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ON THE PROPAGATION OF REGULARITY FOR SOLUTIONS OF THE DISPERSION GENERALIZED BENJAMIN–ONO EQUATION

ARGENIS J. MENDEZ

To my parents

We study some properties of propagation of regularity of solutions of the dispersive generalized Benjamin–Ono (BO) equation. This model defines a family of dispersive equations that can be seen as a dispersive interpolation between the Benjamin–Ono equation and the Korteweg–de Vries (KdV) equation.

Recently, it has been shown that solutions of the KdV and BO equations satisfy the following property: if the initial data has some prescribed regularity on the right-hand side of the real line, then this regularity is propagated with infinite speed by the flow solution.

In this case the nonlocal term present in the dispersive generalized Benjamin–Ono equation is more challenging that the one in the BO equation. To deal with this a new approach is needed. The new ingredient is to combine commutator expansions into the weighted energy estimate. This allows us to obtain the property of propagation and explicitly the smoothing effect.

1. Introduction 2399
2. Notation 2404
3. Preliminaries 2405
4. The linear problem 2410
5. Proof of Theorem A 2414
6. Proof of Theorem B 2417
Acknowledgments 2439
References 2439

1. Introduction

The aim of this work is to study some special regularity properties of solutions to the initial value problem (IVP) associated to the *dispersive generalized Benjamin–Ono equation*

$$\begin{cases} \partial_t u - D_x^{\alpha+1}\partial_x u + u\,\partial_x u = 0, \quad x,t\in\mathbb{R},\ 0<\alpha<1, \\ u(x,0)=u_0(x), \end{cases} \tag{1.1}$$

where D_x^s, denotes the homogeneous derivative of order $s\in\mathbb{R}$,

$$D_x^s = (-\partial_x^2)^{\frac{s}{2}} \quad \text{thus} \quad D_x^s f = c_s(|\xi|^s\hat{f}(\xi))^{\vee},$$

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which in its polar form is decomposed as $D_x^s = (\mathcal{H}\partial_x)^s$, where $\mathcal{H}$ denotes the *Hilbert transform*

$$\mathcal{H}f(x) = \frac{1}{\pi} \lim_{\epsilon\to 0^+} \int_{|y|\geq\epsilon} \frac{f(x-y)}{y}\,\mathrm{d}y = (-i\,\mathrm{sgn}(\xi)\hat{f}(\xi))^{\vee}(x),$$

where $\hat{}$ denotes the Fourier transform and $^{\vee}$ denotes its inverse. These equations model vorticity waves in the coastal zone; see [Molinet et al. 2001].

Our starting point is a property established by Isaza, Linares and Ponce [Isaza et al. 2015] concerning the solutions of the IVP associated to the k-generalized KdV equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k\,\partial_x u = 0, & x, t \in \mathbb{R},\ k \in \mathbb{N}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.2}$$

It was shown in [Isaza et al. 2015] that the unidirectional dispersion of the k-generalized KdV equation gives the following propagation of regularity phenomena.

Theorem 1.3 [Isaza et al. 2015]. *If $u_0 \in H^{3/4^+}(\mathbb{R})$ and for some $l \in \mathbb{Z}$, $l \geq 1$ and $x_0 \in \mathbb{R}$*

$$\|\partial_x^l u_0\|^2_{L^2((x_0,\infty))} = \int_{x_0}^{\infty} |\partial_x^l u_0(x)|^2\,\mathrm{d}x < \infty, \tag{1.4}$$

then the solution of the IVP associated to (1.2) satisfies that for any $v > 0$ and $\epsilon > 0$

$$\sup_{0\leq t\leq T} \int_{x_0+\epsilon-vt}^{\infty} (\partial_x^j u)^2(x,t)\,\mathrm{d}x < c \tag{1.5}$$

for $j = 0, 1, 2, \ldots, l$ with $c = c(l; \|u_0\|_{H^{3/4^+}(\mathbb{R})}; \|\partial_x^l u_0\|_{L^2((x_0,\infty))}; v; \epsilon; T)$. In particular, for all $t \in (0, T]$, the restriction of $u(\cdot, t)$ to any interval (x_0, ∞) belongs to $H^l((x_0, \infty))$.

Moreover, for any $v \geq 0$, $\epsilon > 0$ and $R > 0$

$$\int_0^T \int_{x_0+\epsilon-vt}^{x_0+R-vt} (\partial_x^{l+1} u)^2(x,t)\,\mathrm{d}x\,\mathrm{d}t < c,$$

with $c = c(l; \|u_0\|_{H^{3/4^+}(\mathbb{R})}; \|\partial_x^l u_0\|_{L^2((x_0,\infty))}; v; \epsilon; R; T)$.

The proof of Theorem 1.3 is based on weighted energy estimates. In particular, the iterative process in the induction argument is based on a property discovered originally by T. Kato [1983] in the context of the KdV equation. More precisely, he showed that solution of the KdV equation satisfies

$$\int_0^T \int_{-R}^{R} (\partial_x u)^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \leq c(R; T; \|u_0\|_{L^2_x}), \tag{1.6}$$

where this is the fundamental fact in his proof of existence of the global weak solutions of (1.2) for $k = 1$ and initial data in $L^2(\mathbb{R})$.

This result was also obtained for the Benjamin–Ono equation [Isaza et al. 2016a] but it does not follow as the KdV case because of the presence of the Hilbert transform.

Later on, [Kenig et al. 2018] extended the results in Theorem 1.3 to the case when the local regularity of the initial data u_0 in (1.4) is measured with fractional indices. The scope of this case is much more involved, and its proof is mainly based in weighted energy estimates combined with techniques involving pseudodifferential operators and singular integrals. The property described in Theorem 1.3 is intrinsic to suitable solutions of some nonlinear dispersive models; see also [Linares et al. 2017]. In the context of two-dimensional models, analogous results for the Kadomtsev–Petviashvili II equation [Isaza et al. 2016b] and the Zakharov–Kuznetsov [Linares and Ponce 2018] equation were proved.

Before stating our main result we will give an overview of the local well-posedness of the IVP (1.1).

Following [Kato 1983] we have that the initial value problem IVP (1.1) is *locally well-posed* (LWP) in the Banach space X if for every initial condition $u_0 \in X$ there exists $T > 0$ and a unique solution $u(t)$ satisfying

$$u \in C([0, T] : X) \cap A_T, \tag{1.7}$$

where A_T is an auxiliary function space. Moreover, the solution map $u_0 \mapsto u$ is continuous from X into the class (1.7). If T can be taken arbitrarily large, one says that the IVP (1.1) is *globally well-posed* (GWP) in the space X.

It is natural to study the IVP (1.1) in the Sobolev space

$$H^s(\mathbb{R}) = (1 - \partial_x^2)^{-\frac{s}{2}} L^2(\mathbb{R}), \quad s \in \mathbb{R}.$$

There exist remarkable differences between the KdV (1.2) and the IVP (1.1). In case of KdV, e.g., it possesses infinite conserved quantities, defines a Hamiltonian system, has multisoliton solutions and is a completely integrable system by the inverse scattering method [Coifman and Wickerhauser 1990; Fokas and Ablowitz 1983]. Instead, in the case of the IVP (1.1) there is no integrability, but three conserved quantities (see [Sidi et al. 1986]), specifically

$$\mathrm{I}[u](t) = \int_{\mathbb{R}} u \, \mathrm{d}x, \quad \mathrm{M}[u](t) = \int_{\mathbb{R}} u^2 \, \mathrm{d}x, \quad \mathrm{H}[u](t) = \frac{1}{2} \int_{\mathbb{R}} |D_x^{\frac{1+\alpha}{2}} u|^2 \, \mathrm{d}x - \frac{1}{6} \int_{\mathbb{R}} u^3 \, \mathrm{d}x,$$

are satisfied at least for smooth solutions.

Another property in which these two models differ resides in the fact that one can obtain a local existence theory for the KdV equation in $H^s(\mathbb{R})$, based on the contraction principle. On the contrary, this cannot be done in the case of the IVP (1.1). This is a consequence of the fact that dispersion is not enough to deal with the nonlinear term. In this direction, Molinet, Saut and Tzvetkov [Molinet et al. 2001] showed that for $0 \le \alpha < 1$ the IVP (1.1) with the assumption $u_0 \in H^s(\mathbb{R})$ is not enough to prove local well-posedness by using fixed-point arguments or the Picard iteration method.

Nevertheless, Molinet and Ribaud [2006] proved global well-posedness by considering initial data in a weighted low-frequency Sobolev space. Later, using suitable spaces of Bourgain type, Herr [2007] proved local well-posedness for initial data in $H^s(\mathbb{R}) \cap \dot{H}^{-\omega}(\mathbb{R})$ for any $s > -\frac{3\alpha}{4}$, $\omega = \frac{1}{\alpha+1} - \frac{1}{2}$, where $\dot{H}^{-\omega}(\mathbb{R})$ is a weighted low-frequency Sobolev space (for more details see [Herr 2007]); next by using a conservation law, these results are extended to global well-posedness in $H^s(\mathbb{R}) \cap \dot{H}^{-\omega}(\mathbb{R})$, for $s \ge 0$, $\omega = \frac{1}{\alpha+1} - \frac{1}{2}$. In this sense, an improvement was obtained by Herr, Ionescu, Kenig and Koch [Herr et al. 2010], who showed that the IVP (1.1) is globally well-posed in the space of the real-valued $L^2(\mathbb{R})$-functions by using a renormalization method to control the strong low-high frequency interactions. However, it is not clear that these results described above can be used to establish our main result. Thus a local theory obtained by using energy estimates in addition to dispersive properties of the smooth solutions is required.

In the first step, we obtain the following a priori estimate for solutions of IVP (1.1):

$$\|u\|_{L^\infty_T H^s_x} \lesssim \|u_0\|_{H^s_x} \,\mathrm{e}^{c\|\partial_x u\|_{L^1_T L^\infty_x}};$$

part of this estimate is based on the Kato–Ponce commutator estimate [1988].

The inequality above reads as follows: in order for the solution u to lie in the Sobolev space $H^s(\mathbb{R})$, continuously in time, we must control the term $\|\partial_x u\|_{L^1_T L^\infty_x}$.

First, we use results of Kenig, Ponce and Vega [Kenig et al. 1991a] concerning oscillatory integrals in order to obtain the classical Strichartz estimates associated to the group $S(t) = \mathrm{e}^{tD_x^{\alpha+1}\partial_x}$, corresponding to the linear part of the equation in (1.1).

Additionally, the technique introduced in [Koch and Tzvetkov 2003] related to the refined Strichartz estimate is fundamental in our analysis. Specifically, their method is mainly based in a decomposition of the time interval into small pieces whose lengths depends on the spatial frequencies of the solution. This approach allowed Koch and Tzvetkov to prove local well-posedness for the Benjamin–Ono equation in $H^{5/4^+}(\mathbb{R})$. Then, Kenig and Koenig [2003] enhanced this estimate, which led to proving local well-posedness for the Benjamin–Ono equation in $H^{9/8^+}(\mathbb{R})$.

Several issues arise when handling the nonlinear part of the equation in (1.1); nevertheless, following the work of Kenig, Ponce and Vega [Kenig et al. 1993], we manage the loss of derivatives by combining the local smoothing effect and a maximal function estimate of the group $S(t) = \mathrm{e}^{tD_x^{\alpha+1}\partial_x}$.

These observations lead us to present our first result.

Theorem A. *Let $0 < \alpha < 1$. Set $s(\alpha) = \frac{9}{8} - \frac{3\alpha}{8}$ and assume that $s > s(\alpha)$. Then, for any $u_0 \in H^s(\mathbb{R})$, there exists a positive time $T = T(\|u_0\|_{H^s(\mathbb{R})}) > 0$ and a unique solution u satisfying (1.1) such that*

$$u \in C([0,T] : H^s(\mathbb{R})) \quad \text{and} \quad \partial_x u \in L^1([0,T] : L^\infty(\mathbb{R})). \tag{1.8}$$

Moreover, for any $r > 0$, the map $u_0 \mapsto u(t)$ is continuous from the ball $\{u_0 \in H^s(\mathbb{R}) : \|u_0\|_{H^s(\mathbb{R})} < r\}$ to $C([0,T] : H^s(\mathbb{R}))$.

Theorem A is the base result to describe the propagation of regularity phenomena. As we mentioned above, the propagation of regularity phenomena is satisfied by the BO and KdV equations. These two models correspond to particular cases of the IVP (1.1), specifically by taking $\alpha = 0$ and $\alpha = 1$.

A question that arises naturally is to determine whether the propagation of regularity phenomena is satisfied for a model with an intermediate dispersion between these two models mentioned above.

Our main result gives answer to this problem and it is summarized in the following:

Theorem B. *Let $u_0 \in H^s(\mathbb{R})$, with $s = \frac{3-\alpha}{2}$, and $u = u(x,t)$ be the corresponding solution of the IVP (1.1) provided by Theorem A.*

If for some $x_0 \in \mathbb{R}$ and for some $m \in \mathbb{Z}^+$, $m \geq 2$,

$$\partial_x^m u_0 \in L^2(\{x \geq x_0\}), \tag{1.9}$$

then for any $v \geq 0,\ T > 0,\ \epsilon > 0$ *and* $\tau > \epsilon$

$$\sup_{0\le t\le T}\int_{x_0+\epsilon-vt}^{\infty} (\partial_x^j u)^2(x,t)\,\mathrm{d}x \\ + \int_0^T\int_{x_0+\epsilon-vt}^{x_0+\tau-vt} (D_x^{\frac{\alpha+1}{2}}\partial_x^j u)^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{x_0+\epsilon-vt}^{x_0+\tau-vt} (D_x^{\frac{\alpha+1}{2}}\mathcal{H}\partial_x^j u)^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \le c \tag{1.10}$$

for $j = 1, 2, \dots, m$, *with* $c = c(T;\epsilon; v; \alpha; \|u_0\|_{H^s}; \|\partial_x^m u_0\|_{L^2((x_0,\infty))}) > 0$.

If in addition to (1.9) *there exists* $x_0 \in \mathbb{R}^+$ *with*

$$D_x^{\frac{1-\alpha}{2}}\partial_x^m u_0 \in L^2(\{x \ge x_0\}) \tag{1.11}$$

then for any $v \geq 0,\ \epsilon > 0$ *and* $\tau > \epsilon$

$$\sup_{0\le t\le T}\int_{x_0+\epsilon-vt}^{\infty} (D_x^{\frac{1-\alpha}{2}}\partial_x^m u)^2(x,t)\,\mathrm{d}x \\ + \int_0^T\int_{x_0+\epsilon-vt}^{x_0+\tau-vt} (\partial_x^{m+1} u)^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{x_0+\epsilon-vt}^{x_0+\tau-vt} (\partial_x^{m+1}\mathcal{H}u)^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \le c, \tag{1.12}$$

with $c = c(T;\epsilon; v; \alpha; \|u_0\|_{H^s}; \|D_x^{(1-\alpha)/2}\partial_x^m u_0\|_{L^2((x_0,\infty))}) > 0$.

Although the argument of the proof of Theorem B follows in spirit that of KdV, i.e., an induction process combined with weighted energy estimates, the presence of the nonlocal operator $D_x^{\alpha+1}\partial_x$ in the term providing the dispersion, makes the proof much harder. More precisely, two difficulties appear, the most important of which is to obtain explicitly the Kato smoothing effect [1983], which in the proof of Theorem 1.3 is fundamental.

In contrast to the KdV equation, the gain of the local smoothing in solutions of the dispersive generalized Benjamin–Ono equation is just $\frac{\alpha+1}{2}$ derivatives, so as occurs in the case of the Benjamin–Ono equation [Isaza et al. 2016a], the iterative argument in the induction process is carried out in two steps, one for positive integers m and another one for $m + \frac{1-\alpha}{2}$ derivatives.

In the case of the BO equation [Isaza et al. 2016a], the authors obtain the smoothing effect basing their analysis on several commutator estimates, such as the extension of Calderón's first commutator for the Hilbert transform [Baishanski and Coifman 1967]. However, their method of proof does not allow them to obtain explicitly the local smoothing as in [Kato 1983].

The advantage of our method is that it allows us to obtain explicitly the smoothing effect for any $\alpha \in (0, 1)$ in the IVP (1.1). Roughly, we rewrite the term modeling the dispersive part of the equation in (1.1) in terms of an expression involving $[\mathcal{H}D_x^{\alpha+2}; \chi^2_{\epsilon,b}]$. At this point, we incorporate results of [Ginibre and Velo 1991] about commutator decomposition. This allows us to obtain explicitly the smoothing effect as in [Kato 1983] at every step of the induction process in the energy estimate. Additionally, this approach allows us to study the propagation of regularity phenomena in models where the dispersion is lower in comparison with that of IVP (1.1). We address this issue in a forthcoming work; specifically we study the propagation of regularity phenomena in real solutions of the model

$$\partial_t u - D_x^{\alpha}\partial_x u + u\partial_x u = 0, \quad x, t \in \mathbb{R},\ 0 < \alpha < 1.$$

As a direct consequence of Theorem B, one has that for an appropriate class of initial data the singularity of the solution travels with infinite speed to the left as time evolves. Also, the time reversibility property implies that the solution cannot have had some regularity in the past.

Concerning the nonlinear part of IVP (1.1) in the weighted energy estimate, several issues arise. Nevertheless, following the approach of [Kenig et al. 2018], combined with [Kato and Ponce 1988; Li 2019] on the generalization of several commutator estimates, allows us to overcome these difficulties.

Remark 1.13. (I) It will be clear from our proof that the requirement on the initial data, that is, $u_0 \in H^{(3-\alpha)/2}(\mathbb{R})$ in Theorem B, can be lowered to $H^{((9-3\alpha)/8)+}(\mathbb{R})$.

(II) Also it is worth highlighting that the proof of Theorem B can be extended to solutions of the IVP

$$\begin{cases} \partial_t u - D_x^{\alpha+1}\partial_x u + u^k\,\partial_x u = 0, & x, t \in \mathbb{R},\ 0<\alpha<1,\ k \in \mathbb{Z}^+, \\ u(x,0) = u_0(x). \end{cases} \tag{1.14}$$

(III) The results in Theorem B still hold for solutions of the defocusing generalized dispersive Benjamin–Ono equation

$$\begin{cases} \partial_t u - D_x^{\alpha+1}\partial_x u - u\,\partial_x u = 0, & x, t \in \mathbb{R},\ 0<\alpha<1, \\ u(x,0) = u_0(x). \end{cases}$$

This can be seen applying Theorem B to the function $v(x,t) = u(-x,-t)$, where $u(x,t)$ is a solution of (1.1). In short, Theorem B remains valid, backward in time for initial data u_0 satisfying (1.9) and (1.11).

Next, we present some immediate consequences of Theorem B.

Corollary 1.15. *Let $u \in C([-T,T] : H^{(3-\alpha)/2}(\mathbb{R}))$ be a solution of the equation in (1.1) described by Theorem B. If there exist $n, m \in \mathbb{Z}^+$ with $m \le n$ such that for some $\tau_1, \tau_2 \in \mathbb{R}$ with $\tau_1 < \tau_2$*

$$\int_{\tau_2}^{\infty} |\partial_x^n u_0(x)|^2\,dx < \infty \quad \textit{but} \quad \partial_x^m u_0 \notin L^2((\tau_1,\infty)),$$

then for any $t \in (0,T)$ and any $v > 0$ and $\epsilon > 0$

$$\int_{\tau_2+\epsilon-vt}^{\infty} |\partial_x^n u(x,t)|^2\,dx < \infty,$$

and for any $t \in (-T,0)$ and any $\tau_3 \in \mathbb{R}$

$$\int_{\tau_3}^{\infty} |\partial_x^m u(x,t)|^2\,dx = \infty.$$

The rest of the paper is organized as follows: in the Section 2 we fix the notation to be used throughout the document. Section 3 contains a brief summary of commutator estimates involving fractional derivatives. Section 4 deals with the local well-posedness. Finally, in Sections 5 and 6 we prove Theorems A and B.

2. Notation

The following notation will be used extensively throughout this article. The operator $J^s = (1-\partial_x^2)^{s/2}$ denotes the Bessel potentials of order $-s$.

For $1 \le p \le \infty$, $L^p(\mathbb{R})$ is the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$; additionally for $s \in \mathbb{R}$, we consider the Sobolev space $H^s(\mathbb{R})$ is defined via its usual norm $\|f\|_{H^s} = \|J^s f\|_{L^2}$. In this context, we define

$$H^\infty(\mathbb{R}) = \bigcap_{s \ge 0} H^s(\mathbb{R}).$$

Let $f = f(x,t)$ be a function defined for $x \in \mathbb{R}$ and t in the time interval $[0,T]$, with $T > 0$, or in the whole line $\mathbb{R}$. Then if A denotes any of the spaces defined above, we define the spaces $L^p_T A_x$ and $L^p_t A_x$ by the norms

$$\|f\|_{L^p_T A_x} = \left(\int_0^T \|f(\cdot,t)\|_A^p \, \mathrm{d}t\right)^{\frac{1}{p}} \quad \text{and} \quad \|f\|_{L^p_t A_x} = \left(\int_{\mathbb{R}} \|f(\cdot,t)\|_A^p \, \mathrm{d}t\right)^{\frac{1}{p}}$$

for $1 \le p \le \infty$ with the natural modification in the case $p = \infty$. Moreover, we use similar definitions for the mixed spaces $L^q_x L^p_t$ and $L^q_x L^p_T$ with $1 \le p, q \le \infty$.

For two quantities A and B, we write $A \lesssim B$ if $A \le cB$ for some constant $c > 0$. Similarly, $A \gtrsim B$ if $A \ge cB$ for some $c > 0$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. The dependence of the constant c on other parameters or constants is usually clear from the context and we will often suppress this dependence whenever possible.

For a real number a we will write a^+ instead of $a + \epsilon$ whenever ϵ is a positive number whose value is small enough.

3. Preliminaries

In this section, we state several inequalities to be used in the next sections.

First, we have an extension of the Calderón commutator theorem [1965] established in [Baishanski and Coifman 1967].

Theorem 3.1. *For any $p \in (1,\infty)$ and any $l, m \in \mathbb{Z}^+ \cup \{0\}$ there exists $c = c(p; l; m) > 0$ such that*

$$\|\partial_x^l [\mathcal{H}; \psi] \partial_x^m f\|_{L^p} \le c \|\partial_x^{m+l} \psi\|_{L^\infty} \|f\|_{L^p}. \tag{3.2}$$

For a different proof see [Dawson et al. 2008, Lemma 3.1].

In our analysis the Leibniz rule for fractional derivatives, established in [Grafakos and Oh 2014; Kato and Ponce 1988; Kenig et al. 1994], will be crucial. Even though most of these estimates are valid in several dimensions, we will restrict our attention to the one-dimensional case.

Lemma 3.3. *For $s > 0$, $p \in [1,\infty)$,*

$$\|D^s(fg)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|D^s g\|_{L^{p_2}} + \|g\|_{L^{p_3}} \|D^s f\|_{L^{p_4}}, \tag{3.4}$$

with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \qquad p_j \in (1,\infty], \ j = 1,2,3,4.$$

Also, we will state the fractional Leibniz rule proved by Kenig, Ponce and Vega [Kenig et al. 1993].

Lemma 3.5. *Let $s = s_1 + s_2 \in (0, 1)$, with $s_1, s_2 \in (0, s)$, and $p, p_1, p_2 \in (1, \infty)$, satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Then,

$$\|D^s(fg) - fD^s g - gD^s f\|_{L^p} \lesssim \|D^{s_1} f\|_{L^{p_1}} \|D^{s_2} g\|_{L^{p_2}}. \tag{3.6}$$

Moreover, the case $s_2 = 0$ and $p_2 = \infty$ is allowed.

A natural question about Lemma 3.5 is to investigate the possible generalization of the estimate (3.6) when $s \geq 1$. The answer to this question was given recently by D. Li [2019]; he established new fractional Leibniz rules for the nonlocal operator D^s, $s > 0$, and related ones, including various endpoint situations.

Theorem 3.7. *Let $s > 0$ and $1 < p < \infty$. Then for any $s_1, s_2 \geq 0$ with $s = s_1 + s_2$, and any $f, g \in \mathcal{S}(\mathbb{R}^n)$, the following hold:*

(1) *If $1 < p_1, p_2 < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, then*

$$\left\| D^s(fg) - \sum_{\alpha \leq s_1} \frac{1}{\alpha!} \partial_x^\alpha f D^{s,\alpha} g - \sum_{\beta \leq s_2} \frac{1}{\beta!} \partial_x^\beta g D^{s,\beta} f \right\|_{L^p} \lesssim \|D^{s_1} f\|_{L^{p_1}} \|D^{s_2} g\|_{L^{p_2}}. \tag{3.8}$$

(2) *If $p_1 = p$, $p_2 = \infty$, then*

$$\left\| D^s(fg) - \sum_{\alpha < s_1} \frac{1}{\alpha!} \partial_x^\alpha f D^{s,\alpha} g - \sum_{\beta \leq s_2} \frac{1}{\beta!} \partial_x^\beta g D^{s,\beta} f \right\|_{L^p} \lesssim \|D^{s_1} f\|_{L^p} \|D^{s_2} g\|_{\mathrm{BMO}},$$

where $\|\cdot\|_{\mathrm{BMO}}$ denotes the norm in the BMO space.[1]

(3) *If $p_1 = \infty$, $p_2 = p$, then*

$$\left\| D^s(fg) - \sum_{\alpha \leq s_1} \frac{1}{\alpha!} \partial^\alpha f D^{s,\alpha} g - \sum_{\beta < s_2} \frac{1}{\beta!} \partial^\beta g D^{s,\beta} f \right\|_{L^p} \lesssim \|D^{s_1} f\|_{\mathrm{BMO}} \|D^{s_2} g\|_{L^p}.$$

The operator $D^{s,\alpha}$ is defined via Fourier transform[2]

$$\widehat{D^{s,\alpha} g}(\xi) = \widehat{D}^{s,\alpha}(\xi)\hat{g}(\xi),$$
$$\widehat{D}^{s,\alpha}(\xi) = i^{-\alpha} \partial_\xi^\alpha (|\xi|^s).$$

Remark 3.9. As usual empty summation (such as $\sum_{0 \leq \alpha < 0}$) is defined as zero.

Proof. For a detailed proof of this theorem and related results, see [Li 2019]. □

Next we have the following commutator estimates involving nonhomogeneous fractional derivatives, established by Kato and Ponce.

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[1] For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the BMO seminorm is given by $\|f\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |f(y) - (f)_Q| \, dy$, where $(f)_Q$ is the average of f on Q, and the supreme is taken over all cubes Q in $\mathbb{R}^n$.

[2] The precise form of the Fourier transform does not matter.

Lemma 3.10 [Kato and Ponce 1988]. *Let $s > 0$ and $p, p_2, p_3 \in (1, \infty)$ and $p_1, p_4 \in (1, \infty]$ be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then,

$$\|[J^s; f]g\|_{L^p} \lesssim \|\partial_x f\|_{L^{p_1}} \|J^{s-1} g\|_{L^{p_2}} + \|J^s f\|_{L^{p_3}} \|g\|_{L^{p_4}} \tag{3.11}$$

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|J^s g\|_{L^{p_3}} \|f\|_{L^{p_4}}. \tag{3.12}$$

There are many other reformulations and generalizations of the Kato–Ponce commutator inequalities; see [Bényi and Oh 2014]. Recently Li [2019] has obtained a family of refined Kato–Ponce-type inequalities for the operator D^s. In particular he showed that:

Lemma 3.13. *Let $1 < p < \infty$. Let $1 < p_1, p_2, p_3, p_4 \leq \infty$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Therefore:

(a) *If $0 < s \leq 1$, then*

$$\|D^s(fg) - fD^s g\|_{L^p} \lesssim \|D^{s-1}\partial_x f\|_{L^{p_1}} \|g\|_{L^{p_2}}.$$

(b) *If $s > 1$, then*

$$\|D^s(fg) - fD^s g\|_{L^p} \lesssim \|D^{s-1}\partial_x f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|\partial_x f\|_{L^{p_3}} \|D^{s-1} g\|_{L^{p_4}}. \tag{3.14}$$

For a more detailed exposition on these estimates see [Li 2019, Section 5].

In addition, we have the following inequality of Gagliardo–Nirenberg type:

Lemma 3.15. *Let $1 < q, p < \infty$, $1 < r \leq \infty$ and $0 < \alpha < \beta$. Then,*

$$\|D^\alpha f\|_{L^p} \lesssim c\|f\|_{L^r}^{1-\theta} \|D^\beta f\|_{L^q}^{\theta}$$

with

$$\frac{1}{p} - \alpha = (1-\theta)\frac{1}{r} + \theta\Big(\frac{1}{q} - \beta\Big), \quad \theta \in \Big[\frac{\alpha}{\beta}, 1\Big].$$

Proof. See [Bergh and Löfström 1976, Chapter 4]. □

Now, we present a result that will help us to establish the propagation of regularity of solutions of (1.1). A previous result [Kenig et al. 2018, Corollary 2.1] was proved using the fact that J^r, $r \in \mathbb{R}$, can be seen as a pseudodifferential operator. Thus, this approach allows us to obtain an expression for J^r in terms of a convolution with a certain kernel $k(x, y)$ which enjoys some properties on localized regions in $\mathbb{R}^2$. In fact, this is known as the singular integral realization of a pseudodifferential operator, whose proof can be found in [Stein 1993, Chapter 4].

The estimate we consider here involves the nonlocal operator D^s instead of J^s.

Lemma 3.16. *Let $m \in \mathbb{Z}^+$ and $s \geq 0$. If $f \in L^2(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, $2 \leq p \leq \infty$, with*

$$\operatorname{dist}(\operatorname{supp}(f), \operatorname{supp}(g)) \geq \delta > 0. \tag{3.17}$$

Then

$$\|g\, \partial_x^m D^s f\|_{L^2} \lesssim \|g\|_{L^p} \|f\|_{L^2}.$$

Proof. Let f, g be functions in the Schwartz class satisfying (3.17).

Notice that

$$g(x)(D_x^s \partial_x^m f)(x) = \frac{g(x)}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} \mathrm{e}^{ix\xi}\, |\xi|^s \widehat{\partial_x^m f}(\xi)\, \mathrm{d}\xi = \frac{g(x)}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} |\xi|^s (\widehat{\tau_{-x} \partial_x^m f})(\xi)\, \mathrm{d}\xi, \tag{3.18}$$

where τ_h is the translation operator.[3]

Moreover, the last expression in (3.18) defines a tempered distribution for s in a suitable class, which will be specified later. Indeed, for $z \in \mathbb{C}$ with $-1 < \operatorname{Re}(z) < 0$

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} |\xi|^z (\widehat{\tau_{-x} \partial_x^m \varphi})(\xi)\, \mathrm{d}\xi = c(z) \int_{\mathbb{R}} \frac{(\tau_{-x} \partial_x^m \varphi)(y)}{|y|^{1+z}}\, \mathrm{d}y \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}), \tag{3.19}$$

with $c(z)$ is independent of φ. In fact, evaluating $\varphi(x) = \mathrm{e}^{-x^2/2}$ in (3.19) yields

$$c(z) = \frac{2^z\, \Gamma\left(\frac{z+1}{2}\right)}{\pi^{\frac{1}{2}} \Gamma\left(-\frac{z}{2}\right)}.$$

Thus, for every $\varphi \in \mathcal{S}(\mathbb{R})$ the right-hand side in (3.19) defines a meromorphic function for every test function, which can be extended analytically to a wider range of complex numbers z, specifically z with $\operatorname{Im}(z) = 0$ and $\operatorname{Re}(z) = s > 0$, which is the case that pertains to us. By an abuse of notation, we will denote the meromorphic extension and the original as the same.

Thus, combining (3.17), (3.18) and (3.19) it follows that

$$g(x)(D_x^s \partial_x^m f)(x) = c(s) \int_{\mathbb{R}} \frac{g(x)(\tau_{-x} \partial_x^m f)(y)}{|y|^{1+s}}\, \mathrm{d}y = c(s) g(x) \left(f * \frac{\mathbb{1}_{\{|y| \geq \delta\}}}{|y|^{s+m+1}} \right)(x).$$

Notice that the kernel in the integral expression is not anymore singular due to the condition (3.17). In fact, in the particular case that m is even, we obtain after apply integration by parts

$$g(x)(D_x^s \partial_x^m f)(x) = c(s,m) g(x) \left(f * \frac{\mathbb{1}_{\{|y| \geq \delta\}}}{|y|^{s+m+1}} \right)(x)$$

and in the case m being odd

$$g(x)(D_x^s \partial_x^m f)(x) = c(s,m) g(x) \left(f * \frac{y \mathbb{1}_{\{|y| \geq \delta\}}}{|y|^{s+m+2}} \right)(x).$$

Finally, in both cases combining Young's inequality and Hölder's inequality one gets

$$\|g\, \partial_x^m D_x^s f\|_{L^2} \lesssim \|g\|_{L^p} \|f\|_{L^2} \left\| \frac{\mathbb{1}_{\{|y| \geq \delta\}}}{|\cdot|^{s+m+1}} \right\|_{L^r} \lesssim \|g\|_{L^p} \|f\|_{L^2},$$

where the index p satisfies $\frac{1}{2} = \frac{1}{p} + \frac{1}{r}$, which clearly implies $p \in [2, \infty]$, as was required. □

[3] For $h \in \mathbb{R}$ the translation operator τ_h is defined as $(\tau_h f)(x) = f(x-h)$.

Further in the paper we will use extensively some results about the commutator beyond those presented in this section. Next, we will study the smoothing effect for solutions of the dispersive generalized Benjamin–Ono equation (1.1) following Kato's ideas [1983].

3A. ***Commutator expansions.*** In this section we present several new main tools obtained in [Ginibre and Velo 1989; 1991] which will be the cornerstone of the proof of Theorem B. They include commutator expansions together with their estimates. The basic problem is to handle the nonlocal operator D^s for noninteger s and in particular to obtain representations of its commutator with multiplication operators by functions that exhibit as much locality as possible.

Let $a = 2\mu + 1 > 1$, let n be a nonnegative integer and h be a smooth function with suitable decay at infinity, for instance with $h' \in C_0^\infty(\mathbb{R})$.

We define the operator

$$R_n(a) = [HD^a; h] - \tfrac{1}{2}(P_n(a) - HP_n(a)H), \tag{3.20}$$

$$P_n(a) = a \sum_{0 \le j \le n} c_{2j+1}(-1)^j 4^{-j} D^{\mu-j}(h^{(2j+1)} D^{\mu-j}), \tag{3.21}$$

where

$$c_1 = 1, \quad c_{2j+1} = \frac{1}{(2j+1)!} \prod_{0 \le k < j} (a^2 - (2k+1)^2) \quad \text{and} \quad H = -\mathcal{H}.$$

It was shown in [Ginibre and Velo 1989] that the operator $R_n(a)$ can be represented in terms of anticommutators[4] as follows:

$$R_n(a) = \tfrac{1}{2}([H; Q_n(a)]_+ + [D^a; [H; h]]_+), \tag{3.22}$$

where the operator $Q_n(a)$ is represented in the Fourier space variables by the integral kernel

$$Q_n(a) \to (2\pi)^{\frac{1}{2}} \hat{h}(\xi - \xi') |\xi\xi'|^{\frac{a}{2}} 2a q_n(a, t), \tag{3.23}$$

with $|\xi| = |\xi'|\, e^{2t}$ and

$$q_n(a, t) = \frac{1}{a}(a^2 - (2n+1)^2) c_{2n+1} \int_0^t \sinh^{2n+1}\tau \ \sinh((a(t-\tau)))\, d\tau. \tag{3.24}$$

Based on (3.22) and (3.23), Ginibre and Velo [1991] obtained the following properties of boundedness and compactness of the operator $R_n(a)$.

Proposition 3.25. *Let n be a nonnegative integer, $a \ge 1$, and $\sigma \ge 0$ be such that*

$$2n + 1 \le a + 2\sigma \le 2n + 3. \tag{3.26}$$

Then:

(a) *The operator $D^\sigma R_n(a) D^\sigma$ is bounded in L^2 with norm*

$$\|D^\sigma R_n(a) D^\sigma f\|_{L^2} \le C(2\pi)^{-\frac{1}{2}} \|(\widehat{D^{a+2\sigma} h})\|_{L^1_\xi} \|f\|_{L^2}. \tag{3.27}$$

If $a \ge 2n + 1$, one can take $C = 1$.

[4] For any two operators P and Q we denote the anticommutator by $[P; Q]_+ = PQ + QP$.

(b) *Assume in addition that*

$$2n+1 \le a+2\sigma < 2n+3.$$

Then the operator $D^{\sigma} R_n(a) D^{\sigma}$ *is compact in* $L^2(\mathbb{R})$.

Proof. See Proposition 2.2 in [Ginibre and Velo 1991]. □

In fact Proposition 3.25 is a generalization of a previous result, where the derivatives of the operator $R_n(a)$ are not considered; see [Ginibre and Velo 1989, Proposition 1].

The estimate (3.27) yields the following identity of localization of derivatives.

Lemma 3.28. *Assume* $0<\alpha<1$. *Let be* $\varphi \in C^{\infty}(\mathbb{R})$ *with* $\varphi' \in C_0^{\infty}(\mathbb{R})$. *Then,*

$$\int_{\mathbb{R}} \varphi f\, D^{\alpha+1}\partial_x f\, dx = \left(\frac{\alpha+2}{4}\right)\int_{\mathbb{R}} (|D^{\frac{\alpha+1}{2}} f|^2 + |D^{\frac{\alpha+1}{2}}\mathcal{H} f|^2)\varphi'\, dx + \frac{1}{2}\int_{\mathbb{R}} f R_0(\alpha+2) f\, dx. \tag{3.29}$$

Proof. The proof follows the ideas presented in Proposition 2.12 in [Linares et al. 2014]. □

4. The linear problem

The aim of this section is to obtain Strichartz estimates associated to solutions of the IVP (1.1).

First, consider the linear problem

$$\begin{cases} \partial_t u - D_x^{\alpha+1}\partial_x u = 0, & x,t \in \mathbb{R},\ 0<\alpha<1, \\ u(x,0) = u_0(x), \end{cases} \tag{4.1}$$

whose solution is given by

$$u(x,t) = S(t)u_0 = (e^{it|\xi|^{\alpha+1}\xi}\,\hat{u}_0)^{\vee}. \tag{4.2}$$

We begin studying estimates of the unitary group obtained in (4.2).

Proposition 4.3. *Assume that* $0<\alpha<1$. *Let* q,p *satisfy* $\frac{2}{q}+\frac{1}{p}=\frac{1}{2}$ *with* $2 \le p \le \infty$.

Then

$$\|D_x^{\frac{\alpha}{q}} S(t)u_0\|_{L_t^q L_x^p} \lesssim \|u_0\|_{L_x^2} \tag{4.4}$$

for all $u_0 \in L^2(\mathbb{R})$.

Proof. The proof follows as an application of Theorem 2.1 in [Kenig et al. 1991a]. □

Remark 4.5. Notice that the condition on p implies $q \in [4,\infty]$, which in one of the extremal cases $(p,q)=(\infty,4)$ yields

$$\|D_x^{\frac{\alpha}{4}} S(t)u_0\|_{L_t^4 L_x^{\infty}} \lesssim \|u_0\|_{L_x^2},$$

which shows the gain of $\frac{\alpha}{4}$ derivatives globally in time for solutions of (4.1).

Lemma 4.6. *Assume that* $0<\alpha<1$. *Let* ψ_k *be a* $C^{\infty}(\mathbb{R})$ *function supported in the interval* $[2^{k-1}, 2^{k+1}]$, *where* $k \in \mathbb{Z}^+$. *Then, the function* H_k^{α} *defined as*

$$H_k^{\alpha}(x) = \begin{cases} 2^k & \text{if } |x| \le 1, \\ 2^{\frac{k}{2}}|x|^{-\frac{1}{2}} & \text{if } 1 \le |x| \le c2^{k(\alpha+1)}, \\ (1+x^2)^{-1} & \text{if } |x| > c2^{k(\alpha+1)} \end{cases}$$

satisfies

$$\left|\int_{-\infty}^{\infty} e^{i(t\xi|\xi|^{\alpha+1}+x\xi)}\,\psi_k(\xi)\,d\xi\right| \lesssim H_k^{\alpha}(x) \tag{4.7}$$

for $|t| \le 2$, *where the constant* c *does not depend on* t *or* k.

Moreover, we have

$$\sum_{l=-\infty}^{\infty} H_k^{\alpha}(|l|) \lesssim 2^{k(\frac{\alpha+1}{2})}. \tag{4.8}$$

Proof. The proof of estimate (4.7) is given in [Kenig et al. 1991b, Proposition 2.6] and it uses arguments of localization and the classical Van der Corput lemma. Meanwhile, (4.8) follows exactly that of Lemma 2.6 in [Linares et al. 2014]. □

Theorem 4.9. *Assume* $0 < \alpha < 1$. *Let* $s > \frac{1}{2}$. *Then,*

$$\|S(t)u_0\|_{L_x^2 L_t^{\infty}([-1,1])} \le \left(\sum_{j=-\infty}^{\infty} \sup_{|t|\le 1}\ \sup_{j\le x<j+1} |S(t)u_0(x)|^2\right)^{\frac{1}{2}} \lesssim \|u_0\|_{H_x^s}$$

for any $u_0 \in H^s(\mathbb{R})$.

Proof. See Theorem 2.7 in [Kenig et al. 1991b]. □

Next, we recall a maximal function estimate proved by Kenig, Ponce and Vega [Kenig et al. 1991b].

Corollary 4.10. *Assume that* $0 < \alpha < 1$. *Then, for any* $s > \frac{1}{2}$ *and any* $\eta > \frac{3}{4}$

$$\left(\sum_{j=-\infty}^{\infty} \sup_{|t|\le T}\ \sup_{j\le x<j+1} |S(t)v_0|^2\right)^{\frac{1}{2}} \lesssim (1+T)^{\eta}\|v_0\|_{H_x^s}.$$

Proof. See Corollary 2.8 in [Kenig et al. 1991b]. □

4A. ***The nonlinear problem.*** This section is devoted to studying general properties of solutions of the nonlinear problem

$$\begin{cases} \partial_t u - D_x^{\alpha+1}\partial_x u + u\,\partial_x u = 0, & x, t \in \mathbb{R},\ 0 < \alpha < 1, \\ u(x,0) = u_0(x). \end{cases} \tag{4.11}$$

We begin this section by stating the following local existence theorem proved in [Kato 1975; Saut and Temam 1976].

Theorem 4.12. (1) *For any* $u_0 \in H^s(\mathbb{R})$ *with* $s > \frac{3}{2}$ *there exists a unique solution* u *to* (4.11) *in the class* $C([-T,T] : H^s(\mathbb{R}))$ *with* $T = T(\|u_0\|_{H^s}) > 0$.

(2) *For any* $T' < T$ *there exists a neighborhood* V *of* u_0 *in* $H^s(\mathbb{R})$ *such that the map* $\tilde{u}_0 \mapsto \tilde{u}(t)$ *from* V *into* $C([-T',T'] : H^s(\mathbb{R}))$ *is continuous.*

(3) *If* $u_0 \in H^{s'}(\mathbb{R})$ *with* $s' > s$, *then the time of existence* T *can be taken to depend only on* $\|u_0\|_{H^s}$.

Our first goal will be to obtain some energy estimates satisfied by smooth solutions of the IVP (4.11). We firstly present a result that arises as a consequence of commutator estimates.

Lemma 4.13. *Suppose that* $0 < \alpha < 1$. *Let* $u \in C([0,T] : H^{\infty}(\mathbb{R}))$ *be a smooth solution of* (4.11). *If* $s > 0$ *is given, then*

$$\|u\|_{L^\infty_T H^s_x} \lesssim \|u_0\|_{H^s_x} \, \mathrm{e}^{c\|\partial_x u\|_{L^1_T L^\infty_x}}. \tag{4.14}$$

Proof. Let $s > 0$. By a standard energy estimate argument we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (J^s_x u)^2 \, \mathrm{d}x + \int_{\mathbb{R}} [J^s_x ; u]\partial_x u \, J^s_x u \, \mathrm{d}x + \int_{\mathbb{R}} u \, J^s_x u \, J^s_x \partial_x u \, \mathrm{d}x = 0.$$

Hence integration by parts, Gronwall's inequality and the commutator estimate (3.11) lead to (4.14). □

Remark 4.15. In view of the energy estimate (4.14), the key point to obtaining a priori estimates in $H^s_x(\mathbb{R})$ is to control $\|\partial_x u\|_{L^1_T L^\infty_x}$ at the $H^s_x(\mathbb{R})$-level.

In addition to this estimate, we will present the smoothing effect provided by solutions of the dispersive generalized Benjamin–Ono equation. In fact, the smoothing effect was first observed by Kato [1983] in the context of the Korteweg–de Vries equation. Following Kato's approach joint with the commutator expansions, we present a result proved by Kenig, Ponce and Vega [Kenig et al. 1991b, Lemma 5.1].

Proposition 4.16. *Let* φ *denote a nondecreasing smooth function such that* $\operatorname{supp}\varphi' \subset (-1,2)$ *and* $\varphi'|_{[0,1)} = 1$. *For* $j \in \mathbb{Z}$, *we define* $\varphi_j(\cdot) = \varphi(\cdot - j)$. *Let* $u \in C([0,T] : H^{\infty}(\mathbb{R}))$ *be a real smooth solution of* (1.1) *with* $0 < \alpha < 1$. *Assume also that* $s \geq 0$ *and* $r > \frac{1}{2}$. *Then,*

$$\left(\int_0^T \int_{\mathbb{R}} (|D^{s+\frac{\alpha+1}{2}}_x u(x,t)|^2 + |D^{s+\frac{\alpha+1}{2}}_x \mathcal{H}u(x,t)|^2)\varphi'_j(x) \, \mathrm{d}x \, \mathrm{d}t\right)^{\frac{1}{2}} \lesssim (1 + T + \|\partial_x u\|_{L^1_T L^\infty_x} + T\|u\|_{L^\infty_T H^r_x})^{\frac{1}{2}} \|u\|_{L^\infty_T H^s_x}. \tag{4.17}$$

In addition to the smoothing effect presented above, we will need the following localized version of the $H^s(\mathbb{R})$-norm. For this purpose we will consider a cutoff function φ with the same characteristics as those in Proposition 4.16.

Proposition 4.18. *Let* $s \geq 0$. *Then, for any* $f \in H^s(\mathbb{R})$

$$\|f\|_{H^s(\mathbb{R})} \sim \left(\sum_{j=-\infty}^{\infty} \|f\varphi'_j\|^2_{H^s(\mathbb{R})}\right)^{\frac{1}{2}}.$$

Hence our first goal in establishing the local well-posedness of (4.11) is to obtain Strichartz estimates associated to solutions of

$$\partial_t u - D^{1+\alpha}_x \partial_x u = F. \tag{4.19}$$

Proposition 4.20. *Assume that* $0 < \alpha < 1$, $T > 0$ *and* $\sigma \in [0,1]$. *Let* u *be a smooth solution to* (4.19) *defined on the time interval* $[0,T]$. *Then there exist* $0 \leq \mu_1, \mu_2 < \frac{1}{2}$ *such that*

$$\|\partial_x u\|_{L^2_T L^\infty_x} \lesssim T^{\mu_1}\|J^{1-\frac{\alpha}{4}+\frac{\sigma}{4}+\epsilon}u\|_{L^\infty_T L^2_x} + T^{\mu_2}\|J^{1-\frac{\alpha}{4}-\frac{3\sigma}{4}+\epsilon}F\|_{L^2_T L^2_x} \tag{4.21}$$

for any $\epsilon > 0$.

Remark 4.22. The optimal choice in the parameters present in the estimate (4.21) corresponds to $\sigma = \frac{1-\alpha}{2}$. Indeed, as is pointed out by Kenig and Koenig [2003, Proposition 2.8] in the case of the Benjamin–Ono equation (case $\alpha = 0$) given a linear estimate of the form

$$\|\partial_x u\|_{L^2_T L^\infty_x} \lesssim T^{\mu_1} \|J^a u\|_{L^\infty_T L^2_x} + T^{\mu_2} \|J^b F\|_{L^2_T L^2_x}$$

the idea is to apply the smoothing effect (4.17) and absorb as many as derivatives as possible of the function F. Concerning to our case, the approach requires the choice $a = b + \frac{1-\alpha}{2}$; this particular choice, $\sigma = \frac{1-\alpha}{2}$, in the estimate (4.21) provides the regularity $s > \frac{9}{8} - \frac{3\alpha}{8}$ in Theorem A.

Proof. Let $f = \sum_k f_k$ denote the Littlewood–Paley decomposition of a function f. More precisely we choose functions $\eta, \chi \in C^\infty(\mathbb{R})$ with $\operatorname{supp}(\eta) \subseteq \left\{\xi : \frac{1}{2} < |\xi| < 2\right\}$ and $\operatorname{supp}(\chi) \subseteq \{\xi : |\xi| < 2\}$ such that

$$\sum_{k=1}^{\infty} \eta\left(\frac{\xi}{2^k}\right) + \chi(\xi) = 1$$

and $f_k = P_k(f)$, where $\widehat{(P_0 f)}(\xi) = \chi(\xi)\hat{f}(\xi)$ and $\widehat{(P_k f)}(\xi) = \eta(\xi/2^k)\hat{f}(\xi)$ for all $k \geq 1$.

Fix $\epsilon > 0$. Let $p > \frac{1}{\epsilon}$. By Sobolev embedding and the Littlewood–Paley theorem it follows that

$$\|f\|_{L^\infty_x} \lesssim \|J^\epsilon f\|_{L^p_x} \sim \left\| \left(\sum_{k=0}^{\infty} |J^\epsilon P_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p_x} = \left\| \sum_{k=0}^{\infty} |J^\epsilon P_k f|^2 \right\|^{\frac{1}{2}}_{L^{p/2}_x} \lesssim \left(\sum_{k=0}^{\infty} \|J^\epsilon P_k f\|^2_{L^p_x} \right)^{\frac{1}{2}}.$$

Therefore, to obtain (4.21) it enough to prove that for $p > 2$

$$\|\partial_x u_k\|_{L^2_T L^p_x} \lesssim \|D_x^{1-\frac{\alpha}{4}+\frac{\sigma}{4}+\frac{\alpha-\sigma}{2p}} u_k\|_{L^\infty_T L^2_x} + \|D_x^{1-\frac{\alpha}{4}-\frac{3\sigma}{4}+\frac{\alpha-\sigma}{2p}} F_k\|_{L^2_T L^2_x}, \quad k \geq 1.$$

The estimate for the case $k = 0$ follows using Hölder's inequality and (4.4). For such reason we fix $k \geq 1$, and at this level of frequencies we have

$$\partial_t u_k - D_x^{\alpha+1} \partial_x u_k = F_k.$$

Consider a partition of the interval $[0, T] = \bigcup_{j \in J} I_j$, where $I_j = [a_j, b_j]$, and $T = b_j$ for some j. Indeed, we choose a quantity $\sim 2^{k\sigma} T^{1-\mu}$ of intervals, with length $|I_j| \sim 2^{-k\sigma} T^\mu$, where μ is a positive number to be fixed.

Let q be such that

$$\frac{2}{q} + \frac{1}{p} = \frac{1}{2}.$$

Using that u solves the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-t')F(t')\,\mathrm{d}t', \tag{4.23}$$

we deduce that

$$\|\partial_x u_k\|_{L^2_T L^p_x} \lesssim (T^\mu 2^{-k\sigma})^{\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\sum_{j \in J} \|S(t-a_j)\partial_x u_k(a_j)\|^2_{L^q_{I_j} L^p_x} + \left\| \int_{a_j}^t S(t-s)\partial_x F_k(s)\,\mathrm{d}s \right\|^2_{L^q_{I_j} L^p_x} \right)^{\frac{1}{2}}.$$

In this sense, it follows from (4.4) that

$$
\begin{aligned}
\|\partial_x u_k\|_{L^2_T L^p_x} &\lesssim (T^{\mu}2^{-k\sigma})^{(\frac{1}{2}-\frac{1}{q})}\bigg\{\sum_{j\in J}\|D_x^{-\frac{\alpha}{q}}\partial_x u_k(a_j)\|^2_{L^2_x}+\sum_{j\in J}\bigg(\int_{I_j}\|D_x^{-\frac{\alpha}{q}}\partial_x F_k(t)\|_{L^2_x}\,dt\bigg)^2\bigg\}^{\frac{1}{2}}\\
&\lesssim (T^{\mu}2^{-k\sigma})^{(\frac{1}{2}-\frac{1}{q})}\bigg\{\bigg(\sum_{j\in J}\|D_x^{1-\frac{\alpha}{q}}u_k\|^2_{L^\infty_T L^2_x}\bigg)^{\frac{1}{2}}+\bigg(\sum_{j\in J}T^{\mu}2^{-k\sigma}\int_{I_j}\|D_x^{1-\frac{\alpha}{q}}F_k(t)\|^2_{L^2_x}\,dt\bigg)^{\frac{1}{2}}\bigg\}\\
&\lesssim (T^{\mu}2^{-k\sigma})^{(\frac{1}{2}-\frac{1}{q})}(T^{1-\mu}2^{k\sigma})^{\frac{1}{2}}\|D_x^{1-\frac{\alpha}{q}}u_k\|_{L^\infty_T L^2_x}\\
&\qquad\qquad+(T^{\mu}2^{-k\sigma})^{(\frac{1}{2}-\frac{1}{q})}(T^{\mu}2^{-k\sigma})^{\frac{1}{2}}\bigg(\int_0^T\|D_x^{1-\frac{\alpha}{q}}F_k(t)\|^2_{L^2_x}\,dt\bigg)^{\frac{1}{2}}\\
&\lesssim T^{\frac{1}{2}-\frac{\mu}{q}}\|D_x^{1-\frac{\alpha}{q}+\frac{\sigma}{q}}u_k\|_{L^\infty_T L^2_x}+T^{\mu(1-\frac{1}{q})}\|D_x^{1-\frac{\alpha}{q}+\frac{\sigma}{q}-\sigma}F_k\|_{L^2_T L^2_x},
\end{aligned}
$$

since

$$1-\frac{\alpha}{q}+\frac{\sigma}{q}=1-\frac{\alpha}{4}+\frac{\sigma}{4}+\frac{\alpha-\sigma}{2p}\quad\text{and}\quad 1-\frac{\alpha}{q}+\frac{\sigma}{q}-\sigma=1-\frac{\alpha}{4}-\frac{3\sigma}{4}+\frac{\alpha-\sigma}{2p}.$$

We recall that $\epsilon>\frac{1}{p}$, $\sigma\in[0,1]$ and $\alpha\in(0,1)$; then

$$\epsilon+\frac{\alpha-\sigma}{2p}>\frac{\alpha-\sigma+2}{2p}>0.$$

Next, we choose $\mu_1=\frac{1}{2}-\frac{\mu}{q}$, $\mu_2=\mu\big(1-\frac{1}{q}\big)$ with the particular choice $\mu=\frac{1}{2}$.

Gathering the inequalities above, the proposition follows. □

Now we turn our attention to the proof of Theorem A. Our starting point will be the energy estimate (4.14), where, as was remarked above, the key point is to establish a priori control of $\|\partial_x u\|_{L^1_T L^\infty_x}$.

5. Proof of Theorem A

5A. ***A priori estimates.*** First notice that by scaling, it is enough to deal with small initial data in the H^s-norm. Indeed, if $u(x,t)$ is a solution of (1.1) defined on a time interval $[0,T]$, for some positive time T, then, for all $\lambda>0$, $u_\lambda(x,t)=\lambda^{1+\alpha}u(\lambda x,\lambda^{2+\alpha}t)$ is also solution with initial data $u_{0,\lambda}(x)=\lambda^{1+\alpha}u_0(\lambda x)$, and time interval $[0,T/\lambda^{2+\alpha}]$.

For any $\delta>0$, we define $B_\delta(0)$ as the ball with center at the origin in $H^s(\mathbb{R})$ and radius δ.

Since

$$\|u_{0,\lambda}\|_{L^2_x}=\lambda^{\frac{1+2\alpha}{2}}\|u_0\|_{L^2_x}\quad\text{and}\quad\|D^s_x u_{0,\lambda}\|_{L^2_x}=\lambda^{\frac{1}{2}+\alpha+s}\|D^s_x u_0\|_{L^2_x},$$

we have

$$\|u_{0,\lambda}\|_{H^s_x}\lesssim\lambda^{\frac{1}{2}+\alpha}(1+\lambda^s)\|u_0\|_{H^s_x},$$

so we can force $u_\lambda(\cdot,0)$ to belong to the ball $B_\delta(0)$ by choosing the parameter λ with the condition

$$\lambda\sim\min\{\delta^{\frac{2}{1+2\alpha}}\|u_0\|_{H^s_x}^{-\frac{2}{1+2\alpha}},1\}.$$

Thus, the existence and uniqueness of a solution to (1.1) on the time interval $[0,1]$ for small initial data $\|u_0\|_{H^s_x}$ will ensure the existence and uniqueness of a solution to (1.1) for arbitrary large initial data on a time interval $[0,T]$ with

$$T\sim\min\{1,\|u_0\|_{H^s_x}^{-\frac{2(2+\alpha)}{1+2\alpha}}\}.$$

Thus, without loss of generality we will assume that $T \leq 1$ and that

$$\Lambda := \|u_0\|_{L^2} + \|D^s u_0\|_{L^2} \leq \delta,$$

where δ is a small positive number to be fixed later.

We fix s such that $s(\alpha) = \frac{9}{8} - \frac{3\alpha}{8} < s < \frac{3}{2} - \frac{\alpha}{2}$ and set $\epsilon = s - s(\alpha) > 0$.

Next, taking $\sigma = \frac{1-\alpha}{2} > 0$, $F = -u\,\partial_x u$ in (4.21) together with (4.14) yields

$$\begin{aligned}\|\partial_x u\|_{L^2_T L^\infty_x} &\lesssim T^{\mu_1}\|J^s u\|_{L^\infty_T L^2_x} + T^{\mu_2}\|J^{1-\frac{\alpha}{4}-\frac{3\sigma}{4}+\epsilon}(u\partial_x u)\|_{L^2_T L^2_x} \\ &\lesssim \Lambda + \Lambda\, \mathrm{e}^{c\|\partial_x u\|_{L^2_T L^\infty_x}} + \|D_x^{s+\frac{\alpha-1}{2}}(u\partial_x u)\|_{L^2_T L^2_x}. \end{aligned} \tag{5.1}$$

Now, to analyze the product coming from the nonlinear term we use the Leibniz rule for fractional derivatives (3.6) together with the energy estimate (4.14) as follows:

$$\begin{aligned}\|D_x^{s+\frac{\alpha-1}{2}}(u\partial_x u)\|_{L^2_T L^2_x} &\lesssim \|u D_x^{s+\frac{\alpha-1}{2}}\partial_x u\|_{L^2_T L^2_x} + \Big\| \|\partial_x u(t)\|_{L^\infty_x} \|D_x^{s+\frac{\alpha-1}{2}} u(t)\|_{L^2_x}\Big\|_{L^2_T} \\ &\lesssim \|u D_x^{s+\frac{\alpha-1}{2}}\partial_x u\|_{L^2_T L^2_x} + \Lambda\|\partial_x u\|_{L^2_T L^\infty_x}\, \mathrm{e}^{c\|\partial_x u\|_{L^2_T L^\infty_x}}. \end{aligned} \tag{5.2}$$

To handle the first term in the right-hand side above, we incorporate Kato's smoothing effect estimate obtained in (4.17) in the following way:

$$\begin{aligned}\|u D_x^{s+\frac{\alpha-1}{2}}\partial_x u\|_{L^2_T L^2_x} &\leq \left(\sum_{j=-\infty}^{\infty}\int_0^T \|u(t)\|^2_{L^\infty_{[j,j+1)}}\|D_x^{s+\frac{\alpha+1}{2}}\mathcal{H}u(t)\|^2_{L^2_{[j,j+1)}}\,\mathrm{d}t\right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{j=-\infty}^{\infty}\|u\|^2_{L^\infty_T L^\infty_{[j,j+1)}}\right)^{\frac{1}{2}}(1+\Lambda)\Lambda\, \mathrm{e}^{c\|\partial_x u\|_{L^2_T L^\infty_x}}. \end{aligned} \tag{5.3}$$

In summary, gathering the estimates (5.1)–(5.3) yields

$$\|\partial_x u\|_{L^2_T L^\infty_x} \lesssim \Lambda(1+\Lambda)\,\mathrm{e}^{c\|\partial_x u\|_{L^2_T L^\infty_x}}\left(\sum_{j=-\infty}^{\infty}\|u\|^2_{L^\infty_T L^\infty_{[j,j+1)}}\right)^{\frac{1}{2}} + \Lambda + \Lambda\,\mathrm{e}^{c\|\partial_x u\|_{L^2_T L^\infty_x}}. \tag{5.4}$$

Since u is a solution to (4.11), by Duhamel's formula it follows that

$$u(t) = S(t)u_0 - \int_0^t S(t-s)(u\,\partial_x u)(s)\,\mathrm{d}s,$$

where $S(t) = \mathrm{e}^{tD_x^{\alpha+1}\partial_x}$.

Now, we fix $\eta > 0$ such that $\eta < \frac{1+\alpha}{8}$; this choice implies that $\eta + \frac{1}{2} < s + \frac{\alpha-1}{2}$. Hence, Sobolev's embedding, Hölder's inequality and Corollary 4.10 produce

$$\begin{aligned}\left(\sum_{j=-\infty}^{\infty}\|u\|^2_{L^\infty_T L^\infty_{[j,j+1)}}\right)^{\frac{1}{2}} &\lesssim \left(\sum_{j=-\infty}^{\infty}\|S(t)u_0\|^2_{L^\infty_T L^\infty_{[j,j+1)}}\right)^{\frac{1}{2}} + \left(\sum_{j=-\infty}^{\infty}\left\|\int_0^t S(t-s)(u\partial_x u)(s)\,\mathrm{d}s\right\|^2_{L^\infty_T L^\infty_{[j,j+1)}}\right)^{\frac{1}{2}} \\ &\lesssim (1+T)\Lambda + (1+T)\|u\partial_x u\|_{L^1_T H_x^{\eta+1/2}} \\ &\lesssim \Lambda + \|u\partial_x u\|_{L^1_T L^2_x} + \|D_x^{\eta+\frac{1}{2}}(u\,\partial_x u)\|_{L^1_T L^2_x} \\ &\lesssim \Lambda + \Lambda\|\partial_x u\|_{L^2_T L^\infty_x} + \|D_x^{\eta+\frac{1}{2}}(u\,\partial_x u)\|_{L^2_T L^2_x}. \end{aligned} \tag{5.5}$$

Employing an argument similar to the one applied in (5.2) and (5.4) it is possible to bound the last term in the right-hand side as follows:

$$\|D_x^{\eta+\frac{1}{2}}(u\partial_x u)\|_{L^2_T L^2_x} \lesssim \Big(\sum_{j=-\infty}^{\infty} \|u\|^2_{L^\infty_T L^\infty_{[j,j+1)}}\Big)^{\frac{1}{2}} \Lambda(\Lambda+1)\, e^{c\|\partial_x u\|_{L^2_T L^\infty_x}} + \Lambda\, e^{c\|\partial_x u\|_{L^2_T L^\infty_x}}. \tag{5.6}$$

Next, we define

$$\phi(T) = \Big(\int_0^T \|\partial_x u(s)\|^2_{L^\infty_x}\, ds\Big)^{\frac{1}{2}} + \Big(\sum_{j=-\infty}^{\infty} \|u\|^2_{L^\infty_T L^\infty_{[j,j+1)}}\Big)^{\frac{1}{2}},$$

which is a continuous, nondecreasing function of T.

From (5.4), (5.5) and (5.6) it follows that

$$\phi(T) \lesssim \Lambda(\Lambda+1)\phi(T)\, e^{c\phi(T)} + \Lambda\, e^{c\phi(T)}\phi(T) + \Lambda\, e^{c\phi(T)} + \Lambda + \Lambda\phi(T).$$

Now, if we suppose that $\Lambda \le \delta \le 1$, we obtain

$$\phi(T) \le c\Lambda + c\Lambda\, e^{c\phi(T)}$$

for some constant $c > 0$.

To complete the proof we will show that if there exists $\delta > 0$ such that $\Lambda \le \delta$, then $\phi(1) \le A$ for some constant $A > 0$.

To do this, we define the function

$$\Psi(x, y) = x - cy - cy\, e^{cx}. \tag{5.7}$$

First notice that $\Psi(0, 0) = 0$ and $\partial_x\Psi(0, 0) = 1$. Then the implicit function theorem asserts that there exists $\delta > 0$ and a smooth function $\xi(y)$ such that $\xi(0) = 0$, and $\Psi(\xi(y), y) = 0$ for $|y| \le \delta$.

Notice that the condition $\Psi(\xi(y), y) = 0$ implies that $\xi(y) > 0$ for $y > 0$. Moreover, since $\partial_x\Psi(0, 0) = 1$, the function $\Psi(\cdot, y)$ is increasing close to $\xi(y)$ whenever δ is chosen sufficiently small.

Let us suppose that $\Lambda \le \delta$, and set $\lambda = \xi(\Lambda)$. Then, combining interpolation and Proposition 4.18 we obtain

$$\phi(0) = \Big(\sum_{j=-\infty}^{\infty} \big(\sup_{x\in[j,j+1)} |u(x,0)|\big)^2\Big)^{\frac{1}{2}} \lesssim \|u_0\|_{H^s(\mathbb{R})} \le c_1\|u_0\|_{L^2} + c_1\|D_x^s u_0\|_{L^2_x},$$

where we take $c > c_1$.

Therefore

$$\phi(0) \le c_1\Lambda < c\Lambda + c\Lambda\, e^{c\xi(\Lambda)} = \lambda.$$

Suppose that $\phi(T) > \lambda$ for some $T \in (0, 1)$ and define

$$T_0 = \inf\{T \in (0, 1) \mid \phi(T) > \lambda\}.$$

Hence, $T_0 > 0$ and $\phi(T_0) = \lambda$; additionally, there exists a decreasing sequence $\{T_n\}_{n\ge 1}$ converging to T_0 such that $\phi(T_n) > \lambda$. In addition, notice that (5.7) implies $\Psi(\phi(T), \Lambda) \le 0$ for all $T \in [0, 1]$.

Since the function $\Psi(\cdot, \Lambda)$ is increasing near λ, we have

$$\Psi(\phi(T_n), \Lambda) > \Psi(\phi(T_0), \Lambda) = \Psi(\lambda, \Lambda) = \Psi(\xi(\Lambda), \Lambda) = 0$$

for n sufficiently large.

This is a contradiction with the fact that $\phi(T) > \lambda$. So we conclude $\phi(T) \le A$ for all $T \in (0, 1)$, as was claimed. Thus, $\phi(1) \le A$.

In conclusion we have proved that

$$\phi(T) = \left(\int_0^T \|\partial_x u(s)\|^2_{L^\infty_x} \, ds\right)^{\frac{1}{2}} + \left(\sum_{j=-\infty}^{\infty} \|u\|^2_{L^\infty_T L^\infty_{[j,j+1)}}\right)^{\frac{1}{2}} \lesssim \|u_0\|_{H^s_x} \quad \text{for all } T \in [0, 1]. \tag{5.8}$$

At this stage, the existence, uniqueness, and continuous dependence on the initial data follows from the standard compactness and Bona–Smith approximation arguments; see for example [Kenig et al. 1991b; Ponce 1991].

6. Proof of Theorem B

The aim of this section is to prove Theorem B. To achieve this goal is necessary to take into account two important aspects of our analysis: first, the ambient space, which in our case is the Sobolev space where the theorem is valid together with the properties satisfied by the real solutions of the dispersive generalized Benjamin–Ono equation; and second, the auxiliary weight functions involved in the energy estimates, which we will describe in detail.

The following is a summary of the local well-posedness and Kato's smoothing effect presented in the previous sections.

Theorem C. *If* $u_0 \in H^s(\mathbb{R})$, $s \ge \frac{3-\alpha}{2}$, $\alpha \in (0, 1)$, *then there exist a positive time* $T = T(\|u_0\|_{H^s}) > 0$ *and a unique solution of the IVP* (1.1) *such that*

(a) $u \in C([-T, T] : H^s(\mathbb{R}))$,

(b) (*Strichartz*) $\partial_x u \in L^1([-T, T] : L^\infty(\mathbb{R}))$,

(c) (*smoothing effect*) *for* $R > 0$,

$$\int_{-T}^{T}\int_{-R}^{R} \left(|\partial_x D_x^{r+\frac{\alpha+1}{2}} u|^2 + |\mathcal{H}\partial_x D_x^{r+\frac{\alpha+1}{2}} u|^2\right) dx\, dt \le C \tag{6.1}$$

with $r \in \left(\frac{9-3\alpha}{8}, s\right]$ *and* $C = C(\alpha; R; T; \|u_0\|_{H^s_x}) > 0$.

Since we have set the Sobolev space where we will work, the next step is the description of the cutoff functions to be used in the proof.

In this part we consider families of cutoff functions that will be used systematically in the proof of Theorem B. This collection of weight functions was constructed originally in [Isaza et al. 2015; Kenig et al. 2018] in the proof of Theorem 1.3.

More precisely, for $\epsilon > 0$ and $b \ge 5\epsilon$ define the families of functions

$$\chi_{\epsilon,b},\ \phi_{\epsilon,b},\ \tilde{\phi}_{\epsilon,b},\ \psi_\epsilon,\ \eta_{\epsilon,b} \in C^\infty(\mathbb{R})$$

satisfying the following properties:

(1) $\chi'_{\epsilon,b} \ge 0$.

(2) $\chi_{\epsilon,b}(x) = 0$ if $x \le \epsilon$, and $\chi_{\epsilon,b}(x) = 1$ if $x \ge b$.

(3) $\operatorname{supp}(\chi_{\epsilon,b}) \subseteq [\epsilon, \infty)$.

(4) $\chi'_{\epsilon,b}(x) \geq \frac{1}{10(b-\epsilon)} \mathbb{1}_{[2\epsilon, b-2\epsilon]}(x)$.

(5) $\operatorname{supp}(\chi'_{\epsilon,b}) \subseteq [\epsilon, b]$.

(6) There exists a real number c_j such that

$$|\chi^{(j)}_{\epsilon,b}(x)| \leq c_j \chi'_{\frac{\epsilon}{3}, b+\epsilon}(x) \quad \text{for all } x \in \mathbb{R},\ j \in \mathbb{Z}^+.$$

(7) For $x \in (3\epsilon, \infty)$

$$\chi_{\epsilon,b}(x) \geq \frac{1}{2} \frac{\epsilon}{b-3\epsilon}.$$

(8) For $x \in \mathbb{R}$

$$\chi'_{\frac{\epsilon}{3}, b+\epsilon}(x) \leq \frac{\epsilon}{b-3\epsilon}.$$

(9) Given $\epsilon > 0$ and $b \geq 5\epsilon$ there exist $c_1, c_2 > 0$ such that

$$\chi'_{\epsilon,b}(x) \leq c_1 \chi'_{\frac{\epsilon}{3}, b+\epsilon}(x) \chi_{\frac{\epsilon}{3}, b+\epsilon}(x),$$
$$\chi'_{\epsilon,b}(x) \leq c_2 \chi_{\frac{\epsilon}{5}, \epsilon}(x).$$

(10) For $\epsilon > 0$ given and $b \geq 5\epsilon$, we define the function

$$\eta_{\epsilon,b} = \sqrt{\chi_{\epsilon,b} \chi'_{\epsilon,b}}.$$

(11) $\operatorname{supp}(\phi_{\epsilon,b}),\ \operatorname{supp}(\tilde{\phi}_{\epsilon,b}) \subset \left[\frac{\epsilon}{4}, b\right]$.

(12) $\phi_\epsilon(x) = \tilde{\phi}_{\epsilon,b}(x) = 1,\ x \in \left[\frac{\epsilon}{2}, \epsilon\right]$.

(13) $\operatorname{supp}(\psi_\epsilon) \subseteq \left(-\infty, \frac{\epsilon}{2}\right]$.

(14) For $x \in \mathbb{R}$

$$\chi_{\epsilon,b}(x) + \phi_{\epsilon,b}(x) + \psi_\epsilon(x) = 1,$$
$$\chi^2_{\epsilon,b}(x) + \tilde{\phi}^2_{\epsilon,b}(x) + \psi_\epsilon(x) = 1.$$

The family $\{\chi_{\epsilon,b} \mid \epsilon > 0,\ b \geq 5\epsilon\}$ is constructed as follows: let $\rho \in C_0^\infty(\mathbb{R})$, $\rho(x) \geq 0$, even, with $\operatorname{supp}(\rho) \subseteq (-1, 1)$ and $\|\rho\|_{L^1} = 1$.

Then define

$$\nu_{\epsilon,b}(x) = \begin{cases} 0, & x \leq 2\epsilon, \\ \dfrac{x}{b-3\epsilon} - \dfrac{2\epsilon}{b-3\epsilon}, & 2\epsilon \leq x \leq b-\epsilon, \\ 1, & x \geq b-\epsilon, \end{cases}$$

and

$$\chi_{\epsilon,b}(x) = \rho_\epsilon * \nu_{\epsilon,b}(x),$$

where $\rho_\epsilon(x) = \epsilon^{-1} \rho\left(\frac{x}{\epsilon}\right)$.

Now that we have described all the required estimates and tools necessary, we present the proof of our main result.

Proof of Theorem B. Since the argument is translation invariant, without loss of generality we will consider the case $x_0 = 0$.

First, we will describe the formal calculations assuming as much as regularity as possible; later we provide the justification using a limiting process.

The proof will be established by induction; however, in every step of the induction we will subdivide every case into steps, due to the nonlocal nature of the operator involving the dispersive part in the equation in (1.1).

Case $j = 1$. *Step 1*: First we apply one spatial derivative to the equation in (1.1); after that we multiply by $\partial_x u(x,t)\chi^2_{\epsilon,b}(x+vt)$, and finally we integrate in the x-variable to obtain the identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(\partial_x u)^2\chi^2_{\epsilon,b}\,\mathrm{d}x \underbrace{-\frac{v}{2}\int_{\mathbb{R}}(\partial_x u)^2(\chi^2_{\epsilon,b})'\,\mathrm{d}x}_{A_1(t)}\underbrace{-\int_{\mathbb{R}}(\partial_x D_x^{\alpha+1}\partial_x u)\,\partial_x u\,\chi^2_{\epsilon,b}\,\mathrm{d}x}_{A_2(t)}+\underbrace{\int_{\mathbb{R}}\partial_x(u\partial_x u)\,\partial_x u\,\chi^2_{\epsilon,b}\,\mathrm{d}x}_{A_3(t)}=0.$$

Step 1.1: Combining the local theory we obtain the following

$$\int_0^T |A_1(t)|\,\mathrm{d}t \le \frac{v}{2}\int_0^T\int_{\mathbb{R}}(\partial_x u)^2(\chi^2_{\epsilon,b})'\,\mathrm{d}x\,\mathrm{d}t \lesssim \|u\|_{L^\infty_T H_x^{(3-\alpha)/2}}.$$

Step 1.2: Integration by parts and Plancherel's identity allow us to rewrite the term A_2 as

$$A_2(t) = \frac{1}{2}\int_{\mathbb{R}}\partial_x u[D_x^{\alpha+1}\partial_x;\,\chi^2_{\epsilon,b}]\partial_x u\,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}}\partial_x u[\mathcal{H}D_x^{\alpha+2};\,\chi^2_{\epsilon,b}]\partial_x u\,\mathrm{d}x. \tag{6.2}$$

Since $\alpha + 2 > 1$, we have by (3.20) that the commutator $[\mathcal{H}D_x^{\alpha+2};\,\chi^2_{\epsilon,b}]$ can be decomposed as

$$[\mathcal{H}D_x^{\alpha+2};\,\chi^2_{\epsilon,b}] = -\tfrac{1}{2}P_n(\alpha+2)+\tfrac{1}{2}\mathcal{H}P_n(\alpha+2)\mathcal{H} - R_n(\alpha+2) \tag{6.3}$$

for some positive integer n, which will be fixed later.

Inserting (6.3) into (6.2)

$$\begin{aligned}A_2(t) &= \frac{1}{2}\int_{\mathbb{R}}\partial_x u\,R_n(\alpha+2)\partial_x u\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}}\partial_x u\,P_n(\alpha+2)\partial_x u\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}}\partial_x u\,\mathcal{H}P_n(\alpha+2)\mathcal{H}\partial_x u\,\mathrm{d}x\\ &= A_{2,1}(t)+A_{2,2}(t)+A_{2,3}(t).\end{aligned}$$

Now, we proceed to fix the value of n present in the terms $A_{2,1}$, $A_{2,2}$ and $A_{2,3}$, according to a determinate condition.

First, notice that

$$A_{2,1}(t) = \frac{1}{2}\int_{\mathbb{R}}D_x\mathcal{H}u R_n(\alpha+2)D_x\mathcal{H}u\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}}\mathcal{H}u D_x\{R_n(\alpha+2)D_x\mathcal{H}u\}\,\mathrm{d}x.$$

Then we fix n such that $2n+1 \le a+2\sigma \le 2n+3$, which according to the case we are studying ($j=1$), corresponds to $a = \alpha+2$ and $\sigma = 1$. This produces $n = 1$.

For this n in particular we have by Proposition 3.25 that $R_1(\alpha+2)$ maps L^2_x into L^2_x.

Hence,

$$A_{2,1}(t) \lesssim \|\mathcal{H}u(t)\|^2_{L^2_x}\|\widehat{D_x^{4+\alpha}\chi^2_{\epsilon,b}}\|_{L^1_\xi} = c\|u_0\|^2_{L^2_x}\|\widehat{D_x^{4+\alpha}\chi^2_{\epsilon,b}}\|_{L^1_\xi},$$

which after integrating in time yields

$$\int_0^T |A_{2,1}(t)|\,\mathrm{d}t \lesssim \|u_0\|_{L^2_x}^2 \sup_{0\le t\le T} \|\widehat{D_x^{4+\alpha}\chi_{\epsilon,b}^2}\|_{L^1_\xi}.$$

Next, we turn our attention to $A_{2,2}$. Replacing $P_1(\alpha+2)$ into $A_{2,2}$

$$\begin{aligned} A_{2,2}(t) &= \tilde{c}_1 \int_{\mathbb{R}} (D_x^{\frac{\alpha+1}{2}} \partial_x u)^2 (\chi_{\epsilon,b}^2)'\,\mathrm{d}x - \tilde{c}_3 \int_{\mathbb{R}} (D_x^{\frac{\alpha+1}{2}} \mathcal{H}u)^2 (\chi_{\epsilon,b}^2)'''\,\mathrm{d}x \\ &= A_{2,2,1}(t) + A_{2,2,2}(t). \end{aligned}$$

We shall underline that $A_{2,2,1}(t)$ is positive; additionally it represents explicitly the smoothing effect for the case $j=1$.

Regarding $A_{2,2,2}$, the local theory combined with interpolation leads to

$$\int_0^T |A_{2,2,2}(t)|\,\mathrm{d}t \lesssim \|u\|_{L^\infty_T H_x^{(3-\alpha)/2}}. \tag{6.4}$$

After substituting (3.21) into $A_{2,3}$ and using the fact that the Hilbert transform is skew-symmetric

$$A_{2,3}(t) = \tilde{c}_1 \int_{\mathbb{R}} (D_x^{1+\frac{\alpha+1}{2}} u)^2 (\chi_{\epsilon,b}^2)'\,\mathrm{d}x - \tilde{c}_3 \int_{\mathbb{R}} (\mathcal{H} D_x^{\frac{\alpha+1}{2}} u)^2 (\chi_{\epsilon,b}^2)'''\,\mathrm{d}x = A_{2,3,1}(t) + A_{2,3,2}(t).$$

Notice that the term $A_{2,3,1}$ is positive and represents the smoothing effect. In contrast, the term $A_{2,3,2}$ is estimated as we did with $A_{2,2,2}$ in (6.4). So, after integration in the time variable

$$\int_0^T |A_{2,3,2}(t)|\,\mathrm{d}t \lesssim \|u\|_{L^\infty_T H_x^{(3-\alpha)/2}}.$$

Finally, after apply integration by parts

$$A_3(t) = \frac{1}{2}\int_{\mathbb{R}} \partial_x u\,(\partial_x u)^2 \chi_{\epsilon,b}^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} u(\partial_x u)^2 (\chi_{\epsilon,b}^2)'\,\mathrm{d}x = A_{3,1}(t) + A_{3,2}(t).$$

On one hand,

$$|A_{3,1}(t)| \lesssim \|\partial_x u(t)\|_{L^\infty_x} \int_{\mathbb{R}} (\partial_x u)^2 \chi_{\epsilon,b}^2\,\mathrm{d}x,$$

where the integral expression on the right-hand side is the quantity to be estimated by means of Gronwall's inequality.

On the other hand,

$$|A_{3,2}(t) \lesssim \|u(t)\|_{L^\infty_x} \int_0^T (\partial_x u)^2 (\chi_{\epsilon,b}^2)'\,\mathrm{d}x.$$

By Sobolev embedding we have after integrating in time

$$\int_0^T |A_{3,2}(t)|\,\mathrm{d}t \lesssim \big(\sup_{0\le t\le T} \|u(t)\|_{H_x^{s(\alpha)+}}\big) \int_0^T\int_{\mathbb{R}} (\partial_x u)^2 (\chi_{\epsilon,b}^2)'\,\mathrm{d}x\,\mathrm{d}t \le c.$$

Since $\|\partial_x u\|_{L^1_T L^\infty_x} < \infty$, after gathering all estimates above and applying Gronwall's inequality we obtain

$$\sup_{0\le t\le T} \|\partial_x u \chi_{\epsilon,b}\|^2_{L^2_x} + \|D_x^{\frac{\alpha+1}{2}} \partial_x u \eta_{\epsilon,b}\|^2_{L^2_T L^2_x} + \|D_x^{1+\frac{\alpha+1}{2}} u \eta_{\epsilon,b}\|^2_{L^2_T L^2_x} \le c^*_{1,1}, \tag{6.5}$$

where $c^*_{1,1} = c^*_{1,1}(\alpha; \epsilon; T; \|u_0\|_{H_x^{(3-\alpha)/2}}; \|\partial_x u_0 \chi_{\epsilon,b}\|_{L^2_x}) > 0$ for any $\epsilon > 0,\ b \ge 5\epsilon$ and $v \ge 0$.

This estimate finishes step 1 corresponding to the case $j = 1$.

The local smoothing effect obtained above is just $\frac{1+\alpha}{2}$ derivatives; see [Isaza et al. 2016a]. So, the iterative argument is carried out in two steps, the first step for positive integers m and the second one for $m + \frac{1-\alpha}{2}$.

Step 2: After applying the operator $D_x^{(1-\alpha)/2}\partial_x$ to the equation in (1.1) and multiplying the result by $D_x^{(1-\alpha)/2}\partial_x u \chi^2_{\epsilon,b}(x + vt)$ one gets

$$D_x^{\frac{1-\alpha}{2}} \partial_x \partial_t u D_x^{\frac{1-\alpha}{2}} \partial_x u \chi^2_{\epsilon,b} - D_x^{\frac{1-\alpha}{2}} \partial_x D_x^{1+\alpha} \partial_x u D_x^{\frac{1-\alpha}{2}} \partial_x u \chi^2_{\epsilon,b} + D_x^{\frac{1-\alpha}{2}} \partial_x (u\partial_x u) D_x^{\frac{1-\alpha}{2}} \partial_x u \chi^2_{\epsilon,b} = 0,$$

which after integrating in the spatial variable becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} (D_x^{\frac{1-\alpha}{2}} \partial_x u)^2 \chi^2_{\epsilon,b}\,\mathrm{d}x \underbrace{-v \int_{\mathbb{R}} (D_x^{\frac{1-\alpha}{2}} \partial_x u)^2 (\chi^2_{\epsilon,b})'\,\mathrm{d}x}_{A_1(t)}$$
$$\underbrace{-\int_{\mathbb{R}} (D_x^{\frac{1-\alpha}{2}} \partial_x D_x^{1+\alpha} \partial_x u)\, D_x^{\frac{1-\alpha}{2}} \partial_x u \chi^2_{\epsilon,b}\,\mathrm{d}x}_{A_2(t)} + \underbrace{\int_{\mathbb{R}} D_x^{\frac{1-\alpha}{2}} \partial_x (u\partial_x u) D_x^{\frac{1-\alpha}{2}} \partial_x u \chi^2_{\epsilon,b}\,\mathrm{d}x}_{A_3(t)} = 0.$$

Step 2.1: First observe that by the local theory

$$\int_0^T |A_1(t)|\,\mathrm{d}t \le |v| \int_0^T \int_{\mathbb{R}} (D_x^{\frac{1-\alpha}{2}} \partial_x u)^2 (\chi^2_{\epsilon,b})'\,\mathrm{d}x\,\mathrm{d}t \lesssim \|u\|_{L^\infty_T H_x^{(3-\alpha)/2}}.$$

Step 2.2: Concerning the term A_2, integration by parts and Plancherel's identity yield

$$A_2(t) = -\frac{1}{2} \int_{\mathbb{R}} D_x^{\frac{1-\alpha}{2}+1} \mathcal{H}u [\mathcal{H}D_x^{2+\alpha}; \chi^2_{\epsilon,b}] D_x^{\frac{1-\alpha}{2}+1} \mathcal{H}u\,\mathrm{d}x. \tag{6.6}$$

Since $2 + \alpha > 1$, we have by (3.20) that the commutator $[\mathcal{H}D_x^{\alpha+2}; \chi^2_{\epsilon,b}]$ can be decomposed as

$$[\mathcal{H}D_x^{\alpha+2}; \chi^2_{\epsilon,b}] + \tfrac{1}{2} P_n(\alpha+2) + R_n(\alpha+2) = \tfrac{1}{2}\mathcal{H}P_n(\alpha+2)\mathcal{H} \tag{6.7}$$

for some positive integer n, which as in the previous cases will be fixed suitably.

Substituting (6.7) into (6.6)

$$\begin{aligned} A_2(t) &= \frac{1}{2} \int_{\mathbb{R}} D_x^{\frac{3-\alpha}{2}} \mathcal{H}u (R_n(\alpha+2) D_x^{\frac{3-\alpha}{2}} \mathcal{H}u)\,\mathrm{d}x \\ &\qquad + \frac{1}{4} \int_{\mathbb{R}} D_x^{\frac{3-\alpha}{2}} \mathcal{H}u (P_n(\alpha+2) D_x^{\frac{3-\alpha}{2}} \mathcal{H}u)\,\mathrm{d}x - \frac{1}{4} \int_{\mathbb{R}} D_x^{\frac{3-\alpha}{2}} \mathcal{H}u (\mathcal{H}P_n(\alpha+2)\mathcal{H}D_x^{\frac{3-\alpha}{2}} \mathcal{H}u)\,\mathrm{d}x \\ &= A_{2,1}(t) + A_{2,2}(t) + A_{2,3}(t). \end{aligned}$$

Fixing the value of n present in the terms $A_{2,1}$, $A_{2,2}$ and $A_{2,3}$ requires an argument almost similar to the one used in step 1. First, we deal with $A_{2,1}$ where a simple computation produces

$$A_{2,1}(t) = \frac{1}{2}\int_{\mathbb{R}} \mathcal{H}u D_x^{\frac{3-\alpha}{2}}\{R_n(\alpha+2)D_x^{\frac{3-\alpha}{2}}\mathcal{H}u\}\,\mathrm{d}x.$$

We fix $n \in \mathbb{Z}^+$ in such a way that

$$2n+1 \le a+2\sigma \le 2n+3,$$

where $a=\alpha+2$ and $\sigma = \frac{3-\alpha}{2}$ in order to obtain $n=1$ or $n=2$. For the sake of simplicity we choose $n=1$.

Hence, by construction $R_1(\alpha+2)$ satisfies the hypothesis of Proposition 3.25, and

$$|A_{2,1}(t)| \lesssim \|\mathcal{H}u(t)\|_{L^2_x}\|\widehat{D_x^5(\chi^2_{\epsilon,b})}\|_{L^1_\xi} \lesssim \|u_0\|_{L^2_x}\|\widehat{D_x^5(\chi^2_{\epsilon,b})}\|_{L^1_\xi}.$$

Thus

$$\int_0^T |A_{2,1}(t)|\,\mathrm{d}t \lesssim \|u_0\|_{L^2_x}\sup_{0\le t\le T}\|\widehat{D_x^5(\chi^2_{\epsilon,b})}\|_{L^1_\xi}.$$

Next, after replacing $P_1(\alpha+2)$ in $A_{2,2}$

$$\begin{aligned}A_{2,2}(t) &= \left(\frac{\alpha+2}{4}\right)\int_{\mathbb{R}}(\mathcal{H}\partial_x^2u)^2(\chi^2_{\epsilon,b})'\,\mathrm{d}x - c_3\left(\frac{\alpha+2}{16}\right)\int_{\mathbb{R}}(\partial_x u)^2(\chi^2_{\epsilon,b})'''\,\mathrm{d}x\\ &= A_{2,2,1}(t)+A_{2,2,2}(t).\end{aligned}$$

The smoothing effect corresponds to the term $A_{2,2,1}$ and it will be bounded after integrating in time. In contrast, bounding $A_{2,2,2}$ requires only the local theory; in fact

$$\int_0^T |A_{2,2,2}(t)|\,\mathrm{d}t \lesssim \|u\|_{L^\infty_T H_x^{(3-\alpha)/2}}.$$

Concerning the term $A_{2,3}$ we have after replacing $P_1(\alpha+2)$ and using the properties of the Hilbert transform that

$$\begin{aligned}A_{2,3}(t) &= \left(\frac{\alpha+2}{4}\right)\int_{\mathbb{R}}(\partial_x^2u)^2(\chi^2_{\epsilon,b})'\,\mathrm{d}x - c_3\left(\frac{\alpha+2}{16}\right)\int_{\mathbb{R}}(D_x u)^2(\chi^2_{\epsilon,b})'''\,\mathrm{d}x\\ &= A_{2,3,1}(t)+A_{2,3,2}(t).\end{aligned}$$

As before, $A_{2,3,1}\ge 0$ and represents the smoothing effect. Additionally, the local theory and interpolation yield

$$\int_0^T |A_{2,3,2}(t)|\,\mathrm{d}t \lesssim \|u\|_{L^\infty_T H_x^{(3-\alpha)/2}}.$$

Step 2.3: It only remains to handle the term A_3. We can write

$$\begin{aligned}D_x^{\frac{1-\alpha}{2}}\partial_x(u\partial_x u)\chi_{\epsilon,b} &= -\tfrac{1}{2}[D_x^{\frac{1-\alpha}{2}}\partial_x;\chi_{\epsilon,b}]\,\partial_x((\chi_{\epsilon,b}u)^2+(\tilde{\phi}_{\epsilon,b}u)^2+(\psi_\epsilon u^2))\\ &\quad + [D_x^{\frac{1-\alpha}{2}}\partial_x;u\chi_{\epsilon,b}]\,\partial_x((\chi_{\epsilon,b}u)+(u\phi_{\epsilon,b})+(u\psi_\epsilon)) + u\chi_{\epsilon,b}D_x^{\frac{1-\alpha}{2}}\partial_x^2u\\ &= \widetilde{A}_{3,1}(t)+\widetilde{A}_{3,2}(t)+\widetilde{A}_{3,3}(t)+\widetilde{A}_{3,4}(t)+\widetilde{A}_{3,5}(t)+\widetilde{A}_{3,6}(t)+\widetilde{A}_{3,7}(t).\end{aligned}\tag{6.8}$$

First, we rewrite $\widetilde{A}_{3,1}$ as

$$\widetilde{A}_{3,1}(t) = c_\alpha \mathcal{H}[D_x^{1+\frac{1-\alpha}{2}}; \chi_{\epsilon,b}]\,\partial_x((u\chi_{\epsilon,b})^2) + c_\alpha[\mathcal{H}; \chi_{\epsilon,b}]\,D_x^{1+\frac{1-\alpha}{2}}\partial_x((\chi_{\epsilon,b}u)^2),$$

where c_α denotes a non-null constant. Next, combining (3.4), (3.14) and Lemma 3.15 one gets

$$\|\widetilde{A}_{3,1}(t)\|_{L_x^2} \lesssim \|D_x^{1+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L_x^2}\|u\|_{L_x^\infty} + \|u_0\|_{L_x^2}\|u\|_{L_x^\infty},$$
$$\|\widetilde{A}_{3,2}(t)\|_{L_x^2} \lesssim \|D_x^{1+\frac{1-\alpha}{2}}(u\tilde{\phi}_{\epsilon,b})\|_{L_x^2}\|u\|_{L_x^\infty} + \|u_0\|_{L_x^2}\|u\|_{L_x^\infty}.$$

Next, we recall that by construction

$$\operatorname{dist}(\operatorname{supp}(\chi_{\epsilon,b}), \operatorname{supp}(\psi_\epsilon)) \geq \frac{\epsilon}{2},$$

so, by Lemma 3.16

$$\|\widetilde{A}_{3,3}(t)\|_{L_x^2} = \|[D_x^{\frac{1-\alpha}{2}}\partial_x; \chi_\epsilon]\,\partial_x(\psi_\epsilon u^2)\|_{L_x^2} \lesssim \|u_0\|_{L_x^2}\|u\|_{L_x^\infty}.$$

We can rewrite $\widetilde{A}_{3,4}$ as

$$\widetilde{A}_{3,4}(t) = c\mathcal{H}[D_x^{1+\frac{1-\alpha}{2}}; u\chi_{\epsilon,b}]\,\partial_x(u\chi_{\epsilon,b}) - c[\mathcal{H}; u\chi_{\epsilon,b}]\,\partial_x D_x^{1+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})$$

for some non-null constant c.

Thus, by the commutator estimates (3.2) and Lemma 3.13

$$\|\widetilde{A}_{3,4}(t)\|_{L_x^2} \lesssim \|\partial_x(u\chi_{\epsilon,b})\|_{L_x^\infty}\|D_x^{1+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L_x^2}.$$

Applying the same procedure to $\widetilde{A}_{3,5}$ yields

$$\|\widetilde{A}_{3,5}(t)\|_{L_x^2} \lesssim \|\partial_x(u\chi_{\epsilon,b})\|_{L_x^\infty}\|D_x^{1+\frac{1-\alpha}{2}}(u\phi_{\epsilon,b})\|_{L_x^2} + \|\partial_x(u\phi_{\epsilon,b})\|_{L_x^\infty}\|D_x^{1+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L_x^2}.$$

Since the supports of $\chi_{\epsilon,b}$ and ψ_ϵ are separated, we obtain by Lemma 3.16

$$\|\widetilde{A}_{3,6}(t)\|_{L_x^2} = \|u\chi_{\epsilon,b}\partial_x^2 D_x^{1+\frac{1-\alpha}{2}}(u\psi_\epsilon)\|_{L_x^2} \lesssim \|u_0\|_{L_x^2}\|u\|_{L_x^\infty}.$$

To finish with the estimates above we use the relation

$$\chi_{\epsilon,b}(x) + \phi_{\epsilon,b}(x) + \psi_\epsilon(x) = 1 \quad \text{for all } x \in \mathbb{R}.$$

Then

$$\begin{aligned} D_x^{1+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b}) &= D_x^{1+\frac{1-\alpha}{2}}u\chi_{\epsilon,b} + [D_x^{1+\frac{1-\alpha}{2}}; \chi_{\epsilon,b}](u\chi_{\epsilon,b} + u\phi_{\epsilon,b} + u\psi_\epsilon) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Notice that $\|I_1\|_{L_x^2}$ is the quantity to estimate. In contrast, $\|I_2\|_{L_x^2}$ and $\|I_3\|_{L_x^2}$ can be handled by Lemma 3.13 combined with the local theory. Meanwhile I_3 can be bounded by using Lemma 3.16.

We notice that the gain of regularity obtained in the step 1 implies that $\|D_x^{1+(1+\alpha)/2}(u\phi_{\epsilon,b})\|_{L_x^2} < \infty$. To show this we use Theorem 3.7 and Hölder's inequality as follows:

$$\|D_x^{1+\frac{1+\alpha}{2}}(u\phi_{\epsilon,b})\|_{L_x^2} \lesssim \|u_0\|_{L_x^2} + \|u\|_{L_x^\infty} + \|\mathbb{1}_{[\frac{\epsilon}{4},b]}D_x^{1+\frac{1+\alpha}{2}}u\|_{L_x^2} + \|\mathbb{1}_{[\frac{\epsilon}{4},b]}\mathcal{H}D_x^{\frac{1+\alpha}{2}}u\|_{L_x^2}. \tag{6.9}$$

The second term on the right-hand side after we integrate in time is controlled by using Sobolev's embedding. Meanwhile, the third term can be handled after integrating in time and using (6.5) with $(\epsilon, b) = \left(\frac{\epsilon}{24}, b + \frac{7\epsilon}{24}\right)$.

The fourth term in the right-hand side can be bounded by combining the local theory and interpolation.

Hence, after integration in time

$$\| D_x^{1+\frac{1+\alpha}{2}} (u\phi_{\epsilon,b})\|_{L^2_T L^2_x} < \infty, \tag{6.10}$$

which clearly implies $\| D_x^{1+(1-\alpha)/2}(u\phi_{\epsilon,b})\|_{L^2_T L^2_x} < \infty$, as required. We can handle $\| D_x^{1+(1-\alpha)/2}(u\tilde{\phi}_{\epsilon,b})\|_{L^2_x}$ similarly.

Finally,

$$\begin{aligned} A_{3,7}(t) &= -\frac{1}{2}\int_{\mathbb{R}} \partial_x u \chi^2_{\epsilon,b} (D_x^{\frac{1-\alpha}{2}} \partial_x u)^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} u(\chi^2_{\epsilon,b})' (D_x^{\frac{1-\alpha}{2}} \partial_x u)^2 \,\mathrm{d}x \\ &= A_{3,7,1}(t) + A_{3,7,2}(t). \end{aligned}$$

We have

$$|A_{3,7,1}(t)| \lesssim \|\partial_x u(t)\|_{L^\infty_x} \int_{\mathbb{R}} (D_x^{\frac{1-\alpha}{2}} \partial_x u)^2 \chi^2_{\epsilon,b} \,\mathrm{d}x,$$

where the right-hand side can be estimated using Gronwall's inequality and the local theory $\|\partial_x u\|_{L^1_T L^\infty_x} < \infty$.

Sobolev's embedding leads us to

$$\int_0^T |A_{3,7,2}(t)| \,\mathrm{d}t \lesssim \Big(\sup_{0\le t\le T} \|u(t)\|_{H_x^{s(\alpha)+}}\Big) \int_0^T\int_{\mathbb{R}} \chi_{\epsilon,b}\chi'_{\epsilon,b} (D_x^{\frac{1-\alpha}{2}} \partial_x u)^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Gathering all the information corresponding to this step combined with Gronwall's inequality yields

$$\sup_{0\le t\le T} \|D_x^{\frac{1-\alpha}{2}} \partial_x u \chi_{\epsilon,b}\|^2_{L^2_x} + \|\partial_x^2 u \eta_{\epsilon,b}\|^2_{L^2_T L^2_x} + \|\mathcal{H}\partial_x u \eta_{\epsilon,b}\|^2_{L^2_T L^2_x} \le c^*_{1,2}, \tag{6.11}$$

with $c^*_{1,2} = c^*_{1,2}(\alpha; \epsilon; T; v; \|u_0\|_{H_x^{(3-\alpha)/2}}; \|D_x^{(1-\alpha)/2}\partial_x u_0 \chi_{\epsilon,b}\|_{L^2_x})$ for any $\epsilon > 0$, $b \ge 5\epsilon$ and $v \ge 0$.

This finishes step 2, corresponding to the case $j = 1$ in the induction process.

Next, we present the case $j = 2$, to show how we proceed in the case j even.

Case $j = 2$. *Step 1*: First we apply two spatial derivatives to the equation in (1.1); after that we multiply by $\partial_x^2 u(x,t)\chi^2_{\epsilon,b}(x+vt)$, and finally we integrate in the x-variable to obtain the identity

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (\partial_x^2 u)^2 \chi^2_{\epsilon,b} \,\mathrm{d}x \\ &\quad \underbrace{-\frac{v}{2}\int_{\mathbb{R}} (\partial_x^2 u)^2 (\chi^2_{\epsilon,b})' \,\mathrm{d}x}_{A_1(t)} \underbrace{- \int_{\mathbb{R}} (\partial_x^2 D_x^{\alpha+1}\partial_x u)\, \partial_x^2 u \chi^2_{\epsilon,b} \,\mathrm{d}x}_{A_2(t)} + \underbrace{\int_{\mathbb{R}} \partial_x^2(u\partial_x u)\partial_x^2 u \chi^2_{\epsilon,b} \,\mathrm{d}x}_{A_3(t)} = 0. \end{aligned} \tag{6.12}$$

As was done in the previous steps, we first proceed to estimate A_1.

Step 1.1: By (6.11) it follows that

$$\int_0^T |A_1(t)| \,\mathrm{d}t \le \int_0^T\int_{\mathbb{R}} (\partial_x^2 u)^2 (\chi^2_{\epsilon,b})' \,\mathrm{d}x\,\mathrm{d}t \le c^*_{1,2}. \tag{6.13}$$

Step 1.2: To extract information from the term A_2 we use integration by parts and Plancherel's identity to obtain

$$A_2(t) = \frac{1}{2}\int_{\mathbb{R}} \partial_x^2 u[D_x^{\alpha+1}\partial_x;\, \chi_{\epsilon,b}^2]\, \partial_x^2 u \,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}} \partial_x^2 u[\mathcal{H}D_x^{\alpha+2};\, \chi_{\epsilon,b}^2]\, \partial_x^2 u \,\mathrm{d}x. \tag{6.14}$$

Although this stage of the process is related to the one performed in step 1 (for $j=1$),we will use again the commutator expansion in (3.20), taking into account in this case that $a=\alpha+2>1$ and n is a nonnegative integer whose value will be fixed later.

Then,

$$\begin{aligned} A_2(t) &= \frac{1}{2}\int_{\mathbb{R}} \partial_x^2 u\, R_n(s+2)\, \partial_x^2 u\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}} \partial_x^2 u\, P_n(s+2)\, \partial_x^2 u\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}} \partial_x^2 u\, \mathcal{H}P_n(s+2)\mathcal{H}\partial_x^2 u\,\mathrm{d}x \\ &= A_{2,1}(t) + A_{2,2}(t) + A_{2,3}(t). \end{aligned}$$

Essentially, the key term which allows us to fix the value of n is $A_{2,1}$. Indeed, after some integration by parts

$$A_{2,1}(t) = \frac{1}{2}\int_{\mathbb{R}} u\, \partial_x^2 R_n(\alpha+2)\, \partial_x^2 u\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}} u\, \partial_x^2\{R_n(\alpha+2)\, \partial_x^2 u\}\,\mathrm{d}x.$$

We fix n such that it satisfies

$$2n+1 \le a+2\sigma \le 2n+3.$$

In this case with $a=\alpha+2>1$ and $\sigma=2$, we obtain $n=2$.

Hence by construction Proposition 3.25 guarantees that $D_x^2 R_2(\alpha+2) D_x^2$ is bounded in L_x^2.

Thus

$$|A_{2,1}(t)| \lesssim \|u(t)\|_{L_x^2}^2 \|D_x^2 R_2(\alpha+2) D_x^2 u\|_{L_x^2} \le c\|u_0\|_{L_x^2}^2 \|\widehat{D_x^{\alpha+6}(\chi_{\epsilon,b}^2)}\|_{L_\xi^1}.$$

Since we fixed $n=2$, we proceed to handle the contribution coming from $A_{2,2}$ and $A_{2,3}$.

Next,

$$\begin{aligned} A_{2,2}(t) &= \tilde{c}_1\int_{\mathbb{R}} (D_x^{\frac{\alpha+1}{2}}\partial_x^2 u)^2(\chi_{\epsilon,b}^2)'\,\mathrm{d}x - \tilde{c}_3\int_{\mathbb{R}} (D_x^{1+\frac{\alpha+1}{2}} u)^2(\chi_{\epsilon,b}^2)^{(3)}\,\mathrm{d}x + c_5\left(\frac{\alpha+2}{64}\right)\int_{\mathbb{R}} (D_x^{\frac{\alpha+1}{2}} u)^2(\chi_{\epsilon,b}^2)^{(5)}\,\mathrm{d}x \\ &= A_{2,2,1}(t) + A_{2,2,2}(t) + A_{2,2,3}(t). \end{aligned}$$

Notice that $A_{2,2,1}\ge 0$ represents the smoothing effect.

We recall that

$$|\chi_{\epsilon,b}^{(j)}(x)| \lesssim \chi'_{\frac{\epsilon}{3},b+\epsilon}(x) \quad \text{for all } x\in\mathbb{R},\ j\in\mathbb{Z}^+.$$

Then

$$\int_0^T |A_{2,2,2}(t)|\,\mathrm{d}t \lesssim \int_0^T\int_{\mathbb{R}} (D_x^{1+\frac{1+\alpha}{2}} u)^2 \chi'_{\frac{\epsilon}{3},b+\epsilon}\,\mathrm{d}x\,\mathrm{d}t.$$

Taking $(\epsilon,b)=\left(\frac{\epsilon}{9}, b+\frac{10\epsilon}{9}\right)$ in (6.5) combined with the properties of the cutoff function we have

$$\int_0^T |A_{2,2,2}(t)|\,\mathrm{d}t \lesssim c_{1,1}^*.$$

To finish the terms that make A_2 we proceed to estimate $A_{2,2,3}$.

As usual the low regularity is controlled by interpolation and the local theory. Therefore

$$\int_0^T |A_{2,2,3}(t)|\,\mathrm{d}t \lesssim \|u\|_{L^\infty_T H^{(3-\alpha)/2}_x}.$$

Next,

$$\begin{aligned} A_{2,3}(t) &= \tilde{c}_1 \int_{\mathbb{R}} (\mathcal{H} D_x^{\frac{\alpha+1}{2}} \partial_x^2 u)^2 (\chi^2_{\epsilon,b})'\,\mathrm{d}x \\ &\qquad - \tilde{c}_3 \int_{\mathbb{R}} (D_x^{\frac{\alpha+1}{2}} \partial_x u)^2 (\chi^2_{\epsilon,b})^{(3)}\,\mathrm{d}x + \left(\frac{\alpha+2}{64}\right) c_5 \int_{\mathbb{R}} (D_x^{\frac{\alpha+1}{2}} \mathcal{H} u)^2 (\chi^2_{\epsilon,b})^{(5)}\,\mathrm{d}x \\ &= A_{2,3,1}(t) + A_{2,3,2}(t) + A_{2,3,3}(t). \end{aligned}$$

$A_{2,3,1}$ is positive and it will provide the smoothing effect after being integrated in time.

The terms $A_{2,3,2}$ and $A_{2,3,3}$ can be handled exactly in the same way that we treated $A_{2,2,2}$ and $A_{2,2,3}$ respectively, so we will omit the proof.

Step 1.3: Finally,

$$\begin{aligned} A_3(t) &= \frac{5}{2} \int_{\mathbb{R}} \partial_x u (\partial_x^2 u)^2 \chi^2_{\epsilon,b}\,\mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}} u (\partial_x^2 u)^2 (\chi^2_{\epsilon,b})'\,\mathrm{d}x \\ &= A_{3,1}(t) + A_{3,2}(t). \end{aligned}$$

First,

$$|A_{3,1}(t)| \lesssim \|\partial_x u(t)\|_{L^\infty_x} \int_{\mathbb{R}} (\partial_x^2 u)^2 \chi^2_{\epsilon,b}\,\mathrm{d}x, \tag{6.15}$$

by the local theory $\partial_x u \in L^1([0,T] : L^\infty_x(\mathbb{R}))$ (see Theorem C(b)), and the integral expression is the quantity we want estimate.

Next,

$$|A_{3,2}(t)| \lesssim \|u(t)\|_{L^\infty_x} \int_{\mathbb{R}} (\partial_x^2 u)^2 (\chi^2_{\epsilon,b})'\,\mathrm{d}x. \tag{6.16}$$

After applying the Sobolev embedding and integrating in the time variable we obtain

$$\int_0^T |A_{3,2}(t)|\,\mathrm{d}t \lesssim \Big(\sup_{0\le t\le T} \|u(t)\|_{H^{s(\alpha)+}_x}\Big) \int_0^T \int_{\mathbb{R}} (\partial_x^2 u)^2 (\chi^2_{\epsilon,b})'\,\mathrm{d}x\,\mathrm{d}t,$$

and the integral term in the right-hand side was estimated previously in (6.13).

Thus, after grouping all the terms and applying Gronwall's inequality we obtain

$$\sup_{0\le t\le T} \|\partial_x^2 u \chi_{\epsilon,b}\|^2_{L^2_x} + \|D_x^{\frac{\alpha+1}{2}} \partial_x^2 u \eta_{\epsilon,b}\|^2_{L^2_T L^2_x} + \|D_x^{\frac{\alpha+1}{2}} \mathcal{H} \partial_x^2 u \eta_{\epsilon,b}\|^2_{L^2_T L^2_x} \le c^*_{2,1}, \tag{6.17}$$

where $c^*_{2,1} = c^*_{2,1}(\alpha; \epsilon; T; v; \|u_0\|_{H^{(3-\alpha)/2}_x}; \|\partial_x^2 u_0 \chi_{\epsilon,b}\|_{L^2_x})$ for any $\epsilon > 0$, $b \ge 5\epsilon$ and $v \ge 0$.

Step 2: From equation in (1.1) one gets after applying the operator $D_x^{(1-\alpha)/2}\partial_x^2$ and multiplying the result by $D_x^{(1-\alpha)/2}\partial_x^2 u \chi^2_{\epsilon,b}(x+vt)$

$$D_x^{\frac{1-\alpha}{2}} \partial_x^2 \partial_t u D_x^{\frac{1-\alpha}{2}} \partial_x^2 u \chi^2_{\epsilon,b} - D_x^{\frac{1-\alpha}{2}} \partial_x^2 D_x^{1+\alpha} \partial_x u D_x^{\frac{1-\alpha}{2}} \partial_x^2 u \chi^2_{\epsilon,b} + D_x^{\frac{1-\alpha}{2}} \partial_x^2 (u\partial_x u) D_x^{\frac{1-\alpha}{2}} \partial_x^2 u \chi^2_{\epsilon,b} = 0,$$

which after integration in the spatial variable becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)^2\chi_{\epsilon,b}^2\,\mathrm{d}x\underbrace{-\frac{v}{2}\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)^2(\chi_{\epsilon,b}^2)'\,\mathrm{d}x}_{A_1(t)}$$
$$\underbrace{-\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^2 D_x^{1+\alpha}\partial_x u)(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)\chi_{\epsilon,b}^2\,\mathrm{d}x}_{A_2(t)}+\underbrace{\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^2(u\partial_x u))(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)\chi_{\epsilon,b}^2\,\mathrm{d}x}_{A_3(t)}=0.$$

To estimate A_1 we will use different techniques from the ones implemented to bound A_1 in the previous step. The main difficulty we have to face is dealing with the nonlocal character of the operator D_x^s for $s\in\mathbb{R}^+\setminus 2\mathbb{N}$; the case $s\in 2\mathbb{N}$ is less complicated because D_x^s becomes local, so we can integrate by parts.

The strategy to solve this issue will be the following. In (6.17) we proved that u has a gain of $\frac{\alpha+1}{2}$ derivatives (local), which in total sum to $2+\frac{1+\alpha}{2}$. This suggests that if we can find an appropriate channel where we can localize the smoothing effect, we shall be able to recover all the local derivatives r with $r\le 2+\frac{1+\alpha}{2}$.

Henceforth we will employ recurrently a technique of localization of the commutator used by Kenig, Linares, Ponce and Vega [Kenig et al. 2018] in the study of propagation of regularity (fractional) for solutions of the k-generalized KdV equation. Indeed, the idea consists in constructing an appropriate system of smooth partitions of unit length, localizing the regions where the information obtained in the previous cases is available.

We recall that for $\epsilon>0$ and $b\ge 5\epsilon$

$$\eta_{\epsilon,b}=\sqrt{\chi_{\epsilon,b}\chi_{\epsilon,b}'}\quad\text{and}\quad\chi_{\epsilon,b}+\phi_{\epsilon,b}+\psi_\epsilon=1. \tag{6.18}$$

Step 2.1: We claim

$$\|D_x^{\frac{1+\alpha}{2}}\partial_x^2(u\eta_{\epsilon,b})\|_{L_T^2L_x^2}<\infty. \tag{6.19}$$

Combining the commutator estimate (3.14), (6.18), Hölder's inequality and (6.17) yields

$$\begin{aligned}&\|D_x^{\frac{1+\alpha}{2}}\partial_x^2(u\eta_{\epsilon,b})\|_{L_T^2L_x^2}\\ &\le\|D_x^{2+\frac{1+\alpha}{2}}u\eta_{\epsilon,b}\|_{L_T^2L_x^2}+\|[D_x^{2+\frac{1+\alpha}{2}};\eta_{\epsilon,b}](u\chi_{\epsilon,b}+u\phi_{\epsilon,b}+u\psi_\epsilon)\|_{L_T^2L_x^2}\\ &\lesssim(c_{2,1}^*)^2+\underbrace{\|D_x^{1+\frac{1+\alpha}{2}}(u\chi_{\epsilon,b})\|_{L_T^2L_x^2}}_{B_1}+\underbrace{\|D_x^{1+\frac{1+\alpha}{2}}(u\phi_{\epsilon,b})\|_{L_T^2L_x^2}}_{B_2}+\|u_0\|_{L_x^2}+\underbrace{\|\eta_{\epsilon,b}D_x^{2+\frac{1+\alpha}{2}}(u\psi_\epsilon)\|_{L_T^2L_x^2}}_{B_3}.\end{aligned} \tag{6.20}$$

Since $\chi_{\epsilon/5,\epsilon}=1$ on the support of $\chi_{\epsilon,b}$ we have

$$\chi_{\epsilon,b}(x)\chi_{\frac{\epsilon}{5},\epsilon}(x)=\chi_{\epsilon,b}(x)\quad\text{for all } x\in\mathbb{R}.$$

Thus, combining Lemma 3.15 and Young's inequality we obtain

$$\|D_x^{1+\frac{1+\alpha}{2}}(u\chi_{\epsilon,b})\|_{L_x^2}\lesssim\|\partial_x^2u\chi_{\epsilon,b}\|_{L_x^2}+\|\partial_xu\chi_{\frac{\epsilon}{5},\epsilon}\|_{L_x^2}+\|u_0\|_{L_x^2}. \tag{6.21}$$

Then, an application of (6.17) adapted to every case yields

$$B_1\lesssim\|\partial_x^2u\chi_{\epsilon,b}\|_{L_T^\infty L_x^2}+\|\partial_xu\chi_{\frac{\epsilon}{5},\epsilon}\|_{L_T^\infty L_x^2}+\|u_0\|_{L_x^2}\lesssim c_{2,1}^*+c_{1,1}^*+\|u_0\|_{L_x^2}. \tag{6.22}$$

Notice that B_2 was estimated in the case $j=1$, step 2, see (6.10), so we will omit the proof. Next, we recall that by construction

$$\operatorname{dist}(\operatorname{supp}(\eta_{\epsilon,b}),\operatorname{supp}(\psi_\epsilon))\geq\frac{\epsilon}{2}.$$

Hence by Lemma 3.16

$$B_3=\|\eta_{\epsilon,b}D_x^{2+\frac{1+\alpha}{2}}(u\psi_\epsilon)\|_{L^2_TL^2_x}\lesssim\|\eta_{\epsilon,b}\|_{L^\infty_TL^\infty_x}\|u_0\|_{L^2_x}. \tag{6.23}$$

The claim follows by gathering the calculations above.

At this point we have proved that locally in the interval $[\epsilon,b]$ there exist $2+\frac{\alpha+1}{2}$ derivatives. By Lemma 3.15 we get

$$\|D_x^{2+\frac{1-\alpha}{2}}(u\eta_{\epsilon,b})\|_{L^2_TL^2_x}\lesssim\|D_x^{2+\frac{1+\alpha}{2}}(u\eta_{\epsilon,b})\|_{L^2_TL^2_x}+\|u_0\|_{L^2_x}<\infty.$$

As before

$$D_x^{2+\frac{1-\alpha}{2}}u\eta_{\epsilon,b}=D_x^{2+\frac{1-\alpha}{2}}(u\eta_{\epsilon,b})-[D_x^{2+\frac{1-\alpha}{2}};\eta_{\epsilon,b}](u\chi_{\epsilon,b}+u\phi_{\epsilon,b}+u\psi_\epsilon).$$

The argument used in the proof of the claim yields

$$\|D_x^{2+\frac{1-\alpha}{2}}u\eta_{\epsilon,b}\|_{L^2_TL^2_x}<\infty.$$

Therefore,

$$\int_0^T|A_1(t)|\,\mathrm{d}t\leq|v|\int_0^T\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^2u)^2(\chi_{\epsilon,b}^2)'\,\mathrm{d}x\,\mathrm{d}t\lesssim\|D_x^{2+\frac{1-\alpha}{2}}u\eta_{\epsilon,b}\|^2_{L^2_TL^2_x}<\infty. \tag{6.24}$$

Step 2.2: Now we focus our attention on the term A_2. Notice that after integration by parts and Plancherel's identity

$$A_2(t)=-\frac{1}{2}\int_{\mathbb{R}}D_x^{\frac{5-\alpha}{2}}u[\mathcal{H}D_x^{2+\alpha};\chi_{\epsilon,b}^2]D_x^{\frac{5-\alpha}{2}}u\,\mathrm{d}x. \tag{6.25}$$

The procedure to decompose the commutator will be similar to that in the previous step; the main difference relies on the fact that the quantity of derivatives is higher in comparison with step 1.

Concerning this, we notice that $2+\alpha>1$ and by (3.20) the commutator $[\mathcal{H}D_x^{\alpha+2};\chi_{\epsilon,b}^2]$ can be decomposed as

$$[\mathcal{H}D_x^{\alpha+2};\chi_{\epsilon,b}^2]+\tfrac{1}{2}P_n(\alpha+2)+R_n(\alpha+2)=\tfrac{1}{2}\mathcal{H}P_n(\alpha+2)\mathcal{H} \tag{6.26}$$

for some positive integer n. We shall fix the value of n satisfying a suitable condition.

Substituting (6.26) into (6.25) produces

$$\begin{aligned}A_2(t)&=\frac{1}{2}\int_{\mathbb{R}}D_x^{\frac{5-\alpha}{2}}u(R_n(\alpha+2)D_x^{\frac{5-\alpha}{2}}u)\,\mathrm{d}x\\&\qquad+\frac{1}{4}\int_{\mathbb{R}}D_x^{\frac{5-\alpha}{2}}u(P_n(\alpha+2)D_x^{\frac{5-\alpha}{2}}u)\,\mathrm{d}x-\frac{1}{4}\int_{\mathbb{R}}D_x^{\frac{5-\alpha}{2}}u(\mathcal{H}P_n(\alpha+2)\mathcal{H}D_x^{\frac{5-\alpha}{2}}u)\,\mathrm{d}x\\&=A_{2,1}(t)+A_{2,2}(t)+A_{2,3}(t).\end{aligned} \tag{6.27}$$

Now we proceed to fix the value of n present in $A_{2,1}$, $A_{2,2}$ and $A_{2,3}$.

First we deal with the term that determines the value n in the decomposition associated to A_2. In this case it corresponds to $A_{2,1}$.

Applying Plancherel's identity, $A_{2,1}$ becomes

$$A_{2,1}(t) = \frac{1}{2}\int_{\mathbb{R}} u D_x^{\frac{5-\alpha}{2}}\{R_n(\alpha+2) D_x^{\frac{5-\alpha}{2}} u\}\,\mathrm{d}x.$$

We fix n such that it satisfies (3.26); i.e.,

$$2n+1 \le a + 2\sigma \le 2n+3,$$

with $a = \alpha + 2$ and $\sigma = \frac{5-\alpha}{2}$, which produces $n = 2$ or $n = 3$. Nevertheless, for the sake of simplicity we take $n = 2$.

Hence, by construction $R_2(\alpha+2)$ is bounded in L^2_x (see Proposition 3.25).

Thus,

$$\int_0^T |A_{2,1}(t)|\,\mathrm{d}t \le c\int_0^T \|u(t)\|^2_{L^2_x} \|\widehat{D_x^7(\chi^2_{\epsilon,b}(\,\cdot\,+vt))}\|_{L^1_\xi}\,\mathrm{d}t \lesssim \|u_0\|^2_{L^2_x} \sup_{0\le t\le T} \|\widehat{D_x^7(\chi^2_{\epsilon,b})}\|_{L^1_\xi}.$$

Since we have fixed $n = 2$, we obtain, after substituting $P_2(\alpha+2)$ into $A_{2,2}$,

$$\begin{aligned} A_{2,2}(t) &= \tilde{c}_1 \int_{\mathbb{R}} (\mathcal{H}\partial_x^3 u)^2 (\chi^2_{\epsilon,b})'\,\mathrm{d}x - \tilde{c}_3\int_{\mathbb{R}} (\partial_x^2 u)^2 (\chi^2_{\epsilon,b})^{(3)}\,\mathrm{d}x + \tilde{c}_5 \int_{\mathbb{R}} (\mathcal{H}\partial_x u)^2 (\chi^2_{\epsilon,b})^{(5)}\,\mathrm{d}x \\ &= A_{2,2,1}(t) + A_{2,2,2}(t) + A_{2,2,3}(t). \end{aligned}$$

We underline that $A_{2,2,1}$ is positive and represents the smoothing effect.

On the other hand, by (6.11) with $(\epsilon, b) = \left(\frac{\epsilon}{5}, \epsilon\right)$ we have

$$\int_0^T |A_{2,2,2}(t)|\,\mathrm{d}t = c\int_0^T\!\!\int_{\mathbb{R}} (\partial_x^2 u)^2 \chi^2_{\frac{\epsilon}{5},\epsilon} (\chi^2_{\epsilon,b})'''\,\mathrm{d}x\,\mathrm{d}t \lesssim \sup_{0\le t\le T}\int_{\mathbb{R}} (\partial_x^2 u)^2 \chi^2_{\frac{\epsilon}{5},\epsilon}\,\mathrm{d}x \lesssim c^*_{1,2}. \tag{6.28}$$

Next, by the local theory

$$\int_0^T |A_{2,2,3}(t)|\,\mathrm{d}t \lesssim \|u\|_{L^\infty_T H_x^{(3-\alpha)/2}}. \tag{6.29}$$

After replacing $P_2(\alpha+2)$ into $A_{2,3}$, and using the fact that the Hilbert transform is skew adjoint

$$\begin{aligned} A_{2,3}(t) &= \left(\frac{\alpha+2}{4}\right)\int_{\mathbb{R}} (\partial_x^3 u)^2 (\chi^2_{\epsilon,b})'\,\mathrm{d}x \\ &\qquad - c_3\left(\frac{\alpha+2}{16}\right)\int_{\mathbb{R}} (\mathcal{H}\partial_x^2 u)^2 (\chi^2_{\epsilon,b})'''\,\mathrm{d}x + c_5\left(\frac{\alpha+2}{64}\right)\int_{\mathbb{R}} (\partial_x u)^2 (\chi^2_{\epsilon,b})^{(5)}\,\mathrm{d}x \\ &= A_{2,3,1}(t) + A_{2,3,2}(t) + A_{2,3,3}(t). \end{aligned}$$

Notice that $A_{2,3,1} \ge 0$ and it represents the smoothing effect. However, $A_{2,3,2}$ can be handled if we take $(\epsilon, b) = \left(\frac{\epsilon}{5}, \epsilon\right)$ in (6.5) as follows:

$$A_{2,3,3}(t) = \int_{\mathbb{R}} (\partial_x^2 u)^2 \chi^2_{\frac{\epsilon}{5},\epsilon} (\chi^2_{\epsilon,b})'''\,\mathrm{d}x \lesssim \int_{\mathbb{R}} (\partial_x^2 u)^2 \chi^2_{\frac{\epsilon}{5},\epsilon}\,\mathrm{d}x.$$

Thus,

$$\int_0^T |A_{2,3,3}(t)|\,\mathrm{d}t \lesssim \sup_{0\le t\le T}\int_{\mathbb{R}} (\partial_x u)^2 \chi^2_{\frac{\epsilon}{5},\epsilon}\,\mathrm{d}x \lesssim c^*_{1,1}.$$

To finish the estimate of A_2 it only remains to bound $A_{2,3,2}$. To do this we recall that

$$|\chi^{(j)}_{\epsilon,b}(x)| \lesssim \chi'_{\frac{\epsilon}{3},b+\epsilon}(x) \quad \text{for all } x\in\mathbb{R},\ j\in\mathbb{Z}^+,$$

which together with property (9) of $\chi_{\epsilon,b}$ yields

$$\int_0^T\int_{\mathbb{R}} (\mathcal{H}\partial_x^2 u)^2\chi'_{\frac{\epsilon}{3},b+\epsilon}\,\mathrm{d}x\,\mathrm{d}t \lesssim \|\mathcal{H}\partial_x^2 u\eta_{\frac{\epsilon}{9},b+\frac{10\epsilon}{9}}\|^2_{L^2_TL^2_x} \lesssim c^*_{1,2},$$

where the last inequality is obtained taking $(\epsilon,b)=\left(\frac{\epsilon}{9},b+\frac{10\epsilon}{9}\right)$ in (6.11). The term $A_{2,3,3}$ can be handled by interpolation and the local theory.

Step 2.3: Finally we turn our attention to A_3.We start rewriting the nonlinear part as

$$\begin{aligned} D_x^{\frac{1-\alpha}{2}}\partial_x^2(u\,\partial_x u)\chi_{\epsilon,b} &= -\tfrac{1}{2}[D_x^{\frac{1-\alpha}{2}}\partial_x^2;\,\chi_{\epsilon,b}]\,\partial_x((u\chi_{\epsilon,b})^2+(u\tilde{\phi}_{\epsilon,b})^2+(\psi_\epsilon u^2))\\ &\quad + [D_x^{\frac{1-\alpha}{2}}\partial_x^2;\,u\chi_{\epsilon,b}]\,\partial_x((u\chi_{\epsilon,b})+(u\phi_{\epsilon,b})+(u\psi_\epsilon))+u\chi_{\epsilon,b}\,D_x^{\frac{1-\alpha}{2}}\partial_x^3 u\\ &= \widetilde{A}_{3,1}(t)+\widetilde{A}_{3,2}(t)+\widetilde{A}_{3,3}(t)+\widetilde{A}_{3,4}(t)+\widetilde{A}_{3,5}(t)+\widetilde{A}_{3,6}(t)+\widetilde{A}_{3,7}(t). \end{aligned} \tag{6.30}$$

Hence, after substituting (6.29) into A_3 and applying Hölder's inequality

$$\begin{aligned} A_3(t) &= \sum_{1\le m\le 6}\int_{\mathbb{R}}\widetilde{A}_{3,m}(t)\,D_x^{\frac{1-\alpha}{2}}\partial_x^2 u\chi_{\epsilon,b}\,\mathrm{d}x + \int_{\mathbb{R}}\widetilde{A}_{3,7}(t)\,D_x^{\frac{1-\alpha}{2}}\partial_x^2 u\chi_{\epsilon,b}\,\mathrm{d}x\\ &\le \sum_{1\le m\le 6}\|\widetilde{A}_{3,m}(t)\|_{L^2_x}\|D_x^{2+\frac{1-\alpha}{2}}u(t)\chi_{\epsilon,b}(\,\cdot\,+vt)\|_{L^2_x} + \int_{\mathbb{R}}\widetilde{A}_{3,7}(t)\,D_x^{\frac{1-\alpha}{2}}\partial_x^2 u\chi_{\epsilon,b}\,\mathrm{d}x\\ &= \|D_x^{2+\frac{1-\alpha}{2}}u(t)\chi_{\epsilon,b}(\,\cdot\,+vt)\|_{L^2_x}\sum_{1\le m\le 6}A_{3,m}(t)+A_{3,7}(t). \end{aligned}$$

Notice that the first factor in the right-hand side is the quantity to be estimated by Gronwall's inequality. So, we shall focus on establishing control of the remaining terms.

First,combining (3.4), (3.14) and Lemma 3.15 one gets that

$$\|\widetilde{A}_{3,1}(t)\|_{L^2_x} \lesssim \|D_x^{2+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}\|u\|_{L^\infty_x}+\|u_0\|_{L^2_x}\|u\|_{L^\infty_x}, \tag{6.31}$$

$$\|\widetilde{A}_{3,2}(t)\|_{L^2_x} \lesssim \|D_x^{2+\frac{1-\alpha}{2}}(u\tilde{\phi}_{\epsilon,b})\|_{L^2_x}\|u\|_{L^\infty_x}+\|u_0\|_{L^2_x}\|u\|_{L^\infty_x}. \tag{6.32}$$

To finish with the quadratic terms, we employ Lemma 3.16:

$$\|\widetilde{A}_{3,3}(t)\|_{L^2_x} \lesssim \|u_0\|_{L^2_x}\|u\|_{L^\infty_x}.$$

Combining (3.2) and (3.14) we obtain

$$\|\widetilde{A}_{3,4}(t)\|_{L^2_x} \lesssim \|\partial_x(u\chi_{\epsilon,b})\|_{L^\infty_x}\|D_x^{2+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}.$$

Meanwhile,

$$\|\widetilde{A}_{3,5}(t)\|_{L^2_x} \lesssim \|\partial_x(u\chi_{\epsilon,b})\|_{L^\infty_x}\|D_x^{2+\frac{1-\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_x}+\|\partial_x(u\phi_{\epsilon,b})\|_{L^\infty_x}\|D_x^{2+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}.$$

Next, we recall that by construction

$$\operatorname{dist}(\operatorname{supp}(\chi_{\epsilon,b}), \operatorname{supp}(\psi_\epsilon)) \geq \frac{\epsilon}{2}.$$

Thus by Lemma 3.16

$$\|\widetilde{A}_{3,6}(t)\|_{L^2_x} \lesssim \|u_0\|_{L^2_x}\|u\|_{L^\infty_x}.$$

To complete the estimates in (6.31)–(6.32) it only remains for us to bound $\|D_x^{2+(1-\alpha)/2}(u\chi_{\epsilon,b})\|_{L^2_x}$, $\|D_x^{2+(1-\alpha)/2}(u\tilde{\phi}_{\epsilon,b})\|_{L^2_x}$, and $\|D_x^{2+(1-\alpha)/2}(u\phi_{\epsilon,b})\|_{L^2_x}$.

For the first term we proceed by writing

$$\begin{aligned} D_x^{2+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b}) &= D_x^{2+\frac{1-\alpha}{2}}u\chi_{\epsilon,b} + [D_x^{2+\frac{1-\alpha}{2}}; \chi_{\epsilon,b}](u\chi_{\epsilon,b} + u\phi_{\epsilon,b} + u\psi_\epsilon) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Notice that $\|I_1\|_{L^2_x}$ is the quantity to be estimated by Gronwall's inequality. Meanwhile, $\|I_2\|_{L^2_x}$, $\|I_3\|_{L^2_x}$ and $\|I_4\|_{L^2_x}$ were estimated previously in the case $j = 1$, step 2.

Next, we focus on estimating the term $\|D_x^{2+(1+\alpha)/2}(u\phi_{\epsilon,b})\|_{L^2_x}$, which will be treated by means of Hölder's inequality and Theorem 3.7 as follows:

$$\|D_x^{2+\frac{1+\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_x} \lesssim \|u_0\|_{L^2_x}^{\frac{1}{2}}\|u\|_{L^\infty_x}^{\frac{1}{2}} + \|\eta_{\frac{\epsilon}{24},b+\frac{7\epsilon}{24}}D_x^{2+\frac{1+\alpha}{2}}u\|_{L^2_x} + \|\eta_{\frac{\epsilon}{24},b+\frac{7\epsilon}{24}}D_x^{\frac{1+\alpha}{2}}\partial_x u\|_{L^2_x} + \|D_x^{\frac{1+\alpha}{2}}u\|_{L^2_x}.$$

After integrating in time, the second and third terms on the right-hand side can be estimated taking $(\epsilon, b) = \left(\frac{\epsilon}{24}, b + \frac{7\epsilon}{24}\right)$ in (6.17) and (6.5) respectively. Hence, after integrating in time it follows by interpolation that $\|D_x^{2+(1-\alpha)/2}(u\phi_{\epsilon,b})\|_{L^2_T L^2_x} < \infty$.

We can bound $\|D_x^{2+(1-\alpha)/2}(u\tilde{\phi}_{\epsilon,b})\|_{L^2_x}$ analogously.

Finally, after integrating by parts

$$\begin{aligned} A_{3,7}(t) &= -\frac{1}{2}\int_{\mathbb{R}} \partial_x u\chi_{\epsilon,b}^2(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)^2\,dx - \int_{\mathbb{R}} u\chi_{\epsilon,b}\chi_{\epsilon,b}'(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)^2\,dx \\ &= A_{3,7,1}(t) + A_{3,7,2}(t). \end{aligned}$$

First,

$$|A_{3,7,1}(t)| \lesssim \|\partial_x u(t)\|_{L^\infty_x}\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)^2\chi_{\epsilon,b}^2\,dx,$$

where the last integral is the quantity that will be estimated using Gronwall's inequality, and the other factor will be controlled after integration in time.

After integration in time and Sobolev's embedding it follows that

$$\begin{aligned} \int_0^T |A_{3,7,2}(t)|\,dt &\lesssim \int_0^T\int_{\mathbb{R}} u\,(\chi_{\epsilon,b}^2)'(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)^2\,dx\,dt \\ &\lesssim \Big(\sup_{0\leq t\leq T}\|u(t)\|_{H_x^{s(\alpha)+}}\Big)\int_0^T\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)^2\,(\chi_{\epsilon,b}^2)'\,dx\,dt \end{aligned}$$

and the last term was already estimated in (6.24).

Thus, after collecting all the information in this step and applying Gronwall's inequality together with hypothesis (1.11), we obtain

$$\sup_{0\leq t\leq T}\|D_x^{\frac{1-\alpha}{2}}\partial_x^2 u\chi_{\epsilon,b}\|_{L^2_x}^2+\|\partial_x^3 u\eta_{\epsilon,b}\|_{L^2_TL^2_x}^2+\|\mathcal{H}\partial_x^3 u\eta_{\epsilon,b}\|_{L^2_TL^2_x}^2\leq c_{2,2}^*,$$

where $c_{2,2}^*=c_{2,2}^*(\alpha;\epsilon;T;v;\|u_0\|_{H_x^{(3-\alpha)/2}};\|D_x^{(1-\alpha)/2}\partial_x^2u_0\chi_{\epsilon,b}\|_{L^2_x})$ for any $\epsilon>0,\ b\geq 5\epsilon$ and $v>0$.

According to the induction argument we shall assume that (1.12) holds for $j\leq m$ with $j\in\mathbb{Z}$ and $j\geq 2$; i.e.,

$$\sup_{0\leq t\leq T}\|\partial_x^j u\chi_{\epsilon,b}\|_{L^2_x}^2+\|D_x^{\frac{1+\alpha}{2}}\partial_x^j u\eta_{\epsilon,b}\|_{L^2_TL^2_x}^2+\|\mathcal{H}D_x^{\frac{1+\alpha}{2}}\partial_x^j u\eta_{\epsilon,b}\|_{L^2_TL^2_x}^2\leq c_{j,1}^* \tag{6.33}$$

for $j=1,2,\dots,m$ with $m\geq 1$, for any $\epsilon>0,\ b\geq 5\epsilon\ \ v\geq 0$.

Step 3: We will assume j an even integer. The case where j is odd follows by an argument similar to the case $j=1$.

By reasoning analogous to that employed in the case $j=2$ it follows that

$$D_x^{\frac{1-\alpha}{2}}\partial_x^j\partial_t u\, D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}^2-D_x^{\frac{1-\alpha}{2}}\partial_x^j D_x^{1+\alpha}\partial_x u\, D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}^2+D_x^{\frac{1-\alpha}{2}}\partial_x^j(u\,\partial_x u)\, D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}^2=0,$$

which after integrating in time yields the identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^j u)^2\chi_{\epsilon,b}^2\,\mathrm{d}x\underbrace{-\frac{v}{2}\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^j u)^2(\chi_{\epsilon,b}^2)'\,\mathrm{d}x}_{A_1(t)}$$
$$\underbrace{-\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^j D_x^{1+\alpha}\partial_x u)(D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}^2)\,\mathrm{d}x}_{A_2(t)}+\underbrace{\int_{\mathbb{R}}D_x^{\frac{1-\alpha}{2}}\partial_x^j(u\,\partial_x u)(D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}^2)\,\mathrm{d}x}_{A_3(t)}=0. \tag{6.34}$$

Step 3.1: We claim that

$$\|D_x^{j+\frac{1+\alpha}{2}}(u\eta_{\epsilon,b})\|_{L^2_TL^2_x}<\infty. \tag{6.35}$$

We proceed as in the case $j=2$. A combination of the commutator estimate (3.14), (6.18), Hölder's inequality and (6.33) yields

$$\begin{aligned}\|D_x^{\frac{1+\alpha}{2}}\partial_x^j(u\eta_{\epsilon,b})\|_{L^2_TL^2_x}&\leq\|D_x^{j+\frac{1+\alpha}{2}}u\eta_{\epsilon,b}\|_{L^2_TL^2_x}+\|[D_x^{j+\frac{1+\alpha}{2}};\eta_{\epsilon,b}](u\chi_{\epsilon,b}+u\phi_{\epsilon,b}+u\psi_\epsilon)\|_{L^2_TL^2_x}\\&\lesssim(c_{j,1}^*)^2+\underbrace{\|D_x^{j-1+\frac{1+\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_TL^2_x}}_{B_1}\\&\quad+\|u_0\|_{L^2_x}+\underbrace{\|D_x^{j-1+\frac{1+\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_TL^2_x}}_{B_2}+\underbrace{\|\eta_{\epsilon,b}D_x^{j+\frac{1+\alpha}{2}}(u\psi_\epsilon)\|_{L^2_TL^2_x}}_{B_3}.\end{aligned} \tag{6.36}$$

Since $\chi_{\epsilon/5,\epsilon}=1$ on the support of $\chi_{\epsilon,b}$ we have

$$\chi_{\epsilon,b}(x)\chi_{\frac{\epsilon}{5},\epsilon}(x)=\chi_{\epsilon,b}(x)\quad\text{for all } x\in\mathbb{R}.$$

Combining Lemma 3.15 and Young's inequality

$$\|D_x^{j+\frac{\alpha-1}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}$$
$$\lesssim\|\partial_x^j u\chi_{\epsilon,b}\|_{L^2_x}^2+\sum_{2\leq k\leq j-1}\gamma_{k,j}\|\chi_{\epsilon,b}^{(j-k)}\|_{L^\infty_x}\|\partial_x^k u\chi_{\frac{\epsilon}{5},\epsilon}\|_{L^2_x}+\|u\|_{L^\infty_TH_x^{(3-\alpha)/2}}+\|u_0\|_{L^2_x}. \tag{6.37}$$

Hence, taking $(\epsilon, b) = \left(\frac{\epsilon}{5}, \epsilon\right)$ in (6.33) yields

$$B_1 \lesssim c^*_{j,1} + \sum_{2\le k\le j-1} \gamma_{k,j} c^*_{k,1} + \|u\|_{L^\infty_T H_x^{(3-\alpha)/2}} + \|u_0\|_{L^2_x}. \tag{6.38}$$

B_2 can be estimated as in step 2 of the case $j = 1$, so it is bounded by the induction hypothesis.

Next, since

$$\operatorname{dist}(\operatorname{supp}(\eta_{\epsilon,b}), \operatorname{supp}(\psi_\epsilon)) \ge \frac{\epsilon}{2},$$

we have by Lemma 3.16

$$\|\eta_{\epsilon,b} D_x^{j+\frac{\alpha+1}{2}}(u\psi_\epsilon)\|_{L^2_x} = \|\eta_{\epsilon,b} D_x^{j+\frac{1+\alpha}{2}}(u\psi_\epsilon)\|_{L^2_x} \lesssim \|\eta_{\frac{\epsilon}{8},b+\epsilon}\|_{L^\infty_x}\|u_0\|_{L^2_x}.$$

Gathering the estimates above, (6.35) follows.

We have proved that locally in the interval $[\epsilon, b]$ there exist $j + \frac{\alpha+1}{2}$ derivatives. So, by Lemma 3.15 we obtain

$$\|D_x^{j+\frac{1-\alpha}{2}}(u\eta_{\epsilon,b})\|_{L^2_T L^2_x} \lesssim \|D_x^{j+\frac{1+\alpha}{2}}(u\eta_{\epsilon,b})\|_{L^2_T L^2_x} + \|u_0\|_{L^2_x};$$

then, as before

$$D_x^{j+\frac{1-\alpha}{2}} u\eta_{\epsilon,b} = c_j D_x^{j+\frac{1-\alpha}{2}}(u\eta_{\epsilon,b}) - c_j[D_x^{j+\frac{1-\alpha}{2}}; \eta_{\epsilon,b}](u\chi_{\epsilon,b} + u\phi_{\epsilon,b} + u\psi_\epsilon),$$

where c_j is a constant depending only on j.

Hence, if we proceed as in the proof of the claim (6.35) above, we have

$$\|D_x^{j+\frac{1-\alpha}{2}} u\eta_{\epsilon,b}\|_{L^2_T L^2_x} < \infty. \tag{6.39}$$

Therefore

$$\int_0^T |A_1(t)|\,\mathrm{d}t = v\|D_x^{j+\frac{1-\alpha}{2}} u\eta_{\epsilon,b}\|^2_{L^2_T L^2_x} < \infty.$$

Step 3.2: To handle the term A_2 we use the same procedure as in the previous steps. First,

$$A_2(t) = -\frac{1}{2}\int_{\mathbb{R}} D_x^{\frac{2j+1-\alpha}{2}} u[\mathcal{H}D_x^{2+\alpha}; \chi^2_{\epsilon,b}] D_x^{\frac{2j+1-\alpha}{2}} u\,\mathrm{d}x \tag{6.40}$$

since

$$[\mathcal{H}D_x^{\alpha+2}; \chi^2_{\epsilon,b}] + \tfrac{1}{2}P_n(\alpha+2) + R_n(\alpha+2) = \tfrac{1}{2}\mathcal{H}P_n(\alpha+2)\mathcal{H} \tag{6.41}$$

for some positive integer n. Substituting (6.41) into (6.40) produces

$$\begin{aligned} A_2(t) &= \frac{1}{2}\int_{\mathbb{R}} D_x^{\frac{2j+1-\alpha}{2}} u(R_n(\alpha+2) D_x^{\frac{2j+1-\alpha}{2}} u)\,\mathrm{d}x \\ &\quad + \frac{1}{4}\int_{\mathbb{R}} D_x^{\frac{2j+1-\alpha}{2}} u(P_n(\alpha+2) D_x^{\frac{2j+1-\alpha}{2}} u)\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}} D_x^{\frac{2j+1-\alpha}{2}} u(\mathcal{H}P_n(\alpha+2)\mathcal{H} D_x^{\frac{2j+1-\alpha}{2}} u)\,\mathrm{d}x \\ &= A_{2,1}(t) + A_{2,2}(t) + A_{2,3}(t). \end{aligned} \tag{6.42}$$

As above we deal first with the crucial term in the decomposition associated to A_2, that is, $A_{2,1}$.

Applying Plancherel's identity yields

$$A_{2,1}(t) = \frac{1}{2}\int_{\mathbb{R}} u D_x^{\frac{2j+1-\alpha}{2}}\{R_n(\alpha+2) D_x^{\frac{2j+1-\alpha}{2}} u\}\,\mathrm{d}x.$$

We fix n such that (3.26) is satisfied. In this case we have to take $a = \alpha+2$ and $\sigma = \frac{2j+1-\alpha}{2}$ to get $n = j$. As occurs in the previous cases it is possible for $n = j+1$.

Thus, by construction $R_j(\alpha+2)$ is bounded in L^2_x (see Proposition 3.25).

Then

$$|A_{2,1}(t)| \lesssim \|u_0\|_{L^2_x}^2 \|D_x^{2j+3}\widehat{(\chi_{\epsilon,b}^2)}\|_{L^1_\xi}$$

and

$$\int_0^T |A_{2,1}(t)|\,\mathrm{d}t \lesssim \|u_0\|_{L^2_x}^2 \sup_{0\le t\le T} \|D_x^{2j+3}\widehat{(\chi_{\epsilon,b}^2)}\|_{L^1_\xi}.$$

Substituting $P_j(\alpha+2)$ into $A_{2,2}$

$$\begin{aligned}
A_{2,2}(t) &= \left(\frac{\alpha+2}{4}\right)\int_{\mathbb{R}} (\mathcal{H}\partial_x^{j+1}u)^2(\chi_{\epsilon,b}^2)'\,\mathrm{d}x + \left(\frac{\alpha+2}{2}\right)\sum_{l=1}^{j} c_{2l+1}(-1)^l 4^{-l}\int_{\mathbb{R}} (D_x^{j-l+1}u)^2(\chi_{\epsilon,b}^2)^{(2l+1)}\,\mathrm{d}x \\
&= A_{2,2,1}(t) + \sum_{l=1}^{j-1} A_{2,2,l}(t) + A_{2,2,j}(t).
\end{aligned}$$

Note that $A_{2,2,1}$ is positive and it gives the smoothing effect after integration in time, and $A_{2,2,j}$ is bounded by using the local theory. To handle the remainder terms we recall that by construction

$$|(\chi_{\epsilon,b})^{(j)}(x)| \lesssim \chi'_{\frac{\epsilon}{3},b+\epsilon}(x) \lesssim \chi_{\frac{\epsilon}{9},b+\frac{10\epsilon}{9}}(x)\chi'_{\frac{\epsilon}{9},b+\frac{10\epsilon}{9}}(x) \tag{6.43}$$

for $x \in \mathbb{R},\ j \in \mathbb{Z}^+$.

Hence for $j > 2$

$$\begin{aligned}
\int_0^T |A_{2,2,l}(t)|\,\mathrm{d}t &\lesssim \int_0^T\int_{\mathbb{R}} (D_x^{j-l+1}u)^2 \chi'_{\frac{\epsilon}{3},b+\epsilon}\,\mathrm{d}x\,\mathrm{d}t \\
&\lesssim \int_0^T\int_{\mathbb{R}} (D_x^{j-l+1}u)^2 \chi_{\frac{\epsilon}{9},b+\frac{10\epsilon}{9}}\chi'_{\frac{\epsilon}{9},b+\frac{10\epsilon}{9}}\,\mathrm{d}x\,\mathrm{d}t;
\end{aligned} \tag{6.44}$$

thus if we apply (6.33) with $\left(\frac{\epsilon}{9}, b+\frac{4\epsilon}{3}\right)$ instead of (ϵ, b) we obtain

$$\int_0^T\int_{\mathbb{R}} (D_x^{j-l+1}u)^2 (\chi_{\frac{\epsilon}{9},b+\frac{10\epsilon}{9}}\chi'_{\frac{\epsilon}{9},b+\frac{10\epsilon}{9}})\,\mathrm{d}x\,\mathrm{d}t \le c^*_{l,2}$$

for $l = 1, 2, \ldots, j-1$.

Meanwhile,

$$\begin{aligned}
A_{2,3}(t) &= \left(\frac{\alpha+2}{4}\right)\int_{\mathbb{R}} (\partial_x^{j+1}u)^2(\chi_{\epsilon,b}^2)'\,\mathrm{d}x + \left(\frac{\alpha+2}{4}\right)\sum_{l=1}^{j} c_{2l+1}(-1)^l 4^{-l}\int_{\mathbb{R}} (\mathcal{H}D_x^{j-l+1}u)^2(\chi_{\epsilon,b}^2)^{(2l+1)}\,\mathrm{d}x \\
&= A_{2,3,1}(t) + \sum_{l=1}^{j-1} A_{2,3,l}(t) + A_{2,3,j}(t).
\end{aligned} \tag{6.45}$$

As we can see $A_{2,3,1} \ge 0$ and it represents the smoothing effect. Additionally, applying an argument similar to that employed in (6.43)–(6.44), it is possible to bound the remainder terms in (6.45). Anyway,

$$\int_0^T |A_{2,3,l}(t)|\,\mathrm{d}t \lesssim c^*_{l,2}, \quad 1 \le l \le j-1.$$

Step 3.3: It only remains to estimate A_3 to finish step 3.

$$\begin{aligned}
D_x^{\frac{1-\alpha}{2}}\partial_x^j(u\partial_x u)\chi_{\epsilon,b} &= -\tfrac{1}{2}[D_x^{\frac{1-\alpha}{2}}\partial_x^j;\, \chi_{\epsilon,b}]\partial_x((u\chi_{\epsilon,b})^2 + (u\tilde{\phi}_{\epsilon,b})^2 + (\psi_\epsilon u^2)) \\
&\quad + [D_x^{\frac{1-\alpha}{2}}\partial_x^j;\, u\chi_{\epsilon,b}]\partial_x((u\chi_{\epsilon,b}) + (u\phi_{\epsilon,b}) + (u\psi_\epsilon)) + u\chi_{\epsilon,b}D_x^{\frac{1-\alpha}{2}}\partial_x^j(\partial_x u) \\
&= \widetilde{A}_{3,1}(t) + \widetilde{A}_{3,2}(t) + \widetilde{A}_{3,3}(t) + \widetilde{A}_{3,4}(t) + \widetilde{A}_{3,5}(t) + \widetilde{A}_{3,6}(t) + \widetilde{A}_{3,7}(t).
\end{aligned} \tag{6.46}$$

Substituting (6.46) into A_3 and applying Hölder's inequality

$$\begin{aligned}A_3(t) &= \sum_{1\le k\le 6}\int_{\mathbb{R}} \widetilde{A}_{3,k}(t)\, D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}\,\mathrm{d}x + \int_{\mathbb{R}} \widetilde{A}_{3,7}(t)\, D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}\,\mathrm{d}x\\ &\le \sum_{1\le k\le 6}\|\widetilde{A}_{3,k}(t)\|_{L^2_x}\|D_x^{j+\frac{1-\alpha}{2}}u(t)\chi_{\epsilon,b}(\cdot+vt)\|_{L^2_x} + \int_{\mathbb{R}} \widetilde{A}_{3,7}(t) D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}\,\mathrm{d}x.\\ &= \|D_x^{j+\frac{1-\alpha}{2}}u(t)\chi_{\epsilon,b}(\cdot+vt)\|_{L^2_x}\sum_{1\le m\le 6} A_{3,k}(t)+A_{3,7}(t).\end{aligned}$$

The first factor on the right-hand side is the quantity to be estimated.

We will start by estimating the easiest term:

$$\begin{aligned}A_{3,7}(t) &= -\frac{1}{2}\int_{\mathbb{R}} \partial_x u\chi^2_{\epsilon,b}\,(D_x^{\frac{1-\alpha}{2}}\partial_x^j u)^2\,\mathrm{d}x - \int_{\mathbb{R}} u\chi_{\epsilon,b}\chi'_{\epsilon,b}(D_x^{\frac{1-\alpha}{2}}\partial_x^j u)^2\,\mathrm{d}x\\ &= A_{3,7,1}(t)+A_{3,7,2}(t).\end{aligned}$$

We have that

$$|A_{3,7,1}(t)| \lesssim \|\partial_x u(t)\|_{L^\infty_x}\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^j u)^2\chi^2_{\epsilon,b}\,\mathrm{d}x,$$

where the last integral is the quantity that we want to estimate, and the another factor will be controlled after integration in time.

After integration in time and Sobolev's embedding

$$\begin{aligned}\int_0^T |A_{3,7,2}(t)|\,\mathrm{d}t &\lesssim \int_0^T\int_{\mathbb{R}} u\,(\chi^2_{\epsilon,b})'(D_x^{\frac{1-\alpha}{2}}\partial_x^j u)\mathrm{d}x\,\mathrm{d}t\\ &\lesssim \Big(\sup_{0\le t\le T}\|u(t)\|_{H_x^{s(\alpha)+}}\Big)\int_0^T\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^j u)^2\,(\chi^2_{\epsilon,b})'\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where the integral expression on the right-hand side was already estimated in (6.39).

To handle the contribution coming from $\widetilde{A}_{3,1}$ and $\widetilde{A}_{3,2}$, we apply a combination of (3.4), (3.14) and Lemma 3.15 to obtain

$$\begin{aligned}\|\widetilde{A}_{3,1}(t)\|_{L^2_x} &\lesssim \|D_x^{j+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}\|u\|_{L^\infty_x}+\|u_0\|_{L^2_x}\|u\|_{L^\infty_x},\\ \|\widetilde{A}_{3,2}(t)\|_{L^2_x} &\lesssim \|D_x^{j+\frac{1-\alpha}{2}}(u\widetilde{\phi}_{\epsilon,b})\|_{L^2_x}\|u\|_{L^\infty_x}+\|u_0\|_{L^2_x}\|u\|_{L^\infty_x}.\end{aligned}\tag{6.47}$$

The condition on the supports of $\chi_{\epsilon,b}$ and ψ_ϵ combined with Lemma 3.16 implies

$$\|\widetilde{A}_{3,3}(t)\|_{L^2_x} \lesssim \|u_0\|_{L^2_x}\|u\|_{L^\infty_x}.$$

By using (3.2) and (3.14)

$$\begin{aligned}\|\widetilde{A}_{3,4}(t)\|_{L^2_x} &\lesssim \|\partial_x(u\chi_{\epsilon,b})\|_{L^\infty_x}\|D_x^{j+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x},\\ \|\widetilde{A}_{3,5}(t)\|_{L^2_x} &\lesssim \|\partial_x(u\chi_{\epsilon,b})\|_{L^\infty_x}\|D_x^{j+\frac{1-\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_x}+\|\partial_x(u\phi_{\epsilon,b})\|_{L^\infty_x}\|D_x^{j+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}.\end{aligned}$$

An application of Lemma 3.16 leads to

$$\|\widetilde{A}_{3,6}(t)\|_{L^2_x} = \|u\chi_{\epsilon,b}\partial_x D_x^{j+\frac{1-\alpha}{2}}(u\psi_\epsilon)\|_{L^2_x} \lesssim \|u_0\|_{L^2_x}\|u\|_{L^\infty_x}. \tag{6.48}$$

To complete the estimate in (6.47)–(6.48) we write

$$\chi_{\epsilon,b}(x)+\phi_{\epsilon,b}(x)+\psi_\epsilon(x)=1 \quad \text{for all } x\in\mathbb{R};$$

then

$$\begin{aligned} D_x^{j+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b}) &= D_x^{j+\frac{1-\alpha}{2}}u\chi_{\epsilon,b} + [D_x^{j+\frac{1-\alpha}{2}};\chi_{\epsilon,b}](u\chi_{\epsilon,b}+u\phi_{\epsilon,b}+u\psi_\epsilon) \\ &= I_1+I_2+I_3+I_4. \end{aligned}$$

Notice that $\|I_1\|_{L^2_x}$ is the quantity to be estimated. In contrast, I_4 is handled by using Lemma 3.16. In regards to $\|I_2\|_{L^2_x}$ and $\|I_3\|_{L^2_x}$, Lemma 3.13 combined with the local theory, and the step 2 in the case $j=1$ produces the required bounds.

By Theorem 3.7 and Hölder's inequality

$$\begin{aligned} &\|D_x^{j+\frac{1+\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_x} \\ &\quad\lesssim \|u\|_{L^4_x}\|D_x^{j+\frac{1+\alpha}{2}}\phi_{\epsilon,b}\|_{L^4_x} + \left\|\sum_{\beta\le j}\frac{1}{\beta!}\partial_x^\beta\phi_{\epsilon,b}D_x^{s,\beta}u\right\|_{L^2_x} \\ &\quad\lesssim \|u_0\|_{L^2_x}^{1/2}\|u\|_{L^\infty_x}^{1/2} + \sum_{\beta\in\mathbb{Q}_1(j)}\frac{1}{\beta!}\|\partial_x^\beta\phi_{\epsilon,b}D_x^{j-\beta+\frac{\alpha+1}{2}}u\|_{L^2_x} + \sum_{\beta\in\mathbb{Q}_2(j)}\frac{1}{\beta!}\|\partial_x^\beta\phi_{\epsilon,b}\mathcal{H}D_x^{j-\beta+\frac{\alpha+1}{2}}u\|_{L^2_x}, \end{aligned} \tag{6.49}$$

where $\mathbb{Q}_1(j)$, $\mathbb{Q}_2(j)$ denote odd integers and even integers in $\{0,1,\dots,j\}$ respectively.

To estimate the second term in (6.49), note that $\partial_x^\beta\phi_{\epsilon,b}$ is supported in $\left[\frac{\epsilon}{4},b\right]$; then

$$\begin{aligned} \sum_{\beta\in\mathbb{Q}_1(j)}\frac{1}{\beta!}\|\partial_x^\beta\phi_{\epsilon,b}D_x^{j-\beta+\frac{\alpha+1}{2}}u\|_{L^2_x} &\lesssim \sum_{\beta\in\mathbb{Q}_1(j)}\frac{1}{\beta!}\|\mathbb{1}_{[\frac{\epsilon}{8},b]}D_x^{j-\beta+\frac{\alpha+1}{2}}u\|_{L^2_x} \\ &\lesssim \sum_{\beta\in\mathbb{Q}_1(j)}\frac{1}{\beta!}\|\eta_{\frac{\epsilon}{24},b+\frac{7\epsilon}{24}}D_x^{j-\beta+\frac{\alpha+1}{2}}u\|_{L^2_x}. \end{aligned}$$

Hence, after integrating in time and applying (6.33) with $(\epsilon,b)=\left(\frac{\epsilon}{24},b+\frac{7\epsilon}{24}\right)$ we obtain

$$\sum_{\beta\in\mathbb{Q}_1(j)}\frac{1}{\beta!}\|\eta_{\frac{\epsilon}{24},b+\frac{7\epsilon}{24}}D_x^{j-\beta+\frac{\alpha+1}{2}}u\|_{L^2_TL^2_x} \lesssim \sum_{\beta\in\mathbb{Q}_1(j)}(c^*_{j-\beta,1})^{\frac{1}{2}}<\infty$$

by the induction hypothesis.

Analogously, we can handle the third term in (6.49):

$$\sum_{\beta\in\mathbb{Q}_2(j),\beta\ne j}\frac{1}{\beta!}\|\partial_x^\beta\phi_{\epsilon,b}\mathcal{H}D_x^{j-\beta+\frac{\alpha+1}{2}}u\|_{L^2_TL^2_x} \lesssim \sum_{\beta\in\mathbb{Q}_2(j),\beta\ne j}(c^*_{j-\beta,1})^{\frac{1}{2}}+\|u\|_{L^\infty_TH_x^{(3-\alpha)/2}}<\infty.$$

Therefore, after integrating in time and applying Hölder's inequality we have

$$\|D_x^{j+\frac{1+\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_TL^2_x}<\infty.$$

Next, by interpolation and Young's inequality

$$\|D_x^{j+\frac{1-\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_TL^2_x} \lesssim \|D_x^{j+\frac{1+\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_TL^2_x}+\|u_0\|_{L^2_x}<\infty. \tag{6.50}$$

If we apply (6.49)–(6.50) then

$$\|D_x^{j+\frac{1-\alpha}{2}}(u\tilde{\phi}_{\epsilon,b})\|_{L^2_T L^2_x} < \infty.$$

Finally, after collecting all of the information and applying Gronwall's inequality we obtain

$$\sup_{0\le t\le T} \|D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}\|^2_{L^2_x} + \|\partial_x^{j+1}u\eta_{\epsilon,b}\|^2_{L^2_T L^2_x} + \|\mathcal{H}\partial_x^{j+1}u\eta_{\epsilon,b}\|^2_{L^2_T L^2_x} \le c^*_{j,2},$$

where $c^*_{j,2} = c^*_{j,2}(\alpha;\epsilon;T;v;\|u_0\|_{H_x^{(3-\alpha)/2}};\|D_x^{(1-\alpha)/2}\partial_x^j u_0\chi_{\epsilon,b}\|_{L^2_x})$ for any $\epsilon > 0$, $b \ge 5\epsilon$ and $v \ge 0$.

This finishes the induction process.

To justify the previous estimates we shall follow the following argument of regularization. For arbitrary initial data $u_0 \in H^s(\mathbb{R})$ $s > \frac{3-\alpha}{2}$, we consider the regularized initial data $u_0^\mu = \rho_\mu * u_0$ with $\rho \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\rho \subset (-1,1)$, $\rho \ge 0$, $\|\rho\|_{L^1} = 1$ and

$$\rho_\mu(x) = \mu^{-1}\rho\left(\frac{x}{\mu}\right) \quad \text{for } \mu > 0.$$

The solution u^μ of the IVP (1.1) corresponding to the smoothed data $u_0^\mu = \rho_\mu * u_0$ satisfies

$$u^\mu \in C([0,T] : H^\infty(\mathbb{R}));$$

we note that the time of existence is independent of μ.

Therefore, the smoothness of u^μ allows us to conclude that

$$\sup_{0\le t\le T} \|\partial_x^m u^\mu\chi_{\epsilon,b}\|^2_{L^2_x} + \|D_x^{m+\frac{1+\alpha}{2}}u^\mu\|^2_{L^2_T L^2_x} + \|\mathcal{H}D_x^{m+\frac{1+\alpha}{2}}u^\mu\eta_{\epsilon,b}\|^2_{L^2_T L^2_x} \le c^*,$$

where $c^* = c^*(\alpha;\epsilon;T;v;\|u_0^\mu\|_{H_x^{(3-\alpha)/2}};\|\partial_x^m u_0^\mu\chi_{\epsilon,b}\|_{L^2_x})$. In fact our next task is to prove that the constant c^* is independent of the parameter μ.

The independence from the parameter $\mu > 0$ can be reached first noticing that

$$\|u_0^\mu\|_{H_x^{(3-\alpha)/2}} \le \|u_0\|_{H_x^{(3-\alpha)/2}}\|\hat{\rho}_\mu\|_{L^\infty_\xi} = \|u_0\|_{H_x^{(3-\alpha)/2}}\|\rho_\mu\|_{L^1_x} = \|u_0\|_{H_x^{(3-\alpha)/2}}.$$

Next, since $\chi_{\epsilon,b}(x) = 0$ for $x \le \epsilon$, restricting to $\mu \in (0,\epsilon)$ it follows by Young's inequality

$$\int_\epsilon^\infty (\partial_x^m u_0^\mu)^2\,\mathrm{d}x \le \|\rho_\mu\|_{L^1_\xi}\|\partial_x^m u_0\|_{L^2_x((0,\infty))} = \|\partial_x^m u_0\|_{L^2_x((0,\infty))}.$$

Using the continuous dependence of the solution upon the data we have that

$$\sup_{t\in[0,T]} \|u^\mu(t) - u(t)\|_{H_x^{(3-\alpha)/2}} \xrightarrow[\mu\to 0]{} 0.$$

Combining this fact with the independence of the constant c^* from the parameter μ, weak compactness and Fatou's lemma, the theorem holds for all $u_0 \in H^s(\mathbb{R})$, $s > \frac{3-\alpha}{2}$. □

Remark 6.51. The proof of Theorem B remains valid for the defocusing dispersive generalized Benjamin–Ono equation

$$\begin{cases} \partial_t u - D_x^{\alpha+1}\partial_x u - u\,\partial_x u = 0, & x,t \in \mathbb{R},\ 0<\alpha<1, \\ u(x,0) = u_0(x). \end{cases}$$

In this direction, the propagation of regularity holds for $u(-x,-t)$, where $u(x,t)$ a solution of (1.1). In other words, this means that for initial data satisfying the conditions (1.9) and (1.11) on the left-hand side of the real line, Theorem B remains valid backward in time.

A consequence of the Theorem B is the following corollary, which describes the asymptotic behavior of the function in (1.10).

Corollary 6.52. *Let $u \in C([-T,T] : H^{(3-\alpha)/2}(\mathbb{R}))$ be a solution of the equation in (1.1) described by Theorem B.*

Then, for any $t \in (0,T]$ and $\delta > 0$

$$\int_{-\infty}^{\infty} \frac{1}{\langle x_- \rangle^{j+\delta}} (\partial_x^j u)^2(x,t)\,\mathrm{d}x \le \frac{c}{t}, \tag{6.53}$$

where $x_- = \max\{0,-x\}$, c is a positive constant and $\langle x\rangle := \sqrt{1+x^2}$.

For the proof of (6.53) we use the following lemma provided in [Segata and Smith 2017].

Lemma 6.54. *Let $f : [0,\infty) \to [0,\infty)$ be a continuous function. If for $a > 0$*

$$\int_0^a f(x)\,\mathrm{d}x \le ca^p,$$

then for every $\delta > 0$

$$\int_0^\infty \frac{f(x)}{\langle x\rangle^{p+\delta}}\,\mathrm{d}x \le c(p).$$

Proof. The proof follows by using a smooth dyadic partition of unit of $\mathbb{R}^+$. □

Remark 6.55. Observe that the lemma also applies when integrating a nonnegative function on the interval $[-(a+\epsilon), -\epsilon]$, implying decay on the left half-line.

Proof of Corollary 6.52. We shall recall that Theorem B with $x_0 = 0$ asserts that any $\epsilon > 0$

$$\sup_{t\in[0,T]} \int_{\epsilon - vt}^{\infty} (\partial_x^j u)^2(x,t)\,\mathrm{d}x \le c^*.$$

For fixed $t \in [0,T]$ we split the integral term as follows:

$$\int_{\epsilon-vt}^{\infty} (\partial_x^j u)^2(x,t)\,\mathrm{d}x = \int_{\epsilon-vt}^{\epsilon} (\partial_x^j u)^2(x,t)\,\mathrm{d}x + \int_{\epsilon}^{\infty} (\partial_x^j u)^2(x,t)\,\mathrm{d}x.$$

The second term in the right-hand side is easily bounded by using Theorem B with $v = 0$. Hence, we just need to estimate the first integral in the right-hand side.

Notice that after making a change of variables,

$$\int_{\epsilon-vt}^{\epsilon} (\partial_x^j u)^2(x,t)\,\mathrm{d}x = \int_{-(\epsilon-vt)}^{-\epsilon} (\partial_x^j u)^2(x+2\epsilon,t)\,\mathrm{d}x \le c^*.$$

Thus by using Lemma 6.54 and Remark 6.55 we find

$$\int_{-\infty}^{-\epsilon} \frac{1}{\langle x+2\epsilon\rangle^{j+\delta}} (\partial_x^j u)^2(x+2\epsilon,t)\,\mathrm{d}x = \int_{-\infty}^{\epsilon} \frac{1}{\langle x\rangle^{j+\delta}} (\partial_x^j u)^2(x,t)\,\mathrm{d}x \le \frac{c^*}{t^j}.$$

In summary, we have proved that for all $j \in \mathbb{Z}^+,\ j \geq 2$ and any $\delta > 0$

$$\int_{-\infty}^{\epsilon} \frac{1}{\langle x \rangle^{j+\delta}} (\partial_x^j u)^2(x,t)\,\mathrm{d}x \leq \frac{c^*}{t}, \tag{6.56}$$

$$\int_{\epsilon}^{\infty} (\partial_x^j u)^2(x,t)\,\mathrm{d}x \leq c^*. \tag{6.57}$$

If we apply the Lemma 6.54 to (6.57) we obtain extra decay in the right-hand side. This allow us to obtain a uniform expression that combines (6.56) and (6.57); that is, there exists a constant c such that for any $t \in (0, T]$ and $\delta > 0$

$$\int_{-\infty}^{\infty} \frac{1}{\langle x_- \rangle^{j+\delta}} (\partial_x^j u)^2(x,t)\,\mathrm{d}x \leq \frac{c}{t}. \qquad \square$$

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ANALYSIS & PDE

Volume 13 No. 8 2020

Propagation properties of reaction-diffusion equations in periodic domains 2259
ROMAIN DUCASSE

An elementary approach to free entropy theory for convex potentials 2289
DAVID JEKEL

Parametrix for a semiclassical subelliptic operator 2375
HART F. SMITH

On the propagation of regularity for solutions of the dispersion generalized Benjamin–Ono equation 2399
ARGENIS J. MENDEZ

Optimal regularity in time and space for the porous medium equation 2441
BENJAMIN GESS, JONAS SAUER and EITAN TADMOR