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ON THE REGULARITY OF MINIMIZERS FOR SCALAR INTEGRAL FUNCTIONALS WITH (p, q)-GROWTH

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We revisit the question of regularity for minimizers of scalar autonomous integral functionals with so-called (p, q)-growth. In particular, we establish Lipschitz regularity under the condition $\frac{q}{p} < 1 + \frac{2}{n-1}$ for $n \geq 3$, improving a classical result due to Marcellini (*J. Differential Equations* **90**:1 (1991), 1–30).

1. Introduction and main results

In this note, we consider the problem of regularity for local minimizers of

$$\mathcal{F}[u] := \int_{\Omega} f(\nabla u)\, dx, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain and $f : \mathbb{R}^n \to \mathbb{R}$ is a sufficiently smooth integrand satisfying (p, q)-growth of the form:

Assumption 1. There exist $0 < m \leq M < \infty$ such that $f \in C^2(\mathbb{R}^n)$ satisfies for all $z, \lambda \in \mathbb{R}^n$

$$\begin{cases} m|z|^p \leq f(z) \leq M(1 + |z|^q), \\ m(1 + |z|^2)^{\frac{p-2}{2}} |\lambda|^2 \leq \langle D^2 f(z)\lambda, \lambda\rangle \leq M(1 + |z|^2)^{\frac{q-2}{2}} |\lambda|^2. \end{cases} \tag{2}$$

Regularity properties of local minimizers of (1) in the case $p = q$ are classical; see, e.g., [Giusti 2003]. A systematic regularity theory in the case $p < q$ was initiated in [Marcellini 1989; 1991]. In particular, Marcellini [1991] proved:

(A) If $2 \leq p < q$ and $\frac{q}{p} < 1 + \frac{2}{n-2}$ if $n \geq 3$, then every local minimizer $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ of (1) satisfies $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.

(B) If $2 \leq p < q$ and $\frac{q}{p} < 1 + \frac{2}{n}$, then every local minimizer $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of (1) satisfies $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.

We emphasize that establishing Lipschitz-regularity is the crucial point in the regularity theory for functionals with (p, q)-growth in the form (2). Indeed, local boundedness of the gradient implies that the nonstandard growth of f and $D^2 f$ becomes irrelevant and higher regularity (depending on the smoothness of f) follows by standard arguments; see, e.g., [Marcellini 1989, Chapter 7] and Corollary 7 below.

By now there is a large and quickly growing literature on regularity results for minimizers of functionals with (p, q)-growth and more general nonstandard growth; we refer to [Mingione 2006] for an overview.

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Under additional structural assumptions on the growth of f, for example anisotropic growth of the form

$$m\sum_{i=1}^{n}|z_i|^{p_i} \le f(z) \le M\sum_{i=1}^{n}(1+|z_i|^q),$$

more precise and sharp assumptions on the involved exponents that ensure higher regularity are available in the literature; see, e.g., [Cupini et al. 2015; Fusco and Sbordone 1993]. Regularity results under general structural assumptions beyond polynomial growth can be found, e.g., in [Lieberman 1991; Marcellini 1993]; see also the recent result [Eleuteri et al. 2020], where convexity is only imposed "at infinity". Moreover, rather sharp conditions are known for certain nonautonomous functionals; see, e.g., [Baroni et al. 2018; Colombo and Mingione 2015; De Filippis and Mingione 2020; Esposito et al. 2004], where also Hölder-continuity of the integrand f in the space variable has to be balanced with p, q, and n. In [Carozza et al. 2014; Esposito et al. 1999] higher integrability results for autonomous integral functionals can be found that are also valid in the case of systems.

To the best of our knowledge, there is no improvement of the results (A) and (B) with respect to the relation between the exponents p, q and the dimension n available in the literature (without any additional structure assumption or further a priori assumptions on the minimizer, e.g., boundedness as in [Bousquet and Brasco 2020; Carozza et al. 2011]). In the present paper, we give such an improvement in the case $n \ge 3$. Before we state the results, we recall a standard notion of local minimizer in the context of functionals with (p, q)-growth.

Definition 2. We call $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ a local minimizer of $\mathcal{F}$ given in (1) if and only if

$$f(\nabla u) \in L^1_{\mathrm{loc}}(\Omega)$$

and

$$\int_{\operatorname{supp}\varphi} f(\nabla u)\,dx \le \int_{\operatorname{supp}\varphi} f(\nabla u + \nabla\varphi)\,dx$$

for any $\varphi \in W^{1,1}(\Omega)$ satisfying $\operatorname{supp}\varphi \Subset \Omega$.

The main results of the present paper can be summarized as:

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, and suppose Assumption 1 is satisfied with $2 \le p \le q < \infty$ such that*

$$\frac{q}{p} < 1 + \frac{2}{n-3} \quad \text{if } n \ge 4. \tag{3}$$

Let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then, $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$ and suppose Assumption 1 is satisfied with $2 \le p \le q < \infty$ such that*

$$\frac{q}{p} < 1 + \min\left\{1, \frac{2}{n-1}\right\}. \tag{4}$$

Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then, $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.

Remark 5. Notice that Theorems 3 and 4 improve the results (A) and (B) with respect to the assumptions on $\frac{q}{p}$ in dimensions $n \ge 3$. The results in [Marcellini 1991] apply to more general situations in the sense that (smooth) spatial dependence of f is allowed, a bounded right-hand side is included and nonlinear elliptic equations that not need to be Euler–Lagrange equations of integral functionals of the type (1) are considered. In order to present the new ingredients in the simplest setting we focus on the case of autonomous integral functionals with no right-hand side (as in [Marcellini 1989]). Very recently [Beck and Mingione 2020] sharp criteria for Lipschitz-regularity of minimizers of variational integrals with respect to the right-hand side were obtained under the assumption $\frac{q}{p} < 1 + \frac{2}{n}$. It would be of interest to see whether such results can be extended to the case $\frac{q}{p} < 1 + \frac{2}{n-1}$ if $n \geq 3$.

Remark 6. We do not know whether assumptions (3) and (4) are respectively optimal in Theorems 3 and 4. It is known that Lipschitz-regularity and even boundedness of minimizers fail if $\frac{q}{p}$ is too large depending on the dimension n. In particular it is known that in order to ensure boundedness it is necessary that $\frac{q}{p} \to 1$ if $n \to \infty$; see [Giaquinta 1987; Hong 1992; Marcellini 1989; 1991] for related counterexamples. In particular, it is shown in [Hong 1992] that the functional

$$\int_\Omega |\nabla u|^2 + |u_{x_n}|^4 \, dx,$$

which satisfies (2) with $p = 2$ and $q = 4$, admits an unbounded minimizer if $n \geq 6$. Clearly, this does not match condition (4) in Theorem 4 and even not condition (3).

As already mentioned, once boundedness of the gradient is established, higher regularity follows by standard arguments; see, e.g., [Marcellini 1989, Proof of Theorem D]. Let us state (without proof) a rather direct consequence of Theorem 4.

Corollary 7. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and suppose Assumption 1 is satisfied with $2 \leq p \leq q < \infty$ such that (4) holds. Moreover, suppose that $z \mapsto f(z)$ is of class $C^{k,\alpha}_{\mathrm{loc}}$ for some integer $k \geq 2$ and $\alpha \in (0,1)$. Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then, $u \in C^{k+2,\alpha}_{\mathrm{loc}}(\Omega)$.*

The proofs of Theorems 3 and 4 are in several aspects similar to the approach of [Marcellini 1989; 1991]. Following [Marcellini 1991], we prove Theorem 3 appealing to the difference quotient method in order to differentiate the Euler–Lagrange equation and use a variant of Moser's iteration argument [1960] to prove boundedness of the gradient. The improvement compared to the previous results lies in a recent refinement of Moser's iteration argument in the context of the linear nonuniformly elliptic equation, obtained by us in [Bella and Schäffner 2019] (see [Bella and Schäffner 2020] for an application to finite difference equations and stochastic analysis). In order to illustrate the relation between Theorem 3 and local boundedness results for nonuniformly elliptic equation, we suppose for the moment that f satisfies (2) with $2 = p < q$. A local minimizer $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ of (1) satisfies the Euler–Lagrange equation

$$\nabla \cdot Df(\nabla u) = 0$$

and thus, by differentiating,

$$\nabla \cdot D^2 f(\nabla u)\nabla(\partial_j u) = 0 \quad \text{for } j = 1, \dots, n. \tag{5}$$

The coefficient $D^2 f(\nabla u)$ is nonuniformly elliptic and we have by (2) and the assumption $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$

$$m|\lambda|^2 \leq \langle D^2 f(\nabla u)\lambda, \lambda\rangle \leq \mu|\lambda|^2, \quad \text{where } \mu := M(1 + |\nabla u|^2)^{\frac{q-2}{2}} \in L^{\frac{q}{q-2}}_{\mathrm{loc}}(\Omega)$$

(recall $p = 2$). Classic regularity results for linear nonuniformly elliptic equations, due to [Murthy and Stampacchia 1968; Trudinger 1971], yield local boundedness of $\partial_j u$ if

$$\frac{q-2}{q} < \frac{2}{n} \quad \Longrightarrow \quad \frac{q}{2} < \frac{n}{n-2} = 1 + \frac{2}{n-2},$$

which is precisely Marcellini's condition (A) (in the case $p = 2$). Very recently, we improved in [Bella and Schäffner 2019] the assumptions of [Murthy and Stampacchia 1968; Trudinger 1971] and established local boundedness and validity of the Harnack inequality for linear elliptic equations under essentially optimal assumptions on the integrability of the coefficients; see [Franchi et al. 1998] for related counterexamples. Applied to (5), the results of [Bella and Schäffner 2019] yield local boundedness of $\partial_j u$ if

$$\frac{q-2}{q} < \frac{2}{n-1} \quad \Longrightarrow \quad \frac{q}{2} < \frac{n-1}{n-3} = 1 + \frac{2}{n-3},$$

which is precisely condition (3). For $p > 2$ the results of [Bella and Schäffner 2019] applied to (5) do not give the claimed condition (3) and thus we need to combine the reasoning of [Marcellini 1991] with arguments of [Bella and Schäffner 2019] and provide an essentially self-contained proof of Theorem 3. Theorem 4 follows from Theorem 3 by a combination of an interpolation argument (similar to [Marcellini 1991, Theorem 3.1]) and a suitable approximation procedure (inspired by [Esposito et al. 1999]).

The paper is organized as follows: In Section 2, we recall some results from [Marcellini 1991] and present a technical lemma which is used to derive an improved version of the Caccioppoli inequality, which plays a prominent role in the proof of Theorem 3. In Section 3, we prove Theorem 3 and provide a useful a priori estimate via interpolation; see Corollary 12. Finally, in Section 4, we establish Theorem 4 as a consequence of Corollary 12 and an approximation argument.

2. Preliminary lemmas

For $\alpha \geq 2$ and $k > 0$, let $g_{\alpha,k} : \mathbb{R} \to \mathbb{R}$ be the unique $C^1(\mathbb{R})$-function satisfying

$$g_{\alpha,k}(t) = t(1+t^2)^{\frac{\alpha-2}{2}} \quad \text{for } |t| \leq k, \tag{6}$$

and which is affine on $\mathbb{R} \setminus \{|t| \leq k\}$. Moreover, we set

$$G_{\alpha,k}(t) := \frac{g_{\alpha,k}^2(t)}{g'_{\alpha,k}(t)}. \tag{7}$$

The following bounds on $G_{\alpha,k}$ are derived in [Marcellini 1991]

Lemma 8 [Marcellini 1991, Lemma 2.6]. *For every $\alpha \in [2,\infty)$ and $k > 0$ there exists $c = c(\alpha,k) \in [1,\infty)$ such that for all $t \in \mathbb{R}$*

$$G_{\alpha,k}(t) \leq c_{\alpha,k}(1+t^2), \tag{8}$$

$$G_{\alpha,k}(t) \leq 2\left(\frac{1+k^2}{k^2}\right)^{\frac{\alpha-2}{2}} (1+t^2)^{\frac{\alpha}{2}}. \tag{9}$$

Appealing to the difference quotient method, it was proven in [Marcellini 1991] that local minimizers of (1) satisfying $W^{1,q}_{\mathrm{loc}}(\Omega)$ integrability enjoy higher differentiability:

Lemma 9. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and suppose Assumption 1 is satisfied with $2 \leq p \leq q < \infty$. Let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then, $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$. Moreover, for every $\eta \in C^1_c(\Omega)$, any $s \in \{1,\dots,n\}$ and any $\alpha \geq 2$,*

$$\int_\Omega \eta^2 g'_{\alpha,k}(u_{x_s})(1+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u_{x_s}|^2\,dx \leq \frac{4M}{m}\int_\Omega |\nabla\eta|^2 G_{\alpha,k}(u_{x_s})(1+|\nabla u|^2)^{\frac{q-2}{2}}\,dx. \tag{10}$$

Lemma 9 is essentially proven in [Marcellini 1991]. However, estimate (10), which is the starting point for our analysis, is not explicitly stated in that work (as mentioned above, that work deals with more general equations, and additional terms appear on the right-hand side to which our methods do not directly apply) and thus we sketch the proof of Lemma 9 following the reasoning of [Marcellini 1991].

Proof of Lemma 9. First, we note that since $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ and $|Df(z)| \leq c(1+|z|)^{q-1}$ for some $c = c(M,n,q) \in [1,\infty)$ (by (2)), we obtain that u solves the Euler–Lagrange equation

$$\int_\Omega \langle Df(\nabla u), \nabla\varphi\rangle\,dx = 0 \quad \text{for all } \varphi \in W^{1,q}(\Omega) \text{ with } \operatorname{supp}\varphi \Subset \Omega. \tag{11}$$

For $s \in \{1,\dots,n\}$, we consider the difference quotient operator

$$\tau_{s,h}v := \frac{1}{h}(v(\cdot + he_s) - v), \quad \text{where } v \in L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

Fix $\eta \in C^1_c(\Omega)$. Testing (11) with $\varphi := \tau_{s,-h}(\eta^2 g_{\alpha,k}(\tau_{s,h}u))$, we obtain

$$\begin{aligned}(I) &:= \int_\Omega \eta^2 g'_{\alpha,k}(\tau_{s,h}u)\langle \tau_{s,h}Df(\nabla u), \tau_{s,h}\nabla u\rangle\,dx \\ &= -2\int_\Omega \eta g_{\alpha,k}(\tau_{s,h}u)\langle \tau_{s,h}Df(\nabla u), \nabla\eta\rangle\,dx =: (II).\end{aligned}$$

Writing $\tau_{s,h}Df(\nabla u) = \frac{1}{h}Df(\nabla u + th\tau_{s,h}\nabla u)\big|_{t=0}^{t=1}$, the fundamental theorem of calculus yields

$$\begin{aligned}&\int_\Omega\int_0^1 \eta^2 g'_{\alpha,k}(\tau_{s,h}u)\langle D^2f(\nabla u + th\tau_{s,h}\nabla u))\tau_{s,h}\nabla u, \tau_{s,h}\nabla u\rangle\,dt\,dx \\ &\qquad = (I) = (II) \\ &\qquad = -2\int_\Omega\int_0^1 \eta g_{\alpha,k}(\tau_{s,h}u)\langle D^2f(\nabla u + th\tau_{s,h}\nabla u)\tau_{s,h}\nabla u, \nabla\eta\rangle\,dt\,dx.\end{aligned} \tag{12}$$

Young's inequality and the definition of $G_{\alpha,k}$, see (7), then yield

$$|(II)| \leq \tfrac{1}{2}(I) + 2(III), \tag{13}$$

where

$$(III) := \int_\Omega\int_0^1 G_{\alpha,k}(\tau_{s,h}u)\langle D^2f(\nabla u + th\tau_{s,h}\nabla u)\nabla\eta, \nabla\eta\rangle\,dt\,dx.$$

Combining (12), (13) with the assumptions on $D^2 f$, see (2), we obtain for all $\alpha \geq 2$

$$
\begin{aligned}
m \int_\Omega \int_0^1 \eta^2 g'_{\alpha,k}(\tau_{s,h}u)(1+|\nabla u + th\tau_{s,h}\nabla u|^2)^{\frac{p-2}{2}} |\tau_{s,h}\nabla u|^2 \, dx& \\
\leq (I) \leq 4(III)& \\
\leq 4M \int_\Omega \int_0^1 G_{\alpha,k}(\tau_{s,h}u)(1+|\nabla u + th\tau_{s,h}\nabla u|^2)^{\frac{q-2}{2}} |\nabla \eta|^2 \, dx.& \qquad (14)
\end{aligned}
$$

Estimate (14) with $\alpha = 2$ (and thus $g_{2,k} = t$, $g'_{2,k} = 1$ and $G_{2,k}(t) = t^2$; see (6), (7)), the assumption $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ and the arbitrariness of $\eta \in C^1_c(\Omega)$ and $s \in \{1, \dots, n\}$ yield $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$. Finally, by sending h to zero in (14) we obtain the desired estimate (10) (for this we use that $G_{\alpha,k}$ is quadratic for every $k > 0$, see (8), and thus $G_{\alpha,k}(\tau_{s,h}u) \to G_{\alpha,k}(u_{x_s})$ in $L^{\frac{q}{2}}(\Omega')$ for any $\Omega' \Subset \Omega$). □

To this point, we essentially recalled notation and statements from [Marcellini 1991]. Following that work, we will combine (10) with a Moser-iteration-type argument to establish the desired Lipschitz-estimate. In contrast to [Marcellini 1991], we optimize estimate (10) with respect to η, which will enable us to use Sobolev inequality on spheres instead of balls. The following lemma captures the needed improvement due to a suitable choice of the cut-off function η:

Lemma 10. *Fix $n \geq 2$. For given $0 < \rho < \sigma < \infty$ and $v \in L^1(B_\sigma)$ consider*

$$
J(\rho, \sigma, v) := \inf\left\{ \int_{B_\sigma} |v||\nabla \eta|^2 \, dx \;\middle|\; \eta \in C^1_0(B_\sigma),\ \eta \geq 0,\ \eta = 1 \text{ in } B_\rho \right\}.
$$

Then for every $\delta \in (0, 1]$

$$
J(\rho, \sigma, v) \leq (\sigma - \rho)^{-(1+\frac{1}{\delta})} \left(\int_\rho^\sigma \left(\int_{S_r} |v| \right)^\delta dr \right)^{\frac{1}{\delta}}. \qquad (15)
$$

Proof of Lemma 10. Estimate (15) follows directly by minimizing among radial symmetric cut-off functions. Indeed, we obviously have for every $\varepsilon \geq 0$

$$
J(\rho, \sigma, v) \leq \inf\left\{ \int_\rho^\sigma \eta'(r)^2 \left(\int_{S_r} |v| + \varepsilon \right) dr \;\middle|\; \eta \in C^1(\rho, \sigma),\ \eta(\rho) = 1,\ \eta(\sigma) = 0 \right\} =: J_{1\mathrm{d},\varepsilon}.
$$

For $\varepsilon > 0$, the one-dimensional minimization problem $J_{1\mathrm{d},\varepsilon}$ can be solved explicitly and we obtain

$$
J_{1\mathrm{d},\varepsilon} = \left(\int_\rho^\sigma \left(\int_{S_r} |v| + \varepsilon \right)^{-1} dr \right)^{-1}. \qquad (16)
$$

Let us give an argument for (16). First we observe that using the assumption $v \in L^1(B_\sigma)$ and a simple approximation argument we can replace $\eta \in C^1(\rho, \sigma)$ with $\eta \in W^{1,\infty}(\rho, \sigma)$ in the definition of $J_{1\mathrm{d},\varepsilon}$. Let $\tilde{\eta} : [\rho, \sigma] \to [0, \infty)$ be given by

$$
\tilde{\eta}(r) := 1 - \left(\int_\rho^\sigma b(r)^{-1} \, dr \right)^{-1} \int_\rho^r b(r)^{-1} \, dr, \quad \text{where } b(r) := \int_{S_r} |v| + \varepsilon.
$$

Clearly, $\tilde\eta \in W^{1,\infty}(\rho,\sigma)$ (since $b \geq \varepsilon > 0$), $\tilde\eta(\rho)=1$, $\tilde\eta(\sigma)=0$, and thus

$$J_{1\mathrm{d},\varepsilon} \leq \int_\rho^\sigma \tilde\eta'(r)^2 b(r)\,dr = \left(\int_\rho^\sigma b(r)^{-1}\,dr\right)^{-1}.$$

The reverse inequality follows by Hölder's inequality: For every $\eta \in W^{1,\infty}(\rho,\sigma)$ satisfying $\eta(\rho)=1$ and $\eta(\sigma)=0$, we have

$$1 = \left(\int_\rho^\sigma \eta'(r)\,dr\right)^2 \leq \int_\rho^\sigma \eta'(r)^2 b(r)\,dr \int_\rho^\sigma b(r)^{-1}\,dr.$$

Clearly, the last two displayed formulas imply (16).

Next, we deduce (15) from (16). For every $s>1$, we obtain by Hölder inequality

$$\sigma-\rho = \int_\rho^\sigma \left(\frac{b}{b}\right)^{\frac{s-1}{s}} \leq \left(\int_\rho^\sigma b^{s-1}\right)^{\frac{1}{s}} \left(\int_\rho^\sigma \frac{1}{b}\right)^{\frac{s-1}{s}},$$

with b as above, and by (16) that

$$J_{1\mathrm{d},\varepsilon} \leq (\sigma-\rho)^{-\frac{s}{s-1}} \left(\int_\rho^\sigma \left(\int_{S_r} |v|+\varepsilon\right)^{s-1} dr\right)^{\frac{1}{s-1}}.$$

Sending ε to zero, we obtain (15) with $\delta = s-1>0$. □

3. Proof of Theorem 3

The main result of this section is the following

Theorem 11. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and suppose Assumption 1 is satisfied with $2 \leq p < q < \infty$ such that (3) holds. Fix*

$$\theta = \frac{2q}{(n-1)p-(n-3)q} \quad \textit{if } n \geq 4 \qquad \textit{and} \qquad \theta > \frac{q}{p} \quad \textit{if } n=3. \tag{17}$$

Let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then, there exists $c = c(n,m,M,p,q,\theta) \in [1,\infty)$ such that for every $B_R(x_0) \Subset \Omega$ and any $\rho \in (0,1)$

$$\|(1+|\nabla u|^2)^{\frac12}\|_{L^\infty(B_{\rho R}(x_0))} \leq c((1-\rho)R)^{-n\frac{\theta}{q}} \|(1+|\nabla u|^2)^{\frac12}\|^{\theta}_{L^q(B_R(x_0))}. \tag{18}$$

Proof of Theorem 3. Theorem 11 contains the claim of Theorem 3 in the case $n \geq 3$ and $2 \leq p < q$. The remaining case $n=2$ is contained in [Marcellini 1991, Theorem 2.1] and the statement is classic for $p=q$. □

Proof of Theorem 11. Throughout the proof we write $\lesssim$ if $\leq$ holds up to a positive constant which depends only on n,m,M,p and q.

Step 1: One step improvement. Suppose that $B_2 \Subset \Omega$. We claim that for every

$$\gamma \in (0,1] \quad \text{satisfying} \quad \frac{n-3}{n-1} \leq \gamma \tag{19}$$

there exists $c=c(\gamma,n,m,M,p,q)\in[1,\infty)$ such that for every $\frac12\le\rho<\sigma\le1$ and any $\alpha\ge2$

$$\|\phi_{\alpha+p-2}\|^2_{W^{1,2}(B_\rho)}\le c\alpha^2(\sigma-\rho)^{-(1+\frac{1}{\gamma})}\|\phi_{(\alpha+q-2)\gamma}\|^{\frac{2}{\gamma}}_{W^{1,2}(B_\sigma)}, \tag{20}$$

where we use the shorthand

$$\phi_\beta:=\sum_{j=1}^n(1+u_{x_j}^2)^{\frac{\beta}{4}}\quad\text{for }\beta>0. \tag{21}$$

Moreover, there exists $c=c(n,m,M,p,q)\in[1,\infty)$ such that for every $0<\rho<\sigma\le2$ and any $\alpha\ge2$

$$\|\nabla\phi_{\alpha+p-2}\|^2_{L^2(B_\rho)}\le c\alpha^2(\sigma-\rho)^{-2}\|\phi_{\alpha+q-2}\|^2_{L^2(B_\sigma)}. \tag{22}$$

Substep 1.1: We claim that there exists $c=c(\gamma,n,q)\in[1,\infty)$ such that for every $k>0$, $\alpha\ge2$, $s\in\{1,\dots,n\}$, and $\frac12\le\rho<\sigma\le1$

$$\begin{aligned}I_{\alpha,k,s}(\rho,\sigma)&:=\inf\bigg\{\int_{B_\sigma}|\nabla\eta|^2G_{\alpha,k}(u_{x_s})(1+|\nabla u|^2)^{\frac{q-2}{2}}\;\bigg|\;\eta\in C^1_0(B_\sigma),\ \eta=1\text{ in }B_\rho\bigg\}\\&\le c(\sigma-\rho)^{-(1+\frac{1}{\gamma})}\Big(\frac{1+k^2}{k^2}\Big)^{\frac{\alpha-2}{2}}\|\phi_{(q-2+\alpha)\gamma}\|^{\frac{2}{\gamma}}_{W^{1,2}(B_\sigma)}.\end{aligned} \tag{23}$$

Assumption $u\in W^{1,q}(B_1)$ and estimate (8) imply that $v:=G_{\alpha,k}(u_{x_s})(1+|\nabla u|^2)^{\frac{q-2}{2}}\in L^1(B_1)$. Hence, Lemma 10 and (9) yield for every $\delta\in(0,1]$

$$I_{\alpha,k,s}(\rho,\sigma)\le 2(\sigma-\rho)^{-(1+\frac{1}{\delta})}\Big(\frac{1+k^2}{k^2}\Big)^{\frac{\alpha-2}{2}}\bigg(\int_\rho^\sigma\Big(\int_{S_r}(1+u_{x_s}^2)^{\frac{\alpha}{2}}(1+|\nabla u|^2)^{\frac{q-2}{2}}\Big)^\delta dr\bigg)^{\frac{1}{\delta}}.$$

Appealing to Young's inequality, we find $c=c(n)\in[1,\infty)$ such that

$$\sum_{j=1}^n(1+u_{x_j}^2)^{\frac{\alpha}{2}}\sum_{j=1}^n(1+u_{x_j}^2)^{\frac{q-2}{2}}\le c\sum_{j=1}^n(1+u_{x_j}^2)^{\frac{\alpha+q-2}{2}} \tag{24}$$

(in fact (24) is valid with $c=1+\frac12 n(n-1)$; see [Marcellini 1991, Lemma 2.9]) and thus

$$\begin{aligned}(1+u_{x_s}^2)^{\frac{\alpha}{2}}(1+|\nabla u|^2)^{\frac{q-2}{2}}&\le n^{\max\{\frac{q-2}{2}-1,0\}}\sum_{j=1}^n(1+u_{x_j}^2)^{\frac{\alpha}{2}}\sum_{j=1}^n(1+u_{x_j}^2)^{\frac{q-2}{2}}\\&\overset{(24)}{\le} cn^{\max\{\frac{q-2}{2}-1,0\}}\sum_{j=1}^n(1+u_{x_j}^2)^{\frac{\alpha+q-2}{2}}\\&\le cn^{\max\{\frac{q-2}{2}-1,0\}}\phi^{\frac{2}{\gamma}}_{(\alpha+q-2)\gamma},\end{aligned}$$

where in the first inequality we use Jensen's inequality in the case $\frac{q-2}{2}\ge1$ and the discrete ℓ_s-ℓ_1 estimate, with $s\ge1$, for $\frac{q-2}{2}\in(0,1)$, and the third inequality is again the discrete ℓ_s-ℓ_1 estimate, with $s\ge1$. Hence, we find $c=c(n,q)\in[1,\infty)$ such that

$$I_{\alpha,k,s}(\rho,\sigma)\le c(\sigma-\rho)^{-(1+\frac{1}{\delta})}\Big(\frac{1+k^2}{k^2}\Big)^{\frac{\alpha-2}{2}}\bigg(\int_\rho^\sigma\Big(\int_{S_r}\phi^{\frac{2}{\gamma}}_{(\alpha+q-2)\gamma}\Big)^\delta dr\bigg)^{\frac{1}{\delta}}\quad\text{for all }\delta\in(0,1]. \tag{25}$$

To estimate the right-hand side in (25) we use the Sobolev inequality on spheres; i.e., for all $\gamma \in (0,1]$ there exists $c = c(n,\gamma) \in [1,\infty)$ such that for every $r > 0$

$$\Big(\int_{S_r} |\varphi|^{\frac{2}{\gamma}}\Big)^{\frac{\gamma}{2}} \le c\bigg(\Big(\int_{S_r} |\nabla\varphi|^{(\frac{2}{\gamma})_*}\Big)^{\frac{1}{(2/\gamma)*}} + \frac{1}{r}\Big(\int_{S_r} |\varphi|^{(\frac{2}{\gamma})_*}\Big)^{\frac{1}{(2/\gamma)*}}\bigg), \quad \text{where } \frac{1}{(\frac{2}{\gamma})_*} = \frac{\gamma}{2} + \frac{1}{n-1}. \tag{26}$$

Estimate (26) and assumption (19) in the form

$$\frac{1}{(\frac{2}{\gamma})_*} = \frac{\gamma}{2} + \frac{1}{n-1} \overset{(19)}{\ge} \frac{n-3}{2(n-1)} + \frac{1}{n-1} \ge \frac{1}{2}$$

yield

$$\begin{aligned}
&\bigg(\int_\rho^\sigma \Big(\int_{S_r} \phi_{(\alpha+q-2)\gamma}^{\frac{2}{\gamma}}\Big)^{\delta} dr\bigg)^{\frac{1}{\delta}} \\
&\quad \le c\bigg(\int_\rho^\sigma \bigg[\Big(\int_{S_r} |\nabla\phi_{(\alpha+q-2)\gamma}|^{(\frac{2}{\gamma})_*}\Big)^{\frac{1}{(2/\gamma)*}} + \frac{1}{r}\Big(\int_{S_r} \phi_{(\alpha+q-2)\gamma}^{(\frac{2}{\gamma})_*}\Big)^{\frac{1}{(2/\gamma)*}}\bigg]^{\delta\frac{2}{\gamma}} dr\bigg)^{\frac{1}{\delta}} \\
&\quad \le c\bigg(\int_\rho^\sigma |S_r|^{(\frac{1}{(2/\gamma)*}-\frac{1}{2})\frac{2\delta}{\gamma}} \bigg[\Big(\int_{S_r} |\nabla\phi_{(\alpha+q-2)\gamma}|^2\Big)^{\frac{1}{2}} + \frac{1}{r}\Big(\int_{S_r} \phi_{(\alpha+q-2)\gamma}^2\Big)^{\frac{1}{2}}\bigg]^{\delta\frac{2}{\gamma}} dr\bigg)^{\frac{1}{\delta}},
\end{aligned} \tag{27}$$

where $c = c(\gamma,n) \in [1,\infty)$. Combining (25) and (27) with the choice $\delta = \gamma$, we obtain the claimed estimate (23) (we can ignore the factors $|S_r|$ and $\frac{1}{r}$ in (27) by assumption $\frac{1}{2} \le \rho < \sigma \le 1$).

<u>Substep 1.2</u>: Proof of (20). Lemma 9 and estimate (23) yield for every $s \in \{1,\dots,n\}$

$$\int_{B_\rho} g'_{\alpha,k}(u_{x_s})(1+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u_{x_s}|^2\,dx \le c(\sigma-\rho)^{-(1+\frac{1}{\gamma})}\Big(\frac{1+k^2}{k^2}\Big)^{\frac{\alpha-2}{2}}\|\phi_{(q-2+\alpha)\gamma}\|_{W^{1,2}(B_\sigma)}^{\frac{2}{\gamma}},$$

where $c = c(\gamma,n,m,M,p,q) \in [1,\infty)$. Sending k to infinity and summing over s from 1 to n, we obtain (using $\lim_{k\to\infty} g'_{\alpha,k}(t) \ge (1+t^2)^{\frac{\alpha-2}{2}}$)

$$\int_{B_\rho} \sum_{j=1}^n (1+u_{x_j}^2)^{\frac{\alpha+p-4}{2}}|\nabla u_{x_j}|^2\,dx \le c(\sigma-\rho)^{-(1+\frac{1}{\gamma})}\|\phi_{(q-2+\alpha)\gamma}\|_{W^{1,2}(B_\sigma)}^{\frac{2}{\gamma}}.$$

Combining the above estimate with the pointwise inequality

$$|\nabla\phi_{\alpha+p-2}| \le \frac{\alpha+p-2}{2}\sum_{j=1}^n (1+u_{x_j}^2)^{\frac{\alpha+p-4}{4}}|\nabla u_{x_j}|, \tag{28}$$

we obtain that there exists $c = c(\gamma,n,m,M,p,q) \in [1,\infty)$ such that for all $\frac{1}{2} \le \rho < \sigma \le 1$ and $\alpha \ge 2$

$$\|\nabla\phi_{\alpha+p-2}\|_{L^2(B_\rho)}^2 \le c\alpha^2(\sigma-\rho)^{-(1+\frac{1}{\gamma})}\|\phi_{(q-2+\alpha)\gamma}\|_{W^{1,2}(B_\sigma)}^{\frac{2}{\gamma}}. \tag{29}$$

It remains to estimate $\|\phi_{\alpha+p-2}\|_{L^2(B_\rho)}$. For this, we use a version of the Poincaré inequality: for every $\varepsilon > 0$ there exists $c = c(\varepsilon,n) \in [1,\infty)$ such that for all $r > 0$ and $v \in H^1(B_r)$

$$\Big(\fint_{B_r} |v|^2\Big)^{\frac{1}{2}} \le c\bigg(r\Big(\fint_{B_r} |\nabla v|^2\Big)^{\frac{1}{2}} + \Big(\fint_{B_r} |v|^{\varepsilon}\Big)^{\frac{1}{\varepsilon}}\bigg). \tag{30}$$

We recall a proof of (30) at the end of this step. Inequality (30) with $v = \phi_{\alpha+p-2}$ and

$$\varepsilon = 2\gamma \frac{\alpha+q-2}{\alpha+p-2}$$

together with the inequality

$$1 \le \phi_{\alpha+p-2} \le n^{\max\{0, 1-\frac{\alpha+p-2}{(\alpha+q-2)\gamma}\}} \phi_{(\alpha+q-2)\gamma}^{\frac{1}{\gamma}\frac{\alpha+p-2}{\alpha+q-2}}$$

(the second inequality follows by Jensens inequality if $(\alpha+q-2)\gamma/(\alpha+p-2) \ge 1$ and the discrete ℓ_s-ℓ_1 inequality, with $s \ge 1$, otherwise) yield

$$\|\phi_{\alpha+p-2}\|^2_{L^2(B_\rho)} \le c(\|\nabla\phi_{\alpha+p-2}\|^2_{L^2(B_\rho)} + \|\phi_{(\alpha+q-2)\gamma}\|^{\frac{2}{\gamma}\frac{\alpha+p-2}{\alpha+q-2}}_{L^2(B_\rho)}), \tag{31}$$

where $c = c(n, \gamma, p, q) \in [1, \infty)$ (note that $\rho \in [\frac{1}{2}, 1]$ and $\varepsilon \in [2\gamma, \frac{q}{p}2\gamma]$). The first term on the right-hand side in (31) can be estimated by (29) and the second term (using $p \le q$ and $\phi_\beta \ge 1$ for all $\beta > 0$) by

$$\|\phi_{(\alpha+q-2)\gamma}\|^{\frac{2}{\gamma}\frac{\alpha+p-2}{\alpha+q-2}}_{L^2(B_\rho)} \le c\|\phi_{(\alpha+q-2)\gamma}\|^{\frac{2}{\gamma}}_{L^2(B_\rho)}. \tag{32}$$

A combination of (29), (31) and (32) yield (20).

Finally, we recall an argument for (30): Clearly it suffices to proof the statement for $r = 1$. Given $\varepsilon > 0$, set

$$U_\varepsilon := \{x \in B_1 \mid |v(x)| \le \lambda_\varepsilon\}, \quad \text{where } \lambda_\varepsilon := \left(2 \fint_{B_1} |v|^\varepsilon\right)^{\frac{1}{\varepsilon}}.$$

The choice of λ_ε and the Markov inequality yield

$$|B_1 \setminus U_\varepsilon| \le \lambda^{-\varepsilon} \int_{B_1} |v|^\varepsilon \le \tfrac{1}{2}|B_1|$$

and thus $|U_\varepsilon| \ge \frac{1}{2}|B_1|$. Hence, by a suitable version of the Poincaré inequality, see, e.g., [Gilbarg and Trudinger 1998, (7.45), p. 164], there exists $c = c(n) \in [1, \infty)$ such that

$$\int_{B_1} \left| v - \fint_{U_\varepsilon} v \right|^2 \le c \int_{B_1} |\nabla v|^2.$$

The above inequality, the triangle inequality and

$$\fint_{U_\varepsilon} |v| \le 2\lambda_\varepsilon^{1-\varepsilon} \fint_{B_1} |v|^\varepsilon \le 2^{\frac{1}{\varepsilon}} \left(\fint_{B_1} |v|^\varepsilon\right)^{\frac{1}{\varepsilon}}$$

imply (30).

Substep 1.3: Proof of (22). This estimate is an intermediate step in the proof of [Marcellini 1991, Lemma 2.10], but for completeness we recall the argument. Lemma 9 with η being the affine cutoff function for B_ρ in B_σ yields for every $s \in \{1, \dots, n\}$

$$\int_{B_\rho} g'_{\alpha,k}(u_{x_s})(1+|\nabla u|^2)^{\frac{p-2}{2}} |\nabla u_{x_s}|^2 \, dx \lesssim (\sigma-\rho)^{-2} \int_{B_\sigma} G_{\alpha_k}(u_{x_s})(1+|\nabla u|^2)^{\frac{q-2}{2}} \, dx$$

and by summing s from 1 to n and sending $k \to \infty$, we obtain

$$\int_{B_\rho} \sum_{j=1}^{n} (1+u_{x_j}^2)^{\frac{\alpha+p-4}{2}} |\nabla u_{x_j}|^2 \, dx \lesssim (\sigma-\rho)^{-2} \int_{B_\sigma} \phi_{\alpha+q-2}^2. \tag{33}$$

Estimate (22) is a consequence of (28) and (33).

Step 2: Iteration. Fix θ as in (17). We claim that there exists $c = c(n,m,M,p,q,\theta) \in [1,\infty)$ such that

$$\|(1+|\nabla u|^2)^{\frac{1}{2}}\|_{L^\infty(B_{\frac{1}{2}})} \le c\|(1+|\nabla u|^2)^{\frac{1}{2}}\|_{L^q(B_2)}^{\theta}. \tag{34}$$

Set

$$\gamma = \frac{n-3}{n-1} \quad \text{if } n \ge 4 \quad \text{and} \quad \gamma = \frac{\frac{p}{q}\theta - 1}{\theta - 1} \quad \text{if } n = 3. \tag{35}$$

Note that the assumptions $p < q$ and $\theta > \frac{q}{p}$ yield

$$0 < \gamma < \frac{p}{q} \quad \text{if } n = 3. \tag{36}$$

We define a sequence $\{a_k\}_{k\in\mathbb{N}_0}$ by

$$\alpha_0 := 2, \qquad \alpha_k := \frac{1}{\gamma}(\alpha_{k-1} + p - 2) - (q-2) \quad \text{for all } k \in \mathbb{N}.$$

By induction one sees that

$$\alpha_k = 2 + \left(\frac{p}{\gamma} - q\right) \sum_{i=0}^{k-1} \gamma^{-i} = 2 + \left(\frac{p}{\gamma} - q\right) \frac{\gamma^{-k}-1}{\gamma^{-1}-1} = 2 + p\frac{\gamma^{-k}-1}{1-\gamma}\left(1 - \gamma\frac{q}{p}\right) \quad \text{for all } k \in \mathbb{N}.$$

The choice of γ in (35), assumption (3), and (36) together with $p < q$ imply $1 - \gamma\frac{q}{p} > 0$ and $\gamma^{-1} > 1$; hence

$$\alpha_k \to \infty \quad \text{as } k \to \infty.$$

For $k \in \mathbb{N}$, set

$$\rho_k = \frac{1}{2} + \frac{1}{2^{k+1}}, \quad \sigma_k := \rho_k + \frac{1}{2^{k+1}} = \rho_{k-1}$$

(where $\rho_0 := 1$), and

$$A_k := \|\phi_{\alpha_k+p-2}\|_{W^{1,2}(B_{\rho_k})}^{\frac{2}{\alpha_k+p-2}} \quad \text{for all } k \in \mathbb{N}_0,$$

where ϕ_β, $\beta \ge 0$ is defined in (21). Since $\alpha_{k-1} + p - 2 = (\alpha_k + q - 2)\gamma$, estimate (20) for $\alpha = \alpha_k$ implies

$$A_k \le (c2^{(k+1)(1+\frac{1}{\gamma})}\alpha_k^2)^{\frac{1}{\alpha_k+p-2}} A_{k-1}^{\frac{1}{\gamma}\frac{\alpha_{k-1}+p-2}{\alpha_k+p-2}} \quad \text{for every } k \in \mathbb{N},$$

where $c = c(\gamma,n,m,M,p,q) \in [1,\infty)$ as in (20) and thus by iteration

$$A_k \le A_0^{\gamma^{-k}\prod_{i=1}^{k}\frac{\alpha_{i-1}+p-2}{\alpha_i+p-2}} \prod_{i=1}^{k} (c2^{(i+1)(1+\frac{1}{\gamma})}\alpha_i^2)^{\frac{1}{\alpha_i+p-2}}. \tag{37}$$

Note that for every $k \in \mathbb{N}$

$$\prod_{i=1}^{k} (c2^{(i+1)(1+\frac{1}{\gamma})}\alpha_i^2)^{\frac{1}{\alpha_i+p-2}} \le \exp\left(\sum_{i=1}^{\infty} \frac{\log(c2^{(i+1)(1+\frac{1}{\gamma})}\alpha_i^2)}{\alpha_i + p - 2}\right) = c(\gamma,n,m,M,p,q) < \infty$$

and

$$\gamma^{-k}\prod_{i=1}^{k}\frac{\alpha_{i-1}+p-2}{\alpha_i+p-2}=\gamma^{-k}\frac{\alpha_0+p-2}{\alpha_k+p-2}$$
$$=\gamma^{-k}\frac{p}{p\frac{\gamma^{-k}-1}{1-\gamma}\big(1-\gamma\frac{q}{p}\big)+p}=\Big(\frac{\gamma^{-k}}{\gamma^{-k}-1}\Big)\Big(\frac{1-\gamma}{1-\gamma\frac{q}{p}+\frac{1-\gamma}{\gamma^{-k}-1}}\Big).$$

Hence, sending $k\to\infty$ in (37), we obtain that there exists $c=c(n,m,M,p,q,\theta)\in[1,\infty)$ (note $\gamma=\gamma(n,p,q,\theta)<1$) such that

$$\|(1+|\nabla u|^2)^{\frac12}\|_{L^\infty(B_{\frac12})}\le cA_0^{\frac{1-\gamma}{1-\gamma(q/p)}}=c\|\phi_p\|_{W^{1,2}(B_1)}^{\frac{2(1-\gamma)}{p-\gamma q}}. \tag{38}$$

Estimate (22) and $2\le p\le q$ together with $\phi_\beta\ge 1$ for all $\beta\ge 0$ yield

$$\|\phi_p\|_{W^{1,2}(B_1)}^{\frac{2(1-\gamma)}{p-\gamma q}}\lesssim\|\phi_q\|_{L^2(B_2)}^{\frac{2(1-\gamma)}{p-\gamma q}}\lesssim\|(1+|\nabla u|^2)^{\frac12}\|_{L^q(B_2)}^{\frac{q}{p}\frac{1-\gamma}{1-\gamma\frac{q}{p}}}. \tag{39}$$

Estimates (38), (39) and the choice of γ in (35) imply (34).

<u>Step 3</u>: Conclusion. Fix $\rho\in(0,1)$ and $B_R(x_0)\Subset\Omega$. By scaling and translation, we deduce from Step 2 that

$$\|(1+|\nabla u|^2)^{\frac12}\|_{L^\infty(B_{\frac{R}{4}}(x_0))}\le cR^{-n\frac{\theta}{q}}\|(1+|\nabla u|^2)^{\frac12}\|_{L^q(B_R(x_0))}^{\theta}, \tag{40}$$

where $c=c(n,m,M,p,q,\theta)\in[1,\infty)$ is the same as in (34). Applying for every $y\in B_{\rho R}(x_0)$ estimate (40) with $B_R(x_0)$ replaced by $B_{(1-\rho)R}(y)\subset\Omega$, we obtain

$$\|(1+|\nabla u|^2)^{\frac12}\|_{L^\infty(B_{\frac{1-\rho}{4}R}(y))}\le c((1-\rho)R)^{-n\frac{\theta}{q}}\|(1+|\nabla u|^2)^{\frac12}\|_{L^q(B_R(x_0))}^{\theta}$$

and thus the claimed estimate (18) follows. □

By the same interpolation argument as in [Marcellini 1991, Theorem 3.1], we deduce from Theorem 11:

Corollary 12. *Let $\Omega\subset\mathbb{R}^n$, $n\ge 3$, and suppose Assumption 1 is satisfied with $2\le p<q<\infty$ such that (4) holds. Let θ be given as in (17) with the additional constraint $\theta<\frac{q}{q-p}$ for $n=3$ and set*

$$\alpha:=\frac{\theta\frac{p}{q}}{1-\theta\big(1-\frac{p}{q}\big)}. \tag{41}$$

Let $u\in W^{1,q}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then, there exists $c=c(n,m,M,p,q,\theta)\in[1,\infty)$ such that for every $B_{2R}(x_0)\Subset\Omega$

$$\|(1+|\nabla u|^2)^{\frac12}\|_{L^\infty(B_{\frac{R}{2}}(x_0))}\le cR^{-n\frac{\alpha}{p}}\|(1+|\nabla u|^2)^{\frac12}\|_{L^p(B_R(x_0))}^{\alpha}. \tag{42}$$

Remark 13. A direct calculation yields

$$\alpha=\frac{2p}{(n+1)p-(n-1)q}\quad\text{if } n\ge 4.$$

For $n=3$, the assumption on θ in Corollary 12 reads $\theta\in\left(\frac{q}{p},\frac{q}{q-p}\right)$. Since $2\le p<q$, we have

$$\frac{q}{p}<\frac{q}{q-p}\quad\Longleftrightarrow\quad\frac{q}{p}<2,$$

where the second inequality is ensured by (4) (for $n=3$).

Proof of Corollary 12. We prove the statement for $x_0=0$ and $R=1$; the general claim follows by scaling and translation. Throughout the proof we write $\lesssim$ if $\le$ holds up to a positive constant which depends only on n,m,M,p,q and θ.

For $\nu\in\mathbb{N}\cup\{0\}$, we set

$$\rho_\nu=1-\frac{1}{2^{1+\nu}}.$$

Combining the elementary interpolation inequality

$$\|(1+|\nabla u|^2)^{\frac12}\|_{L^q(B_{\rho_\nu})}\le\|(1+|\nabla u|^2)^{\frac12}\|_{L^p(B_{\rho_\nu})}^{\frac{p}{q}}\|(1+|\nabla u|^2)^{\frac12}\|_{L^\infty(B_{\rho_\nu})}^{1-\frac{p}{q}}\tag{43}$$

with estimate (18), we obtain for every $\nu\in\mathbb{N}$

$$\begin{aligned}\|(1+|\nabla u|^2)^{\frac12}\|_{L^\infty(B_{\rho_{\nu-1}})}&\overset{(18)}{\lesssim}\left(1-\frac{\rho_{\nu-1}}{\rho_\nu}\right)^{-n\frac{\theta}{q}}\|(1+|\nabla u|^2)^{\frac12}\|_{L^q(B(\rho_\nu))}^{\theta}\\&\overset{(43)}{\le}\left(1-\frac{\rho_{\nu-1}}{\rho_\nu}\right)^{-n\frac{\theta}{q}}\|(1+|\nabla u|^2)^{\frac12}\|_{L^p(B(\rho_\nu))}^{\frac{p}{q}\theta}\|(1+|\nabla u|^2)^{\frac12}\|_{L^\infty(B(\rho_\nu))}^{(1-\frac{p}{q})\theta}\\&\le c2^{(1+\nu)n\frac{\theta}{q}}\|(1+|\nabla u|^2)^{\frac12}\|_{L^p(B_1)}^{\frac{p}{q}\theta}\|(1+|\nabla u|^2)^{\frac12}\|_{L^\infty(B_{\rho_\nu})}^{(1-\frac{p}{q})\theta},\end{aligned}\tag{44}$$

where $c=c(n,n,m,M,p,q,\theta)\in[1,\infty)$. Iterating (44) from $\nu=1$ to $\hat\nu$, we obtain

$$\begin{aligned}&\|(1+|\nabla u|^2)^{\frac12}\|_{L^\infty(B_{\frac12})}\\&=\|(1+|\nabla u|^2)^{\frac12}\|_{L^\infty(B_{\rho_0})}\\&\overset{(44)}{\le}2^{n\frac{\theta}{q}\sum_{\nu=0}^{\hat\nu-1}(\nu+1)((1-\gamma)\theta)^\nu}\left(c\|(1+|\nabla u|^2)^{\frac12}\|_{L^p(B_1)}^{\frac{p}{q}\theta}\right)^{\sum_{\nu=0}^{\hat\nu-1}((1-\frac{p}{q})\theta)^\nu}\|(1+|\nabla u|^2)^{\frac12}\|_{L^\infty(B_1)}^{((1-\frac{p}{q})\theta)^{\hat\nu}}.\end{aligned}\tag{45}$$

The choice of θ and assumption (4) imply

$$0<\left(1-\frac{p}{q}\right)\theta<1.\tag{46}$$

Indeed, (46) is ensured for $n=3$ by the assumption $\theta<\frac{q}{q-p}$ and for $n\ge4$ by

$$0<\left(1-\frac{p}{q}\right)\theta\overset{(17)}{=}\frac{2(q-p)}{(n-1)p-(n-3)q}=1-\frac{(n+1)p-(n-1)q}{(n-1)p-(n-3)q}\overset{(4)}{<}1.$$

Hence,

$$\sum_{\nu=0}^{\infty}(\nu+1)\left(\left(1-\frac{p}{q}\right)\theta\right)^\nu\lesssim1\quad\text{and}\quad\sum_{\nu=0}^{\infty}\left(\left(1-\frac{p}{q}\right)\theta\right)^\nu=\frac{1}{1-\theta\left(1-\frac{p}{q}\right)}.$$

Thus, estimates (18) and (45) yield for every $\hat{\nu} \in \mathbb{N}$

$$\|(1+|\nabla u|^2)^{\frac12}\|_{L^\infty(B_{\frac12})} \overset{(45)}{\lesssim} \|(1+|\nabla u|^2)^{\frac12}\|_{L^p(B_1)}^{\frac{\theta(p/q)}{1-\theta(1-p/q)}} \|(1+|\nabla u|^2)^{\frac12}\|_{L^\infty(B_1)}^{((1-\frac{p}{q})\theta)^{\hat{\nu}}}$$
$$\overset{(18)}{\lesssim} \|(1+|\nabla u|^2)^{\frac12}\|_{L^p(B_1)}^{\alpha} \|(1+|\nabla u|^2)^{\frac12}\|_{L^q(B_2)}^{\theta((1-\frac{p}{q})\theta)^{\hat{\nu}}}.$$

Assumptions $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ and $B_2 \Subset \Omega$ imply $\|(1+|\nabla u|^2)^{\frac12}\|_{L^q(B_2)} < \infty$ and thus we find $\hat{\nu} \in \mathbb{N}$ such that

$$\|(1+|\nabla u|^2)^{\frac12}\|_{L^q(B_2)}^{\theta((1-\frac{p}{q})\theta)^{\hat{\nu}}} \le 2,$$

which finishes the proof. □

4. Proof of Theorem 4

The main result of this section is:

Theorem 14. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 3$, and suppose Assumption 1 is satisfied with $2 \le p < q < \infty$ such that (4) holds. Let θ be given as in (17) with the additional constraint $\theta < \frac{q}{q-p}$ for $n = 3$. Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then there exists $c = c(n, m, M, p, q, \theta) \in [1, \infty)$ such that for every $B_{2R}(x_0) \Subset \Omega$*

$$\|\nabla u\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \le c\left(\fint_{B_R(x_0)} f(\nabla u)\,dx + 1\right)^{\frac{\alpha}{p}},$$

where α is given in (41).

Proof of Theorem 4. Theorem 14 contains the claim of Theorem 4 in the case $n \ge 3$ and $2 \le p < q$. The remaining case $n = 2$ follows from a combination of [Marcellini 1991, Theorem 2.1] and [Esposito et al. 1999, Theorem 2.1], and the result is classic for $p = q$. □

Appealing to the a priori estimate of Corollary 12, the statement of Theorem 14 follows from by now well-established approximation arguments. Below, we present a proof of Theorem 14 that closely follows [Esposito et al. 1999, proof of Theorem 2.1, Step 3].

Proof of Theorem 14. Throughout the proof we write $\lesssim$ if $\le$ holds up to a positive constant which depends only on n, m, M, p, q and θ.

We assume $B_2 \Subset \Omega$ and show

$$\|\nabla u\|_{L^\infty(B_{\frac18})} \lesssim \left(\fint_{B_1} f(\nabla u)\,dx + 1\right)^{\frac{\alpha}{p}}. \tag{47}$$

Clearly the general claim follows by standard scaling, translation and covering arguments.

Following [Esposito et al. 1999], we introduce two small parameters $\sigma, \varepsilon \in (0, 1)$. Parameter $\sigma > 0$ is related to a perturbation f_σ of the integrand f

$$f_\sigma(\xi) := f(\xi) + \sigma|\xi|^q \quad \text{for every } \xi \in \mathbb{R}^n. \tag{48}$$

Since f satisfies (2) and $\sigma \in (0,1)$, the function f_σ satisfies (2) with M replaced by M' depending on M and q. The second parameter $\varepsilon > 0$ corresponds to a regularization u_ε of u, where $u_\varepsilon := u * \varphi_\varepsilon$ with $\varphi_\varepsilon := \varepsilon^{-n}\varphi\big(\frac{\cdot}{\varepsilon}\big)$ and φ being a nonnegative, radially symmetric mollifier; i.e., it satisfies

$$\varphi \geq 0, \quad \operatorname{supp}\varphi \subset B_1, \quad \int_{\mathbb{R}^n} \varphi(x)\,dx = 1, \quad \varphi(\cdot) = \tilde{\varphi}(|\cdot|) \quad \text{for some } \tilde{\varphi} \in C^\infty(\mathbb{R}).$$

Given $\varepsilon, \sigma \in (0,1)$, we denote by $v_{\varepsilon,\sigma} \in u_\varepsilon + W_0^{1,q}(B_1)$ the unique function satisfying

$$\int_{B_1} f_\sigma(\nabla v_{\varepsilon,\sigma})\,dx \leq \int_{B_1} f_\sigma(\nabla v)\,dx \quad \text{for all } v \in u_\varepsilon + W_0^{1,q}(B_1). \tag{49}$$

In view of Corollary 12, we have

$$\begin{aligned}
\|\nabla v_{\varepsilon,\sigma}\|_{L^\infty(B_{\frac{1}{8}})} &\overset{(42)}{\lesssim} \left(\int_{B_{\frac{1}{4}}} |\nabla v_{\varepsilon,\sigma}|^p\,dx + 1\right)^{\frac{\alpha}{p}} \\
&\overset{(2)}{\lesssim} \left(\int_{B_1} f_\sigma(\nabla v_{\varepsilon,\sigma})\,dx + 1\right)^{\frac{\alpha}{p}} \\
&\overset{(48),(49)}{\leq} \left(\int_{B_1} f(\nabla u_\varepsilon) + \sigma|\nabla u_\varepsilon|^q\,dx + 1\right)^{\frac{\alpha}{p}} \\
&\leq \left(\int_{B_{1+\varepsilon}} f(\nabla u)\,dx + \sigma\int_{B_1} |\nabla u_\varepsilon|^q\,dx + 1\right)^{\frac{\alpha}{p}},
\end{aligned} \tag{50}$$

where we used Jensen's inequality and the convexity of f in the last step. Similarly,

$$\begin{aligned}
m\int_{B_1} |\nabla v_{\varepsilon,\sigma}|^p\,dx &\overset{(2)}{\leq} \int_{B_1} f(\nabla v_{\varepsilon,\sigma})\,dx \overset{(48),(49)}{\leq} \int_{B_1} f(\nabla u_\varepsilon) + \sigma|\nabla u_\varepsilon|^q\,dx \\
&\leq \int_{B_{1+\varepsilon}} f(\nabla u)\,dx + \sigma\int_{B_1} |\nabla u_\varepsilon|^q\,dx.
\end{aligned} \tag{51}$$

Fix $\varepsilon \in (0,1)$. In view of (50) and (51), we find $w_\varepsilon \in u_\varepsilon + W_0^{1,p}(B_1)$ such that as $\sigma \to 0$, up to subsequence,

$$\begin{aligned}
v_{\varepsilon,\sigma} &\rightharpoonup w_\varepsilon && \text{weakly in } W^{1,p}(B_1), \\
\nabla v_{\varepsilon,\sigma} &\overset{*}{\rightharpoonup} \nabla w_\varepsilon && \text{weakly* in } L^\infty(B_{\frac{1}{8}}).
\end{aligned}$$

Hence, a combination of (50), (51) with the weak/weak* lower-semicontinuity of convex functionals yields

$$\|\nabla w_\varepsilon\|_{L^\infty(B_{\frac{R}{8}})} \leq \liminf_{\sigma\to 0} \|\nabla v_{\varepsilon,\sigma}\|_{L^\infty(B_{\frac{1}{8}})} \lesssim \left(\int_{B_{1+\varepsilon}} f(\nabla u)\,dx + 1\right)^{\frac{\alpha}{p}}, \tag{52}$$

$$m\int_{B_1} |\nabla w_\varepsilon|^p\,dx \leq \int_{B_1} f(\nabla w_\varepsilon)\,dx \leq \int_{B_{1+\varepsilon}} f(\nabla u)\,dx. \tag{53}$$

Since $w_\varepsilon \in u_\varepsilon + W_0^{1,q}(B_1)$ and $u_\varepsilon \to u$ in $W^{1,p}(B_1)$, we find by (53) a function $w \in u + W_0^{1,p}(B_1)$ such that, up to subsequence,

$$\nabla w_\varepsilon \rightharpoonup \nabla w \quad \text{weakly in } L^p(B_1).$$

Appealing to the bounds (52), (53) and lower semicontinuity, we obtain

$$\|\nabla w\|_{L^\infty(B_{\frac{1}{8}})} \lesssim \left(\int_{B_{1+\varepsilon}} f(\nabla u)\,dx + 1 \right)^{\frac{\alpha}{p}}, \tag{54}$$

$$\int_{B_1} f(\nabla w)\,dx \le \int_{B_1} f(\nabla u)\,dx. \tag{55}$$

Inequality (55), the strong convexity of f and the fact $w \in u + W_0^{1,p}(B_1)$ imply $w = u$ and thus the claimed estimate (47) is a consequence of (54). □

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