_{\gamma\lambda}$ in the coordinates given by Φ^{-1}. By [Kim and McCann 2010, Lemma 4.1] the Christoffel symbols of h in Euclidean coordinates are identically zero unless all three of the indices are simultaneously between 1 and n, or between $n+1$ and $2n$. Thus the standard transformation law shows that in the coordinates given by Φ^{-1}, the only Christoffel symbols that can be nonzero are those where either the upper index is not barred and both lower indices are, or the upper index is barred and both lower indices are not. Since Ω is c-convex with respect to $\bar{\Omega}$, there is an n-dimensional cone $K(x_0)$ of directions that point inward to $[\Omega]_{T(x_0)}$ from the boundary point $-\bar{D}c(x_0, T(x_0))$. By [Kim and McCann 2010, Lemma 4.4], for any such direction v in this cone $K(x_0)$, any segment of the form $s \mapsto \Phi^{-1}(sv \oplus -Dc(x_0, T(x_0)))$ is a geodesic for h for small $s > 0$. Thus plugging such a segment into the geodesic equations in Φ^{-1} coordinates yields for any fixed $\bar{i}$, at $(x_0, T(x_0))$,

$$0 = \widehat{\Gamma}^{\bar{i}}_{jk} v^j v^k.$$

Suppose $\{v_l\}_{l=1}^n$ is a linearly independent collection of vectors in $K(x_0)$; then for any $1 \le l_1 \ne l_2 \le n$ we have

$$0 = \widehat{\Gamma}^{\bar{i}}_{jk}(v^j_{l_1} + v^j_{l_2})(v^k_{l_1} + v^k_{l_2}) = \widehat{\Gamma}^{\bar{i}}_{jk} v^j_{l_1} v^k_{l_1} + \widehat{\Gamma}^{\bar{i}}_{jk} v^j_{l_2} v^k_{l_2} + \widehat{\Gamma}^{\bar{i}}_{jk} v^j_{l_1} v^k_{l_2} + \widehat{\Gamma}^{\bar{i}}_{jk} v^j_{l_2} v^k_{l_1} = 2\widehat{\Gamma}^{\bar{i}}_{jk} v^j_{l_1} v^k_{l_2},$$

which implies all Christoffel symbols of the form $\widehat{\Gamma}^{\bar{i}}_{jk}$ are also zero. A similar argument reversing the roles of Ω and $\overline{\Omega}$ yields that *all* Christoffel symbols of h are zero in the Φ^{-1} coordinates at the point $(x_0, T(x_0))$.

Now using (39), we see that the coordinates of the 1-form $(\beta \oplus DT(x_0)\beta)^\flat$ in Φ^{-1} are equal to the Euclidean coordinates of the tangent vector $\frac{1}{2}(\beta \oplus DT(x_0)\beta)$. Also we can calculate, for $i = 1$ or 2,

$$(\widehat{\tau_i \oplus DT(x_0)\tau_i})^j = \frac{\partial \Phi^j}{\partial X^{\bar{k}}}(x_0, T(x_0))(\tau_i \oplus DT(x_0)\tau_i)^{\bar{k}} = -c_{j\bar{k}}(DT(x_0)\tau_i)^k = \hat{\bar{\tau}}^j_i,$$
$$(\widehat{\tau_i \oplus DT(x_0)\tau_i})^{\bar{j}} = \frac{\partial \Phi^{\bar{j}}}{\partial X^k}(x_0, T(x_0))(\tau_i \oplus DT(x_0)\tau_i)^k = -c_{k\bar{j}}\tau^k_i = \hat{\tau}^j_i,$$

where we have identified $T^*_{x_0}\Omega$ and $T^*_{T(x_0)}\overline{\Omega}$ with $\mathbb{R}^n$ to write the vectors $\hat{\tau}_i$ and $\hat{\bar{\tau}}_i$ defined in the statement of the theorem in Euclidean coordinates. Combining this fact with (39), we can write (38) in the coordinates given by Φ^{-1} as

$$-\frac{1}{2}|\beta|^{-1}_w\left(\hat{\bar{\tau}}^j_1 \sum_{i=1}^n \beta^i(\partial_{\hat{x}^j}\hat{\bar{\tau}}^i_2) + \hat{\tau}^l_1 \sum_{k=1}^n (DT(x_0)\beta)^k(\partial_{\hat{x}^l}\hat{\tau}^k_2)\right). \tag{40}$$

Now we can see that the function $h^*(Y(x_0, \cdot))$ is a defining function for the set $[\overline{\Omega}]_{x_0}$; hence identifying $T^*_{x_0}\Omega$ with $\mathbb{R}^n$ and differentiating yields that $\nabla_p h^*(Y(x_0, p))$ is in the outward normal direction for $p \in \partial[\overline{\Omega}]_{x_0}$. In particular, the unit outward normal vector to $\partial[\overline{\Omega}]_{x_0}$ at $-Dc(x_0, T(x_0))$ has coordinates given by $\beta^i/|\beta|$. A similar calculation involving $h(X(T(x_0), \cdot))$ yields that the coordinates of the unit outward normal vector to $\partial[\Omega]_{T(x_0)}$ at $-\bar{D}c(x_0, T(x_0))$ are given by $(DT(x_0)\beta)^k/|DT(x_0)\beta|$. Additionally, since each τ_i is tangent to $\partial\Omega$, we see that $\hat{\tau}_i$ and $\hat{\bar{\tau}}_i$ are respectively tangent to $\partial[\overline{\Omega}]_{x_0}$ and $\partial[\Omega]_{T(x_0)}$. Thus we calculate

$$\begin{aligned}
\mathrm{II}^{\partial[\overline{\Omega}]_{x_0}}_{-Dc(x_0,T(x_0))}(\hat{\bar{\tau}}_1, \hat{\bar{\tau}}_2) &= \left\langle \nabla_{\hat{\bar{\tau}}_1}\frac{\beta}{|\beta|}, \hat{\bar{\tau}}_2 \right\rangle = |\beta|^{-1}\langle \nabla_{\hat{\bar{\tau}}_1}\beta, \hat{\bar{\tau}}_2\rangle + D_{\hat{\bar{\tau}}_1}\left(\frac{1}{|\beta|}\right)\langle \beta, \hat{\bar{\tau}}_2\rangle \\
&= |\beta|^{-1}\langle \nabla_{\hat{\bar{\tau}}_1}\beta, \hat{\bar{\tau}}_2\rangle = |\beta|^{-1}(D_{\hat{\bar{\tau}}_1}\langle \beta, \hat{\bar{\tau}}_2\rangle - \langle \beta, \nabla_{\hat{\bar{\tau}}_1}\hat{\bar{\tau}}_2\rangle) = -|\beta|^{-1}\langle \beta, \nabla_{\hat{\bar{\tau}}_1}\hat{\bar{\tau}}_2\rangle \\
&= -|\beta|^{-1}\hat{\bar{\tau}}^j_1 \sum_{i=1}^n \beta^i(\partial_{\hat{x}^j}\hat{\bar{\tau}}^i_2)
\end{aligned}$$

and likewise

$$\begin{aligned}
\mathrm{II}^{\partial[\Omega]_{T(x_0)}}_{-\bar{D}c(x_0,T(x_0))}(\hat{\tau}_1, \hat{\tau}_2) &= \left\langle \nabla_{\hat{\tau}_1}\frac{DT(x_0)\beta}{|DT(x_0)\beta|}, \hat{\tau}_2 \right\rangle = -|DT(x_0)\beta|^{-1}\langle DT(x_0)\beta, \nabla_{\hat{\tau}_1}\hat{\tau}_2\rangle \\
&= -|DT(x_0)\beta|^{-1}\hat{\tau}^l_1 \sum_{k=1}^n (DT(x_0)\beta)^k(\partial_{\hat{x}^l}\hat{\tau}^k_2).
\end{aligned}$$

Comparing this with (40) completes the proof of the theorem. □

The relevance of the above theorem to our current exponential convergence result is as follows. In terms of the metric w, we see that the β-directional derivative of the first term in the function F defined by (19) is given by (at x_0)

$$\begin{aligned}D_\beta(w(\nabla^w f, \nabla^w f)) &= 2w(\nabla^w_\beta \nabla^w f, \nabla^w f) = \operatorname{Hess} f(\beta, \nabla^w f)\\ &= \operatorname{Hess} f(\nabla^w f, \beta) = 2w(\nabla^w_{\nabla^w f} \nabla^w f, \beta) = -2w(\nabla^w_{\nabla^w f}\beta, \nabla^w f)\\ &= -2|\beta|_w \mathrm{II}^w(\nabla^w f, \nabla^w f).\end{aligned}$$

Here we repeatedly used that $\nabla^w f$ is tangent to $\partial\Omega$ (due to the boundary condition $D_\beta v = 0$ and since $f = \log v$), while β is normal in the metric w, and we have used (38) in the last line. Under the c- and c^*-convexity conditions (8) and (9), the two terms on the right-hand side of (37) are nonnegative; hence, by Theorem 5.1, $D_\beta w(\nabla^w f, \nabla^w f)$ is nonpositive. Thus in order to obtain a contradiction with the Hopf lemma as in Section 4, all that remains is to evaluate the last term $-\alpha D_\beta(\partial_t f)$. Obtaining a sign on this term depends on the specific choice of the function v, as in Section 4.

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