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We prove that $\mathcal{Z}$-stable, simple, separable, nuclear, nonunital C*-algebras have nuclear dimension at most 1. This completes the equivalence between finite nuclear dimension and $\mathcal{Z}$-stability for simple, separable, nuclear, nonelementary C*-algebras.

Introduction

The Elliott classification programme, a 40-year endeavour involving generations of researchers, asks the following question: when are K-theory and traces a complete invariant for simple, separable, nuclear C*-algebras?

Fundamentally, there are two cases to consider: the *unital* case and the *stably projectionless* case. (This dichotomy follows from Brown's theorem [1977] and is discussed further below.) Recall that a C*-algebra A is said to be stably projectionless if there are no nonzero projections in the matrix amplification $M_n(A)$ for any $n \in \mathbb{N}$. Stably projectionless, simple, separable, nuclear C*-algebras arise naturally as crossed products [Kishimoto and Kumjian 1996] and can also be constructed using inductive limits with a wide variety of K-theoretic and tracial invariants occurring [Blackadar 1980; Razak 2002; Tsang 2005; Jacelon 2013; Gong and Lin 2016; 2017; Elliott et al. 2017; 2020].

In the unital case, a definitive answer for when K-theory and traces form a complete invariant is now known [Kirchberg 1995; Phillips 2000; Gong et al. 2015; Elliott et al. 2015; Tikuisis et al. 2017; Winter 2014]. Firstly, Rosenberg and Schochet's universal coefficient theorem [1987] must hold for the C*-algebras concerned. Secondly, the C*-algebras must have *finite nuclear dimension* [Winter and Zacharias 2010]. This second condition has a geometric flavour and generalises the notation of finite covering dimension for topological spaces. Recent results [Gong and Lin 2016; 2017; Elliott et al. 2017; 2020] are now converging on a similar classification result in the stably projectionless case; the most recent developments will be discussed below.

A major programme of research now focuses on providing methods for verifying finite nuclear dimension in practice. In the unital setting, a recent result of the authors together with Tikuisis, White and Winter [Castillejos et al. 2019] shows that finite nuclear dimension can be accessed through the tensorial absorption condition known as $\mathcal{Z}$-*stability*, where $\mathcal{Z}$ is the *Jiang–Su algebra* (discussed in more detail below).

In concrete examples, it can be very hard to prove directly that a C*-algebra has finite nuclear dimension. The strategy of verifying $\mathcal{Z}$-stability instead has recently been used to prove that certain unital, simple, separable, nuclear C*-algebras coming from dynamical systems are classifiable [Conley et al. 2018; Kerr and Szabó 2020]. However, since this strategy relies on [Castillejos et al. 2019], it has until now only been available in the unital setting.

In this paper, we consider and overcome the conceptual and technical challenges unique to the nonunital setting, allowing us prove the following:

Theorem A. *Let A be a simple, separable, nuclear, $\mathcal{Z}$-stable C*-algebra. Then A has nuclear dimension at most* 1.

For the following reasons, the nonunital case is harder than the unital case and needs new methods. Obviously, we cannot just unitise our C*-algebras because this breaks both simplicity and $\mathcal{Z}$-stability. A more fundamental issue is that nonunital, simple C*-algebras need not actually be *algebraically simple*. There can be nontrivial (nonclosed) ideals. Examples of such ideals are the domains of *unbounded traces*, which may now exist and must therefore be taken into account. Furthermore, [Castillejos et al. 2019] builds on the foundations of [Bosa et al. 2019a], which has a global assumption of unitality and makes explicit use of the unit at a number of critical points in the argument (an example is the 2×2 matrix trick of [Bosa et al. 2019a, Section 2], which is inspired by ideas of Connes).

To understand how we circumvent the issues associated to unbounded traces, it will be helpful to first discuss the folklore result, alluded to above, that Brown's theorem [1977] implies a dichotomy for simple C*-algebras between the unital and the stably projectionless cases.

Writing $\mathbb{K}$ for the C*-algebra of compact operators (on a separable, infinite-dimensional Hilbert space), recall that C*-algebras A, B are *stably isomorphic* if $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$. Suppose now that A is a simple, separable C*-algebra that is not stably projectionless. Then there exists a nonzero projection $p \in A \otimes \mathbb{K}$, and so the hereditary subalgebra $p(A \otimes \mathbb{K})p$ is unital. By [Brown 1977, Theorem 2.8], $p(A \otimes \mathbb{K})p$ is stably isomorphic to $A \otimes \mathbb{K}$, and hence stably isomorphic to A (see Section 2 for more details).

Crucial to proving Theorem A in general is the observation that the hypotheses and the conclusion depend only on the stable isomorphism class of A.[1] Hence, by [Castillejos et al. 2019, Theorem B] and the folklore result above based on Brown's theorem, it suffices to prove Theorem A in the stably projectionless case.

However, this folklore reduction is not enough for us. We go a step further and pass to a hereditary subalgebra $A_0 \subseteq A \otimes \mathbb{K}$ on which all tracial functionals are bounded and the set of tracial states $T(A_0)$ is weak* compact. The existence of such hereditary subalgebra follows from the Cuntz semigroup computation of [Elliott et al. 2011] for $\mathcal{Z}$-stable C*-algebras, and Brown's theorem assures us that A_0 is stably isomorphic to A. This second reduction puts us in a position where a similar proof strategy to that of [Bosa et al. 2019a] can be implemented, and where the key new ingredient from [Castillejos et al. 2019], *complemented partitions of unity* (CPoU), is also available.

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[1]That is, A is a simple, separable, nuclear and $\mathcal{Z}$-stable C*-algebra if and only if $A \otimes \mathbb{K}$ is likewise, and $\dim_{\mathrm{nuc}} A \leq 1$ if and only if $\dim_{\mathrm{nuc}}(A \otimes \mathbb{K}) \leq 1$; see Proposition 2.3 for details and references.

Of course, we still have to deal with the global assumption of unitality in [Bosa et al. 2019a]. A key tool in this endeavour is our unitisation lemma for order-zero maps into ultrapowers (Lemma 4.2), which allows us to assume the domains of certain maps are unital in places where simplicity and $\mathcal{Z}$-stability are only really needed on the codomain side in [Bosa et al. 2019a].

We now turn to the broader context of Theorem A and its applications. As alluded to above, *nuclear dimension* for C*-algebras is a noncommutative dimension theory that reduces to the covering dimension of the spectrum in the commutative case. *Finite nuclear dimension* has proven to be a technically useful strengthening of nuclearity, that is both necessary for classification [Villadsen 1999; Rørdam 2003; Toms 2008; Giol and Kerr 2010] and a vital ingredient of the most recent classification theorems [Kirchberg 1995; Phillips 2000; Gong et al. 2015; Elliott et al. 2015; Tikuisis et al. 2017; Winter 2014].

The *Jiang–Su algebra* $\mathcal{Z}$ [1999] is a simple C*-algebra which plays a fundamental role in the classification of simple C*-algebras since A and $A \otimes \mathcal{Z}$ have the same K-theory and traces under mild hypotheses. A C*-algebra is said to be $\mathcal{Z}$-*stable* if $A \cong A \otimes \mathcal{Z}$. Moreover, any C*-algebra can be $\mathcal{Z}$-stabilised by tensoring with the Jiang–Su algebra because $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$. In many ways, the Jiang–Su algebra is the C*-algebraic analogue of the hyperfinite II_1 factor $\mathcal{R}$ [Murray and von Neumann 1943], with $\mathcal{Z}$-stability analogous to the McDuff property [1970].

Combining Theorem A with the main results of [Winter 2012; Tikuisis 2014], we arrive at the following relationship between finite nuclear dimension and $\mathcal{Z}$-stability, which was conjectured in [Toms and Winter 2009].

Theorem B. *Let A be a nonelementary, simple, separable, nuclear C*-algebra. The following are equivalent*:

(i) *A has finite nuclear dimension.*

(ii) *A is $\mathcal{Z}$-stable.*

One striking consequence of Theorems A and B is that nuclear dimension can only attain three different values in the simple setting.

Corollary C. *The nuclear dimension of a simple C*-algebra is 0, 1 or ∞.*

This is in stark contrast to the commutative case, where all nonnegative integers can occur. Moreover, we remark that the C*-algebras of nuclear dimension zero are known to be precisely the approximately finite-dimensional C*-algebras [Winter and Zacharias 2010, Remark 2.2.(iii)].[2]

Whilst Corollary C is interesting in its own right, we believe the main applications of our results will be in classification of simple, stably projectionless C*-algebras. Theorem A opens up a new pathway to proving that concrete examples of stably projectionless, simple, separable, nuclear C*-algebras, such as C*-algebras coming from flows on C*-algebras or from actions of more general locally compact groups, have finite nuclear dimension: it now suffices to verify $\mathcal{Z}$-stability.

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[2]In the nonseparable case, there are different and inequivalent definitions of *approximately finite-dimensional* [Farah and Katsura 2010]. The one required here is that any finite set is approximately contained in a finite-dimensional subalgebra; see for example [Castillejos 2017, Definition 2.2].

We end this introduction with a discussion of recent developments in the classification of simple, stably projectionless C*-algebras. As mentioned above, there has been impressive progress in recent years [Gong and Lin 2016; 2017; Elliott et al. 2020]. As in the unital case, the classification is via a functor constructed from the K-theory and the tracial data of the C*-algebra; this functor is called the *Elliott invariant* and is typically denoted Ell($\cdot$); see [Gong and Lin 2017, Definition 2.9] for a precise definition.

By combining Theorem A with [Gong and Lin 2017, Theorem 1.2], one obtains a classification of simple, separable, nuclear C*-algebras in the UCT class that tensorially absorb the C*-algebra $\mathcal{Z}_0$ — a stably projectionless analogue of the Jiang–Su algebra introduced in [Gong and Lin 2017, Definition 8.1].

Corollary D. *Let A and B be simple, separable, nuclear C*-algebras which satisfy the UCT. Then*

$$A \otimes \mathcal{Z}_0 \cong B \otimes \mathcal{Z}_0 \quad \text{if and only if} \quad \text{Ell}(A \otimes \mathcal{Z}_0) \cong \text{Ell}(B \otimes \mathcal{Z}_0).$$

Corollary D reduces to the celebrated Kirchberg–Phillips classification [Kirchberg 1995; Phillips 2000] in the traceless case and is otherwise a result about stably projectionless C*-algebras. For these C*-algebras, the difference between $\mathcal{Z}_0$-stability and $\mathcal{Z}$-stability, roughly speaking, comes down to how complex the interaction between the K-theory and traces is allowed to be; see [Gong and Lin 2017] for more details.

Structure of paper. Section 1 reviews the necessary preliminary material as appropriate to the nonunital setting. Section 2 is concerned with the invariance of C*-algebraic properties under stable isomorphism and the reduction argument outlined above. The next three sections generalise the necessary technical machinery from [Bosa et al. 2019a; Castillejos et al. 2019]. Section 3 concerns the existence of an order-zero embedding $\Phi : A \to A_\omega$ with appropriate finite-dimensional approximations. Section 4 contains the aforementioned unitisation lemma for order-zero maps into ultrapowers. Section 5 is devoted to a uniqueness theorem for maps into ultrapowers, which we shall use to compare (unitisations of) Φ and the canonical embedding $A \to A_\omega$. Theorem A and its corollaries are proved in Section 6, with analogous results for decomposition rank (a forerunner of nuclear dimension) proved in Section 7. Since some preliminary lemmas from [Bosa et al. 2019a] are stated only in the unital case, we include an Appendix with their nonunital versions.

1. Preliminaries

In this section, we recall the most important definitions and results that will be used in the sequel, and we introduce the notation used in this paper.

We write $\mathbb{K}$ to denote the C*-algebra of compact operators (on a separable, infinite-dimensional Hilbert space). Given a C*-algebra A, we write A_+ for the positive elements of A and $A_{+,1}$ for the positive contractions; we write Ped(A) for the Pedersen ideal of A, which is the minimal dense ideal of A (see [Pedersen 1979, Section 5.6]), and we write $A^\sim$ for the unitisation of A. Our convention is that, if A is already unital, then we adjoin a new unit, so $A^\sim \cong A \oplus \mathbb{C}$ as C*-algebras. For $S \subseteq A$ self-adjoint, we set $S^\perp := \{a \in A : ab = ba = 0 \text{ for all } b \in S\}$. For $\epsilon > 0$ and $a, b \in A$, the notation $a \approx_\epsilon b$ means $\|a - b\| < \epsilon$. For $a, b \in A$ with b self-adjoint, we write $a \lhd b$ to mean that $ab = ba = a$.

We use the common abbreviation c.p.c. for completely positive and contractive maps between C*-algebras. A c.p.c. map $\phi : A \to B$ is *order-zero* if it preserves orthogonality in the sense that, for $a, b \in A_+$, $\phi(a)\phi(b) = 0$ whenever $ab = 0$.

Following [Winter and Zacharias 2010, Definition 2.1], a C*-algebra A has *nuclear dimension at most n* if there is a net $(F_i, \psi_i, \phi_i)_{i \in I}$, where F_i is a finite-dimensional C*-algebra, $\psi_i : A \to F_i$ is a c.p.c. map, and $\phi_i : F_i \to A$ is a c.p. map, such that $\phi_i \circ \psi_i(a) \to a$ for all $a \in A$ and, moreover, each F_i decomposes into $n+1$ ideals $F_i = F_i^{(0)} \oplus \cdots \oplus F_i^{(n)}$ for which the restrictions $\phi_i|_{F_i^{(k)}}$ are c.p.c. order-zero. The *nuclear dimension* of A, denoted by $\dim_{\mathrm{nuc}} A$, is defined to be the smallest such n (and to be ∞, if no such n exists). The *decomposition rank*, a forerunner of nuclear dimension, is obtained if one additionally requires ϕ_i to be a c.p.c. map [Kirchberg and Winter 2004, Definition 3.1]. We shall denote the decomposition rank of a C*-algebra A by $\mathrm{dr}(A)$.

By a *trace* on a C*-algebra we will typically mean a tracial state, i.e., a positive linear functional $\tau : A \to \mathbb{C}$ of operator norm 1 such that $\tau(ab) = \tau(ba)$ for all $a, b \in A$. We write $T(A)$ for the set of tracial states on A endowed with the weak*-topology. More general notions of traces are discussed in Section 1A below. By a *cone* we will mean a convex subset C of a locally convex space that satisfies $C + C \subseteq C$, $\lambda C \subset C$ for $\lambda > 0$, and $C \cap (-C) = \{0\}$. A *base* for a cone C is a closed, convex, and bounded subset X such that for any nonzero $c \in C$ there exist unique $\lambda > 0$ and $x \in X$ such that $c = \lambda x$. By [Alfsen 1971, Theorem II.2.6], a cone is locally compact if and only if it has a compact base. A map $f : C \to D$ between cones is *linear* if $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$ for $\lambda, \mu \geq 0$ and $x, y \in C$. If X is a compact base for the cone C, then any continuous affine map $X \to D$ extends uniquely to a continuous linear map $C \to D$.

1A. *Generalised traces.*

In this preliminary section, we briefly discuss the generalisations of traces that arise in the general theory of C*-algebras.

Definition 1.1 (cf. [Blanchard and Kirchberg 2004, Definition 2.22]). A *quasitrace*[3] on a C*-algebra A is a function $\tau : A_+ \to [0, \infty]$ with $\tau(0) = 0$ such that

(i) $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$,

(ii) $\tau(a + b) = \tau(a) + \tau(b)$ for all commuting elements $a, b \in A_+$,

(iii) τ extends to a function $\tau_2 : M_2(A)_+ \to [0, \infty]$ for which (i) and (ii) hold.

The quasitrace τ is *additive* if (ii) holds for all $a, b \in A_+$.[4] Setting $\mathrm{Dom}_{1/2}(\tau) := \{a \in A : \tau(a^*a) < \infty\}$, we say that τ is *densely defined* if $\mathrm{Dom}_{1/2}(\tau)$ is dense in A, and that τ is *bounded* if $\mathrm{Dom}_{1/2}(\tau) = A$.

We write $Q\widetilde{T}(A)$ for the cone of densely defined, lower-semicontinuous quasitraces; $\widetilde{T}(A)$ for the cone of densely defined, lower-semicontinuous, additive quasitraces; and $\widetilde{T}_b(A)$ for the cone of bounded, additive quasitraces. The topology on these cones is given by pointwise convergence on $\mathrm{Ped}(A)$.

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[3]Strictly speaking, a *2-quasitrace*; however, we shall not need this terminology.

[4]We use the terminology *additive quasitrace* because we are reserving the word *trace* for tracial states. For additive quasitraces, condition (iii) is automatic with τ_2 given by the usual formula.

Since the traces on a C*-algebra will play a crucial role in the arguments of this paper, the following existence theorem of Blackadar–Cuntz is fundamental.

Theorem 1.2 [Blackadar and Cuntz 1982, Theorem 1.2]. *Let A be a simple C*-algebra such that $A \otimes \mathbb{K}$ contains no infinite projections. Then $Q\widetilde{T}(A) \neq 0$.*

It is an open question whether $Q\widetilde{T}(A) = \widetilde{T}(A)$ in general. However, when A is exact, this is a famous result of Haagerup; for the unital case see [Haagerup 2014] and for deducing the general case from that work see [Blanchard and Kirchberg 2004, Remark 2.29(i)].

Every $\tau \in Q\widetilde{T}(A)$ has a unique extension to a densely defined, lower-semicontinuous quasitrace on $A \otimes \mathbb{K}$ which is additive whenever τ is additive [Blanchard and Kirchberg 2004, Remark 2.27(viii)]. Therefore, we have canonical isomorphisms $Q\widetilde{T}(A) \cong Q\widetilde{T}(A \otimes \mathbb{K})$ and $\widetilde{T}(A) \cong \widetilde{T}(A \otimes \mathbb{K})$, which we treat as identifications. Furthermore, every $\tau \in \widetilde{T}_b(A)$ has a unique extension to a positive linear functional on A, which we also denote τ, satisfying the trace condition $\tau(ab) = \tau(ba)$ for all $a, b \in A$.

Let $a, b \in A_+$. If there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in A such that $b = \sum_{n=1}^{\infty} x_n^* x_n$ and $\sum_{n=1}^{\infty} x_n x_n^* \leq a$, then b is said to be *Cuntz–Pedersen subequivalent* to a [1979]. Our notation for this subequivalence will be $b \preccurlyeq a$. The following proposition is proven by the same method as [Cuntz and Pedersen 1979, Proposition 4.7]. For the benefit of the reader, we give full details.

Proposition 1.3. *Let A be a simple, separable C*-algebra and $B \subseteq A$ a nonzero hereditary subalgebra. The restriction map $\rho : \widetilde{T}(A) \to \widetilde{T}(B)$ is a linear homeomorphism of cones.*

Proof. Since $\mathrm{Ped}(B) \subseteq \mathrm{Ped}(A)$, the restriction of a densely defined quasitrace on A is a densely defined quasitrace on B. Restriction also preserves additivity and lower-semicontinuity. Hence, ρ is well-defined. Continuity of ρ follows immediately from the fact that $\mathrm{Ped}(B) \subseteq \mathrm{Ped}(A)$, and it is clear that ρ is linear.

We now turn to proving that ρ is surjective. Let $\sigma \in \widetilde{T}(B)$. Define $\tau : A_+ \to [0, \infty]$ by $\tau(a) := \sup\{\sigma(b) : b \in B_+, b \preccurlyeq a\}$. The following properties of τ are easy to verify:

$$\tau(0) = 0, \tag{1-1}$$

$$\tau(a^*a) = \tau(aa^*), \qquad a \in A, \tag{1-2}$$

$$\tau(\lambda a) = \lambda\tau(a), \qquad \lambda \geq 0, \ a \in A_+, \tag{1-3}$$

$$\tau(a_1 + a_2) \geq \tau(a_1) + \tau(a_2), \quad a_1, a_2 \in A_+. \tag{1-4}$$

Let $a_1, a_2 \in A_+$. Suppose $b \in B_+$ and $b \preccurlyeq a_1 + a_2$. By [Pedersen 1969, Corollary 1.2], there exist $b_1, b_2 \in A_+$ with $b = b_1 + b_2$ such that $b_1 \preccurlyeq a_1$ and $b_2 \preccurlyeq a_2$. Since B is a hereditary subalgebra, $b_1, b_2 \in B_+$. Hence,

$$\sigma(b) = \sigma(b_1) + \sigma(b_2) \leq \tau(a_1) + \tau(a_2). \tag{1-5}$$

Taking the supremum, we get $\tau(a_1 + a_2) \leq \tau(a_1) + \tau(a_2)$. Therefore, we have $\tau(a_1 + a_2) = \tau(a_1) + \tau(a_2)$. This completes the proof that τ is an additive quasitrace.

Since B is a hereditary subalgebra of A, the restriction of the Cuntz–Pedersen subequivalence relation on A to B is the same as the Cuntz–Pedersen subequivalence relation on B. It follows that $\tau|_{B_+}$ is σ. As σ is densely defined on B and A is simple, τ is densely defined.

Let $\tilde{\tau}(a) := \sup_{\epsilon > 0} \tau((a - \epsilon)_+)$ be the *lower-semicontinuous regularisation* of τ; see [Blanchard and Kirchberg 2004, Remark 2.27(iv)] and [Elliott et al. 2011, Lemma 3.1]. Then $\tilde{\tau}$ is a densely defined, lower-semicontinuous, additive quasitrace on A, and we still have $\tilde{\tau}|_{B_+} = \sigma$ because σ is lower-semicontinuous. Therefore, ρ is surjective.

We now prove that ρ is injective. Let σ, τ, and $\tilde{\tau}$ be as above. Suppose $\psi \in \widetilde{T}(A)$ also satisfies $\psi|_B = \sigma$. Since $\psi(b) \le \psi(a)$ whenever $b \precsim a$, we must have $\tau \le \psi$. Since taking lower-semicontinuous regularisations is order-preserving, we have $\tilde{\tau} \le \psi$. By [Elliott et al. 2011, Proposition 3.2], there exists $\phi \in \widetilde{T}(A)$ such that $\psi = \tilde{\tau} + \phi$. However, $\psi|_{B_+} = \tilde{\tau}|_{B_+} = \sigma$ and so ϕ vanishes on B_+. Since A is simple, it follows that $\phi = 0$ and so $\psi = \tilde{\tau}$. Therefore, ρ is injective.

Finally, we prove ρ that is a homeomorphism. Fix $b \in \mathrm{Ped}(B) \setminus \{0\}$. Note that b is also in $\mathrm{Ped}(A)$ and is full in both A and B by simplicity. Set $X_A := \{\tau \in \widetilde{T}(A) : \tau(b) = 1\}$ and $X_B := \{\tau \in \widetilde{T}(B) : \tau(b) = 1\}$. By [Tikuisis and Toms 2015, Proposition 3.4], X_A is a compact base for the cone $\widetilde{T}(A)$ and X_B is a compact base for the cone $\widetilde{T}(B)$. Since $b \in B$, we have that $\rho(X_A) = X_B$. Hence, ρ defines a continuous, affine bijection from X_A to X_B. Since X_A and X_B are compact Hausdorff spaces, ρ in fact defines an affine homeomorphism between compact bases for the cones $\widetilde{T}(A)$ and $\widetilde{T}(B)$. Therefore, ρ is a linear homeomorphism of the cones $\widetilde{T}(A)$ and $\widetilde{T}(B)$. □

1B. *Strict comparison.*

We first recall the definition of *Cuntz subequivalence*. Let A be a C*-algebra and $a, b \in A_+$. Then $a \precsim b$ if and only if there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in A such that

$$\lim_{n \to \infty} \|x_n^* b x_n - a\| = 0. \tag{1-6}$$

If $a \precsim b$ and $b \precsim a$, then a is said to be *Cuntz equivalent* to b. We shall write $[a]$ for the Cuntz equivalence class of a.

The *Cuntz semigroup* $\mathrm{Cu}(A)$ is the ordered abelian semigroup obtained by considering the Cuntz equivalence classes of positive elements in $A \otimes \mathbb{K}$ under orthogonal addition and the order induced by Cuntz subequivalence; see [Coward et al. 2008]. If one only considers the Cuntz equivalence classes of positive elements in $\bigcup_{k=1}^{\infty} M_k(A)$, then one obtains the *classical Cuntz semigroup* $W(A)$; see [Ara et al. 2011].

Informally, a C*-algebra A has *strict comparison* if traces determine the Cuntz comparison theory. In order to formalise this notion, we need to recall the *rank function* associated to a lower-semicontinuous quasitrace. Suppose $\tau : A_+ \to [0, \infty]$ is a lower-semicontinuous quasitrace. Then the rank function $d_\tau : (A \otimes \mathbb{K})_+ \to [0, \infty]$ is given by

$$d_\tau(a) = \lim_{n \to \infty} \tau(a^{1/n}), \tag{1-7}$$

where we have made use of the unique extension of τ to $A \otimes \mathbb{K}$. We have $d_\tau(a) \le d_\tau(b)$ whenever $a, b \in (A \otimes \mathbb{K})_+$ satisfy $a \precsim b$ by [Blackadar and Handelman 1982, Theorem II.2.2]. Strict comparison can be viewed as a partial converse.

Since we will be adapting the methods of [Bosa et al. 2019a], we shall be working with the same definition of strict comparison that is used there.

Definition 1.4 [Bosa et al. 2019a, Definition 1.5]. A C*-algebra A has *strict comparison* (*of positive elements, with respect to bounded traces*) if

$$[\text{for all } \tau \in T(A), \ \ d_\tau(a) < d_\tau(b)] \implies a \precsim b \tag{1-8}$$

for $k \in \mathbb{N}$ and $a, b \in M_k(A)_+$.

We alert the reader to two facts about this definition. Firstly, it only concerns positive elements in $\bigcup_{k=1}^\infty M_k(A)$, so it is a property of the classical Cuntz semigroup $W(A)$. Secondly, we only require the condition $d_\tau(a) < d_\tau(b)$ to be shown when τ is a tracial state.

In light of the potential confusion that could arise from the variety of definitions of strict comparison that appear in the literature, we include a proof that $\mathcal{Z}$-stability implies strict comparison in the sense of Definition 1.4 for the benefit of the reader. The key ingredient in the proof is that $W(A)$ is almost unperforated whenever A is $\mathcal{Z}$-stable, which is due to [Rørdam 2004].

Proposition 1.5. *Let A be a simple, separable, $\mathcal{Z}$-stable C*-algebra with $Q\widetilde{T}(A) = \widetilde{T}_b(A) \neq 0$. Then A has strict comparison of positive elements with respect to bounded traces.*

Proof. As A is $\mathcal{Z}$-stable, so is $A \otimes \mathbb{K}$. Hence, by [Rørdam 2004, Theorem 4.5], $\mathrm{Cu}(A) = W(A \otimes \mathbb{K})$ is almost unperforated. Applying [Elliott et al. 2011, Propositions 6.2 and 4.4], we find that A has strict comparison in the following sense: for all $a, b \in (A \otimes \mathbb{K})_+$ with $[a] \leq \infty[b]$ in $\mathrm{Cu}(A)$ if $d_\tau(a) < d_\tau(b)$ for all lower-semicontinuous quasitraces with $d_\tau(b) = 1$ then $[a] \leq [b]$ in $\mathrm{Cu}(A)$.

We show that under our hypothesis on A this implies that A has strict comparison in the sense of Definition 1.4. Consider $a, b \in M_k(A)_+$ and let $\epsilon > 0$ and $f_\epsilon : [0, 1] \to [0, 1]$ be the function that is 0 on $[0, \epsilon]$, affine on $[\epsilon, 2\epsilon]$ and 1 on $[2\epsilon, 1]$. Since $M_k(A)$ is simple, there exists $n \in \mathbb{N}$ such that $[f_\epsilon(a)] \leq n[b] \leq \infty[b]$ in $\mathrm{Cu}(A)$ by [Blackadar 2006, Corollary II.5.2.12]. As ϵ is arbitrary, we have $[a] \leq \infty[b]$.

Since $Q\widetilde{T}(A) = \widetilde{T}_b(A)$, if $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(A)$, then $a \precsim b$ in $A \otimes \mathbb{K}$. As $M_k(A)$ is a hereditary subalgebra of $A \otimes \mathbb{K}$, we have $a \precsim b$ in $M_k(A)$ by [Kirchberg and Rørdam 2000, Lemma 2.2(iii)]. $\qquad \square$

Remark 1.6. By replacing $M_k(A)$ with $A \otimes \mathbb{K}$ in the proof of Proposition 1.5, we see that (1-8) holds for all $a, b \in (A \otimes \mathbb{K})_+$. Therefore, A also has strict comparison by traces in the sense of [Ng and Robert 2016, Definition 3.1] under the hypotheses of Proposition 1.5.

1C. *Ultraproducts and Kirchberg's epsilon test.* Let ω be a free ultrafilter on $\mathbb{N}$, which we regard as fixed for the entirety of the paper. The *ultraproduct* $\prod_{n \to \omega} A_n$ of a sequence of C*-algebras $(A_n)_{n \in \mathbb{N}}$ is defined by

$$\prod_{n \to \omega} A_n := \frac{\prod_{n \in \mathbb{N}} A_n}{\left\{ (a_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} A_n : \lim_{n \to \omega} \|a_n\| = 0 \right\}}. \tag{1-9}$$

The *ultrapower* A_ω of a C*-algebra A is the ultraproduct of the constant sequence $(A_n)_{n \in \mathbb{N}}$ with $A_n = A$ for all $n \in \mathbb{N}$. We identify A with the subalgebra of A_ω given by constant sequences $(a)_{n \in \mathbb{N}}$.

Every sequence $(\tau_n)_{n\in\mathbb{N}}$, where $\tau_n \in T(A_n)$, defines a tracial state on the ultrapower $\prod_{n\to\omega} A_n$ via $(a_n) \mapsto \lim_{n\to\omega} \tau_n(a_n)$. Tracial states of this form are known as *limit traces*. The set of all limit traces will be denoted by $T_\omega\big(\prod_{n\to\omega} A_n\big)$.

Not all traces on an ultraproduct are limit traces but we have the following density result due to [Ng and Robert 2016, Theorem 1.2] (generalising an earlier result of [Ozawa 2013, Theorem 8]).

Theorem 1.7 [Ng and Robert 2016; Ozawa 2013]. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of simple, separable, $\mathcal{Z}$-stable C*-algebras with $Q\widetilde{T}(A_n) = \widetilde{T}_b(A_n)$ for all $n \in \mathbb{N}$. Then $T_\omega\big(\prod_{n\to\omega} A_n\big)$ is weak*-dense in $T\big(\prod_{n\to\omega} A_n\big)$.*

Proof. By Proposition 1.5 and Remark 1.6, each A_n has strict comparison by traces in the sense of [Ng and Robert 2016, Definition 3.1]. The result now follows by Theorem 1.2 of the same work. $\square$

We shall also need uniform tracial ultraproducts. Recall that any trace $\tau \in T(A)$ defines a 2-seminorm $\|a\|_{2,\tau} := \tau(a^*a)^{1/2}$. The uniform 2-seminorm is then defined by

$$\|a\|_{2,T(A)} := \sup_{\tau\in T(A)} \|a\|_{2,\tau} = \sup_{\tau\in T(A)} \tau(a^*a)^{1/2}. \tag{1-10}$$

We can then define the *uniform tracial ultraproduct* of a sequence of C*-algebras $(A_n)_{n\in\mathbb{N}}$ by

$$\prod^{n\to\omega} A_n := \frac{\prod_{n\in\mathbb{N}} A_n}{\big\{(a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n : \lim_{n\to\omega} \|a_n\|_{2,T(A_n)} = 0\big\}}. \tag{1-11}$$

The *uniform tracial ultrapower* A^ω of a C*-algebra A, which can be defined as the uniform tracial ultraproduct of the constant sequence $(A_n)_{n\in\mathbb{N}}$ with $A_n = A$ for all $n \in \mathbb{N}$, was introduced in [Castillejos et al. 2019]. We identify A with the subalgebra of A^ω given by constant sequences $(a)_{n\in\mathbb{N}}$.

Since $\|a\|_{2,T(A)} \le \|a\|$ for all $a \in A$, there exists a canonical surjection from the ultraproduct to the uniform tracial ultraproduct. The kernel of this *-homomorphism is the *trace kernel ideal* given by

$$J_{(A_n)} := \left\{(a_n)_{n\in\mathbb{N}} \in \prod_{n\to\omega} A_n : \lim_{n\to\omega} \|a_n\|_{2,T(A_n)} = 0\right\}$$

$$= \left\{x \in \prod_{n\to\omega} A_n : \|x\|_{2,\tau} = 0, \ \tau \in T_\omega\bigg(\prod_{n\to\omega} A_n\bigg)\right\}. \tag{1-12}$$

It follows that limit traces also induce traces on the uniform tracial ultraproduct. In the ultrapower case, we therefore use a unified notation $T_\omega(A)$ for the limit traces on A_ω or the induced traces on A^ω.

An important tool for working with ultrapowers are reindexing arguments, which allow one to find elements of the ultrapower exactly satisfying some given condition provided one can find elements of the ultrapower which approximately satisfy the condition for any given tolerance. A precise and very general formulation of such reindexing arguments is *Kirchberg's epsilon test*, which we state below.

Lemma 1.8 (Kirchberg's epsilon test [2006, Lemma A.1]). *Let $X_1, X_2, \ldots$ be a sequence of nonempty sets, and for each $k, n \in \mathbb{N}$, let $f_n^{(k)} : X_n \to [0, \infty)$ be a function. Define $f_\omega^{(k)} : \prod_{n=1}^\infty X_n \to [0, \infty]$ by $f_\omega^{(k)}((s_n)_{n=1}^\infty) := \lim_{n\to\omega} f_n^{(k)}(s_n)$ for $(s_n) \in \prod_{n=1}^\infty X_n$. Suppose that for all $m \in \mathbb{N}$ and $\epsilon > 0$, there exists $(s_n)_{n=1}^\infty \in \prod_{n=1}^\infty X_n$ with $f_\omega^{(k)}((s_n)) < \epsilon$ for $k = 1, \ldots, m$. Then there exists $(t_n)_{n=1}^\infty \in \prod_{n=1}^\infty X_n$ such that $f_\omega^{(k)}((t_n)) = 0$ for all $k \in \mathbb{N}$.*

1D. *Stable rank one.* A unital C*-algebra A is said to have *stable rank one* if the invertible elements form a dense subset. In this paper, we shall make use of the following nonunital generalisation.

Definition 1.9. Let A be a C*-algebra. We say that A has *stable rank one in $A^\sim$* if every element of A is a limit of invertible elements in $A^\sim$.

In the unital case, $A^\sim \cong A \oplus \mathbb{C}$, so A has stable rank one in $A^\sim$ if and only if A has stable rank one. In the nonunital case, A having stable rank one in $A^\sim$ is weaker than requiring that $A^\sim$ itself have stable rank one; see [Robert 2016, Example 3.4].

A related notion is Robert's *almost stable rank one* [2016, Definition 3.1], which requires that, for all hereditary subalgebras $B \subseteq A$, B has stable rank one in $B^\sim$. Robert proved the following.

Theorem 1.10 [Robert 2016, Corollary 3.2]. *Let A be a $\mathcal{Z}$-stable, projectionless C*-algebra. Then A has almost stable rank one. In particular, A has stable rank one in $A^\sim$.*

We now prove that having stable rank one in the unitisation passes to ultraproducts. We employ the notation $[(a_n)]$ for the element of the ultraproduct defined by the bounded sequence (a_n). First, let us record that taking unitisations commutes with taking the ultraproduct. The proof of this lemma is straightforward and we omit it.

Lemma 1.11. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of C*-algebras. The canonical inclusion $\prod_{n\to\omega} A_n \to \prod_{n\to\omega} A_n^\sim$ extends to an isomorphism*

$$\left(\prod_{n\to\omega} A_n\right)^\sim \cong \prod_{n\to\omega} A_n^\sim. \tag{1-13}$$

We now proceed to show that having stable rank one in the unitisation passes to ultraproducts.

Proposition 1.12. *Let (A_n) be a sequence of C*-algebras. Suppose for each $n \in \mathbb{N}$, A_n has stable rank one in $A_n^\sim$. Then $A_\omega := \prod_{n\to\omega} A_n$ has stable rank one in $A_\omega^\sim$.*

Proof. Let $x \in A_\omega$ and say $x = [(a_n)]$. By Theorem 1.10 and [Bosa et al. 2019a, Lemma 1.20], for each $n \in \mathbb{N}$ there is a unitary $u_n \in A_n^\sim$ such that $a_n \approx_{1/n} u_n|a_n|$. We then have $x = [(u_n)]|x| \in \prod_{n\to\omega} A_n^\sim$. By [Bosa et al. 2019a, Lemma 1.20] once more, x is a norm limit of invertible elements in $\prod_{n\to\omega} A_n^\sim$. By Lemma 1.11, $\prod_{n\to\omega} A_n^\sim$ is just $A_\omega^\sim$. □

1E. *Complemented partitions of unity.* The key technical tool in [Castillejos et al. 2019] was the complemented partitions of unity technique which enabled Theorem A to be proven in the unital case. This property is best formulated in terms of the tracial ultrapower A^ω of a separable C*-algebra with $T(A)$ nonempty and compact. These assumptions imply that A^ω is unital, with any sequential approximate identity representing the unit [loc. cit., Proposition 1.11]. We refer to [loc. cit., Definition G] of the same work for a detailed explanation of the ideas behind this definition.

Definition 1.13. Let A be a separable C*-algebra with $Q\widetilde{T}(A) = \widetilde{T}_b(A) \neq 0$ and $T(A)$ compact. We say that A has *complemented partitions of unity* (CPoU) if for every $\|\cdot\|_{2,T_\omega(A)}$-separable subset S of A^ω,

every family $a_1, \ldots, a_k \in (A^\omega)_+$, and any scalar

$$\delta > \sup_{\tau \in T_\omega(A)} \min_{i=1,\ldots,k} \tau(a_i), \tag{1-14}$$

there exist orthogonal projections $p_1, \ldots, p_k \in A^\omega \cap S'$ such that

$$p_1 + \cdots + p_k = 1_{A^\omega} \quad \text{and} \quad \tau(a_i p_i) \le \delta\tau(p_i), \quad \tau \in T_\omega(A), \ i = 1, \ldots, k. \tag{1-15}$$

The following theorem gives sufficient conditions for a C*-algebra to have complemented partitions of unity. Although not necessary for our purposes, the hypothesis of $\mathcal{Z}$-stability can be weakened to *uniform property* Γ; see [Castillejos et al. 2019, Section 2] for more details.

Theorem 1.14 [Castillejos et al. 2019, Theorem I]. *Let A be a separable, nuclear, $\mathcal{Z}$-stable C*-algebra with $Q\widetilde{T}(A) = \widetilde{T}_b(A) \ne 0$ and $T(A)$ compact. Then A has complemented partitions of unity.*

2. Reductions

In this section, we show how Brown's theorem [1977, Theorem 2.8] can be used to reduce the task of proving Theorem A in general to proving it for unital C*-algebras and for stably projectionless C*-algebras with a compact trace space. We begin with the general statement of Brown's theorem.

Theorem 2.1 [Brown 1977, Theorem 2.8]. *Let B be a full hereditary subalgebra of a C*-algebra A. Suppose both A and B are σ-unital. Then B is stably isomorphic to A.*

In our applications, we shall be working with C*-algebras that are simple and separable. Hence, the fullness and σ-unitality conditions will be satisfied. We shall therefore use the following form of Brown's theorem.

Corollary 2.2. *Let B be a nonzero hereditary subalgebra of a simple, separable C*-algebra A. Then B is stably isomorphic to A.*

The utility of Brown's theorem for this paper derives from the fact that the hypotheses and conclusion of Theorem A are invariant under stable isomorphism. We state this formally below.

Proposition 2.3 [Kirchberg and Winter 2004; Winter and Zacharias 2010; Toms and Winter 2007]. *Let A be a C*-algebra. Then*:

(i) *A is simple if and only if $A \otimes \mathbb{K}$ is simple.*

(ii) *A is separable if and only if $A \otimes \mathbb{K}$ is separable.*

(iii) *A is nuclear if and only if $A \otimes \mathbb{K}$ is nuclear.*

(iv) *$\mathrm{dr}(A) = \mathrm{dr}(A \otimes \mathbb{K})$.*

(v) *$\dim_{\mathrm{nuc}}(A) = \dim_{\mathrm{nuc}}(A \otimes \mathbb{K})$.*

(vi) *A is separable and $\mathcal{Z}$-stable if and only if $A \otimes \mathbb{K}$ is separable and $\mathcal{Z}$-stable.*

Proof. Properties (i)–(iii) are well known; see for example [Blackadar 2006, Chapter IV.3]. Part (iv) is [Kirchberg and Winter 2004, Corollary 3.9]. Part (v) is [Winter and Zacharias 2010, Corollary 2.8]. Part (vi) is [Toms and Winter 2007, Corollary 3.2]. □

Next, we recall that a C*-algebra A is *stably projectionless* if there are no nonzero projections in $A \otimes \mathbb{K}$. By definition, this property is preserved under stable isomorphism. Stably projectionless C*-algebras can be viewed as highly nonunital C*-algebras. Indeed, the following folklore result establishes a dichotomy for simple, separable C*-algebras.

Proposition 2.4. *Let A be a nonzero, simple, separable* C*-*algebra. Then exactly one of the following holds*:

(a) *A is stably isomorphic to a unital* C*-*algebra.*

(b) *A is stably projectionless.*

Proof. Let A be a simple, separable C*-algebra A. Then $A \otimes \mathbb{K}$ is simple and separable by Proposition 2.3. Suppose that A is not stably projectionless. Then there exists a nonzero projection $p \in A \otimes K$. Set $B := p(A \otimes \mathbb{K})p$. Then B is a unital C*-algebra with unit $1_B = p$. Moreover, B is a nonzero hereditary subalgebra of $A \otimes \mathbb{K}$. Therefore, B is stably isomorphic to $A \otimes \mathbb{K}$ by Corollary 2.2, and hence is stably isomorphic to A.

Now suppose that A is stably isomorphic to a unital C*-algebra B. Then there exists an isomorphism $\phi : B \otimes \mathbb{K} \to A \otimes \mathbb{K}$. Writing 1_B for the unit of B and e_{ij} for the matrix units of $\mathbb{K}$, we have that $\phi(1_B \otimes e_{ii})$ is a nonzero projection in $A \otimes \mathbb{K}$. Hence, A cannot be stably projectionless. □

This dichotomy justifies the terminology *stably unital* for the nonstably projectionless, simple, separable C*-algebras. The stably unital case of Theorem A follows immediately from [Castillejos et al. 2019, Theorem B] together with Propositions 2.4 and 2.3. The stably projectionless case on the other hand requires a further reduction and a technical modification of the methods of [Bosa et al. 2019a]. The purpose of the additional reduction is to pass to the case where the trace space is compact, and it is based on the following folklore result.

Lemma 2.5. *Let A be a simple, separable* C*-*algebra with $\widetilde{T}(A) \neq 0$. Let $A_0 := \overline{a(A \otimes \mathbb{K})a}$ be the hereditary subalgebra generated by a nonzero positive contraction $a \in (A \otimes \mathbb{K})_{+,1}$ for which the function $\tau \mapsto d_\tau(a)$ is continuous and finite-valued. Then $\widetilde{T}(A_0) = \widetilde{T}_b(A_0) \neq 0$ and $T(A_0)$ is compact.*

Proof. By Proposition 1.3, the restriction map $\rho : \widetilde{T}(A \otimes \mathbb{K}) \to \widetilde{T}(A_0)$ is a linear homeomorphism. Let $\sigma \in \widetilde{T}(A_0)$. Then σ has an extension $\tau := \rho^{-1}(\sigma) \in \widetilde{T}(A \otimes \mathbb{K})$. By [Tikuisis 2014, Proposition 2.4], we have $\|\sigma\|_{A_0^*} = d_\tau(a) < \infty$. Therefore, $\widetilde{T}(A_0) = \widetilde{T}_b(A_0) \neq 0$. Since $\sigma \mapsto d_{\rho^{-1}(\sigma)}(a)$ is continuous, $T(A_0)$ is a weak*-closed subspace of the unit ball of A_0^*. Therefore, $T(A_0)$ is compact. □

We now explain how the results of [Elliott et al. 2011] can be used to prove the existence of positive contractions with continuous-rank functions under suitable hypotheses.

Proposition 2.6. *Let A be a simple, separable, $\mathcal{Z}$-stable* C*-*algebra with $Q\widetilde{T}(A) = \widetilde{T}(A) \neq 0$. Let $f : \widetilde{T}(A) \to [0, \infty)$ be a strictly positive, continuous, linear function. Then there exists a nonzero positive contraction $a \in (A \otimes \mathbb{K})_{+,1}$ with $d_\tau(a) = f(\tau)$ for all $\tau \in \widetilde{T}(A)$.*

Proof. Following [Elliott et al. 2011, Section 4.1], we write $F(\mathrm{Cu}(A))$ for the space of functionals on the Cuntz semigroup of A. In [loc. cit., Theorem 4.4], it is shown that all functionals on the Cuntz semigroup are of the form d_τ for some lower-semicontinuous quasitrace τ on A. However, we alert the reader to the fact that quasitraces are not assumed to be densely defined in [loc. cit.]. Since A is simple, this means that either $\tau \in Q\widetilde{T}(A)$ or τ is the *trivial quasitrace*, which satisfies $\tau(0) = 0$ and is infinite otherwise.

We now consider the topology on $F(\mathrm{Cu}(A))$, defined in general in [loc. cit., Section 4.1], and its relation to the topology on $\widetilde{T}(A)$, which is given by pointwise convergence on $\mathrm{Ped}(A)$. By [loc. cit., Theorem 4.4] and our assumption that all quasitraces are additive, the topology on $F(\mathrm{Cu}(A))$ agrees with the topology on set $\widehat{T}(A)$ of (not necessarily densely defined) lower-semicontinuous, additive quasitraces defined in [loc. cit., Section 3.2].[5] This topology is shown to be compact and Hausdorff in [loc. cit., Theorem 3.7]. By [loc. cit., Theorem 3.10], the restriction of this topology to $\widetilde{T}(A)$ is pointwise convergence on $\mathrm{Ped}(A)$. Since $\widehat{T}(A) \setminus \widetilde{T}(A)$ is just one point, it follows that the topology on $\widehat{T}(A)$ is simply the one point compactification of the topology on $\widetilde{T}(A)$.

By [Tikuisis and Toms 2015, Proposition 3.4], the cone $\widetilde{T}(A)$ has a compact base K. Since f is strictly positive and continuous, $\inf_{\tau \in K} f(\tau) > 0$. Hence, we may extend f to the one-point compactification of $\widetilde{T}(A)$ by setting $f(\infty) = \infty$ and the resulting map is still continuous. It follows that f defines an element of the dual cone $L(F(\mathrm{Cu}(A)))$; see [Elliott et al. 2011, Section 5.1]. Therefore, as A is $\mathcal{Z}$-stable, there exists $a \in (A \otimes \mathbb{K})_{+,1}$ such that $f(\tau) = d_\tau(a)$ for all $\tau \in \widetilde{T}(A)$ by [loc. cit., Theorem 6.6]. $\square$

We end this section with the following summary of all the reductions.

Theorem 2.7. *Let A be a nonzero, simple, separable, exact, $\mathcal{Z}$-stable C*-algebra. Then one of the following holds*:

(a) *A is stably isomorphic to a unital C*-algebra.*

(b) *A is stably isomorphic to a stably projectionless C*-algebra A_0 with $Q\widetilde{T}(A_0) = \widetilde{T}_b(A_0) \neq 0$ and $T(A_0)$ is compact.*

Proof. Suppose (a) does not hold. Then A is stably projectionless by Proposition 2.4. By Theorem 1.2, $Q\widetilde{T}(A) \neq 0$. Since A is exact, the nonunital version of Haagerup's theorem gives $Q\widetilde{T}(A) = \widetilde{T}(A) \neq 0$; see [Haagerup 2014; Blanchard and Kirchberg 2004, Remark 2.29(i)]. Since $\widetilde{T}(A)$ is a cone with a compact base, there exists a strictly positive, continuous, linear function $f : \widetilde{T}(A) \to [0, \infty)$. By Proposition 2.6, there is a positive contraction $a \in (A \otimes \mathbb{K})_{+,1}$ such that $f(\tau) = d_\tau(a)$ for all $\tau \in \widetilde{T}(A)$. Set $A_0 := \overline{a(A \otimes \mathbb{K})a}$. By Lemma 2.5, $Q\widetilde{T}(A_0) = \widetilde{T}_b(A_0) \neq 0$ and $T(A_0)$ compact. By Corollary 2.2, A is stably isomorphic to A_0. Hence, A_0 is stably projectionless. $\square$

3. Existence

Let A be a separable, nuclear C*-algebra with complemented partitions of unity (CPoU). In this section, we will construct a sequence of maps $A \xrightarrow{\theta_n} F_n \xrightarrow{\eta_n} A$, where F_n are finite-dimensional C*-algebras,

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[5]In [Elliott et al. 2011], the notation $T(A)$ is used instead of $\widehat{T}(A)$, but this clashes with the notation for the tracial states used in this paper.

θ_n are c.p.c. maps and η_n are c.p.c. order-zero maps, which induces a *-homomorphism $A \to A^\omega$ that agrees with the diagonal inclusion $A \to A^\omega$.

We will do this in two steps. First, we will fix a trace τ and produce maps $A \to F \to A$ that approximate the identity map on A in $\| \cdot \|_{2,\tau}$-norm. We shall then construct the required sequence of maps using complemented partitions of unity (CPoU).

The following lemma will be deduced from [Castillejos et al. 2019, Lemma 5.1], but it can also be proved by directly applying the methods of [Brown et al. 2017, Lemma 2.5].

Lemma 3.1. *Let A be a separable, nuclear C*-algebra and let $\tau \in T(A)$. For any finite subset $\mathcal{F} \subseteq A$ and $\epsilon > 0$ there exist a finite-dimensional C*-algebra F, a c.p.c. map $\theta : A \to F$, and a c.p.c. order-zero map $\eta : F \to A$ such that*

$$\|\theta(a)\theta(b)\| < \epsilon \quad \text{for } a, b \in \mathcal{F} \text{ such that } ab = 0, \tag{3-1}$$

$$\|\eta \circ \theta(a) - a\|_{2,\tau} < \epsilon \quad \text{for } a \in \mathcal{F}. \tag{3-2}$$

If all traces are quasidiagonal,[6] one can additionally arrange that

$$\|\theta(ab) - \theta(a)\theta(b)\| < \epsilon, \quad a, b \in \mathcal{F}. \tag{3-3}$$

Proof. The trace τ extends to a trace on $A^\sim$. By [Castillejos et al. 2019, Lemma 5.1] applied to $A^\sim$, there exist a finite-dimensional F, a c.p.c. map $\tilde{\theta} : A^\sim \to F$, and a c.p.c. order-zero map $\tilde{\eta} : F \to A^\sim$ such that

$$\|\tilde{\theta}(a)\tilde{\theta}(b)\| < \tfrac{1}{2}\epsilon \quad \text{for } a, b \in \mathcal{F} \text{ satisfying } ab = 0, \tag{3-4}$$

$$\|\tilde{\eta} \circ \tilde{\theta}(a) - a\|_{2,\tau} < \tfrac{1}{2}\epsilon \quad \text{for } a \in \mathcal{F}. \tag{3-5}$$

Let $(e_n)_{n \in \mathbb{N}}$ be an approximate identity of A. Then $e_n \nearrow 1_{A^\sim}$ in $\| \cdot \|_{2,\tau}$. Hence, the c.p.c. maps $\hat{\eta}_n : F \to A$ given by $\hat{\eta}_n(x) = e_n \tilde{\eta}(x) e_n$ converge to $\tilde{\eta}$ in the point-$\| \cdot \|_{2,\tau}$ topology. The sequence $\hat{\eta}_n$ is asymptotically order-zero in $\| \cdot \|_{2,\tau}$. Since F is finite-dimensional, we can make use of order-zero lifting to obtain a sequence of c.p.c. order-zero maps $\eta_n : F \to A$ converging to $\tilde{\eta}$ in the point-$\| \cdot \|_{2,\tau}$ topology.[7]

Set $\theta := \tilde{\theta}|_A$. Choose $n \in \mathbb{N}$ such that $\|\eta_n(\theta(a)) - \tilde{\eta}(\theta(a))\|_{2,\tau} < \tfrac{1}{2}\epsilon$ for all $a \in \mathcal{F}$. We then have

$$\|\eta \circ \theta(a) - a\|_{2,\tau} < \|\tilde{\eta} \circ \theta(a) - a\|_{2,\tau} + \tfrac{1}{2}\epsilon < \epsilon, \quad a \in \mathcal{F}. \tag{3-6}$$

If all traces are quasidiagonal, the map $\tilde{\theta}$ given by [Castillejos et al. 2019, Lemma 5.1] is approximately a *-homomorphism. Hence, so is θ. $\qquad\square$

With the previous lemma in hand, we can now utilise complemented partitions of unity (CPoU) to prove the following.

Lemma 3.2. *Let A be a separable, nuclear C*-algebra with $Q\widetilde{T}(A) = \widetilde{T}_b(A) \neq 0$ and $T(A)$ compact. Suppose A has CPoU. Then there exists a sequence of c.p.c. maps $\phi_n : A \to A$ which factor through*

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[6]See Definition 7.1.

[7]Indeed, let $J_\tau := \{(a_n)_{n \in \mathbb{N}} \in \ell^\infty(A) : \lim_{n \to \infty} \tau(a_n^* a_n) = 0\}$ and consider the diagonal map $(\hat{\eta}_n) : F \to \ell^\infty(A)/J_\tau$. This map is c.p.c. order-zero and so has a c.p.c. order-zero lift $(\eta_n) : F \to \ell^\infty(A)$ by [Winter 2009, Proposition 1.2.4].

finite-dimensional algebras F_n as

$$A \xrightarrow{\phi_n} A \tag{3-7}$$

with θ_n and η_n mapping to F_n below.

with θ_n c.p.c. and η_n c.p.c. order-zero, in such a way that the induced map $(\theta_n)_{n=1}^{\infty} : A \to \prod_{\omega} F_n$ is order-zero, and the induced map $\overline{\Phi} = (\phi_n)_{n=1}^{\infty} : A \to A^{\omega}$ agrees with the diagonal inclusion $A \to A^{\omega}$.

*If all traces on A are quasidiagonal, then we may arrange that the induced map $(\theta_n)_{n=1}^{\infty} : A \to \prod_{\omega} F_n$ is a *-homomorphism.*

Proof. As in [Castillejos et al. 2019, Lemma 5.2], by a standard application of Kirchberg's epsilon test, it suffices to show that for a finite set $\mathcal{F} \subseteq A$ and a tolerance $\epsilon > 0$, there is a sequence (F_n, θ_n, η_n) such that $\theta_n : A \to F_n$ is approximately order-zero (or approximately multiplicative if all traces are quasidiagonal), $\eta_n : F_n \to A$ is an order-zero map, and the induced map $\overline{\Phi}_{\epsilon} = (\eta_n \circ \theta_n)_{n=1}^{\infty} : A \to A^{\omega}$ satisfies $\|a - \overline{\Phi}_{\epsilon}(a)\|_{2, T_{\omega}(A)} < \epsilon$ for all $a \in \mathcal{F}$. In fact, we will arrange for all the F_n to be the same finite-dimensional algebra F, and all the θ_n to be the same map θ.

Let $\mathcal{F} \subseteq A$ be a finite subset and $\epsilon > 0$. By Lemma 3.1, for any $\tau \in T(A)$ there exist a finite-dimensional C*-algebra F_{τ}, a c.p.c. map $\theta_{\tau} : A \to F_{\tau}$, and an order-zero map $\eta_{\tau} : F_{\tau} \to A$ such that

$$\|\theta(a)\theta(b)\| < \epsilon, \qquad a, b \in \mathcal{F} \text{ such that } ab = 0, \tag{3-8}$$

$$\|\eta_{\tau} \circ \theta_{\tau}(x) - x\|_{2, \tau}^2 < \frac{\epsilon^2}{|\mathcal{F}|}, \quad x \in \mathcal{F}. \tag{3-9}$$

Set $a_{\tau} := \sum_{x \in \mathcal{F}} |x - \eta_{\tau}\tau \circ \theta_{\tau}(a)|^2$. By the compactness of $T(A)$, there exist $\tau_1, \ldots, \tau_k \in T(A)$ such that for all $\tau \in T(A)$ there is some τ_i such that $\tau(a_{\tau_i}) < \epsilon^2$.

By CPoU, there exist pairwise orthogonal projections $p_1, \ldots, p_k \in A^{\omega} \cap A'$ adding up to $1_{A^{\omega}}$ such that $\tau(a_i p_i) \le \epsilon^2 \tau(p_i)$ for all $\tau \in T_{\omega}(A)$. Set $F := \bigoplus_{i=1}^{k} F_{\tau_i}$, $\theta : A \to F$ and $\eta : F \to A^{\omega}$ by

$$\theta(a) := (\theta_{\tau_1}(a), \ldots, \theta_{\tau_k}(a)), \quad \eta(x_1, \ldots, x_k) := \sum_{i=1}^{k} \eta_{\tau_i}(x_i) p_i, \tag{3-10}$$

where $a \in A$ and $x_i \in F_{\tau_i}$. By construction (see [Castillejos et al. 2019, Lemma 5.2, equation (5.16)]), we obtain

$$\|a - \eta \circ \theta(a)\|_{2, T_{\omega}(A)} < \epsilon, \quad a \in \mathcal{F}. \tag{3-11}$$

By [Winter 2009, Proposition 1.2.4], $\eta : F \to A^{\omega}$ can be lifted to a sequence of order-zero maps $\eta_n : F \to A$. Thus the sequence (F, θ, η_n) is the required sequence.

Finally, if all traces are quasidiagonal, the map θ is approximately multiplicative by Lemma 3.1. Combining the previous argument with Kirchberg's epsilon test yields that the induced map $(\theta_n) : A \to \prod_{\omega} F_n$ is a *-homomorphism. $\square$

4. Unitisation

In this section, we prove that a c.p.c. order-zero map $\phi : A \to B_\omega$ from a separable C*-algebra into a C*-ultrapower extends to a c.p.c. order-zero map $\phi^\sim : A^\sim \to B_\omega$. Moreover, under appropriate conditions, Dini's theorem can be used to construct an extension for which the tracial behaviour of $\phi^\sim(1_{A^\sim})$ is determined by ϕ. These results were inspired by the structure theory for order-zero maps developed in [Winter and Zacharias 2009] and the existence of *supporting order-zero maps* proved in [Bosa et al. 2019a, Lemma 1.14]. We begin with a technical lemma.

Lemma 4.1. *Let $\phi : A \to B$ be a c.p.c. order-zero map between C*-algebras. Suppose that $h \in B$ is a positive contraction such that*

$$\phi(a)\phi(b) = h\phi(ab), \quad a, b \in A_+. \tag{4-1}$$

Then the map $\phi^\sim : A^\sim \to B$ defined by $\phi^\sim(a + \lambda 1_{A^\sim}) := \phi(a) + \lambda h$ is c.p.c. order-zero.

Proof. By [Winter and Zacharias 2009, Corollary 4.1], there exists a *-homomorphism $\pi : C_0(0, 1] \otimes A \to B_\omega$ such that $\phi(a) = \pi(t \otimes a)$ for all $a \in A$, where t denotes the canonical generator of the cone. In terms of π, equation (4-1) gives $h\pi(t \otimes ab) = \pi(t^2 \otimes ab)$ for $a, b \in A_+$, from which we deduce that

$$h\pi(t \otimes a) = \pi(t^2 \otimes a), \quad a \in A, \tag{4-2}$$

since $(A_+)^2 = A_+$ and A_+ spans A. It then follows that $h^n\pi(t^m \otimes a) = \pi(t^{n+m} \otimes a)$ for $a \in A$ and for all $n, m \in \mathbb{N}_{\geq 1}$, from which we obtain

$$g(h)\pi(f \otimes a) = \pi(gf \otimes a), \quad a \in A, \ f, g \in C_0(0, 1], \tag{4-3}$$

since $\mathrm{span}\{t^n : n \in \mathbb{N}_{\geq 1}\}$ is dense in $C_0(0, 1]$. Taking adjoints, we also obtain $\pi(f \otimes a)g(h) = \pi(fg \otimes a)$ for all $a \in A$, $f, g \in C_0(0, 1]$.

We now define a map $\pi^\sim : C_0(0, 1] \odot A^\sim \to B$ from the algebraic tensor product by setting $\pi^\sim(f \otimes (a + \lambda 1_{A^\sim})) := \pi(f \otimes a) + \lambda f(h)$ on elementary tensors. A straightforward computation using (4-3) and its adjoint shows that $\pi^\sim$ is a *-homomorphism. Hence, $\pi^\sim$ extends to a map $C_0(0, 1] \otimes A^\sim \to B$. Finally, define $\phi^\sim : A^\sim \to B$ by $\phi^\sim(x) := \pi^\sim(t \otimes x)$. Then $\phi^\sim$ is a c.p.c. order-zero map and $\phi^\sim(a + \lambda 1_{A^\sim}) = \phi(a) + \lambda h$ as required. $\qquad\square$

We now prove the unitisation lemma for order-zero maps.

Lemma 4.2. *Let A, B be C*-algebras with A separable and let $\phi : A \to B_\omega$ be a c.p.c. order-zero map:*

(a) *There exists a c.p.c. order-zero map $\phi^\sim : A^\sim \to B_\omega$ which extends ϕ.*

(b) *Suppose now that $T(B)$ is compact and nonempty. Let $(e_n)_{n \in \mathbb{N}}$ be an approximate unit for A and suppose the function*

$$\theta : \overline{T_\omega(B)}^{w*} \to [0, 1], \quad \tau \mapsto \lim_{n \to \infty} \tau(\phi(e_n)),$$

is continuous. Then there exists a c.p.c. order-zero map $\phi^\sim : A^\sim \to B_\omega$ which extends ϕ and satisfies $\tau(\phi^\sim(1_{A^\sim})) = \theta(\tau)$ for all $\tau \in \overline{T_\omega(B)}^{w*}$.

Proof. (a) Let $(e_n)_{n \in \mathbb{N}}$ be an approximate unit for A. By [Winter and Zacharias 2009, Corollary 4.1], there exists a *-homomorphism $\pi : C_0(0, 1] \otimes A \to B_\omega$ such that $\phi(a) = \pi(t \otimes a)$ for all $a \in A$, where t denotes the canonical generator of the cone. For any $a, b \in A_+$, we have

$$\lim_{n \to \infty} \phi(e_n)\phi(ab) = \lim_{n \to \infty} \pi(t^2 \otimes e_n ab)$$
$$= \pi(t^2 \otimes ab) = \phi(a)\phi(b). \tag{4-4}$$

We shall now prove the existence of a positive contraction $h \in B_\omega$ such that (4-1) holds for all $a, b \in A_+$ by an application of Kirchberg's epsilon test (Lemma 1.8).

Let $X_n := B_{+,1}$ for all $n \in \mathbb{N}$. Let $\phi_n : A_+ \to B_+$ be a sequence of functions such that $(\phi_n(a))_{n \in \mathbb{N}}$ is a representative for $\phi(a)$ for all $a \in A_+$. Fix a dense sequence $(a_n)_{n \in \mathbb{N}}$ in A_+. Define $f_n^{(r,s)} : X_n \to [0, \infty]$ for $r, s \in \mathbb{N}$ by

$$f_n^{(r,s)}(x) := \|x\phi_n(a_r a_s) - \phi_n(a_r)\phi_n(a_s)\|. \tag{4-5}$$

Then define $f_\omega^{(r,s)} : \prod_{n=1}^\infty X_n \to [0, \infty]$ by $(x_n)_{n \in \mathbb{N}} \mapsto \lim_{n \to \omega} f_n^{(r,s)}(x_n)$.

Let $m \in \mathbb{N}$ and $\epsilon > 0$. By (4-4), there is $k \in \mathbb{N}$ such that

$$\|\phi(e_k)\phi(a_r a_s) - \phi(a_r)\phi(a_s)\| < \epsilon, \quad 1 \le r, s \le m. \tag{4-6}$$

Let $x = (x_n)_{n \in \mathbb{N}}$ be a sequence of positive contractions in B representing $\phi(e_k)$. Then $f_\omega^{(r,s)}(x) < \epsilon$ whenever $1 \le r, s \le m$. By Kirchberg's epsilon test, there exists a sequence of positive contractions $y = (y_n)_{n \in \mathbb{N}}$ in B such that $f_\omega^{(r,s)}(y) = 0$ for all $r, s \in \mathbb{N}$. Let h be the positive contraction in B_ω represented by $(y_n)_{n \in \mathbb{N}}$. Then h satisfies (4-1) for all $a, b \in \{a_n : n \in \mathbb{N}\}$. By density, h satisfies (4-1) for all $a, b \in A_+$. The result now follows by Lemma 4.1.

(b) By Dini's theorem, $\tau(\phi(e_n)) \nearrow \theta(\tau)$ uniformly for $\tau \in \overline{T_\omega(B)}^{w*}$.[8] For each $l \in \mathbb{N}$, set

$$\gamma_l := \max_{\tau \in \overline{T_\omega(B)}^{w*}} (\theta(\tau) - \tau(\phi(e_l))). \tag{4-7}$$

Then $\gamma_l \ge 0$ as $\tau(\phi(e_n))$ increases with n, and $\lim_{l \to \infty} \gamma_l = 0$ as the convergence is uniform.

We shall now prove the existence of a positive contraction $h \in B_\omega$ such that (4-1) holds for all $a, b \in A_+$ and that

$$\tau(h) = \lim_{n \to \infty} \tau(\phi(e_n)), \quad \tau \in \overline{T_\omega(B)}^{w*}. \tag{4-8}$$

Once again, we use Kirchberg's epsilon test (Lemma 1.8).

Let X_n, ϕ_n, $f_n^{(r,s)}$, and $f_\omega^{(r,s)}$ be as in (a). Define $g_n^{(l,+)}, g_n^{(l,-)} : X_n \to [0, \infty]$ for $l \in \mathbb{N}$ by

$$g_n^{(l,+)}(x) := \max\big(\sup_{\tau \in T(B)} (\tau(x) - \tau(\phi_n(e_l))) - \gamma_l, 0\big), \tag{4-9}$$

$$g_n^{(l,-)}(x) := \max\big(\sup_{\tau \in T(B)} (\tau(\phi_n(e_l)) - \tau(x)), 0\big). \tag{4-10}$$

Then define $g_\omega^{(l,+)}, g_\omega^{(l,-)} : \prod_{n=1}^\infty X_n \to [0, \infty]$ by $(x_n)_{n \in \mathbb{N}} \mapsto \lim_{n \to \omega} g_n^{(l,+)}(x_n)$ and $(x_n)_{n \in \mathbb{N}} \mapsto \lim_{n \to \omega} g_n^{(l,-)}(x_n)$ respectively.

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[8]Our convention is that approximate units for C*-algebras are by default assumed to be increasing.

The key observation is that a sequence $x = (x_n)_{n \in \mathbb{N}}$ representing a positive contraction $b \in B_\omega$ satisfies $g_\omega^{(l,+)}(x) = g_\omega^{(l,-)}(x) = 0$ if and only if

$$\tau(\phi(e_l)) \leq \tau(b) \leq \tau(\phi(e_l)) + \gamma_l, \quad \tau \in \overline{T_\omega(B)}^{w*}. \tag{4-11}$$

Let $m \in \mathbb{N}$ and $\epsilon > 0$. By (4-4), there is $k > m$ such that

$$\|\phi(e_k)\phi(a_r a_s) - \phi(a_r)\phi(a_s)\| < \epsilon, \quad 1 \leq r, s \leq m. \tag{4-12}$$

Let $x = (x_n)_{n \in \mathbb{N}}$ be a sequence of positive contractions in B representing $\phi(e_k)$. Then $f_\omega^{(r,s)}(x) < \epsilon$ whenever $1 \leq r, s \leq m$. Furthermore, as $k > m$, we have by (4-7) that for any $l \leq m$

$$\tau(\phi(e_l)) \leq \tau(\phi(e_k)) \leq \theta(\tau) \leq \tau(\phi(e_l)) + \gamma_l, \quad \tau \in \overline{T_\omega(B)}^{w*}. \tag{4-13}$$

Therefore, $g_\omega^{(l,+)}(x) = g_\omega^{(l,-)}(x) = 0$ for $l \leq m$.

By Kirchberg's epsilon test, there exists a sequence of positive contractions $y = (y_n)_{n \in \mathbb{N}}$ in B such that $f_\omega^{(r,s)}(y) = g_\omega^{(l,+)}(y) = g_\omega^{(l,-)}(y) = 0$ for all $r, s, l \in \mathbb{N}$. Let h be the positive contraction in B_ω represented by $(y_n)_{n \in \mathbb{N}}$. Then h satisfies (4-1) for all $a, b \in A_+$ as in (a) and

$$\tau(\phi(e_l)) \leq \tau(h) \leq \tau(\phi(e_l)) + \gamma_l, \quad \tau \in \overline{T_\omega(B)}^{w*}, \, l \in \mathbb{N}. \tag{4-14}$$

Letting $l \to \infty$, we obtain (4-8) because $\lim_{l \to \infty} \gamma_l = 0$. The result now follows by Lemma 4.1. $\square$

5. The uniqueness theorem

In this section, we establish the uniqueness theorem for maps from a C*-algebra into a C*-ultrapower, which will be used to bound the nuclear dimension of $\mathcal{Z}$-stable C*-algebras. This theorem is a nonunital version of [Castillejos et al. 2019, Lemma 4.8] which in turn builds on [Bosa et al. 2019a, Theorem 5.5]. For notational convenience, we work with ultrapowers throughout rather than general ultraproducts.

Theorem 5.1 (cf. [Bosa et al. 2019a, Theorem 5.5]). *Let B be a simple, separable, $\mathcal{Z}$-stable C*-algebra with CPoU, stable rank one in $B^\sim$, $Q\widetilde{T}(B) = \widetilde{T}_b(B) \neq 0$, and $T(B)$ compact. Let A be a unital, separable, nuclear C*-algebra, let $\phi_1 : A \to B_\omega$ be a c.p.c. order-zero map such that $\phi_1(a)$ is full for all nonzero $a \in A$ and the induced map $\bar\phi_1 : A \to B^\omega$ is a *-homomorphism, and let $\phi_2 : A \to B_\omega$ be a c.p.c. order-zero map such that*

$$\tau \circ \phi_1 = \tau \circ \phi_2^m, \quad \tau \in T(B_\omega), \, m \in \mathbb{N}, \tag{5-1}$$

where order-zero functional calculus is used to interpret ϕ_2^m.[9] Let $k \in \mathcal{Z}_+$ be a positive contraction with spectrum $[0, 1]$, and define c.p.c. order-zero maps $\psi_i : A \to (B \otimes \mathcal{Z})_\omega$ by $\psi_i(a) := \phi_i(a) \otimes k$. Then ψ_1 and ψ_2 are unitarily equivalent in $(B \otimes \mathcal{Z})_\omega^\sim$.

The proof of Theorem 5.1 follows by a careful adaptation of the arguments from [Bosa et al. 2019a; Castillejos et al. 2019] to handle the potential nonunitality of B. In the subsections that follow, we shall first review the key ingredients of the proof of [Castillejos et al. 2019, Lemma 4.8] and [Bosa et al. 2019a,

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[9]Suppose $\phi_2(x) = \pi_2(t \otimes x)$ where $\pi_2 : C_0(0, 1] \otimes A \to B_\omega$ is a *-homomorphism and t is the canonical generator of $C_0(0, 1]$. Then $\phi_2^m(x) = \pi_2(t^m \otimes x)$; see [Winter and Zacharias 2009, Corollary 4.2].

Theorem 5.5] and explain clearly the modifications needed in the nonunital setting. We shall then return to the proof of Theorem 5.1.

5A. *The 2 × 2 matrix trick.* We begin by reviewing the 2×2 matrix trick, which converts the problem of unitary equivalence of maps into the problem of unitary equivalence of positive elements. The version stated below is very similar to [Bosa et al. 2019a, Lemma 2.3]; however, for our applications, we must weaken the stable rank one assumption and we have no need for the Kirchberg algebra case.

Proposition 5.2 (cf. [Bosa et al. 2019a, Lemma 2.3]). *Let A be a separable, unital C*-algebra and B be a separable C*-algebra. Let $\phi_1, \phi_2 : A \to B_\omega$ be c.p.c. order-zero maps and $\hat{\phi}_1, \hat{\phi}_2 : A \to B_\omega$ be supporting order-zero maps (as in (A-1)). Suppose that B_ω has stable rank one in $B_\omega^\sim$. Let $\pi : A \to M_2(B_\omega)$ be given by*

$$\pi(a) := \begin{pmatrix} \hat{\phi}_1(a) & 0 \\ 0 & \hat{\phi}_2(a) \end{pmatrix}, \quad a \in A, \tag{5-2}$$

and set $C := M_2(B_\omega) \cap \pi(A)' \cap \{1_{M_2(B_\omega^\sim)} - \pi(1_A)\}^\perp$. If

$$\begin{pmatrix} \phi_1(1_A) & 0 \\ 0 & 0 \end{pmatrix} \quad and \quad \begin{pmatrix} 0 & 0 \\ 0 & \phi_2(1_A) \end{pmatrix} \tag{5-3}$$

are unitarily equivalent in $C^\sim$, then ϕ_1 and ϕ_2 are unitarily equivalent in $B_\omega^\sim$.

Proof. Let

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in C^\sim \tag{5-4}$$

be a unitary implementing the unitary equivalence of the positive elements. Since B_ω has stable rank one in $B_\omega^\sim$, we have that $u_{21}^* \phi_2(1_A)$ is the limit of invertibles in $B_\omega^\sim$. Hence, by [Bosa et al. 2019a, Lemma 1.20] and Kirchberg's epsilon test, there is a unitary $w \in B_\omega^\sim$ with $u_{21}^* \phi_2(1_A) = w|u_{21}^* \phi_2(1_A)|$. Arguing exactly as in the proof of [Bosa et al. 2019a, Lemma 2.3], we obtain that $\phi_1(a) = w\phi_2(a)w^*$ for all $a \in A$. $\square$

5B. *Property* (SI). Our goal in this section is to show that c.p.c. order-zero maps from separable, unital C*-algebras into ultrapowers of C*-algebras with compact trace space satisfy property (SI).

The following definition is a variant of [Bosa et al. 2019a, Definition 4.2], which in turn goes back to [Matui and Sato 2012], that allows us to handle cases when the codomain is not unital.

Definition 5.3. Let B be a simple, separable, C*-algebra with $Q\widetilde{T}(B) = \widetilde{T}_b(B) \neq 0$. Write J_{B_ω} for the trace kernel ideal. Let A be a separable, unital C*-algebra, let $\pi : A \to B_\omega$ be a c.p.c. order-zero map, and define

$$C := B_\omega \cap \pi(A)' \cap \{1_{B_\omega^\sim} - \pi(1_A)\}^\perp. \tag{5-5}$$

The map π has *property (SI)* if the following holds. For all $e, f \in C_+$ such that $e \in J_{B_\omega}$, $\|f\| = 1$ and f has the property that, for every nonzero $a \in A_+$, there exists $\gamma_a > 0$ such that

$$\tau(\pi(a)f^n) > \gamma_a, \quad \tau \in T_\omega(B), \ n \in \mathbb{N}, \tag{5-6}$$

there exists $s \in C$ such that

$$s^*s = e \quad \text{and} \quad fs = s. \tag{5-7}$$

The main result of this subsection is that under certain hypotheses, c.p.c. maps $A \to B_\omega$ have property (SI). This result is a nonunital version of [Bosa et al. 2019a, Lemma 4.4] and its proof is almost identical to the original proof. Since this result is one of the most delicate parts of this work, we include its proof.

Proposition 5.4 (cf. [Bosa et al. 2019a, Lemma 4.4]). *Let B be a simple, separable, $\mathcal{Z}$-stable C^*-algebra with $Q\widetilde{T}(B) = \widetilde{T}_b(B) \neq 0$. Let A be a separable, unital, nuclear C^*-algebra. Then every c.p.c. order-zero map $\pi : A \to B_\omega$ has property (SI).*

Proof. Let $\pi : A \to B_\omega$ be a c.p.c. order-zero map with A and B as in the statement. Let C be as in (5-5) and set $\bar{C} := C/(C \cap J_{B_\omega})$. Let $e, f \in C_+$ and γ_a be as in the definition of property (SI). As in the proof of [Bosa et al. 2019a, Lemma 4.4], it is enough to exhibit an element $s \in B_\omega$ approximately satisfying

$$s^*\pi(a)s = \pi(a)e \quad \text{for all } a \in A \text{ and } fs = s. \tag{5-8}$$

Let $\mathcal{F} \subseteq A$ be a finite subset of contractions and $\epsilon > 0$. Since B is $\mathcal{Z}$-stable, using Lemma A.6(ii) we can find a c.p.c. order-zero map $\alpha : \mathcal{Z} \to B_\omega \cap \pi(A)' \cap \{e, f\}'$ such that $\alpha(1_{\mathcal{Z}})$ acts as unit on $\pi(A)$. Therefore, we may define a new c.p.c. map $\tilde{\pi} : A \otimes \mathcal{Z} \to B_\omega$ by setting $\tilde{\pi}(a \otimes z) := \pi(a)\alpha(z)$. It follows by construction that $\pi(a) = \tilde{\pi}(a \otimes 1_{\mathcal{Z}})$ for $a \in A$. By [Winter and Zacharias 2009, Corollary 4.3], $\tilde{\pi}$ is a c.p.c. order-zero map and note that e and f are elements of the relative commutant $B_\omega \cap \tilde{\pi}(A \otimes \mathcal{Z})' \cap \{1_{B_\omega^{\sim}} - \tilde{\pi}(1_{A \otimes \mathcal{Z}})\}^\perp$.

Arguing as in the proof of [Bosa et al. 2019a, Lemma 4.4], for any $b \in (A \otimes \mathcal{Z})_+$, there exists a positive constant $\tilde{\gamma}_b$ such that

$$\tau(\tilde{\pi}(b)f^n) > \tilde{\gamma}_b, \quad \tau \in T_\omega(B_\omega), \ n \in \mathbb{N}. \tag{5-9}$$

Next, we will apply [loc. cit., Lemmas 4.7 and 4.8] to the unital, separable, nuclear C^*-algebra A. Set $\mathcal{G} := \{x \otimes 1_{\mathcal{Z}} : x \in \mathcal{F}\} \subseteq A \otimes \mathcal{Z}$. Since no irreducible representation of $A \otimes \mathcal{Z}$ contains any compact operator, by [loc. cit., Lemma 4.8] there exist $L, N \in \mathbb{N}$, pairwise inequivalent pure states $\lambda_1, \ldots, \lambda_L$ on $A \otimes \mathcal{Z}$ and elements $c_i, d_{i,l} \in A \otimes \mathcal{Z}$ for $i = 1, \ldots, N$, $l = 1, \ldots, L$ such that

$$x \approx_\epsilon \sum_{l=1}^{L} \sum_{i,j=1}^{N} \lambda_l(d_{i,l}^* x d_{j,l}) c_i^* c_j, \quad x \in \mathcal{G}. \tag{5-10}$$

By [loc. cit., Lemma 4.7], applied to the set $\{d_{i,l}^* x d_{j,l'} : x \in \mathcal{G}, \ i, j = 1, \ldots, N, \ l, l' = 1, \ldots, L\}$, there exist positive contractions $a_1, \ldots, a_L \in (A \otimes \mathcal{Z})_+$ such that for $l = 1, \ldots, L$, $\lambda_l(a_l) = 1$ and

$$a_l d_{i,l}^* x d_{j,l} a_l \approx_\delta \lambda_l(d_{i,l}^* x d_{j,l}) a_l^2, \quad x \in \mathcal{G}, \ i, j = 1, \ldots, N, \tag{5-11}$$

and for $l \neq l'$,

$$a_l d_{i,l}^* x d_{j,l'} a_{l'} \approx_\delta 0, \quad x \in \mathcal{G}, \ i, j = 1, \ldots, N, \tag{5-12}$$

with $\delta := \epsilon/(N^2 L \max_k \|c_k\|^2)$. Note, the condition $\lambda_l(a_l) = 1$ ensures that the a_l have norm 1.

By hypothesis, B is simple, separable, $\mathcal{Z}$-stable and $Q\widetilde{T}(B) = \widetilde{T}_b(B) \neq 0$. Hence, by Proposition 1.5, B has strict comparison of positive elements by bounded traces. Thus, for $l = 1, \ldots, L$, we may apply Lemma A.11 with a_l in place of a. Let $S_l \subseteq (A \otimes \mathcal{Z})_+ \setminus \{0\}$ denote the countable subset such that the conclusion of Lemma A.11 is satisfied with a_l in place of a.

Let $\hat{\pi} : A \otimes \mathcal{Z} \to B_\omega \cap \{f\}'$ be a supporting c.p.c. order-zero map for $\tilde{\pi}$. As in [loc. cit., Lemma 4.4], using (5-9) and Lemma A.4 twice (taking $x := 0$ and with $S_0 := \tilde{\pi}(S_1 \cup \cdots \cup S_L)$), we find $t, h \in B_\omega \cap \hat{\pi}(A \otimes \mathcal{Z})' \cap \tilde{\pi}(A \otimes \mathcal{Z})'$ satisfying $h \lhd t \lhd f$ and, for every $b \in S_1 \cup \cdots \cup S_L$,

$$\tau(\tilde{\pi}(b)h^n) \geq \tilde{\gamma}_b, \quad \tau \in T_\omega(B), \ n \in \mathbb{N}. \tag{5-13}$$

By Lemma A.11 (with $\tilde{\pi}$ in place of π), there is a contraction $r_l \in B_\omega$ such that $\tilde{\pi}(a_l)r_l = tr_l = r_l$ and $r_l^* r_l = e$. Using $t \lhd f \lhd \tilde{\pi}(1_A)$, we obtain $\tilde{\pi}(1_A)r_l = r_l$ for each l, and hence

$$r_l^* \tilde{\pi}(a_l^2)r_l = \tilde{\pi}(1_A)^{1/2} e \tilde{\pi}(1_A)^{1/2}. \tag{5-14}$$

Set

$$s := \sum_{l=1}^{L} \sum_{i=1}^{N} \hat{\pi}(d_{i,l}a_l)r_l\hat{\pi}(c_i) \in B_\omega. \tag{5-15}$$

Using $r_l = tr_l$, $t \lhd f$ and that t commutes with the image of $\hat{\pi}$, we can obtain $fs = s$. For $x \in \mathcal{F}$, the calculations of [loc. cit., Lemma 4.4, equation 4.46] show

$$s^*\pi(x)s = \pi(x)e, \tag{5-16}$$

as required. Then Kirchberg's epsilon test produces an element $s \in B_\omega$ that exactly satisfies (5-8). As in the proof of [loc. cit., Lemma 4.10], $s \in C$. $\qquad\square$

5C. *Structural results for relative commutants.*

Combining property (SI) with complemented partitions of unity (CPoU), one can now prove important structural properties for the relative commutant algebras $C := B_\omega \cap \pi(A)' \cap \{1_{\widetilde{B}_\omega} - \pi(1_A)\}^\perp$ arising from the 2×2 matrix trick.

Theorem 5.5 (cf. [Castillejos et al. 2019, Lemma 4.7]). *Let B be a simple, separable, $\mathcal{Z}$-stable C*-algebra with $Q\widetilde{T}(B) = \widetilde{T}_b(B) \neq 0$ and $T(B)$ compact. Suppose additionally that B has CPoU. Let A be a separable unital nuclear C*-algebra and $\pi : A \to B_\omega$ a c.p.c. order-zero map which induces a *-homomorphism $\bar{\pi} : A \to B^\omega$. Let*

$$C := B_\omega \cap \pi(A)' \cap \{1_{\widetilde{B}_\omega} - \pi(1_A)\}^\perp, \quad \overline{C} := C/(C \cap J_{B_\omega}). \tag{5-17}$$

Then:

(i) *All traces on C factor through $\overline{C}$.*

(ii) *C has strict comparison of positive elements by bounded traces.*

(iii) *The traces on C are the closed convex hull of traces of the form $\tau(\pi(a) \cdot)$ for $\tau \in T(B_\omega)$ and $a \in A_+$ with $\tau(\pi(a)) = 1$.*

First, we discuss two preliminary results, which originate from [Bosa et al. 2019a, Lemmas 3.20 and 3.22] and were generalised in [Castillejos et al. 2019, Lemmas 4.3 and 4.6], where the newly

discovered CPoU was used in place of the earlier methods that required further assumptions on $T(B)$. Both results are proven by checking that these lemmas approximately hold for $\pi_\tau(B^\omega)''$ for any trace in $\tau \in \overline{T_\omega(B)}^{w*}$, which in turn follows from the fact that $\pi_\tau(B^\omega)''$ is a finite von Neumann algebra, and then using CPoU to patch local solutions together. In [loc. cit.], these results are stated for B unital, but the proofs do not make use of the unit. They only require that $T(B)$ is compact, as this guarantees that B^ω is unital [loc. cit., Proposition 1.11].

Lemma 5.6 (cf. [Castillejos et al. 2019, Lemma 4.3]). *Let B be a separable C*-algebra with $Q\widetilde{T}(B) = \widetilde{T}_b(B) \neq 0$ and $T(B)$ compact. Suppose B has CPoU. Let $S \subseteq B^\omega$ be a $\| \cdot \|_{2,T_\omega(B)}$-separable and self-adjoint subset, and let p be a projection in the centre of $B^\omega \cap S'$. Then $p(B^\omega \cap S')$ has strict comparison of positive elements by bounded traces.*

Proposition 5.7 (cf. [Castillejos et al. 2019, Lemma 4.6]). *Let B be a separable C*-algebra with $Q\widetilde{T}(B) = \widetilde{T}_b(B) \neq 0$ and $T(B)$ compact. Suppose B has CPoU. Let A be a separable, unital, nuclear C*-algebra and $\phi : A \to B^\omega$ a *-homomorphism. Set $C := B^\omega \cap \phi(A)' \cap \{1_{B^\omega} - \phi(1_A)\}^\perp$. Define T_0 to be the set of all traces on C of the form $\tau(\phi(a) \cdot)$, where $\tau \in T(B^\omega)$ and $a \in A_+$ satisfies $\tau(\phi(a)) = 1$.*

Suppose $z \in C$ is a contraction and $\delta > 0$ satisfies $|\rho(z)| \leq \delta$ for all $\rho \in T_0$. Write $K := 12 \cdot 12 \cdot (1+\delta)$. Then there exist contractions $w, x_1, \ldots, x_{10}, y_1, \ldots, y_{10} \in C$ such that

$$z = \delta w + K \sum_{i=1}^{10} [x_i, y_i]. \tag{5-18}$$

In particular, $T(C)$ is the closed convex hull of T_0.

With these preparatory results now established, we explain how to adapt the original proof of [Bosa et al. 2019a, Theorem 4.1] to prove Theorem 5.5.

Proof of Theorem 5.5. For (i), the proof of [loc. cit., Theorem 4.1(i)] still works in our situation with the following minor modifications. We use Lemma A.4 instead of [loc. cit., Lemma 1.18], Proposition 5.4 in place of [loc. cit., Lemma 4.4] and Lemma A.5 in place of [loc. cit., Lemma 1.19].

Similarly, for (ii) we use the proof from [loc. cit., Lemma 3.20] with the following modifications. Since B is $\mathcal{Z}$-stable, any matrix algebra embeds into $B^\omega \cap \bar{\pi}(A)' \cap \{\bar{c}\}'$ [Castillejos et al. 2019, Proposition 2.3]. We use Lemma 5.6 to see that $\bar{C}$ has strict comparison of positive elements by traces in place of [Bosa et al. 2019a, Lemma 3.20], and [Castillejos et al. 2019, Lemma 1.8] in place of [Bosa et al. 2019a, Lemma 3.10].

In the same vein, (iii) follows from (i), [Castillejos et al. 2019, Lemma 1.5], and Proposition 5.7. $\square$

5D. *Unitary equivalence of totally full positive elements.* The main theorem of this section is a nonunital version of the classification of totally full positive elements up to unitary equivalence in relative commutant sequence algebras obtained in [Bosa et al. 2019a, Lemma 5.1].[10]

Let us begin by stating the following lemma which can be proved in exactly the same way as [Bosa et al. 2019a, Lemma 5.3] since the Robert–Santiago argument [2010] at the core of the proof has no

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[10]Recall that a nonzero $h \in C_+$ is *totally full* if $f(h)$ is full in C for every nonzero $f \in C_0((0, \|h\|])_+$ [Bosa et al. 2019a, Definition 1.1].

unitality hypothesis. All that is required is to formally replace all occurrences of 1_{B_ω} with $1_{B_\omega^\sim}$, and replace [Bosa et al. 2019a, Lemma 1.17] with Lemma A.3, [loc. cit., Lemma 2.2] with Lemma A.8, and [loc. cit., Lemma 5.4] with Lemma A.9.

Lemma 5.8 (cf. [Bosa et al. 2019a, Lemma 5.3]). *Let B be a separable, $\mathcal{Z}$-stable C*-algebra and let A be a separable, unital C*-algebra. Let $\pi : A \to B_\omega$ be a c.p.c. order-zero map such that*

$$C := B_\omega \cap \pi(A)' \cap \{1_{B_\omega^\sim} - \pi(1_A)\}^\perp \tag{5-19}$$

is full in B_ω.

Assume that every full hereditary subalgebra D of C satisfies the following: if $x \in D$ is such that there exist totally full elements $e_l, e_r \in D_+$ such that $e_l x = x e_r = 0$, then there exists a full element $s \in D$ such that $sx = xs = 0$.

Let $a, b \in C_+$ be totally full positive contractions. Then a and b are unitarily equivalent in $C^\sim$ if and only if for every $f \in C_0(0,1]_+$, $f(a)$ is Cuntz equivalent to $f(b)$ in C.

With this lemma in hand, we can now prove the main theorem of this section.

Theorem 5.9 (cf. [Bosa et al. 2019a, Theorem 5.1]). *Let B be a separable, $\mathcal{Z}$-stable C*-algebra with $Q\widetilde{T}(B) = \widetilde{T}_b(B) \neq 0$. Let A be a separable, unital C*-algebra and let $\pi : A \to B_\omega$ be a c.p.c. order-zero map such that*

$$C := B_\omega \cap \pi(A)' \cap \{1_{B_\omega^\sim} - \pi(1_A)\}^\perp \tag{5-20}$$

is full in B_ω and has strict comparison of positive elements with respect to bounded traces.

Let $a, b \in C_+$ be totally full positive elements. Then a and b are unitarily equivalent in $C^\sim$ if and only if $\tau(a^k) = \tau(b^k)$ for every $\tau \in T(C)$ and $k \in \mathbb{N}$.

Proof. Let $a, b \in C_+$ be totally full positive elements satisfying $\tau(a^k) = \tau(b^k)$ for every $\tau \in T(C)$ and $k \in \mathbb{N}$. Without loss of generality, assume that a and b are contractions.

After replacing [Bosa et al. 2019a, Lemma 1.22(iv)] with Lemma A.6(iv), part (i) of the proof of [loc. cit., Theorem 5.1] shows that the technical hypothesis of Lemma 5.8 is satisfied for every full hereditary subalgebra $D \subseteq C$. The argument of part (ii) of the proof of [loc. cit., Theorem 5.1] then shows that $f(a)$ is Cuntz equivalent to $f(b)$ for all $f \in C_0(0,1]_+$. (This part of the proof of [loc. cit., Theorem 5.1] does not make any use of the unit; only strict comparison is needed.) By Lemma 5.8, a and b are unitarily equivalent by unitaries in $C^\sim$. The converse is straightforward. $\qquad\square$

5E. *Proof of the uniqueness theorem.* We now have all the ingredients we need for the proof of Theorem 5.1.

Proof of Theorem 5.1. By hypothesis, $\bar{\phi}_1(1_A) \in B^\omega$ is a projection. Hence $d_\tau(\phi_1(1_A)) = \tau(\phi_1(1_A))$ and we immediately can conclude that the map $\tau \mapsto d_\tau(\phi_1(1_A))$ is continuous. Similarly, by (5-1), the map $\tau \mapsto d_\tau(\phi_2(1_A))$ is continuous. Hence, by Lemma A.1, there exist supporting order-zero maps $\hat{\phi}_1, \hat{\phi}_2 : A \to B_\omega$ such that

$$\tau(\hat{\phi}_i(a)) = \lim_{m \to \infty} \tau(\phi_i^{1/m}(a)), \quad a \in A, \ \tau \in T_\omega(B), \ i = 1, 2, \tag{5-21}$$

and the maps $\bar{\hat{\phi}}_i : A \to B^\omega$ are *-homomorphisms. In particular,

$$\tau(\hat{\phi}_2(a)) \overset{(5\text{-}1)}{=} \tau(\phi_1(a)), \quad a \in A, \ \tau \in T_\omega(B). \tag{5-22}$$

By Proposition 1.12, B_ω has stable rank one in $B_\omega^\sim$. Thus, we may use the 2×2 matrix trick (Proposition 5.2). Recall $\psi_i(a) := \phi_i(a) \otimes k$ and define $\hat{\psi}_1, \hat{\psi}_2 : A \to (B \otimes \mathcal{Z})_\omega$ by $\hat{\psi}_i(a) := \hat{\phi}_i(a) \otimes 1_{\mathcal{Z}}$, with $i = 1, 2$. It is immediate that $\hat{\psi}_i$ is a supporting order-zero map for ψ_i. Then define $\pi : A \to M_2(B_\omega) \subseteq M_2((B \otimes \mathcal{Z})_\omega)$ by

$$\pi(a) := \begin{pmatrix} \hat{\psi}_1(a) & 0 \\ 0 & \hat{\psi}_2(a) \end{pmatrix}, \quad a \in A, \tag{5-23}$$

and set $C := M_2((B \otimes \mathcal{Z})_\omega) \cap \pi(A)' \cap \{1_{M_2((B \otimes \mathcal{Z})_\omega^\sim)} - \pi(1_A)\}^\perp$. We will show that

$$h_1 := \begin{pmatrix} \psi_1(1_A) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad h_2 := \begin{pmatrix} 0 & 0 \\ 0 & \psi_2(1_A) \end{pmatrix}$$

are unitarily equivalent in $C^\sim$. For nonzero $a \in A$, observe that

$$0 \le \begin{pmatrix} \psi_1(a) & 0 \\ 0 & 0 \end{pmatrix} \le \begin{pmatrix} \hat{\psi}_1(a) & 0 \\ 0 & \hat{\psi}_2(a) \end{pmatrix} = \pi(a), \tag{5-24}$$

and using that $\psi_1(a)$ is full in $(B \otimes \mathcal{Z})_\omega$ since $\phi_1(a)$ is full, we conclude that $\pi(a)$ is full in $M_2((B \otimes \mathcal{Z})_\omega)$. By construction, the induced map $\bar{\pi} : A \to M_2(B^\omega)$ is a *-homomorphism. Thus, by Theorem 5.5, C has strict comparison.

Notice that $h_1 \in C$ is full in $M_2(B_\omega)$, and hence C is also full in $M_2(B_\omega)$. Let ρ be a trace on C of the form $\tau(\pi(x) \cdot)$, where $\tau \in T(M_2((B \otimes \mathcal{Z})_\omega))$, $x \in A_+$ and $\tau(\pi(x)) = 1$. Set a trace $\tilde{\tau}$ on B_ω by $\tilde{\tau}(b) := \tau(1_{M_2} \otimes b \otimes 1_{\mathcal{Z}})$. Thus, as in [Bosa et al. 2019a, Theorem 5.5, equation (5.41)],

$$\rho(h_1^m) = \tfrac{1}{2}\tau_{\mathcal{Z}}(k^m) = \rho(h_2^m), \quad m \in \mathbb{N}. \tag{5-25}$$

By Theorem 5.5, equation (5-25) holds for any trace on C.

A standard strict comparison argument shows that $f(h_1)$ and $f(h_2)$ are full in C for any $f \in C_0(0, 1]_+$, so h_1 and h_2 are totally full. By Theorem 5.9, h_1 is unitarily equivalent to h_2 in $C^\sim$. By the 2×2 matrix trick (Proposition 5.2), ψ_1 and ψ_2 are unitarily equivalent in $(B \otimes \mathcal{Z})_\omega^\sim$. $\qquad \square$

6. Nuclear dimension and $\mathcal{Z}$-stability

In this section, we prove Theorems A and B, and deduce Corollaries C and D.

Theorem 6.1. *Let A be a simple, separable, nuclear and $\mathcal{Z}$-stable C*-algebra. Then $\dim_{\mathrm{nuc}} A \le 1$.*

Proof. By Theorem 2.7, either A is stably isomorphic to a unital C*-algebra B, or A is stably isomorphic to a stably projectionless C*-algebra A_0 with $Q\tilde{T}(A_0) = \tilde{T}_b(A_0) \ne 0$ and $T(A_0)$ compact.

The stably unital case follows immediately from [Castillejos et al. 2019, Theorem B] together with Proposition 2.3. Indeed, if A is stably isomorphic to a unital C*-algebra B, then B is also simple, separable,

nuclear and $\mathcal{Z}$-stable by Proposition 2.3. Hence, $\dim_{\text{nuc}} B \leq 1$ by [loc. cit., Theorem B]. Therefore, $\dim_{\text{nuc}} A \leq 1$ by a second application of Proposition 2.3.

We now consider the case when A is stably isomorphic to a stably projectionless C*-algebra A_0 with $Q\widetilde{T}(A_0) = \widetilde{T}_b(A_0) \neq 0$ and $T(A_0)$ compact. By Proposition 2.3, A_0 is simple, separable, nuclear and $\mathcal{Z}$-stable. Since A_0 is stably projectionless and $\mathcal{Z}$-stable, A_0 has stable rank one in $A_0^\sim$ by Theorem 1.10. Furthermore, A_0 has CPoU by Theorem 1.14.

In light of Proposition 2.3, it suffices to prove that $\dim_{\text{nuc}} A_0 \leq 1$. We now show this using the same fundamental strategy of [Bosa et al. 2019a] (taking into account the modification introduced in [Castillejos et al. 2019]). We shall estimate the nuclear dimension of the first factor embedding $j : A_0 \to A_0 \otimes \mathcal{Z}$, $j(x) = x \otimes 1_{\mathcal{Z}}$, in the sense of [Tikuisis and Winter 2014, Definition 2.2]. Since A_0 is $\mathcal{Z}$-stable and $\mathcal{Z}$ is strongly self-absorbing, we have $\dim_{\text{nuc}}(A_0) = \dim_{\text{nuc}}(j)$; see [loc. cit., Proposition 2.6].

Let $\iota : A_0 \to (A_0)_\omega$ be the canonical embedding. Let h be a strictly positive contraction in A_0, and let $(e_n)_{n \in \mathbb{N}}$ be the approximate identity given by $e_n := h^{1/n}$. Then $\lim_{n \to \infty} \tau(e_n) = 1$ for all $\tau \in T(A_0)$. Since $T(A_0)$ is compact, $\tau \circ \iota \in T(A_0)$ for all $\tau \in T_\omega(A_0)$ and so for all $\tau \in \overline{T_\omega(A_0)}^{w*}$. It follows that

$$\lim_{n \to \infty} \tau(\iota(e_n)) = 1, \quad \tau \in \overline{T_\omega(A_0)}^{w*}. \tag{6-1}$$

Thus applying Lemma 4.1, we obtain a c.p.c. order-zero extension $\iota^\sim : A_0^\sim \to (A_0)_\omega$ with $\tau(\iota^\sim(1_{A_0^\sim})) = 1$ for all $\tau \in \overline{T_\omega(A_0)}^{w*}$. Writing $\overline{\iota^\sim} : A_0^\sim \to A_0^\omega$ for the induced map into the uniform tracial ultrapower, we observe that $1_{A_0^\omega} - \overline{\iota^\sim}(1_{A_0^\sim})$ is a positive element in A_0^ω that vanishes on all limit traces and so must be zero. Hence, $\overline{\iota^\sim}$ is a unital c.p.c. order zero map and so must be a unital *-homomorphism.

Let $(\phi_n : A_0 \to A_0)_{n=1}^\infty$ be the sequence of c.p.c. maps constructed in Lemma 3.2, which factorize as $\eta_n \circ \theta_n$ through finite-dimensional algebras F_n as in (3-7). By construction, the induced map $\Phi : A_0 \to (A_0)_\omega$ is c.p.c. order-zero and the induced map $\overline{\Phi} : A_0 \to A_0^\omega$ agrees with the diagonal inclusion $\overline{\iota} : A_0 \to A_0^\omega$. It follows that $\tau \circ \Phi = \tau \circ \iota$ for all $\tau \in \overline{T_\omega(A)}^{w*}$. Hence,

$$\lim_{n \to \infty} \tau(\Phi(e_n)) = 1, \quad \tau \in \overline{T_\omega(A_0)}^{w*}. \tag{6-2}$$

Therefore, applying Lemma 4.1 again, we obtain a c.p.c. order-zero extension $\Phi^\sim : A_0^\sim \to (A_0)_\omega$ with $\tau(\Phi^\sim(1_{A_0^\sim})) = 1$ for all $\tau \in \overline{T_\omega(A_0)}^{w*}$. Arguing as before, $\overline{\Phi^\sim} : A_0^\sim \to A_0^\omega$ is a unital *-homomorphism. In fact, we have $\overline{\Phi^\sim} = \overline{\iota^\sim}$ since both maps agree on A_0 by construction and are unital.

We are almost ready to apply Theorem 5.1 to the c.p.c. order-zero maps $\iota^\sim$ and $\Phi^\sim$. We observe that A_0 is a simple, separable, $\mathcal{Z}$-stable with CPoU, stable rank one in $A_0^\sim$, $Q\widetilde{T}(A_0) = \widetilde{T}_b(A_0) \neq 0$, and $T(A_0)$ compact; that $A_0^\sim$ is unital, separable and nuclear; and that both maps induce a unital *-homomorphism $\overline{\iota^\sim} = \overline{\Phi^\sim} : A_0^\sim \to A_0^\omega$. Since $\overline{\iota^\sim} = \overline{\Phi^\sim}$ and both maps are *-homomorphisms, we have

$$\tau \circ \iota = \tau \circ \Phi^m, \quad \tau \in \overline{T_\omega(A_0)}^{w*}, \, m \in \mathbb{N}. \tag{6-3}$$

The tracial condition (5-1) follows because $T_\omega(A_0)$ is dense in $T((A_0)_\omega)$ by Theorem 1.7.

Before we may apply Theorem 5.1, we must show that $\iota^\sim(x)$ is full for all nonzero $x \in A_0^\sim$. By Proposition 1.5, A_0 has strict comparison by bounded traces because A_0 is simple, separable, $\mathcal{Z}$-stable and $Q\widetilde{T}(A_0) = \widetilde{T}_b(A_0) \neq 0$. Hence, $(A_0)_\omega$ has strict comparison in the sense of Lemma A.10.

Using that A_0 is simple and $T(A_0)$ is compact, the minimum $\gamma_a := \min_{\tau \in T(A)} \tau(a)$ exists and is strictly positive for any nonzero $a \in (A_0)_{+,1}$. Since $\tau \circ \iota \in T(A_0)$ for any $\tau \in \overline{T_\omega(A_0)}^{w*}$, we have $d_\tau(\iota(a)) \geq \gamma_a$ for any $\tau \in \overline{T_\omega(A_0)}^{w*}$. Hence, $\iota(a)$ is full in $(A_0)_\omega$ using Lemma A.10.

For any nonzero $x \in A_0^\sim$, the ideal I_x of $A_0^\sim$ generated by x contains a nonzero positive contraction $a \in A_{+,1}$. A simple computation using supporting order-zero maps shows that the ideal of $(A_0)_\omega$ generated by $\iota^\sim(x)$ contains $\iota^\sim(I_x)$, which is full since it contains the full element $\iota^\sim(a)$. Hence, $\iota^\sim(x)$ is full in $(A_0)_\omega$.

Fix a positive contraction $k \in \mathcal{Z}_+$ of full spectrum. Applying Theorem 5.1 to the maps $\iota^\sim$ and $\Phi^\sim$, we obtain unitaries $w^{(0)}, w^{(1)} \in (A_0 \otimes \mathcal{Z})_\omega^\sim$ such that

$$x \otimes k = w^{(0)}(\Phi(x) \otimes k)w^{(0)*}, \tag{6-4}$$

$$x \otimes (1_\mathcal{Z} - k) = w^{(1)}(\Phi(x) \otimes (1_\mathcal{Z} - k))w^{(1)*}, \quad x \in A. \tag{6-5}$$

Choose representing sequences $(w_n^{(0)})_{n=1}^\infty$ and $(w_n^{(1)})_{n=1}^\infty$ of unitaries in $(A_0 \otimes \mathcal{Z})^\sim$ for $w^{(0)}$ and $w^{(1)}$, respectively. We have c.p.c. maps $\theta_n \oplus \theta_n : A_0 \to F_n \oplus F_n$ and $\tilde{\eta}_n : F_n \oplus F_n \to A_0 \otimes \mathcal{Z}$, where

$$\tilde{\eta}_n(y_0, y_1) := w_n^{(0)}(\eta_n(y_0) \otimes k)w_n^{(0)*} + w_n^{(1)}(\eta_n(y_1) \otimes (1_\mathcal{Z} - k))w_n^{(1)*}. \tag{6-6}$$

Hence, $j(x)$ is the limit, as $n \to \omega$, of $(\tilde{\eta}_n \circ (\theta_n \oplus \theta_n)(x))_{n=1}^\infty$ and, since $\tilde{\eta}_n$ is the sum of two c.p.c. order-zero maps, $\dim_{\mathrm{nuc}}(j) \leq 1$. $\square$

Theorem 6.2. *Let A be a nonelementary, simple, separable, nuclear C*-algebra. Then A has finite nuclear dimension if and only if it is $\mathcal{Z}$-stable.*

Proof. Let A be a nonelementary, simple, separable, nuclear C*-algebra. If A is $\mathcal{Z}$-stable, then $\dim_{\mathrm{nuc}}(A) \leq 1 < \infty$ by Theorem 6.1. Conversely, if $\dim_{\mathrm{nuc}}(A) < \infty$, then A is $\mathcal{Z}$-stable by [Tikuisis 2014, Theorem 8.5] $\square$

Corollary 6.3. *The nuclear dimension of a simple C*-algebra is 0, 1 or ∞.*

Proof. Let A be a simple, separable C*-algebra with finite nuclear dimension. Then, in particular, A is nuclear. If A is elementary, then $\dim_{\mathrm{nuc}}(A) = 0$; otherwise, A is $\mathcal{Z}$-stable by Theorem 6.2. Hence, $\dim_{\mathrm{nuc}}(A) \leq 1$ by Theorem 6.1. The nonseparable case follows from the separable one as in the proof of [Castillejos et al. 2019, Corollary C]. $\square$

In [Elliott 1996, Theorem 5.2.2], a stably projectionless, simple, separable, nuclear C*-algebra with a unique trace, $K_0 = \mathbb{Z}$ and $K_1 = 0$ is constructed as a limit of 1-dimensional noncommutative CW complexes. By [Gong and Lin 2017, Theorem 1.4], there is a unique C*-algebra with these properties that has finite nuclear dimension and satisfies the UCT. This C*-algebra is denoted by $\mathcal{Z}_0$ [Gong and Lin 2017, Definition 8.1], reflecting its role as a stably projectionless analogue of the Jiang–Su algebra $\mathcal{Z}$. An important further property of $\mathcal{Z}_0$, which follows from its construction, is that $\mathcal{Z}_0$ is $\mathcal{Z}$-stable [Gong and Lin 2017, Remark 7.3, Definition 8.1].

It has recently been shown that simple, separable C*-algebras which satisfy the UCT and have finite nuclear dimension are classified up to stabilisation with $\mathcal{Z}_0$ by the Elliott invariant [Gong and Lin

2017, Theorem 1.2]. The appropriate form of the Elliott invariant in this setting is detailed in [loc. cit., Definition 2.9]. In light of the main result of this paper, we can weaken the hypothesis of finite nuclear dimension in [loc. cit., Theorem 1.2] to that of nuclearity.

Corollary 6.4 (cf. [Gong and Lin 2017, Theorem 1.2]). *Let A and B be simple, separable, nuclear C*-algebras which satisfy the UCT. Then*

$$A \otimes \mathcal{Z}_0 \cong B \otimes \mathcal{Z}_0 \quad \text{if and only if} \quad \text{Ell}(A \otimes \mathcal{Z}_0) \cong \text{Ell}(B \otimes \mathcal{Z}_0).$$

Proof. Since A and $\mathcal{Z}_0$ are simple, separable and nuclear, so is $A \otimes \mathcal{Z}_0$. Using that $\mathcal{Z}_0$ is $\mathcal{Z}$-stable, it follows that $A \otimes \mathcal{Z}_0$ is $\mathcal{Z}$-stable. Therefore, $\dim_{\text{nuc}} A \otimes \mathcal{Z}_0 \leq 1$ by Theorem A. Similarly, $\dim_{\text{nuc}} B \otimes \mathcal{Z}_0 \leq 1$. The result now follows from [Gong and Lin 2017, Theorem 1.2]. □

7. Decomposition rank and $\mathcal{Z}$-stability

Using the machinery developed to prove Theorem A, we can also prove similar results for the decomposition rank of simple $\mathcal{Z}$-stable C*-algebras under suitable finiteness and quasidiagonality assumptions. To this end, we recall the definition of quasidiagonality for tracial states.

Definition 7.1 (cf. [Brown 2006, Definition 3.3.1]). Let A be a C*-algebra. A tracial state $\tau \in T(A)$ is *quasidiagonal* if there exists a net[11] of c.p.c. maps $\phi_n : A \to M_{k_n}(\mathbb{C})$ with $\|\phi_n(ab) - \phi_n(a)\phi_n(b)\| \to 0$ and $\text{tr}_{k_n}(\phi_n(a)) \to \tau(a)$.

In the unital case, the c.p.c. maps in Definition 7.1 can be taken to be unital (see the proof of [Brown and Ozawa 2008, Lemma 7.1.4]). Moreover, a trace $\tau \in T(A)$ is quasidiagonal if and only if its extension to $A^\sim$ is quasidiagonal [Brown 2006, Proposition 3.5.10]. We write $T_{QD}(A)$ for the set of all quasidiagonal tracial states on A.

We can now state a decomposition-rank version of Theorem A.

Theorem 7.2. *Let A be a simple, separable, nuclear and $\mathcal{Z}$-stable C*-algebra. Suppose further that A is stably finite and that $T(B) = T_{QD}(B)$ for all nonzero hereditary subalgebras $B \subseteq A \otimes \mathbb{K}$. Then $\text{dr}(A) \leq 1$.*

Proof. By Theorem 2.7, either A is stably isomorphic to a unital C*-algebra B, or A is stably isomorphic to a stably projectionless C*-algebra A_0 with $Q\widetilde{T}(A_0) = \widetilde{T}_b(A_0) \neq 0$ and $T(A_0)$ compact.

In the first case, B is simple, separable, nuclear and $\mathcal{Z}$-stable by Proposition 2.3. Moreover, B is finite and $T(B) = T_{QD}(B)$ by our additional hypotheses on A. Hence $\text{dr}(B) \leq 1$ by [Castillejos et al. 2019, Theorem B]. By Proposition 2.3 once more, $\text{dr}(A) \leq 1$.

In the second case, we have that $T(A_0) = T_{QD}(A_0)$ by our additional hypotheses on A, so in the proof of Theorem 6.1 the maps θ_n from Lemma 3.2 can be taken to be approximately multiplicative. Therefore, $\text{dr}(A_0) \leq 1$ by [Bosa et al. 2019b, Lemma 1.9]. Hence, $\text{dr}(A) \leq 1$ by Proposition 2.3. □

Remark 7.3. If A is a simple, separable, nuclear C*-algebra in the UCT class, then $T(B) = T(B)_{QD}$ for all hereditary subalgebras $B \subseteq A \otimes \mathbb{K}$ by [Tikuisis et al. 2017, Theorem A] since the UCT class is closed under stable isomorphism.

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[11]When A is separable, one can work with sequences instead of general nets.

As with nuclear dimension, we obtain a trichotomy result for decomposition rank as a corollary of Theorem 7.2.

Corollary 7.4. *The decomposition rank of a simple C*-algebra is* 0, 1 *or* ∞.

Proof. Elementary C*-algebras have decomposition rank zero and so are covered by this result.

Let A be a nonelementary, simple, separable C*-algebra with finite decomposition rank. Then A has finite nuclear dimension, and so is $\mathcal{Z}$-stable by [Tikuisis 2014, Corollary 8.6]. Since $\mathrm{dr}(A) < \infty$, A is stably finite and $T(A) = T_{QD}(A)$.[12] Moreover, by Corollary 2.2 and Proposition 2.3, $\mathrm{dr}(B) = \mathrm{dr}(A) < \infty$ for any nonzero hereditary subalgebra $B \subseteq A \otimes \mathbb{K}$. Therefore, we have $T(B) = T_{QD}(B)$. Now, $\mathrm{dr}(A) \leq 1$ by Theorem 7.2.

The nonseparable case follows from the separable case as in the proof of [Castillejos et al. 2019, Corollary C] since the proof of [Winter and Zacharias 2010, Proposition 2.6] works equally well for decomposition rank. $\qquad\square$

Appendix: Nonunital lemmas

The purpose of this appendix is to state appropriate nonunital versions of the technical lemmas from [Bosa et al. 2019a]. In cases where substantial modifications to the proof are required, we give full details. In cases where the modifications are trivial, we refer the reader to the proof of the corresponding result from [loc. cit.] and explain the modifications in a remark.

We begin with the existence of supporting order-zero maps.

Lemma A.1 (cf. [Bosa et al. 2019a, Lemma 1.14]). *Let* A, B_n *be* C*-algebras with A separable and unital, set $B_\omega := \prod_\omega B_n$, and suppose that $S \subseteq B_\omega$ is separable and self-adjoint. Let $\phi : A \to B_\omega \cap S'$ be a c.p.c. order-zero map. Then there exists a c.p.c. order-zero map $\hat{\phi} : A \to B_\omega \cap S'$ such that

$$\phi(ab) = \hat{\phi}(a)\phi(b) = \phi(a)\hat{\phi}(b), \quad a, b \in A. \tag{A-1}$$

Suppose now that $T(B_n)$ *is nonempty for all* $n \in \mathbb{N}$. *If the map* $\tau \mapsto d_\tau(\phi(1_A))$ *from* $\overline{T_\omega(B_\omega)}^{w*}$ *to* $[0, 1] \subseteq \mathbb{R}$ *is continuous* (*with respect to the weak*-topology*) *then we can, in addition, arrange that*

$$\tau(\hat{\phi}(a)) = \lim_{m \to \infty} \tau(\phi^{1/m}(a)), \quad a \in A_+, \ \tau \in T_\omega(B_\omega), \tag{A-2}$$

where order-zero map functional calculus is used to interpret $\phi^{1/m}$. *In this case, the induced map* $\overline{\hat{\phi}} : A \to B^\omega$ *is a *-homomorphism.*

Remarks. The proof of [Bosa et al. 2019a, Lemma 1.14] only actually requires continuity of $\tau \mapsto d_\tau(\phi(1_A))$ on $\overline{T_\omega(B_\omega)}^{w*}$ (as opposed to $T(B_\omega)$) and $\overline{T_\omega(B_\omega)}^{w*}$ is compact in the nonunital case too. There is no further use of the unitality of the B_n in the proof of [loc. cit., Lemma 1.14]. $\qquad\square$

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[12]One can reduce to the unital case because $\mathrm{dr}(A) = \mathrm{dr}(A^\sim)$ [Kirchberg and Winter 2004, Proposition 3.4]. Then $T(A) = T_{QD}(A)$ by [Bosa et al. 2019a, Proposition 8.5]. Stably finiteness of A follows from [Kirchberg and Winter 2004, Proposition 5.1] and [Brown and Ozawa 2008, Theorem 7.1.15] for example.

We now record some more straightforward applications of the Kirchberg's epsilon test. These results are almost identical to those proven in [Bosa et al. 2019a, Section 1]. However, we shall need slightly more general statements because we wish to apply them to the algebras of the form $B_\omega \cap S' \cap \{1_{B_\omega^\sim} - d\}^\perp$.

Lemma A.2 (cf. [Bosa et al. 2019a, Lemma 1.16]). *Let $(B_n)_{n=1}^\infty$ be a sequence of C*-algebras and set $B_\omega := \prod_\omega B_n$. Let S_1, S_2 be separable self-adjoint subsets of $B_\omega^\sim$, and let T be a separable subset of $B_\omega \cap S_1' \cap S_2^\perp$. Then there exists a contraction $e \in (B_\omega \cap S_1' \cap S_2^\perp)_+$ that acts as a unit on T, i.e., such that $et = te = t$ for every $t \in T$.*

Remarks. The only change to the statement is that S_1, S_2 are subsets of $B_\omega^\sim$ (as opposed to B_ω). The proof is not affected. □

Lemma A.3 (cf. [Bosa et al. 2019a, Lemma 1.17]). *Let $(B_n)_{n=1}^\infty$ be a sequence of C*-algebras and set $B_\omega := \prod_\omega B_n$. Let S_1, S_2 be separable self-adjoint subsets of $B_\omega^\sim$, and set $C := B_\omega \cap S_1' \cap S_2^\perp$.*

(i) *Let $h_1, h_2 \in C_+$. Then h_1 and h_2 are unitarily equivalent via a unitary from $C^\sim$ if and only if they are approximately unitarily equivalent; i.e., for any $\epsilon > 0$ there exists a unitary $u \in C^\sim$ with $uh_1u^* \approx_\epsilon h_2$.*

(ii) *Let $a \in C$. Then there exists a unitary $u \in C^\sim$ with $a = u|a|$ if and only if for each $\epsilon > 0$ there exists a unitary $u \in C^\sim$ with $a \approx_\epsilon u|a|$.*

(iii) *Let $h_1, h_2 \in C_+$. Then h_1 and h_2 are Murray–von Neumann equivalent if and only if they are approximately Murray–von Neumann equivalent; i.e., for any $\epsilon > 0$ there exists $x \in C$ with $xx^* \approx_\epsilon h_1$ and $x^*x \approx_\epsilon h_2$.*

Remarks. The statement of [Bosa et al. 2019a, Lemma 1.17] uses the convention that $C^\sim := C$ when C is already unital. In this paper, we use the convention that a new unit is still adjoined, so $C^\sim \cong C \oplus \mathbb{C}$ when C is unital. The choice of convention does not affect the validity of the lemma.[13] Apart from this, the only change to the statement is that S_1, S_2 are subsets of $B_\omega^\sim$ (as opposed to B_ω), which does not affect the proof. □

Lemma A.4 (cf. [Bosa et al. 2019a, Lemma 1.18]). *Let $(B_n)_{n=1}^\infty$ be a sequence of C*-algebras with $T(B_n)$ nonempty for each $n \in \mathbb{N}$. Write $B_\omega := \prod_\omega B_n$. Let S_0 be a countable self-adjoint subset of $(B_\omega)_+$ and let T be a separable self-adjoint subset of B_ω. If x, $f \in (B_\omega \cap S_0' \cap T')_+$ are contractions with $x \lhd f$ and with the property that for all $a \in S_0$ there exists $\gamma_a \geq 0$ such that $\tau(af^m) \geq \gamma_a$ for all $m \in \mathbb{N}$, $\tau \in T_\omega(B_\omega)$, then there exists a contraction $f' \in (B_\omega \cap S_0' \cap T')_+$ such that $x \lhd f' \lhd f$ and $\tau(a(f')^m) \geq \gamma_a$ for all $m \in \mathbb{N}$, $\tau \in T_\omega(B_\omega)$, and $a \in S_0$.*

If each B_n is simple, separable, $\mathcal{Z}$-stable and $Q\widetilde{T}(B_n) = \widetilde{T}_b(B_n) \neq 0$ for all $n \in \mathbb{N}$, then the above statement holds with $T(B_\omega)$ in place of $T_\omega(B_\omega)$.

Remarks. The only change to the proof of [Bosa et al. 2019a, Lemma 1.18] is to replace $\min_{\tau \in T(B_n)}$ with $\inf_{\tau \in T(B_n)}$ in [loc. cit., equation (1.34)], as the minimum need not exist in the nonunital case. The final sentence follows since $T_\omega(B_\omega)$ is weak*-dense in $T_\omega(B_\omega)$ under the additional hypotheses by Theorem 1.7. □

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[13]For example, if C is unital, h_1, h_2 are unitary equivalent in C if and only if they are unitary equivalent in $C \oplus \mathbb{C}$.

Lemma A.5 (cf. [Bosa et al. 2019a, Lemma 1.19]). *Let $(B_n)_{n=1}^{\infty}$ be a sequence of separable C*-algebras with $T(B_n) \neq \varnothing$ for each $n \in \mathbb{N}$ and set $B_{\omega} := \prod_{\omega} B_n$. Let A be a separable, unital C*-algebra and let $\pi : A \to B_{\omega}$ be a c.p.c. order-zero map such that $\pi(1_A)$ is full and the induced map $\bar{\pi} : A \to B^{\omega}$ is a *-homomorphism. Define $C := B_{\omega} \cap \pi(A)' \cap \{1_{B^{\sim}_{\omega}} - \pi(1_A)\}^{\perp}$. Let $S \subseteq C$ be a countable self-adjoint subset and let $\bar{S}$ denote the image of S in B^{ω}:*

(i) *Then the image of $C \cap S'$ in B^{ω} is precisely*

$$\bar{\pi}(1_A)(B^{\omega} \cap \bar{\pi}(A)' \cap \bar{S}') = B^{\omega} \cap \bar{\pi}(A)' \cap \bar{S}' \cap \{1_{(B^{\omega})^{\sim}} - \bar{\pi}(1_A)\}^{\perp}, \tag{A-3}$$

   a C*-subalgebra of B^{ω} with unit $\bar{\pi}(1_A)$.

(ii) *Let $\tau \in T_{\omega}(B_{\omega})$ be a limit trace and $a \in A_+$ and form the tracial functional $\rho := \tau(\pi(a) \cdot)$ on C. Then $\|\rho\| = \tau(\pi(a))$. If each B_n is additionally simple, $\mathcal{Z}$-stable and $Q\widetilde{T}(B_n) = \widetilde{T}_b(B_n) \neq 0$ for all $n \in \mathbb{N}$, then this holds for all traces $\tau \in T(B_{\omega})$.*

Remarks. The proof of [Bosa et al. 2019a, Lemma 1.19] does not need the B_n to be unital. Note that the notation $\{1_{B^{\sim}_{\omega}} - \pi(1_A)\}^{\perp}$ is just an alternative notation for subalgebra on which $\pi(1_A)$ acts as a unit, and similarly for $\{1_{(B^{\omega})^{\sim}} - \bar{\pi}(1_A)\}^{\perp}$. The final sentence of (ii) follows since $T_{\omega}(B_{\omega})$ is weak*-dense in $T_{\omega}(B_{\omega})$ under the additional hypotheses by Theorem 1.7. $\qquad\square$

Next, we consider some properties of ultraproducts of separable, $\mathcal{Z}$-stable C*-algebras.

Lemma A.6 (cf. [Bosa et al. 2019a, Lemma 1.22]). *Let $(B_n)_{n \in \mathbb{N}}$ be a sequence of separable, $\mathcal{Z}$-stable C*-algebras and set $B_{\omega} := \prod_{\omega} B_n$. Then:*

(i) *If $S \subseteq B_{\omega}$ is separable, then there exist isomorphisms $\phi_n : B_n \to B_n \otimes \mathcal{Z}$ such that the induced isomorphism $\Phi : B_{\omega} \to \prod_{\omega}(B_n \otimes \mathcal{Z})$ maps $x \in S$ to $x \otimes 1_{\mathcal{Z}} \in \left(\prod_{\omega} B_n\right) \otimes \mathcal{Z} \subseteq \prod_{\omega}(B_n \otimes \mathcal{Z})$.*

(ii) *Let $S_1, S_2 \subseteq B^{\sim}_{\omega}$ be separable and self-adjoint. For any separable subset $T \subseteq B_{\omega} \cap S_1' \cap S_2^{\perp}$, there exists a c.p.c. order-zero map $\psi : \mathcal{Z} \to B_{\omega} \cap S_1' \cap S_2^{\perp} \cap T'$ such that $\psi(1_{\mathcal{Z}})$ acts as a unit on T.*

(ii') *Let $S_1, S_2 \subseteq B^{\sim}_{\omega}$ be separable and self-adjoint. For any separable subalgebra $C \subseteq B_{\omega} \cap S_1' \cap S_2^{\perp}$, there exists a *-homomorphism $\Psi : C \otimes \mathcal{Z} \to B_{\omega} \cap S_1' \cap S_2^{\perp}$ such that $\Psi(x \otimes 1_{\mathcal{Z}}) = x$ for all $x \in C$.*

(iii) *If each B_n is projectionless, then B_{ω} has stable rank one in $B^{\sim}_{\omega}$.*

(iv) *If $S \subseteq B_{\omega}$ is separable and self-adjoint, and $b \in (B_{\omega} \cap S')_+$, then for any $n \in \mathbb{N}$ there exists $c \in (B_{\omega} \cap S')_+$ with $c \leq b$ such that $n[c] \leq [b] \leq (n+1)[c]$ in $W(B_{\omega} \cap S')$.*

Proof. Observe that (i) is the same as in [Bosa et al. 2019a, Lemma 1.22(i)] and follows as $\mathcal{Z}$ is strongly self-absorbing.

For (ii), by Lemma A.2, there exists a positive contraction $h \in B_{\omega} \cap S_1' \cap S_2^{\perp}$ that acts as a unit on T. Set $S := S_1 \cup S_2 \cup T \cup \{h\}$. Let $\Phi : B_{\omega} \to \prod_{\omega}(B_n \otimes \mathcal{Z})$ be the isomorphism from (i) with $\Phi(x) = x \otimes 1_{\mathcal{Z}}$ for all $x \in S$. Define a c.p.c. order-zero map $\psi' : \mathcal{Z} \to \prod_{\omega}(B_n \otimes \mathcal{Z})$ by $\psi'(z) := h \otimes z$. By the choice of h, $\psi(1_{\mathcal{Z}})$ acts as a unit on $T \otimes 1_{\mathcal{Z}}$ and the image of ψ lies in $(B_{\omega} \otimes 1_{\mathcal{Z}}) \cap (S_1 \otimes 1_{\mathcal{Z}})' \cap (S_2 \otimes 1_{\mathcal{Z}})^{\perp} \cap (T \otimes 1_{\mathcal{Z}})'$. Now set $\psi := \Phi^{-1} \circ \psi'$.

For (ii′), by part (ii), there exists a c.p.c. order-zero map $\psi : \mathcal{Z} \to B_\omega \cap S'_1 \cap S_2^\perp \cap T'$ such that $\psi(1_{\mathcal{Z}})$ acts as a unit on C. Since $\mathcal{Z}$ is nuclear, we can define a c.p.c. order-zero map $\Psi : C \otimes \mathcal{Z} \to B_\omega \cap S'_1 \cap S_2^\perp$ by $x \otimes z \mapsto x\psi(z)$. Let $x_1, x_2 \in C$ and $z_1, z_2 \in \mathcal{Z}$. Then

$$
\begin{aligned}
\Psi(x_1 \otimes z_1)\Psi(x_1 \otimes z_1) &= x_1\psi(z_1)x_2\psi(z_2) \\
&= x_1 x_2 \psi(1_{\mathcal{Z}})\psi(z_1 z_2) \\
&= x_1 x_2 \psi(z_1 z_2) \\
&= \Psi(x_1 x_2 \otimes z_1 z_2),
\end{aligned} \tag{A-4}
$$

where we have used the order-zero identity in the second line. Hence, Ψ is in fact a *-homomorphism. Moreover, we have $\Psi(x \otimes 1_{\mathcal{Z}}) = x\psi(1_{\mathcal{Z}}) = x$ for all $x \in C$.

Part (iii) follows by combining Theorem 1.10 with Proposition 1.12.

For (iv), let C be the C*-algebra generated by b. By (ii′), there is a *-homomorphism $\Psi : C \otimes \mathcal{Z} \to B_\omega \cap S'$. By [Rørdam 2004, Lemma 4.2], there exists $e_n \in \mathcal{Z}_{+,1}$ with $n[e_n] \leq [1_{\mathcal{Z}}] \leq (n+1)[e_n]$. Set $c := \Psi(b \otimes e_n)$. $\square$

The following lemmas are crucial to the results of [Bosa et al. 2019a, Section 5]. The proof of the first needs to be adapted slightly to the nonunital setting.

Lemma A.7 (cf. [Bosa et al. 2019a, Lemma 2.1]). *Let $(B_n)_{n=1}^\infty$ be a sequence of $\mathcal{Z}$-stable C*-algebras and set $B_\omega := \prod_\omega B_n$. Let $S \subseteq B_\omega$ be separable and self-adjoint, and let $d \in (B_\omega \cap S')_+$ be a contraction. Suppose that $x, f \in C := B_\omega \cap S' \cap \{1_{B_\omega^\sim} - d\}^\perp$ are such that $xf = fx = 0$, $f \geq 0$ and f is full in C. Then x is approximated by invertibles in $C^\sim$.*

Proof. By Lemma A.2 (with $T := \{x^*x, xx^*\}$, $S_1 := S$, $S_2 := \{1_{B_\omega^\sim} - d, f\}$), we obtain a contraction $e \in C_+$ such that $xx^*, x^*x \lhd e$ and $ef = 0$. Polar decomposition yields $ex = xe = x$. As in the proof of [Bosa et al. 2019a, Lemma 2.1], we may find a separable subalgebra C_0 of C containing x, e, and f such that f is full in C_0. By Lemma A.6(ii′), there is a *-homomorphism $\Psi : C_0 \otimes \mathcal{Z} \to C$ such that $\Psi(x \otimes 1_{\mathcal{Z}}) = x$ for all $x \in C_0$. By [Robert 2016, Lemma 2.1], $x \otimes 1_{\mathcal{Z}}$ is a product of two nilpotent elements $n_1, n_2 \in C_0 \otimes \mathcal{Z}$. It follows that $x = \Psi(x \otimes 1_{\mathcal{Z}}) = \Psi(n_1)\Psi(n_2)$ is the product of two nilpotent elements in C. If $y \in C$ is nilpotent and $\epsilon > 0$, the operator $y + \epsilon 1_{C^\sim}$ is invertible in $C^\sim$ (with inverse $-\sum_{k=1}^N (-\epsilon)^{-k} y^{k-1}$, where $N \in \mathbb{N}$ satisfies $y^N = 0$). Therefore, x can be approximated by invertible elements in $C^\sim$. $\square$

Lemma A.8 (cf. [Bosa et al. 2019a, Lemma 2.2]). *Let $(B_n)_{n=1}^\infty$ be a sequence of $\mathcal{Z}$-stable C*-algebras and set $B_\omega := \prod_\omega B_n$. Let $S \subseteq B_\omega$ be separable and self-adjoint, and let $d \in (B_\omega \cap S')_+$ be a contraction. Suppose that $x, s \in C := B_\omega \cap S' \cap \{1_{B_\omega^\sim} - d\}^\perp$ are such that $xs = sx = 0$ and s is full in C. Then x is approximated by invertibles in $C^\sim$.*

Proof. Let C_0 be the C*-subalgebra of C generated by x and s. By Lemma A.6(ii′), there exists a *-homomorphism $\Psi : C_0 \otimes \mathcal{Z} \to C$ with $\Psi(y \otimes 1_{\mathcal{Z}}) = y$ for all $y \in C_0$. Let $z_1, z_2 \in \mathcal{Z}_+$ be nonzero orthogonal elements. Set $s' := \Psi(s \otimes z_1)$ and $f := \Psi(|s| \otimes z_2)$. As in the proof of [Bosa et al. 2019a, Lemma 2.2], it follows by Lemma A.7 (with s' in place of x) that s' is approximated by invertibles

in $C^\sim$. We finish the proof exactly as in the proof of [loc. cit., Lemma 2.2], where we replace [loc. cit., Lemma 1.17] with Lemma A.3, and [loc. cit., Lemma 2.1] with Lemma A.7. □

Lemma A.9 (cf. [Bosa et al. 2019a, Lemma 5.4] and [Robert and Santiago 2010, Lemma 2]). *Let B be a separable, $\mathcal{Z}$-stable C^*-algebra and let A be a separable, unital C^*-algebra. Let $\pi : A \to B_\omega$ be a c.p.c. order-zero map such that*

$$C := B_\omega \cap \pi(A)' \cap \{1_{B_{\tilde{\omega}}} - \pi(1_A)\}^\perp \tag{A-5}$$

is full in B_ω. Assume that every full hereditary subalgebra D of C satisfies the following: if $x \in D$ is such that there exist totally full elements $e_l, e_r \in D_+$ such that $e_l x = x e_r = 0$, then there exists a full element $s \in D$ such that $sx = xs = 0$. Let $e, f, f', \alpha, \beta \in C_+$ be such that

$$\alpha \lhd e, \quad \alpha \sim \beta \lhd f, \quad and \quad f \sim f' \lhd e. \tag{A-6}$$

Suppose also that there exist $d_e, d_f \in C_+$ that are totally full such that

$$\begin{aligned} d_e \lhd e, \quad d_e \alpha = 0, \\ d_f \lhd f, \quad d_f \beta = 0. \end{aligned} \tag{A-7}$$

Then there exists $e' \in C_+$ such that

$$\alpha \lhd e' \lhd e \quad and \quad \alpha + e' \sim \beta + f. \tag{A-8}$$

Remarks. The proof of [Robert and Santiago 2010, Lemma 2] does not assume unitality. The proof from [Bosa et al. 2019a] is still valid after replacing [loc. cit., Lemma 1.17] with Lemma A.3, [loc. cit., Lemma 2.2] with Lemma A.8, and using $1_{B_{\tilde{\omega}}}$ in place of 1_{B_ω}. □

The following lemma concerns the interplay between strict comparison and ultraproducts in the nonunital setting.

Lemma A.10 (cf. [Bosa et al. 2019a, Lemma 1.23]). *Let $(B_n)_{n=1}^\infty$ be a sequence of C^*-algebras with $T(B_n)$ nonempty and set $B_\omega := \prod_\omega B_n$. Suppose each B_n has strict comparison of positive elements with respect to bounded traces. Then B_ω has strict comparison of positive elements with respect to limit traces, in the following sense: if $a, b \in M_k(B_\omega)_+$ for some $k \in \mathbb{N}$ satisfy $d_\tau(a) < d_\tau(b)$ for all τ in the weak*-closure of $T_\omega(B_\omega)$, then $a \preceq b$.*

Remarks. The proof is identical to that of [Bosa et al. 2019a, Lemma 1.23]. However, it is important to note that, in the nonunital case, we do not necessarily have that $\overline{T_\omega(B_\omega)}^{w*} \subseteq T(B_\omega)$, as the later need not be closed. Indeed, we may have $0 \in \overline{T_\omega(B_\omega)}^{w*}$ in which case $d_\tau(a) < d_\tau(b)$ cannot hold for all $\tau \in \overline{T_\omega(B_\omega)}^{w*}$. □

Finally, we record a technical lemma needed for the proof of the main theorem of the property (SI) section.

Lemma A.11 (cf. [Bosa et al. 2019a, Lemma 4.9]). *Let B be a simple, separable, C^*-algebra with $Q\widetilde{T}(B) = \widetilde{T}_b(B) \neq 0$. Suppose B has strict comparison of positive elements by bounded traces. Let A be a separable, unital C^*-algebra and let $\pi : A \to B_\omega$ be a c.p.c. order-zero map. Let $a \in A_+$ be a positive*

contraction of norm 1. *Then there exists a countable set* $S \subseteq A_+ \setminus \{0\}$ *such that the following holds*: *if* $e, t, h \in (B_\omega \cap \pi(A)' \cap \{1_{B_{\tilde{\omega}}} - \pi(1_A)\}^\perp)_+$ *are contractions such that*

$$e \in J_{B_\omega} \quad and \quad h \lhd t, \tag{A-9}$$

and if for all $b \in S$, *there exists* $\gamma_b > 0$ *such that*

$$\tau(\pi(b)h) > \gamma_b, \quad \tau \in T_\omega(B_\omega), \tag{A-10}$$

then there exists a contraction $r \in B_\omega$ *such that*

$$\pi(a)r = tr = r \quad and \quad r^*r = e. \tag{A-11}$$

Remarks. The proof of [Bosa et al. 2019a, Lemma 4.9] works in our situation using Lemma A.10 in place of [loc. cit., Lemma 1.23]. □

Acknowledgements

References

[Alfsen 1971] E. M. Alfsen, *Compact convex sets and boundary integrals*, Ergebnisse der Mathematik (3) **57**, Springer, 1971. MR Zbl

[Ara et al. 2011] P. Ara, F. Perera, and A. S. Toms, "*K*-theory for operator algebras: classification of *C**-algebras", pp. 1–71 in *Aspects of operator algebras and applications*, edited by P. Ara et al., Contemp. Math. **534**, Amer. Math. Soc., Providence, RI, 2011. MR Zbl

[Blackadar 1980] B. E. Blackadar, "A simple *C**-algebra with no nontrivial projections", *Proc. Amer. Math. Soc.* **78**:4 (1980), 504–508. MR Zbl

[Blackadar 2006] B. Blackadar, *Operator algebras: theory of C**-algebras and von Neumann algebras, operator algebras and non-commutative geometry, III*, Encycl. Math. Sci. **122**, Springer, 2006. MR Zbl

[Blackadar and Cuntz 1982] B. E. Blackadar and J. Cuntz, "The structure of stable algebraically simple *C**-algebras", *Amer. J. Math.* **104**:4 (1982), 813–822. MR Zbl

[Blackadar and Handelman 1982] B. Blackadar and D. Handelman, "Dimension functions and traces on *C**-algebras", *J. Funct. Anal.* **45**:3 (1982), 297–340. MR Zbl

[Blanchard and Kirchberg 2004] E. Blanchard and E. Kirchberg, "Non-simple purely infinite *C**-algebras: the Hausdorff case", *J. Funct. Anal.* **207**:2 (2004), 461–513. MR Zbl

[Bosa et al. 2019a] J. Bosa, N. P. Brown, Y. Sato, A. Tikuisis, S. White, and W. Winter, *Covering dimension of C**-algebras and 2-coloured classification*, Mem. Amer. Math. Soc. **1233**, Amer. Math. Soc., Providence, RI, 2019. MR Zbl

[Bosa et al. 2019b] J. Bosa, J. Gabe, A. Sims, and S. White, "The nuclear dimension of $\mathcal{O}_\infty$-stable *C**-algebras", preprint, 2019. arXiv

[Brown 1977] L. G. Brown, "Stable isomorphism of hereditary subalgebras of *C**-algebras", *Pacific J. Math.* **71**:2 (1977), 335–348. MR Zbl

[Brown 2006] N. P. Brown, *Invariant means and finite representation theory of C^*-algebras*, Mem. Amer. Math. Soc. **865**, 2006. MR Zbl

[Brown and Ozawa 2008] N. P. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Grad. Studies in Math. **88**, Amer. Math. Soc., Providence, RI, 2008. MR Zbl

[Brown et al. 2017] N. P. Brown, J. R. Carrión, and S. White, "Decomposable approximations revisited", pp. 45–65 in *Operator algebras and applications*, edited by T. M. Carlsen et al., Abel Symp. **12**, Springer, 2017. MR Zbl

[Castillejos 2017] J. Castillejos, "Decomposable approximations and approximately finite dimensional C^*-algebras", *Math. Proc. Cambridge Philos. Soc.* **162**:1 (2017), 1–12. MR Zbl

[Castillejos et al. 2019] J. Castillejos, S. Evington, A. Tikuisis, S. White, and W. Winter, "Nuclear dimension of simple C^*-algebras", preprint, 2019. To appear in *Invent. Math.* arXiv

[Conley et al. 2018] C. T. Conley, S. C. Jackson, D. Kerr, A. S. Marks, B. Seward, and R. D. Tucker-Drob, "Følner tilings for actions of amenable groups", *Math. Ann.* **371**:1-2 (2018), 663–683. MR Zbl

[Coward et al. 2008] K. T. Coward, G. A. Elliott, and C. Ivanescu, "The Cuntz semigroup as an invariant for C^*-algebras", *J. Reine Angew. Math.* **623** (2008), 161–193. MR Zbl

[Cuntz and Pedersen 1979] J. Cuntz and G. K. Pedersen, "Equivalence and traces on C^*-algebras", *J. Funct. Anal.* **33**:2 (1979), 135–164. MR Zbl

[Elliott 1996] G. A. Elliott, "An invariant for simple C^*-algebras", pp. 61–90 in *Canadian Mathematical Society: 1945–1995, III*, edited by J. B. Carrell and R. Murty, Canadian Math. Soc., Ottawa, 1996. MR Zbl

[Elliott et al. 2011] G. A. Elliott, L. Robert, and L. Santiago, "The cone of lower semicontinuous traces on a C^*-algebra", *Amer. J. Math.* **133**:4 (2011), 969–1005. MR Zbl

[Elliott et al. 2015] G. A. Elliott, G. Gong, H. Lin, and Z. Niu, "On the classification of simple amenable C^*-algebras with finite decomposition rank, II", preprint, 2015. arXiv

[Elliott et al. 2017] G. A. Elliott, G. Gong, H. Lin, and Z. Niu, "The classification of simple separable KK-contractible C^*-algebras with finite nuclear dimension", preprint, 2017. arXiv

[Elliott et al. 2020] G. A. Elliott, G. Gong, H. Lin, and Z. Niu, "Simple stably projectionless C*-algebras with generalized tracial rank one", *J. Noncommut. Geom.* **14**:1 (2020), 251–347. MR

[Farah and Katsura 2010] I. Farah and T. Katsura, "Nonseparable UHF algebras, I: Dixmier's problem", *Adv. Math.* **225**:3 (2010), 1399–1430. MR Zbl

[Giol and Kerr 2010] J. Giol and D. Kerr, "Subshifts and perforation", *J. Reine Angew. Math.* **639** (2010), 107–119. MR Zbl

[Gong and Lin 2016] G. Gong and H. Lin, "On classification of non-unital simple amenable C^*-algebras, I", preprint, 2016. arXiv

[Gong and Lin 2017] G. Gong and H. Lin, "On classification of simple non-unital amenable C^*-algebras, II", preprint, 2017. arXiv

[Gong et al. 2015] G. Gong, H. Lin, and Z. Niu, "Classification of finite simple amenable $\mathscr{Z}$-stable C^*-algebras", preprint, 2015. arXiv

[Haagerup 2014] U. Haagerup, "Quasitraces on exact C^*-algebras are traces", *C. R. Math. Acad. Sci. Soc. R. Can.* **36**:2-3 (2014), 67–92. MR Zbl

[Jacelon 2013] B. Jacelon, "A simple, monotracial, stably projectionless C^*-algebra", *J. Lond. Math. Soc.* (2) **87**:2 (2013), 365–383. MR Zbl

[Jiang and Su 1999] X. Jiang and H. Su, "On a simple unital projectionless C^*-algebra", *Amer. J. Math.* **121**:2 (1999), 359–413. MR Zbl

[Kerr and Szabó 2020] D. Kerr and G. Szabó, "Almost finiteness and the small boundary property", *Comm. Math. Phys.* **374**:1 (2020), 1–31. MR

[Kirchberg 1995] E. Kirchberg, "Exact C^*-algebras, tensor products, and the classification of purely infinite algebras", pp. 943–954 in *Proceedings of the International Congress of Mathematicians, II* (Zürich, 1994), edited by S. D. Chatterji, Birkhäuser, Basel, 1995. MR Zbl

[Kirchberg 2006] E. Kirchberg, "Central sequences in *C*-*algebras and strongly purely infinite algebras", pp. 175–231 in *Operator algebras: the Abel symposium* (Oslo, 2004), edited by S. Neshveyev and C. Skau, Abel Symp. **1**, Springer, 2006. MR Zbl

[Kirchberg and Rørdam 2000] E. Kirchberg and M. Rørdam, "Non-simple purely infinite *C*-*algebras", *Amer. J. Math.* **122**:3 (2000), 637–666. MR Zbl

[Kirchberg and Winter 2004] E. Kirchberg and W. Winter, "Covering dimension and quasidiagonality", *Int. J. Math.* **15**:1 (2004), 63–85. MR Zbl

[Kishimoto and Kumjian 1996] A. Kishimoto and A. Kumjian, "Simple stably projectionless *C*-*algebras arising as crossed products", *Canad. J. Math.* **48**:5 (1996), 980–996. MR Zbl

[Matui and Sato 2012] H. Matui and Y. Sato, "𝒵-stability of crossed products by strongly outer actions", *Comm. Math. Phys.* **314**:1 (2012), 193–228. MR Zbl

[McDuff 1970] D. McDuff, "Central sequences and the hyperfinite factor", *Proc. Lond. Math. Soc.* (3) **21** (1970), 443–461. MR Zbl

[Murray and von Neumann 1943] F. J. Murray and J. von Neumann, "On rings of operators, IV", *Ann. of Math.* (2) **44** (1943), 716–808. MR Zbl

[Ng and Robert 2016] P. W. Ng and L. Robert, "Sums of commutators in pure *C*-*algebras", *Münster J. Math.* **9**:1 (2016), 121–154. MR Zbl

[Ozawa 2013] N. Ozawa, "Dixmier approximation and symmetric amenability for *C*-*algebras", *J. Math. Sci. Univ. Tokyo* **20**:3 (2013), 349–374. MR Zbl

[Pedersen 1969] G. K. Pedersen, "Measure theory for *C** algebras, III", *Math. Scand.* **25** (1969), 71–93. Zbl

[Pedersen 1979] G. K. Pedersen, *C*-algebras and their automorphism groups*, Lond. Math. Soc. Monogr. **14**, Academic Press, London, 1979. MR Zbl

[Phillips 2000] N. C. Phillips, "A classification theorem for nuclear purely infinite simple *C*-*algebras", *Doc. Math.* **5** (2000), 49–114. MR Zbl

[Razak 2002] S. Razak, "On the classification of simple stably projectionless *C*-*algebras", *Canad. J. Math.* **54**:1 (2002), 138–224. MR Zbl

[Robert 2016] L. Robert, "Remarks on 𝒵-stable projectionless *C*-*algebras", *Glasg. Math. J.* **58**:2 (2016), 273–277. MR Zbl

[Robert and Santiago 2010] L. Robert and L. Santiago, "Classification of *C*-*homomorphisms from $C_0(0, 1]$ to a *C*-*algebra", *J. Funct. Anal.* **258**:3 (2010), 869–892. MR Zbl

[Rørdam 2003] M. Rørdam, "A simple *C*-*algebra with a finite and an infinite projection", *Acta Math.* **191**:1 (2003), 109–142. MR Zbl

[Rørdam 2004] M. Rørdam, "The stable and the real rank of 𝒵-absorbing *C*-*algebras", *Int. J. Math.* **15**:10 (2004), 1065–1084. MR Zbl

[Rosenberg and Schochet 1987] J. Rosenberg and C. Schochet, "The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor", *Duke Math. J.* **55**:2 (1987), 431–474. MR Zbl

[Tikuisis 2014] A. Tikuisis, "Nuclear dimension, 𝒵-stability, and algebraic simplicity for stably projectionless *C*-*algebras", *Math. Ann.* **358**:3-4 (2014), 729–778. MR Zbl

[Tikuisis and Toms 2015] A. P. Tikuisis and A. Toms, "On the structure of Cuntz semigroups in (possibly) nonunital *C*-*algebras", *Canad. Math. Bull.* **58**:2 (2015), 402–414. MR Zbl

[Tikuisis and Winter 2014] A. Tikuisis and W. Winter, "Decomposition rank of 𝒵-stable *C*-*algebras", *Anal. PDE* **7**:3 (2014), 673–700. MR Zbl

[Tikuisis et al. 2017] A. Tikuisis, S. White, and W. Winter, "Quasidiagonality of nuclear *C*-*algebras", *Ann. of Math.* (2) **185**:1 (2017), 229–284. MR Zbl

[Toms 2008] A. S. Toms, "On the classification problem for nuclear *C*-*algebras", *Ann. of Math.* (2) **167**:3 (2008), 1029–1044. MR Zbl

[Toms and Winter 2007] A. S. Toms and W. Winter, "Strongly self-absorbing *C*-*algebras", *Trans. Amer. Math. Soc.* **359**:8 (2007), 3999–4029. MR Zbl

[Toms and Winter 2009] A. S. Toms and W. Winter, "The Elliott conjecture for Villadsen algebras of the first type", *J. Funct. Anal.* **256**:5 (2009), 1311–1340. MR Zbl

[Tsang 2005] K.-W. Tsang, "On the positive tracial cones of simple stably projectionless C^*-algebras", *J. Funct. Anal.* **227**:1 (2005), 188–199. MR Zbl

[Villadsen 1999] J. Villadsen, "On the stable rank of simple C^*-algebras", *J. Amer. Math. Soc.* **12**:4 (1999), 1091–1102. MR Zbl

[Winter 2009] W. Winter, "Covering dimension for nuclear C^*-algebras, II", *Trans. Amer. Math. Soc.* **361**:8 (2009), 4143–4167. MR Zbl

[Winter 2012] W. Winter, "Nuclear dimension and $\mathscr{Z}$-stability of pure C^*-algebras", *Invent. Math.* **187**:2 (2012), 259–342. MR Zbl

[Winter 2014] W. Winter, "Localizing the Elliott conjecture at strongly self-absorbing C^*-algebras", *J. Reine Angew. Math.* **692** (2014), 193–231. MR Zbl

[Winter and Zacharias 2009] W. Winter and J. Zacharias, "Completely positive maps of order zero", *Münster J. Math.* **2** (2009), 311–324. MR Zbl

[Winter and Zacharias 2010] W. Winter and J. Zacharias, "The nuclear dimension of C^*-algebras", *Adv. Math.* **224**:2 (2010), 461–498. MR Zbl

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