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We study the phenomenon of *invasion* for heterogeneous reaction-diffusion equations in periodic domains with monostable and combustion reaction terms. We give an answer to a question raised by Berestycki, Hamel and Nadirashvili concerning the connection between the speed of invasion and the critical speed of fronts. To do so, we extend the classical Freidlin–Gärtner formula to such equations and we derive some bounds on the speed of invasion using estimates on the heat kernel. We also give geometric conditions on the domain that ensure that the spreading occurs at the critical speed of fronts.

1. Introduction and results

1.1. ***Introduction.*** This paper deals with the spreading properties of the reaction-diffusion equation

$$\begin{cases} \partial_t u = \operatorname{div}(A(x)\nabla u) + q(x)\cdot\nabla u + f(x,u), & t>0,\ x\in\Omega, \\ \nu\cdot A(x)\nabla u = 0, & t>0,\ x\in\partial\Omega. \end{cases} \tag{1}$$

Throughout the paper, the domain Ω and the coefficients are assumed to be periodic. Here, ν stands for the exterior normal. Reaction-diffusion equations arise in the study of various phenomena in biology (propagation of genes, epidemics), physics (combustion), and more recently in social sciences (rioting models). A particular emphasis is given here to the case where the equation is homogeneous but the domain is not the whole space:

$$\begin{cases} \partial_t u = \Delta u + f(u), & t>0,\ x\in\Omega, \\ \partial_\nu u = 0, & t>0,\ x\in\partial\Omega. \end{cases}$$

In such a case, we provide an answer to a question asked by Berestycki, Hamel and Nadirashvili [Berestycki et al. 2005] concerning the relation between the speed of invasion and the speed of fronts for this problem.

Reaction-diffusion equations have been extensively studied since the seminal paper of Kolmogorov, Petrovski and Piskunov [Kolmogorov et al. 1937]. There, the authors dealt with the homogeneous equation

$$\partial_t u = \Delta u + f(u), \quad t>0,\ x\in\mathbb{R}^N, \tag{2}$$

with $f(u)=u(1-u)$. The results of [Kolmogorov et al. 1937] were extended in [Aronson and Weinberger 1978] to more general *reaction terms* f. The basic assumption is that $f(0)=f(1)=0$, so that the constant

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states $u \equiv 0$ and $u \equiv 1$ are stationary solutions. We shall pay a particular attention to the following two types of nonlinearities:

(*monostable*) $f > 0$ in $(0, 1)$.

(*combustion*) There exists $\theta \in (0, 1)$ such that $f = 0$ in $[0, \theta]$, $f > 0$ in $(\theta, 1)$.

These two notions extend to the case where f can depend on x; see Definition 1 below. Two important features of reaction-diffusion equations were derived in [Aronson and Weinberger 1978]. First, (2) admits particular solutions called *traveling fronts*. These are positive entire (i.e., defined for all $t \in \mathbb{R}$) solutions of the form $u(t, x) = \phi(x \cdot e - ct)$, for some $e \in \mathbb{S}^{N-1}$, $c \in \mathbb{R}$, ϕ decreasing and satisfying $\phi(s) \to 1$ as $s \to -\infty$ and $\phi(s) \to 0$ as $s \to +\infty$. The unit vector e is the *direction of propagation*, c is the *speed of propagation* and ϕ is the *profile* of the traveling front. More specifically, there exists a quantity $c^\star$ such that there are fronts with speed c for every $c \geq c^\star$ if f is of *monostable* type, whereas there are traveling fronts only with speed $c = c^\star$ if f is of *combustion* type. Of course, the homogeneity of (2) implies that the quantity $c^\star$ does not depend on the direction of the fronts e. We mention that, if f is of *KPP* type (i.e., if it is monostable and satisfies $f'(0) > 0$ and $f(u) \leq f'(0)u$ for $u \in [0, 1]$), then it is proved in [Kolmogorov et al. 1937] that $c^\star = 2\sqrt{f'(0)}$. The quantity $c^\star$ is called the *critical* (*or minimal*) *speed of fronts*. We consider this quantity in a more general context in Section 1.2.

The second important feature of reaction-diffusion equations is the property of *invasion*. If $u(t, x)$ is a solution of (2) arising from the initial datum u_0 such that

$$u(t, x) \xrightarrow[t \to +\infty]{} 1 \quad \text{locally uniformly in } x,$$

we say that invasion occurs for the initial datum u_0. Of course, this depends on the nonlinearity f. For instance, if f is of combustion type, and if u_0 is a compactly supported nonnegative initial datum and is such that $u_0 \leq \theta$, then the problem (2) boils down to the heat equation, and then $u(t, x) \to 0$ as t goes to $+\infty$ uniformly in x. However, it is shown in [Aronson and Weinberger 1978] that, for every $\eta \in (\theta, 1)$, there is $R > 0$ such that any initial datum such that $u_0(x) \geq \eta \mathbb{1}_{B_R}$ (where B_R is the ball of center 0 and of radius R) satisfies the invasion property. In contrast, if f is of KPP type, then invasion occurs for any nonnegative nonzero initial datum.

Once we know that invasion occurs for some initial data, we can define the *speed of invasion*. We say that $w(e) > 0$ is the speed of invasion for (2) in the direction $e \in \mathbb{S}^{N-1}$ if, for any solution $u(t, x)$ of (2) emerging from a compactly supported nonnegative initial datum that converges to 1 as t goes to $+\infty$, locally uniformly in x, the following holds:

$$\begin{aligned} &\text{for all } c > w(e), && u(t, x + cte) \to 0 \quad \text{as } t \to +\infty, \\ &\text{for all } c \in [0, w(e)), && u(t, x + cte) \to 1 \quad \text{as } t \to +\infty, \end{aligned}$$

locally uniformly in $x \in \mathbb{R}^N$. The homogeneity of (2) yields that the speed of invasion is actually independent of the direction e. Moreover, if f is of KPP type, it is proved in [Kolmogorov et al. 1937] that $w(e) = 2\sqrt{f'(0)}$ for all $e \in \mathbb{S}^{N-1}$. Hence, in this case $c^\star \equiv w$. In other terms, this means that the invasion occurs at the critical speed of fronts in every direction.

One of the main motivations behind the present paper is to understand to what extent this is still satisfied in more general domains, in which case closed formulas for the speeds are not available. Berestycki, Hamel and Nadirashvili [Berestycki et al. 2005] conjectured that the geometry of the domain could give that the invasion does not occur at the critical speed of fronts in every direction (see Question 4 below). We shall construct such domains. We shall also give geometric conditions on the domain that ensure that the invasion speed coincides with the critical speed of fronts in some directions.

In order to state our main results, we first present how the notions of fronts and invasion extend to the case of spatially periodic heterogeneous equations.

1.2. ***Pulsating traveling fronts.*** The notion of *pulsating traveling fronts* was first introduced in dimension $N=1$ in periodic media by Shigesada, Kawasaki and Teramoto [Shigesada et al. 1986] to generalize the notion of traveling fronts available in the homogenous case. Berestycki and Hamel [2002] extended this notion to the more general framework of (1). Throughout the paper, we assume that A, q, f, Ω are periodic, with the same period; i.e, there are $L_1, \dots, L_N > 0$ such that

$$\text{for all } k \in \prod_{i=1}^{N} L_i\mathbb{Z}, \quad \Omega + \{k\} = \Omega,$$

$$\text{for all } k \in \prod_{i=1}^{N} L_i\mathbb{Z}, \quad f(\,\cdot\, + k, \cdot\,) = f, \quad q(\,\cdot\, + k) = q, \quad A(\,\cdot\, + k) = A.$$

We shall denote by $\mathcal{C} := \prod_{i=1}^{N}[0, L_i)$ the periodicity cell. Typical examples of such domains Ω are domains with "obstacles": if $K \subset \mathbb{R}^N$ is a smooth compact set, we can define the periodic domain $\Omega := (K + L\mathbb{Z}^N)^c$, with $L > 0$ large enough so that the resulting domain is smooth and connected. This domain can be seen as the whole space with K-shaped obstacles periodically distributed.

To simplify the notation, unless otherwise stated, we shall assume that the period is 1, i.e., $L_1 = \dots = L_N = 1$. In order to apply the results of [Berestycki and Hamel 2002], we make the following assumptions on the domain:

$$\Omega \text{ is a periodic, connected open subset of } \mathbb{R}^N \text{of class } C^3, \tag{3}$$

and the following hypotheses on the coefficients:

$$\begin{gathered} A \in C^3(\overline{\Omega}) \text{ is symmetric and uniformly elliptic and periodic,} \\ q \in C^{1,\alpha}(\overline{\Omega}) \text{ for some } \alpha \in (0,1), \quad \operatorname{div} q = 0, \quad \textstyle\int_{\mathcal{C}\cap\Omega} q = 0, \quad q \text{ is periodic,} \\ f : \overline{\Omega} \times [0,1] \mapsto \mathbb{R} \text{ is of class } C^{1,\alpha} \text{ for some } \alpha \in (0,1). \end{gathered} \tag{4}$$

We also assume that the nonlinearity f satisfies the following:

$$\begin{aligned} \text{for all } x \in \Omega, \quad & f(x,0) = f(x,1) = 0, \\ \text{there exists } S \in (0,1) \text{ such that for all } x \in \overline{\Omega}, \quad & f(x,\cdot) \text{ is nonincreasing in } [S,1], \\ \text{for all } s \in (0,1), \quad & f(\,\cdot\,, s) \text{ is periodic.} \end{aligned} \tag{5}$$

By analogy with the homogeneous case $f = f(u)$, we define monostable, KPP and combustion nonlinearities $f(x,u)$:

Definition 1. We say that f is of *monostable* type if

$$\text{for all } s \in (0,1), \quad \min_{x\in\overline{\Omega}} f(x,s) \geq 0, \quad \max_{x\in\overline{\Omega}} f(x,s) > 0. \tag{6}$$

Among monostable nonlinearities, there is the special class of *KPP* nonlinearities. In addition to being monostable, they satisfy

$$\text{for all } x \in \overline{\Omega}, \text{ for all } s \in [0,1], \quad f(x,s) \leq \partial_s f(x,0)s. \tag{7}$$

We say that f is of *combustion* type if

$$\begin{aligned} &\text{there exists } \theta \in (0,1) \text{ such that for all } (x,s) \in \Omega \times [0,\theta], \quad f(x,s) = 0, \\ &\text{for all } s \in (\theta,1), \quad \min_{x\in\overline{\Omega}} f(x,s) \geq 0, \quad \max_{x\in\overline{\Omega}} f(x,s) > 0. \end{aligned} \tag{8}$$

The important difference between combustion and monostable nonlinearities (from which stems the nonuniqueness of speeds of fronts for monostable equation) is that, when f is of combustion type,

$$\text{there exists } \theta \in (0,1] \text{ such that for all } x \in \Omega, \quad f(x,\cdot) \text{ is nonincreasing in } [0,\theta]. \tag{9}$$

In the periodic framework, the notion of traveling fronts can be generalized by *pulsating traveling fronts*.

Definition 2. A pulsating traveling front in the direction $e \in \mathbb{S}^{N-1}$ of speed $c \in \mathbb{R}\backslash\{0\}$ connecting 1 to 0 is an entire (i.e., defined for all $t \in \mathbb{R}$) solution v of (1) satisfying

$$\begin{cases} \text{for all } k \in \mathbb{Z}^N \quad \text{for all } x \in \Omega, & v(t+(k\cdot e)/c, x) = v(t, x-k), \\ v(t,x) \to 1 \quad \text{as } x\cdot e \to -\infty, & v(t,x) \to 0 \quad \text{as } x \cdot e \to +\infty. \end{cases}$$

Such fronts are known to exist in several situations. For instance, it is proved in [Berestycki and Hamel 2002] that, under hypotheses (4)–(5), for every $e \in \mathbb{S}^{N-1}$, there is $c^\star(e) > 0$, called again the *critical* (*or minimal*) *speed of fronts* in direction e, such that pulsating traveling fronts in the direction e with speed c exist if, and only if, $c \geq c^\star(e)$ when f is of monostable type (6) or only if $c = c^\star(e)$ when f is of combustion type (8); see [Berestycki and Hamel 2002, Theorems 1.13–1.14].

1.3. ***The speed of invasion.*** The results of Kolmogorov, Petrovski, Piskunov [Kolmogorov et al. 1937] and Aronson and Weinberger [1978] concerning the invasion have also been extended to a more general framework than the homogeneous one. First, consider the periodic equation on $\mathbb{R}^N$

$$\partial_t u = \operatorname{div}(A(x)\nabla u) + q(x)\cdot\nabla u + f(x,u), \quad t > 0,\ x \in \mathbb{R}^N. \tag{10}$$

Then, one can define the speed of invasion w as a function from the unit sphere $\mathbb{S}^{N-1}$ to $\mathbb{R}^+$ such that, for every u solution of (1) arising from a compactly supported nonnegative initial datum which converges to 1 as t goes to $+\infty$, locally uniformly in $x \in \mathbb{R}^N$, we have, for $e \in \mathbb{S}^{N-1}$,

$$\begin{aligned} &\text{for all } c > w(e), && u(t, x+cte) \to 0 \quad \text{as} \quad t \to +\infty, \\ &\text{for all } c \in [0, w(e)), && u(t, x+cte) \to 1 \quad \text{as} \quad t \to +\infty, \end{aligned}$$

locally uniformly in $x \in \mathbb{R}^N$.

Using probabilistic techniques, Gärtner and Freidlin [1979] showed the existence of a speed of invasion for (10) when f is of KPP type (7) and A, q, f are x-periodic. They showed that invasion occurs for every nonnegative nonnull compactly supported initial datum and proved what is now known as the *Freidlin–Gärtner formula*:

$$w(e) := \min_{\substack{\xi\in\mathbb{R}^N\\ e\cdot\xi>0}} \frac{k(\xi)}{e\cdot\xi}, \tag{11}$$

where $k(\xi)$ is the periodic principal eigenvalue of the operator

$$L_\xi u := \operatorname{div}(A\nabla u) - 2\xi\cdot A\nabla u + q\cdot\xi u + (-\operatorname{div}(A\xi) - q\cdot\xi + \xi\cdot A\xi + \partial_u f(x,0))u.$$

This formula was also proved by Berestycki, Hamel and Nadin [Berestycki et al. 2008] using a PDE approach. Similar properties of spreading for heterogeneous reaction-diffusion equations have been studied with other approaches: the viscosity solution/singular perturbation method was adopted by Evans and Souganidis [1989] and Barles, Soner and Souganidis [1993]. Weinberger [2002] used an abstract discrete system approach.

Berestycki, Hamel and Nadirashvili [Berestycki et al. 2005] showed that, if one considers KPP nonlinearities, the critical speed of pulsating traveling fronts in the direction e for (10) is given by $c^\star(e) = \min_{\lambda>0} k(\lambda e)/\lambda$, where k is the principal eigenvalue introduced before (if the equation were set on a periodic domain Ω instead of $\mathbb{R}^N$, this relation still holds true with k being the periodic principal eigenvalue of the same operator but with the additional boundary condition $\nu\cdot A\nabla u = \lambda(\nu\cdot e)u$ on $\partial\Omega$, see [Berestycki et al. 2005] for the details). Consequently, in the KPP case, the Freidlin–Gärtner formula (11) can be rewritten as

$$w(e) = \min_{e\cdot\xi>0} \frac{c^\star(\xi)}{e\cdot\xi}. \tag{12}$$

The fact that pulsating traveling fronts exist not only in the KPP case but also for other reaction terms, and hence that the formula (12) could make sense in more general frameworks than the KPP one, led Rossi [2017] to extend the Freidlin–Gärtner formula to much more general equations in the whole space, essentially, all those for which pulsating traveling fronts are known to exist.

In this paper, we deal with invasion in domains Ω that are not necessarily $\mathbb{R}^N$. In this case, it is convenient to introduce the notion of *asymptotic set of spreading*.

Definition 3. Let $\mathcal{W}\subset\mathbb{R}^N$ be a closed set coinciding with the closure of its interior. We say that $\mathcal{W}$ is the asymptotic set of spreading for a reaction-diffusion equation if, for any bounded solution $u(t,x)$ emerging from a nonnegative compactly supported initial datum such that $u(t,x)\to 1$ as $t\to+\infty$, locally uniformly in $x\in\overline{\Omega}$, we have

$$\text{for all } K \text{ compact, } K\subset\operatorname{int}(\mathcal{W}), \quad \inf_{x\in tK} u(t,x)\to 1 \quad \text{as } t\to+\infty, \tag{13}$$

$$\text{for all } C \text{ closed, } C\cap\mathcal{W}=\varnothing, \quad \sup_{x\in tC} u(t,x)\to 0 \quad \text{as } t\to+\infty. \tag{14}$$

If only (13) holds, $\mathcal{W}$ is said to be an asymptotic subset of spreading, and if only (14) holds, $\mathcal{W}$ is said to be an asymptotic superset of spreading.

The asymptotic set of spreading relates to the notion of speed of invasion previously described. Indeed, assume that $\mathcal{W}$ is an asymptotic set of spreading and that we can write $\mathcal{W} = \{r\xi : \xi \in \mathbb{S}^{N-1},\ 0 \le r \le w(\xi)\}$ with w a continuous function. Then, if $\Omega = \mathbb{R}^N$, $w(e)$ is the speed of spreading in the direction e, as defined before. For example, if f is a KPP nonlinearity independent of x, then the asymptotic set of spreading associated with the homogeneous equation (2) is the ball of center 0 and of radius $2\sqrt{f'(0)}$.

Observe that, for the definition of the asymptotic set of spreading to be meaningful, it is necessary that there are compactly supported initial data u_0 for which the invasion property holds. Rossi and the author [Ducasse and Rossi 2018] gave necessary and sufficient conditions to have invasion for (1). In particular, we showed there that, if f is of monostable or combustion type, in the sense of Definition 1, and if the drift term q is "not too large" (see [Ducasse and Rossi 2018] for the details), then, setting

$$\theta := \max\{s \in [0, 1) : \text{there exists } x \in \overline{\Omega} \text{ such that } f(x, s) = 0\},$$

we have that, for all $\eta \in (\theta, 1)$, there is $r > 0$ such that any solution of (1) with an initial datum u_0 satisfying

$$u_0 > \eta \quad \text{in } \Omega \cap B_r$$

converges to 1 as t goes to $+\infty$, locally uniformly in $x \in \overline{\Omega}$.

1.4. ***Statement of the main results.*** One of the main motivations behind the present paper is to answer the following question, raised by Berestycki, Hamel and Nadirashvili [Berestycki et al. 2005]:

Question 4. *Consider the homogeneous equation set on a periodic domain* Ω

$$\begin{cases} \partial_t u - \Delta u = f(u), & t > 0,\ x \in \Omega, \\ \partial_\nu u = 0, & t > 0,\ x \in \partial\Omega. \end{cases} \tag{15}$$

Are there domains Ω *such that* $c^\star \not\equiv w$?

We recall that $c^\star$ is the critical speed of pulsating traveling fronts and w is the speed of invasion. Originally, this question was asked for f of KPP type (7), but it also makes sense if f is a monostable (6) or a combustion (8) nonlinearity.

As we already mentioned, if the domain Ω is the whole space $\mathbb{R}^N$ and if f is of KPP type, then w and $c^\star$ are independent of the direction and are both equal to $2\sqrt{df'(0)}$. In general periodic domains, the propagation may not be isotropic anymore: w and $c^\star$ can depend on the direction.

Let us mention that, if we considered the equation with general coefficients (1), it would be actually much easier to have $c^\star \not\equiv w$ with $\Omega = \mathbb{R}^N$. For instance, in dimension 2, when the Laplace operator is replaced by $a\partial^2_{xx} + b\partial^2_{yy}$, with $a, b > 0$, and when the nonlinearity f is of KPP type, one could explicitly compute $c^\star$ and w, see [Berestycki et al. 2005, Remark 1.12], and one could observe that, if $a \neq b$, then $c^\star \not\equiv w$. What was not known is whether the geometry of the domain alone could give that $c^\star \not\equiv w$. We prove that this is the case.

Theorem 5. *Let* f *be a monostable* (6) *or a combustion* (8) *nonlinearity independent of* x. *There are smooth periodic domains* Ω *such that the critical speed of pulsating traveling fronts* $c^\star$ *and the invasion speed* w *for* (15) *are not everywhere equal*, *that is*, $c^\star \not\equiv w$.

This provides a positive answer to Question 4. When the nonlinearity f is of monostable or combustion type, then the domains we exhibit are L-periodic, with L large enough. If f is a KPP nonlinearity, then we can construct domains with any periodicity.

Let us emphasize that Theorem 5 does not say that we can construct domains where $c^\star(e) \neq w(e)$ for every $e \in \mathbb{S}^{N-1}$: we shall explain after why this is actually impossible. Finding directions where the two speeds coincide is the object of Theorem 7 below.

A first step in proving Theorem 5 will be to give a formula for the speed of invasion. We show that the Freidlin–Gärtner formula (12) still holds true for the general equation (1) in the periodic domain Ω with combustion and monostable nonlinearities.

Theorem 6. *Let A, q, f, Ω be periodic, satisfying* (4)–(5). *Assume that f is a monostable* (6) *or a combustion* (8) *nonlinearity. Then,* (1) *has the asymptotic set of spreading*

$$\mathcal{W} = \{r\xi : \xi \in \mathbb{S}^{N-1},\ 0 \leq r \leq w(\xi)\}, \tag{16}$$

where $w(\xi) := \inf_{e\cdot\xi>0} c^\star(e)/(e\cdot\xi)$, and $c^\star(e)$ is the critical speed of pulsating traveling fronts in the direction e.

This result extends the one by Rossi to the case where the domain is not $\mathbb{R}^N$ anymore. We shall follow the same strategy of proof. As this result is crucial to carry through our investigations, and as the result is of independent interest, we shall prove it in detail in Section 2.

Once Theorem 6 is established, we employ it to derive a simple criterion ensuring that $c^\star \not\equiv w$. We show that if $c^\star \equiv w$, then $c^\star$ and w are necessarily constant; see Proposition 12 below. To answer Question 4 is then tantamount to finding domains where w or $c^\star$ are not constant. Intuitively, we may think that, if a domain is "very obstructed" in a direction, then the speed should be small in this direction.

In order to make this intuition rigorous, we derive new estimates on the invasion speed that take into account the geometry of the domain. This is the subject of Section 3.3. The main tool is an upper bound on the heat kernel in Ω. Once we have these estimates at hand, we are able to construct domains where $c^\star$ and w are not constant, and hence are different. This is done in Section 3.4, proving then Theorem 5.

The remainder of the paper is dedicated to giving conditions under which $c^\star$ and w coincide in some directions. Indeed, observe that, though we can construct domains Ω where $c^\star \not\equiv w$, there is always at least one direction $e \in \mathbb{S}^{N-1}$ such that $c^\star(e) = w(e)$: it is readily seen from the Freidlin–Gärtner formula (12) that any direction $e_{\min}$ that minimizes $c^\star$ satisfies $c^\star(e_{\min}) = w(e_{\min})$. The only other characterization of directions e where $c^\star(e) = w(e)$ we are aware of holds true in the KPP case: it is proved in [Berestycki et al. 2005] that $c^\star(e_{\text{inv}}) = w(e_{\text{inv}})$ if Ω is invariant in the direction e_{inv} (i.e., $\Omega + \{\lambda e_{\text{inv}}\} = \Omega$ for every $\lambda \in \mathbb{R}$).

Our next theorem, proved in Section 4, shows that, if the domain Ω is symmetric, then there are directions where $c^\star$ and w coincide. This result requires $u \mapsto f(u)/u$ to be nonincreasing (strong KPP property).

Theorem 7. *Assume that f satisfies* (4), (5) *and that $u \mapsto f(u)/u$ is nonincreasing. Let $c^\star$ and w be the critical speed of fronts and the speed of invasion for* (15). *Then, assume that there is an orthogonal transformation T such that*:

- *T leaves Ω invariant*; *i.e.*, $T\Omega = \Omega$.
- *There is $e \in \mathbb{S}^{N-1}$ such that $Te = e$, and* $\mathrm{Ker}(T - I_N) = \mathbb{R}e$.

Then

$$c^\star(e) = w(e).$$

This result implies for instance that, if a periodic domain in $\mathbb{R}^2$ is symmetric with respect to an axis, then $c^\star$ and w coincide in the direction of this axis; see Corollary 24 for more examples.

As we shall explain in Remark 25, the hypothesis that $\mathrm{Ker}(T - I_N) = \mathbb{R}e$ is necessary.

Let us conclude this section with some questions that are still open. The set $\mathcal{W}$ given by (16) is sometimes called the *Wulff shape* associated with the surface tension $c^\star$. It appears in crystallography and in isoperimetric problems. A natural question is whether the function w parametrizing the boundary of $\mathcal{W}$ is regular. Rossi [2017] proved that it is continuous. We are not aware of further regularity results. We conjecture that, at least in the KPP case (where $c^\star$ is known to be smooth), w is smooth.

Theorem 5 states that there are domains Ω such that $c^\star \not\equiv w$. One may wonder on the contrary if there are periodic domains $\Omega \neq \mathbb{R}^N$ such that $c^\star \equiv w$. Thanks to our Proposition 12 below, this is equivalent to finding domains where $c^\star$ is constant. As far as we know, the existence of such domains is still open.

Let us also mention that, although the construction of the domain Ω where $c^\star \not\equiv w$ in Theorem 5 will be explicit, our proof will not tell in which direction(s) the two speeds indeed differ. We leave this as an open question.

Remark 8. In addition to the monostable and combustion cases, there is another class of reaction terms f that is widely studied in the literature, namely the *bistable nonlinearities*. The prototype is $f(u) = u(1-u)(u-a)$, with $a \in (0, 1)$. In this paper, we *do not* consider such nonlinearities; indeed, the main tool we use is the existence of pulsating traveling fronts with *positive speed*. If $\Omega = \mathbb{R}^N$, there are results in some particular cases; see [Ducrot 2016; Xin 1991a; 1991b] for instance. If $\Omega \neq \mathbb{R}^N$, the situation is yet to be explored, and the geometry of the domain can yield phenomena that do not appear in the combustion or monostable case. For instance, Rossi and the author showed in [Ducasse and Rossi 2018] that invasion can occur in some directions but not in others. However, we mention that the strategy used to derive Theorem 5 still applies if f is bistable, provided there exist pulsating traveling fronts with positive speeds in every direction $e \in \mathbb{S}^N$.

2. Freidlin–Gärtner formula for a periodic domain

This section is dedicated to the proof of Theorem 6; i.e., we show that the Freidlin–Gärtner formula (12) relating the critical speeds of fronts to the speed of invasion still holds true when the domain is not $\mathbb{R}^N$ but a periodic domain Ω and with monostable or combustion nonlinearities. Our proof is based on the same strategy as the one used in [Rossi 2017]. We start by stating some preliminary technical results. For simplicity, we assume throughout this section that the domain and the coefficients are 1-periodic, i.e., $L_1 = \cdots = L_N = 1$.

2.1. ***Preliminary results.*** In the proof of Theorem 6, we will need some technical lemmas. They generalize those of [Rossi 2017, Section 2.1] to the case where the domain is not $\mathbb{R}^N$ anymore. The main technical difficulty is that Ω is not invariant under translations in general. The proofs follow the same lines as in [Rossi 2017] and can be found for completeness in the Appendix. We say that u is a subsolution (respectively supersolution) if it satisfies (1) with the symbols $=$ replaced by $\leq$ (respectively $\geq$).

The first lemma states that every entire solution that is "large enough" in some direction is actually "front-like" in this direction.

Lemma 9. *Let $\gamma > 0$. Assume that* (4) *and* (5) *hold. Let $u \in C^{1+\alpha/2,2+\alpha}(\mathbb{R} \times \Omega)$ for some $\alpha \in (0,1)$ be an entire supersolution of* (1) *such that*

$$\inf_{\substack{t<0\\ x\cdot e<\gamma t\\ x\in\Omega}} u(t,x) > S,$$

where S is defined in (5). *Then*

$$\liminf_{\delta\to+\infty} \inf_{\substack{t<0\\ x\cdot e<\gamma t-\delta\\ x\in\Omega}} u(t,x) \geq 1.$$

The following lemma is a comparison principle for front-like solutions.

Lemma 10. *Assume that* (4) *and* (5) *hold. Let $\bar{u}$, $\underline{u} \in C^{1+\alpha/2,2+\alpha}(\mathbb{R} \times \Omega)$, for some $\alpha \in (0,1)$, be respectively an entire supersolution and subsolution of* (1). *Assume that there are $e \in \mathbb{S}^{N-1}$, $\gamma > 0$ such that*

$$\bar{u} > 0, \quad \liminf_{\delta\to+\infty} \inf_{\substack{t<0\\ x\cdot e<\gamma t-\delta\\ x\in\Omega}} \bar{u}(t,x) \geq 1. \tag{17}$$

Moreover, assume that $\underline{u} \leq 1$ and that there is $\eta > 0$ such that the following hold:

- *The nonlinearity f is of combustion type* (8) *and*

$$\text{for all } s > 0, \text{ there exists } L \in \mathbb{R},\ \underline{u}(t,x) \leq s \text{ such that if } t \leq 0, \text{ then } x\cdot e \geq (\gamma+\eta)t + L,\ x \in \Omega, \tag{18}$$

  or

- *the nonlinearity f is of monostable type* (6) *and*

$$\text{there exists } L \in \mathbb{R},\ \underline{u}(t,x) \leq 0 \text{ such that if } t \leq 0, \text{ then } x\cdot e \geq (\gamma+\eta)t + L,\ x \in \Omega. \tag{19}$$

Then, the following comparison result holds:

$$\underline{u}(t,x) \leq \bar{u}(t,x), \quad \text{for all } t \in \mathbb{R}, \text{ for all } x \in \Omega.$$

In addition to those two technical lemmas, we shall need the following result, stating that, in our framework, the speed of invasion w is a continuous function:

Lemma 11. *Let A, q, f, Ω be periodic, satisfying* (4)–(5). *Assume that f is of monostable type* (6) *or of combustion type* (8). *Let w be defined by* (12). *Then w is a continuous function from the sphere $\mathbb{S}^{N-1}$ to $\mathbb{R}_+$.*

This lemma is proved in [Rossi 2017] in the case $\Omega = \mathbb{R}^N$, but the proof directly works in our case, so we omit it. It relies on the fact that $c^\star$ is lower semicontinuous. Additionally, we mention that it is proved in [Alfaro and Giletti 2016] that $c^\star$ is actually continuous.

2.2. ***Proof of Theorem 6.*** This section is dedicated to the proof Theorem 6. We show that the Freidlin–Gärtner formula (12) still holds in the context of periodic domains Ω considered in this paper. The proof is divided into several steps. We use a geometric argument, introduced in [Rossi 2017]. The idea is to argue by contradiction: we will consider a solution that invades space, and we will translate our solution in time and space to keep track with the transition zone. Our solution will converge to a "fast" front-like solution, which we shall compare to a pulsating traveling front to get a contradiction.

Proof. We start to prove that $\mathcal{W}$, defined by (16), is an asymptotic subset of spreading. We argue by contradiction. We assume that $\mathcal{W}$ is not an asymptotic subset of spreading; then, there is a compact set $K \subset \text{int}(W)$ such that (13) does not hold. Now, we take $W \subset \mathcal{W}$, W star-shaped with respect to the origin, compact and C^∞ such that $K \subset \text{int}(W)$. We assume that W is the graph of a function $\tilde{w}$, i.e., $W = \{r\xi : \xi \in \mathbb{S}^{N-1},\ 0 \le r \le \tilde{w}(\xi)\}$, with $\tilde{w}$ smooth and $\tilde{w} < w$, so that W is strictly contained in $\mathcal{W}$. We take $\tilde{w}$ strictly positive. This is possible because the function w is continuous thanks to Lemma 11.

The set W satisfies the uniform interior ball estimates

there exists $\rho > 0$ such that for all $x \in \partial W$, there exists $y \in W$ such that $\bar{B}_\rho(y) \subset W$ and $x \in \partial B_\rho(y)$,

where $B_\rho(y)$ is the ball of center y and of radius ρ. In the course of the proof, $u(t, x)$ denotes a solution of (1) arising from a nonnegative, compactly supported initial datum such that invasion occurs; i.e., $u(t, x) \to 1$ as t goes to $+\infty$, locally uniform in $x \in \bar{\Omega}$.

Step 1: Definition of $\mathcal{R}^\eta$. Let $0 < \eta < 1$. We define

$$\mathcal{R}^\eta(t) := \sup\{r \ge 0 : \text{for all } x \in (rW) \cap \bar{\Omega},\ u(t, x) > \eta\}.$$

For $t \ge 0$, this quantity is well-defined because $u(t, x)$ decays to zero as $|x|$ goes to $+\infty$ (this is readily seen by comparison with pulsating traveling fronts) implying that $\mathcal{R}^\eta(t) < +\infty$. Moreover, we have that $\mathcal{R}^\eta(t) \to +\infty$ as t goes to $+\infty$ (because of the assumption that $u(t, x) \to 1$ locally uniformly in x when $t \to +\infty$).

Remembering that we assumed, by contradiction, that there is a compact set $K \subset \text{int}(W)$ such that (13) does not hold, we can infer that there are $\eta, k \in (0, 1)$ such that

$$\liminf_{t \to +\infty} \frac{\mathcal{R}^\eta(t)}{t} < k. \tag{20}$$

Indeed, if this were not the case, then for every $\eta \in (0, 1)$, we would have $\liminf_{t \to +\infty} \mathcal{R}^\eta(t)/t \ge 1$. Hence, taking $h \in (0, 1)$ such that $K \subset hW$, we have

$$\eta \le \liminf_{t \to +\infty} \inf_{x \in R^\eta(t) W} u(t, x) \le \liminf_{t \to +\infty} \inf_{x \in htW} u(t, x) \le \liminf_{t \to +\infty} \inf_{x \in tK} u(t, x).$$

If this were true for each $\eta \in (0, 1)$, it would yield that K satisfies (13), which we assumed not to be the case. Hence, (20) holds. Observe that (20) still holds if we increase η. We do so, and in the following we assume that $\eta \in (S, 1)$, where S is defined in (5).

From now on, we simplify our notation by writing $\mathcal{R}$ instead of $\mathcal{R}^\eta$. Observe that $\mathcal{R}$ is lower semicontinuous. Indeed, let t_n be a sequence such that $t_n \to t_0$ as n goes to $+\infty$ and such that $\mathcal{R}(t_n) \to R \in \mathbb{R}$.

Consider $r > R$. Then, for n large enough, we have that $r > \mathcal{R}(t_n)$, and, by the definition of $\mathcal{R}(t_n)$, there is $x_n \in (rW) \cap \overline{\Omega}$ such that $u(t_n, x_n) \le \eta$. By the continuity of u, there is some $x_0 \in (rW) \cap \overline{\Omega}$ such that $u(t_0, x_0) \le \eta$. This implies that $\mathcal{R}(t_0) \le r$, and then that $\mathcal{R}(t_0) \le R$ by the arbitrariness of $r > R$, and hence the semicontinuity.

Step 2: Shifting the function. By definition of $\mathcal{R}$ we have that $\liminf_{t\to+\infty}(\mathcal{R}(t) - kt) = -\infty$. We define, for $n \in \mathbb{N}$,

$$t_n := \inf\{t \ge 0 : \mathcal{R}(t) - kt \le -n\}.$$

The lower semicontinuity of $\mathcal{R}$, proved in the first step, gives us that the above infimum is a minimum, i.e., that $\mathcal{R}(t_n) - kt_n \le -n < \mathcal{R}(t) - kt$ for all $t < t_n$, and that $t_n \to +\infty$ as n goes to $+\infty$. Hence, the sequence $(t_n)_{n\in\mathbb{N}}$ satisfies

$$\lim_{n\to+\infty} t_n = +\infty \qquad \text{and} \qquad \text{for all } n \in \mathbb{N}, \text{ for all } t \in [0, t_n), \quad \mathcal{R}(t_n) - k(t_n - t) < \mathcal{R}(t).$$

Now, by the definition of $\mathcal{R}(t)$, we have that for all $r > \mathcal{R}(t)$ there exists $x_r \in (rW \cap \overline{\Omega}) \setminus ((\mathcal{R}(t)W) \cap \overline{\Omega})$ such that $u(t, x_r) \le \eta$. Up to extraction, we can assume that $x_r \to x_\infty$ as r goes to $\mathcal{R}(t)$, where $x_\infty \in \overline{\Omega} \cap \partial(\mathcal{R}(t)W)$. By continuity, we have that $u(t, x_\infty) = \eta$.

Hence, we can consider a sequence $(x_n)_{n\in\mathbb{N}} \in \overline{\Omega}$ such that $u(t_n, x_n) = \eta$, with the additional property that $x_n \in \partial(\mathcal{R}(t_n)W)$. Clearly, $|x_n| \to +\infty$ as n goes to $+\infty$. If $x \in \partial W$, let $\tilde{\nu}(x)$ be the outer unit normal to W at the point x. We define

$$\hat{x}_n = \frac{x_n}{\mathcal{R}(t_n)} \quad \text{and} \quad y_n = \hat{x}_n - \rho\tilde{\nu}(\hat{x}_n).$$

By definition, $\hat{x}_n \in \partial W$ and y_n is the center of the interior ball tangent to W at the point $\hat{x}_n$, of radius ρ (we recall that W satisfies the uniform interior ball estimate with radius ρ).

For every n, we define $k_n \in \mathbb{Z}^N$ and $z_n \in [0, 1)^N$ by $x_n = k_n + z_n$. Up to extraction, we can assume that there is $z \in [0, 1]^N$ such that $z_n \to z$ as $n \to +\infty$. We also assume that there is $\hat{x}$ such that $\hat{x}_n$ converges to $\hat{x}$, whence $\tilde{\nu}(\hat{x}_n)$ converges to $\tilde{\nu}(\hat{x})$. We now define, for $n \in \mathbb{N}$, the translated functions

$$u_n(t, x) = u(t + t_n, x + k_n).$$

Thanks to the periodicity and regularity hypotheses on Ω, we can apply the usual interior and portion boundary parabolic estimates (see, for instance [Ladyzhenskaya et al. 1968, Theorems 5.2, 5.3]) to get that u_n converges uniformly locally to an entire solution $u^\star$ of (1). Moreover $u^\star(0, z) = \eta$.

Step 3: Properties of $u^\star$. We show here that $u^\star$ is a front-like solution, in the sense that it satisfies, writing $H_T := \{x \in \Omega : x \cdot \tilde{\nu}(\hat{x}) < -k\hat{x} \cdot \tilde{\nu}(\hat{x})T\}$,

$$\text{for all } T \ge 0, \text{ for all } x \in H_T + \{z\}, \quad u^\star(-T, x) \ge \eta. \tag{21}$$

To show this, take $T \in [0, t_n]$ and $x \in (\mathcal{R}(t_n) - kT)W \cap \Omega$. As $\mathcal{R}(t_n) - kT \le \mathcal{R}(t_n - T)$, we have that $x \in \mathcal{R}(t_n - T)W \cap \Omega$. Therefore, by the definition of $\mathcal{R}$, we have $u(t_n - T, x) \ge \eta$. Then, we have

$$\text{for all } T \in [0, t_n], \text{ for all } x \in ((\mathcal{R}(t_n) - kT)W) \cap \Omega - \{k_n\}, \quad u_n(-T, x) \ge \eta.$$

From that, we infer

$$\text{for all } T \geq 0, \text{ for all } x \in \Omega \cap \bigcup_{M\in\mathbb{N}} \bigcap_{n\geq M} ((\mathcal{R}(t_n) - kT)W - \{k_n\}), \quad u^\star(-T, x) \geq \eta.$$

To prove (21), it suffices to show that $H_T + \{z\} \subset \Omega \cap \bigcup_{M\in\mathbb{N}} \bigcap_{n\geq M}((\mathcal{R}(t_n) - kT)W - \{k_n\})$. To see this, take $x \in H_T + \{z\}$. We start by computing

$$\begin{aligned}\left|\frac{x+k_n}{\mathcal{R}(t_n)-kT} - y_n\right| &= \left|\frac{x+k_n-(\mathcal{R}(t_n)-kT)(\hat{x}_n - \rho\tilde{\nu}(\hat{x}_n))}{\mathcal{R}(t_n)-kT}\right| \\ &= \left|\frac{x+kT\hat{x}_n+(k_n-x_n)+(\mathcal{R}(t_n)-kT)\rho\tilde{\nu}(\hat{x}_n)}{\mathcal{R}(t_n)-kT}\right| \\ &= \left|\rho\tilde{\nu}(\hat{x}_n) + \frac{x+kT\hat{x}_n - z_n}{\mathcal{R}(t_n)-kT}\right|.\end{aligned}$$

Let us write

$$w_n := \frac{x+kT\hat{x}_n - z_n}{\mathcal{R}(t_n)-kT}.$$

This goes to zero as n goes to infinity. The last term in the above equality can be rewritten

$$|\rho\tilde{\nu}(\hat{x}_n) + w_n| = \sqrt{\rho^2 + |w_n|\left(\frac{2\rho\tilde{\nu}(\hat{x}_n)\cdot w_n}{|w_n|+|w_n|}\right)}.$$

Now, observe that

$$\lim_{n\to+\infty} \frac{2\rho\tilde{\nu}(\hat{x}_n)\cdot w_n}{|w_n|+|w_n|} = 2\rho\tilde{\nu}(\hat{x}) \cdot \frac{x+kT\hat{x}-z}{|x+kT\hat{x}-z|}.$$

This limit is strictly negative. Indeed, if $x \in H_T + \{z\}$, then $(x-z) \cdot \tilde{\nu}(\hat{x}) < -kT\hat{x} \cdot \tilde{\nu}(\hat{x})$. Therefore, we have, for n large enough,

$$\left|\frac{x+k_n}{\mathcal{R}(t_n)-kT} - y_n\right| < \rho,$$

which means $(x+k_n)/(\mathcal{R}(t_n)-kT) \in W$ by the definition of y_n and ρ. That is, $x \in (\mathcal{R}(t_n)-kT)W-\{k_n\}$, which concludes this step.

Step 4: Comparison. We now compare the function $u^\star$ constructed in the previous steps to the pulsating traveling front in the direction $\tilde{\nu}(\hat{x})$ with critical speed $c^\star(\tilde{\nu}(\hat{x}))$. Combining Lemma 9 and (21), we have

$$\liminf_{\delta\to+\infty} \inf_{\substack{t<0 \\ x\cdot\tilde{\nu}(\hat{x})<\gamma t-\delta \\ x\in\Omega}} u^\star(t,x) \geq 1,$$

with $\gamma := k\hat{x}\cdot\tilde{\nu}(\hat{x}) > 0$. Hence $u^\star$ satisfies the hypotheses of Lemma 10. Observe that we have

$$\gamma = k\hat{x}\cdot\tilde{\nu}(\hat{x}) = k\frac{\hat{x}}{|\hat{x}|}\cdot\tilde{\nu}(\hat{x})\tilde{w}\left(\frac{\hat{x}}{|\hat{x}|}\right) < \frac{\hat{x}}{|\hat{x}|}\cdot\tilde{\nu}(\hat{x})w\left(\frac{\hat{x}}{|\hat{x}|}\right) \leq c^\star(\tilde{\nu}(\hat{x})),$$

where the last inequality follows from the definition of w in Theorem 6.

Assume first that f is of combustion type (8). Let v be a pulsating traveling front in the direction $\nu(\hat{x})$, with critical speed $c^\star(\nu(\hat{x}))$. Up to a time translation, we normalize it so that $v(0,0) > u^\star(0,0)$. Then, v

satisfies the hypotheses of Lemma 10 (with $\eta = c^\star(\nu(\hat{x})) - \gamma$ in the hypotheses of Lemma 10), giving $v \leq u^\star$, which is in contradiction with the fact that $v(0,0) > u^\star(0,0)$.

Now, if the nonlinearity is of monostable type (6), we have to construct a function v satisfying (19) to apply Lemma 10. This can be done exactly as in [Rossi 2017, Proposition 2.6]; the fact that the domain is not $\mathbb{R}^N$ adds no difficulty here. This proves that W is an asymptotic subset of spreading, and then, so is $\mathcal{W}$. Now, we show that it is an asymptotic superset of spreading.

Step 5: Superset of spreading. Let C be a closed set such that $\mathcal{W} \cap C = \varnothing$. Then, because w is continuous, we can find $\varepsilon > 0$ so that $\mathcal{W}_\varepsilon := \{r\xi : \xi \in \mathbb{S}^{N-1},\ 0 \leq r \leq w(\xi) + \varepsilon\}$ is such that $\mathcal{W}_\varepsilon \cap C = \varnothing$. To prove that $\mathcal{W}$ is an asymptotic superset of spreading, it is sufficient to show that $\sup_{x \in t\mathcal{W}_\varepsilon^c} u(t,x) \to 0$ as t goes to $+\infty$. To do so, we take a sequence $(t_n)_{n\in\mathbb{N}} \in (\mathbb{R}^+)^{\mathbb{N}}$ such that t_n goes to infinity as n goes to infinity and a sequence $x_n \in t_n\mathcal{W}_\varepsilon^c$ such that

$$u(t_n, x_n) \geq \tfrac{1}{2} \sup_{x \in t_n \mathcal{W}_\varepsilon^c} u(t_n, x).$$

Up to extraction, we take $e \in \mathbb{S}^{N-1}$ such that $x_n/|x_n| \to e$ as n goes to $+\infty$. Let $\xi \in \mathbb{S}^{N-1}$ be such that $w(e) = c^\star(\xi)/(\xi \cdot e)$, and let v be a pulsating traveling front in the direction ξ with critical speed $c^\star(\xi)$. Up to some translation in time, we can assume, thanks to the parabolic comparison principle, that $u(t,x) \leq v(t,x)$ for all $t \geq 0$, for all $x \in \Omega$. Let us show that $v(t_n, x_n)$ goes to zero as $n \to +\infty$.

We write $x_n := (x_n/|x_n| \cdot \xi)|x_n|\xi + d_n$, where d_n is orthogonal to ξ. Because $x_n/|x_n| \to e$ as n goes to $+\infty$, using the continuity of w, for n large enough, we have

$$\left(\frac{x_n}{|x_n|} \cdot \xi\right)|x_n| \geq \left(\frac{x_n}{|x_n|} \cdot \xi\right)\left(w\left(\frac{x_n}{|x_n|}\right) + \varepsilon\right)t_n \geq \left(c^\star(\xi) + (e \cdot \xi)\frac{\varepsilon}{2}\right)t_n.$$

So, we get that, for $n \in \mathbb{N}$ large enough, there is some λ_n such that $\lambda_n \geq c^\star(\xi) + (e \cdot \xi)(\varepsilon/2)$ and $x_n = \lambda_n \xi t_n + d_n$. Now, observe that the definition of the pulsating traveling fronts, Definition 2, implies that $v(t_n, \lambda_n t_n \xi + d_n) \to 0$ as n goes to $+\infty$; hence

$$\lim_{n\to+\infty} \tfrac{1}{2} \sup_{x \in t_n \mathcal{W}_\varepsilon^c} u(t_n, x) \leq \lim_{n\to+\infty} u(t_n, x_n) \leq \lim_{n\to+\infty} v(t_n, x_n) = 0,$$

which implies the result. □

Now that we have the Freidlin–Gärtner formula (12) at our disposal, we use it to answer Question 4.

3. Invasion and the critical speed of fronts

This whole section is dedicated to the proof of Theorem 5. We consider here the problem (15), with nonlinearity f independent of x of monostable or combustion type. In the following, for f and Ω given, we denote by $c^\star$ and w the critical speed of fronts and the speed of invasion respectively, for (15).

The proof of Theorem 5 is done in several steps: first, we show that $w \equiv c^\star$ is equivalent to saying that w and $c^\star$ are actually constant. This is the object of Section 3.1. Then, we exhibit in Section 3.3 some estimates on the spreading speed that take into account the geometry of the domain. Gathering all this, we will be able to prove Theorem 5.

3.1. ***Comparison between w and $c^\star$.*** This section is dedicated to proving that, if the critical speed of fronts $c^\star$ and the speed of invasion w are everywhere equal, then they are constant. This uses only the Freidlin–Gärtner formula (12) proved in Section 2.

Proposition 12. *Assume that Ω is a smooth periodic domain satisfying* (3) *and that f is a nonlinearity satisfying* (5) *of the monostable* (6) *or combustion* (8) *type. Assume that $c^\star \equiv w$. Then, the functions w and $c^\star$ are constant.*

Proof. Because of the hypotheses on Ω and f, we can apply Theorem 6 to get that for all $e \in \mathbb{S}^{N-1}$ $w(e) = \inf_{\xi\cdot e>0} c^\star(\xi)/(\xi\cdot e)$. Assume that $w \equiv c^\star$ and take ξ_0, $\xi \in \mathbb{S}^{N-1}$ so that $\xi_0\cdot\xi > 0$, and let ω be the angle between those two vectors. Let us take $M \in \mathbb{N}$. We define a sequence $(\xi_k)_{k\in[\![0,M]\!]} \in \mathbb{S}^{N-1}$ to be equidistributed on the arc joining ξ_0 to ξ on the sphere; i.e., $\xi_k\cdot\xi_{k+1} = \cos(\omega/M)$ and $\xi_M = \xi$. Then, we have

$$w(\xi_0) \le w(\xi_1)\frac{1}{\xi_0\cdot\xi_1} \le w(\xi_2)\frac{1}{\xi_2\cdot\xi_1}\frac{1}{\xi_1\cdot\xi_0}.$$

Iterating and using that $\xi_k\cdot\xi_{k+1} = \cos(\omega/M)$, we get

$$w(\xi_0) \le w(\xi)\prod_{k=0}^{M-1}\frac{1}{\xi_{k+1}\cdot\xi_k} = w(\xi)\frac{1}{\cos(\omega/M)^M}.$$

Because $1/\cos(\omega/M)^M \to 1$ as M goes to $+\infty$, passing to the limit yields

$$w(\xi_0) \le w(\xi).$$

Inverting the roles of ξ_0 and ξ, we get $w(\xi_0) = w(\xi)$. Hence, w is constant, and so is $c^\star$. □

Observe that, in the course of the proof, we did not use the particular form of (15), only the Freidlin–Gärtner formula; hence Proposition 12 holds true also for the general equation (1).

As mentioned in the Introduction, we shall use this result to construct domains where $c^\star \not\equiv w$. Indeed, Proposition 12 reduces the problem to finding domains where w or $c^\star$ are not constants. Intuitively, it seems that, if in a certain direction e there are many obstacles, then the speeds w and $c^\star$ should be small. On the contrary, if in a certain direction, there are few obstacles, then the speeds should be larger. Hence, if the domain Ω is very "obstructed" in some direction and not in another, then the speeds should not be constants, and so they would be different.

To construct such domains is actually quite easy if f is KPP and if the dimension is greater than or equal to 3: in this case, domains that are invariants in one direction provide an answer to Question 4. We shall focus on such domains in the next Section 3.2. There, we shall also prove a lemma that will be useful in Section 4. If the nonlinearity is not KPP or if the dimension is equal to 2, things are much more involved. To overcome this difficulty, we introduce estimates for w that do take into account the geometry of the domain. This is done in Section 3.3.

3.2. ***Invasion in domains that are invariant in one direction.*** In this section, f is a KPP nonlinearity independent of x and Ω is invariant in the direction $e \in \mathbb{S}^{N-1}$; i.e., for all $\lambda \in \mathbb{R}$, we have $\Omega + \lambda e = \Omega$. Let us answer Question 4 in this specific case by proving the following:

Proposition 13. *Let Ω be a periodic domain in $\mathbb{R}^N$, $N \geq 3$, satisfying* (3) *and suppose that there is $e \in \mathbb{S}^{N-1}$ such that Ω is invariant in the direction e. Let f satisfying* (5) *be a KPP nonlinearity independent of x. Denoting by $c^\star$ and w the critical speed of fronts and the speed of invasion respectively for problem* (15), *we have*

$$w \equiv c^\star \quad \iff \quad \Omega = \mathbb{R}^N.$$

This comes directly by combining our Proposition 12 with the following result from [Berestycki et al. 2005]:

Theorem 14. *Let $c^\star$ be the critical speed of fronts for the problem* (15) *with f KPP independent of x. Then $c^\star(e) \leq 2\sqrt{f'(0)}$ and the equality holds if and only if Ω is invariant in the direction e.*

If Ω is a periodic domain satisfying hypothesis (3) and invariant in a direction $\Omega \neq \mathbb{R}^N$, then this theorem implies that $c^\star$ is not a constant function of the direction. Then, Proposition 12 implies that $c^\star \not\equiv w$. This answers Question 4 in the special case where f is KPP and the dimension greater than 3. The general setting is more involved and is addressed after.

Before ending this section, we prove a result concerning domains invariant in a direction that will be useful in Section 4. When considering such domains, we can actually give further information about the shape of the asymptotic set of spreading $\mathcal{W}$. The next result shows that, if Ω is invariant in the direction e, then the spreading speed in a direction orthogonal to e only depends on the part of the domain orthogonal to e.

Proposition 15. *Let Ω be a periodic domain satisfying* (3), *invariant in the direction $e \in \mathbb{S}^{N-1}$. Let $\mathcal{W}$ be the asymptotic set of spreading of* (15) *set on Ω with f satisfying* (5) *and such that $u \mapsto f(u)/u$ is decreasing* (*this implies that f is KPP*). *Let $\mathcal{H}$ be the hyperplane in $\mathbb{R}^N$ orthogonal to e. Then, if $\mathcal{W}_{\mathcal{H}\cap\Omega}$ is the asymptotic set of spreading for the equation restricted to $\mathcal{H} \cap \Omega$, i.e.,*

$$\begin{cases} \partial_t u - \Delta u = f(u), & t > 0,\ x \in \mathcal{H} \cap \Omega, \\ \quad \partial_{\nu'} u = 0, & t > 0,\ x \in \partial(\mathcal{H} \cap \Omega), \end{cases} \tag{22}$$

where $\nu' \in \mathbb{S}^{N-2}$ denotes the exterior normal to $\mathcal{H} \cap \Omega$, we have

$$\mathcal{W}_{\mathcal{H}\cap\Omega} = \mathcal{W} \cap \mathcal{H}.$$

Proof. To simplify the notation, we denote by w_N the spreading speed for the Fisher-KPP equation (15) set on $\Omega \subset \mathbb{R}^N$ and w_{N-1} the spreading speed for (22) set on $\mathcal{H} \cap \Omega \subset \mathbb{R}^{N-1}$. Similarly, we denote by $c^\star_N$ and $c^\star_{N-1}$ the critical speeds of fronts for (15) and (22) respectively. Up to some rotation of the coordinates, we write the points of Ω in the form (x, y), where $x \in \mathcal{H} \cap \Omega$ and $y \in \mathbb{R}$.

Step 1: $\mathcal{W} \cap \mathcal{H} \subset \mathcal{W}_{\mathcal{H}\cap\Omega}$. We start by showing that, for each $\zeta \in \mathbb{S}^{N-2}$, we have $w_N((\zeta, 0)) \leq w_{N-1}(\zeta)$. To do so, take $\xi \in \mathbb{S}^{N-2}$ such that $\xi \cdot \zeta > 0$. Let $\phi_\xi(t, x)$ be a pulsating traveling front solution of (22) in the direction ξ with critical speed $c^\star_{N-1}(\xi)$. For $(x, y) \in \Omega$, we define $\Phi(t, x, y) := \phi_\xi(t, x)$. Then Φ is solution of (15) on the whole of Ω. If $u_0(x, y)$ is a nonnegative compactly supported initial datum and if $u(t, x, y)$ is the solution of (15) arising from it, we can assume that (up to translation) $u_0(x, y) \leq \Phi(0, x, y)$. Hence, the parabolic comparison principle gives us that

$$u(t, x, y) \leq \Phi(t, x, y) \quad \text{for all } t \geq 0, \text{ for all } (x, y) \in \Omega.$$

Observe that Φ moves in the direction $(\zeta, 0) \in \mathbb{S}^{N-1}$ with speed $c^\star_{N-1}(\xi)/(\xi \cdot \zeta)$. This means that $w_N((\zeta, 0)) \leq c^\star_{N-1}(\xi)/(\xi \cdot \zeta)$, and because this is true for all ξ such that $\xi \cdot \zeta > 0$, Theorem 6 implies that $w_N((\zeta, 0)) \leq w_{N-1}(\zeta)$.

Step 2: $\mathcal{W}_{\mathcal{H}\cap\Omega} \subset \mathcal{W} \cap \mathcal{H}$. We now prove the reverse inequality. To start, let $\varepsilon > 0$ be fixed such that $\varepsilon^2 < f'(0)$. We define a KPP nonlinearity $f_\varepsilon(u) := f(u) - \varepsilon^2 u$. Let $u_0(x)$ be a smooth, nonnegative, compactly supported function in $\mathcal{H} \cap \Omega$. Let $u_\varepsilon(t, x)$ be the solution arising from u_0 of (15) but *with f replaced by f_ε*.

Define the cut-off function

$$\phi(y) := \begin{cases} \cos(\varepsilon y) & \text{for } |y| \leq \pi/(2\varepsilon), \\ 0 & \text{for } |y| \geq \pi/(2\varepsilon). \end{cases}$$

Now, let $v(t, x, y) := u_\varepsilon(t, x)\phi(y)$. Let us show that v is a (generalized) subsolution. An easy computation shows that, for $(x, y) \in \Omega$ such that $v(t, x, y) > 0$, we have

$$\begin{aligned} \partial_t v - \Delta v - f(v) &= f_\varepsilon(u_\varepsilon)\phi(y) + \varepsilon^2 u_\varepsilon \phi(y) - f(u_\varepsilon \phi) \\ &= \left(\frac{f_\varepsilon(u_\varepsilon)}{u_\varepsilon} - \frac{f(u_\varepsilon \phi)}{u_\varepsilon \phi} + \varepsilon^2 \right) u_\varepsilon \phi \leq 0. \end{aligned}$$

The last inequality comes from the fact that $z \mapsto f(z)/z$ is decreasing. One can then check that $\partial_\nu v = 0$ on $\partial\Omega$. This comes from $\partial_{\nu'} u_\varepsilon = 0$ on $\partial(\Omega \cap \mathcal{H})$ together with the fact that Ω is invariant in the direction e.

Hence, $u_\varepsilon \phi$ is a (generalized) subsolution of (15) (with nonlinearity f). We can observe that u_ε spreads in $\Omega \cap \mathcal{H}$ in the direction $\zeta \in \mathbb{S}^{N-2}$ with speed $w_{N-1}(\zeta) - \varepsilon^2$. Hence, by comparison, we get that $w_{N-1}(\zeta) - \varepsilon^2 \leq w_N((\zeta, 0))$. Taking the limit $\varepsilon \to 0$ yields the result. □

Observe that the same result holds in what concerns the critical speed of fronts: using the same notation as in the proof, we can prove that $c^\star_N((e, 0)) = c^\star_{N-1}(e)$ for every $e \in \mathbb{S}^{N-2}$: one inequality is proved in the first step, and the second inequality can be proved as in the second step just by taking u_ε to be front.

Now, we turn to the full proof of Theorem 5, answering then Question 4.

3.3. ***Geodesic estimates.*** This aim of this section is to establish estimates on $w(e)$ that do take into account the geometry of the domain. The key tool is an estimate on the heat kernel from [Berestycki et al. 2010], following from general results on the heat kernel from [Davies 1989; Grigoryan 1997]. This estimate is valid for domains satisfying the *extension property*. Denoting by $W^{1,p}(\Omega)$ the usual Sobolev space over Ω, a nonempty subset of $\mathbb{R}^N$ satisfies the extension property if, for all $1 \leq p \leq +\infty$, there is a bounded linear map E from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^N)$ such that $E(f)$ is an extension of f from Ω to $\mathbb{R}^N$ for all $f \in W^{1,p}(\Omega)$. For our purpose, we mention that the smooth periodic domains we consider here satisfy the extension property; see [Stein 1970].

Proposition 16. *Let Ω be a locally C^2 nonempty connected open subset of $\mathbb{R}^N$ satisfying the extension property. Let $p(t, x, y)$ be the heat kernel in $\overline{\Omega}$ with Neumann boundary condition on $\partial\Omega$. Then, for every $\varepsilon > 0$, there are two positive constants C and δ such that*

$$\text{for all } t > 0, \text{ for all } (z, x) \in \overline{\Omega} \times \overline{\Omega}, \quad p(t, z, x) \leq C(1 + \delta t^{-N/2}) \exp\left(-\frac{d_\Omega(z, x)^2}{(4 + \varepsilon)t} \right), \tag{23}$$

where $d_\Omega(z, x)$ denotes the geodesic distance in $\overline{\Omega}$.

See [Berestycki et al. 2010, Proposition 2.5] for the proof. We use this to get upper estimates on the spreading speed $w(e)$. To do so, we introduce the following coefficient for $e \in \mathbb{S}^{N-1}$:

$$C_\Omega(e) := \liminf_{\lambda \to +\infty} \frac{\lambda}{d_\Omega(0, \lambda e)}. \tag{24}$$

For notational simplicity and without loss of generality, we assume in the following that the point 0 is in Ω (this is always possible up to translation).

This coefficient represents *how much the domain is obstructed* in the direction e. The geodesic distance d_Ω is always greater than the euclidean distance; hence $C_\Omega(e) \le 1$.

Proposition 17. *Let Ω be a domain satisfying* (3) *and f a monostable* (6) *or a combustion* (8) *nonlinearity independent of x. We denote by w the speed of invasion associated to problem* (15). *Then, we have*

$$w(e) \le 2C_\Omega(e)\sqrt{\max_{u\in[0,1]} \frac{f(u)}{u}}. \tag{25}$$

Observe that, if f is a KPP nonlinearity, then this formula boils down to $w(e) \le 2C_\Omega(e)\sqrt{f'(0)}$. In the case where $\Omega = \mathbb{R}^N$, the upper bound is actually the KPP speed $2\sqrt{f'(0)}$.

Proof. Let us observe that it is sufficient to prove the result in the KPP case. Indeed, if f is a monostable or a combustion nonlinearity, then we can find a KPP nonlinearity $\overline{f}$ such that $\overline{f}'(0) = \max_{u\in[0,1]} f(u)/u$ and $\overline{f} \ge f$. If u_0 is an initial datum, denoting by u, respectively $\bar{u}$, the solution of (15) with nonlinearity f, respectively $\overline{f}$, arising from u_0, the parabolic comparison principle tells us that

$$u(t,x) \le \bar{u}(t,x) \quad \text{for all } t \ge 0, \text{ for all } x \in \Omega.$$

Then, $w(e) \le \overline{w}(e)$, for all $e \in \mathbb{S}^{N-1}$, where w, respectively $\overline{w}$, is the invasion speed for (15) with nonlinearity f, respectively $\overline{f}$. Then, it is sufficient to prove the estimate (25) for $\overline{f}$ because $\max_{u\in[0,1]} \overline{f}(u)/u = \max_{u\in[0,1]} f(u)/u$. Hence, in the rest of the proof, we assume that f is KPP, and then $\max_{u\in[0,1]} f(u)/u = f'(0)$.

Let $u(t,x)$ be the solution of the parabolic problem (15) arising from a compactly supported nonnegative initial smooth datum u_0. Let K be a compact set of Ω such that the support of u_0 is in K. We denote by $p(t,x,z)$ the heat kernel with Neumann condition on Ω. Then, we first observe that

$$u(t,x) \le e^{f'(0)t} \int_\Omega p(t,x,z) u_0(z)\,dz. \tag{26}$$

Indeed, $e^{f'(0)t}\int_\Omega p(t,x,z)u_0(z)\,dz$ is the solution of the linearized problem

$$\begin{cases} \partial_t v - \Delta v = f'(0)v, & t>0,\ x\in\Omega, \\ \partial_\nu v = 0, & t>0,\ x \in \partial\Omega, \\ v(0,x) = u_0(x), & x\in\Omega, \end{cases} \tag{27}$$

and hence is a supersolution of (15), thanks to the KPP property. Then, the inequality (26) follows by the parabolic comparison principle. Now, let $\varepsilon > 0$ be fixed. Using the estimate (23) in (26), we get

$$u(t,x) \le C(1+\delta t^{-N/2}) e^{f'(0)t} \int_\Omega \exp\Big(-\frac{d_\Omega(z,x)^2}{(4+\varepsilon)t}\Big) u_0(z)\,dz \tag{28}$$

for some positive constants C and δ (depending on ε). This gives us

$$u(t,x) \le C\|u_0\|_{L^1}(1+\delta t^{-N/2})\exp\left(\left(f'(0) - \frac{(\min_{z\in K} d_\Omega(z,x))^2}{(4+\varepsilon)t^2}\right)t\right). \tag{29}$$

Now, take $e \in \mathbb{S}^{N-1}$ and $\omega > 0$ such that $\omega < w(e)$. Then, $u(t, \omega t e) \to 1$ as $t \to +\infty$, by the definition of $w(e)$. Then, necessarily, we have

$$\limsup_{t\to+\infty} \frac{\min_{z\in K} d_\Omega(z, \omega t e)}{t} \le \sqrt{(4+\varepsilon) f'(0)},$$

if this were not the case, up to subsequence the right-hand term of (29) would go to zero along some time sequence $(t_n)_{n\in\mathbb{N}}$, $t_n \to +\infty$ as n goes to $+\infty$, which would be in contradiction with the fact that $u(t_n, \omega t_n e)$ goes to 1. Using the triangular inequality for d_Ω and the fact that K is compact we eventually get

$$\omega \le \frac{\sqrt{(4+\varepsilon) f'(0)}}{\limsup_{t\to+\infty} d_\Omega(0, \omega t e)/(\omega t)}.$$

Recalling the definition of $C_\Omega(e)$ and that the above inequality is true for every $\varepsilon > 0$, we have

$$\omega \le 2C_\Omega(e)\sqrt{f'(0)},$$

and the result follows. □

We are now in position to answer Question 4.

3.4. *Domains where $c^\star \not\equiv w$.* In this section, we construct periodic domains Ω such that $c^\star \not\equiv w$. If f is a KPP nonlinearity, we exhibit a 1-periodic domain (but the periodicity can be chosen arbitrary). If f is a monostable or a combustion nonlinearity, we construct an L-periodic domain, where $L > 0$ can be large. For clarity, we do this in dimension $N = 2$, but these constructions can be easily generalized to larger dimensions.

In the following, we define $e_x := (1, 0)$, $e_y := (0, 1) \in \mathbb{S}^1$ the unit vectors of the canonical basis of $\mathbb{R}^2$. Moreover, we define $e_d := (1/\sqrt{2})(1, 1) \in \mathbb{S}^1$.

3.4.1. *The KPP case.* We show here the following:

Proposition 18. *Let f be a KPP nonlinearity* (7). *There is a smooth periodic domain $\Omega \subset \mathbb{R}^2$ such that*

$$c^\star(e_x) > w(e_d),$$

where $c^\star$ and w are the critical speed of fronts and the speed of invasion respectively for (15) *set in Ω with nonlinearity f.*

We see that in this domain, it is not possible that $w = c^\star$, thanks to Proposition 12. Hence, this answers Question 4 in the KPP case.

Proof. For $\alpha \in \left(\frac{1}{2}, 1\right)$, $\beta \in \left(0, \frac{1}{2}\right)$, we define $\Omega_{\alpha,\beta}$ to be a smooth periodic domain such that

$$\mathbb{Z}^2 + (1-\alpha, \alpha) \times [\beta, 1-\beta] \subset \Omega^c_{\alpha,\beta} \subset \mathbb{Z}^2 + \left(\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right) \times [\beta, 1-\beta]. \tag{30}$$

This domain is simply $\mathbb{R}^2$ with "almost square" obstacles. For α, β given we denote by $c^\star_{\alpha,\beta}(e)$ the critical speed of fronts in this domain in the direction e. If β is fixed and if we let $\alpha \to 1$, then the domain "converges" in some sense to an array of parallel disconnected strips in the direction e_x. This observation is made rigorous by [Berestycki et al. 2005, Theorem 1.4], where it is proved that

$$c^\star_{\alpha,\beta}(e_x) \xrightarrow[\alpha\to 1]{} 2\sqrt{f'(0)}.$$

Now, let $\kappa \in (1, \sqrt{2})$ and take α close enough to 1 so that $c^\star_{\alpha,\beta}(e_x) > (1/\kappa)2\sqrt{f'(0)}$.

Take $n \in \mathbb{N}$. Denoting by $d_{\Omega_{\alpha,\beta}}$ the geodesic distance in $\Omega_{\alpha,\beta}$, it is easy to see that $d_{\Omega_{\alpha,\beta}}(0, n\sqrt{2}e_d) \geq 2n(\alpha-\beta)$. Plotting this in (24) yields $C_{\Omega_{\alpha,\beta}}(e_d) \leq 1/(\sqrt{2}(\alpha-\beta))$. Taking β small enough, and increasing α if needed, we can assume that $C_{\Omega_{\alpha,\beta}}(e_d) \leq 1/\kappa$. Denoting by $w_{\alpha,\beta}$ the speed of invasion in the domain $\Omega_{\alpha,\beta}$, Proposition 17 implies that $w_{\alpha,\beta}(e_d) \leq (1/\kappa)2\sqrt{f'(0)}$. Hence, $c^\star_{\alpha,\beta}(e_x) > w_{\alpha,\beta}(e_d)$ when α is close enough to 1 and β close enough to 0. This yields the result. □

3.4.2. *Combustion and monostable case.* Now, we answer Question 4 in the case where f is a combustion or a monostable nonlinearity. We do it for f combustion first, and then we explain how this yields the result for monostable nonlinearities.

Proposition 19. *Let f be a combustion nonlinearity* (8). *Then, there are $L > 0$ and a family of smooth L-periodic domains $(\Omega_\alpha)_{\alpha\in(0,1)}$ such that $w_\alpha(e_x) \geq K$, where $K > 0$ is independent of α, and $w_\alpha(e_y) \to 0$ as α goes to* 0.

If $\alpha > 0$ is chosen small enough so that $w_\alpha(e_y) < K$, we see that w_α cannot be constant, and then Proposition 12 implies that $c^\star \not\equiv w$ on Ω_α for α small. This answers Question 4 and proves Theorem 5 when f is a combustion nonlinearity.

Before turning to the proof of Proposition 19, we state the following technical lemma. We recall that we denote by B_R the ball of radius R and of center 0.

Lemma 20. *Let f be a combustion nonlinearity* (8) *independent of x. Then, there are $R, c > 0$ and $\phi \in W^{2,\infty}(\mathbb{R}^2)$, $\phi > 0$ in B_R and $\phi = 0$ on ∂B_R such that, on B_R we have*

$$\Delta\phi + c\partial_x\phi + f(\phi) \geq 0.$$

Proof. We construct ϕ to be radial. We set $\phi(x) := h(|x|)$. Now, take R_1, R_2, $R_3 > 0$ to be chosen after, such that $R_1 < R_2 < R_3$. We set $\tilde{c} := c + 1/R_1$. Let $C \in (\theta, 1)$, and α, $\beta > 0$. We define h as follows:

$$h(r) = \begin{cases} C, & r \in [0, R_1], \\ h(r) - (\alpha/2)(r-R_1)^2 + C, & r \in [R_1, R_2], \\ h(r)\beta(e^{-\tilde{c}(r-R_3)} - 1), & r \in [R_2, R_3]. \end{cases} \tag{31}$$

We can choose $R_1, R_2, R_3, c, \alpha, \beta, C$ such that

$$\begin{gathered} h \in W^{2,\infty}(\mathbb{R}^+), \\ h(R_2) = K, \quad \text{where } K \in (\theta, C) \text{ will be chosen after}, \\ h''(r) + \tilde{c}h'(r) + f(h(r)) \geq 0 \quad \text{for } r \geq 0. \end{gathered} \tag{32}$$

The existence of such a function proves our result, indeed

$$\Delta\phi + c\partial_x\phi + f(\phi) \geq h'' + \left(c + \frac{1}{r}\right)h' + f(h)$$
$$\geq h'' + \left(c + \frac{1}{R_1}\right)h' + f(h).$$

We used the fact that h is nonincreasing and $h'(r) = 0$ if $r \in [0, R_1]$ here.

Let us define

$$F := \inf_{s\in(K,C)} f(s) > 0.$$

Because $h(R_2) = K$, we can bound from behind $f(h(r))$ by F when $r \in [R_1, R_2]$ and by 0 elsewhere. Some easy computations show that (32) boils down to verifying the following algebraic system:

$$\begin{cases} \beta(e^{\tilde{c}(R_3-R_2)} - 1) = K, \\ (\alpha/2)(R_2 - R_1)^2 = C - K, \\ \alpha(R_2 - R_1) = \beta\tilde{c}e^{\tilde{c}(R_3-R_2)}, \\ F \geq \alpha(1 + \tilde{c}(R_2 - R_1)). \end{cases} \tag{33}$$

Up to some computations, it is readily seen that (33) admits positive solutions, for instance

$$\alpha = \frac{F}{1 + (C-K)/(2K)}, \quad R_1 = \frac{1}{c},$$
$$c = \frac{1}{8}\frac{\sqrt{2\alpha(C-K)}}{K}, \qquad R_2 = \sqrt{\frac{2(C-K)}{\alpha}} + R_1,$$
$$\beta = \frac{\sqrt{\alpha(C-K)}}{2c\sqrt{2}} - K, \qquad R_3 = \frac{1}{2c}\ln\left(1 + \frac{K}{\beta}\right) + R_2.$$

Hence, $\phi(x) := h(|x|)$ satisfies the lemma with $R := R_3$. □

Now, we use this lemma to prove Proposition 19.

Proof of Proposition 19. Step 1: Construction of the domain. Let $R > 0$ be large enough, so that we can apply Lemma 20. Let $\alpha \in (0, 1)$, $\varepsilon \in [0, \alpha R/2]$ and define

$$\widetilde{K}_\alpha^\varepsilon := \{(x, y) \in \mathbb{R}^2 \text{ such that } \alpha x + R + \varepsilon \leq y \leq \alpha x + (1+\alpha)R - \varepsilon,\ y \in [R, 2R]\}.$$

Now, let K_α be a smooth connected compact set such that

$$\widetilde{K}_\alpha^{\alpha R/4} \subset K_\alpha \subset \widetilde{K}_\alpha^0.$$

We define Ω_α to be the smooth $3R$-periodic domain

$$\Omega_\alpha := (K_\alpha + 3R\mathbb{Z}^2)^c;$$

see Figure 1. Observe that, if $k, l \in \mathbb{Z}^2$ are such that $k \neq l$, then $(K_\alpha + 3Rk) \cap (K_\alpha + 3Rl) = \varnothing$.

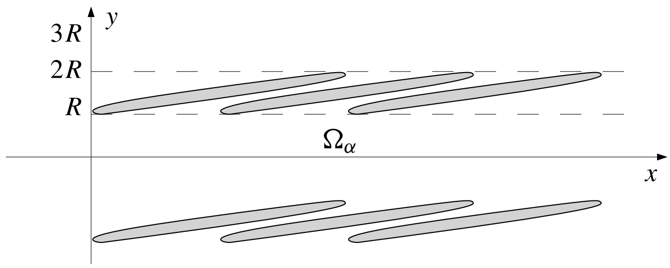

Figure 1. The domain Ω_α used in the proof of Proposition 19.

Step 2: Lower bound on $w_\alpha(e_x)$. For $\alpha > 0$ given, we denote by w_α the speed of invasion for (15) set on the smooth periodic domain Ω_α. Let us show that there is $K > 0$ independent of α such that

$$w_\alpha(e_x) \geq K. \tag{34}$$

Because of the choice of R, we can apply Lemma 20 to find $c > 0$ and $\phi \in W^{2,\infty}(B_R)$, $\phi > 0$ on B_R and $\phi = 0$ on ∂B_R such that $\Delta\phi + c\partial_x\phi + f(\phi) \geq 0$. Now, we define

$$v(t, x, y) := \begin{cases} \phi(x - ct, y) & \text{if } (x, y) \in B_R(cte_x), \\ 0 & \text{elsewhere.} \end{cases}$$

Then, the support of $v(t, \cdot, \cdot)$ never intersects the boundary of Ω_α, and because

$$\partial_t v - \Delta v - f(v) = -c\partial_x\phi - \Delta\phi - f(\phi) \leq 0 \text{ for } (x, y) \in \operatorname{supp} v(t, \cdot, \cdot),$$

we have that v is a nonnegative compactly supported generalized subsolution of (15).

Now, take u_0 a compactly supported initial datum such that $u_0(x, y) \geq \phi(x, y)$ and such that the solution arising from it, say $u(t, x, y)$, converges to 1 (as we mentioned earlier, such initial datum always exists, see [Ducasse and Rossi 2018]). The parabolic comparison principle yields

$$u(t, x, y) \geq \phi(x - ct, y) \quad \text{for all } t \geq 0, \text{ for all } (x, y) \in \Omega_\alpha.$$

By the definition of $w_\alpha(e_x)$, this implies that $w_\alpha(e_x) \geq c$, where c, given by Lemma 20, is independent of α. Hence, (34) holds with $K := c$.

Step 3: Upper bound on $w_\alpha(e_y)$. We now show that $w_\alpha(e_y) \to 0$ as α goes to 0. To do so, we first apply Proposition 17, to get

$$w_\alpha(e_y) \leq 2C_{\Omega_\alpha}(e_y)\sqrt{\max_{u\in[0,1]} \frac{f(u)}{u}}.$$

Let us estimate $C_{\Omega_\alpha}(e_y)$. If we take $n \in \mathbb{N}$, we see that, if α is small enough, $d_{\Omega_\alpha}(0, 4Rne_y) \geq 2Rn\sqrt{1 + (1 - 1/\alpha)^2}$. Then, if α is small enough, $C_{\Omega_\alpha}(e_y) \leq 3\alpha$. Thus

$$w_\alpha(e_y) \leq 6\alpha\sqrt{\max_{u\in[0,1]} \frac{f(u)}{u}} \xrightarrow[\alpha\to 0]{} 0,$$

and hence the result. $\square$

Now, Proposition 19 is proved, and answers Question 4 in the combustion case: in Ω_α, $c^\star \not\equiv w$, for $\alpha > 0$ small enough.

Let us now explain how this also answers Question 4 in the monostable case. Take f to be a monostable nonlinearity and let $\underline{f}$ be a combustion nonlinearity and let $\overline{f}$ be a KPP nonlinearity, both independent of x, such that

$$\underline{f} \le f \le \overline{f}.$$

Let $\overline{w}_\alpha$, w_α, $\underline{w}_\alpha$ be the invasion speed for the problem (15) with nonlinearity $\overline{f}$, f, $\underline{f}$ respectively. Then, by comparison, we have

$$\text{for all } e \in \mathbb{S}^1, \quad \underline{w}_\alpha(e) \le w_\alpha(e) \le \overline{w}_\alpha(e). \tag{35}$$

Now, we can apply Lemma 20 to find $c > 0$ and $\phi \in W^{2,\infty}(\mathbb{R}^2)$, $\phi > 0$ in B_R and $\phi = 0$ on ∂B_R such that, on B_R we have

$$\Delta\phi + c\partial_x\phi + \underline{f}(\phi) \ge 0.$$

Then, consider the domain Ω_α constructed in the proof of Proposition 19, but with this new $R > 0$.

On this domain, we have a lower bound on $\underline{w}_\alpha(e_x)$ independent of α. Moreover, we can show that $\overline{w}_\alpha(e_y)$ goes to zero as α goes to 0, as in the proof of Proposition 19.

Hence, (35) yields that there is $K > 0$ independent of α such that $w_\alpha(e_x) \ge K$, and $w_\alpha(e_y) \to 0$ as α goes to 0. This means that Proposition 19 still holds if f is monostable; hence this answers Question 4 in the monostable case and concludes the proof of Theorem 5.

4. Symmetries of the domain and relation with $c^\star$ and w

This section is dedicated to the proof of Theorem 7. As we mentioned earlier, even in a domain Ω where $c^\star \not\equiv w$, the Freidlin–Gärtner formula yields that any direction $e \in \mathbb{S}^{N-1}$ minimizing $c^\star$ satisfies the equality $c^\star(e) = w(e)$. Theorem 7 gives a geometrical condition that ensures the existence of directions where $c^\star$ and w coincide. To prove it, we first state the following lemma:

Lemma 21. *Let $c^\star$ and w be respectively the critical speed of fronts and the speed of invasion for* (15) *with the nonlinearity f satisfying* (4), (5) *and such that $u \mapsto f(u)/u$ is nonincreasing. For any $k \in \mathbb{N}$ and $e \in \mathbb{S}^{N-1}$, $(\xi_i)_{i\in[\![1,k]\!]} \in (\mathbb{S}^{N-1})^k$ such that*

$$e \in \left\{ x \in \mathbb{R}^N : x = \sum_{i=1}^{k} \lambda_i \xi_i,\ \lambda_i \ge 0 \right\},$$

the following holds:

$$c^\star(e) \le \max_{i\in[\![1,k]\!]} \frac{c^\star(\xi_i)}{e \cdot \xi_i}.$$

Proof. For $i \in [\![1,k]\!]$, we denote by $\phi_{\xi_i}(t,x)$ a pulsating traveling front solution of (1) in the direction ξ_i with critical speed $c^\star(\xi_i)$. Let

$$v(t,x) := \sum_{i=1}^{k} \phi_{\xi_i}(t,x).$$

Now, the hypotheses on f imply that $f(v) \le \sum_{i=1}^{k} f(\phi_{\xi_i})$, and then v is a supersolution of (1).

Now, for $\varepsilon > 0$, let f_ε be a combustion nonlinearity satisfying

$$\begin{aligned} 0 \le f_\varepsilon(x,u) &\le f(x,u) && \text{for all } u \in [0,1], \text{ for all } x \in \Omega, \\ f_\varepsilon(x,u) &= f(x,u) && \text{for all } u \in [0,1-2\varepsilon], \text{ for all } x \in \Omega, \\ f_\varepsilon(x,u) &= 0 && \text{for all } u \in [-\varepsilon,0], \text{ for all } x \in \Omega, \\ f_\varepsilon(x,1-\varepsilon) &= 0 && \text{for all } x \in \Omega. \end{aligned}$$

Now, take $e \in \mathbb{S}^{N-1}$ such that $e = \sum_{i=1}^k \lambda_i \xi_i$, with $\lambda_i \ge 0$ for all $i \in [\![1,k]\!]$, and let ϕ_e^ε be a pulsating traveling front connecting $1-\varepsilon$ to $-\varepsilon$, a solution of (1) with the combustion nonlinearity f_ε, in the direction e with critical speed $c_\varepsilon^\star(e)$.

Up to some translation in time, we can assume that $\phi_e^\varepsilon(0,x) < 0$ if $x \cdot e > 0$ and, for all $i \in [\![1,k]\!]$, $\phi_{\xi_i}(0,x) \ge 1-\varepsilon$ if $x \cdot \xi_i < 0$.

Moreover, if $x \in \Omega$ is such that $x \cdot e < 0$, then there is at least one of the ξ_i such that $x \cdot \xi_i < 0$. Hence, $v(0,x) > 1-\varepsilon$ if $x \cdot e < 0$. If $x \cdot e \ge 0$, we have $v(0,x) > 0 \ge \phi_e^\varepsilon(0,x)$. Hence

$$v(0,x) \ge \phi_e^\varepsilon(0,x) \quad \text{for all } x \in \Omega.$$

Because $f_\varepsilon \le f$, the parabolic comparison principle yields

$$v(t,x) \ge \phi_e^\varepsilon(t,x) \quad \text{for all } t \ge 0, \text{ for all } x \in \Omega. \tag{36}$$

Now, if we take $\bar{c} > \max_{i \in [\![1,N]\!]} c^\star(\xi_i)/(e \cdot \xi_i)$, we have that $v(t, \bar{c}te) \to 0$ as t goes to $+\infty$. It then follows from (36) that $c_\varepsilon^\star(e) \le \max_{i \in [\![1,N]\!]} c^\star(\xi_i)/(e \cdot \xi_i)$. Now, it is classical that $c_\varepsilon^\star(e) \to c^\star(e)$ as ε goes to 0 (see, for instance, [Rossi 2017, Proposition 2.6]). Taking the limit $\varepsilon \to 0$ then yields the result. □

Remark 22. Lemma 21 yields a very strong geometrical condition on $c^\star$, and prevents it from being any arbitrary function. Consider

$$\mathcal{C} := \{ r(\xi)\xi \in \mathbb{R}^2 : r(\xi) \in [0, c^\star(\xi)] \}.$$

In the case of (15) with $\Omega = \mathbb{R}^N$, $c^\star$ is constant and then $\mathcal{C}$ is a ball. In general, it is not clear what "shapes" $\mathcal{C}$ can adopt. Lemma 21 prevents it from being some natural candidates; for instance, $\mathcal{C}$ cannot be an ellipse with eccentricity larger than $1/\sqrt{2}$. We recall that an ellipse of equation $x^2/a^2 + y^2/b^2 = 1$, with $a > b$, has eccentricity $\sqrt{1 - b^2/a^2}$.

Before turning to the proof of Theorem 7, we need another technical lemma.

Lemma 23. *Let Ω be a periodic domain, and let T be an orthogonal transformation that leaves T invariant; i.e., $T\Omega = \Omega$. Then, at least one of the two possibilities below holds true:*

(i) *T is of finite order; i.e., there is $m \in \mathbb{N}^\star$ such that $T^m = I_N$, where I_N is the identity matrix.*

(ii) *The domain Ω is invariant in a direction orthogonal to the eigenvectors associated with the eigenvalue 1.*

Proof. Assume that T leaves the domain Ω invariant and that it is not of finite order. Then, there is at least one vector e of the canonical basis of $\mathbb{R}^N$ such that

$$T^k(e) \neq e \quad \text{for all } k \in \mathbb{Z}^\star.$$

It is then readily seen that each point of the set $\{T^k(e) : k \in \mathbb{Z}\}$ is a point of accumulation. Therefore, up to extraction,

$$u_k := T^k(e) - e \xrightarrow[k\to+\infty]{} 0.$$

Moreover, because Ω is left invariant by T and because $\Omega + e = \Omega$, there holds

$$\Omega + T^k(e) = \Omega + e \quad \text{for all } k \in \mathbb{N},$$

i.e.,

$$\Omega + u_k = \Omega \quad \text{for all } k \in \mathbb{N}.$$

Now, we can find $v \in \mathbb{S}^{N-1}$ such that $u_k/|u_k|$ converges up to another extraction to v. It is then readily seen that

$$\Omega + \lambda v = \Omega \quad \text{for all } \lambda \in \mathbb{R};$$

i.e., Ω is invariant in the direction v. Observe now that, if y is an eigenvector associated to the eigenvalue 1, then $u_k \cdot y = 0$, from which we get that $v \cdot y = 0$, and this concludes the result. □

We are now in position to prove Theorem 7.

Proof of Theorem 7. Let T be an orthogonal transformation as in the theorem and let $e \in \mathbb{S}^{N-1}$ be such that $Te = e$.

Step 1: Reduction to the case of a finite-order orthogonal transformation. Assume that T is not of finite order. Then, owing to Lemma 23, the domain Ω is invariant in at least one direction orthogonal to e. We denote by $\mathcal{S}$ the set of all such directions. It is a subspace of $\mathbb{R}^N$ orthogonal to e such that $T(S) = S$. We define

$$\widetilde{\Omega} := \Omega \cap \mathcal{S}^\perp,$$

where $\mathcal{S}^\perp$ denotes the orthogonal of $\mathcal{S}$. Then, $e \in \widetilde{\Omega}$ and

$$T(\widetilde{\Omega}) = \widetilde{\Omega}.$$

Consider now the problem

$$\begin{cases} \partial_t u - \Delta u = f(u), & t > 0,\ x \in \widetilde{\Omega}, \\ \partial_\nu u = 0, & t > 0,\ x \in \partial\widetilde{\Omega}. \end{cases} \tag{37}$$

We denote by $\mathcal{W}$ the asymptotic set of spreading for (15). Then, owing to Proposition 15, the asymptotic set of spreading for (37) is $\mathcal{W} \cap \mathcal{S}^\perp$. In particular, the speed of invasion and the critical speed of fronts in the direction e for (15) and for (37) are the same. It is then sufficient to prove our result in the domain $\widetilde{\Omega}$, which is not invariant in any direction orthogonal to e. Because $\widetilde{\Omega}$ is left invariant by T and owing to Lemma 23, the restriction of T to $\widetilde{\Omega}$ is of finite order.

Step 2: Proof when T is of finite order. Let us now restrict our attention to the case where T is of finite order; i.e, there is $m \in \mathbb{N}^\star$ such that $T^m = I_N$.

Let ξ_0 be such that $w(e) = c^\star(\xi_0)/(\xi_0 \cdot e)$ and $\xi_0 \cdot e > 0$. If $\xi_0 = e$, then $w(e) = c^\star(e)$ and we are done. If not, we define

$$\xi_k := T^k \xi_0 \quad \text{for } k \in [\![0, m-1]\!].$$

Let us show that the vector e is in the positive cone spanned by the $(\xi_k)_{k \in [\![0,m-1]\!]}$. To do so, observe first that

$$(T^m - I_N)(\xi_0) = (T - I_N)\Bigg(\sum_{k=0}^{k=m-1} T^k \xi_0\Bigg) = (T - I_N)\Bigg(\sum_{k=0}^{k=m-1} \xi_k\Bigg) = 0.$$

Owing to the hypotheses on T, this implies that there is $\lambda \in \mathbb{R}$ such that

$$\sum_{k=0}^{k=m-1} \xi_k = \lambda e. \tag{38}$$

Moreover, because T is orthogonal, we see that, for $k \in [\![0, m-1]\!]$,

$$\xi_k \cdot e = \xi_0 \cdot e > 0, \tag{39}$$

and then, (38) yields that $\lambda > 0$; i.e., e is in the positive cone spanned by the $(\xi_k)_{k \in [\![0,m-1]\!]}$.

Now, owing to Lemma 21, we have

$$c^\star(e) \le \max_{k \in [\![0,m-1]\!]} \frac{c^\star(\xi_k)}{\xi_k \cdot e}. \tag{40}$$

Observe that, because $T\Omega = \Omega$, we have

$$c^\star(\xi_0) = c^\star(\xi_1) = \cdots = c^\star(\xi_{m-1}). \tag{41}$$

Indeed, if $\phi(t, x)$ is a pulsating traveling front solution of (15) in the direction $\xi \in \mathbb{S}^{N-1}$ with speed $c^\star(\xi)$, then $\phi(t, Tx)$ is a pulsating traveling front solution of (15) in the direction $T\xi$ with speed $c^\star(\xi)$. Then, by definition of the critical speed, $c^\star(T\xi) \le c^\star(\xi)$. The same reasoning but with $T\xi$ instead of ξ and with T^{-1} instead of T yields the reverse inequality and then $c^\star(\xi) = c^\star(T\xi)$. Hence, (41) follows from the definition of the $\xi_k,\ k \in [\![0, m-1]\!]$.

Now, combining (39) and (41) with (40), we see that

$$c^\star(e) \le \frac{c^\star(\xi_0)}{\xi_0 \cdot e} = w(e).$$

Because $w(e) \le c^\star(e)$, thanks to the Freidlin–Gärtner formula (12), we finally get

$$c^\star(e) = w(e),$$

and hence the result. □

We can deduce from this theorem the following:

Corollary 24. *Assume that f satisfies* (4)–(5) *and that $u \mapsto f(u)/u$ is nonincreasing. Let $\Omega \subset \mathbb{R}^N$ be a periodic domain. Then, $c^\star(e) = w(e)$ in the following cases*:

- *If $N=2$ and if Ω is symmetric with respect to the line $\mathbb{R}e$.*
- *If $N=3$ and if Ω is stable with respect to the rotation of angle π and of axis directed by e.*
- *If $N\in\mathbb{N}$ and if Ω is symmetric with respect to $N-1$ hyperplanes whose intersection is the line directed by e.*

The cases $N=2$ and $N=3$ are straightforward. For the general case $N\in\mathbb{N}$, one may observe that the composition of $N-1$ symmetries whose stable hyperplanes have a one-dimensional intersection satisfies the hypotheses of Theorem 7.

A typical domain to which we could apply Corollary 24 is the whole space with ball-shaped obstacles, i.e.,

$$\Omega := (\bar{B}_{1/4}+\mathbb{Z}^N)^c.$$

In this domain, Corollary 24 yields that $w(e)=c^\star(e)$ for any $e\in\mathbb{S}^{N-1}$ in the canonical basis. We conclude this section with two remarks.

Remark 25. Let us observe that the hypothesis $\operatorname{Ker}(T-I_N)=\mathbb{R}e$ is necessary in Theorem 7. Indeed, consider $\Omega\subset\mathbb{R}^2$ to be the periodic domain constructed in Proposition 18 and define

$$\widetilde{\Omega} := \Omega\times\mathbb{R}.$$

Let T be the symmetry with respect to the hyperplane orthogonal to $u:=(0,0,1)$. The domain $\widetilde{\Omega}$ is invariant in the direction $u:=(0,0,1)$; therefore we can apply Proposition 15 to see that there are directions orthogonal to u where $c^\star$ and w do not coincide, although these directions are left invariant by T.

Remark 26. Observe that, if one considers the general equation (1), then Theorem 7 still holds provided the coefficients satisfy the same symmetry as the domain; i.e., we need the coefficients to satisfy

$$A(Tx)=TA(x)T^\star,\quad q(Tx)=Tq(x)\quad\text{and}\quad f(Tx,\cdot)=f(x,\cdot),$$

where T is the transformation considered in Theorem 7.

Appendix

Proof of Lemma 9. As we mentioned, Lemma 9 is the natural extension of [Rossi 2017, Lemma 2.1], in the case of a periodic domain.

Proof. Let u be taken as in the lemma. We define

$$h := \liminf_{\delta\to+\infty}\ \inf_{\substack{t<0\\ x\cdot e<\gamma t-\delta\\ x\in\Omega}} u(t,x).$$

Assume that, by contradiction, $h\in(S,1)$. We can find two sequences $(x_n)_n\in\Omega^{\mathbb{N}}$, $(t_n)_n\in(-\infty,0)^{\mathbb{N}}$ such that $x_n\cdot e-\gamma t_n\to-\infty$ and $u(t_n,x_n)\to h$ as n goes to $+\infty$. Let us define $k_n\in\mathbb{Z}^N$, $z_n\in[0,1)^N$

so that $x_n = k_n + z_n$. Up to extraction, we assume that $z_n \to z$ as n goes to $+\infty$ for some $z \in [0, 1]^N$. Consider the sequence of translated functions

$$u_n := u(\cdot + t_n, \cdot + k_n).$$

These functions are supersolutions of (1), by periodicity of the domain. As before, we can use the usual parabolic estimates to get local uniform convergence of the sequence $(u_n)_n$ to a function u_∞ supersolution of (1). Moreover, we have

$$u_\infty(0, z) = h \le u_\infty(t, x) \quad \text{for all } t \le 0, \text{ for all } x \in \Omega. \tag{42}$$

Indeed, for $t \le 0$, $x \in \Omega$, we have

$$u_n(t, x) = u(t + t_n, x + k_n) \ge \inf_{\substack{\tau<0 \\ y\cdot e-\gamma\tau\le\tilde{\delta}_n}} u(\tau, y), \tag{43}$$

where $\tilde{\delta}_n := x \cdot e - \gamma t - z_n \cdot e + x_n \cdot e - \gamma t_n$ goes to $-\infty$ as n goes to $+\infty$. Hence, passing to the limit $n \to +\infty$ in (43) yields (42). Because $f \ge 0$, it follows from the parabolic maximum principle and Hopf principle that u_∞ is actually equal to h if $t \le 0$ and $x \in \Omega$. This implies that $f(x, h) = 0$, which is in contradiction with the fact that $h \in (S, 1)$ together with hypothesis (5); hence the result. □

Proof of Lemma 10. We now turn to the proof of Lemma 10. Again, it is the natural extension of [Rossi 2017, Lemma 2.2] to the case of a periodic domain.

Proof. Let us define $\bar{u}_\varepsilon := \bar{u} + \varepsilon$, where $\varepsilon > 0$. The hypotheses on $\bar{u}$ yield that there is $\delta > 0$ such that $\bar{u}_\varepsilon(t, x) \ge 1 + \varepsilon/2$ if $t < 0$ and $x \cdot e < \gamma t - \delta$, $x \in \Omega$. The hypotheses on $\underline{u}$ give us that there is $L > 0$ such that $\underline{u}(t, x) \le \varepsilon$ if $t < 0$ and $x \cdot e \ge (\gamma + \eta)t + L$. Hence, there is $T_\varepsilon \le 0$ such that $\bar{u}_\varepsilon(t, x) > \underline{u}(t, x)$ for $t < T_\varepsilon$, for all $x \in \Omega$. Indeed, if t is negative enough, we have $\eta t + L < -\delta$; hence we can take $T_\varepsilon := (-\delta - L)/\eta$.

In order to prove the result, we shall argue by contradiction. Hence, we will assume that there is $\varepsilon_0 > 0$ such that

$$\text{for all } \varepsilon \in (0, \varepsilon_0) \text{ there exist } \tau \in (T_\varepsilon, 0) \text{ and } x_\tau \in \Omega \text{ such that } \bar{u}_\varepsilon(\tau, x_\tau) < \underline{u}(\tau, x_\tau). \tag{44}$$

Indeed, if (44) does not hold, our result follows by letting $\varepsilon \to 0$. Now, we define $t_\varepsilon \in [T_\varepsilon, 0)$ to be the infimum of all the τ such that (44) holds true. Hence

$$\bar{u}_\varepsilon(t, x) \ge \underline{u}(t, x) \quad \text{for all } t \le t_\varepsilon, \text{ for all } x \in \Omega,$$

and by continuity we have

$$\inf_{x\in\Omega} (\bar{u}_\varepsilon - \underline{u})(t_\varepsilon, x) = 0.$$

Thanks to the hypotheses, we can find $\rho_\varepsilon \in \mathbb{R}$ such that

$$\inf_{x\cdot e=\rho_\varepsilon} (\bar{u}_\varepsilon - \underline{u})(t_\varepsilon, x) = 0.$$

Depending on the behavior of ρ_ε, we now consider three cases.

First case: $(\rho_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$ is bounded. We can find a sequence of points $(x_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$, with $x_\varepsilon \in \Omega$ such that

$$x_\varepsilon \cdot e = \rho_\varepsilon \quad \text{and} \quad \bar{u}_\varepsilon(t_\varepsilon, x_\varepsilon) - \underline{u}(t_\varepsilon, x_\varepsilon) < \varepsilon.$$

We define $k_\varepsilon \in \mathbb{Z}^N$, $y_\varepsilon \in [0,1)^N$ to be such that $x_\varepsilon = k_\varepsilon + y_\varepsilon$. Up to extraction,we can find $y \in [0,1)^N$ such that $y_\varepsilon \to y$ as ε goes to 0.

We now consider the translated functions $\bar{u}_\varepsilon(t+t_\varepsilon, x+k_\varepsilon)$, $\underline{u}(t+t_\varepsilon, x+k_\varepsilon)$. Using parabolic estimates and extracting, these functions converge locally uniformly as ε goes to 0 to $\bar{u}_\infty$, $\underline{u}_\infty$, a supersolution and a subsolution respectively of (1).

Moreover, $\bar{u}_\infty$, $\underline{u}_\infty$ satisfy

$$\bar{u}_\infty(0,y) = \underline{u}_\infty(0,y) \qquad \text{and} \qquad \bar{u}_\infty(t,x) \geq \underline{u}_\infty(t,x) \quad \text{for } t \leq 0,\ x \in \Omega.$$

Hence, the strong comparison principle and the Hopf lemma (see [Protter and Weinberger 1967, Chapter 3]) imply that $\bar{u}_\infty = \underline{u}_\infty$ for $t \leq 0$. But the boundedness of $x_\varepsilon \cdot e = \rho_\varepsilon$ implies that we still have

$$\liminf_{\delta\to+\infty} \inf_{\substack{t<0 \\ x\cdot e<\gamma t-\delta \\ x\in\Omega}} \bar{u}_\infty(t,x) \geq 1.$$

However, the hypotheses on $\underline{u}$ yield that there is $K \in \mathbb{R}$ such that

$$\underline{u}_\infty(t,x) \leq \tfrac{1}{2} \quad \text{for all } t<0, \text{ for all } x \in \Omega \text{ such that } x\cdot e \geq (\gamma+\eta)t + K.$$

Taking $t < 0$ small enough yields a contradiction.

Second case: $\inf_{\varepsilon\in(0,\varepsilon_0)} \rho_\varepsilon = -\infty$. Let us take ε such that $-\rho_\varepsilon$ is large enough to have

$$\inf_{\substack{t<0 \\ x\cdot e-\gamma t<\rho_\varepsilon}} \bar{u}(t,x) > S.$$

Because $f(x,\cdot)$ is decreasing in $(S,1)$, we have that $\bar{u}_\varepsilon = \bar{u}+\varepsilon$ is a supersolution of (1) for $\{(t,x) \in \mathbb{R}\times\Omega : x\cdot e - \gamma t < \rho_\varepsilon\}$.

We can find a sequence $(x_n)_n \in \Omega^{\mathbb{N}}$ such that $x_n \cdot e = 0$ and

$$\lim_{n\to+\infty} (\bar{u}_\varepsilon - \underline{u})(t_\varepsilon, \rho_\varepsilon e + x_n) = 0.$$

We write as before $x_n = k_n + y_n$, where $k_n \in \mathbb{Z}^N$ and $y_n \in [0,1)^N$, and up to extraction we can find $y \in [0,1]^N$ such that $y_n \to y$ as n goes to $+\infty$.

We define $\bar{u}_n^\varepsilon(t,x) := \bar{u}_\varepsilon(t, x+k_n)$ and $\underline{u}_n(t,x) := \underline{u}(t,x+k_n)$. Observe that $\bar{u}_n^\varepsilon$ is a supersolution in $\{(t,x) \in \mathbb{R}\times\Omega : x\cdot e - \gamma t < \rho_\varepsilon - 1\}$. Again, using parabolic estimates and extracting as n goes to $+\infty$, we get two functions $\bar{u}_\infty^\varepsilon$ and $\underline{u}_\infty$ that are respectively a supersolution and a subsolution of (1) on the same set. Moreover, they satisfy $\bar{u}_\infty^\varepsilon(t_\varepsilon, \rho_\varepsilon e + y) = \underline{u}_\infty(t_\varepsilon, \rho_\varepsilon e + y)$; we have a contact point.

Observe that $(t_\varepsilon, \rho_\varepsilon e + y)$ is in $\{(t,x) \in \mathbb{R}\times\Omega : x\cdot e - \gamma t < \rho_\varepsilon - 2\}$. Hence, we can apply the Hopf lemma [Protter and Weinberger 1967, Theorem 6] to (1) on $\{(t,x) \in \mathbb{R}\times\Omega : x\cdot e - \gamma t < \rho_\varepsilon - 1\}$ to get that $(t_\varepsilon, \rho_\varepsilon e + y)$ is not on a boundary point. Therefore, it is an interior contact point and the parabolic comparison principle yields that $\bar{u}_\infty^\varepsilon(t,x) = \underline{u}_\infty(t,x)$ on $\{(t,x) \in \mathbb{R}\times\Omega : x\cdot e - \gamma t < \rho_\varepsilon - 1\}$. But this is not possible, because the hypotheses on $\bar{u}$ imply that there is δ large enough so that $\bar{u}^{\varepsilon}_{\infty}(t,x) \geq 1+\varepsilon/2$ if $x \cdot e - \gamma t < -\delta$. Because $\underline{u} \leq 1$, we are led to a contradiction.

Third case: $\sup_{\varepsilon \in (0,\varepsilon_0)} \rho_\varepsilon = +\infty$. If we are in the case (19), this cannot happen because $\bar{u}_\varepsilon \geq 0$ and $\underline{u}(t_\varepsilon, x) < 0$ if $x \cdot e$ is large enough. Then, we are left to assume that f satisfies (9) and $\underline{u}$ satisfies (18). In particular, we can take ε small enough so that ρ_ε is large enough to have $\underline{u}(t,x) \leq \theta$ on $\{(t,x) \in \mathbb{R} \times \Omega : x \cdot e - \gamma t > \rho_\varepsilon\}$, where θ is from (9). Hence, $\underline{u}_\varepsilon := \underline{u} - \varepsilon$ is a subsolution of (1) on this set. Arguing as in the previous case, we get a contradiction, and hence the result. □

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