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MICROLOCAL ANALYSIS NEAR NULL INFINITY IN ASYMPTOTICALLY FLAT SPACETIMES

PETER HINTZ AND ANDRÁS VASY

We present a novel approach to the analysis of regularity and decay for solutions of wave equations in a neighborhood of null infinity in asymptotically flat spacetimes of any dimension. The classes of metrics and wave-type operators we consider near null infinity include those arising in nonlinear stability problems for Einstein's field equations in 1+3 dimensions. In a neighborhood of null infinity, in an appropriate compactification of the spacetime to a manifold with corners, the wave operators are of edge type at null infinity and totally characteristic at spacelike and future timelike infinity. On a corresponding scale of Sobolev spaces, we demonstrate how microlocal regularity propagates across or into null infinity via a sequence of radial sets. As an application, inspired by work of the second author with Baskin and Wunsch, we prove regularity and decay estimates for forward solutions of wave-type equations on asymptotically flat spacetimes which are asymptotically homogeneous with respect to scaling in the forward timelike cone and have an appropriate structure at null infinity. These estimates are new even for the wave operator on Minkowski space.

The results obtained here are also used as black boxes in a global theory of wave-type equations on asymptotically flat and asymptotically stationary spacetimes developed by the first author.

1. Introduction 2
1.1. Prior work on microlocal analysis near null infinity 5
1.2. The edge-b-perspective near null infinity 8
1.3. Outline of the paper 11
2. Edge-b-geometry and analysis 12
2.1. Vector fields, differential operators, bundles 13
2.2. Symbols and normal operators of edge-b-differential operators 16
2.3. Edge-b-pseudodifferential operators 21
2.4. Sobolev spaces 23
2.5. Invariant edge-b-operators and edge-b-Sobolev spaces 25
2.6. Semiclassical 0-analysis 27
3. Geometric and analytic setting 37
3.1. Admissible metrics 37
3.2. Admissible operators 44
4. Microlocal edge-b-regularity theory near $\mathscr{I}^+$ 46
4.1. Structure of the null-bicharacteristic flow near $\mathscr{I}^+$ 47
4.2. Microlocal estimates 51

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Keywords: wave equation, asymptotically flat spacetimes, null infinity, microlocal analysis, edge operators, totally characteristic operators, null-bicharacteristic flow, propagation estimates, radial sets.

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5. Higher b-regularity 61
5.1. Commutator b-vector fields 61
5.2. Edge,b;b-operators and Sobolev spaces 64
5.3. Propagation estimates near $\mathscr{I}^+$ 67
6. Energy estimates and solvability near I^0 and $\mathscr{I}^+ \setminus I^+$ 67
7. Control of edge-b-decay at $\mathscr{I}^+$ 79
7.1. Analysis of the edge normal operator 79
7.2. Leading-order control at $\mathscr{I}^+$ 84
8. Microlocal estimates for the I^+-normal operator 90
8.1. Bounded frequency estimates 91
8.2. High frequency estimates 94
9. Application to linear waves on asymptotically flat spacetimes 97
Acknowledgments 103
References 103

1. Introduction

We introduce a new point of view for the analysis of linear waves on asymptotically flat spacetimes near null infinity. The main novelty is the fully microlocal nature of our approach (apart from a simple energy estimate). This allows one to combine the estimates proved here in the usual modular microlocal fashion with microlocal estimates far from null infinity; a first implementation of this is given in [Hintz 2023d].

Our regularity theory appears to be new even on Minkowski space. Let us work on $\mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x$ (with $n \geq 1$) equipped with the Minkowski metric $g_0 = -\mathrm{d}t^2 + \sum_{j=1}^n (\mathrm{d}x^j)^2$ and volume density $|\mathrm{d}g_0| = |\mathrm{d}t\, \mathrm{d}x^1 \cdots \mathrm{d}x^n|$, and introduce polar coordinates $x = r\omega,\ r \geq 0,\ \omega \in \mathbb{S}^{n-1}$, on $\mathbb{R}^n$. In the exterior domain

$$\Omega = \{(t, x) \in \mathbb{R}^{1+n} : 0 \leq t \leq r - 1\},$$

and for weights $\alpha_0, \alpha_{\mathscr{I}} \in \mathbb{R}$, we set[1]

$$\begin{gathered} H_{\mathrm{e,b}}^{0,(\alpha_0,2\alpha_{\mathscr{I}})}(\Omega) = \rho_0^{\alpha_0} x_{\mathscr{I}}^{2\alpha_{\mathscr{I}}} L^2(\Omega, |\mathrm{d}g_0|) = \{\rho_0^{\alpha_0} x_{\mathscr{I}}^{2\alpha_{\mathscr{I}}} u : u \in L^2(\Omega, |\mathrm{d}g_0|)\}, \\ \rho_0 := \frac{1}{r-t}, \quad x_{\mathscr{I}} := \sqrt{\frac{r-t}{r}}. \end{gathered} \tag{1-1}$$

Define the vector fields (which we will refer to as *edge-b-vector fields*)[2]

$$\begin{aligned} V_0 &= -\rho_0 \partial_{\rho_0} = t\partial_t + r\partial_r && \text{(scaling)}, \\ V_1 &= -\tfrac{1}{2} x_{\mathscr{I}} \partial_{x_{\mathscr{I}}} = r(\partial_t + \partial_r) && \text{(weighted derivative along outgoing light cones)}, \\ V_a &= x_{\mathscr{I}} \Omega_a \ \ (a = 2, \dots, N) && \text{(where the } \Omega_a \text{ span } \mathcal{V}(\mathbb{S}^{n-1}) \text{ over } \mathcal{C}^\infty(\mathbb{S}^{n-1})). \end{aligned} \tag{1-2}$$

We then define *weighted edge-b-Sobolev spaces* for $s \in \mathbb{N}_0$ by

$$H_{\mathrm{e,b}}^{s,(\alpha_0,2\alpha_{\mathscr{I}})}(\Omega) = \{u \in H_{\mathrm{e,b}}^{0,(\alpha_0,2\alpha_{\mathscr{I}})}(\Omega) : V^\beta u \in H_{\mathrm{e,b}}^{0,(\alpha_0,2\alpha_{\mathscr{I}})}(\Omega)\ \forall\, \beta \in \mathbb{N}_0^N,\ |\beta| \leq s\}.$$

[1] The factor of 2 in the second order is introduced so that $\alpha_{\mathscr{I}}$ corresponds to the amount of r-decay at null infinity ($|t-r| \lesssim 1$, $r \to \infty$) as measured by powers of r^{-1}.

[2] One can replace V_0 here by $2V_0 - \frac{t+r}{r} V_1 = (t-r)(\partial_t - \partial_r)$, which is a weighted derivative along incoming light cones.

Theorem 1.1 (edge-b-regularity and decay of waves in the exterior domain). *Suppose*

$$\alpha_{\mathscr{I}} < \min\left(-\tfrac{1}{2}, \alpha_0 + \tfrac{1}{2}\right),$$

and let $s \in \mathbb{N}$. *Let* $f \in H_{\mathrm{e,b}}^{s-1,(\alpha_0+2,2\alpha_{\mathscr{I}}+2)}(\Omega)$. *Then the forward solution* u *of* $\Box_{g_0} u = f$ *(that is, the unique solution of* $\left(-D_t^2 + \sum_{j=1}^n D_{x^j}^2\right)u = f$ *with* $u|_{t<0} = 0$*) satisfies* $u \in H_{\mathrm{e,b}}^{s,(\alpha_0,2\alpha_{\mathscr{I}})}(\Omega)$.

As we will discuss in Section 1.2, a key property of the operator $\Box_{g_0}$ used in Theorem 1.1 is the fact that, up to an overall weight $\rho_0^2 x_{\mathscr{I}}^2$, it is, to leading order, a nondegenerate (Lorentzian signature) quadratic form in the edge-b-vector fields (1-2), uniformly as $r \to \infty$ in Ω; see already (1-8). We also remark that these vector fields are related to a parabolic scaling near null hypersurfaces; the null hypersurface of interest here being null infinity in an appropriate compactification of Ω; see Remark 1.6.

The restriction $\alpha_{\mathscr{I}} < -\frac{1}{2}$ in Theorem 1.1 is sharp, as it precisely guarantees that the edge-b-Sobolev space for u permits the well-known pointwise $r^{-(n-1)/2}$ decay along light cones $t - r = \text{const.}$ towards null infinity $\mathscr{I}^+$ (given in the coordinates (1-1) by $x_{\mathscr{I}} = 0$). The restriction $\alpha_{\mathscr{I}} < \alpha_0 + \frac{1}{2}$ is likewise necessary since a forcing term f which is large (i.e., $\alpha_0 < -1$, roughly corresponding to less than pointwise $r^{-(n-1)/2}$ decay) near spacelike infinity I^0 (given by $\rho_0 = 0$) (i.e., f is large when $\frac{t}{r} \in [0, 1)$ with r large) produces a wave u which is large at I^0 and $\mathscr{I}^+$. Roughly speaking, u has two orders of decay (as measured by powers of r^{-1}) less than f at $I^0 \setminus \mathscr{I}^+$ and one order of decay less than f at $\mathscr{I}^+ \setminus I^0$.

Theorem 1.1 remains valid on a large class of generalizations of Minkowski space, including those arising in $n + 1 = 3 + 1$ dimensions as solutions of the Einstein field equations in the context of the nonlinear stability of Minkowski space, or more generally in the context of the existence of a piece of null infinity for asymptotically flat data sets [Christodoulou and Klainerman 1993; Klainerman and Nicolò 2003; Lindblad and Rodnianski 2010; Hintz and Vasy 2020]. (Whether the spacetimes constructed in [Bieri and Zipser 2009; Bieri and Chruściel 2017] under minimal assumptions on the initial data lie in this class is not clear at this point.) In fact, the companion paper [Hintz 2023b] revisits this latter problem from the edge-b-perspective. We discuss such geometric generalizations in Section 1.2; see also Theorem 9.3.

Our proof of Theorem 1.1 proceeds via an energy estimate in the case $s = 1$ (using a weighted linear combination of V_0 and V_1 as the vector field multiplier) — see Theorem 4.5 — and follows for general s (including real orders above a certain negative threshold) from results on the microlocal propagation of edge-b-regularity; see Section 1.2. If the forcing term f remains in the stated space upon application of up-to-$k \in \mathbb{N}_0$ vector fields V_0, V_1, Ω_a (we shall refer to this as k orders of *b-regularity*[3]), then the solution u has the same extra regularity. Discarding the edge-b-regularity information on u and taking the Ω_a to be generators of rotations (which are thus symmetries of the spacetime), this is essentially the original starting point of the vector field method of Klainerman [1985]; see also [Dafermos and Rodnianski 2010; Moschidis 2016] for recent developments. A minor novelty of our approach is that we can prove b-regularity using arbitrary spherical vector fields; correspondingly, the underlying metric or operator under study need not have any (asymptotic) spherical symmetry at $\mathscr{I}^+$.

[3]This is the general terminology for regularity under repeated application of smooth vector fields, on a manifold with corners, which are tangent to all boundary hypersurfaces. The relevant manifold in the present setting is depicted in Figure 1.2 below.

We remark here that there is a significant difference between edge-b- and b-regularity (besides b-regularity being stronger as far as regularity in the spherical directions is concerned): roughly speaking, edge-b-regularity can be tracked *microlocally* using by now essentially off-the-shelf microlocal techniques, namely, symbolic positive commutator arguments; see Section 4. On the other hand, a satisfactory microlocal framework for b-regularity at $\mathscr{I}^+$ remains elusive; we explain the structural reason in Remark 1.9.

In light of the success of vector field methods, we shall attempt to provide further justification for our insistence on developing a microlocal approach. In order to do so, we turn from the exterior region Ω to the forward causal cone. (There, we can define edge-b-Sobolev spaces in an analogous manner, now using weights in $\rho_+ := 1/(t-r)$ and $x_{\mathscr{I}} := \sqrt{(t-r)/r}$ and testing regularity using $\rho_+\partial_{\rho_+}$, $x_{\mathscr{I}}\partial_{x_{\mathscr{I}}}$, and $x_{\mathscr{I}}\Omega_a$.) Suppose that we are given a metric g on $\mathbb{R}^{1+n}$ which is (approximately) equal to g_0 in a neighborhood $\frac{r}{t} \in (1-\epsilon, 1+\epsilon)$, $r \gg 1$, of $\mathscr{I}^+$. Crucially note then that one *cannot* solve the wave equation $\Box_g u = f$ *locally* near future affine complete subsets of $\mathscr{I}^+$, i.e., in regions of the form $t - r \geq 1$, $\frac{r}{t} \in (1-\epsilon, 1]$. The reason is that the behavior of u in such a region is global in character: it depends on the spacetime geometry (or more generally on the coefficients of the wave-type operator under consideration), and on the forcing f, in a full neighborhood of future timelike infinity I^+ — which includes regions far from $\mathscr{I}^+$.

Remark 1.2 (global character of analysis in $\frac{r}{t} < 1-\epsilon$, $t \gg 1$). We give three examples illustrating this global character.

(1) When solving $\Box_g u = f$ on Minkowski space — with a source term $f(t,x) = f_0(t)f_1(x)$, where $f_0|_{t<0} = 0$ and $f_0|_{t>1} = e^{\mu t}$, $\mu > 0$, and $0 \neq f_1 \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n_x)$ — the solution u typically behaves like $r^{-1}e^{\mu(t-r)}$ for $t - r > 1$: its bound $u \lesssim r^{-1}$ near compact subsets of $(\mathscr{I}^+)^\circ$ (i.e., where $t-r$ is bounded) is determined by local information on the metric (and the source term) near $\mathscr{I}^+$, but its behavior $u \sim r^{-1}e^{\mu(t-r)}$ as $t - r \to \infty$ (i.e., near future affine complete subsets of $\mathscr{I}^+$) is influenced by the source term f far from $\mathscr{I}^+$ (cf. the compact spatial support of f_1).

(2) Taking g to be the Minkowski metric still and supposing $V \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n_x)$ is a potential such that $\Delta + V$ admits a negative eigenvalue, solutions of $(\Box_g + V)u = (-D_t^2 + \Delta + V)u = f$ grow exponentially fast as $t - r \to \infty$ for generic $f \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^{1+n}_{t,x})$ (of compact support *in space and time*).

(3) Even regularity cannot be controlled locally or microlocally near $\mathscr{I}^+$, since past-directed null-geodesics starting at points p with $r \gg t - r \gg 1$ pass through points q with $r \sim 1$, $t \gg 1$. (Thus, if f is singular at q, then typically u is singular at p, where "singular" means a failure of uniform integrability of u or its derivatives with respect to appropriate weights.) This happens for example on Minkowski space and also on spacetimes such as Schwarzschild and Kerr which feature trapping in regions of bounded r.

If even *just one* part of the analysis of $\Box_g$ near future timelike infinity (i.e., in a region where $\frac{r}{t} < 1-\epsilon$ and $t \gg 1$) uses microlocal tools, it is desirable to have a microlocal perspective in a full neighborhood of I^+, including near (a future affine complete part of) $\mathscr{I}^+$. More specifically, microlocal regularity results near critical or invariant sets of the null-geodesic flow (lifted to the cotangent bundle), such as

radial points or normally hyperbolic trapping, take the following form: *if regularity is known on the stable manifold of the critical/invariant set, then it holds at the set itself and thus also on its unstable manifold*. (In the noncompact setting of interest here, "regularity" entails uniform (L^2-)integrability of u and its derivatives.) A microlocal perspective near $\mathscr{I}^+$ is exactly what allows for a clean separation of those null-geodesics, lifted to phase space, which are incoming (i.e., which are on their way to a region far away from $\mathscr{I}^+$ where they may encounter, say, a black hole and normally hyperbolic trapping) and those which are outgoing (i.e., which tend towards $\mathscr{I}^+$).

With this in mind, we briefly describe the microlocal propagation of edge-b-regularity through $\mathscr{I}^+$ in Section 1.2. In order to illustrate the way in which the microlocal analysis near $\mathscr{I}^+$ fits into global analysis in the forward cone, we prove an extension of Theorem 1.1 which proves the membership of waves in weighted edge-b-spaces also near I^+ under suitable assumptions on the global geometry away from $\mathscr{I}^+$ (which are satisfied by the Minkowski metric); see Theorem 9.3. The assumptions on the spacetime away from $\mathscr{I}^+$ are inspired by work of the second author with Baskin and Wunsch [Baskin et al. 2015]; we discuss the relationship to our work in detail in Section 1.1. In this introduction, we content ourselves with the Minkowski setting of Theorem 9.3 (see also Example 9.1) and with extra b-regularity.

Theorem 1.3 (global edge-b-regularity of waves). *Let* $\Omega = \{t \geq 0\}$. *Suppose*

$$\alpha_+ + \tfrac{1}{2} < \alpha_{\mathscr{I}} < \min\big(-\tfrac{1}{2}, \alpha_0 + \tfrac{1}{2}\big),$$

and let $s \in \mathbb{N}$. *Let* $f \in H_{\mathrm{e,b}}^{s-1,(\alpha_0+2,2\alpha_{\mathscr{I}}+2,\alpha_++2)}(\Omega)$. *Then the forward solution* u *of the wave equation* $\Box_{g_0} u = f$ *on Minkowski space satisfies* $u \in H_{\mathrm{e,b}}^{s,(\alpha_0,2\alpha_{\mathscr{I}},\alpha_+)}(\Omega)$. *If* f *enjoys additional* k *orders of b-regularity*,[4] *then* u *enjoys additional* k *degrees of b-regularity as well.*

The best pointwise decay that follows directly from this result via Sobolev embedding is the bound $|u| \lesssim \langle t\rangle^{-(n-1)/2+\epsilon}$ for all $\epsilon > 0$ (provided f has appropriate decay itself); see Remark 9.5.

A more elaborate setting in which the results of the present paper play a crucial role is described in [Hintz 2023d]: the spacetimes considered there have asymptotically stationary regions, in which case the analysis far from $\mathscr{I}^+$ requires yet different microlocal tools which are developed in [Hintz 2023d; 2023e].

1.1. ***Prior work on microlocal analysis near null infinity.*** The radial compactification of $\mathbb{R}^{n+1}$ is the smooth manifold

$$\overline{\mathbb{R}^{n+1}} = (\mathbb{R}^{n+1} \sqcup ([0,\infty) \times \mathbb{S}^n))/\sim, \qquad \mathbb{R}^{n+1} \setminus \{0\} \ni z = R\varpi \sim (R^{-1}, \varpi) \in (0,\infty) \times \mathbb{S}^n, \tag{1-3}$$

with boundary "at infinity" given by $\partial\overline{\mathbb{R}^{n+1}} = \{\tilde\varrho = 0\} \cong \mathbb{S}^n$, $\tilde\varrho = R^{-1}$. (Here, $R^2 = |z|^2$ is defined using any fixed positive definite quadratic form on $\mathbb{R}^{n+1}$, such as the Euclidean one.) All future null-geodesics

[4] That is, $V_1 \cdots V_j f \in H_{\mathrm{e,b}}{}^{s-1,(\alpha_0+2,2\alpha_{\mathscr{I}}+2,\alpha_++2)}(\Omega)$ for all $j = 0, \dots, k$, where the V_i are b-vector fields on the manifold with corners $\widetilde{M}$ defined in (1-6): in $r > 1$ and $0 \leq t \leq 2r$, a basis of these is given by $\langle t-r\rangle(\partial_t - \partial_r)$, $r(\partial_t + \partial_r)$, Ω_a; and in $r = |x| \leq \frac{1}{2}t$, one can take $\langle t\rangle\partial_t$ and $\langle t\rangle\partial_x$.

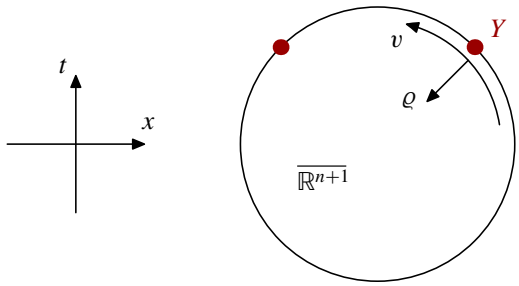

Figure 1.1. The radial compactification of $\mathbb{R}^{n+1}$, and the light cone at infinity $Y \cong \mathbb{S}^{n-1}$, in the case $n = 1$.

on Minkowski space $(\mathbb{R}^{n+1}, g_0)$ limit to the same codimension 1 submanifold $Y \subset \partial\overline{\mathbb{R}^{n+1}}$; in local coordinates

$$\varrho = (t+r)^{-1}, \quad v = \frac{t-r}{t+r}, \quad \omega \in \mathbb{S}^{n-1} \tag{1-4}$$

on $\overline{\mathbb{R}^{n+1}}$ near $\frac{t}{r} \in (0,\infty)$, $t > 0$, this submanifold is given by $\{\varrho = v = 0\}$. See Figure 1.1.

In [Vasy 2013; Baskin et al. 2015, §5], the analysis of the wave operator $\Box_{g_0}$ on Minkowski space $(\mathbb{R}^{n+1}, g_0)$ focuses on its homogeneity of degree -2 under spacetime dilations $(t, x) \mapsto (\lambda t, \lambda x)$, $\lambda > 0$. More specifically, the rescaling $\tilde\varrho^2\Box_{g_0}$ is invariant under dilations in $\tilde\varrho$ in the collar neighborhood $[0,1)_{\tilde\varrho} \times \mathbb{S}^n$ of $\partial\overline{\mathbb{R}^{n+1}}$. In fact, it is a hyperbolic totally characteristic operator, or *b-differential operator* in the terminology of [Melrose 1993], which in the local coordinates ϱ, v, ω from (1-4) means that it is constructed from the b-vector fields $\varrho\partial_\varrho$, ∂_v, ∂_ω (spherical vector fields) — or more invariantly (and globally) smooth vector fields tangent to the boundary of $\overline{\mathbb{R}^{n+1}}$ — and smooth functions on $\overline{\mathbb{R}^{n+1}}$. In the coordinates (1-4) and using $\varrho \sim \tilde\varrho$ instead of $\tilde\varrho$, one has (to leading differential order, and to leading order at $v = 0$)

$$\varrho^{-2}\Box_{g_0} \simeq 4(\varrho D_\varrho + v D_v)D_v + 4\Delta_{\mathbb{S}^{n-1}}; \tag{1-5}$$

see also Example 3.10. One can more generally consider metrics g on $\mathbb{R}^{n+1}$ (and the corresponding rescaled wave operators $\tilde\varrho^2\Box_g$) which have an *asymptotic* homogeneity under dilations and an appropriate structure near Y; these are called *Lorentzian scattering metrics* in [Baskin et al. 2015].

Working with the Minkowski case for definiteness, the operator $\tilde\varrho^2\Box_{g_0}$ is analyzed in [Baskin et al. 2015] (see also [Vasy 2013, §5; 2014]) using microlocal techniques in the b-cotangent bundle, which is an extension of $T^*\mathbb{R}^{n+1}$ to a bundle ${}^{\mathrm{b}}T^*\overline{\mathbb{R}^{n+1}} \to \overline{\mathbb{R}^{n+1}}$ with smooth frame given by the 1-forms dual to the aforementioned b-vector fields. In the characteristic set of $\tilde\varrho^2\Box_{g_0}$, the Hamiltonian vector field of the principal symbol (i.e., the generator of the lifted null-geodesic flow) has a sink over Y, corresponding to the outgoing null-geodesics which tend to Y. Null-bicharacteristics over $\partial\overline{\mathbb{R}^{n+1}}$ may instead also cross Y first[5] and only at a later time tend to Y. Importantly, one can track microlocal b-regularity[6] in the

[5]That is, they are incoming null-geodesics lifted to phase space, or more precisely limits of families of such geodesics. A concrete example, projected to the base $\overline{\mathbb{R}^{n+1}}$, is the limit of $(0,1) \ni s \mapsto (t, r, \omega) = (Ts, T - Ts, \omega_0)$ as $T \nearrow \infty$.

[6]This is b-regularity on $\overline{\mathbb{R}^{1+n}}$ and thus differs from b-regularity on the manifold $\widetilde{M}$ in (1-6) and Figure 1.2 near null infinity. In the coordinates ϱ, v, ω from above, b-regularity on $\overline{\mathbb{R}^{1+n}}$ tests for regularity using $\varrho\partial_\varrho$, ∂_v, ∂_ω, whereas b-regularity on $\widetilde{M}$ uses $\varrho\partial_\varrho$, $\varrho\partial_v$, $v\partial_v$, ∂_ω; note that the v-derivatives here come with a prefactor that vanishes at null infinity.

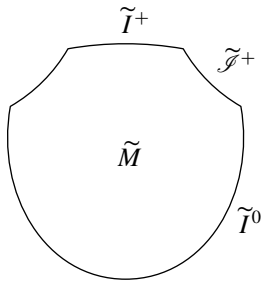

Figure 1.2. The blow-up $\widetilde{M}$ of $\overline{\mathbb{R}^{n+1}}$ at Y and labels for the boundary hypersurfaces.

b-phase space over a full neighborhood of Y, as demonstrated in [Baskin et al. 2015, §4]; the amount of b-regularity at Y is necessarily limited (the issue being limited regularity upon differentiation along the weighted incoming vector field $\partial_v \sim r(\partial_t - \partial_r)$). One can furthermore obtain additional (integer amounts of) *module regularity* at Y (using techniques going back to [Hassell et al. 2008]).[7] In [Vasy and Wrochna 2018; Hintz and Vasy 2015], such a regularity theory on $\overline{\mathbb{R}^{n+1}}$ was used to describe asymptotic data for Feynman propagators and to solve semilinear wave equations.

Equipped with this b- and module regularity, the second author with Baskin and Wunsch [Baskin et al. 2015] (see [Baskin et al. 2018] for a more general class of metrics) obtains a full asymptotic expansion (i.e., the polyhomogeneity) of solutions u of the wave equation on a resolution of $\overline{\mathbb{R}^{n+1}}$ defined by blowing up Y [Melrose 1996]. Recall that passage to the blow-up

$$\widetilde{M} := [\overline{\mathbb{R}^{n+1}}; Y] \tag{1-6}$$

of $\overline{\mathbb{R}^{n+1}}$ at Y amounts to the introduction of polar coordinates around Y; see Figure 1.2 for an illustration. Local coordinates near the interior of the front face[8] $\widetilde{\mathscr{I}}^+$ are $\varrho = (t+r)^{-1}$, $\frac{v}{\varrho} = t - r$, and $\omega \in \mathbb{S}^{n-1}$; at $\varrho = 0$, this is the usual parametrization of null infinity. The asymptotic expansion of u in particular captures its radiation field, which is (a derivative of) the restriction to $\widetilde{\mathscr{I}}^+$ of the rescaling $r^{(n-1)/2}u$. Coordinates near the past boundary $\widetilde{\mathscr{I}}^+ \cap \widetilde{I}^0$ of null infinity are

$$\rho_0 = \frac{1}{r-t} \geq 0, \quad \rho_{\mathscr{I}} = \frac{r-t}{r}, \quad \omega; \tag{1-7}$$

thus, level sets of ρ_0 are outgoing null cones, whereas level sets $\rho_{\mathscr{I}} = c \in (0, 1)$ are spacelike hypersurfaces with boundary at infinity contained in $\widetilde{I}^0$. See [Hintz and Vasy 2020, §1.1.1] for an extensive discussion.

Now, spacetime metrics g arising from the solution of *quasilinear* wave equations on Minkowski space — a key example being the solutions of the Einstein vacuum equations $\mathrm{Ric}(g) = 0$ with initial data close to those of Minkowski space [Christodoulou and Klainerman 1993; Lindblad and Rodnianski 2010; Hintz and Vasy 2020] — typically are approximately dilation-invariant on spacetime *except* near null infinity; there, the quasilinear waves or perturbed metrics g are instead regular only when described on the resolution $\widetilde{M}$ since the radiation field couples back into the metric. Put differently, rescaled wave operators

[7] This module regularity is in fact equivalent to the b-regularity on $\widetilde{M}$ mentioned earlier.

[8] We use tildes here for consistency with the notation used in Section 1.2 and in the main part of the paper.

$\tilde{\varrho}^2\Box_g$ associated with such $(\mathbb{R}^{n+1}, g)$ (arising, e.g., via linearization of the quasilinear equation) have highly *singular* coefficients if one regards them as b-differential operators on $\overline{\mathbb{R}^{n+1}}$. As a toy example, consider instead of the Minkowski metric $g_0 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \not{g}_{ab}(\omega)\,\mathrm{d}\omega^a\,\mathrm{d}\omega^b$ (where $\not{g}$ denotes the standard metric on $\mathbb{S}^{n-1}$) the metric

$$g = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\not{g}_{ab} + r^{-1}h_{ab}(t-r))\,\mathrm{d}\omega^a\,\mathrm{d}\omega^b;$$

here $h_{ab} \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$. (Such metrics capture features of gravitational radiation.) Analogously to (1-5), the corresponding wave operator is of the schematic form

$$\varrho^{-2}\Box_g \simeq 4(\varrho D_\varrho + v D_v)D_v + 4\Delta_{\not{g}} + 8\varrho h_{ab}\Big(\frac{v}{\varrho}\Big)D_{\omega^a}D_{\omega^b}.$$

The coefficient $\varrho h_{ab}(v/\varrho)$ fails to be smooth precisely at $Y = \{\varrho = v = 0\}$.

Such operators with singular coefficients at $Y \subset \overline{\mathbb{R}^{n+1}}$ are not b-microlocal at Y, and indeed they typically create extra singularities at Y (in the sense of b-wave front sets), rendering a precise microlocal regularity theory very delicate, if not impossible.[9] The point of the present paper is thus to describe a point of view which, unlike the b-setting on $\overline{\mathbb{R}^{n+1}}$, *is* microlocal near null infinity.

Remark 1.4 (geometric singular analysis on $\widetilde{M}$). The analysis of the Einstein equations by Wang [2010] and the authors [Hintz and Vasy 2020; Hintz 2023b], while inspired by the perspective of geometric singular analysis (in particular [Melrose 1993; Mazzeo 1991]) and taking place on $\widetilde{M}$, is not microlocal at $\mathscr{I}^+$ but rather fully relies on energy estimates and adaptations of the vector field method.

Remark 1.5 (Schwarzschild and Kerr metrics). Near $Y \subset \partial\overline{\mathbb{R}^{n+1}}$, Schwarzschild metrics and the more general Kerr metrics are examples of Lorentzian scattering metrics, as shown in [Baskin et al. 2018, Appendix A]. Thus, the b-techniques of [Baskin et al. 2015; 2018] apply there. We emphasize that the question of how to deal with wave equations for metrics that are regular only on $\widetilde{M}$ — which we address in the present paper — is entirely decoupled from the problem of how to deal with the black hole aspects (trapping, horizons) of Kerr. See Example 3.4 and also Remark 9.2.

1.2. ***The edge-b-perspective near null infinity.*** The geometric setup for Theorem 1.1 involves the smooth manifold M with corners which is obtained from $\widetilde{M} = [\overline{\mathbb{R}^{n+1}}; Y]$ via performing a *square root blow-up* of its front face $\widetilde{\mathscr{I}}^+$; that is, $M = \widetilde{M}$ as sets, but a defining function of $\mathscr{I}^+ = \widetilde{\mathscr{I}}^+$ is now given by the square root of a defining function of $\widetilde{\mathscr{I}}^+ \subset \widetilde{M}$. In the local coordinates (1-7) on $\widetilde{M}$, this amounts to regarding $\rho_0 = 1/(r-t) \geq 0$, $x_{\mathscr{I}} = \sqrt{\rho_{\mathscr{I}}} = \sqrt{(r-t)/r} \geq 0$, $\omega \in \mathbb{S}^{n-1}$ as smooth local coordinates on M; cf. (1-1).

Following Melrose [1993] and Mazzeo [1991], we then write $\mathcal{V}_{\mathrm{e,b}}(M)$ for the Lie algebra of smooth vector fields on M which are tangent to all boundary hypersurfaces and which at $\mathscr{I}^+$ are in addition tangent to the fibers of the (blow-down) map $\mathscr{I}^+ \to Y$ given in local coordinates by $(\rho_0, \omega) \mapsto \omega$. This Lie algebra is spanned, over $\mathcal{C}^\infty(M)$, by the vector fields in (1-2). The corresponding classes of (pseudo)differential operators and Sobolev spaces are discussed in Section 2.

[9] This is analogous to how, for example, the solution of a wave equation on $\mathbb{R}^{1+n}$, even if it is initially smooth, typically develops singularities at places where the coefficients of the wave operator are singular.

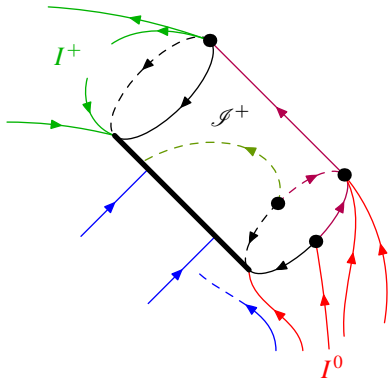

Figure 1.3. The structure of the null-bicharacteristic flow near null infinity in 2+1 spacetime dimensions. The cross sections of the cylinder are cross sections of the future light cones inside of each fiber of the eb-phase space over $\mathscr{I}^+$. The thick black sets are the radial sets (the two antipodal points over $\mathscr{I}^+ \cap I^0$ forming a connected radial set in higher dimensions). See Section 4.1 and Figure 4.1 for details.

An explicit calculation shows that the wave operator on Minkowski space satisfies

$$2\rho_0^{-2} x_{\mathscr{I}}^{-2} \Box_{g_0} \equiv (x_{\mathscr{I}} \partial_{x_{\mathscr{I}}} - (n-1))(x_{\mathscr{I}} \partial_{x_{\mathscr{I}}} - 2\rho_0 \partial_{\rho_0}) + 2x_{\mathscr{I}}^2 \Delta_{g_{\mathbb{S}^{n-1}}}, \tag{1-8}$$

where $\Delta_{g_{\mathbb{S}^{n-1}}}$ is the nonnegative Laplacian on the standard $(n-1)$-sphere, and where we write "$\equiv$" for equality modulo the space $x_{\mathscr{I}} \operatorname{Diff}^2_{\mathrm{e,b}}(M)$ of linear combinations of up-to-twofold products of the vector fields (1-2), with coefficients vanishing at $x_{\mathscr{I}} = 0$. Intimately related to this is the fact that the rescaled Minkowski metric $\rho_0^2 x_{\mathscr{I}}^2 g_0$ is a Lorentzian edge-b-metric, i.e., a nondegenerate Lorentzian signature quadratic form in the 1-forms $\mathrm{d}\rho_0/\rho_0$, $\mathrm{d}x_{\mathscr{I}}/x_{\mathscr{I}}$, $\mathrm{d}\omega/x_{\mathscr{I}}$ dual to the vector fields (1-2). (See (3-7b) for the computation of the dual metric.)

We perform a systematic microlocal analysis of the operator (1-8) in Section 4. The null-bicharacteristic flow of its (edge-b-)principal symbol — which is the same as the lifted null-geodesic flow for the metric $\rho_0^2 x_{\mathscr{I}}^2 g_0$ — has a rather intricate structure involving three radial sets having different source/sink/saddle point structures. (There is a fourth radial set over the future boundary $\mathscr{I}^+ \cap I^+$.) See Figure 1.3. Crucially, the linearization of the Hamiltonian vector field at each of these radial sets is nondegenerate in the normal directions, which allows for a proof of microlocal radial point propagation estimates through (or into) them by means of standard positive commutator methods; see [Melrose 1994, §9; Vasy 2013, §2; Dyatlov and Zworski 2019, Appendix E.4]. Combined with a simple energy estimate on edge-b-spaces, which appeared already in [Hintz and Vasy 2020, §4.1] (albeit in less generality and without identifying the underlying singular geometric structure), we can then prove Theorem 1.1 and its generalizations in Section 6.

The class of operators (or metrics) to which our analysis applies is the natural generalization of $\rho_0^{-2} x_{\mathscr{I}}^{-2} \Box_{g_0}$ within the class of edge-b-operators, in that one can allow for the operators (or metrics) to have additional lower-order terms (in the sense of decay at $x_{\mathscr{I}} = 0$). We more generally consider operators acting on sections of vector bundles, as long as their principal part is that of a scalar wave operator still.

Furthermore, one can allow for the presence of subprincipal terms at $x_{\mathscr{I}} = 0$; a particularly important example for applications to nonlinear stability problems (see for example [Hintz 2023b, Proposition 3.29 and §3.6]) is the replacement of $n-1$ in (1-8) by another constant (or bundle endomorphism), this constant (or the eigenvalues) governing the decay rate(s) towards $\mathscr{I}^+ = \{x_{\mathscr{I}} = 0\}$. (Note that the constant $n-1$ is exactly the exponent in the $x_{\mathscr{I}}^{n-1} \sim r^{-(n-1)/2}$ (for $|t-r| \lesssim 1$) decay rate towards null infinity of waves on Minkowski space.) Our general setup encompasses a large class of asymptotically Minkowskian spacetimes, allowing in particular for the presence of long-range mass terms (as present in Schwarzschild or Kerr metrics) and radiation field-type decay for certain metric coefficients in $3+1$ dimensions compatible with nonlinear stability problems; see also Example 3.4.

Remark 1.6 (parabolic scaling). Via the coordinate change of Friedlander [1980] or via the conformal embedding of Minkowski space into the Einstein cylinder $(\mathbb{R}_s \times \mathbb{S}^n, -\mathrm{d}s^2 + g_{\mathbb{S}^n})$, the Minkowski metric can be conformally rescaled to a metric on $\widetilde{M}$ which is smooth and nondegenerate down to $(\widetilde{\mathscr{I}}^+)^\circ$, with $(\widetilde{\mathscr{I}}^+)^\circ$ becoming a null hypersurface. The edge perspective at null infinity of Minkowski space is then directly related to the following general setup: consider a null hypersurface $\mathcal{I}$ in an $(n+1)$-dimensional Lorentzian manifold. Near a point $p \in \mathcal{I}$, one can then find local coordinates u, v, y such that u, v are null, the hypersurface $\mathcal{I}$ is given by $u = 0$, and at p the spacetime metric is $g_p = \mathrm{d}u\,\mathrm{d}v + \mathrm{d}y^2$. Thus, g_p is homogeneous of degree 2 under the $\mathcal{I}$-preserving parabolic scaling $(u, v, y) \mapsto (\lambda^2 u, v, \lambda y)$. The rescaling $u^{-1} g_p = \frac{\mathrm{d}u}{u}\,\mathrm{d}v + u^{-1}\mathrm{d}y^2$ is scaling-invariant and, passing to $x := \sqrt{u}$ in $u > 0$, gives the metric

$$x^{-2} g_p = 2\frac{\mathrm{d}x}{x}\mathrm{d}v + \frac{\mathrm{d}y^2}{x^2},$$

which is invariant under the homogeneous scaling $(x, v, y) \mapsto (\lambda x, v, \lambda y)$. Thus, in $u > 0$, the rescaling $x^{-2} g_p$ is an edge metric $\big($i.e., a nondegenerate Lorentzian signature expression in $\frac{\mathrm{d}x}{x}$, $\mathrm{d}v$, $\frac{\mathrm{d}y}{x}\big)$ on the manifold $[0, 1)_x \times \mathbb{R}_v \times \mathbb{R}_y^{n-1}$, with the boundary $\mathcal{I} = x^{-1}(0)$ fibered via $(v, y) \mapsto y$.

Remark 1.7 (conormal coefficients at null infinity). The conformal perspective on Minkowski space mentioned in Remark 1.6 largely breaks down when applied to perturbations of Minkowski space in the context of the stability problem in $3+1$ dimensions. (See [Christodoulou 2002; Dafermos 2013] and [Hintz and Vasy 2020, Remark 8.12], but see also [Friedrich 1986] for a more restrictive setting in which a conformal approach does succeed.) However, the conformally rescaled metric typically does have some conormal regularity down to $\widetilde{\mathscr{I}}^+$. It is thus conceivable that one can adapt the methods used in [de Hoop et al. 2015] (see also [Gannot and Wunsch 2023]) for the diffraction of singularities by mildly singular timelike boundaries to the lightlike case.

Remark 1.8 (Klein–Gordon equation). The structure near null infinity of the Klein–Gordon operator $\Box_{g_0} + m^2$, with $m \in \mathbb{R} \setminus \{0\}$, is altogether different; for example, unlike in (1-8) one cannot factor out $\rho_0^{-2} x_{\mathscr{I}}^{-2}$ from $\Box_{g_0} + m^2$. Sussman [2023] thus develops a different microlocal framework for studying regularity and decay of solutions of the Klein–Gordon equation near null infinity (and also globally) on Minkowski space and generalizations thereof; this framework is based on the Lie algebra of "double edge, scattering" vector fields, which have an additional order of vanishing at I^0, $\mathscr{I}^+$, and I^+ compared to the edge-b-vector fields used in the present paper.

The edge-b-microlocal analysis of $\Box_{g_0}$ and its generalizations is supplemented in Section 5 with propagation estimates on edge-b-spaces which carry an additional integer amount of b-regularity on $\widetilde{M}$ (or equivalently on M). This extra b-regularity is captured via testing with vector fields, much as in the aforementioned Klainerman vector field method.

Remark 1.9 (b-perspective). No matter how one changes the smooth structure of $\widetilde{M}$, the operator $\Box_{g_0}$ is not a nondegenerate (weighted) b-differential operator. (For example, the spherical Laplacian term in (1-8) is of lower order in the sense of decay at $x_{\mathscr{I}} = 0$ than the first term.) This means that a b-microlocal analysis of the operator $\Box_{g_0}$ and its perturbations, if possible at all, is delicate due to the degenerate nature of the operator as a b-differential operator.

Besides the microlocal edge-b-regularity theory and the solvability theory away from future timelike infinity I^+, we also show the invertibility of the *edge normal operator* of $\Box_{g_0}$ at $\mathscr{I}^+$, which is an invariant (cf. Remark 1.6) model at each fiber of $\mathscr{I}^+$; see Section 7. The inversion of normal operators is the main ingredient in the development of Fredholm theory for elliptic operators [Mazzeo and Melrose 1987; 1998; Melrose 1993; Mazzeo 1991] and also for nonelliptic operators [Hintz and Vasy 2015, §2; Gell-Redman et al. 2016].

Edge-b-operators such as (a weighted version of) $\Box_{g_0}$ have a dilation-invariant normal operator also at the boundary hypersurface $I^+ \subset M$. In Section 8, we prove microlocal estimates for this normal operator which only use the structure of the underlying wave-type operator near $\mathscr{I}^+$. We indicate how these edge normal operator inverses and normal operator estimates, together with our microlocal regularity theory, can be put to use in the *global* analysis of a wave equation on a class of asymptotically Minkowski spaces in (the proof of) Theorem 9.3. (A significantly more elaborate setting is discussed in [Hintz 2023d].)

1.3. ***Outline of the paper.*** In Section 2, we introduce edge-b-vector fields, (pseudo)differential operators, Sobolev spaces, and related concepts.

In Section 3, we describe the Minkowski metric and its generalizations, called *admissible metrics* in this paper (Definition 3.2), from the edge-b-perspective and also introduce the class of wave-type operators that our methods can handle (Definition 3.7).

The microlocal heart of the paper is described in Sections 4–5.

- In Section 4.1, we describe the null-bicharacteristic flow (the lift of the null-geodesic flow to the edge-b-phase space) near null infinity in detail, identify the critical points of the Hamiltonian vector field (which form smooth invariant submanifolds of radial points lying over null infinity, see Lemma 4.1), and determine the global structure of the null-bicharacteristic flow on the characteristic set (Figure 4.1).
- In Section 4.2, we analyze the admissible operators of Section 3 from an edge-b-microlocal point of view. Concretely, near each of the critical sets $\mathcal{R}$ identified in Section 4.1, we prove a "black box" result (Lemmas 4.8–4.11) on the microlocal propagation of regularity through $\mathcal{R}$ which states, in a quantitative fashion, that (under appropriate conditions on regularity and decay orders) microlocal regularity of a solution u of a wave-type equation $Pu = f$ on the stable manifold of $\mathcal{R}$ implies

microlocal regularity at $\mathcal{R}$ itself (and thus also along the unstable manifold via standard propagation estimates [Duistermaat and Hörmander 1972]). The concatenation of these results gives semiglobal regularity results (Theorems 4.5 and 4.6).

- Edge-b-microlocal estimates on function spaces which capture additional (integer amounts of) b-regularity are proved in Section 5 via simple commutation arguments (Propositions 5.14–5.16). The relevant space of *commutator b-vector fields* (more general than those useable in the traditional vector field method) are identified in Definition 5.1; see also Proposition 5.3.

In Section 6, we prove the solvability of wave equations away from the future boundary of $\mathscr{I}^+$, i.e., up to a spacelike hypersurface transversal to $\mathscr{I}^+ \setminus I^+$ (with Theorem 1.1 being a special case). Energy estimates produce solutions with edge-b-regularity order 1, and the microlocal propagation results from Section 4 generalize this to general orders which in particular are allowed to be arbitrarily large (Theorem 6.4). Using commutations with commutator b-vector fields, we also prove higher b-regularity (Corollary 6.6).

The next two sections (Sections 7–8) develop material which is needed for controlling the decay of solutions of wave equations near future timelike infinity I^+. Since our assumptions on metrics and operators only concern a neighborhood of $\mathscr{I}^+$, we only prove localized estimates near $\mathscr{I}^+$. Concretely, in Section 7, we prove the invertibility of the edge normal operators of the wave operators of interest at (the fibers of) $\mathscr{I}^+ \setminus I^0$ (Proposition 7.1). In Section 8, we study the b-normal operator at I^+ which, roughly speaking, is dilation-invariant in $\frac{1}{t-r}$ and can thus be studied using the Mellin transform; we then prove regularity results for the Mellin-transformed normal operator family at bounded (Theorem 8.2) and high (Theorem 8.5) Mellin-dual frequencies.

An application to the global existence, regularity, and decay of waves on a class of asymptotically Minkowskian spacetimes is given in Section 9. In particular, the concrete setting described there complements the assumptions near $\mathscr{I}^+$ made in all previous sections by assumptions on the metric and wave operator far from $\mathscr{I}^+$, in particular near I^+. For simplicity, we only discuss metrics with approximate dilation-invariance near $(I^+)^\circ$, which includes asymptotically Minkowski spacetimes such as those arising in [Hintz and Vasy 2020] but excludes spacetimes which settle down to stationary spacetimes in spatially compact regions. (The latter are discussed in [Hintz 2023d].)

2. Edge-b-geometry and analysis

We begin by recalling, in Section 2.1, the Lie algebra of edge-b-vector fields on a manifold with corners and a fibered boundary hypersurface. The corresponding spaces of differential operators are defined in Section 2.1 as well, and their normal operators are described in Section 2.2. The algebra of edge-b-pseudodifferential operators is recalled in Section 2.3. We will use these operator algebras for the (microlocal) analysis near null infinity. Sobolev spaces are discussed in Section 2.4, and various notions required for the study of certain normal operators of edge-b-operators are defined in Sections 2.5–2.6. The material in this section is largely based on [Melrose 1993; Mazzeo 1991; Melrose et al. 2013].

2.1. ***Vector fields, differential operators, bundles.*** Let M denote an n-dimensional manifold with corners. Denote by $H_1, \dots, H_N \subset \partial M$, $N \in \mathbb{N}$, the collection of its boundary hypersurfaces which we require to be embedded submanifolds of M. Recall then:

Definition 2.1 (Lie algebras of vector fields). Denote by $\mathcal{V}(M) = \mathcal{C}^\infty(M; TM)$ the Lie algebra of smooth vector fields on M.

(1) The space $\mathcal{V}_{\mathrm{b}}(M) \subset \mathcal{V}(M)$ of *b-vector fields* [Melrose 1981; 1993] consists of all $V \in \mathcal{V}(M)$ which are tangent to ∂M, i.e., to H_j for each $j = 1, \dots, N$.

(2) If $N = 1$, then the space $\mathcal{V}_0(M) \subset \mathcal{V}(M)$ of *0-vector fields* [Mazzeo and Melrose 1987] is defined as $\mathcal{V}_0(M) = \{\rho V : V \in \mathcal{V}(M)\}$, where $\rho \in \mathcal{C}^\infty(M)$ is a boundary-defining function (that is, $\rho \geq 0$ vanishes only at ∂M and has nonvanishing differential there).

(3) If $N = 1$, then we define the space $\mathcal{V}_{\mathrm{sc}}(M) \subset \mathcal{V}_{\mathrm{b}}(M)$ of *scattering vector fields* [Melrose 1994] as $\mathcal{V}_{\mathrm{sc}}(M) = \{\rho V : V \in \mathcal{V}_{\mathrm{b}}(M)\}$, where ρ is a boundary-defining function.

(4) If $N = 1$ and $\partial M = H_1$ is the total space of a smooth fibration $Z - \partial M \to Y$, then the space $\mathcal{V}_{\mathrm{e}}(M) \subset \mathcal{V}_{\mathrm{b}}(M)$ of *edge vector fields* [Mazzeo 1991] consists of all b-vector fields which at ∂M are tangent to the fibers of ∂M.

(5) If $N \geq 2$ and some (but not all) H_j are total spaces of fibrations, the space $\mathcal{V}_{\mathrm{e,b}}(M) \subset \mathcal{V}_{\mathrm{b}}(M)$ of *edge-b-vector fields* consists of all b-vector fields which at each fibered boundary hypersurface are tangent to the fibers; see also [Melrose et al. 2013; Albin and Gell-Redman 2023; Hintz 2023e].

In local coordinates $x \in [0, \infty)^k$ and $y \in \mathbb{R}^{n-k}$ near a point inside a codimension k corner, with ∂M locally given by $x = 0$, b-vector fields are linear combinations with smooth coefficients of

$$x^i \partial_{x^i} \ (i = 1, \dots, k), \quad \partial_{y^j} \ (j = 1, \dots, n-k). \tag{2-1}$$

When $N = 1$ and $k = 1$, this frame becomes $x\partial_x$, ∂_{y^j}; the space of scattering vector fields is then spanned by

$$x^2 \partial_x, \quad x\partial_{y^j} \ (j = 1, \dots, n-1). \tag{2-2}$$

Correspondingly, there are smooth vector bundles

$${}^{\mathrm{b}}TM \to M, \quad {}^{\mathrm{sc}}TM \to M,$$

and bundle maps ${}^{\mathrm{b}}TM \to TM$, ${}^{\mathrm{sc}}TM \to TM$, which over the interior M° are isomorphisms, such that $\mathcal{V}_{\mathrm{b}}(M) = \mathcal{C}^\infty(M; {}^{\mathrm{b}}TM)$ and $\mathcal{V}_{\mathrm{sc}}(M) = \mathcal{C}^\infty(M; {}^{\mathrm{sc}}TM)$; a smooth basis of the fibers of ${}^{\mathrm{b}}TM$ and ${}^{\mathrm{sc}}TM$ is given, in the respective settings, by (2-1) and (2-2). The dual bundles are denoted by ${}^{\mathrm{b}}T^*M$ and ${}^{\mathrm{sc}}T^*M$, and the corresponding density bundles are denoted by ${}^{\mathrm{b}}\Omega M$ and ${}^{\mathrm{sc}}\Omega M$.

As an important special case, consider the radial compactification $\overline{\mathbb{R}^n}$ of $\mathbb{R}^n$, defined in (1-3). Denoting by $x^1, \dots, x^n$ standard coordinates on $\mathbb{R}^n$, then in the closure of the region where, say, x^1 is relatively large, meaning $x^1 \geq c|x^j|$ for $j = 2, \dots, n$ and some $c > 0$, we can use $\rho = 1/x^1$, $y^j = x^j/x^1$ as smooth local coordinates on $\overline{\mathbb{R}^n}$; since $\partial_{x^1} = -\rho^2\partial_\rho - \rho \sum_{j=2}^n y^j \partial_{y^j}$, $\partial_{x^j} = \rho\partial_{y^j}$, one finds that translation-invariant vector fields on $\mathbb{R}^n$ are special cases of scattering vector fields on $\overline{\mathbb{R}^n}$, and indeed $\mathcal{V}_{\mathrm{sc}}(\overline{\mathbb{R}^n})$ is spanned

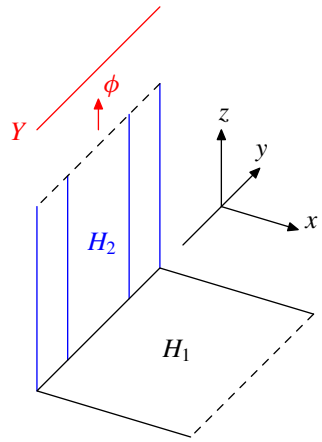

Figure 2.1. A neighborhood of $H_1 \cap H_2$ inside M and the local coordinates x, y, z used in (2-3b). The fibration ϕ of H_2 and the base Y are indicated in red and the fibers of H_2 in blue.

over $\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ by such translation-invariant vector fields. Similarly then, the coordinate differentials $\mathrm{d}x^1, \dots, \mathrm{d}x^n$ extend by continuity from $\mathbb{R}^n$ to give a basis of ${}^{\mathrm{sc}}T^*\overline{\mathbb{R}^n}$, and a basis of ${}^{\mathrm{sc}}\Omega\overline{\mathbb{R}^n}$ is given by the Euclidean volume density $|\mathrm{d}x^1 \cdots \mathrm{d}x^n|$.

For the sake of notational simplicity, we discuss the edge-b-setting only in the special case of interest in this paper. Thus, we assume that $H_2 \subset M$ is the total space of a fibration $Z - H_2 \xrightarrow{\phi} Y$, where $Z \cong [-1, 1]$ is a closed interval and Y is a compact $(n-2)$-dimensional manifold without boundary. Moreover, we assume that $H_j \cap H_2 = \varnothing$ except for $j = 1, 3$, and H_j for $j \neq 2$ is not fibered.[10] See Figure 2.1. A local coordinate description is as follows:

(1) Near the interior H_2° of H_2, we can choose local coordinates $x \in [0, \infty)$, $y \in \mathbb{R}^{n-2}$, $z \in \mathbb{R}$ on M, with y, z local coordinates on Y, Z, so that H_2 is locally given by $x^{-1}(0)$ and the fibration of H_2 takes the form $(y, z) \mapsto y$. Edge-b-vector fields on M are then smooth linear combinations of

$$x\partial_x, \quad x\partial_{y^j} \ (j = 1, \dots, n-2), \quad \partial_z. \tag{2-3a}$$

(2) Near $H_1 \cap H_2$ (and analogously near $H_3 \cap H_2$), we can choose local coordinates $x, z \in [0, \infty)$ and $y \in \mathbb{R}^{n-2}$ with the same properties as above, and so that in addition H_1 is locally given by $z = 0$. Since b-vector fields are spanned by $x\partial_x$, ∂_{y^j}, $z\partial_z$, the space of edge-b-vector fields is now spanned by

$$x\partial_x, \quad x\partial_{y^j} \ (j = 1, \dots, n-2), \quad z\partial_z. \tag{2-3b}$$

Again, we conclude that $\mathcal{V}_{\mathrm{e,b}}(M)$ is equal to the space of sections of a smooth vector bundle ${}^{\mathrm{e,b}}TM \to M$, with local frames near H_2 given by (2-3a)–(2-3b). The *edge-b-cotangent bundle* ${}^{\mathrm{e,b}}T^*M$ will play a key role in the present paper as the phase space for the microlocal analysis of edge-b-differential operators.

[10] Equivalently, H_j, $j \neq 2$, is equipped with the trivial fibration whose base is a singleton set. But we prefer to speak of b-behavior at H_j, $j \neq 2$, rather than of an extreme type of edge behavior.

The spaces of vector fields in Definition 2.1 are Lie algebras; we use the notation

$$\bullet \in \{\mathrm{b},\ 0,\ \mathrm{sc},\ \mathrm{e},\ \mathrm{e,b}\}.$$

The space of locally finite linear combinations of up-to-m-fold compositions of elements of $\mathcal{V}_\bullet(M)$ is denoted by $\mathrm{Diff}^m_\bullet(M)$; we put $\mathrm{Diff}_\bullet(M) = \bigoplus_{m\in\mathbb{N}_0} \mathrm{Diff}^m_\bullet(M)$. Given a weight $\alpha \in \mathbb{R}$, or in the case that M has $N \geq 2$ boundary hypersurfaces a vector $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$ of weights, we put

$$\mathrm{Diff}^{m,\alpha}_\bullet(M) := \{\rho^{-\alpha} P : P \in \mathrm{Diff}^m_\bullet(M)\},$$

where ρ is a boundary-defining function when $N = 1$ and a collection $\rho = (\rho_1, \ldots, \rho_N)$, where ρ_j is a defining function of H_j, when $N \geq 2$; in the latter case, we use the notation $\rho^{-\alpha} := \prod_{j=1}^N \rho_j^{-\alpha_j}$. Since for any b-vector field $V \in \mathcal{V}_\mathrm{b}(M)$ one has $\rho^\alpha V(\rho^{-\alpha}) \in \mathcal{C}^\infty(M)$ (cf. (2-1)), it follows that one can compose elements of $\bigoplus_{m,\alpha} \mathrm{Diff}^{m,\alpha}_\bullet(M)$, and the orders of a composition are the sums of the orders of the individual factors.

More generally, given a weight (vector) α, we can consider the space of L^∞-conormal functions on M,

$$\mathcal{A}^\alpha(M) := \{u \in \rho^\alpha L^\infty(M^\circ) : Pu \in \rho^\alpha L^\infty(M^\circ)\ \forall\ P \in \mathrm{Diff}_\mathrm{b}(M)\}.$$

One can then define the space

$$\mathcal{A}^{-\alpha}\,\mathrm{Diff}^m_\bullet(M)$$

of $\bullet$-differential operators with coefficients in $\mathcal{A}^{-\alpha}(M)$ to consist of locally finite linear combinations of operators of the form wP, where $w \in \mathcal{A}^{-\alpha}(M)$ and $P \in \mathrm{Diff}_\bullet(M)$. Since $\rho^{-\alpha} \in \mathcal{A}^{-\alpha}(M)$, we have $\mathcal{A}^{-\alpha}\,\mathrm{Diff}^m_\bullet(M) \supset \mathrm{Diff}^{m,\alpha}_\bullet(M)$. The space $\bigoplus_{\alpha,m} \mathcal{A}^{-\alpha}\,\mathrm{Diff}^m_\bullet(M)$ is an algebra, and the orders are additive under composition.

We shall encounter further variants of the spaces $\mathcal{A}^\alpha(M)$: at some hypersurfaces $H_{i_1}, \ldots, H_{i_k}$, we may require classical conormality, i.e., smoothness upon multiplication by $\rho_i^{-\alpha_i}$. Ordering indices so that $i_j = j,\ j = 1, \ldots, k$, we thus introduce the notation

$$\mathcal{A}^{((\alpha_1,0),(\alpha_2,0),\ldots,(\alpha_k,0),\alpha_{k+1},\ldots,\alpha_N)}(M) := \left(\prod_{j=1}^N \rho_j^{\alpha_j}\right)\mathcal{A}^{((0,0),\ldots,(0,0),0,\ldots,0)}(M) \tag{2-4}$$

for spaces of mixed conormal and classical conormal functions; here $\mathcal{A}^{((0,0),\ldots,(0,0),0,\ldots,0)}(M) \subset \mathcal{A}^0(M)$ consists of all $u \in \mathcal{A}^0(M)$ such that $V_1 \cdots V_J u \in L^\infty(M)$ for all $J \in \mathbb{N}$ and $V_j \in \mathcal{V}(M)$ which are tangent to $H_{k+1}, \ldots, H_N$ (but not necessarily to $H_1, \ldots, H_k$). Thus, elements of $\mathcal{A}^{((0,0),\ldots,(0,0),0,\ldots,0)}(M)$ are smooth down to H_j for $j = 1, \ldots, k$ and bounded conormal at H_j for $j = k+1, \ldots, N$.[11] One can then also consider spaces of differential operators with coefficients in these mixed conormal spaces.

[11] One can consider even more general spaces, e.g., $\mathcal{A}^{(\mathcal{E}_1,\ldots,\mathcal{E}_k,\alpha_{k+1},\ldots,\alpha_N)}(M)$, where the $\mathcal{E}_j \subset \mathbb{C}\times\mathbb{N}_0$ are index sets. Elements of this space are polyhomogeneous at the boundary hypersurfaces $H_1, \ldots, H_k$ and conormal at $H_{k+1}, \ldots, H_N$. The spaces (2-4) arise as special cases where the index sets are $\mathcal{E}_j = \{(\alpha_j + l, 0) : l \in \mathbb{N}_0\}$. Since we only need the spaces (2-4), we do not go into the details of more general partially polyhomogeneous spaces here.

Finally, if $E, F \to M$ are two smooth vector bundles, one can consider the space

$$\mathrm{Diff}_{\bullet}^{m}(M; E, F) = \mathrm{Diff}_{\bullet}^{m}(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{C}^\infty(M; \mathrm{Hom}(E, F))$$

of differential operators mapping sections of E into sections of F; in local trivializations of E and F, these are simply (rank F) $\times$ (rank E) matrices of elements of $\mathrm{Diff}_{\bullet}^{m}(M)$. Versions of these spaces with weights or conormal coefficients are defined similarly.

2.2. ***Symbols and normal operators of edge-b-differential operators.*** We now let M denote a compact n-dimensional manifold with corners, with embedded hypersurfaces $H_1, \dots, H_N$, $N \geq 3$, and with H_2 the total space of a fibration $Z - H_2 \to Y$, where $Z \cong [-1, 1]$ and Y is compact without boundary, and with $H_j \cap H_2 = \varnothing$ unless $j = 1, 3$. This is the setting discussed around (2-3a)–(2-3b) and illustrated in Figure 2.1. Let us denote defining functions of H_j by $\rho_j \in \mathcal{C}^\infty(M)$. Note that the boundary hypersurfaces of the edge-b-cotangent bundle ${}^{\mathrm{e,b}}T^*M$ are ${}^{\mathrm{e,b}}T^*_{H_j}M$, and the fibration $\phi : H_2 \to Y$ induces a fibration ${}^{\mathrm{e,b}}T^*_{H_2}M \to Y$ (with fibers given by the restriction of ${}^{\mathrm{e,b}}T^*M$ to the fibers of ϕ) via composition with the projection map ${}^{\mathrm{e,b}}T^*_{H_2}M \to H_2$; thus, there is a natural notion of edge-b-vector fields on ${}^{\mathrm{e,b}}T^*M$.

Let us work in the local coordinates (2-3b), so x and z are defining functions of H_2 and H_1, respectively, and y denotes local coordinates on Y. A local frame of ${}^{\mathrm{e,b}}T^*M$ is thus

$$\frac{\mathrm{d}x}{x}, \quad \frac{\mathrm{d}y^j}{x} \ (j = 1, \dots, n-2), \quad \frac{\mathrm{d}z}{z}.$$

Writing the canonical 1-form on T^*M° in the form

$$\xi \frac{\mathrm{d}x}{x} + \sum_{j=1}^{n-2} \eta_j \frac{\mathrm{d}y^j}{x} + \zeta \frac{\mathrm{d}z}{z} \tag{2-5}$$

defines smooth fiber-linear coordinates on T^*M° which extend by continuity to fiber-linear coordinates on ${}^{\mathrm{e,b}}T^*M$ over the local coordinate patch. Edge-b-vector fields on ${}^{\mathrm{e,b}}T^*M$ are linear combinations, with $\mathcal{C}^\infty({}^{\mathrm{e,b}}T^*M)$ coefficients, of the vector fields (2-3b) and ∂_ξ, ∂_{η_j}, ∂_ζ. We first elucidate the symplectic structure of T^*M° from the edge-b-perspective.

Lemma 2.2 (Hamiltonian vector field on ${}^{\mathrm{e,b}}T^*M$). *Let $p \in \mathcal{C}^\infty({}^{\mathrm{e,b}}T^*M)$. Then the Hamiltonian vector field H_p satisfies $H_p \in \mathcal{V}_{\mathrm{e,b}}({}^{\mathrm{e,b}}T^*M)$. The map $p \mapsto H_p$ is a first-order edge-b-differential operator,*

$$H_{(-)} \in \mathrm{Diff}^1_{\mathrm{e,b}}({}^{\mathrm{e,b}}T^*M; \underline{\mathbb{C}}, {}^{\mathrm{e,b}}T({}^{\mathrm{e,b}}T^*M)), \tag{2-6}$$

where $\underline{\mathbb{C}}$ denotes the trivial bundle $M \times \mathbb{C}$. In the local coordinates (2-5), we have

$$\begin{aligned} H_p = (\partial_\xi p)\Big(x\partial_x + \sum_{j=1}^{n-2} \eta_j \partial_{\eta_j}\Big) + \sum_{j=1}^{n-2} (\partial_{\eta_j} p) x \partial_{y^j} + (\partial_\zeta p) z \partial_z \\ - \Big(\Big(x\partial_x + \sum_{j=1}^{n-2} \eta_j \partial_{\eta_j}\Big) p\Big)\partial_\xi - \sum_{j=1}^{n-2} (x\partial_{y^j} p)\partial_{\eta_j} - (z\partial_z p)\partial_\zeta. \end{aligned} \tag{2-7}$$

Proof. Taking the exterior derivative of (2-5), the symplectic form ω on T^*M° is

$$\omega = \left(\mathrm{d}\xi + \sum_{j=1}^{n-2} \frac{\eta_j \,\mathrm{d}y^j}{x}\right) \wedge \frac{\mathrm{d}x}{x} + \sum_{j=1}^{n-2} \mathrm{d}\eta_j \wedge \frac{\mathrm{d}y^j}{x} + \mathrm{d}\zeta \wedge \frac{\mathrm{d}z}{z}.$$

The definition

$$\omega(-, H_p) = \mathrm{d}p = (x\partial_x p)\frac{\mathrm{d}x}{x} + \sum (x\partial_{y^j} p)\frac{\mathrm{d}y^j}{x} + (z\partial_z p)\frac{\mathrm{d}z}{z} + (\partial_\xi p)\,\mathrm{d}\xi + (\partial_\eta p)\,\mathrm{d}\eta + (\partial_\zeta p)\,\mathrm{d}\zeta$$

of H_p then gives the expression (2-7). This expression also implies (2-6); more invariantly, (2-6) follows from the facts that $\omega \in \mathcal{C}^\infty({}^{\mathrm{e,b}}T^*M;\, \Lambda^2\, {}^{\mathrm{e,b}}T^*({}^{\mathrm{e,b}}T^*M))$ is a nondegenerate edge-b-2-form and the exterior derivative d satisfies $\mathrm{d} \in \mathrm{Diff}^1_{\mathrm{e,b}}(M;\, \underline{\mathbb{C}},\, {}^{\mathrm{e,b}}T^*M)$. □

We pause to describe H_p for homogeneous p; for this purpose, we denote by $\overline{{}^{\mathrm{e,b}}T^*}M$ the fiber-radial compactification of ${}^{\mathrm{e,b}}T^*M$. (Since the $\mathrm{GL}(n)$-action on $\mathbb{R}^n$ extends to an action on $\overline{\mathbb{R}^n}$ by diffeomorphisms, $\overline{{}^{\mathrm{e,b}}T^*}M$ carries the structure of a smooth closed ball bundle over M.) The only one of its boundary hypersurfaces which we regard as the total space of a fibration is $\overline{{}^{\mathrm{e,b}}T^*_{H_2}}M$; the boundary ${}^{\mathrm{e,b}}S^*M$ at fiber infinity is not fibered, i.e., edge-b-vector fields on $\overline{{}^{\mathrm{e,b}}T^*}M$ are merely required to be tangent to ${}^{\mathrm{e,b}}S^*M$.

Corollary 2.3 (rescaled Hamiltonian vector field). *Let $\rho_\infty \in \mathcal{C}^\infty(\overline{{}^{\mathrm{e,b}}T^*}M)$ be a defining function of ${}^{\mathrm{e,b}}S^*M$, and suppose $p \in \mathcal{C}^\infty({}^{\mathrm{e,b}}T^*M \setminus o)$ is homogeneous of degree s with respect to (positive) dilations in the fibers. Then $\rho_\infty^{s-1} H_p \in \mathcal{V}_{\mathrm{e,b}}(\overline{{}^{\mathrm{e,b}}T^*}M \setminus o)$.*

Proof. The conclusion is independent of the choice of ρ_∞. We shall work in local coordinates in which (2-7) is valid, and in a region where ξ is relatively large, i.e., $\xi > c(|\eta| + |\zeta|)$ for some $c > 0$. (Regions where some η_j or ζ are relatively large are analyzed similarly.) There, we shall take

$$\rho_\infty = \xi^{-1}, \quad \hat\eta = \frac{\eta}{\xi}, \quad \hat\zeta = \frac{\zeta}{\xi}.$$

Writing $p = \rho_\infty^{-s} p_0$, where p_0 is a smooth function of $(x, y, z, \hat\eta, \hat\zeta)$, we then compute

$$\begin{aligned}\rho_\infty^{s-1} H_p = -\big(((\rho_\infty\partial_{\rho_\infty} - s) + \hat\eta\partial_{\hat\eta} + \hat\zeta\,\partial_{\hat\zeta})p_0\big)(x\partial_x + \hat\eta\partial_{\hat\eta}) + (\partial_{\hat\eta} p_0)\cdot x\partial_y + (\partial_{\hat\zeta} p_0) z\partial_z \\ + ((x\partial_x + \hat\eta\partial_{\hat\eta})p_0)(\rho_\infty\partial_{\rho_\infty} + \hat\eta\partial_{\hat\eta} + \hat\zeta\,\partial_{\hat\zeta}) - (x\partial_y p_0)\cdot\partial_{\hat\eta} - (z\partial_z p_0)\partial_{\hat\zeta}.\end{aligned}$$

Since $\mathcal{V}_{\mathrm{e,b}}(\overline{{}^{\mathrm{e,b}}T^*}M)$ is locally spanned by the vector fields $x\partial_x$, ∂_y, $z\partial_z$, $\rho_\infty\partial_{\rho_\infty}$, $\partial_{\hat\eta}$, and $\partial_{\hat\zeta}$, this proves the corollary. □

Let now $P \in \mathrm{Diff}^m_{\mathrm{e,b}}(M)$. In the above local coordinates, we have

$$P = \sum_{j+k+|\alpha|\le m} a_{jk\alpha}(x, y, z)(xD_x)^j (xD_y)^\alpha (zD_z)^k, \quad a_{jk\alpha} \in \mathcal{C}^\infty(M). \tag{2-8}$$

It has a well-defined (independent of the choice of local coordinates x, y, z) principal symbol

$${}^{\mathrm{e,b}}\sigma^m(P) := \sum_{j+k+|\alpha|=m} a_{jk\alpha}(x, y, z)\xi^j\eta^\alpha\zeta^k \in P^{[m]}({}^{\mathrm{e,b}}T^*M),$$

where $P^{[m]}({}^{\mathrm{e,b}}T^*M)$ denotes the space of smooth functions on ${}^{\mathrm{e,b}}T^*M$ which are homogeneous polynomials of degree m in the fibers. We have a short exact sequence

$$0 \to \mathrm{Diff}^{m-1}_{\mathrm{e,b}}(M) \hookrightarrow \mathrm{Diff}^{m}_{\mathrm{e,b}}(M) \xrightarrow{{}^{\mathrm{e,b}}\sigma^m} P^{[m]}({}^{\mathrm{e,b}}T^*M) \to 0. \tag{2-9}$$

Given $A_j \in \mathrm{Diff}^{m_j}_{\mathrm{e,b}}(M)$ with principal symbols $a_j = {}^{\mathrm{e,b}}\sigma^{m_j}(A_j)$, we have the usual properties

$${}^{\mathrm{e,b}}\sigma^{m_1+m_2}(A_1A_2) = a_1a_2, \quad {}^{\mathrm{e,b}}\sigma^{m_1+m_2-1}(i[A_1, A_2]) = H_{a_1}a_2.$$

Note that, in general, the symbol of a commutator is nonzero at ∂M. Thus, just like the b- and edge-algebras, the algebra of edge-b-differential operators is commutative to leading order only in the differential order sense, but not in the sense of decay at any of the boundary hypersurfaces of ∂M. We thus proceed to recall the definitions of the normal operators of $P \in \mathrm{Diff}^m_{\mathrm{e,b}}(M)$. We shall only discuss the normal operators at H_1 and H_2; the normal operator at H_3 is defined exactly as the one at H_1.

The normal operator at H_1 is defined by freezing coefficients at H_1; in terms of the local coordinates used in (2-8), this means setting

$$N_{H_1}(P) := \sum_{j+k+|\alpha|\leq m} a_{jk\alpha}(x, y, 0)(xD_x)^j(xD_y)^\alpha(zD_z)^k. \tag{2-10}$$

This can be defined invariantly as an element $N_{H_1}(P) \in \mathrm{Diff}^m_{\mathrm{b},I}({}^+NH_1)$, where ${}^+NH_1$ is the (nonstrictly) inward pointing normal bundle of H_1, and the subscript "I" restricts to operators which are invariant under the dilation action in the fibers of ${}^+NH_1$. Indeed, $N_{H_1}(P)$ can be defined for general b-differential operators $P \in \mathrm{Diff}^m_{\mathrm{b}}(M)$ as the multiplicative extension of the map [Melrose 1993, §4.15]

$$\mathcal{V}_{\mathrm{b}}(M) \to \mathcal{V}_{\mathrm{b}}(M)/z\mathcal{V}_{\mathrm{b}}(M) = \mathcal{C}^\infty(H_1; {}^{\mathrm{b}}T_{H_1}M) \cong \mathcal{V}_{\mathrm{b},I}({}^+NH_1).$$

In order to sharpen the description of $N_{H_1}(P)$ for edge-b-differential operators, note that the restriction of the fibration $H_2 \to Y$ to $H_1 \cap H_2$ is a diffeomorphism $H_1 \cap H_2 \xrightarrow{\cong} Y$, and hence the fibers of $H_1 \cap H_2$ are points. Correspondingly, the boundary hypersurface ${}^+N_{H_1\cap H_2}H_1$ of ${}^+NH_1$ is the total space of a fibration given by the base projection

$${}^+N_{H_1\cap H_2}H_1 \to H_1 \cap H_2 \cong Y.$$

We can thus consider the spaces

$$\mathcal{V}_{\mathrm{e,b},I}({}^+NH_1), \quad \mathrm{Diff}^m_{\mathrm{e,b},I}({}^+NH_1)$$

of dilation-invariant edge-b-vector fields and edge-b-operators. The local coordinate expression (2-10) shows directly the following.

Definition 2.4 (b-normal operator at H_1). The map $N_{H_1} : \mathrm{Diff}_{\mathrm{e,b}}(M) \to \mathrm{Diff}_{\mathrm{e,b},I}({}^+NH_1)$ is an algebra homomorphism and for each $m \in \mathbb{N}_0$ fits into the short exact sequence

$$0 \to \rho_1 \mathrm{Diff}^m_{\mathrm{e,b}}(M) \hookrightarrow \mathrm{Diff}^m_{\mathrm{e,b}}(M) \xrightarrow{N_{H_1}} \mathrm{Diff}^m_{\mathrm{e,b},I}({}^+NH_1) \to 0, \tag{2-11}$$

where we recall that $\rho_1 \in \mathcal{C}^\infty(M)$ is a defining function of H_1.

Upon fixing a trivialization

$$ {}^{+}NH_1 \cong H_1 \times [0,\infty)_{\rho_1}, \tag{2-12} $$

we can consider the action of $N_{H_1}(P)$ on functions of the form $\rho_1^{i\zeta}u$, $u \in \mathcal{C}^\infty(H_1)$,

$$ \widehat{N_{H_1}}(P,\zeta)u := (\rho_1^{-i\zeta} N_{H_1}(P)(\rho_1^{i\zeta}u))|_{\rho_1=0}. $$

In the local coordinates (2-10) and with $\rho_1 = z$, we have

$$ \widehat{N_{H_1}}(P,\zeta) = \sum_{j+k+|\alpha|\le m} a_{jk\alpha}(x,y,0)\zeta^k (xD_x)^j (xD_y)^\alpha \in \mathrm{Diff}_0^m(H_1), $$

which is thus a 0-differential operator [Mazzeo and Melrose 1987] (i.e., an edge differential operator with respect to the fibration of the boundary given by the identity map $\partial H_1 \to \partial H_1$).

Definition 2.5 (Mellin-transformed normal operator family). Fix a trivialization (2-12). For $P \in \mathrm{Diff}^m_{\mathrm{e,b}}(M)$, the family $\widehat{N_{H_1}}(P,\zeta)$, $\zeta \in \mathbb{C}$, of 0-differential operators on H_1 is the *Mellin-transformed normal operator family* of P at H_1.

We now turn to the edge normal operator at H_2, which is a family of operators on the fiber Z, parametrized by the base Y of the fibration $\phi : H_2 \to Y$. It is defined by freezing coefficients at a fiber $\phi^{-1}(y_0)$, $y_0 \in Y$; here we are following [Mazzeo 1991, (2.17)]. In local coordinates as in (2-8), this means setting

$$ {}^{\mathrm{e}}N_{H_2,y_0}(P) := \sum_{j+k+|\alpha|\le m} a_{jk\alpha}(0,y_0,z)(xD_x)^j (xD_y)^\alpha (zD_z)^k. \tag{2-13} $$

Here (x,y) is now allowed to range over all of $[0,\infty)\times\mathbb{R}^{n-2}$; note that ${}^{\mathrm{e}}N_{H_2,y_0}(P)$ is invariant under dilations in (x,y) and translations in y. To define ${}^{\mathrm{e}}N_{H_2,y_0}(P)$ invariantly, consider the (nonstrictly) inward pointing normal bundle

$$ {}^{+}N\phi^{-1}(y_0) = {}^{+}T_{\phi^{-1}(y_0)}M / T\phi^{-1}(y_0). $$

Note that ${}^{+}N\phi^{-1}(y_0)$ is equipped with a natural $\mathbb{R}_+$-action given by dilations in the fibers; moreover, the image of $T_{\phi^{-1}(y_0)}H_2 \subset {}^{+}T_{\phi^{-1}(y_0)}M$ in the quotient ${}^{+}N\phi^{-1}(y_0)$ acts by translations. Lastly, we can lift $\phi : H_2 \to Y$ to a fibration $TH_2 \to TY$ and restrict to the tangent bundle over $\phi^{-1}(y_0)$, giving a fibration $T_{\phi^{-1}(y_0)}H_2 \to T_{y_0}Y$; since $T\phi^{-1}(y_0)$ lies in the kernel of this fibration, this descends to a fibration ${}^{+}N\phi^{-1}(y_0) \to T_{y_0}Y$.

Given this edge-b-structure on ${}^{+}N\phi^{-1}(y_0)$, there is a natural bundle isomorphism

$$ {}^{\mathrm{e,b}}T_pM \cong {}^{\mathrm{e,b}}T_{(p,0)}({}^{+}N\phi^{-1}(y_0)), \tag{2-14} $$

where $(p,0)$ denotes the unique point in the zero section of ${}^{+}N_p\phi^{-1}(y_0)$; in local coordinates in which

$$ \begin{aligned} {}^{+}T_{\phi^{-1}(y_0)}M &= \{(z,\dot{x}\partial_x + \dot{y}\partial_y + \dot{z}\partial_z) : (y_0,z) \in \phi^{-1}(y_0),\ \dot{x} \ge 0\},\\ {}^{+}N\phi^{-1}(y_0) &= \{(z,\dot{x}\partial_x + \dot{y}\partial_y) : (y_0,z) \in \phi^{-1}(y_0),\ \dot{x} \ge 0\}, \end{aligned} $$

this isomorphism takes $x\partial_x$, $x\partial_y$, $z\partial_z$ to $\dot{x}\partial_{\dot{x}}$, $\dot{x}\partial_{\dot{y}}$, $z\partial_z$.

We may furthermore consider the space $\mathcal{V}_{\mathrm{e,b},I}({}^{+}N\phi^{-1}(y_0))$ of edge-b-vector fields on ${}^{+}N\phi^{-1}(y_0)$ (with "b"-behavior at the normal bundle ${}^{+}N_{\partial\phi^{-1}(y_0)}\phi^{-1}(y_0)$ over the boundary of the fiber, which in our setting has two connected components) which are invariant under both the dilation and translation actions; we then have an isomorphism

$$\mathcal{C}^\infty(\phi^{-1}(y_0); {}^{\mathrm{e,b}}T_{\phi^{-1}(y_0)}M) \cong \mathcal{V}_{\mathrm{e,b},I}({}^{+}N\phi^{-1}(y_0)) \tag{2-15}$$

given by applying (2-14) and passing to the unique invariant extension. The multiplicative extension of this map gives invariant meaning to

$${}^{\mathrm{e}}N_{H_2,y_0}(P) \in \mathrm{Diff}^m_{\mathrm{e,b},I}({}^{+}N\phi^{-1}(y_0)),$$

where the subscript "I" restricts to dilation- and translation-invariant operators.

Definition 2.6 (edge normal operator at H_2). For $y_0 \in Y$, the map

$${}^{\mathrm{e}}N_{H_2,y_0} : \mathrm{Diff}_{\mathrm{e,b}}(M) \to \mathrm{Diff}_{\mathrm{e,b},I}({}^{+}N\phi^{-1}(y_0))$$

is an algebra homomorphism and for each $m \in \mathbb{N}_0$ fits into the short exact sequence

$$0 \to \mathcal{I}_{\phi^{-1}(y_0)}\,\mathrm{Diff}^m_{\mathrm{e,b}}(M) \hookrightarrow \mathrm{Diff}^m_{\mathrm{e,b}}(M) \xrightarrow{{}^{\mathrm{e}}N_{H_2,y_0}} \mathrm{Diff}^m_{\mathrm{e,b},I}({}^{+}N\phi^{-1}(y_0)) \to 0, \tag{2-16}$$

where $\mathcal{I}_{\phi^{-1}(y_0)} \subset \mathcal{C}^\infty(M)$ is the ideal of smooth functions vanishing along $\phi^{-1}(y_0)$. The collection ${}^{\mathrm{e}}N_{H_2} = ({}^{\mathrm{e}}N_{H_2,y_0})_{y_0\in Y}$ fits into the short exact sequence

$$0 \to \rho_2\,\mathrm{Diff}^m_{\mathrm{e,b}}(M) \hookrightarrow \mathrm{Diff}^m_{\mathrm{e,b}}(M) \xrightarrow{{}^{\mathrm{e}}N_{H_2}} \mathrm{Diff}^m_{\mathrm{e,b},I}({}^{+}N\phi) \to 0,$$

where ${}^{+}N\phi = \bigsqcup_{y\in Y} {}^{+}N_p(\phi^{-1}(y))$ is the total space of a fibration ${}^{+}N\phi \to TY$ and the space $\mathrm{Diff}^m_{\mathrm{e,b},I}({}^{+}N\phi)$ consists of differential operators (with smooth coefficients) which are tangent to each fiber ${}^{+}N\phi^{-1}(y_0)$, $y_0 \in Y$, and whose restriction to ${}^{+}N\phi^{-1}(y_0)$ is an element of $\mathrm{Diff}^m_{\mathrm{e,b},I}({}^{+}N\phi^{-1}(y_0))$.[12]

A less precise normal operator can be defined by first regarding P simply as a b-differential operator, $P \in \mathrm{Diff}^m_{\mathrm{b}}(M)$, and then considering the b-normal operator

$${}^{\mathrm{b}}N_{H_2}(P) \in \mathrm{Diff}^m_{\mathrm{b},I}({}^{+}NH_2). \tag{2-17}$$

In local coordinates as in (2-8), this means

$${}^{\mathrm{b}}N_{H_2}(P) := \sum_{j+k\le m} a_{jk0}(0, y, z)(xD_x)^j(zD_z)^k.$$

We remark that ${}^{\mathrm{b}}N_{H_2}(P)$ is uniquely determined by ${}^{\mathrm{e}}N_{H_2}(P)$. (The conjugation of ${}^{\mathrm{b}}N_{H_2}(P)$ by the Mellin transform in x is the indicial family in the terminology of [Mazzeo 1991, Definition (2.18)].) It is often the case that the properties of the simpler model ${}^{\mathrm{b}}N_{H_2}(P)$ determine the asymptotic behavior of solutions of P near H_2; see [Hintz and Vasy 2020, §1.1.1] or [Hintz 2023b, §3.6].

[12] Thus, $\mathrm{Diff}^m_{\mathrm{e,b},I}({}^{+}N\phi)$ consists of collections of invariant edge-b-operators, indexed by $y_0 \in Y$, which depend *smoothly* on the parameter y_0.

We observe that the extension of the above definitions to operators acting on sections of vector bundles, $P \in \mathrm{Diff}^m_{\mathrm{e,b}}(M; E, F)$, only requires notational changes: for $P \in \mathrm{Diff}^m_{\mathrm{e,b}}(M; E, F)$, we have

$$\begin{aligned} {}^{\mathrm{e,b}}\sigma^m(P) &\in P^{[m]}({}^{\mathrm{e,b}}T^*M; \pi^*\operatorname{Hom}(E, F)),\\ N_{H_1}(P) &\in \mathrm{Diff}^m_{\mathrm{e,b},I}({}^+NH_1; \pi^*E|_{H_1}, \pi^*F|_{H_1}),\\ {}^{\mathrm{e}}N_{H_2,y_0} &\in \mathrm{Diff}^m_{\mathrm{e,b},I}({}^+N\phi^{-1}(y_0); \pi^*E|_{\phi^{-1}(y_0)}, \pi^*F|_{\phi^{-1}(y_0)}), \quad y_0 \in Y, \end{aligned}$$

where in each line π denotes the relevant base projection, and we have corresponding short exact sequences mirroring (2-9), (2-11), and (2-16).

2.3. ***Edge-b-pseudodifferential operators.*** We now restrict attention to the case of compact M for simplicity. We denote by $S^s({}^{\mathrm{e,b}}T^*M)$ the space of symbols of order $s \in \mathbb{R}$. In the local coordinates (2-5), this means that $a \in S^s({}^{\mathrm{e,b}}T^*M)$ if and only if

$$|\partial_x^j \partial_y^\alpha \partial_z^k \partial_\xi^p \partial_\eta^\beta \partial_\zeta^q a(x, y, z, \xi, \eta, \zeta)| \le C_{j\alpha kp\beta q}(1 + |\xi| + |\eta| + |\zeta|)^{s-(p+|\beta|+q)} \tag{2-18}$$

for all $j, k, p, q \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0^{n-2}$. The condition (2-18) can be phrased invariantly using the radial compactification $\overline{{}^{\mathrm{e,b}}T^*}M$: let $\rho_\infty \in \mathcal{C}^\infty(\overline{{}^{\mathrm{e,b}}T^*}M)$ denote a defining function of ${}^{\mathrm{e,b}}S^*M$; then we have $a \in S^s({}^{\mathrm{e,b}}T^*M)$ if and only if $Pa \in \rho_\infty^{-s}L^\infty({}^{\mathrm{e,b}}T^*M)$ for all $P \in \mathrm{Diff}(\overline{{}^{\mathrm{e,b}}T^*}M)$ which are linear combinations of up-to-m-fold compositions of vector fields on $\overline{{}^{\mathrm{e,b}}T^*}M$ which are tangent to ${}^{\mathrm{e,b}}S^*M$. We also need to consider weighted symbols. Recall the notation $\rho = (\rho_1, \dots, \rho_N)$, with ρ_j a defining function of H_j; for $\alpha \in \mathbb{R}^N$, we then put

$$S^{s,\alpha}({}^{\mathrm{e,b}}T^*M) := \rho^{-\alpha} S^s({}^{\mathrm{e,b}}T^*M).$$

More generally still, we can consider symbols with conormal regularity in the base; thus, we define the space

$$\mathcal{A}^{-\alpha} S^s({}^{\mathrm{e,b}}T^*M)$$

to consist of all smooth functions a on T^*M° such that $Pa \in \rho^{-\alpha}\rho_\infty^{-s}L^\infty(T^*M^\circ)$ for all $P \in \mathrm{Diff}_{\mathrm{b}}(\overline{{}^{\mathrm{e,b}}T^*}M)$ (with b-behavior at *all* boundary hypersurfaces of $\overline{{}^{\mathrm{e,b}}T^*}M$, not just at ${}^{\mathrm{e,b}}S^*M$). This contains the space of finite linear combinations of functions on T^*M° of the form wa where $w \in \mathcal{A}^{-\alpha}(M)$ (pulled back to ${}^{\mathrm{e,b}}T^*M$ along the base projection) and $a \in S^s({}^{\mathrm{e,b}}T^*M)$.

Our main tool for the study of edge-b-differential operators will be the algebra

$$\Psi_{\mathrm{e,b}}(M) = \bigcup_{s\in\mathbb{R}} \Psi^s_{\mathrm{e,b}}(M)$$

of edge-b-pseudodifferential operators (or edge-b-ps.d.o.s for brevity), which we describe geometrically following [Melrose et al. 2013, Appendix B], followed further below by a description in terms of quantization maps. Recall that H_2 is equipped with a fibration $\phi : H_2 \to Y$. By [Melrose et al. 2013, Lemma B.1], the fiber diagonal

$$(H_2)^2_\phi := H_2 \times_\phi H_2 = \{(p, p') \in H_2 \times H_2 : \phi(p) = \phi(p')\}$$

is a p-submanifold of M^2; it is moreover transversal to, or disjoint from, the remaining boundary diagonals H_j^2, $j \neq 2$. Collect the latter in the set $\mathcal{B} := \{H_j^2 : j \neq 2\}$. The edge-b-double space is then the iterated blow-up

$$M^2_{\mathrm{e,b}} := [M^2; (H_2)^2_\phi; \mathcal{B}]. \tag{2-19}$$

Let $\beta : M^2_{\mathrm{e,b}} \to M^2$ denote the blow-down map, let $\pi_{L/R} : M^2 \to M$ denote the projection onto the left/right factor, and denote by $\pi_{\mathrm{e,b},L/R} = \pi_{L/R} \circ \beta$ the stretched projection. Denote by $\mathrm{diag}_{\mathrm{e,b}} \subset M^2_{\mathrm{e,b}}$ the edge-b-diagonal, defined as the lift of the diagonal in M^2. Then the space $\Psi^s_{\mathrm{e,b}}(M)$ is defined on the level of Schwartz kernels by

$$\Psi^s_{\mathrm{e,b}}(M) = \{\kappa \in I^s(M^2_{\mathrm{e,b}}, \mathrm{diag}_{\mathrm{e,b}}; (\pi_{\mathrm{e,b},R})^*({}^{\mathrm{e,b}}\Omega M)) : \kappa \equiv 0 \text{ at } \partial M^2_{\mathrm{e,b}} \setminus \mathrm{ff}(\beta)\},$$

where I^s denotes the space of conormal distributions [Hörmander 1971a] (with smooth coefficients down to $\partial\,\mathrm{diag}_{\mathrm{e,b}}$), "$\equiv 0$" denotes equality in Taylor series (i.e., infinite order of vanishing), and $\mathrm{ff}(\beta)$ denotes the union of all boundary hypersurfaces produced by the blow-ups in the definition (2-19) of $M^2_{\mathrm{e,b}}$.

Operators between vector bundles $E, F \to M$ arise by tensoring the bundle in which the conormal distributions take values with $\beta^*\,\mathrm{Hom}(\pi_R^* E, \pi_L^* F)$. Schwartz kernels of weighted edge-b-ps.d.o.s are defined by

$$\Psi^{s,\alpha}_{\mathrm{e,b}}(M) = \rho^{-\alpha}\Psi^s_{\mathrm{e,b}}(M) := \{(\pi^*_{\mathrm{e,b},L}\rho^{-\alpha})\kappa : \kappa \in \Psi^s_{\mathrm{e,b}}(M)\}.$$

More generally, one can consider spaces of ps.d.o.s with conormal coefficients,

$$\mathcal{A}^{-\alpha}\Psi^s_{\mathrm{e,b}}(M),$$

by requiring their Schwartz kernels to be conormal distributions whose symbols are conormal (rather than smooth) down to $\mathrm{diag}_{\mathrm{e,b}}$, with weight $-\alpha_j$ at the lift of $(H_j)^2$ ($j \neq 2$) or $(H_2)^2_\phi$ ($j = 2$).

Turning to the explicit description, $\Psi^s_{\mathrm{e,b}}(M)$ can be defined in terms of quantizations of symbols $a \in S^s({}^{\mathrm{e,b}}T^*M)$. Concretely, in the local coordinates (2-5) and for u with support in the local coordinate patch, a typical element of $\Psi^s_{\mathrm{e,b}}(M)$ is $A = \mathrm{Op}_{\mathrm{e,b}}(a)$, acting on u via

$$Au(x,y,z) = (2\pi)^{-n}\int \exp(i((x-x')\xi + (y-y')\cdot\eta + (z-z')\zeta))\chi\left(\frac{x-x'}{x}\right)\chi\left(\frac{|y-y'|}{x}\right)\chi\left(\frac{z-z'}{z}\right)$$
$$\times a(x,y,z;x\xi,x\eta,z\zeta)u(x',y',z')\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z';$$

here $\chi \in \mathcal{C}^\infty_{\mathrm{c}}((-1,1))$ is identically 1 near 0. In general, one defines $\mathrm{Op}_{\mathrm{e,b}}(a)$ using a partition of unity. Thus,

$$\mathrm{Op}_{\mathrm{e,b}} : S^s({}^{\mathrm{e,b}}T^*M) \to \Psi^s_{\mathrm{e,b}}(M),$$

and this map is surjective modulo the space $\Psi^{-\infty}_{\mathrm{e,b}}(M)$ of residual operators. (The space $\Psi^{-\infty}_{\mathrm{e,b}}(M)$ consists of all smooth right edge-b-densities on $M^2_{\mathrm{e,b}}$ which vanish to infinite order at $\partial M^2_{\mathrm{e,b}} \setminus \mathrm{ff}(\beta)$.) In fact, any finite number of compositions of quantizations is again a quantization upon enlarging the size of the cutoffs; the space $\Psi^{-\infty}_{\mathrm{e,b}}(M)$ is merely a simple device to capture all (symbolically trivial) off-diagonal terms. Spaces of edge-b-ps.d.o.s with weighted or conormal coefficients, and possibly acting on vector bundles, can be defined similarly.

The principal symbol of edge-b-ps.d.o.s fits into the short exact sequence

$$0 \to \Psi^{s-1}_{\mathrm{e,b}}(M) \hookrightarrow \Psi^{s}_{\mathrm{e,b}}(M) \xrightarrow{{}^{\mathrm{e,b}}\sigma^{s}} (S^s/S^{s-1})({}^{\mathrm{e,b}}T^*M) \to 0;$$

it is defined by ${}^{\mathrm{e,b}}\sigma^s(\mathrm{Op}_{\mathrm{e,b}}(a)) = [a]$ and ${}^{\mathrm{e,b}}\sigma^s|_{\Psi^{-\infty}_{\mathrm{e,b}}(M)} = 0$. The principal symbol map is multiplicative; moreover, for $A_j \in \Psi^{s_j}_{\mathrm{e,b}}(M)$ with principal symbol a_j, we have

$${}^{\mathrm{e,b}}\sigma^{s_1+s_2-1}(i[A_1, A_2]) = H_{a_1}a_2.$$

As usual, for $A \in \Psi^{s}_{\mathrm{e,b}}(M)$, we denote by[13]

$$\mathrm{Ell}_{\mathrm{e,b}}(A) \subset {}^{\mathrm{e,b}}T^*M \setminus o, \quad (\text{resp. } \mathrm{WF}'_{\mathrm{e,b}}(A) \subset {}^{\mathrm{e,b}}T^*M \setminus o)$$

the elliptic set (resp. operator wave front set) consisting of points $\varpi \in {}^{\mathrm{e,b}}T^*M \setminus o$ so that ${}^{\mathrm{e,b}}\sigma^s(A)$ is invertible in a conic neighborhood of ϖ (resp. the full symbol of A (in any local coordinate chart) is not of order $-\infty$ (at fiber infinity) in any conic neighborhood of ϖ). Identifying these sets with their boundaries at fiber infinity inside $\overline{{}^{\mathrm{e,b}}T^*M}$, we shall often take the following point of view:

$$\mathrm{Ell}_{\mathrm{e,b}}(A),\ \mathrm{WF}'_{\mathrm{e,b}}(A) \subset {}^{\mathrm{e,b}}S^*M.$$

(Carefully note that $\mathrm{WF}'_{\mathrm{e,b}}(A) = \varnothing$ implies $A \in \Psi^{-\infty}_{\mathrm{e,b}}(M)$, which is thus trivial in the differential order sense, but not in the sense of decay at ∂M.) We furthermore write $\mathrm{Char}_{\mathrm{e,b}}(A) = {}^{\mathrm{e,b}}S^* \setminus \mathrm{Ell}_{\mathrm{e,b}}(A)$ for the characteristic set.

2.4. ***Sobolev spaces.*** We continue assuming that M is compact. Fixing a weight vector $w \in \mathbb{R}^N$, we shall work with a weighted edge-b-density

$$\mu \in \rho^w \mathcal{C}^\infty(M; {}^{\mathrm{e,b}}\Omega M)$$

which is positive, i.e., $\rho^{-w}\mu > 0$ as an edge-b-density. In the local coordinates (2-3b), this means

$$\mu = a(x, y, z) z^{w_1} x^{w_2} \left| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{x^{n-2}} \frac{\mathrm{d}z}{z} \right|, \quad 0 < a \in \mathcal{C}^\infty.$$

We then set

$$H^0_{\mathrm{e,b}}(M, \mu) := L^2(M, \mu).$$

Whenever the density μ is clear from the context, we shall omit it from the notation. For an integer $s \in \mathbb{N}_0$, we define $H^s_{\mathrm{e,b}}(M)$ to consist of all $u \in H^0_{\mathrm{e,b}}(M)$ such that $Pu \in H^0_{\mathrm{e,b}}(M)$ for all $P \in \mathrm{Diff}^s_{\mathrm{e,b}}(M)$. For $s \in \mathbb{R}$, we can define $H^s_{\mathrm{e,b}}(M)$ via duality and interpolation. Alternatively, for $s \geq 0$, we have $H^s_{\mathrm{e,b}}(M) = \{u \in H^0_{\mathrm{e,b}}(M) : Au \in H^0_{\mathrm{e,b}}(M)\}$, where $A \in \Psi^s_{\mathrm{e,b}}(M)$ is any fixed elliptic operator; this is thus a Hilbert space with squared norm

$$\|u\|^2_{H^s_{\mathrm{e,b}}(M)} = \|u\|^2_{H^0_{\mathrm{e,b}}(M)} + \|Au\|^2_{H^0_{\mathrm{e,b}}(M)}.$$

[13]The elliptic set of A depends on the order of the space of ps.d.o.s of which one regards A as an element of; since this order is always clear from the context, we shall write $\mathrm{Ell}_{\mathrm{e,b}}(A)$ simply instead of the more cumbersome $\mathrm{Ell}_{\mathrm{e,b}}{}^{s}(A)$.

By elliptic regularity, any two choices of A give equivalent norms. For $s < 0$, we have $H^s_{\mathrm{e,b}}(M) = \{u_1 + Au_2 : u_1, u_2 \in H^0_{\mathrm{e,b}}(M)\}$, where $A \in \Psi^{-s}_{\mathrm{e,b}}(M)$ is elliptic; since this is isomorphic to $(H^{-s}_{\mathrm{e,b}}(M))^*$ via the $L^2(M,\mu)$-pairing, it is a Hilbert space as well. Weighted versions of these spaces are defined by

$$H^{s,\alpha}_{\mathrm{e,b}}(M) := \rho^\alpha H^s_{\mathrm{e,b}}(M) = \{\rho^\alpha u : u \in H^s_{\mathrm{e,b}}(M)\}.$$

These are Hilbert spaces as well, and $(H^{s,\alpha}_{\mathrm{e,b}}(M))^* = H^{-s,-\alpha}_{\mathrm{e,b}}(M)$ with respect to $L^2(M)$. When $\Omega \subset M$ is an open set, we denote spaces of extendible and supported distributions by

$$\begin{aligned}\overline{H}^{s,\alpha}_{\mathrm{e,b}}(\Omega) &:= \{u|_\Omega : u \in H^{s,\alpha}_{\mathrm{e,b}}(M)\},\\ \dot{H}^{s,\alpha}_{\mathrm{e,b}}(\overline{\Omega}) &:= \{u \in H^{s,\alpha}_{\mathrm{e,b}}(M) : \operatorname{supp} u \subset \overline{\Omega}\},\end{aligned}$$

following the notation of [Hörmander 2007, Appendix B].

We also need spaces of sections of vector bundles $E \to M$: for $s = \alpha = 0$, and fixing any positive definite fiber metric on E, we set $H^{0,0}_{\mathrm{e,b}}(M; E) = H^0_{\mathrm{e,b}}(M; E) = L^2(M, \mu; E)$, any two choices of fiber metrics giving the same space (up to equivalence of norms) since M is compact. For general s and α, the space $H^{s,\alpha}_{\mathrm{e,b}}(M; E)$ is then defined as above.

Edge-b-ps.d.o.s $A \in \Psi^0_{\mathrm{e,b}}(M)$ act continuously on $H^0_{\mathrm{e,b}}(M)$; see [Melrose et al. 2013, Appendix B]. Using the algebra properties of $\Psi_{\mathrm{e,b}}(M)$, this can be shown to imply that every $A \in \Psi^{m,\beta}_{\mathrm{e,b}}(M)$ defines a bounded linear map $A : H^{s,\alpha}_{\mathrm{e,b}}(M) \to H^{s-m,\alpha-\beta}_{\mathrm{e,b}}(M)$.

We define

$$H^{-\infty,\alpha}_{\mathrm{e,b}}(M) := \bigcup_{s\in\mathbb{R}} H^{s,\alpha}_{\mathrm{e,b}}(M), \quad H^{\infty,\alpha}_{\mathrm{e,b}}(M) := \bigcap_{s\in\mathbb{R}} H^{s,\alpha}_{\mathrm{e,b}}(M).$$

Given a distribution $u \in H^{-\infty,\alpha}_{\mathrm{e,b}}(M)$, we can define its wave front set in the usual manner: for $s \in \mathbb{R}$, we define

$$\mathrm{WF}^{s,\alpha}_{\mathrm{e,b}}(u) \subset {}^{\mathrm{e,b}}S^*M$$

as the complement of the set of $\varpi \in {}^{\mathrm{e,b}}S^*M$ for which there exists an operator $A \in \Psi^s_{\mathrm{e,b}}(M)$ which is elliptic at ϖ and such that $Au \in H^{0,\alpha}_{\mathrm{e,b}}(M) = \rho^\alpha H^0_{\mathrm{e,b}}(M)$. We stress that we need to assume a priori that u lies in an edge-b-Sobolev space with weight α in order to guarantee that the membership $Au \in H^{0,\alpha}_{\mathrm{e,b}}(M)$ implies also that $A'u \in H^{0,\alpha}_{\mathrm{e,b}}(M)$ for all $A' \in \Psi^s_{\mathrm{e,b}}(M)$ with $\mathrm{WF}'_{\mathrm{e,b}}(A') \subset \mathrm{Ell}_{\mathrm{e,b}}(A)$. In particular,

$$u \in H^{-\infty,\alpha}_{\mathrm{e,b}}(M), \ \ \mathrm{WF}^{s,\alpha}_{\mathrm{e,b}}(u) = \varnothing \quad \Longrightarrow \quad u \in H^{\infty,\alpha}_{\mathrm{e,b}}(M);$$

i.e., the wave front set controls edge-b-regularity but not decay.

We shall also need to use Sobolev spaces associated with other Lie algebras, in particular b-Sobolev spaces $H^{s,\alpha}_{\mathrm{b}}(M)$ on manifolds with boundary or corners, scattering Sobolev spaces $H^{s,\alpha}_{\mathrm{sc}}(M)$ on manifolds with boundary, and 0-Sobolev spaces $H^{s,\alpha}_0(M)$ on manifolds with boundary; these spaces are defined in complete analogy with the edge-b-spaces above. Detailed discussions of these spaces (as well as of the corresponding calculi of pseudodifferential operators) can be found in [Mazzeo 1991; Melrose 1993; 1994; Mazzeo and Melrose 1998; Hintz 2023e].

2.5. ***Invariant edge-b-operators and edge-b-Sobolev spaces.*** We now turn to notions related to the analysis of edge normal operators of edge-b-differential operators. Fix $y_0 \in Y$, and identify $Z = \phi^{-1}(y_0)$ with the zero section of

$$\mathcal{N} := {}^{+}N\phi^{-1}(y_0) = {}^{+}T_{\phi^{-1}(y_0)}M/T\phi^{-1}(y_0).$$

Recall on $\mathcal{N}$ the space $\mathcal{V}_{\mathrm{e,b},I}(\mathcal{N})$ of edge-b-vector fields which are invariant with respect to the dilation and translation actions introduced in the paragraph preceding (2-15). In terms of coordinates $x \geq 0$, $y \in \mathbb{R}^{n-2}$ on the fibers of

$$\pi : \mathcal{N} \to \phi^{-1}(y_0) = Z,$$

so $\mathcal{N} = Z \times [0,\infty)_x \times \mathbb{R}^{n-2}_y$, the space $\mathcal{V}_{\mathrm{e,b},I}(\mathcal{N})$ is thus spanned over $\mathcal{C}^\infty(Z)$ by $x\partial_x$, $x\partial_y$, $\mathcal{V}_{\mathrm{b}}(Z)$, with the dilation action given by $(x, y, z) \mapsto (\lambda x, \lambda y, z)$, $\lambda \in \mathbb{R}_+$, and the translation action given by $(x, y, z) \mapsto (x, y + y', z)$, $y' \in \mathbb{R}^{n-2}$. Here, we identify $\mathcal{C}^\infty(Z) \subset \mathcal{C}^\infty(\mathcal{N})$ via pullback. (Thus, $\mathcal{C}^\infty(Z)$ is the space of dilation- and translation-invariant smooth functions.) We also recall the notation $\mathrm{Diff}^m_{\mathrm{e,b},I}(\mathcal{N})$ for up-to-m-fold compositions of elements of $\mathcal{V}_{\mathrm{e,b},I}(\mathcal{N})$.

Following Mazzeo [1991, §5], one may analyze elements of $\mathrm{Diff}^m_{\mathrm{e,b},I}(\mathcal{N})$ (see (2-13) for a local coordinate expression) by passing to the Fourier transform in y, with dual momentum denoted by η, and then changing variables to $(\hat{x}, \hat{\eta}, z)$, where $\hat{x} = x|\eta|$ and $\hat{\eta} = \eta/|\eta|$, thus obtaining a smooth family (in $(y_0, \hat{\eta}) \in S^*Y$) of operators of Bessel type on $[0,\infty)_{\hat{x}} \times Z$ (or indeed of weighted b-scattering type on $[0,\infty]_{\hat{x}} \times Z$, with scattering behavior at the boundary hypersurface $\{\infty\} \times Z$). In this paper, we instead work directly with invariant edge-b-operators, as in the wave equation setting of interest here this makes the proofs of their mapping and regularity properties straightforward modifications of the proofs for the original edge-b-operator. We thus proceed to define spaces and ps.d.o.s for the analysis of invariant edge-b-operators on $\mathcal{N}$.

The principal symbol ${}^{\mathrm{e,b}}\sigma^m(P) \in P^{[m]}({}^{\mathrm{e,b}}T^*\mathcal{N})$ of $P \in \mathrm{Diff}^m_{\mathrm{e,b},I}(\mathcal{N})$ is invariant under the lifts of the dilation and translation actions; in local coordinates on Z and using fiber-linear coordinates ξ, η_j, ζ as in (2-5), this simply means that ${}^{\mathrm{e,b}}\sigma^m(P)$ is independent of x, y. Thus, the principal symbol short exact sequence reads

$$0 \to \mathrm{Diff}^{m-1}_{\mathrm{e,b},I}(\mathcal{N}) \hookrightarrow \mathrm{Diff}^m_{\mathrm{e,b},I}(\mathcal{N}) \xrightarrow{{}^{\mathrm{e,b},I}\sigma^m} P^{[m]}_I({}^{\mathrm{e,b}}T^*\mathcal{N}) \to 0,$$

where the subscript "I" restricts to invariant symbols. (Equivalently, restriction to Z gives an isomorphism $P^{[m]}_I({}^{\mathrm{e,b}}T^*\mathcal{N}) \cong P^{[m]}({}^{\mathrm{e,b}}T^*_Z\mathcal{N})$ with the inverse given by invariant extension.)

Note moreover that, since $Z \cong [-1, 1]$, the manifold $\mathcal{N}$ has three boundary hypersurfaces which we denote by

$$\mathcal{H}_j := {}^{+}N_{\phi^{-1}(y_0)\cap H_j}\phi^{-1}(y_0), \quad j = 1, 3, \qquad \mathcal{H}_2 := T_{\phi^{-1}(y_0)}H_2/T\phi^{-1}(y_0).$$

Thus, $\mathcal{H}_1$ and $\mathcal{H}_3$ are the fibers of $\mathcal{N} \to \phi^{-1}(y_0)$ over the two points of ∂Z, while $\mathcal{H}_2$ is the fibered boundary of $\mathcal{N}$ (given in the above coordinates by $x = 0$). Any two defining functions ρ_2, ρ_2' of $\mathcal{H}_2$ which are invariant are related via $\rho_2' = a\rho_2$, where $0 < a \in \mathcal{C}^\infty(Z)$. Invariant defining functions of $\mathcal{H}_1$ and $\mathcal{H}_3$ are pullbacks of defining functions of the two boundary points of Z.

Definition 2.7 (invariant edge-b-Sobolev spaces). Denote by $\rho_1, \rho_2, \rho_3 \in \mathcal{C}^\infty(\mathcal{N})$ invariant defining functions of $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, and let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$. Fix an invariant edge-b-density $0 < \mu_0 \in \mathcal{C}^\infty(\mathcal{N}; {}^{\mathrm{e,b}}\Omega\mathcal{N})$, a weight $w \in \mathbb{R}^3$, and fix $\mu = \rho^w \mu_0$. Then $H^0_{\mathrm{e,b},I}(\mathcal{N}, \mu) := L^2(\mathcal{N}, \mu)$. We drop μ from the notation from now on. For $s \in \mathbb{N}$, we set $H^s_{\mathrm{e,b},I}(\mathcal{N}) = \{u \in L^2(\mathcal{N}) : Pu \in L^2(\mathcal{N}) \ \forall\, P \in \mathrm{Diff}^k_{\mathrm{e,b},I}(\mathcal{N})\}$,[14] and we define $H^s_{\mathrm{e,b},I}(\mathcal{N})$ for general $s \in \mathbb{R}$ via interpolation and duality. Finally, we set

$$H^{s,\alpha}_{\mathrm{e,b},I}(\mathcal{N}) := \rho^\alpha H^s_{\mathrm{e,b},I}(\mathcal{N}).$$

Given a vector bundle $E \to M$, invariant weighted edge-b-Sobolev spaces of sections of the bundle $\pi^* E|_{\phi^{-1}(y_0)} \to \mathcal{N}$ are defined analogously.[15]

In compact subsets of $\mathcal{N}$, these are standard weighted edge-b-Sobolev spaces. The only reason for including the subscript "I" in the notation for these spaces is that $\mathcal{N}$ is noncompact; invariant edge-b-Sobolev spaces provide a means to measure edge-b-regularity and square integrability uniformly on $\mathcal{N}$. Given an open subset $\mathcal{U} \subset Z$, the preimage $\Omega := \pi^{-1}(\mathcal{U}) \subset \mathcal{N}$ is invariant; we shall then consider spaces of supported and extendible distributions on Ω,

$$\overline{H}^{s,\alpha}_{\mathrm{e,b},I}(\Omega) := \{u|_\Omega : u \in H^{s,\alpha}_{\mathrm{e,b},I}(\mathcal{N})\} \quad \text{and} \quad \dot{H}^{s,\alpha}_{\mathrm{e,b},I}(\overline{\Omega}) := \{u \in H^{s,\alpha}_{\mathrm{e,b},I}(\mathcal{N}) : \operatorname{supp} u \subset \overline{\Omega}\},$$

equipped as usual with the quotient and subspace topology, respectively.

We shall also use invariant edge-b-pseudodifferential operators, i.e., edge-b-ps.d.o.s on $\mathcal{N}$ which are invariant under the translation and dilation actions. From a Schwartz kernel perspective, they are thus uniquely determined by the restriction of their Schwartz kernels to the edge front face (the lift of $(\mathcal{H}_2)^2_\phi$ to $\mathcal{N}^2_{\mathrm{e,b}} = [\mathcal{N}^2; (\mathcal{H}_2)^2_\phi; \mathcal{H}_1^2, \mathcal{H}_3^2]$). Explicitly, consider a symbol $a \in S^s({}^{\mathrm{e,b}}T^*_Z\mathcal{N})$, or equivalently an invariant symbol $a \in S^s_I({}^{\mathrm{e,b}}T^*\mathcal{N})$ (the subspace of $S^s({}^{\mathrm{e,b}}T^*\mathcal{N})$ consisting of invariant elements); in local coordinates $z \geq 0$ on Z, we thus have $a = a(z, \xi, \eta, \zeta)$, and the quantization of a as an invariant edge-b-ps.d.o. is

$$(\mathrm{Op}_{\mathrm{e,b}}(a)u)(x, y, z) = (2\pi)^{-n} \int \exp\left(i\left(\frac{x-x'}{x}\xi + \frac{y-y'}{x}\cdot\eta + (z-z')\zeta\right)\right)\chi\left(\frac{x-x'}{x}\right)\chi\left(\frac{|y-y'|}{x}\right)\chi\left(\frac{z-z'}{z}\right)$$
$$\times\, a(z; \xi, \eta, z\zeta)u(x', y', z')\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\frac{\mathrm{d}x'}{x}\,\frac{\mathrm{d}y'}{x^{n-2}}\,\mathrm{d}z'.$$

Denoting the space of invariant edge-b-ps.d.o.s by $\Psi^s_{\mathrm{e,b},I}(\mathcal{N})$, we correspondingly have a principal symbol short exact sequence

$$0 \to \Psi^{s-1}_{\mathrm{e,b},I}(\mathcal{N}) \hookrightarrow \Psi^s_{\mathrm{e,b},I}(\mathcal{N}) \xrightarrow{{}^{\mathrm{e,b},I}\sigma^s} (S^s_I/S^{s-1}_I)({}^{\mathrm{e,b}}T^*\mathcal{N}) \to 0.$$

One can also consider classes of weighted operators

$$\Psi^{s,\alpha}_{\mathrm{e,b},I}(\mathcal{N}) = \rho^{-\alpha}\Psi^s_{\mathrm{e,b},I}(\mathcal{N}),$$

[14] Simply put, the space $H^s_{\mathrm{e,b},I}(\mathcal{N})$ consists of all elements $u = u(x, y, z) \in L^2(\mathcal{N})$ (where $x \geq 0,\ y \in \mathbb{R}^{n-2},\ z \in Z$) so that its up-to-$s$-fold derivatives along $x\partial_x,\ x\partial_y$, and elements of $\mathcal{V}(Z)$ remain in $L^2(\mathcal{N})$.

[15] This requires the choice of a connection on $E|_{\phi^{-1}(y_0)}$, but any two choices give the same space. Note that over any fiber $\pi^{-1}(z_0),\ z_0 \in \phi^{-1}(y_0)$, of $\mathcal{N}$, the bundle $\pi^* E|_{\phi^{-1}(y_0)}$ is canonically trivial, and hence sections of $\pi^* E|_{\phi^{-1}(y_0)} \to \mathcal{N}$ can be differentiated along the generators of the translation and dilation actions without any further choices.

whose Schwartz kernels are homogeneous of degree $-\alpha_2$ with respect to the dilation action and invariant under the translation action. (More generally, one can allow for conormal behavior at $\mathcal{H}_1, \mathcal{H}_3$.) Their principal symbols are elements of $(S_I^{s,\alpha}/S_I^{s-1,\alpha})({}^{\mathrm{e,b}}T^*\mathcal{N})$, where $S_I^{s,\alpha}({}^{\mathrm{e,b}}T^*\mathcal{N})$ is the subspace of $S^{s,\alpha}({}^{\mathrm{e,b}}T^*\mathcal{N})$ consisting of symbols which are homogeneous of degree $-\alpha_2$ with respect to the dilation action and invariant under the translation action.

One can then characterize the spaces $H^{s,\alpha}_{\mathrm{e,b},I}(\mathcal{N})$ via testing with invariant edge-b-ps.d.o.s, analogously to the standard edge-b-setting in Section 2.4. Moreover, an element $A \in \Psi^{m,\beta}_{\mathrm{e,b},I}(\mathcal{N})$ defines a bounded linear map $H^{s,\alpha}_{\mathrm{e,b},I}(\mathcal{N}) \to H^{s-m,\alpha-\beta}_{\mathrm{e,b},I}(\mathcal{N})$. We define the invariant wave front set

$$\mathrm{WF}^{s,\alpha}_{\mathrm{e,b},I}(u) \subset {}^{\mathrm{e,b}}S^*_Z\mathcal{N} \tag{2-20}$$

(which one can equivalently regard as an invariant subset of ${}^{\mathrm{e,b}}S^*\mathcal{N}$) for $u \in H^{-\infty,\alpha}_{\mathrm{e,b},I}(\mathcal{N})$ in the usual manner by testing with elliptic-invariant edge-b-ps.d.o.s.

2.6. ***Semiclassical* 0*-analysis.*** The following material will be used only in Section 8.

Definition 2.8 (semiclassical 0-differential operators, Sobolev spaces). Let X denote an n-dimensional manifold with boundary. Then $\mathrm{Diff}^m_{0,\hbar}(X)$ denotes the space of h-dependent differential operators P on X°, $h \in (0,1)$, which in local coordinates $[0,\infty)_x \times \mathbb{R}^{n-1}_y$ on X are of the form

$$\sum_{j+|\alpha|\le m} a_{j\alpha}(h,x,y)(hxD_x)^j(hxD_y)^\alpha, \quad a_{j\alpha} \in \mathcal{C}^\infty([0,1)_h \times [0,\infty) \times \mathbb{R}^{n-1}).$$

If X is compact and equipped with a positive weighted b-density (omitted from the notation), and if $\rho \in \mathcal{C}^\infty(X)$ denotes a boundary-defining function, then we define semiclassical Sobolev spaces $H^{s,\alpha}_{0,h}(X) = \rho^\alpha H^s_{0,h}(X)$ to be equal to $H^{s,\alpha}_0(X)$ as sets, but with the norm for $s \in \mathbb{N}_0$ given by

$$\|u\|^2_{H^{s,\alpha}_{0,h}} := \sum_j \|P_j u\|^2_{L^2},$$

where $\{P_j\} \subset \mathrm{Diff}^s_{0,\hbar}(X)$ is a finite spanning set of $\mathrm{Diff}^s_{0,h}(X)$ over $\mathcal{C}^\infty([0,1)\times X)$. For real s, the norm on $H^{s,\alpha}_{0,h}(X)$ is defined via duality and interpolation.

For u supported in a coordinate patch and for $s \in \mathbb{N}_0$, an equivalent norm on $H^{s,\alpha}_{0,h}(X)$ is given by

$$\|u\|^2_{H^{s,\alpha}_{0,h}} := \sum_{j+|\alpha|\le s} \|x^{-\alpha}(hxD_x)^j(hxD_y)^\alpha u\|^2_{L^2}.$$

That is, every 0-derivative is weighted by a factor of h. There is an associated algebra of *semiclassical* 0*-pseudodifferential operators* which we describe in Section 2.6.1 below.

There is a close relationship between 0-analysis on X and edge-b-analysis on

$$M = [0,\infty)_{\rho_+} \times X,$$

where the boundary hypersurface $[0,\infty)\times\partial X$ of M is fibered by $[0,\infty)\times\partial X \to \partial X$.

Lemma 2.9 (the Mellin transform, edge-b-Sobolev spaces on M, and 0-Sobolev spaces on X). *Fix a positive b-density μ_{b} on X and the b-density $|\mathrm{d}\rho_+/\rho_+|\mu_{\mathrm{b}}$ on M. Let $s, \gamma_{\mathscr{I}}, \gamma_+ \in \mathbb{R}$. Then the Mellin transform in ρ_+, defined by*

$$(\mathcal{M}u)(\lambda, x_{\mathscr{I}}, y) = \int_0^\infty \rho_+^{-i\lambda} u(\rho_+, x_{\mathscr{I}}, y)\frac{\mathrm{d}\rho_+}{\rho_+} \tag{2-21}$$

is an isomorphism

$$\mathcal{M} : H_{\mathrm{e,b}}^{s,(2\gamma_{\mathscr{I}},\gamma_+)}(M) \xrightarrow{\cong} L^2(\{\operatorname{Im}\lambda = -\gamma_+\}; \langle\lambda\rangle^{-s} H_{0,\langle\lambda\rangle^{-1}}^{s,2\gamma_{\mathscr{I}}}(X)),$$

where $H_{\mathrm{e,b}}^{s,(2\gamma_{\mathscr{I}},\gamma_+)}(M) = x_{\mathscr{I}}^{2\gamma_{\mathscr{I}}}\rho_+^{\gamma_+} H_{\mathrm{e,b}}^s(M)$, with $x_{\mathscr{I}} \in \mathcal{C}^\infty(X)$ denoting a boundary-defining function of X.

Proof. It suffices to consider the case $\gamma_{\mathscr{I}} = \gamma_+ = 0$. Via interpolation and duality, it suffices to consider the case $s \in \mathbb{N}_0$. For $s = 0$, the claim is then a restatement of Plancherel's theorem. For $s = 1$, observe that, in local coordinates (x, y) on X, the Mellin transforms intertwines the operators 1, xD_x, xD_y, $\rho_+D_{\rho_+}$ with 1, xD_x, xD_y, λ. Therefore, writing $\hat{u}(\lambda; x, y) = (\mathcal{M}u)(\lambda; x, y)$, we have

$$\begin{aligned}\|u\|_{L^2(M)}^2 + \|xD_xu\|_{L^2(M)}^2 + \|xD_yu\|_{L^2(M)}^2 + \|\rho_+D_{\rho_+}u\|_{L^2(M)}^2 \\ = (1+|\lambda|^2)\big(\|\hat{u}(\lambda)\|_{L^2(\mathbb{R}_\lambda;L^2(X))}^2 + \|\langle\lambda\rangle^{-1}xD_x\hat{u}(\lambda)\|_{L^2(\mathbb{R}_\lambda;L^2(X))}^2 + \|\langle\lambda\rangle^{-1}xD_y\hat{u}(\lambda)\|_{L^2(\mathbb{R}_\lambda;L^2(X))}^2\big).\end{aligned}$$

This proves the lemma for $s = 1$; the case of general $s \in \mathbb{N}$ is similar. See also [Vasy 2013, §3.1]. □

There is a close relationship also on the phase space level. This is discussed in [Hintz 2015, §3.3.4] in the simpler setting where X is a closed manifold. We now describe this relationship from the perspective of differential operators.

Lemma 2.10 (phase space relationship). *Let $P \in \mathrm{Diff}_{\mathrm{e,b}}^m(M)$.*[16] *Define the semiclassical rescaling of its Mellin-transformed normal operator family as $P_{h,\hat{\lambda}} := h^m\widehat{N_X}(P, h^{-1}\hat{\lambda})$, where we identify X with the boundary hypersurface $\{0\} \times X$ of M. Then $P_{h,\hat{\lambda}} \in \mathrm{Diff}_{0,\hbar}^m(X)$, and its principal symbol $p_{\hat{\lambda}} = \sigma_{0,\hbar}^m(P_{h,\hat{\lambda}}) \in P^m({}^0T^*X)$ is related to $p = {}^{\mathrm{e,b}}\sigma^m(P)$ by*

$$p_{\hat{\lambda}}(\varpi) = p\left(\hat{\lambda}\frac{\mathrm{d}\rho_+}{\rho_+} + \varpi\right), \quad \varpi \in {}^0T^*X \subset {}^{\mathrm{e,b}}T_X^*M.$$

Here, the inclusion ${}^0T^*X \subset {}^{\mathrm{e,b}}T_X^*M$ is the adjoint of the map ${}^{\mathrm{e,b}}T_XM \to {}^0TX$ induced by restriction of vector fields to X.

Proof. In local coordinates and by linearity, it suffices to consider

$$P = a(\rho_+, x, y)(\rho_+D_{\rho_+})^j(xD_x)^k(xD_y)^\alpha, \tag{2-22}$$

where $j + k + |\alpha| \le m$. Then $\widehat{N_X}(P, \lambda)$ is obtained by restricting a to $\rho_+ = 0$ and formally replacing $\rho_+D_{\rho_+}$ by λ; hence

$$P_{h,\hat{\lambda}} = h^{m-(j+k+|\alpha|)}a(0, x, y)\hat{\lambda}^j(hxD_x)^k(hxD_y)^\alpha. \tag{2-23}$$

[16] One can also consider P acting on sections of a bundle $E \to M$, in which case $P_{h,\hat{\lambda}}$ acts on sections of $E|_X$. We leave the necessary notational changes to the reader.

Thus, unless $j+k+|\alpha|=m$, the respective principal symbols of P and $P_{h,\hat\lambda}$ vanish, whereas, if $j+k+|\alpha|=m$, we have, at a point (x,y) on X and at the corresponding point $(0,x,y)\in\{0\}\times X\subset M$,

$$\sigma^m_{0,\hbar}(P_{h,\hat\lambda})\left(\xi\frac{\mathrm{d}x}{x}+\eta\frac{\mathrm{d}y}{x}\right)=a(0,x,y)\hat\lambda^j\xi^k\eta^\alpha={}^{\mathrm{e,b}}\sigma^m(P)\left(\hat\lambda\frac{\mathrm{d}\rho_+}{\rho_+}+\xi\frac{\mathrm{d}x}{x}+\eta\frac{\mathrm{d}y}{x}\right).\qquad\square$$

The case of main interest is when $\hat\lambda=\pm1+\mathcal{O}(h)$, which arises when $h=\langle\lambda\rangle^{-1}$ and $\hat\lambda=\lambda/\langle\lambda\rangle$ and λ is restricted to a line of constant imaginary part. Since, in the notation of Lemma 2.10, the Hamiltonian vector field H_p is tangent to the level sets of $\rho_+\partial_{\rho_+}$ inside ${}^{\mathrm{e,b}}T^*_XM$ (cf. Lemma 2.2 where z and ζ play the roles of ρ_+ and $\rho_+\partial_{\rho_+}(\cdot)$), the restriction of H_p to ${}^{\mathrm{e,b}}T^*_XM\supset\pm\mathrm{d}\rho_+/\rho_+ + {}^0T^*X\cong{}^0T^*X$ is equal to the Hamiltonian vector field $H_{p_{\pm1}}$.

In particular, $H_{p_{\pm1}}$ has a critical point at $\varpi\in{}^0T^*X$ if H_p has a critical point at $\pm\mathrm{d}\rho_+/\rho_+ + \varpi$, and the linearizations of both vector fields have a simple relationship (namely, one drops the $\rho_+\partial_{\rho_+}$ term of the latter). Microlocal estimates at such critical points require control of subleading terms, as we now record.

Lemma 2.11 (relationship of subprincipal terms). *Let P and $P_{h,\hat\lambda}$ be as in Lemma 2.10. Fix positive b-densities on X and M as in Lemma 2.9. Suppose that P has a real scalar principal symbol, and let $\hat\lambda=\pm1+\mathcal{O}(h)$. Then, for $\varpi\in{}^0T^*X\subset{}^{\mathrm{e,b}}T^*_XM$, we have*

$$\sigma^{m-1}_{0,\hbar}\left(\frac{P_{h,\hat\lambda}-P^*_{h,\hat\lambda}}{2ih}\right)(\varpi)=\left({}^{\mathrm{e,b}}\sigma^{m-1}\left(\frac{P-P^*}{2i}\right)+\frac{\operatorname{Im}\hat\lambda}{h}\rho_+^{-1}H_p\rho_+\right)\Big|_{\pm\mathrm{d}\rho_+/\rho_++\varpi}.\tag{2-24}$$

Proof. It suffices to consider P of the form (2-22). When $j+k+|\alpha|\le m-2$, both sides of (2-24) vanish. When $j+k+|\alpha|=m-1$, the principal symbol of $(2i)^{-1}(P-P^*)$ at a point $\hat\lambda(\mathrm{d}\rho_+/\rho_+)+\varpi$, with $\varpi=\xi(\mathrm{d}x/x)+\eta(\mathrm{d}y/x)$, over $\rho_+=0$ is $\operatorname{Im}(a(0,x,y)\hat\lambda^j)\xi^k\eta^\alpha$, which matches the principal symbol of $(2ih)^{-1}(P_{h,\hat\lambda}-P^*_{h,\hat\lambda})$ at ϖ in view of (2-23).

When $j+k+|\alpha|=m$, we have that a is real-valued by assumption, and we compute

$$\frac{P-P^*}{2i}\equiv(\rho_+D_{\rho_+})^jP_1\mod\rho_+\operatorname{Diff}^{m-1}_0(M),$$
$$P_1=\frac{1}{2i}(a(0,x,y)(xD_x)^k(xD_y)^\alpha-((xD_x)^k(xD_y)^\alpha)^*a(0,x,y))\in\operatorname{Diff}^{m-1}_0(M).$$

On the other hand, writing $\hat\lambda=\pm1+h\kappa$ (thus with $\kappa=\mathcal{O}(1)$ as $h\to0$) and noting that $\hat\lambda^j=(\pm1)^j+(\pm1)^{j-1}jh\kappa+\mathcal{O}(h^2)$, we find

$$\frac{P_{h,\hat\lambda}-P^*_{h,\hat\lambda}}{2ih}\equiv(\pm1)^{j-1}j(\operatorname{Im}\kappa)a(0,x,y)(hxD_x)^k(hxD_y)^\alpha+h^{k+|\alpha|-1}(\pm1)^jP_1\mod h\operatorname{Diff}^{m-1}_{0,\hbar}(M).$$

It then remains to observe that the Hamiltonian vector field of $p=a\xi^k\eta^\alpha\zeta^j$ satisfies $\rho_+^{-1}H_p\rho_+=j\zeta^{-1}p$, which at $\zeta=\pm1$ evaluates to $(\pm1)^{j-1}ja\xi^k\eta^\alpha$. $\square$

2.6.1. *Semiclassical 0-pseudodifferential operators.* Aspects of semiclassical 0-ps.d.o.s were described in [Melrose et al. 2014, §3]; we shall only need the small semiclassical 0-calculus here. We first describe the salient properties and basic applications of this ps.d.o. algebra in geometric terms before giving a local

coordinate description and proving the composition property. We work with compact X for simplicity. The space

$$\Psi^{s,\alpha}_{0,\hbar}(X) = \rho^{-\alpha}\Psi^{s}_{0,\hbar}(X)$$

can then be defined in a geometric manner as follows: Schwartz kernels of elements of $\Psi^{s}_{0,\hbar}(X)$ are distributions on the semiclassical 0-double space $X^2_{0,\hbar} := [[0,1)_h \times X^2; [0,1)\times \mathrm{diag}_{\partial X}; \{0\}\times \mathrm{diag}_X]$ (with $\mathrm{diag}_{\partial X} \subset (\partial X)^2$ and $\mathrm{diag}_X \subset X^2$ denoting the diagonals) which are conormal distributions (with values in the right semiclassical 0-density bundle) of order $s - \frac{1}{4}$ to the lift $\mathrm{diag}_{0,\hbar}$ of $[0,1)\times \mathrm{diag}_X$ (in particular, their restriction to $h = h_0 > 0$ is a distribution on $X^2_0 = [X^2; \mathrm{diag}_{\partial X}]$ which is conormal of order s at the lift of diag_X) and which vanish to infinite order at all boundary hypersurfaces of $X^2_{0,\hbar}$ which are disjoint from $\mathrm{diag}_{0,\hbar}$.

The principal symbol short exact sequence is

$$0 \to h\Psi^{s-1,\alpha}_{0,\hbar}(X) \hookrightarrow \Psi^{s,\alpha}_{0,\hbar}(X) \to (S^{s,\alpha}/hS^{s-1,\alpha})([0,1)\times {}^0T^*X) \to 0.$$

Correspondingly, the elliptic set of an operator $A = (A_h)_{h\in(0,1)} \in \Psi^{s,\alpha}_{0,\hbar}(X)$ is the open subset $\mathrm{Ell}^{s,\alpha}_{0,\hbar}(A) \cup {}^+\mathrm{Ell}^{s,\alpha}_{0,\hbar}(A)$ of $(\{0\}\times\overline{{}^0T^*}X)\cup([0,1]\times {}^0S^*X)$, where $\mathrm{Ell}^{s,\alpha}_{0,\hbar}(A) \subset \{0\}\times\overline{{}^0T^*X}$ and ${}^+\mathrm{Ell}^{s,\alpha}_{0,\hbar}(A) \subset [0,1]\times {}^0S^*X$; the two elliptic sets agree in $\{0\}\times {}^0S^*X$. We shall only be concerned with the elliptic set $\mathrm{Ell}^{s,\alpha}_{0,\hbar}(A)$ over $h = 0$, which we shall thus simply regard as a subset

$$\mathrm{Ell}^{s,\alpha}_{0,\hbar}(A) \subset \overline{{}^0T^*}X;$$

it captures the ellipticity of A to leading order at $h = 0$ and thus arises from the simplified principal symbol map $\sigma^{s,\alpha}_{0,\hbar}$,

$$0 \to h\Psi^{s,\alpha}_{0,\hbar}(X) \hookrightarrow \Psi^{s,\alpha}_{0,\hbar}(X) \xrightarrow{\sigma^{s,\alpha}_{0,\hbar}} S^{s,\alpha}({}^0T^*X) \to 0.$$

Note that if $\varpi \in \mathrm{Ell}^{s,\alpha}_{0,\hbar}(A)$, then also $\{h, \varpi\} \in {}^+\mathrm{Ell}^{s,\alpha}_{0}(A)$ for small $h > 0$. Thus, the elliptic parametrix construction gives, for elliptic A, an element $B \in \Psi^{-s,-\alpha}_{0,\hbar}(X)$, so that $AB - I$, $BA - I \in h^\infty\Psi^{-\infty,\alpha}_{0,\hbar}(X)$ upon restricting h to $(0, h_0)$ for sufficiently small $h_0 > 0$. Restricted to the space $\mathrm{Diff}^{s}_{0,\hbar}(X)$ of *differential* operators, $\sigma^{s}_{0,\hbar} = \sigma^{s,0}_{0,\hbar}$ maps onto the space

$$P^s({}^0T^*X) \subset S^s({}^0T^*X)$$

of symbols which are (not necessarily homogeneous) polynomials of degree s in the fibers.

We also have an operator wave front set $\mathrm{WF}'_{0,\hbar}(A) \subset \overline{{}^0T^*}X$ for $A \in h^{-N}\Psi^{s,\alpha}_{0,\hbar}(X)$ which is the complement of all points in $\overline{{}^0T^*}X$ near which the full symbol of A vanishes to infinite order; thus, $\mathrm{WF}'_{0,\hbar}(A) = \varnothing$ if and only if $A \in h^\infty\Psi^{-\infty,\alpha}_{0,\hbar}(X)$.

Semiclassical 0-ps.d.o.s act on weighted semiclassical 0-Sobolev spaces in the expected manner. We also define the corresponding wave front set.

Definition 2.12 (semiclassical 0-Sobolev wave front set). Let s and α be the differential and decay order, respectively, and suppose that $u = (u_h)_{h\in(0,1)} \in h^{-N}H^{-N,\alpha}_{0,h}(X)$ for some N. Then

$$\mathrm{WF}^{s,\alpha}_{0,\hbar}(u) \subset \overline{{}^0T^*}X$$

is the complement of the set of all $\varpi \in \overline{{}^0T^*X}$ for which there exists $A \in \Psi^{s,\alpha}_{0,\hbar}(X)$, elliptic at ϖ, such that Au is uniformly bounded (as $h \to 0$) in $L^2(X)$.

Thus, if $u \in h^{-N} H^{-N,\alpha}_{0,h}(X)$ satisfies $\mathrm{WF}^{s,\alpha}_{0,\hbar}(u) = \varnothing$, then $u \in H^{s,\alpha}_{0,h}(X)$ upon restricting to $h \in (0, h_0)$ with $h_0 > 0$ sufficiently small. The part $\mathrm{WF}^{s,\alpha}_{0,\hbar}(u) \cap {}^0T^*X$ of the semiclassical 0-wave front set of u at finite 0-momenta is independent of s.

For the commutator of $A_j \in \Psi^{s_j,\alpha_j}_{0,\hbar}(X)$, $j = 1, 2$, with $a_j = \sigma^{s_j,\alpha_j}_{0,\hbar}(A_j)$, we have

$$\frac{i}{h}[A_1, A_2] \in \Psi^{s_1+s_2-1,\alpha_1+\alpha_2}_{0,\hbar}(X), \quad \sigma^{s_1+s_2-1,\alpha_1+\alpha_2}_{0,\hbar}\Big(\frac{i}{h}[A_1, A_2]\Big) = H_{a_1} a_2.$$

Here, in terms of local coordinates on ${}^0T^*X$ defined by writing the canonical 1-form as

$$\xi \frac{\mathrm{d}x}{x} + \sum_{j=1}^{n-1} \eta_j \frac{\mathrm{d}y^j}{x},$$

the Hamiltonian vector field of $p = p(x, y, \xi, \eta)$ takes the form

$$H_a = (\partial_\xi p)\Big(x\partial_x + \sum_{j=1}^{n-1} \eta_j \partial_{\eta_j}\Big) + \sum_{j=1}^{n-1} (\partial_{\eta_j} p) x \partial_{y^j} - \Big(\Big(x\partial_x + \sum_{j=1}^{n-1} \eta_j \partial_{\eta_j}\Big) p\Big)\partial_\xi - \sum_{j=1}^{n-1} (x\partial_{y^j} p)\partial_{\eta_j}.$$

We now proceed to prove the above statements about compositions and principal symbols. With X denoting a manifold with embedded boundary, the 0-double space is the blow-up of X^2 at the boundary $\partial\,\mathrm{diag}_X$ of the diagonal. If x and y are local coordinates on the manifold X, with x a boundary-defining function and $y \in \mathbb{R}^{n-1}$ denoting local coordinates on ∂X, let us write x and y also for their pullbacks to X^2 from the left factor; we write x' and y' for their pullbacks from the right factor. This blow-up is then that of $x = x' = 0$, $y = y'$, and in the interior of the front face, called the 0-*front face*, one can use

$$x, \quad y, \quad \frac{x - x'}{x}, \quad \frac{y - y'}{x}$$

as coordinates, with the lifted diagonal being given by

$$\Big\{\frac{x - x'}{x} = 0, \ \frac{y - y'}{x} = 0\Big\}.$$

Note that near the boundary of the lifted diagonal one can indeed always assume that the coordinates from the left and right factors on ∂X are identical; i.e., one is on the same coordinate chart in both factors since one is in a neighborhood of the diagonal of $\partial X \times \partial X$. Equivalently, we could use coordinates

$$x', \quad y', \quad \frac{x - x'}{x'}, \quad \frac{y - y'}{x'}.$$

For semiclassical families one considers $X^2 \times [0, 1)_h$ and blows up the boundary of the diagonal still, so in the interior of the front face (still called the 0-front face) one can simply add h to the list coordinates; i.e., they are

$$x, \quad y, \quad \frac{x - x'}{x}, \quad \frac{y - y'}{x}, \quad h.$$

The semiclassical 0-double space is the blow-up at the lifted diagonal at $h = 0$, i.e.,

$$\left\{\frac{x-x'}{x} = 0,\ \frac{y-y'}{x} = 0,\ h = 0\right\}.$$

The resulting front face is the *semiclassical front face*, and local coordinates in the interior are

$$x, \quad y, \quad X = \frac{x-x'}{hx}, \quad Y = \frac{y-y'}{hx}, \quad h. \tag{2-25}$$

While these are valid only in the interior, in a neighborhood of the semiclassical front face, smooth functions on the semiclassical double space which vanish to infinite order at the lift of the $h = 0$ face are equivalently smooth functions of the above coordinates which are *Schwartz* in X, Y. Indeed $\langle (X, Y)\rangle^{-1}$ is a defining function of the lift of this face in this region, since, e.g., where $(x - x')/x$ is large relative to $(y - y')/x$ and h, local coordinates are

$$x, \quad y, \quad \frac{x-x'}{x}, \quad \widehat{Y} = \frac{y-y'}{x-x'}, \quad \frac{hx}{x-x'} = X^{-1},$$

with the latter defining the lift of the $h = 0$ face. Thus,

$$\langle (X, Y)\rangle^{-1} = (1 + X^2 + Y^2)^{-1/2} = X^{-1}(1 + X^{-2} + \widehat{Y}^2)^{-1/2}$$

shows that X^{-1} is equivalent to $\langle (X, Y)\rangle^{-1}$ in this region. Instead of (2-25), one can equivalently use

$$x', \quad y', \quad \frac{x-x'}{x'h}, \quad \frac{y-y'}{x'h}, \quad h;$$

infinite-order vanishing at the lift of $h^{-1}(0)$ now corresponds to Schwartz decay in the third and fourth variables.

Semiclassical 0-pseudodifferential operators $\Psi_{0,\hbar}^{s,\alpha,k} = h^{-k}\Psi_{0,\hbar}^{s,\alpha,0} = h^{-k}\Psi_{0,\hbar}^{s,\alpha}$, with the orders being the differential, decay, and semiclassical orders, are defined on the semiclassical 0-double space with Schwartz kernels demanded to be conormal to the lifted diagonal (corresponding to the differential order s) and with rapid decay at all faces but the 0-face (corresponding to the decay order α) and the semiclassical face (corresponding to the semiclassical order k) where they are conormal. In fact, regularity in h is not necessary here and will not be imposed in what follows. However, our arguments below only need the full control of the differential order and semiclassical behavior, i.e., we work modulo $\Psi_{0,\hbar}^{-\infty,\alpha,-\infty}$. Due to this, we may impose that the Schwartz kernels are compactly supported in a neighborhood of the semiclassical front face in the sense that the support does not intersect any of the boundary hypersurfaces except the necessary ones, i.e., the 0-face, the semiclassical face, and the lift of $h = 0$, rapidly vanishing at the latter. Correspondingly, elements of $\Psi_{0,\hbar}^{-\infty,\alpha,k}$ *with the stated support condition* are of the form

$$(Av)(x, y) = h^{-n}\int K(x, y, X, Y, h)v(x', y')\frac{\mathrm{d}x'\,\mathrm{d}y'}{(x')^n},$$

where K is a conormal function (i.e., a symbol) of order α, k, with rapid decay in (X, Y), i.e., it satisfies estimates

$$|(xD_x)^\epsilon D_y^\beta D_X^\gamma D_Y^\delta K(x, y, X, Y, h)| \leq C_{\epsilon\beta\gamma\delta N}h^{-k}x^{-\alpha}\langle (X, Y)\rangle^{-N}, \tag{2-26}$$

with support in

$$\left|\frac{x-x'}{x}\right| < c, \quad \left|\frac{y-y'}{x}\right| < c,$$

where c can be taken small (and in any case should be $< \frac{1}{2}$, so that x'/x is bounded and bounded away from 0, i.e., x and x' are comparable). Note that this can be rewritten using

$$x' = x - hxX, \quad y' = y - hxY, \quad \frac{x'}{x} = 1 - hX \tag{2-27}$$

as

$$(Av)(x, y) = \int K(x, y, X, Y, h)v(x - hxX, y - hxY)(1 - hX)^{-n}\,\mathrm{d}X\,\mathrm{d}Y, \tag{2-28}$$

with K supported in

$$|X| < \frac{c}{h}, \quad |Y| < \frac{c}{h}. \tag{2-29}$$

We remark that, in order to obtain arbitrary elements of $\Psi_{0,\hbar}^{-\infty,\alpha,k}$ from these, we just need to add elements of $\Psi_{0,\hbar}^{-\infty,\alpha,-\infty}$, which in view of the infinite-order vanishing at the semiclassical front face are in fact simply $\mathcal{O}(h^\infty)$ families of elements of $\Psi_0^{-\infty,\alpha}$ and hence have simple properties on semiclassical 0-Sobolev spaces.

Now, for general s, semiclassical 0-pseudodifferential operators $\Psi_{0,\hbar}^{s,\alpha,k}$ can be considered as the sum of $\Psi_{0,\hbar}^{-\infty,\alpha,k}$ and elements of $\Psi_{0,\hbar}^{s,\alpha,k}$ whose support only intersects the faces that intersect the lifted diagonal, i.e., the semiclassical and 0-faces, and thus have compact support in X, Y. Now (standard) symbols a of compact support in x, y, X, Y and of order s, α, k satisfy estimates

$$|(x D_x)^\epsilon D_y^\beta D_X^\gamma D_Y^\delta D_\xi^\mu D_\eta^\nu a(x, y, X, Y, h, \xi, \eta)| \leq C_{\epsilon\beta\gamma\delta\mu\nu} h^{-k} x^{-\alpha} \langle(\xi, \eta)\rangle^{s-|\mu|-|\nu|}, \tag{2-30}$$

so s is the usual symbolic, α is the growth at the boundary, and k the semiclassical order, with the sign convention that the space grows as s, α, k grow. Hence, using the definition of conormal distributions, locally, the operators with Schwartz kernel supported near the diagonal acting on test functions v are of the form

$$(Av)(x, y) = (2\pi h)^{-n} \int e^{i(X\xi+Y\eta)} a(x, y, X, Y, h, \xi, \eta) v(x', y')\,\mathrm{d}\xi\,\mathrm{d}\eta \frac{\mathrm{d}x'\,\mathrm{d}y'}{(x')^n},$$

interpreted as an oscillatory integral (i.e., initially for symbols of sufficiently negative order, then extended by continuity in appropriate seminorms and density). The support may be taken to lie in $|X|, |Y| < C$, $C > 0$, $x < x_0$, and one may even take C, x_0 small (though $C > 0$ small is not important here, while x_0 small is implicit already from the region to which we are localizing in this whole discussion), where the powers of x' and h enter as the normalization so that a symbol of order s, α, k defines a pseudodifferential operator of the same order, as we shall momentarily see. On the support of a, by (2-27), we note that x'/x differs from 1 by an $\mathcal{O}(h)$ term, and similarly y and y' differ by an $\mathcal{O}(hx)$ term. The integral can be rewritten as

$$(Av)(x, y) = (2\pi)^{-n} \int e^{i(X\xi+Y\eta)} a(x, y, X, Y, h, \xi, \eta) \times v(x - hxX, y - hxY)(1 - hX)^{-n}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}X\,\mathrm{d}Y. \tag{2-31}$$

Note that the singular powers of x' and h have disappeared, so an operator of order $(s, 0, 0)$ is indeed a "standard" order[17] $s - \frac{1}{4}$ conormal distribution on the semiclassical 0-double space, explaining the order convention above; in general there is a weight α at the 0-face and k at the semiclassical face.

We can combine the two classes we discussed into a single quantization formula.[18] Namely, if in (2-30) we allow a to be supported in $|X| < \frac{c}{h}$, $|Y| < \frac{c}{h}$, as in (2-29) for the kernel K, without changing the estimate (2-30) itself (so it is simply a uniform estimate in X, Y subject to this support condition, though the precise estimate is not too important, and even polynomial growth in (X, Y) could be allowed for the Schwartzness reason explained below), then the joint quantization formula (2-31) and the kernel action (2-28) show that the distributional kernel is in fact simply the inverse Fourier transform of a in the last two variables, evaluated at (X, Y). Calling the inverse Fourier variables $(\widetilde{X}, \widetilde{Y})$ for a moment for clarity, given the symbolic estimates for a, this inverse Fourier transform is a conormal distribution to $\widetilde{X} = 0$, $\widetilde{Y} = 0$, with Schwartz behavior at infinity in $(\widetilde{X}, \widetilde{Y})$, and thus in $|(\widetilde{X}, \widetilde{Y})| > 1$, say, is C^∞ and satisfies estimates of the form

$$|(xD_x)^\epsilon D_y^\beta D_X^\gamma D_Y^\delta D_{\widetilde{X}}^\mu D_{\widetilde{Y}}^\nu (\mathcal{F}^{-1}_{\xi,\eta} a)(x, y, X, Y, h, \widetilde{X}, \widetilde{Y})| \leq C_{\epsilon\beta\gamma\delta\mu\nu,N} h^{-k} x^{-\alpha} \langle (\widetilde{X}, \widetilde{Y}) \rangle^{-N}. \tag{2-32}$$

Hence under pullback by the map

$$(x, y, X, Y, h) \mapsto (x, y, X, Y, h, \widetilde{X} = X, \widetilde{Y} = Y)$$

away from $X = 0$, $Y = 0$, i.e., the lifted diagonal, the result is that the Schwartz kernel is C^∞ away from the diagonal with estimates

$$|(xD_x)^\epsilon D_y^\beta D_X^\gamma D_Y^\delta (\mathcal{F}^{-1}_{\xi,\eta} a)(x, y, X, Y, h, X, Y)| \leq C_{\epsilon\beta\gamma\delta,N} h^{-k} x^{-\alpha} \langle (X, Y) \rangle^{-N}, \tag{2-33}$$

which is exactly of the form (2-26). On the other hand, in $|(\widetilde{X}, \widetilde{Y})| < 2$, hence after pullback in $|(X, Y)| < 2$, this is a conormal distribution (with $\widetilde{X} D_{\widetilde{X}}$, $\widetilde{X} D_{\widetilde{Y}}$, etc., i.e., vector fields tangent to $\widetilde{X} = 0$, $\widetilde{Y} = 0$, preserving regularity after the inverse Fourier transform, hence XD_X, XD_Y, etc., after the pull back) and hence exactly of the oscillatory integral form discussed above. Conversely, any kernel of the form (2-26) can be divided up by a partition of unity to one supported in $|(X, Y)| > 1$ and one in $|(X, Y)| < 2$. Write the latter piece as $\chi(h(X, Y))\chi_2(X, Y)K$ (with the first factor identically 1 on the support of K); then the Fourier transform of $\chi_2(X, Y)K$ in X, Y gives rise to an amplitude, call it

$$a_2 = \chi(h(X, Y)) \mathcal{F}_{(X,Y)\to(\xi,\eta)}(\chi_2(X, Y)K),$$

which satisfies both the support conditions and the estimates (2-30).

[17] The shift by $-\frac{1}{4}$ being the conventional shift due to the presence of the parametric variable h.

[18] While our quantization formula is not completely global since we localize to the region $|X| < c/h$, $|Y| < c/h$, it could be made global by a slight twist. In order to do this, one should work with

$$\overline{X} = \frac{x - x'}{(xx')^{1/2} h}, \quad \overline{Y} = \frac{y - y'}{(xx')^{1/2} h}$$

in place of X, Y, in which case no cutoff χ is needed below in (2-34). However, the formulae become more cumbersome, and as we do not need the global results, we stick with our choices for the simplicity of presentation.

We can mostly eliminate the X, Y dependence of a in the following sense: if we fix $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ which is identically 1 near 0, we can require

$$a(x, y, X, Y, h, \xi, \eta) = \phi(X, Y)a_0(x, y, h, \xi, \eta) + a_N(x, y, X, Y, h, \xi, \eta),$$

where a_N has N lower order in the differential order sense. Indeed, this follows by simply Taylor-expanding the original a in X, Y around 0 and using that $X^\epsilon Y^\beta$, after integration by parts, converts into $D_\xi^\epsilon D_\eta^\beta$. In particular, to leading order, i.e., modulo $S^{s-1,\alpha,k}$,

$$a_0(x, y, h, \xi, \eta) = a(x, y, 0, 0, h, \xi, \eta),$$

where a on the right-hand side is the original a. An asymptotic summation argument then allows one to replace a_N by a_∞ which is order $-\infty$ in the differential order sense. One can then replace $\phi(X, Y)$ in front of a_0 by $\chi(h(X, Y))$, χ identically 1 near 0, as the above arguments show that the difference is of order $-\infty$ in the differential order sense and thus can be absorbed into a_∞. Finally, the above arguments using the Fourier transform also show that a_∞ gives rise to an operator of order $-\infty$ which locally on $\operatorname{supp}\chi(h(X, Y))$ can be written as the oscillatory integral for a symbol in $S^{-\infty,\alpha,k}$ which in fact is independent of X, Y. Thus, modulo $\Psi_{0,\hbar}^{-\infty,\alpha,-\infty}$, semiclassical zero pseudodifferential operators are of the form

$$\begin{aligned}(Av)(x, y) &= (q_{L,0,\hbar}(a)v)(x, y)\\ &= (2\pi h)^{-n}\int e^{i(X\xi+Y\eta)}\chi(h(X, Y))a(x, y, h, \xi, \eta)v(x', y')\,\mathrm{d}\xi\,\mathrm{d}\eta\frac{\mathrm{d}x'\,\mathrm{d}y'}{(x')^n},\end{aligned} \tag{2-34}$$

where χ is as above, i.e., smooth, compactly supported, and identically 1 near 0.

We next consider the composition of two such operators. For the second operator, we use the second system of local coordinates mentioned previously. Since the product Schwartz kernel needs to be evaluated on X^3, we use x' as the output and x'' as the input variables of the second operator, so

$$Bu(x', y') = (2\pi h)^{-n}\int e^{i(X'\xi'+Y'\eta')}b(x'', y'', X', Y', h, \xi', \eta')u(x'', y'')\,\mathrm{d}\xi'\,\mathrm{d}\eta'\frac{\mathrm{d}x''\,\mathrm{d}y''}{(x'')^n},$$

where

$$X' = \frac{x' - x''}{hx''}, \quad Y' = \frac{y' - y''}{hx''}.$$

Thus, $x'/x'' = 1 + hX'$ differs from 1 by $\mathcal{O}(c)$.

The Schwartz kernel of AB then is of the form

$$\begin{aligned}ABu(x, y) = (2\pi h)^{-2n}\int e^{i(X\xi+Y\eta+X'\xi'+Y'\eta')}a(x, y, X, Y, h, \xi, \eta)b(x'', y'', X', Y', h, \xi', \eta')\\ \times u(x'', y'')\,\mathrm{d}\xi\,\mathrm{d}\eta\frac{\mathrm{d}x'\,\mathrm{d}y'}{(x')^n}\,\mathrm{d}\xi'\,\mathrm{d}\eta'\frac{\mathrm{d}x''\,\mathrm{d}y''}{(x'')^n};\end{aligned}$$

our goal is to show that this indeed is the Schwartz kernel of a pseudodifferential operator of the desired type. First, let us express the integrals in terms of the expected coordinates

$$x, \quad y, \quad X'' = \frac{x - x''}{hx}, \quad Y'' = \frac{y - y''}{hx}, \quad h;$$

note that

$$x'' = x - hxX'', \quad y'' = y - hxY'', \quad \frac{x''}{x} = 1 - hX''.$$

We will keep the variables of integration $X, Y, \xi, \eta, \xi', \eta'$, and express X', Y' in terms of X, Y, X'', Y''. Namely,

$$\frac{x'}{x''} = \frac{x'}{x} \cdot \frac{x}{x''} = \frac{1 - hX}{1 - hX''},$$

so

$$X' = h^{-1}\left(\frac{x'}{x''} - 1\right) = \frac{X'' - X}{1 - hX''}, \quad Y' = \frac{x}{x''}(Y'' - Y) = \frac{Y'' - Y}{1 - hX''}.$$

Moreover, the density $\mathrm{d}x'\,\mathrm{d}y'/(x')^n$ becomes $(1 - hX)^{-n}h^n\,\mathrm{d}X\,\mathrm{d}Y$ as above. Hence, our integral takes the form (suppressing the input test function and writing out the Schwartz kernel)

$$\begin{aligned} K_{AB}(x, y, X'', Y'') = (2\pi)^{-2n}h^{-n}\int \exp\left[i\left(X\xi + Y\eta + \frac{X'' - X}{1 - hX''}\xi' + \frac{Y'' - Y}{1 - hX''}\eta'\right)\right] a(x, y, X, Y, h, \xi, \eta) \\ \times b\left(x - hxX'', y - hxY'', \frac{X'' - X}{1 - hX''}, \frac{Y'' - Y}{1 - hX''}, h, \xi', \eta'\right) \\ \times (1 - hX)^{-n}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}\xi'\,\mathrm{d}\eta'. \end{aligned}$$

On the support of ab, we note that $x'/x,\ x'/x''$ are both $1 + \mathcal{O}(c)$, and hence their ratio x/x'' is also $1 + \mathcal{O}(c)$, so hX'' is indeed $\mathcal{O}(c)$, and thus hY'' is also $\mathcal{O}(c)$. This is *almost* a slightly nonstandard parametrization of a conormal distribution (in the region where X'', Y'' are bounded), except that ab is not a symbol (though it has the desired support properties); rather, it is a product-type symbol. This is, however, easily remedied by noting that the phase is nonstationary where the symbol-type behavior fails, i.e., where (ξ', η') is small relative to (ξ, η), or vice versa. To see this, note that the derivative of the phase in (X, Y) is

$$\left(\xi - \frac{\xi'}{1 - hX''}\right)\mathrm{d}X + \left(\eta - \frac{\eta'}{1 - hX''}\right)\mathrm{d}Y,$$

which, if hX'' is small (to which region we can focus as already noted), is nonzero under either of these two scenarios. Thus, integration by parts in (X, Y) allows us to rewrite the integral — when the integrand is localized to this a priori troublesome region — as one with an amplitude that is in fact rapidly decaying and thus for which one easily sees that the composite operator behaves as desired. Thus, modulo such smoothing operators, the integrand can be assumed to be localized away from this region; hence ab can be regarded as a symbol, and then the integral is a usual parametrization of a Lagrangian distribution, albeit in the slightly broader interpretation that we are localizing to $hX'',\ hY''$ being $\mathcal{O}(c)$, and not to the standard interpretation where X'', Y'' are considered bounded.

One can then rewrite this oscillatory integral in the usual (though again nonstandard as we are working in a larger region as above) conormal parametrization formulation by using the standard stationary phase lemma (rescaling the symbolic variables to formally get into its compactly supported form, which can be done as we may assume based on the above discussion that $|(\xi, \eta)| \sim |(\xi', \eta')|$), noting that the critical

points of the phase in (X, Y, ξ', η') are

$$\xi = \frac{\xi'}{1 - hX''}, \quad \eta = \frac{\eta'}{1 - hX''}, \quad X'' = X, \quad Y'' = Y,$$

while the determinant of the Jacobian is $(1 - hX'')^{-2n}$. This shows that

$$K_{AB}(x, y, X'', Y'') = (2\pi h)^{-n} \int e^{i(X''\xi + Y''\eta)} c(x, y, X'', Y'', h, \xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta,$$

where c is a symbol given by

$$a(x, y, X'', Y'', h, \xi, \eta) b(x, y, 0, 0, h, \xi, \eta)$$

modulo terms with extra $h(X'', Y'')$ vanishing — i.e., keeping in mind that extra (X'', Y'') can be used to lower the symbolic order — at the operator level modulo $\Psi_{0,\hbar}^{s-1,\alpha,k-1}$.

Recall also that a can be arranged to be independent of X'', Y'' up to an overall compactly supported factor $\chi(h(X'', Y''))$; hence if $A = q_{L,0,\hbar}(a)$, $B = q_{L,0,\hbar}(b)$, then $AB = q_{L,0,\hbar}(c)$, where c is given by

$$a(x, y, h, \xi, \eta) b(x, y, h, \xi, \eta)$$

modulo $S^{s-1,\alpha,k-1}$ and thus at the operator level modulo $\Psi_{0,\hbar}^{s-1,\alpha,k-1}$.

3. Geometric and analytic setting

The goal of this section is to define the class of wave-type operators which we shall study in this paper (see Definition 3.7). We discuss the underlying geometry in Section 3.1 before turning to the differential operators of interest in Section 3.2.

3.1. ***Admissible metrics.*** Before describing the general setup in Section 3.1.2, we consider as a model case the Minkowski metric.

3.1.1. *The Minkowski metric near null infinity.* In polar coordinates, the Minkowski metric on $\mathbb{R}^{1+n}$ takes the form

$$g_0 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \not{g}, \tag{3-1}$$

where $t \in \mathbb{R}$, $r > 0$, and $\not{g}$ is the standard metric on $\mathbb{S}^{n-1}$. Since this can equivalently be written as $g_0 = -\mathrm{d}t^2 + \sum_{j=1}^n (\mathrm{d}x^j)^2$, where $(t, x^1, \dots, x^n)$ are standard coordinates on $\mathbb{R}^{1+n}$, we find that g_0 is a Lorentzian scattering metric, by which we mean

$$g_0 \in \mathcal{C}^\infty(\overline{\mathbb{R}^{n+1}}; S^2\,{}^{\mathrm{sc}}T^*\overline{\mathbb{R}^{n+1}}), \quad g_0^{-1} \in \mathcal{C}^\infty(\overline{\mathbb{R}^{n+1}}; S^2\,{}^{\mathrm{sc}}T\overline{\mathbb{R}^{n+1}}).$$

We are interested in the structure of g_0 near the large end of the future light cone; thus, we introduce

$$\varrho := \frac{1}{t+r}, \quad v := \frac{t-r}{t+r}. \tag{3-2}$$

Inserting these into (3-1) gives

$$g_0 = -v\frac{\mathrm{d}\varrho^2}{\varrho^4} + \frac{\mathrm{d}\varrho}{\varrho^2} \otimes_s \frac{\mathrm{d}v}{\varrho} + \Big(\frac{1-v}{2\varrho}\Big)^2 \not{g}. \tag{3-3}$$

The closure of every future light cone in $\mathbb{R}^{n+1}$ intersects $\partial\overline{\mathbb{R}^{n+1}}$ in the $(n-1)$-dimensional set

$$Y := \{\varrho = v = 0\} \subset \overline{\mathbb{R}^{n+1}}.$$

Thus, to resolve different light cones also at infinity, we blow up Y and obtain the manifold with corners

$$\widetilde{M} = [\overline{\mathbb{R}^{n+1}}; Y].$$

This is defined as $(\overline{\mathbb{R}^{n+1}} \setminus Y) \sqcup SN^+Y$, with $SN^+Y = N^+Y/\mathbb{R}_+$ denoting the (nonstrictly) inward pointing spherical normal bundle, and equipped with the minimal smooth structure in which polar coordinates around Y are smooth. We denote the front face (i.e., SN^+Y) of $\widetilde{M}$ by $\widetilde{\mathscr{I}}^+$, the "north cap" — the closure of $\{\varrho = 0, v > 0\}$ — by $\widetilde{I}^+$, and the closure of $\{\varrho = 0, v < 0\}$ by $\widetilde{I}^0$. (The latter terminology is imprecise and should only be taken seriously in $t > -r$, as we are not resolving past null infinity here. See also [Baskin et al. 2015, §3; Hintz and Vasy 2020, §2].) The restriction of the blow-down map $\widetilde{M} \to \overline{\mathbb{R}^{n+1}}$ (the identity on $\overline{\mathbb{R}^{n+1}} \setminus Y$, and the base projection on SN^+Y) restricts to a fibration $\widetilde{\mathscr{I}}^+ \to Y$, with fibers diffeomorphic to a closed interval. See Figure 3.1.

Near the interior of $\widetilde{\mathscr{I}}^+$, local coordinates are given by

$$\varrho = \frac{1}{t+r}, \quad t_* := \frac{v}{\varrho} = t - r, \quad \omega \in \mathbb{S}^{n-1},$$

in which the metric takes the double null form

$$g_0 = \frac{\mathrm{d}\varrho}{\varrho^2} \otimes_s \mathrm{d}t_* + \big(\tfrac{1}{4} + \mathcal{O}(\varrho)\big)\varrho^{-2}\not{g}.$$

Freezing coefficients at a single fiber of $\widetilde{\mathscr{I}}^+$, let us consider the model metric

$$\frac{\mathrm{d}\varrho}{\varrho^2} \otimes_s \mathrm{d}t_* + \frac{\mathrm{d}y^2}{\varrho^2},$$

where $y \in \mathbb{R}^{n-1}$. This is homogeneous of degree -2 under pullback by the parabolic scaling $(0, \infty) \ni \lambda : (\varrho, y) \mapsto (\lambda^2\varrho, \lambda y)$ (see also Remark 1.6); it is thus natural to change the homogeneity of the metric by multiplying it by ϱ and subsequently introducing $x_{\mathscr{I}} := \varrho^{1/2}$. This produces the metric

$$2\frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}} \otimes_s \mathrm{d}t_* + \frac{\mathrm{d}y^2}{x_{\mathscr{I}}^2},$$

which is an edge metric on $\mathbb{R}_{t_*} \times [0, 1)_{x_{\mathscr{I}}} \times \mathbb{R}^{n-1}_y$. To capture this invariantly, one defines the manifold

$$M = \big[\widetilde{M}; \widetilde{\mathscr{I}}^+, \tfrac{1}{2}\big] \tag{3-4}$$

as the *square root blow-up* of $\widetilde{M}$ at $\widetilde{\mathscr{I}}^+$, defined to be $\widetilde{M}$ as a set, but with the square root of a defining function of $\widetilde{\mathscr{I}}^+$ adjoined to the smooth structure; denoting the lifts of $\widetilde{I}^0$, $\widetilde{\mathscr{I}}^+$, $\widetilde{I}^+$ by I^0, $\mathscr{I}^+$, I^+, respectively, null infinity $\mathscr{I}^+$ is the total space of a fibration with base $\mathbb{S}^{n-1}$ and closed intervals as fibers as before, and ϱg_0 is a nondegenerate Lorentzian edge metric on $M \setminus (I^0 \cup I^+)$.

Away from $\mathscr{I}^+ \subset M$, i.e., away from $Y \subset \partial\overline{\mathbb{R}^{n+1}}$ on the other hand, the degree -2 homogeneity of g_0 in (3-3) under scaling in ϱ suggests working with $\varrho^2 g_0$, which is a nondegenerate Lorentzian b-metric

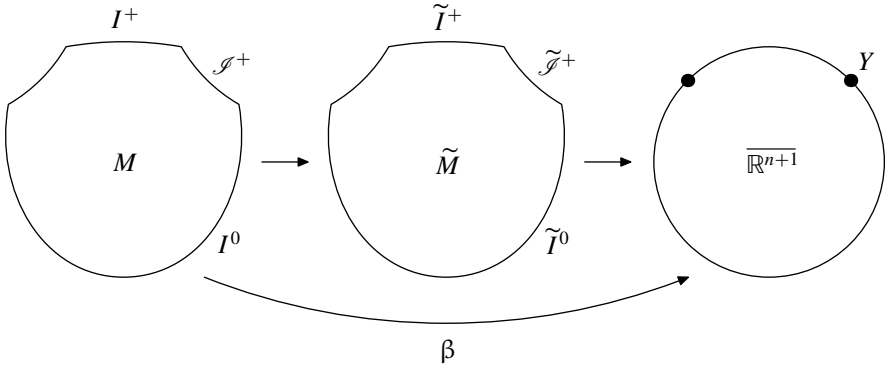

Figure 3.1. The resolution $\widetilde{M}$ of the radial compactification $\overline{\mathbb{R}^{n+1}}$ at the light cone Y at future infinity and the boundary hypersurfaces of $\widetilde{M}$. The manifold M is the square root blow-up of $\widetilde{M}$ at $\widetilde{\mathscr{I}}^+$, and β is the total blow-down map $M \to \overline{\mathbb{R}^{n+1}}$.

on $\overline{\mathbb{R}^{n+1}} \setminus Y$.[19] Globally on M, the rescaling $\varrho^2 x_{\mathscr{I}}^{-2} g_0$, where $x_{\mathscr{I}} \in \mathcal{C}^\infty(M)$ is a defining function of $\widetilde{\mathscr{I}}^+$, is a smooth nondegenerate Lorentzian edge-b-metric on M, as we proceed to demonstrate.

To study g_0 near the transition regions $I^0 \cap \mathscr{I}^+$ and $\mathscr{I}^+ \cap I^+$, we first pass to the coordinates $t_* = t - r$, r, $\omega \in \mathbb{S}^{n-1}$, in which the Minkowski metric and its dual metric take the form

$$g_0 = -\mathrm{d}t_*^2 - 2\,\mathrm{d}t_*\,\mathrm{d}r + r^2 \not{g}, \quad g_0^{-1} = -2\partial_{t_*} \otimes_s \partial_r + \partial_r^2 + r^{-2} \not{g}^{-1}.$$

We then introduce the following.

Definition 3.1 (open sets containing parts of $\mathscr{I}^+$; local coordinates). For $T \in \mathbb{R}$, we define

$$\mathcal{U}_0(T) := \{r > 1,\, t_* < T - 1\}, \quad \mathcal{U}_+(T) := \{r > 1,\, t_* > T + 1\} \subset M.$$

On $\mathcal{U}_0(T)$, we define

$$\rho_0 = \frac{1}{T - t_*}, \quad x_{\mathscr{I}} = \sqrt{\frac{T - t_*}{r}}, \quad \rho_+ = 1, \tag{3-5}$$

while on $\mathcal{U}_+(T)$ we put

$$\rho_0 = 1, \quad x_{\mathscr{I}} = \sqrt{\frac{t_* - T}{r}}, \quad \rho_+ = \frac{1}{t_* - T}. \tag{3-6}$$

Thus, $[0,1)_{\rho_0} \times [0,1)_{x_{\mathscr{I}}} \times \mathbb{S}^{n-1} \subset \mathcal{U}_0(T)$ and $[0,1)_{x_{\mathscr{I}}} \times [0,1)_{\rho_+} \times \mathbb{S}^{n-1} \subset \mathcal{U}_+(T)$ are two coordinate charts on M. The union $\mathcal{U}_0(T) \cup \mathcal{U}_+(T')$ contains a neighborhood of $\mathscr{I}^+$ provided $T' + 1 < T - 1$. The functions ρ_0, $x_{\mathscr{I}}$, and ρ_+ defined in the two charts do not agree on the overlap of the two charts (and which definition we use will always be made explicit); they are local defining functions of I^0, $\mathscr{I}^+$, and I^+, respectively. Note also that ϱ is a smooth positive multiple of $\rho_0 x_{\mathscr{I}}^2 \rho_+ = r^{-1}$ on either coordinate chart. Now, on $\mathcal{U}_0(T)$, we have

$$\partial_{t_*} = \rho_0\big(\rho_0 \partial_{\rho_0} - \tfrac{1}{2} x_{\mathscr{I}} \partial_{x_{\mathscr{I}}}\big), \quad \partial_r = -\tfrac{1}{2}\rho_0 x_{\mathscr{I}}^3 \partial_{x_{\mathscr{I}}}, \tag{3-7a}$$

[19] While the nondegeneracy and homogeneity are valid on all of $\overline{\mathbb{R}^{n+1}}$, we ignore this fact here, as we are ultimately interested in metrics and operators which have smooth or conormal coefficients only on M, but not on $\overline{\mathbb{R}^{n+1}}$ — working on the resolved space M with its edge-b-structure will be the appropriate framework for this task.

and therefore

$$\rho_0^{-2}x_{\mathscr{I}}^{-2}g_0^{-1} \equiv -\tfrac{1}{2}x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} \otimes_s (x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} - 2\rho_0\partial_{\rho_0}) + x_{\mathscr{I}}^2 \not{g}^{-1} \mod x_{\mathscr{I}}\mathcal{C}^\infty(\mathcal{U}_0(T); S^{2\,\mathrm{e,b}}TM), \tag{3-7b}$$

with ${}^{\mathrm{e,b}}TM$ spanned by $x_{\mathscr{I}}\partial_{x_{\mathscr{I}}}$, $\rho_0\partial_{\rho_0}$, $x_{\mathscr{I}}T\mathbb{S}^{n-1}$. On $\mathcal{U}_+(T)$, we similarly compute

$$\partial_{t_*} = -\rho_+\big(\rho_+\partial_{\rho_+} - \tfrac{1}{2}x_{\mathscr{I}}\partial_{x_{\mathscr{I}}}\big), \quad \partial_r = -\tfrac{1}{2}\rho_+ x_{\mathscr{I}}^3\partial_{x_{\mathscr{I}}}, \tag{3-8a}$$

and therefore

$$x_{\mathscr{I}}^{-2}\rho_+^{-2}g_0^{-1} \equiv \tfrac{1}{2}x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} \otimes_s (x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} - 2\rho_+\partial_{\rho_+}) + x_{\mathscr{I}}^2 \not{g}^{-1} \mod x_{\mathscr{I}}\mathcal{C}^\infty(\mathcal{U}_+(T); S^{2\,\mathrm{e,b}}TM), \tag{3-8b}$$

with ${}^{\mathrm{e,b}}TM$ now spanned by $x_{\mathscr{I}}\partial_{x_{\mathscr{I}}}$, $\rho_+\partial_{\rho_+}$, $x_{\mathscr{I}}T\mathbb{S}^{n-1}$. For later use, we also record the form of the metric itself on $\mathcal{U}_0(T)$, to wit,

$$\rho_0^2 x_{\mathscr{I}}^2 g_0 \equiv 2\frac{\mathrm{d}\rho_0}{\rho_0} \otimes_s \left(\frac{\mathrm{d}\rho_0}{\rho_0} + 2\frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}}\right) + x_{\mathscr{I}}^{-2}\not{g} \mod x_{\mathscr{I}}\mathcal{C}^\infty(\mathcal{U}_0(T); S^{2\,\mathrm{e,b}}T^*M). \tag{3-9}$$

3.1.2. *General setup.* We proceed to generalize the above example of the Minkowski metric. Thus, suppose M_0 is an $(n+1)$-dimensional manifold with boundary $\partial M_0 \neq \varnothing$, and suppose $Y \subset \partial M_0$ is a compact and boundaryless embedded 1-codimensional submanifold whose normal bundle (inside ∂M_0) is orientable; thus, there exists a collar neighborhood $[0,\epsilon)_\varrho \times (-\epsilon,\epsilon)_v \times Y$ of Y in M_0. Recalling (3-3), we make the following definition. We define $\widetilde{M} = [M_0; Y]$ and $M = \big[\widetilde{M}; \widetilde{\mathscr{I}}^+, \frac{1}{2}\big]$ (with $\widetilde{\mathscr{I}}^+$ the lift of Y) as in (3-4). We call $I^0 \subset M$ (spacelike infinity) (resp. $I^+ \subset M$ (future timelike infinity)) the boundary hypersurface of M where $v < 0$ (resp. $v > 0$) (and we assume that $I^0 \neq I^+$), and $\mathscr{I}^+ \subset M$ (null infinity) denotes the lift of $\widetilde{\mathscr{I}}^+$.

Definition 3.2 (admissible metrics). Let $\ell_0, \ell_+ \in (0,1]$, and let $\ell_{\mathscr{I}} \in \big(0, \frac{1}{2}\big]$. A Lorentzian metric $g \in \mathcal{C}^\infty(M_0^\circ; S^2T^*M_0^\circ)$ is called an $(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)$-*admissible metric* (or simply *admissible metric*) if

$$\rho_0^2 x_{\mathscr{I}}^2 \rho_+^2 \left[g - \left(-v\frac{\mathrm{d}\varrho^2}{\varrho^4} + \frac{\mathrm{d}\varrho}{\varrho^2} \otimes_s \frac{\mathrm{d}v}{\varrho} + \frac{k(y,\mathrm{d}y)}{4\varrho^2}\right)\right] \in (x_{\mathscr{I}}\mathcal{C}^\infty + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)})(M; S^{2\,\mathrm{e,b}}T^*M), \tag{3-10}$$

where k is a Riemannian metric on Y and ρ_0, $x_{\mathscr{I}}$, and ρ_+ are defining functions of I^0, $\mathscr{I}^+$, and I^+, respectively.

One instance of an admissible metric arising in a nonlinear stability problem is given in Example 3.4 below.

Remark 3.3 (weaker error term: geometry). The remainder of this section as well as the analysis of the null-bicharacteristic flow of admissible metrics in the eb-phase space in Section 4.1 go through with only notational modifications if we relax (3-10) to membership in the space $\mathcal{A}^{(((0,0),\ell_0),2\ell_{\mathscr{I}},((0,0),\ell_+))}(M; S^{2\,\mathrm{e,b}}T^*M)$. Admissible metrics arising in applications (e.g., the one in (3-15)) typically have the stronger form (3-10), which in particular entails the smoothness (rather than mere conormality), as an eb-metric, of the restriction of $\rho_0^2 x_{\mathscr{I}}^2 \rho_+^2 g$ to I^0 (resp. I^+) down to $I^0 \cap \mathscr{I}^+$ (resp. $I^+ \cap \mathscr{I}^+$). See also Remark 3.9.

The inclusion of the normalization factor 2 in the weight at $\mathscr{I}^+$ is merely a matter of convention, motivated by the square root blow-up (so $x_{\mathscr{I}}^{2\ell_{\mathscr{I}}} = \rho_{\mathscr{I}}^{\ell_{\mathscr{I}}}$, where $\rho_{\mathscr{I}} = x_{\mathscr{I}}^2$ is a defining function of $\widetilde{\mathscr{I}}^+ \subset \widetilde{M}$).

The factor $\frac{1}{4}$ is inserted so as to make $k = \not{g}$ in the case of the Minkowski metric (cf. (3-3)). Since the leading-order term in (3-10) is the Minkowski metric from (3-3) (possibly with a different metric on $Y \cong \mathbb{S}^{n-1}$), we see from (3-7b) and (3-8b) that Definition 3.2 can be stated in a number of equivalent ways.

(1) Dual metric: The dual metric g^{-1} is of the form

$$\rho_0^{-2} x_{\mathscr{I}}^{-2} \rho_+^{-2} \big[g^{-1} - \varrho^2 (4(\varrho\partial_\varrho + v\partial_v) \otimes_s \partial_v + 4k^{-1}(y, \partial_y)) \big] \in (x_{\mathscr{I}} \mathcal{C}^\infty + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)})(M; S^{2\,\mathrm{e,b}} TM). \tag{3-11}$$

(2) Near $I^0 \cap \mathscr{I}^+$ (resp. $\mathscr{I}^+ \cap I^+$): On $\mathcal{U}_0(T)$ (resp. $\mathcal{U}_+(T)$), see Definition 3.1, and in the coordinates[20] (3-5) (resp. (3-6)),[21]

$$\begin{aligned} \rho_0^{-2} x_{\mathscr{I}}^{-2} g^{-1} &\equiv -\tfrac{1}{2} x_{\mathscr{I}} \partial_{x_{\mathscr{I}}} \otimes_s (x_{\mathscr{I}} \partial_{x_{\mathscr{I}}} - 2\rho_0 \partial_{\rho_0}) + k^{-1}(y, x_{\mathscr{I}} \partial_y) \\ &\qquad \mathrm{mod}\ (x_{\mathscr{I}} \mathcal{C}^\infty + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}})})(\mathcal{U}_0(T); S^{2\,\mathrm{e,b}} TM), \end{aligned} \tag{3-12}$$

$$\begin{aligned} x_{\mathscr{I}}^{-2} \rho_+^{-2} g^{-1} &\equiv \tfrac{1}{2} x_{\mathscr{I}} \partial_{x_{\mathscr{I}}} \otimes_s (x_{\mathscr{I}} \partial_{x_{\mathscr{I}}} - 2\rho_+ \partial_{\rho_+}) + k^{-1}(y, x_{\mathscr{I}} \partial_y) \\ &\qquad \mathrm{mod}\ (x_{\mathscr{I}} \mathcal{C}^\infty + \mathcal{A}^{(2\ell_{\mathscr{I}}, \ell_+)})(\mathcal{U}_+(T); S^{2\,\mathrm{e,b}} TM). \end{aligned} \tag{3-13}$$

(We drop the weight at I^+ (resp. I^0) from the notation, since $\mathcal{U}_0(T) \cap I^+ = \varnothing$ (resp. $\mathcal{U}_+(T) \cap I^0 = \varnothing$).)

Example 3.4 (double null formulation). Examples of admissible metrics which arise in the context of the nonlinear stability of asymptotically flat spacetimes as solutions of the Einstein vacuum equations are given in [Hintz 2023b, §§3.2–3.3]. Concretely, we write the Minkowski metric (3-1) (which is $(1, 1, 1)$-admissible) in double null form

$$g_0 = -\mathrm{d}x^0\, \mathrm{d}x^1 + r^2 \not{g}, \quad x^0 = t + r = \frac{1}{\varrho}, \quad x^1 = t - r = \frac{v}{\varrho}$$

and work in $v < 0$. In terms of $x_{\mathscr{I}} = \sqrt{-v} \geq 0$, $\rho_0 = -\varrho/v \geq 0$ (thus $x^0 = (\rho_0 x_{\mathscr{I}}^2)^{-1}$, $x^1 = -\rho_0^{-1}$) on $\mathcal{U}_0(0)$ (which are smooth positive multiples of the coordinates ρ_0 and $x_{\mathscr{I}}$ used in Definition 3.1),

$$\begin{aligned} \mathrm{d}x^0 &= -\frac{1}{\rho_0 x_{\mathscr{I}}^2} \left(\frac{\mathrm{d}\rho_0}{\rho_0} + 2\frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}} \right) \in \rho_0^{-1} x_{\mathscr{I}}^{-2} \mathcal{C}^\infty(M; {}^{\mathrm{e,b}}T^*M), \\ \mathrm{d}x^1 &= \rho_0^{-1} \frac{\mathrm{d}\rho_0}{\rho_0} \in \rho_0^{-1} \mathcal{C}^\infty(M; {}^{\mathrm{e,b}}T^*M), \\ r\, \mathrm{d}\omega &= x_{\mathscr{I}} r \frac{\mathrm{d}\omega}{x_{\mathscr{I}}} \in \rho_0^{-1} x_{\mathscr{I}}^{-1} \mathcal{C}^\infty(M; {}^{\mathrm{e,b}}T^*M) \end{aligned} \tag{3-14}$$

for $\mathrm{d}\omega \in \mathcal{C}^\infty(\mathbb{S}^{n-1}; T^*\mathbb{S}^{n-1})$. (We omit the analogous computation on $\mathcal{U}_0(2)$.) Thus, a metric g is $(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)$-admissible if (and only if, provided one fixes the Riemannian metric k on Y to be $k = \not{g}$) it

[20] Here, ρ_0, $x_{\mathscr{I}}$ (resp. $x_{\mathscr{I}}$, ρ_+) are defined using $t_* = t - r$, with t and r in turn determined by ϱ and v via (3-2); that is, $t = (1 + v)/2\varrho$ and $r = (1 - v)/2\varrho$.

[21] In terms of the metric g, this is equivalent to

$$\rho_0^2 x_{\mathscr{I}}^2 g \equiv 2\frac{\mathrm{d}\rho_0}{\rho_0} \otimes_s \left(\frac{\mathrm{d}\rho_0}{\rho_0} + 2\frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}} \right) + k\left(y, \frac{\mathrm{d}y}{x_{\mathscr{I}}} \right) \quad \left(\text{resp. } x_{\mathscr{I}}^2 \rho_+^2 g \equiv -2\frac{\mathrm{d}\rho_+}{\rho_+} \otimes_s \left(\frac{\mathrm{d}\rho_+}{\rho_+} + 2\frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}} \right) + k\left(y, \frac{\mathrm{d}y}{x_{\mathscr{I}}} \right) \right)$$

modulo sections of $S^{2\,\mathrm{e,b}}T^*M$ over $\mathcal{U}_0(T)$ (resp. $\mathcal{U}_+(T)$) of class $x_{\mathscr{I}} \mathcal{C}^\infty + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)}$.

is the sum of g_0 and linear combinations of

$$\begin{aligned} &x_{\mathscr{I}}^2(\mathrm{d}x^0)^2, && \mathrm{d}x^0\otimes_s \mathrm{d}x^1, && x_{\mathscr{I}}\,\mathrm{d}x^0\otimes_s r\,\mathrm{d}\omega,\\ &x_{\mathscr{I}}^{-2}(\mathrm{d}x^1)^2, && x_{\mathscr{I}}^{-1}\,\mathrm{d}x^1\otimes_s r\,\mathrm{d}\omega, && r\,\mathrm{d}\omega\otimes_s r\,\mathrm{d}\omega, \end{aligned} \tag{3-15}$$

where $\mathrm{d}\omega$ may change between any two occurrences, with coefficients in $x_{\mathscr{I}}\mathcal{C}^\infty + \mathcal{A}^{(\ell_0,2\ell_{\mathscr{I}},\ell_+)}$.

Example 3.5 (Schwarzschild). Evidently, the class of metrics g discussed in Example 3.4 in particular includes the Schwarzschild metric $g_{\mathrm{m}} := -\big(1-\frac{2\mathrm{m}}{r}\big)\,\mathrm{d}x^0\,\mathrm{d}x^1 + r^2\not{g}$ (and its perturbations of this class). We can alternatively explicitly relate g_{m} to the form (3-10) with $k=\not{g}$ as follows, working near $I^0\cap\mathscr{I}^+$ (the computations near $I^+\cap\mathscr{I}^+$ being analogous). Recalling the expression (3-3) for the Minkowski metric g_0 and using (3-14), we have

$$\begin{aligned} g_{\mathrm{m}}-g_0 &= \frac{2\mathrm{m}}{r}\,\mathrm{d}x^0\,\mathrm{d}x^1\\ &= -\frac{2\varrho}{1-v}\,\frac{1}{\rho_0^2 x_{\mathscr{I}}^2}\left(\frac{\mathrm{d}\rho_0}{\rho_0}+2\frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}}\right)\frac{\mathrm{d}\rho_0}{\rho_0}, \end{aligned}$$

so $\rho_0^2 x_{\mathscr{I}}^2(g_{\mathrm{m}}-g_0)$ equals $c := -2\rho_0 x_{\mathscr{I}}^2(1+x_{\mathscr{I}}^2)$ times the smooth edge-b-2-tensor

$$\left(\frac{\mathrm{d}\rho_0}{\rho_0}+2\frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}}\right)\frac{\mathrm{d}\rho_0}{\rho_0}.$$

The coefficient c here is, a fortiori, of class $x_{\mathscr{I}}\mathcal{C}^\infty(M)$ (cf. (3-10)). This computation shows, in fact, that the mass term of g_{m} is of subleading order (in the sense of decay) at $\partial M = I^0\cup\mathscr{I}^+\cup I^+$.

We fix a time orientation on (M,g) near $\mathscr{I}^+$ by declaring, at $\mathscr{I}^+$, the causal (for $g_{\mathrm{e,b}} := \rho_0^2 x_{\mathscr{I}}^2\rho_+^2 g$) edge-b tangent vectors

$$\begin{aligned} &\rho_0\partial_{\rho_0} - \tfrac{1}{2}x_{\mathscr{I}}\partial_{x_{\mathscr{I}}},\ -x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} && \text{(in the coordinates (3-5))},\\ &-\rho_+\partial_{\rho_+} + \tfrac{1}{2}x_{\mathscr{I}}\partial_{x_{\mathscr{I}}},\ -x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} && \text{(in the coordinates (3-6))} \end{aligned} \tag{3-16}$$

to be future causal. Using (3-7a) and (3-8a), this matches the standard time orientation in the Minkowski case, where ∂_{t_*} and ∂_r (i.e., in t,r coordinates ∂_t and $\partial_t+\partial_r$) are future timelike and null, respectively.

Lemma 3.6 (hypersurfaces in (M,g)). *Let g be an $(\ell_0,2\ell_{\mathscr{I}},\ell_+)$-admissible metric on M.*

(1) *Let $T\in\mathbb{R}$, and define ρ_0, $x_{\mathscr{I}}$ on $\mathcal{U}_0(T)$ by (3-5). Let $\bar\rho_0\in(0,1)$. Then there exist $\delta_0>0$ and $C>0$ such that, for all $\delta\in(0,\delta_0]$, the hypersurface $\{x_{\mathscr{I}}=\delta,\ \rho_0<\bar\rho_0\}$ is spacelike and the hypersurface $\{x_{\mathscr{I}}<2\delta,\ \rho_0=\rho_{0,0}+Cx_{\mathscr{I}}^{2\ell_{\mathscr{I}}}\}$ is also spacelike for all $\rho_{0,0}\in(0,1-2C(2\delta)^{2\ell_{\mathscr{I}}})$.*

(2) *Let $T\in\mathbb{R}$, and define $x_{\mathscr{I}}$, ρ_+ on $\mathcal{U}_+(T)$ by (3-6). Then there exist $\delta_0>0$, $\bar\rho_+\in(0,1)$, and $C>0$ such that, for all $\delta\in(0,\delta_0]$, the hypersurface $\{x_{\mathscr{I}}=\delta,\ \rho_+<\bar\rho_+\}$ is timelike and the hypersurface $\{x_{\mathscr{I}}<2\delta,\ \rho_+=\rho_{+,0}-Cx_{\mathscr{I}}^{2\ell_{\mathscr{I}}}\}$ is spacelike for all $\rho_{+,0}<1$.*

In the last statement of part (1), the range of $\rho_{0,0}$ is chosen so that the stated hypersurface is contained in the local coordinate chart $\mathcal{U}_0(T)$. See Figure 3.2.

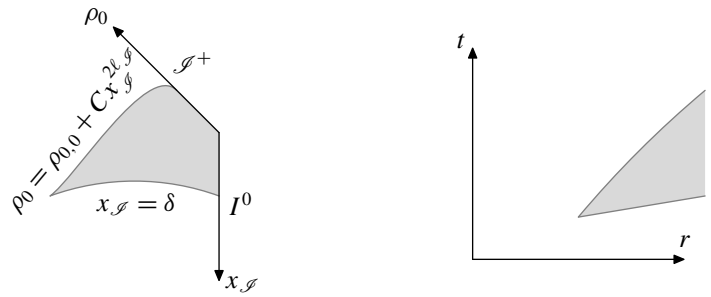

Figure 3.2. Illustration of Lemma 3.6(1). On the left, the perspective of M. On the right, the perspective of (1+1)-dimensional Minkowski space with coordinates (t, r).

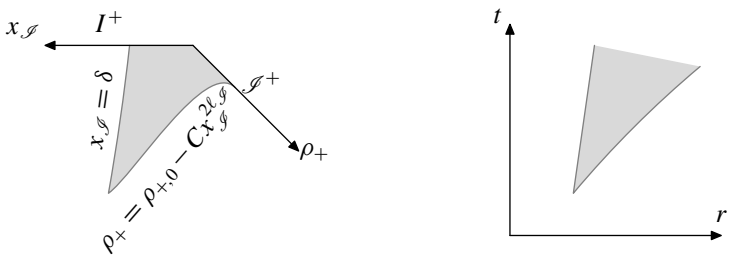

Figure 3.3. Illustration of Lemma 3.6(2). On the left, the perspective of M. On the right, the perspective of (1+1)-dimensional Minkowski space with coordinates (t, r).

Proof of Lemma 3.6. We only prove part (1); the proof of part (2) is analogous. Let $g_{\mathrm{e,b}}^{-1} := \rho_0^{-2} x_{\mathscr{I}}^{-2} g^{-1}$. Using the expression (3-12), we compute

$$g_{\mathrm{e,b}}^{-1}\left(\frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}}, \frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}}\right) \equiv -\tfrac{1}{2} \mod x_{\mathscr{I}} C^\infty + \rho_0^{\ell_0} x_{\mathscr{I}}^{2\ell_{\mathscr{I}}} \mathcal{A}^0,$$

which is thus negative for $0 \le \rho_0 < \bar{\rho}_0$ when $0 < x_{\mathscr{I}} < \delta_0$ for sufficiently small δ_0 (depending on $\bar{\rho}_0$); thus, $\mathrm{d}x_{\mathscr{I}}$ is timelike there.

We next compute the squared norm of

$$\mathrm{d}(\rho_0 - C x_{\mathscr{I}}^{2\ell_{\mathscr{I}}}) = \rho_0 \frac{\mathrm{d}\rho_0}{\rho_0} - \widetilde{C} x_{\mathscr{I}}^{2\ell_{\mathscr{I}}} \frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}},$$

where $\widetilde{C} = 2\ell_{\mathscr{I}} C$, with respect to $g_{\mathrm{e,b}}^{-1}$ using the expression (3-12) to be $\frac{1}{2}$ times

$$x_{\mathscr{I}}^{2\ell_{\mathscr{I}}}(-\widetilde{C}^2 x_{\mathscr{I}}^{2\ell_{\mathscr{I}}} - 2\widetilde{C}\rho_0 + \mathrm{Err}),$$
$$|\mathrm{Err}| \le C(g)(x_{\mathscr{I}}^{1-2\ell_{\mathscr{I}}} + \rho_0^{\ell_0})(\rho_0^2 + \widetilde{C}\rho_0 x_{\mathscr{I}}^{2\ell_{\mathscr{I}}} + \widetilde{C}^2 x_{\mathscr{I}}^{4\ell_{\mathscr{I}}}) \le C'(g)(\rho_0^2 + \widetilde{C}^2 x_{\mathscr{I}}^{4\ell_{\mathscr{I}}}),$$

where the constants $C(g)$ and $C'(g)$ only depend on g; here we use that $\ell_{\mathscr{I}} \le \frac{1}{2}$. Restricting ρ_0 to any compact subinterval of $[0, 1)$, note then that $\widetilde{C}\rho_0$ dominates, for sufficiently large $\widetilde{C}$, the error term $C'(g)\rho_0^2$, and $\widetilde{C}^2 x_{\mathscr{I}}^{2\ell_{\mathscr{I}}}$ then dominates, for $x_{\mathscr{I}} < \delta_0$ and sufficiently small $\delta_0 > 0$, the error term $C'(g)\widetilde{C}^2 x_{\mathscr{I}}^{4\ell_{\mathscr{I}}} \le (C'(g)(2\delta_0)^{2\ell_{\mathscr{I}}})\widetilde{C}^2 x_{\mathscr{I}}^{2\ell_{\mathscr{I}}}$. This proves the lemma. □

3.2. ***Admissible operators.*** The core of our analysis will be local near $\mathscr{I}^+ \subset M$. Thus, we work on the $(n+1)$-dimensional manifold $M_0 = [0,\epsilon)_\varrho \times (-\epsilon,\epsilon)_v \times Y$, put $\widetilde{M} = [M_0; Y]$, and set $M = \big[\widetilde{M}; \widetilde{\mathscr{I}}^+, \frac{1}{2}\big]$ as before, where $\widetilde{\mathscr{I}}^+ \subset \widetilde{M}$ denotes the front face; we denote the closure of $\{\varrho = 0, v < 0\}$ (resp. $\{\varrho = 0, v > 0\}$) by I^0 (resp. $I^+ \subset M$), and the lift of $\widetilde{\mathscr{I}}^+$ by $\mathscr{I}^+ \subset M$. Denote by

$$\beta : M \to M_0$$

the blow-down map, and by ρ_0, $x_{\mathscr{I}}$, ρ_+ defining functions of I^0, $\mathscr{I}^+$, I^+, respectively. We fix an $(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)$-admissible metric g on M (see Definition 3.2); its dual metric is thus of the form (3-11), or equivalently (3-12)–(3-13).

Definition 3.7 (admissible operators). Let $E \to M_0$ denote a smooth vector bundle. Then an operator $P \in \mathrm{Diff}^2(M^\circ; E)$ is called *g-admissible* (or simply *admissible*) if the following conditions are satisfied:

(1) P is principally scalar, with principal symbol equal to the dual metric function

$$G : T^*M^\circ \ni \zeta \mapsto g^{-1}(\zeta,\zeta).$$

(2) P can be written as a sum $P = P_0 + \widetilde{P}$, where

$$\begin{aligned} P_0 &\in \rho_0^2 x_{\mathscr{I}}^2 \rho_+^2 (\mathrm{Diff}^2_{\mathrm{e,b}}(M; \beta^* E) + \mathcal{A}^{(\ell_0,(0,0),\ell_+)} \mathrm{Diff}^1_{\mathrm{e,b}}(M; \beta^* E)), \\ \widetilde{P} &\in \rho_0^2 x_{\mathscr{I}}^2 \rho_+^2 \mathcal{A}^{(\ell_0,2\ell_{\mathscr{I}},\ell_+)} \mathrm{Diff}^2_{\mathrm{e,b}}(M; \beta^* E). \end{aligned} \tag{3-17}$$

(3) There exist

$$p_0 \in \rho_0\rho_+(\mathcal{C}^\infty + \mathcal{A}^{(\ell_0,\ell_+)})(\mathscr{I}^+; \beta^* \mathrm{End}(E)), \qquad p_1 \in (\mathcal{C}^\infty + \mathcal{A}^{(\ell_0,\ell_+)})(\mathscr{I}^+; \beta^* \mathrm{End}(E)),$$

so that in the coordinates (3-5) (with $T = T_0$) (resp. (3-6) (with $T = T_+ < T_0$)), and setting $p_0^0 = \rho_0^{-1} p_0 \in (\mathcal{C}^\infty + \mathcal{A}^{\ell_0})(\mathscr{I}^+ \setminus I^+; \beta^* \mathrm{End}(E))$ and $p_0^+ = \rho_+^{-1} p_0 \in (\mathcal{C}^\infty + \mathcal{A}^{\ell_+})(\mathscr{I}^+ \setminus I^0; \beta^* \mathrm{End}(E))$, the edge normal operator of $2\rho_0^{-2} x_{\mathscr{I}}^{-2} P_0$ (resp. $2x_{\mathscr{I}}^{-2}\rho_+^{-2} P_0$) at the fiber $\mathscr{I}^+_{y_0}$ of $\beta|_{\mathscr{I}^+} : \mathscr{I}^+ \to Y$ over $y_0 \in Y$ is given by[22]

$$\begin{aligned} &-\Big(x_{\mathscr{I}} D_{x_{\mathscr{I}}} - 2i^{-1}\Big(\frac{n-1}{2} + p_1\Big)\Big)(x_{\mathscr{I}} D_{x_{\mathscr{I}}} - 2\rho_0 D_{\rho_0}) + 2k^{ij}(y_0)(x_{\mathscr{I}} D_{y^i})(x_{\mathscr{I}} D_{y^j}) + p_0^0, \\ &\Big(\text{resp. } \Big(x_{\mathscr{I}} D_{x_{\mathscr{I}}} - 2i^{-1}\Big(\frac{n-1}{2} + p_1\Big)\Big)(x_{\mathscr{I}} D_{x_{\mathscr{I}}} - 2\rho_+ D_{\rho_+}) + 2k^{ij}(y_0)(x_{\mathscr{I}} D_{y^i})(x_{\mathscr{I}} D_{y^j}) + p_0^+\Big). \end{aligned} \tag{3-18}$$

Note that the operators (3-18) are invariant edge-b-operators on $[0,1)_{\rho_0} \times [0,\infty)_{x_{\mathscr{I}}} \times \mathbb{R}^{n-1}_y$ (resp. $[0,1)_{\rho_+} \times [0,\infty)_{x_{\mathscr{I}}} \times \mathbb{R}^{n-1}_y$) which act on sections of the fixed vector space E_{y_0}. Thus, the derivatives (of sections of $\beta^* E|_{\mathscr{I}^+_{y_0}}$) appearing in (3-18) are well-defined. The inclusion of the term $\frac{1}{2}(n-1)$ is a convenient normalization of p_1; see Example 3.10 below.

The first-order contribution to P_0 in (3-17) corresponds exactly to the terms p_0 and p_1 in part (3). (That is, if one wishes to treat operators P whose edge normal operators have the structure (3-18), one must allow for $\rho_0 x_{\mathscr{I}}^2 \rho_+^2 \mathcal{A}^{(\ell_0,(0,0),\ell_+)} \mathrm{Diff}^1_{\mathrm{e,b}}$ contributions to P.) A natural example where nontrivial terms p_0 and p_1 (indeed only satisfying conormal bounds at I^0) arise is given in Example 3.12 below.

[22] The zeroth-order terms are consistent: multiplying the first (resp. second) line in (3-18) by ρ_0 (resp. ρ_+), this follows from the definition of p_0^0 and p_0^+ and the fact that $\rho_0^{-1} x_{\mathscr{I}}^{-2} = r^{-1} = x_{\mathscr{I}}^{-2}\rho_+^{-1}$ in the two coordinate systems.

By Definition 3.2, we have $G \in \rho_0^2 x_{\mathscr{I}}^2 \rho_+^2 (S^2({}^{\mathrm{e,b}}T^*M) + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)} S^2({}^{\mathrm{e,b}}T^*M))$; condition (2) thus demands the corresponding membership of P as an edge-b-differential operator and allows for an additional (necessarily subprincipal) term which is of leading order at $\mathscr{I}^+$ in the sense of decay. Condition (1) determines the leading-order part of P in the differential sense, and (3) determines the leading-order part in the sense of decay at $\mathscr{I}^+$. We do not place any restrictions on the structure of P at I^0 or I^+ (beyond (3-17)).

Remark 3.8 (normal operator). One can check that an equivalent phrasing of condition (3) is the requirement

$$P - \Big[4\varrho\Big(\varrho D_\varrho + v D_v - i^{-1}\Big(\frac{n-1}{2} + p_1\Big)\Big)\varrho D_v + 4\varrho^2 \Delta_k + p_0\Big] \\ \in \rho_0^2 x_{\mathscr{I}}^2 \rho_+^2 (x_{\mathscr{I}} \mathcal{C}^\infty + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)}) \operatorname{Diff}^2_{\mathrm{e,b}}(M; E), \quad (3\text{-}19)$$

where $\varrho = \frac{1}{t+r}$ and $v = \frac{t-r}{t+r}$ as in (3-11); indeed, passing to the coordinates (3-5) or (3-6), one computes

$$\varrho\partial_\varrho + v\partial_v \equiv \tfrac{1}{2} x_{\mathscr{I}} \partial_{x_{\mathscr{I}}}, \quad \varrho\partial_v = \rho_0\big(\rho_0\partial_{\rho_0} - \tfrac{1}{2} x_{\mathscr{I}}\partial_{x_{\mathscr{I}}}\big) = -\rho_+\big(\rho_+\partial_{\rho_+} - \tfrac{1}{2} x_{\mathscr{I}}\partial_{x_{\mathscr{I}}}\big)$$

modulo the space $x_{\mathscr{I}} \mathcal{V}_{\mathrm{e,b}}(M)$ of edge-b-vector fields which vanish at $\mathscr{I}^+$; moreover, $\varrho \equiv \frac{1}{2}\rho_0 x_{\mathscr{I}}^2 \equiv \frac{1}{2} x_{\mathscr{I}}^2 \rho_+$ modulo $\rho_0 x_{\mathscr{I}}^3 \rho_+ \mathcal{C}^\infty$, and finally $p_0 = \frac{1}{2}\varrho^{-1}\rho_0^2 x_{\mathscr{I}}^2 p_0^0 = \rho_0 p_0^0$ (resp. $p_0 = \frac{1}{2}\varrho^{-1} x_{\mathscr{I}}^2 \rho_+^2 p_0^+ = \rho_+ p_0^+$) in the coordinates (3-5) (resp. (3-6)) indeed in the notation of Definition 3.7. In (3-19), $\Delta_k \in \operatorname{Diff}^2(Y; E|_Y)$ is principally scalar with principal part $k^{ij}(y) D_{y^i} D_{y^j}$. (Modifications of Δ_k by subprincipal terms, i.e., elements of $\operatorname{Diff}^1(Y; E|_Y)$, contribute terms of class $\rho_0^2 x_{\mathscr{I}}^2 \rho_+^2 \cdot x_{\mathscr{I}} \operatorname{Diff}^1_{\mathrm{e,b}}(M; E)$, which are thus error terms.) The principal part of the operator in parentheses here is prescribed by condition (1) (see (3-11)), so the new pieces of information from condition (3) are the bundle endomorphisms p_0 and p_1.

Remark 3.9 (weaker error term: operator). Mirroring Remark 3.3, a natural and slightly more permissible definition would only require

$$P \in \rho_0^2 x_{\mathscr{I}}^2 \rho_+^2 \mathcal{A}^{(((0,0),\ell_0), 2\ell_{\mathscr{I}}, ((0,0),\ell_+))}(M; \beta^* E) \quad (3\text{-}20)$$

instead of (3-17). The analysis in Sections 4.2, 5, 6, and 7 goes through in this generality with only notational modifications. In Section 8 however, we appeal to an elliptic parametrix construction in the 0-calculus which, in the existing literature, is stated only for smooth coefficient operators; extending this to the case of coefficients which are smooth plus decaying conormal would suffice to treat the case (3-20).

Example 3.10 (Minkowski wave operator). The wave operator on Minkowski space $(\mathbb{R}^4, g_0)$ is an example of a g_0-admissible operator. Indeed, in the coordinates (3-2) used in (3-3), one computes

$$\begin{aligned} \Box_{g_0} &= 4\varrho(\varrho D_\varrho + v D_v)\varrho D_v + \frac{2i(n-1)}{1-v}\varrho(\varrho^2 D_\varrho + \varrho(1+v) D_v) + \frac{4\varrho^2}{(1-v)^2}\Delta_{\not g} \\ &\equiv 4\varrho\Big(\varrho D_\varrho + v D_v - i^{-1}\frac{n-1}{2}\Big)\varrho D_v + 4\varrho^2 \Delta_{\not g} \mod \rho_0^2 x_{\mathscr{I}}^2 \rho_+^2 \cdot x_{\mathscr{I}} \operatorname{Diff}^2_{\mathrm{e,b}}(M), \end{aligned}$$

which is thus indeed of the form (3-19) for $p_0 = p_1 = 0$ and $k = \not g$.

Example 3.11 (wave operator of an admissible metric). Generalizing the previous example, one can show that if g is an admissible metric, then the scalar wave operator $\Box_g$ is an admissible operator. Indeed, the Koszul formula, together with the fact that conjugation by any weight $\rho_0^{\alpha_0} x_{\mathscr{I}}^{2\alpha_{\mathscr{I}}} \rho_+^{\alpha_+}$ preserves the space $\mathcal{V}_{\mathrm{e,b}}(M)$, implies that the Levi-Civita connection of g satisfies

$$\nabla^g \in (\mathcal{C}^\infty + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)}) \operatorname{Diff}^1_{\mathrm{e,b}}(M; {}^{\mathrm{e,b}}TM, {}^{\mathrm{e,b}}T^*M \otimes {}^{\mathrm{e,b}}TM),$$

similarly for the connection acting on sections of tensor products of ${}^{\mathrm{e,b}}T^*M$ and ${}^{\mathrm{e,b}}TM$. This implies $\Box_g \in \rho_0^2 x_{\mathscr{I}}^2 \rho_+^2 \operatorname{Diff}^2_{\mathrm{e,b}}(M)$, and we have (3-19) with $p_0 = p_1 = 0$ since the normal operators of $\Box_g$ at $\mathscr{I}^+$ are equal to those of $\Box_{g_0}$, where g_0 is of the Minkowskian form (3-3) with $\not{g}$ replaced by k.

Example 3.12 (linearization of the gauge-fixed Einstein operator). Consider the case $n+1 = \dim M = 4$; then, in the coordinates ρ_0 and $x_{\mathscr{I}}$, the edge normal operator of $\rho_{\mathscr{I}} \rho^{-3} P \rho = \rho_0^{-2} x_{\mathscr{I}}^{-2} (\rho_0 x_{\mathscr{I}}^2)^{-1} P \rho_0 x_{\mathscr{I}}^2$ is given by the conjugation of the first operator in (3-18) by the weight $(\rho_0 x_{\mathscr{I}}^2)^{-1}$, and thus (modulo the term $k^{ij}(y_0)(x_{\mathscr{I}} D_{y^i})(x_{\mathscr{I}} D_{y^j})$) is equal to

$$-(x_{\mathscr{I}} D_{x_{\mathscr{I}}} - 2i^{-1} p_1)(x_{\mathscr{I}} D_{x_{\mathscr{I}}} - 2\rho_0 D_{\rho_0}) = -4(\rho_{\mathscr{I}} \partial_{\rho_{\mathscr{I}}} - p_1)(\rho_0 \partial_{\rho_0} - \rho_{\mathscr{I}} \partial_{\rho_{\mathscr{I}}}).$$

Thus, the linearized gauge-fixed Einstein operator considered in [Hintz 2023b, Proposition 3.29] is an admissible operator, acting on sections of the bundle $S^2\widetilde{T}^*M$ in the notation of the reference.

4. Microlocal edge-b-regularity theory near $\mathscr{I}^+$

We now fix an $(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)$-admissible metric g (see Definition 3.2) and a g-admissible operator P on the $(n+1)$-dimensional manifold M (see Definition 3.7). We shall assume that $n \geq 2$ for simplicity of notation; the case $n = 1$ can be treated with the same methods upon making some straightforward simplifications and modifications (related to the fact that the "0-sphere" $\mathbb{S}^0 = \{-1, 1\}$ is disconnected); we leave these to the interested reader.

We denote the edge-b-principal symbol of P by G. Fix arbitrary choices of boundary-defining functions $\rho_0,\ x_{\mathscr{I}},\ \rho_+$. In view of (3-11), G satisfies

$$G_{\mathrm{e,b}} := \rho_0^{-2} x_{\mathscr{I}}^{-2} \rho_+^{-2} G \in S^2({}^{\mathrm{e,b}}T^*M) + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)} S^2({}^{\mathrm{e,b}}T^*M).$$

The characteristic set $G_{\mathrm{e,b}}^{-1}(0) \subset {}^{\mathrm{e,b}}T^*M \setminus o$ of P is conic in the fibers; we shall work near fiber infinity and thus define

$$\Sigma \subset \overline{{}^{\mathrm{e,b}}T^*}M \setminus o$$

as the closure of $G_{\mathrm{e,b}}^{-1}(0)$ inside $\overline{{}^{\mathrm{e,b}}T^*}M \setminus o$. With respect to the time orientation of M near $\mathscr{I}^+$ introduced in (3-16), we split Σ into its two connected components

$$\Sigma = \Sigma^+ \cup \Sigma^-,$$

with Σ^+ containing the future null covectors. In Section 4.1, we analyze the dynamical structure of the flow of the Hamiltonian vector field $H_{G_{\mathrm{e,b}}}$ inside Σ. This information is used in Section 4.2 to track the microlocal regularity of solutions of $Pu = f$ near ${}^{\mathrm{e,b}}T^*_{\mathscr{I}^+}M$.

Note that if $0 < f \in \mathcal{C}^\infty(\overline{{}^{\mathrm{e,b}}T^*}M)$, then the restriction of the Hamiltonian vector field

$$H_{fG_{\mathrm{e,b}}} = f H_{G_{\mathrm{e,b}}} + G_{\mathrm{e,b}} H_f$$

to Σ is a positive rescaling of $H_{G_{\mathrm{e,b}}}$. Therefore the properties of the $H_{G_{\mathrm{e,b}}}$-flow inside Σ of interest for the microlocal analysis of P, i.e., critical sets and the signs and ratios of eigenvalues of the linearization at critical sets, are independent of the choice of boundary-defining functions. We can thus make convenient choices in the calculations below; by an abuse of notation, we shall always call them ρ_0, $x_{\mathscr{I}}$, ρ_+ (as already done in the previous section) in order to not overburden the notation. Moreover, since $H_{G_{\mathrm{e,b}}}$ is homogeneous of degree 1 with respect to dilations in the fibers of ${}^{\mathrm{e,b}}T^*M$, we shall work with the rescaled vector field

$$\mathsf{H} := \rho_\infty H_{G_{\mathrm{e,b}}} \in \mathcal{V}_{\mathrm{e,b}}(\overline{{}^{\mathrm{e,b}}T^*}M \setminus o), \tag{4-1}$$

where $\rho_\infty \in \mathcal{C}^\infty(\overline{{}^{\mathrm{e,b}}T^*}M)$ is a defining function of ${}^{\mathrm{e,b}}S^*M$; see Corollary 2.3. Again, we may change ρ_∞ via multiplication with a positive function without altering the relevant properties of H.

4.1. ***Structure of the null-bicharacteristic flow near*** $\mathscr{I}^+$**.** We first work near $I^0 \cap \mathscr{I}^+$, or indeed in a region $\mathcal{U}_0 = \mathcal{U}_0(T)$ for any fixed $T \in \mathbb{R}$ (see Definition 3.1), and use the coordinates (3-5). Writing the canonical 1-form on the cotangent bundle as

$$\xi \frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}} + \sum_{j=1}^{n-1} \eta_j \frac{\mathrm{d}y^j}{x_{\mathscr{I}}} + \zeta \frac{\mathrm{d}\rho_0}{\rho_0}, \tag{4-2}$$

we can read off from (3-18) (using the summation convention and dropping the weight at I^+ from the notation)

$$\begin{aligned} G^0_{\mathrm{e,b}} &:= \rho_0^{-2} x_{\mathscr{I}}^{-2} G = -\tfrac{1}{2}\xi(\xi - 2\zeta) + k^{ij}(y)\eta_i\eta_j + \widetilde{G}^0_{\mathrm{e,b}}, \\ \widetilde{G}^0_{\mathrm{e,b}} &\in x_{\mathscr{I}} S^2({}^{\mathrm{e,b}}T^*_{\mathcal{U}_0}M) + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}})} S^2({}^{\mathrm{e,b}}T^*_{\mathcal{U}_0}M). \end{aligned} \tag{4-3a}$$

Using Lemma 2.2 (where x and z are equal to $x_{\mathscr{I}}$ and ρ_0 in present notation), we compute

$$\begin{aligned} H_{G^0_{\mathrm{e,b}}} &= (\zeta - \xi)(x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} + \eta\partial_\eta) + \xi\rho_0\partial_{\rho_0} - 2|\eta|^2_{k^{-1}}\partial_\xi + \widetilde{H}, \\ \rho_\infty \widetilde{H} &\in (x_{\mathscr{I}}\mathcal{C}^\infty + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}})})\mathcal{V}_{\mathrm{b}}(\overline{{}^{\mathrm{e,b}}T^*_{\mathcal{U}_0}}M). \end{aligned} \tag{4-3b}$$

Note carefully that we regard the error $\rho_\infty \widetilde{H}$ as a b-vector field (rather than an edge-b-vector field); therefore, the terms involving derivatives along $x_{\mathscr{I}}\partial_y$ can be regarded as error terms. At $x_{\mathscr{I}} = 0$, the two components of Σ are given by

$$\Sigma^\pm \cap {}^{\mathrm{e,b}}T^*_{\mathscr{I}^+}M = \{-(\xi - \zeta)^2 + |\zeta|^2 + 2|\eta|^2_{k^{-1}} = 0 : \pm(\xi - \zeta) > 0\}.$$

(Indeed, the coefficient of $x_{\mathscr{I}}\partial_{x_{\mathscr{I}}}$ of $H_{G^0_{\mathrm{e,b}}}$ is then negative on Σ^+, cf. (3-16).)

We restrict attention to Σ^+ and proceed to determine the radial sets of $H_{G^0_{\mathrm{e,b}}}$, i.e., the places where $H_{G^0_{\mathrm{e,b}}}$ is a multiple of the generator of fiber dilations.

• *No radial sets over* $(I^0)^\circ$. At $\rho_0 = 0$ and $0 < x_{\mathscr{I}} < \delta_0$ (with $\delta_0 > 0$ given by Lemma 3.6(1)), we have $H_{G^0_{\mathrm{e,b}}} x_{\mathscr{I}} < 0$, and hence $H_{G^0_{\mathrm{e,b}}}$ is not radial there.

• *Radial sets over $\mathscr{I}^+ \setminus I^+$.* Working at $x_\mathscr{I} = 0$, we ask when $H_{G^0_{\mathrm{e,b}}} = c(\xi\partial_\xi + \eta\partial_\eta + \zeta\partial_\zeta)$ for some $c \in \mathbb{R}$. If $c = 0$, then $\eta = 0$ by inspection of the ∂_ξ-coefficient in (4-3b), and hence $\xi = 0$ or $\xi = 2\zeta$ on Σ^+. At $\eta = 0$, $\xi = 0$, the vector field $H_{G^0_{\mathrm{e,b}}}$ is indeed radial. At $\eta = 0$, $\xi = 2\zeta$ on the other hand, we need $\xi \neq 0$ as we are working away from the zero section of ${}^{\mathrm{e,b}}T^*M$, and then necessarily $\rho_0 = 0$, for otherwise the ∂_{ρ_0} coefficient of $H_{G^0_{\mathrm{e,b}}}$ would be nonzero; this identifies a second radial set. Next, if $c \neq 0$ and still $x_\mathscr{I} = 0$, then necessarily $\zeta = 0$ since $H_{G^0_{\mathrm{e,b}}}$ has a vanishing ∂_ζ-coefficient; thus, $\xi^2 = 2|\eta|^2_{k^{-1}}$ and $c = -\xi$ (by inspection of the ∂_ξ-coefficient), and then the $\eta\partial_\eta$-coefficient of $H_{G^0_{\mathrm{e,b}}}$ is $(\zeta - \xi) = -\xi = c$; the requirement $H_{G^0_{\mathrm{e,b}}}\rho_0 = 0$ forces $\rho_0 = 0$, and we have thus identified a third radial set.

• *Radial sets near $\mathscr{I}^+ \cap I^+$.* Turning to a neighborhood of $\mathscr{I}^+ \cap I^+$, or indeed a region $\mathcal{U}_+ = \mathcal{U}_+(T)$ (see Definition 3.1) for any fixed $T \in \mathbb{R}$, we use the coordinates (3-6) and write the canonical 1-form (repurposing the above notation) as

$$\xi\frac{\mathrm{d}x_\mathscr{I}}{x_\mathscr{I}} + \sum_{j=1}^{n-1}\eta_j\frac{\mathrm{d}y^j}{x_\mathscr{I}} + \zeta\frac{\mathrm{d}\rho_+}{\rho_+}. \tag{4-4}$$

From (3-18) and Lemma 2.2, we find (dropping the weight at I^0 from the notation)

$$G^+_{\mathrm{e,b}} := x_\mathscr{I}^{-2}\rho_+^{-2}G = \tfrac{1}{2}\xi(\xi - 2\zeta) + k^{ij}(y)\eta_i\eta_j + \widetilde{G}^+_{\mathrm{e,b}}, \tag{4-5a}$$

$$H_{G^+_{\mathrm{e,b}}} = (\xi - \zeta)(x_\mathscr{I}\partial_{x_\mathscr{I}} + \eta\partial_\eta) - \xi\rho_+\partial_{\rho_+} - 2|\eta|^2_{k^{-1}}\partial_\xi + 2x_\mathscr{I}k^{ij}\eta_i\partial_{y^j} + \widetilde{H}, \tag{4-5b}$$

where the error terms satisfy

$$\widetilde{G}^+_{\mathrm{e,b}} \in x_\mathscr{I}S^2({}^{\mathrm{e,b}}T^*_{\mathcal{U}_+}M) + \mathcal{A}^{(2\ell_\mathscr{I},\ell_+)}S^2({}^{\mathrm{e,b}}T^*_{\mathcal{U}_+}M), \quad \rho_\infty\widetilde{H} \in (x_\mathscr{I}\mathcal{C}^\infty + \mathcal{A}^{(2\ell_\mathscr{I},\ell_+)})\mathcal{V}_{\mathrm{e,b}}(\overline{{}^{\mathrm{e,b}}T^*_{\mathcal{U}_+}}M).$$

We record $H_{G^+_{\mathrm{e,b}}}$ as an edge-b-vector field here in order to keep nondegenerate track of its ∂_y-component. We moreover have

$$\Sigma^\pm \cap {}^{\mathrm{e,b}}T^*_{\mathscr{I}^+}M = \{-\zeta^2 + (\xi - \zeta)^2 + 2|\eta|^2_{k^{-1}} = 0 : \pm\zeta > 0\} \tag{4-6}$$

by comparison of the $x_\mathscr{I}\partial_{x_\mathscr{I}}$-coefficient of $H_{G^+_{\mathrm{e,b}}}$ with (3-16).

We proceed to locate the radial sets inside Σ^+ in these coordinates. At $\rho_+ = 0$ and $0 < x_\mathscr{I} < \delta_0$, let us work in $\xi^2 + |\eta|^2_{k^{-1}} + \zeta^2 = 1$. The vanishing of $x_\mathscr{I}^{-1}H_{G^+_{\mathrm{e,b}}}y^j = 2k^{ij}\eta_i + \mathcal{O}(x_\mathscr{I})$ forces $|\eta| = \mathcal{O}(x_\mathscr{I})$, and therefore $x_\mathscr{I}^{-1}H_{G^+_{\mathrm{e,b}}}x_\mathscr{I} = (\xi - \zeta) + \mathcal{O}(x_\mathscr{I})$ cannot vanish; indeed, if it did, then $\xi - \zeta = \mathcal{O}(x_\mathscr{I})$ and thus $\zeta = \mathcal{O}(x_\mathscr{I})$ on Σ by (4-6), giving the contradiction $1 = \xi^2 + |\eta|^2 + \zeta^2 = \mathcal{O}(x_\mathscr{I})$ when $\delta_0 > 0$ is sufficiently small. Thus, in a small neighborhood of $\mathscr{I}^+$, the radial sets all lie over $\mathscr{I}^+$. Within $\mathscr{I}^+$ then, suppose $H_{G^+_{\mathrm{e,b}}} = c(\xi\partial_\xi + \eta\partial_\eta + \zeta\partial_\zeta)$. If $c = 0$, then $\eta = 0$, so $\xi = 0$ or $\xi = 2\zeta$; in the former case, we are at a radial set indeed, whereas in the latter case we must have $\xi, \zeta \neq 0$ to stay clear of the zero section, but then the vanishing of $H_{G^+_{\mathrm{e,b}}\rho_+}$ forces $\rho_+ = 0$; and a radial set is indeed located here. If on the other hand $c \neq 0$, then since $H_{G^+_{\mathrm{e,b}}}\zeta = 0$ we must have $c\zeta = 0$ and thus $\zeta = 0$, which cannot happen inside Σ by (4-6).

• *No radial points in M°; summary.* Given any point $p \in M^\circ$ and a covector $0 \neq \varpi \in T^*_pM^\circ$ with $G(\varpi) = 0$, take any function $f \in \mathcal{C}^\infty(M^\circ)$ so that $\mathrm{d}f(p)$ is timelike; then $H_G|_\varpi f = 2g^{-1}(\varpi, \mathrm{d}f) \neq 0$, which implies that the pushforward of $H_G|_\varpi$ to T_pM° is nonzero, and thus $H_G|_\varpi$ is not radial. We may thus summarize the above computations as follows.

Lemma 4.1 (radial sets of $H_{G_{\mathrm{e,b}}}$). *In a sufficiently small neighborhood $\mathcal{U}\subset M$ of $\mathscr{I}^+$, a complete list of the radial sets of $H_{G_{\mathrm{e,b}}}$ inside $\Sigma^\pm\cap({}^{\mathrm{e,b}}T^*_{\mathcal{U}}M\setminus o)$ is as follows*:

$$\mathcal{R}^\pm_{\mathrm{in},-}:=\{(\rho_0,x_\mathscr{I},y;\zeta,\xi,\eta):\rho_0=x_\mathscr{I}=0,\ \eta=0,\ \xi=2\zeta,\ \pm\xi>0\},\tag{4-7}$$

$$\mathcal{R}^\pm_{\mathrm{c}}:=\{(\rho_0,x_\mathscr{I},y;\zeta,\xi,\eta):\rho_0=x_\mathscr{I}=0,\ \zeta=0,\ \xi=\pm\sqrt{2}|\eta|_{k^{-1}}\},\tag{4-8}$$

$$\begin{aligned}\mathcal{R}^\pm_{\mathrm{out}}&:=\{(\rho_0,x_\mathscr{I},y;\zeta,\xi,\eta):x_\mathscr{I}=0,\ \xi=\eta=0,\ \pm\zeta<0\}\quad(\textit{in the coordinates (4-2)})\\&=\{(\rho_+,x_\mathscr{I},y;\zeta,\xi,\eta):x_\mathscr{I}=0,\ \xi=\eta=0,\ \pm\zeta>0\}\quad(\textit{in the coordinates (4-4)}),\end{aligned}\tag{4-9}$$

$$\mathcal{R}^\pm_{\mathrm{in},+}:=\{(\rho_+,x_\mathscr{I},y;\zeta,\xi,\eta):\rho_+=x_\mathscr{I}=0,\ \eta=0,\ \xi=2\zeta,\ \pm\xi>0\}.\tag{4-10}$$

Note now that $H_{G_{\mathrm{e,b}}}$ is radial at a point $\varpi\in{}^{\mathrm{e,b}}T^*M\setminus o$ if and only if the restriction of the rescaled vector field $\mathsf{H}=\rho_\infty H_{G_{\mathrm{e,b}}}$ to ${}^{\mathrm{e,b}}S^*M$ vanishes, as a smooth vector field, at $[\varpi]\in{}^{\mathrm{e,b}}S^*M$ (i.e., the boundary at fiber infinity of the ray $\mathbb{R}_+\varpi$). Explicitly, near $\mathcal{R}^+_{\mathrm{in},-}$ and $\mathcal{R}^+_{\mathrm{out}}$, we use the coordinates (4-2) and, near fiber infinity,

$$\rho_\infty=|\zeta|^{-1},\quad \hat\eta=\frac{\eta}{\zeta},\quad \hat\xi=\frac{\xi}{\zeta}.\tag{4-11a}$$

Thus, $\rho_\infty=0$ at fiber infinity, $\rho_\infty=\zeta^{-1}$ near $\partial\mathcal{R}^+_{\mathrm{in},-}$, and $(\hat\eta,\hat\xi)=(0,2)$ at $\partial\mathcal{R}^+_{\mathrm{in},-}$; and $\rho_\infty=-\zeta^{-1}$ near $\partial\mathcal{R}^+_{\mathrm{out}}$, and in addition $(\hat\eta,\hat\xi)=(0,0)$ at $\partial\mathcal{R}^+_{\mathrm{out}}$. For a submanifold $S\subset{}^{\mathrm{e,b}}S^*M$, denote by $\mathcal{I}_S\subset\mathcal{C}^\infty(\overline{{}^{\mathrm{e,b}}T^*}M)$ the ideal of functions vanishing at S. Inserting the coordinates (4-11a) into (4-3b), we find that

$$\rho_\infty H_{G^0_{\mathrm{e,b}}}\equiv 2\rho_0\partial_{\rho_0}-x_\mathscr{I}\partial_{x_\mathscr{I}}-\hat\eta\partial_{\hat\eta}\mod(\mathcal{I}_{\partial\mathcal{R}^+_{\mathrm{in},-}}+\mathcal{A}^{(\ell_0,2\ell_\mathscr{I})})\mathcal{V}_{\mathrm{b}}(\overline{{}^{\mathrm{e,b}}T^*_{\mathcal{U}_0}M}),\tag{4-11b}$$

$$\rho_\infty H_{G^0_{\mathrm{e,b}}}\equiv -x_\mathscr{I}\partial_{x_\mathscr{I}}-\hat\eta\partial_{\hat\eta}\mod(\mathcal{I}_{\partial\mathcal{R}^+_{\mathrm{out}}}+\mathcal{A}^{(\ell_0,2\ell_\mathscr{I})})\mathcal{V}_{\mathrm{b}}(\overline{{}^{\mathrm{e,b}}T^*_{\mathcal{U}_0}M}).\tag{4-11c}$$

Near $\mathcal{R}^+_{\mathrm{c}}$, we need to work with different coordinates; we choose

$$\rho_\infty=\xi^{-1},\quad \hat\eta=\frac{\eta}{\xi},\quad \hat\zeta=\frac{\zeta}{\xi}\tag{4-12a}$$

and compute (using that $\hat\zeta=0$ and $|\hat\eta|_{k^{-1}}=\frac{1}{\sqrt{2}}$ at $\partial\mathcal{R}^+_{\mathrm{c}}$)

$$\rho_\infty H_{G^0_{\mathrm{e,b}}}\equiv\rho_\infty\partial_{\rho_\infty}+\rho_0\partial_{\rho_0}-x_\mathscr{I}\partial_{x_\mathscr{I}}+\hat\zeta\partial_{\hat\zeta}\mod(\mathcal{I}_{\partial\mathcal{R}^+_{\mathrm{c}}}+\mathcal{A}^{(\ell_0,2\ell_\mathscr{I})})\mathcal{V}_{\mathrm{b}}(\overline{{}^{\mathrm{e,b}}T^*_{\mathcal{U}_0}M}).\tag{4-12b}$$

Finally, near $\mathcal{R}^+_{\mathrm{in},+}$ and $\mathcal{R}^+_{\mathrm{out}}$ and in the coordinates (4-4) and

$$\rho_\infty=|\zeta|^{-1},\quad \hat\eta=\frac{\eta}{\zeta},\quad \hat\xi=\frac{\xi}{\zeta},\tag{4-13a}$$

we have $\rho_\infty=\zeta^{-1}$ near $\partial\mathcal{R}^+_{\mathrm{in},+}$, and in addition $(\hat\eta,\hat\xi)=(0,2)$ at $\partial\mathcal{R}^+_{\mathrm{in},+}$; and $\rho_\infty=\zeta^{-1}$ near $\partial\mathcal{R}^+_{\mathrm{out}}$ and $(\hat\eta,\hat\xi)=(0,0)$ at $\partial\mathcal{R}^+_{\mathrm{out}}$. Inserting this into (4-5b) gives

$$\rho_\infty H_{G^+_{\mathrm{e,b}}}\equiv x_\mathscr{I}\partial_{x_\mathscr{I}}-2\rho_+\partial_{\rho_+}+\hat\eta\partial_{\hat\eta}\mod(\mathcal{I}_{\partial\mathcal{R}^+_{\mathrm{in},+}}+\mathcal{A}^{(2\ell_\mathscr{I},\ell_+)})\mathcal{V}_{\mathrm{b}}(\overline{{}^{\mathrm{e,b}}T^*_{\mathcal{U}_+}M}),\tag{4-13b}$$

$$\rho_\infty H_{G^+_{\mathrm{e,b}}}\equiv -x_\mathscr{I}\partial_{x_\mathscr{I}}-\hat\eta\partial_{\hat\eta}\mod(\mathcal{I}_{\partial\mathcal{R}_{\mathrm{out},+}}+\mathcal{A}^{(2\ell_\mathscr{I},\ell_+)})\mathcal{V}_{\mathrm{b}}(\overline{{}^{\mathrm{e,b}}T^*_{\mathcal{U}_+}M}).\tag{4-13c}$$

From the expressions (4-11b), (4-11c), (4-12b), (4-13b), and (4-13c), one can read off the dynamical nature (source, sink, saddle point) of the respective radial sets; see Figure 4.1. We determine the structure

of the flow between these radial sets by computing the flows of $H_{G^0_{\mathrm{e,b}}}$ and $H_{G^+_{\mathrm{e,b}}}$ explicitly inside the future characteristic set Σ^+. Consider the time s flow of

$$H^0 := H_{G^0_{\mathrm{e,b}}}|_{{}^{\mathrm{e,b}}T^*_{\mathscr{I}^+}M} = (\zeta-\xi)\eta\partial_\eta + \xi\rho_0\partial_{\rho_0} - 2|\eta|^2_{k^{-1}}\partial_\xi$$

inside

$$(G^0_{\mathrm{e,b}})^{-1}(0) = \{-\tfrac{1}{2}\xi(\xi-2\zeta)+|\eta|^2_{k^{-1}} = 0\}.$$

Dropping the coordinates $x_{\mathscr{I}}$ and y (which are constants of integration) from the notation and only keeping (ρ_0,ζ,ξ,η), we have for $\zeta = 0$ (and thus $\xi = \sqrt{2}|\eta|_{k^{-1}} > 0$)

$$e^{sH^0}(\rho_0,0,\xi,\eta) = \left(\rho_0(1+s\xi),\, 0,\, \frac{\xi}{1+s\xi},\, \frac{\eta}{1+s\xi}\right),$$

with domain of definition given by $s \in (-1/\xi, (\rho_0^{-1}-1)/\xi)$ when $\rho_0 > 0$ (the upper bound ensuring that $\rho_0 < 1$, i.e., we stay in the coordinate patch); when $s \searrow -\frac{1}{\xi}$, this tends to a point on $\partial\mathcal{R}^+_{\mathrm{c}}$. In the case $\rho_0 = 0$, the flow exists for all $s > -\frac{1}{\xi}$ and merely dilates the fiber variables.

When $\zeta \neq 0$ on the other hand, we have $\xi \geq \max(0, 2\zeta)$ on Σ^+ and

$$e^{sH^0}(\rho_0,\zeta,\xi,\eta) = \left(\rho_0\left(1+\frac{\xi}{2\zeta}(e^{2\zeta s}-1)\right),\, \zeta,\, \frac{2\xi\zeta}{\xi-(\xi-2\zeta)e^{-2\zeta s}},\, \frac{2\eta\zeta e^{-\zeta s}}{\xi-(\xi-2\zeta)e^{-2\zeta s}}\right).$$

When $\xi = 2\zeta$ (and thus $\eta = 0$ on Σ^+), this tends to $\mathcal{R}^+_{\mathrm{in},-}$ as $s \searrow -\infty$. When $\xi = 0$ (and thus $\eta = 0$, $\zeta < 0$ on Σ^+), i.e., on $\mathcal{R}^+_{\mathrm{out}}$, this is stationary. In the remaining case that $\xi \neq 0,\ 2\zeta$ and $\zeta \neq 0$ (and thus $\eta \neq 0$), one has

$$e^{sH^0}(\rho_0,\zeta,\xi,\eta) \to \partial\mathcal{R}^+_{\mathrm{c}} \quad \text{as } s \searrow \frac{1}{2\zeta}\log\left(\frac{\xi-2\zeta}{\xi}\right)$$

(the infimum of the maximal interval of existence of the flow); the flow leaves the local coordinate chart into $\rho_0 \geq 1$ when s exceeds a positive threshold. For $\rho_0 = 0$ finally, the flow is defined for all $s \geq 0$, and for $\zeta > 0$ (resp. $\zeta < 0$) tends to $\mathcal{R}^+_{\mathrm{in},-}$ (resp. $\mathcal{R}^+_{\mathrm{out}}$) as $s \nearrow \infty$.

We now turn to the time s flow of

$$H^+ := H_{G^+_{\mathrm{e,b}}}|_{{}^{\mathrm{e,b}}T^*_{\mathscr{I}^+}M} = (\xi-\zeta)\eta\partial_\eta - \xi\rho_+\partial_{\rho_+} - 2|\eta|^2_{k^{-1}}\partial_\xi$$

inside Σ^+ over $\mathscr{I}^+$, where $\frac{1}{2}\xi(\xi-2\zeta)+|\eta|^2_{k^{-1}} = 0$. Thus, $\zeta > 0$ and therefore $0 \leq \xi \leq 2\zeta$ in Σ^+, and we find (dropping $x_{\mathscr{I}} \equiv 0$ and the constant y from the notation)

$$e^{sH^+}(\rho_+,\zeta,\xi,\eta) = \left(\rho_+\left(1+\frac{\xi}{2\zeta}(e^{-2\zeta s}-1)\right),\, \zeta,\, \frac{2\xi\zeta}{\xi+(2\zeta-\xi)e^{2\zeta s}},\, \frac{2\eta\zeta e^{\zeta s}}{\xi+(2\zeta-\xi)e^{2\zeta s}}\right).$$

When $\xi = 2\zeta$ and thus $\eta = 0$, the fiber coordinates remain fixed and $\rho_+ \to 0$ as $s \nearrow \infty$, thus this tends to $\mathcal{R}^+_{\mathrm{in},+}$. When $\xi \neq 2\zeta$, this tends to the point $(\rho_+(1-\xi/(2\zeta)),0,0,\zeta) \in \mathcal{R}^+_{\mathrm{out}}$ as $s \nearrow \infty$.

The fact that the set $\xi = 2\zeta,\ \eta = 0$ (in either coordinate system) is preserved under the flow motivates the first half of the following definition.

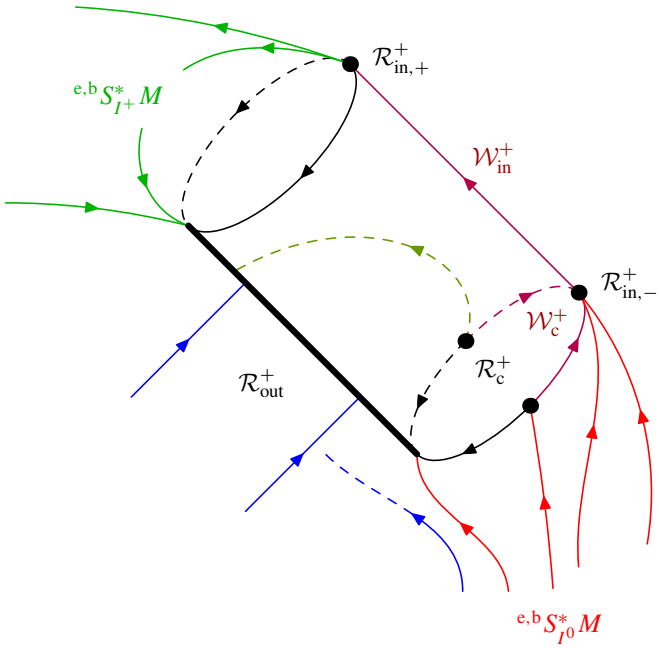

Figure 4.1. Illustration of the null-bicharacteristic flow of the rescaled Hamiltonian vector field H, see (4-1), at fiber infinity over a fiber $\mathscr{I}^+_y$, $y \in Y$, of $\mathscr{I}^+$. We identify conic subsets of ${}^{\mathrm{e,b}}T^*M \setminus o$ with their boundaries at fiber infinity. We draw $\Sigma^+ \cap {}^{\mathrm{e,b}}S^*_{\mathscr{I}^+_y}M$ as a cylinder with base $\mathscr{I}^+_y$ (a closed interval) and cross section $\Sigma^+ \cap {}^{\mathrm{e,b}}S^*_pM \cong \mathbb{S}^{n-1}$, drawn here for $n = 2$. The direction of the flow is indicated by arrows, and the boundaries at fiber infinity of the radial sets defined in Lemma 4.1 are drawn in bold black. We also show the boundary at fiber infinity of the two stable/unstable manifolds from Definition 4.2. Moreover, we sketch null-bicharacteristics in $(I^0)^\circ$ in red, null-bicharacteristics in $(I^+)^\circ$ in green, and null-bicharacteristics lying over M° in blue.

Definition 4.2 (stable/unstable manifolds). We define the following submanifolds of $\Sigma^\pm \cap ({}^{\mathrm{e,b}}T^*_{\mathscr{I}^+}M \setminus o)$:

$$\begin{aligned}\mathcal{W}^\pm_{\mathrm{in}} &:= \{(\rho_0, x_\mathscr{I}, y; \zeta, \xi, \eta) : x_\mathscr{I} = 0,\ \eta = 0,\ \xi = 2\zeta,\ \pm\xi > 0\} \quad \text{(in the coordinates (4-2))}\\ &= \{(\rho_+, x_\mathscr{I}, y; \zeta, \xi, \eta) : x_\mathscr{I} = 0,\ \eta = 0,\ \xi = 2\zeta,\ \pm\xi > 0\} \quad \text{(in the coordinates (4-4))}, \qquad (4\text{-}14)\\ \mathcal{W}^\pm_{\mathrm{c}} &:= \{(\rho_0, x_\mathscr{I}, y; \zeta, \xi, \eta) : \rho_0 = x_\mathscr{I} = 0,\ \pm\zeta \geq 0\} \cap \Sigma^\pm. \qquad (4\text{-}15)\end{aligned}$$

Thus, inside Σ^+, the set $\mathcal{W}^+_{\mathrm{in}}$ is the unstable (resp. stable) manifold of $\mathcal{R}^+_{\mathrm{in},-}$ (resp. $\mathcal{R}^+_{\mathrm{in},+}$), while $\mathcal{W}^+_{\mathrm{c}}$ is the intersection of the unstable manifold of $\mathcal{R}^+_{\mathrm{c}}$ and the stable manifold of $\mathcal{R}^+_{\mathrm{in},-}$. Inside Σ^-, the analogous statements hold with "stable" and "unstable" switched. See Figure 4.1.

4.2. ***Microlocal estimates.*** We continue using the notation introduced at the beginning of Section 4. The structure of the linearizations of the Hamilton flow at the radial sets determines the structure of the corresponding microlocal propagation results, even at a quantitative level as far as the relative sizes of weights and differential orders are concerned. In addition, suitable subprincipal symbols of the operator P affect the precise threshold conditions entering in the radial point estimates at $\mathcal{R}_{\mathrm{c}}$ and $\mathcal{R}_{\mathrm{out}}$; there, the endomorphism p_1 from Definition 3.7(3) enters (see also Remark 4.12). We thus define the following.

Definition 4.3 (minimum of p_1). Given $p_1 \in (\mathcal{C}^\infty + \mathcal{A}^{(\ell_0,\ell_+)})(\mathscr{I}^+; \beta^* \operatorname{End}(E))$, we set

$$\underline{p}_1 := \inf_{p\in\mathscr{I}^+} \{\operatorname{Re}\mu : \mu \in \operatorname{spec} p_1(p)\}.$$

Here, $\operatorname{spec} p_1(p) \subset \mathbb{C}$ is the spectrum (in the sense of linear algebra) of the linear map $p_1(p) : E_y \to E_y$, $y = \beta(p)$. We define $\underline{p}_{1,+}$ by the same expression except for taking the infimum only over $p \in \mathscr{I}^+ \cap I^+$.

We rewrite this in a manner more amenable for use in estimates.

Lemma 4.4 (characterization of $\underline{p}_1$). *Let $p_1 \in (\mathcal{C}^\infty + \mathcal{A}^{(\ell_0,\ell_+)})(\mathscr{I}^+; \beta^* \operatorname{End}(E))$. For a positive definite fiber inner product h_E on $E|_Y$, set*

$$\underline{p}_1(h_E) := \inf_{p\in\mathscr{I}^+} \operatorname{spec}\big(\tfrac{1}{2}(p_1(p) + p_1(p)^*)\big),$$

where $p_1(p)^* \in \operatorname{End}((\beta^* E)_p) = \operatorname{End}(E_y)$, $y = \beta(p)$, is the adjoint of $p_1(p)$ with respect to $h_E(y)$. Then $\underline{p}_1 = \sup_{h_E} \underline{p}_1(h_E)$.

Proof. For any h_E, one has $\underline{p}_1(h_E) \le \underline{p}_1$, and hence the statement is that, for any $\epsilon > 0$, one can find an inner product h_E such that $\underline{p}_1 - \epsilon < \underline{p}_1(h_E) \le \underline{p}_1$. The lemma is a general linear algebra result, which is a minor variant of [Hintz 2023d, Proposition B.1 and Lemma B.2]: we need to use that, due to $\ell_0, \ell_+ > 0$, we have $(\mathcal{C}^\infty + \mathcal{A}^{(\ell_0,\ell_+)})(\mathscr{I}^+) \subset \mathcal{C}^0(\mathscr{I}^+)$, and thus the compactness argument of the proof of [Hintz 2023d, Proposition B.1] still applies. □

We shall identify conic subsets of ${}^{\mathrm{e,b}}T^*M \setminus o$ with their boundaries at fiber infinity. We shall moreover drop the vector bundle $\beta^* E \to M$, in which all distributions below are valued, from the notation. Finally, in order to state our estimates, we need to fix an integration density on M, which we take to be a positive element of

$$\rho_0^{-n-1} x_{\mathscr{I}}^{-2n} \rho_+^{-n-1} \mathcal{C}^\infty(M; {}^{\mathrm{b}}\Omega M). \tag{4-16}$$

The motivation is that the lift of the metric density $|\mathrm{d}g|$ on M° is given by

$$|\mathrm{d}g| = \rho_0^{-n-1} x_{\mathscr{I}}^{-n-1} \rho_+^{-n-1} |\mathrm{d}(\rho_0^2 x_{\mathscr{I}}^2 \rho_+^2 g)| \in \rho_0^{-n-1} x_{\mathscr{I}}^{-n-1} \rho_+^{-n-1} (\mathcal{C}^\infty + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)})(M; {}^{\mathrm{e,b}}\Omega M),$$

and we have $\mathcal{C}^\infty(M; {}^{\mathrm{e,b}}\Omega M) = x_{\mathscr{I}}^{-n+1} \mathcal{C}^\infty(M; {}^{\mathrm{b}}\Omega M)$. (We may as well allow for a conormal term in (4-16) as well, which makes no difference in the subsequent arguments; one could then simply use $|\mathrm{d}g|$ as the metric density, which is of class (4-16) plus a lower-order conormal error term.)

The semiglobal microlocal propagation results near $\mathscr{I}^+$ are then the following.

Theorem 4.5 (propagation through $\mathscr{I}^+$: forward direction). *Let $s \in \mathbb{R}$, and let $\alpha = (\alpha_0, 2\alpha_{\mathscr{I}}, \alpha_+)$ be a vector of weights; put $\alpha' = \alpha + (2,2,2)$. Suppose $u \in H_{\mathrm{e,b}}^{-\infty,\alpha}(M) = \rho_0^{\alpha_0} x_{\mathscr{I}}^{2\alpha_{\mathscr{I}}} \rho_+^{\alpha_+} H_{\mathrm{e,b}}^{-\infty}(M)$, and let $f = Pu \in H_{\mathrm{e,b}}^{-\infty,\alpha'}(M)$. Then $\mathrm{WF}_{\mathrm{e,b}}^{s,\alpha}(u) \subset \Sigma \cup \mathrm{WF}_{\mathrm{e,b}}^{s-2,\alpha'}(f)$. Moreover:*

(1) (propagation through $\mathscr{I}^+$) *Suppose that*[23]

$$\alpha_+ < \alpha_{\mathscr{I}} - \tfrac{1}{2} < \alpha_0, \quad s > s_0 > \tfrac{1}{2} - \alpha_0 + 2\alpha_{\mathscr{I}} - \underline{p}_1. \tag{4-17}$$

[23]The condition on s_0 can be written as $s_0 > -(\alpha_0 - \alpha_{\mathscr{I}}) - \big(-\frac{1}{2} + \underline{p}_1 - \alpha_{\mathscr{I}}\big)$, cf. the first inequality in (4-17) and the threshold condition (4-19).

Assuming that[24]

$$\begin{aligned} \mathrm{WF}^{s,\alpha}_{\mathrm{e,b}}(u) \cap {}^{\mathrm{e,b}}S^*_{(I^0)^\circ} M &= \varnothing, \\ \mathrm{WF}^{s_0,\alpha}_{\mathrm{e,b}}(u) \cap \mathcal{R}_{\mathrm{c}} &= \varnothing, \\ \mathrm{WF}^{s-1,\alpha'}_{\mathrm{e,b}}(f) \cap {}^{\mathrm{e,b}}S^*_{\mathscr{I}^+} M &\subset \mathcal{R}_{\mathrm{out}}, \end{aligned} \tag{4-18}$$

we then have $\mathrm{WF}^{s,\alpha}_{\mathrm{e,b}}(u) \cap {}^{\mathrm{e,b}}S^*_{\mathscr{I}^+} M \subset \mathcal{R}_{\mathrm{out}}$.

(2) (full control at $\mathscr{I}^+$) *Suppose that, in addition to* (4-17)*, we have*

$$\alpha_{\mathscr{I}} < -\tfrac{1}{2} + \underline{p}_1 \tag{4-19}$$

and that, for an open neighborhood $\mathcal{U} \subset {}^{\mathrm{e,b}}S^*M$ *of* $\mathcal{R}_{\mathrm{out}}$*, we have*

$$\begin{aligned} \mathrm{WF}^{s,\alpha}_{\mathrm{e,b}}(u) \cap (\mathcal{U} \setminus {}^{\mathrm{e,b}}S^*_{\mathscr{I}^+} M) &= \varnothing, \\ \mathrm{WF}^{s-1,\alpha'}_{\mathrm{e,b}}(f) \cap {}^{\mathrm{e,b}}S^*_{\mathscr{I}^+} M &= \varnothing. \end{aligned} \tag{4-20}$$

Then $\mathrm{WF}^{s,\alpha}_{\mathrm{e,b}}(u) \cap {}^{\mathrm{e,b}}S^*_{\mathscr{I}^+} M = \varnothing$.

All statements can be microlocalized to either of the two components $\Sigma^\pm$ of Σ. The terminology "forward" refers to the causal nature of the direction in which we are propagating. Relative to the rescaled Hamiltonian vector field, this means propagating regularity of u in the forward direction on Σ^+ and in the backward direction in Σ^-.

If $f \in H^{s-1,\alpha'}_{\mathrm{e,b}}(M)$, an application of part (1) shows that control of u in $(I^0)^\circ$ can be propagated through $\mathscr{I}^+$ and gives $H^{s,\alpha}_{\mathrm{e,b}}$-control away from the outgoing radial set. In particular, one gets control near the ingoing radial set $\mathcal{R}_{\mathrm{in},+}$ at the future boundary of $\mathscr{I}^+$. In order to gain full control of u, one needs to understand how the flow continues in I^+, which depends on the global geometry of the spacetime of interest; examples, including asymptotically stationary Minkowski spacetimes, are discussed in [Hintz 2023d]. Ultimately, if one can propagate control of u (from global propagation through I^+ as well as from initial data) to a punctured neighborhood of $\mathcal{R}_{\mathrm{out}}$, part (2) applies and yields full control of u at $\mathscr{I}^+$ in the differential order sense.

Setting up a solvability theory for P requires estimates for the adjoint P^* — defined with respect to a density (4-16) and any choice of positive definite fiber inner product on E — in which we propagate regularity in the opposite direction:

Theorem 4.6 (propagation through $\mathscr{I}^+$: backward direction). *Let* $\tilde{s} \in \mathbb{R}$*, let* $\tilde{\alpha} = (\tilde{\alpha}_0, 2\tilde{\alpha}_{\mathscr{I}}, \tilde{\alpha}_+)$ *be a vector of weights, and put* $\tilde{\alpha}' = \tilde{\alpha} + (2,2,2)$*. Suppose* $\tilde{u} \in H^{-\infty,\tilde{\alpha}}_{\mathrm{e,b}}(M)$*, and let* $P^*\tilde{u} = \tilde{f} \in H^{-\infty,\tilde{\alpha}'}_{\mathrm{e,b}}(M)$*. Then* $\mathrm{WF}^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b}}(\tilde{u}) \subset \Sigma \cup \mathrm{WF}^{\tilde{s}-2,\tilde{\alpha}'}_{\mathrm{e,b}}(\tilde{f})$*. Moreover*:

(1) (propagation through $\mathscr{I}^+$) *Suppose that*

$$\tilde{\alpha}_{\mathscr{I}} > -\tfrac{1}{2} - \underline{p}_1, \quad \mathrm{WF}^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b}}(\tilde{f}) \cap {}^{\mathrm{e,b}}S^*_{\mathscr{I}^+} M \subset \mathcal{W}_{\mathrm{c}} \cup \mathcal{W}_{\mathrm{in}}. \tag{4-21}$$

Then $\mathrm{WF}^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b}}(\tilde{u}) \cap {}^{\mathrm{e,b}}S^*_{\mathscr{I}^+} M \subset \mathcal{W}_{\mathrm{c}} \cup \mathcal{W}_{\mathrm{in}}$.

[24] Carefully note that the assumption on u at $\mathcal{R}_{\mathrm{c}}$ requires only above-threshold regularity s_0 on u (while the conclusion of the theorem gives $s > s_0$ degrees of edge-b-differentiability at $\mathcal{R}_{\mathrm{c}}$).

(2) (full control at $\mathscr{I}^+$) *Suppose that, in addition to* (4-21), *we have*

$$\tilde{\alpha}_0 < \tilde{\alpha}_{\mathscr{I}} - \tfrac{1}{2} < \tilde{\alpha}_+, \quad \tilde{s} < \tfrac{1}{2} - \tilde{\alpha}_0 + 2\tilde{\alpha}_{\mathscr{I}} + \underline{p}_1, \tag{4-22}$$

and that, for an open neighborhood $\mathcal{U} \subset {}^{\mathrm{e,b}}S^*M$ *of* $\mathcal{R}_{\mathrm{in},+}$, *we have*

$$\mathrm{WF}_{\mathrm{e,b}}^{\tilde{s},\tilde{\alpha}}(\tilde{u}) \cap (\mathcal{U} \cap {}^{\mathrm{e,b}}S^*_{(I^+)^\circ}M) = \varnothing, \quad \mathrm{WF}_{\mathrm{e,b}}^{\tilde{s}-1,\tilde{\alpha}'}(\tilde{f}) \cap {}^{\mathrm{e,b}}S^*_{\mathscr{I}^+}M = \varnothing. \tag{4-23}$$

Then $\mathrm{WF}_{\mathrm{e,b}}^{\tilde{s},\tilde{\alpha}}(\tilde{u}) \cap {}^{\mathrm{e,b}}S^*_{\mathscr{I}^+}M = \varnothing$.

Remark 4.7 (orders and duality). Given s, α, α' as in Theorem 4.5, the dual space of the space $H_{\mathrm{e,b}}^{s-1,\alpha'}$ (in which f is estimated) is $H_{\mathrm{e,b}}^{\tilde{s},\tilde{\alpha}}$, where $\tilde{s} = -s+1$ and $\tilde{\alpha} = (\tilde{\alpha}_0, 2\tilde{\alpha}_{\mathscr{I}}, \tilde{\alpha}_+) := -\alpha' = -\alpha - (2,2,2)$ (thus $\tilde{\alpha}_0 = -\alpha_0 - 2$, $\tilde{\alpha}_{\mathscr{I}} = -\alpha_{\mathscr{I}} - 1$, $\tilde{\alpha}_+ = -\alpha_+ - 2$). The threshold condition in (4-21) is then equivalent to (4-19), and the threshold conditions (4-22) for $\tilde{s}$, $\tilde{\alpha}_0$, $\tilde{\alpha}_{\mathscr{I}}$, $\tilde{\alpha}_+$ are equivalent to (4-17) (except s_0 does not enter in (4-23)).

The wave front set statements are proved using positive commutator arguments and are thus qualitative versions of quantitative statements. The quantitative version of Theorem 4.5(1), restricted to Σ^+ for definiteness, is the following. Suppose $B, E, E_{\mathrm{c}}, W \in \Psi_{\mathrm{e,b}}^0(M)$ satisfy the following conditions:

- $\mathrm{WF}'_{\mathrm{e,b}}(B) \cap \mathcal{R}^+_{\mathrm{out}} = \varnothing$;
- all backward null-bicharacteristics starting at $\mathrm{WF}'_{\mathrm{e,b}}(B) \cap \Sigma^+$ tend to $\mathcal{R}^+_{\mathrm{in},+} \cup \mathcal{R}^+_{\mathrm{in},-} \cup \mathcal{R}^+_{\mathrm{c}}$ or enter $\mathrm{Ell}_{\mathrm{e,b}}(E)$ in finite time, all while remaining inside $\mathrm{Ell}_{\mathrm{e,b}}(W)$;
- W is elliptic also *at* $\mathcal{R}^+_{\mathrm{in},+} \cup \mathcal{R}^+_{\mathrm{in},-} \cup \mathcal{R}^+_{\mathrm{c}}$;
- E_{c} is elliptic at $\mathcal{R}^+_{\mathrm{c}}$.

Let $N \in \mathbb{R}$. Then under the assumptions (4-17), the estimate

$$\|Bu\|_{H_{\mathrm{e,b}}^{s,\alpha}(M)} \leq C\big(\|WPu\|_{H_{\mathrm{e,b}}^{s-1,\alpha'}(M)} + \|Eu\|_{H_{\mathrm{e,b}}^{s,\alpha}(M)} + \|E_{\mathrm{c}}u\|_{H_{\mathrm{e,b}}^{s_0,\alpha}(M)} + \|u\|_{H_{\mathrm{e,b}}^{-N,\alpha}(M)}\big) \tag{4-24}$$

holds in the strong sense that if all quantities on the right are finite, then so is the left-hand side, and the estimate holds. While the estimate (4-24) can be recovered from Theorem 4.5(1) via the closed graph theorem [Vasy 2018, §4], the direct proof via positive commutators provides control on the constant C in terms of norms of the coefficients of P which, in principle, can be made explicit; this is important in applications to quasilinear equations; see, e.g., [Hintz 2016]. We leave statements of quantitative versions of the other parts of the above theorems to the reader.

In order to prove Theorems 4.5 and 4.6, we work microlocally near each of the radial sets separately. Throughout the remainder of this section, we shall assume

$$\begin{aligned} &s \in \mathbb{R}, \quad \alpha = (\alpha_0, 2\alpha_{\mathscr{I}}, \alpha_+), && \alpha' = (\alpha'_0, 2\alpha'_{\mathscr{I}}, \alpha'_+) := \alpha + (2,2,2),\\ &\tilde{s} \in \mathbb{R}, \quad \tilde{\alpha} = (\tilde{\alpha}_0, 2\tilde{\alpha}_{\mathscr{I}}, \tilde{\alpha}_+), && \tilde{\alpha}' = (\tilde{\alpha}'_0, 2\tilde{\alpha}'_{\mathscr{I}}, \tilde{\alpha}'_+) := \tilde{\alpha} + (2,2,2),\\ &\qquad u \in H_{\mathrm{e,b}}^{-\infty,\alpha}(M), && Pu = f \in H_{\mathrm{e,b}}^{-\infty,\alpha'}(M),\\ &\qquad \tilde{u} \in H_{\mathrm{e,b}}^{-\infty,\tilde{\alpha}}(M), && P^*\tilde{u} = \tilde{f} \in H_{\mathrm{e,b}}^{-\infty,\tilde{\alpha}'}(M). \end{aligned} \tag{4-25}$$

For conciseness, we only state the qualitative wave front set versions of the radial point estimates near each of the radial sets.

Lemma 4.8 (propagation at $\mathcal{R}_{\mathrm{c}}$). *Let $\mathcal{U} \subset {}^{\mathrm{e,b}}S^*M$ be an open neighborhood of $\mathcal{R}_{\mathrm{c}}$.*

(1) (forward propagation) *Let $s > s_0 > \frac{1}{2} - \alpha_0 + 2\alpha_{\mathscr{I}} - \underline{p}_1$. Suppose that $\mathrm{WF}_{\mathrm{e,b}}^{s-1,\alpha'}(f) \cap \mathcal{R}_{\mathrm{c}} = \varnothing$ and $\mathrm{WF}_{\mathrm{e,b}}^{s_0,\alpha}(u) \cap \mathcal{R}_{\mathrm{c}} = \varnothing$. If $\mathrm{WF}_{\mathrm{e,b}}^{s,\alpha}(u) \cap (\mathcal{U} \cap {}^{\mathrm{e,b}}S^*_{(I^0)^\circ}M) = \varnothing$, then we obtain $\mathrm{WF}_{\mathrm{e,b}}^{s,\alpha}(u) \cap \mathcal{R}_{\mathrm{c}} = \varnothing$.*

(2) (backward propagation) *Suppose that $\tilde{s} < \frac{1}{2} - \tilde{\alpha}_0 + 2\tilde{\alpha}_{\mathscr{I}} + \underline{p}_1$ and $\mathrm{WF}_{\mathrm{e,b}}^{\tilde{s}-1,\tilde{\alpha}'}(\tilde{f}) \cap \mathcal{R}_{\mathrm{c}} = \varnothing$. If $\mathrm{WF}_{\mathrm{e,b}}^{\tilde{s},\tilde{\alpha}}(\tilde{u}) \cap (\mathcal{U} \cap {}^{\mathrm{e,b}}S^*_{\mathscr{I}^+}M \setminus \mathcal{R}_{\mathrm{c}}) = \varnothing$, then $\mathrm{WF}_{\mathrm{e,b}}^{\tilde{s},\tilde{\alpha}}(\tilde{u}) \cap \mathcal{R}_{\mathrm{c}} = \varnothing$.*

Lemma 4.9 (propagation at $\mathcal{R}_{\mathrm{in},-}$). *Let $\mathcal{U} \subset {}^{\mathrm{e,b}}S^*M$ be an open neighborhood of $\mathcal{R}_{\mathrm{in},-}$.*

(1) (forward propagation) *Suppose that $\alpha_{\mathscr{I}} < \alpha_0 + \frac{1}{2}$ and $\mathrm{WF}_{\mathrm{e,b}}^{s-1,\alpha'}(f) \cap \mathcal{R}_{\mathrm{in},-} = \varnothing$. If $\mathrm{WF}_{\mathrm{e,b}}^{s,\alpha}(u) \cap (\mathcal{U} \cap {}^{\mathrm{e,b}}S^*_{I^0}M \setminus \mathcal{R}_{\mathrm{in},-}) = \varnothing$, then $\mathrm{WF}_{\mathrm{e,b}}^{s,\alpha}(u) \cap \mathcal{R}_{\mathrm{in},-} = \varnothing$.*

(2) (backward propagation) *Suppose that $\tilde{\alpha}_0 < \tilde{\alpha}_{\mathscr{I}} - \frac{1}{2}$ and $\mathrm{WF}_{\mathrm{e,b}}^{\tilde{s}-1,\tilde{\alpha}'}(\tilde{f}) \cap \mathcal{R}_{\mathrm{in},-} = \varnothing$. If $\mathrm{WF}_{\mathrm{e,b}}^{\tilde{s},\tilde{\alpha}}(\tilde{u}) \cap (\mathcal{U} \cap {}^{\mathrm{e,b}}S^*_{(\mathscr{I}^+)^\circ}M) = \varnothing$, then $\mathrm{WF}_{\mathrm{e,b}}^{\tilde{s},\tilde{\alpha}}(\tilde{u}) \cap \mathcal{R}_{\mathrm{in},-} = \varnothing$.*

Lemma 4.10 (propagation at $\mathcal{R}_{\mathrm{in},+}$). *Let $\mathcal{U} \subset {}^{\mathrm{e,b}}S^*M$ be an open neighborhood of $\mathcal{R}_{\mathrm{in},+}$.*

(1) (forward propagation) *Suppose that $\alpha_+ < \alpha_{\mathscr{I}} - \frac{1}{2}$ and $\mathrm{WF}_{\mathrm{e,b}}^{s-1,\alpha'}(f) \cap \mathcal{R}_{\mathrm{in},+} = \varnothing$. If $\mathrm{WF}_{\mathrm{e,b}}^{s,\alpha}(u) \cap (\mathcal{U} \cap {}^{\mathrm{e,b}}S^*_{(\mathscr{I}^+)^\circ}M) = \varnothing$, then $\mathrm{WF}_{\mathrm{e,b}}^{s,\alpha}(u) \cap \mathcal{R}_{\mathrm{in},+} = \varnothing$.*

(2) (backward propagation) *Suppose that $\tilde{\alpha}_{\mathscr{I}} < \tilde{\alpha}_+ + \frac{1}{2}$ and $\mathrm{WF}_{\mathrm{e,b}}^{\tilde{s}-1,\tilde{\alpha}'}(\tilde{f}) \cap \mathcal{R}_{\mathrm{in},+} = \varnothing$. If $\mathrm{WF}_{\mathrm{e,b}}^{\tilde{s},\tilde{\alpha}}(\tilde{u}) \cap (\mathcal{U} \cap {}^{\mathrm{e,b}}S^*_{I^+}M \setminus \mathcal{R}_{\mathrm{in},+}) = \varnothing$, then $\mathrm{WF}_{\mathrm{e,b}}^{\tilde{s},\tilde{\alpha}}(\tilde{u}) \cap \mathcal{R}_{\mathrm{in},+} = \varnothing$.*

Lemma 4.11 (propagation at $\mathcal{R}_{\mathrm{out}}$). (1) (forward propagation) *Suppose that $\alpha_{\mathscr{I}} < -\frac{1}{2} + \underline{p}_1$ and $\mathrm{WF}_{\mathrm{e,b}}^{s-1,\alpha'}(f) \cap \mathcal{R}_{\mathrm{out}} = \varnothing$. Let $\mathcal{U} \subset {}^{\mathrm{e,b}}S^*M$ be an open neighborhood of $\mathcal{R}_{\mathrm{out}}$. If $\mathrm{WF}_{\mathrm{e,b}}^{s,\alpha}(u) \cap (\mathcal{U} \setminus \mathcal{R}_{\mathrm{out}}) = \varnothing$, then $\mathrm{WF}_{\mathrm{e,b}}^{s,\alpha}(u) \cap \mathcal{R}_{\mathrm{out}} = \varnothing$.*

(2) (localized forward propagation near I^0) *Using the coordinates (3-5) for any fixed $T \in \mathbb{R}$, we let $0 < \rho_{0,-} < \rho_{0,+} < 1$ and suppose $\mathcal{U}^\flat, \mathcal{U} \subset {}^{\mathrm{e,b}}S^*M \cap \{\rho_0 < \rho_{0,+}\}$ are open sets containing $\mathcal{R}_{\mathrm{out}} \cap \{\rho_0 < \rho_{0,-}\}$ and $\overline{\mathcal{U}^\flat} \subset \mathcal{U}$. If we have $\alpha_{\mathscr{I}} < -\frac{1}{2} + \underline{p}_1$, $\mathrm{WF}_{\mathrm{e,b}}^{s,\alpha}(u) \cap (\mathcal{U} \setminus \mathcal{R}_{\mathrm{out}}) = \varnothing$, and $\mathrm{WF}_{\mathrm{e,b}}^{s-1,\alpha'}(f) \cap {}^{\mathrm{e,b}}S^*_{\mathscr{I}^+}M \subset \mathcal{R}_{\mathrm{out}} \setminus \overline{\mathcal{U}}$, then $\mathrm{WF}_{\mathrm{e,b}}^{s,\alpha}(u) \cap {}^{\mathrm{e,b}}S^*_{\mathscr{I}^+}M \subset \mathcal{R}_{\mathrm{out}} \setminus \overline{\mathcal{U}^\flat}$.*

(3) (backward propagation) *Suppose that $\tilde{\alpha}_{\mathscr{I}} > -\frac{1}{2} - \underline{p}_1$ and $\mathrm{WF}_{\mathrm{e,b}}^{\tilde{s}-1,\tilde{\alpha}'}(\tilde{f}) \cap \mathcal{R}_{\mathrm{out}} = \varnothing$. Then $\mathrm{WF}_{\mathrm{e,b}}^{\tilde{s},\tilde{\alpha}}(\tilde{u}) \cap \mathcal{R}_{\mathrm{out}} = \varnothing$.*

(4) (localized backward propagation near I^+) *Using the coordinates (3-6) for any fixed $T \in \mathbb{R}$, let $0 < \bar{\rho}_+ < 1$, and let $\mathcal{U} \subset {}^{\mathrm{e,b}}S^*M$ be an open set with $\mathcal{U} \cap \mathcal{R}_{\mathrm{out}} = \{\rho_+ < \bar{\rho}_+\} \cap \mathcal{R}_{\mathrm{out}}$. If $\tilde{\alpha}_{\mathscr{I}} > -\frac{1}{2} - \underline{p}_1$ and $\mathrm{WF}_{\mathrm{e,b}}^{\tilde{s}-1,\tilde{\alpha}'}(\tilde{f}) \cap \mathcal{U} = \varnothing$, then $\mathrm{WF}_{\mathrm{e,b}}^{\tilde{s},\tilde{\alpha}}(\tilde{u}) \cap (\mathcal{U} \cap \mathcal{R}_{\mathrm{out}}) = \varnothing$.*

Remark 4.12 (threshold conditions and $\underline{p}_1$). The threshold conditions for propagation at $\mathcal{R}_{\mathrm{in},-}$ (see Lemma 4.9) (resp. $\mathcal{R}_{\mathrm{in},+}$ (see Lemma 4.10)) only involve the (relative) weights at I^0 and $\mathscr{I}^+$ (resp. $\mathscr{I}^+$ and I^+). This is consistent with the fact that these radial sets control the uniform behavior (i.e., the amplitude) of waves passing *through* (a neighborhood of) $\mathscr{I}^+$ (i.e., they are essentially transported across $\mathscr{I}^+$ without any change to their amplitude). On the other hand, the radial point estimate at $\mathcal{R}_{\mathrm{out}}$ (see Lemma 4.11) naturally *does* involve $\underline{p}_1$, as this quantity determines the decay rate at $\mathscr{I}^+$ of waves tending *towards* $\mathscr{I}^+$ (cf. the first factor in each line of (3-18)). The radial set $\mathcal{R}_{\mathrm{c}}$ (see Lemma 4.8) controls waves featuring both aspects (cf. its unstable manifold in Figure 4.1).

Proof of Theorems 4.5 and 4.6, given Lemmas 4.8–4.11. Elliptic regularity in the edge-b-calculus immediately gives $\mathrm{WF}_{\mathrm{e,b}}^{s,\alpha}(u) \subset \Sigma \cap \mathrm{WF}_{\mathrm{e,b}}^{s-2,\alpha'}(f)$. For part (1), we first apply Lemma 4.8(1) to get control of u microlocally at $\mathcal{R}_{\mathrm{c}}$. We can then propagate this control using standard real principal propagation along the unstable manifolds of $\mathcal{R}_{\mathrm{c}}$, thus controlling u microlocally in ${}^{\mathrm{e,b}}S^*_{\mathscr{I}^+\setminus I^+}M \setminus (\mathcal{R}_{\mathrm{out}} \cup \mathcal{W}_{\mathrm{in}})$. In particular, the assumptions of Lemma 4.9(1) are satisfied for a sufficiently small neighborhood of $\mathcal{R}_{\mathrm{in},-}$. Hence, we obtain control at $\mathcal{R}_{\mathrm{in},-}$, which can be propagated along $\mathcal{W}_{\mathrm{in}}$. We can then apply Lemma 4.10(1) to get control at $\mathcal{R}_{\mathrm{in},+}$; this can then be further propagated over $\mathscr{I}^+ \cap I^+$ to any punctured neighborhood of $\mathcal{R}_{\mathrm{out}}$. Altogether, this proves the absence of $H_{\mathrm{e,b}}^{s,\alpha}$-wave front set of u over $\mathscr{I}^+$ except at $\mathcal{R}_{\mathrm{out}}$. Part (2) of Theorem 4.5 is now an immediate application of Lemma 4.11(1).

We observe that Theorem 4.6(1) follows from Lemma 4.11(3) and real principal-type propagation in ${}^{\mathrm{e,b}}S^*_{\mathscr{I}^+}M \setminus (\mathcal{W}_{\mathrm{in}} \cup \mathcal{W}_{\mathrm{c}})$. For the proof of part (2), one in addition applies Lemma 4.10(2) to get control at $\mathcal{R}_{\mathrm{in},+}$, which propagates to a punctured neighborhood of $\mathcal{R}_{\mathrm{in},-}$ over $\mathscr{I}^+ \setminus I^0$. Propagation into $\mathcal{R}_{\mathrm{in},-}$ is accomplished by Lemma 4.9(2), from where one can then propagate regularity of u to a punctured neighborhood of $\mathcal{R}_{\mathrm{c}}$. An application of Lemma 4.8(2) completes the argument. □

We now turn to the proofs of Lemmas 4.8–4.11.

Proof of Lemma 4.8. We work near $\mathcal{R}_{\mathrm{c}}^+$; the arguments near $\mathcal{R}_{\mathrm{c}}^-$ are completely analogous. We omit the weight at I^+ from the notation. Let $\check{\alpha} = (\check{\alpha}_0, 2\check{\alpha}_{\mathscr{I}}) := \alpha + (1, 1) = (\alpha_0 + 1, 2\alpha_{\mathscr{I}} + 1)$. We consider a commutant

$$\check{A} = \mathrm{Op}_{\mathrm{e,b}}(\check{a}) \in \Psi_{\mathrm{e,b}}^{s-1/2,\check{\alpha}}, \quad A = \check{A}^*\check{A}, \tag{4-26}$$

with $\check{a}$ defined momentarily, and shall estimate the L^2-pairing[25]

$$\mathrm{Im}\langle Pu, Au\rangle = \langle \mathscr{C}u, u\rangle, \quad \mathscr{C} = \frac{i}{2}[P, A] + \frac{P - P^*}{2i}A \in \Psi_{\mathrm{e,b}}^{2s,2\alpha}(M) \tag{4-27}$$

in two different ways. Here, we define the L^2 inner product (and the adjoint P^*) with respect to the volume density (4-16) and a fiber inner product h_E on E which is almost optimal, that is,

$$\underline{p}_1(h_E) \in (\underline{p}_1 - \epsilon_0, \underline{p}_1] \tag{4-28}$$

for any fixed $\epsilon_0 > 0$. Using the Cauchy–Schwarz inequality and elliptic estimates, the left-hand side of (4-27) can be bounded, for any $\delta > 0$, by

$$\mathrm{Im}\langle Pu, Au\rangle \geq -|\langle \check{A}^*Pu, \check{A}u\rangle| \geq -C\delta^{-1}\|WPu\|^2_{H_{\mathrm{e,b}}^{s-1,\alpha'}} - \delta\|\check{A}u\|^2_{H_{\mathrm{e,b}}^{1/2,(-1,-1)}} - C\|u\|^2_{H_{\mathrm{e,b}}^{-N,\alpha}} \tag{4-29}$$

for a constant C (independent of u, f, δ), where $W \in \Psi_{\mathrm{e,b}}^0$ is any fixed operator that is elliptic on $\mathrm{WF}'_{\mathrm{e,b}}(A)$.

We next compute the principal symbol of the "commutator" $\mathscr{C}$. We employ the coordinates ξ, η_j, ζ from (4-2) in the base and the coordinates ρ_∞, $\hat{\eta}$, $\hat{\zeta}$ from (4-12a) in the compactified fibers. Recall $G_{\mathrm{e,b}}^0$ from (4-3a), and define, for $\delta_{\mathscr{I}} \in (0, 1)$ specified below,

$$\check{a} = \rho_\infty^{-s+1/2}\rho_0^{-\check{\alpha}_0}x_{\mathscr{I}}^{-2\check{\alpha}_{\mathscr{I}}}\chi(\rho_0)\chi\left(\frac{x_{\mathscr{I}}}{\delta_{\mathscr{I}}}\right)\chi(|\hat{\zeta}|^2)\chi(\rho_\infty^2 G_{\mathrm{e,b}}^0);$$

[25] Regarding the weights of $\mathscr{C}$, note that $(-2, -2) + 2\check{\alpha} = 2\alpha$ indeed.

here, for $\epsilon > 0$ fixed below, the cutoff function $\chi \in \mathcal{C}_c^\infty((-\epsilon, \epsilon))$ is identically 1 near 0, and on $[0, \epsilon)$ satisfies $\chi' \leq 0$, $\sqrt{-\chi'\chi} \in \mathcal{C}^\infty$. Thus, $\check{a}$ is supported in any given conic neighborhood of $\mathcal{R}_c^+$ upon choosing $\epsilon > 0$ small enough. In view of (4-12b), we have

$$\check{a} H_{G^0_{e,b}} \check{a} = \big(-s + \tfrac{1}{2} - \check{\alpha}_0 + 2\check{\alpha}_{\mathscr{I}} + \mathcal{O}(\epsilon) + \mathcal{O}(\epsilon^{\ell_0}\epsilon^{2\ell_{\mathscr{I}}})\big)\rho_\infty^{-1}\check{a}^2 + \rho_\infty^{-2s}\rho_0^{-2\check{\alpha}_0} x_{\mathscr{I}}^{-4\check{\alpha}_{\mathscr{I}}}(-b_0^2 - b_c^2 + b_{\mathscr{I}}^2 + q\rho_\infty^2 G^0_{e,b}). \quad (4\text{-}30)$$

Here $b_0, b_c, b_{\mathscr{I}}, q \in S^0 + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}})}S^0$ are symbols on ${}^{e,b}T^*M$; the symbol b_0 arises from differentiation of $\chi(\rho_0)$ (which by (4-12b) gives a negative square when $\epsilon > 0$, hence the support of $\check{a}$, is small enough); the symbol b_c from differentiation of $\chi(|\hat{\zeta}|^2)$ (which for sufficiently small $\epsilon > 0$ gives a negative square at $x_{\mathscr{I}} = 0$, and thus on $\operatorname{supp}\check{a}$ when $\delta_{\mathscr{I}}$ is sufficiently small); moreover, $b_{\mathscr{I}}$ arises from differentiation of $\chi(x_{\mathscr{I}}/\delta_{\mathscr{I}})$, and q arises from differentiation of the localizer $\chi(\rho_\infty^2 G^0_{e,b})$ to the characteristic set. In terms of the principal symbol $G = \rho_0^2 x_{\mathscr{I}}^2 G^0_{e,b}$ of P, this means

$${}^{e,b}\sigma^{2s,2\alpha}\Big(\frac{i}{2}[P, A]\Big) = \check{a} H_G \check{a} = \rho_0^2 x_{\mathscr{I}}^2 \check{a} H_{G^0_{e,b}} \check{a} + \rho_\infty^{-2s}\rho_0^{-2\alpha_0} x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}} \tilde{q} \cdot \rho_\infty^2 \rho_0^{-2} x_{\mathscr{I}}^{-2} G, \quad (4\text{-}31)$$

with $\tilde{q} \in S^0 + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}})}S^0$. By definition of $\check{\alpha}$, the expression (4-31) is a symbol of class $S^{2s,2\alpha} + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}})}S^{2s,2\alpha}$.

Turning to the second term of $\mathscr{C}$ in (4-27), we need to evaluate ${}^{e,b}\sigma^{1,(-2,-2,-2)}((P - P^*)/2i)$ near $\mathcal{R}_c^+$. In the notation of Definition 3.7, we may replace P here by P_0, as the operator $\widetilde{P}$ only contributes an element of $\mathcal{A}^{(2+\ell_0, 2+2\ell_{\mathscr{I}})}S^1$. We have $P_0 = \rho_0^2 x_{\mathscr{I}}^2 P_{0,e,b}$, where the normal operator of $2P_{0,e,b}$ is given by the first line of (3-18). Recall from (4-16) that we are working with an integration density

$$\mu \rho_0^{-n-1} x_{\mathscr{I}}^{-2n} \left| \frac{d\rho_0}{\rho_0} \frac{dx_{\mathscr{I}}}{x_{\mathscr{I}}} dy \right|,$$

where $\mu > 0$ is smooth. We now have that the normal operator of $\rho_0^{-2} x_{\mathscr{I}}^{-2}(P_0 - P_0^*)$ is equal to that of $P_{0,e,b} - \rho_0^{-2} x_{\mathscr{I}}^{-2} P^*_{0,e,b} \rho_0^2 x_{\mathscr{I}}^2$ and thus, after a short calculation (see also Example 3.10 and note that $\Box_{g_0}$ is symmetric with respect to $|dg_0|$), given by

$${}^{e}N_{\mathscr{I}^+, y_0}\Big(\rho_0^{-2} x_{\mathscr{I}}^{-2} \frac{P - P^*}{2i}\Big) = -\tfrac{1}{2}(p_1 + p_1^*)(x_{\mathscr{I}} D_{x_{\mathscr{I}}} - 2\rho_0 D_{\rho_0}) + \widetilde{N},$$
$$\widetilde{N} \in (\mathcal{C}^\infty + \mathcal{A}^{\ell_0})(\mathscr{I}^+_{y_0} \setminus I^+; \operatorname{End}(E_{y_0})).$$

Therefore, over $\mathscr{I}^+$,

$$\begin{aligned} {}^{e,b}\sigma^1\Big(\rho_0^{-2} x_{\mathscr{I}}^{-2} \frac{P - P^*}{2i}\Big) &= -\tfrac{1}{2}(p_1 + p_1^*)(\xi - 2\zeta) \\ &= -\tfrac{1}{2}(p_1 + p_1^*)\rho_\infty^{-1}(1 - 2\hat{\zeta}). \end{aligned} \quad (4\text{-}32)$$

Recall that, at $\mathcal{R}_c^+$, we have $\hat{\zeta} = 0$; in combination with (4-30), we therefore obtain

$${}^{e,b}\sigma^{2s,2\alpha}(\mathscr{C}) = \big(-s + \tfrac{1}{2} - \check{\alpha}_0 + 2\check{\alpha}_{\mathscr{I}} - \tfrac{1}{2}(p_1 + p_1^*) + \mathcal{O}(\epsilon) + \mathcal{O}(\epsilon^{\ell_0}\epsilon^{2\ell_{\mathscr{I}}})\big)\rho_0^2 x_{\mathscr{I}}^2 \rho_\infty^{-1}\check{a}^2 + \rho_\infty^{-2s}\rho_0^{-2\alpha_0} x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}}(-b_0^2 - b_c^2 + b_{\mathscr{I}}^2 + (q + \tilde{q})\rho_\infty^2 G^0_{e,b}). \quad (4\text{-}33)$$

The first line is negative (and thus has the same sign as the terms $-b_0^2$ and $-b_{\mathrm{c}}^2$ in the second line) under the condition

$$s+\alpha_0-2\alpha_{\mathscr{I}}>\tfrac{1}{2}-\underline{p}_1 \quad \text{on } \operatorname{supp}\check{a} \tag{4-34}$$

in the sense of quadratic forms on E. By assumption, this condition holds at $\mathcal{R}_{\mathrm{c}}$ provided we work with small enough $\epsilon_0>0$ (see (4-28)), and therefore also in a sufficiently small neighborhood of $\mathcal{R}_{\mathrm{c}}$ and thus on $\operatorname{supp}\check{a}$ if we choose $\epsilon>0$ small enough.

This symbolic calculation can be turned into an estimate in the standard manner; we sketch this briefly. We give ourselves some extra room, $\delta\in\big(0, s+\alpha_0-2\alpha_{\mathscr{I}}-\frac{1}{2}+\underline{p}_1\big)$, and denote by B_0, B_{c}, $B_{\mathscr{I}}$, $Q\in\mathcal{A}^{(-\alpha_0,-2\alpha_{\mathscr{I}})}\Psi_{\mathrm{e,b}}^s$ quantizations of $\rho_\infty^{-s}\rho_0^{-\alpha_0}x_{\mathscr{I}}^{-2\alpha_{\mathscr{I}}}$ times b_0, b_{c}, $b_{\mathscr{I}}$, $q+\tilde{q}$, respectively; we further let $\Lambda\in\Psi_{\mathrm{e,b}}^{1/2,(-1,-1)}$ with principal symbol $\rho_0 x_{\mathscr{I}}\rho_\infty^{-1/2}$ near $\operatorname{supp}\check{a}$. Then we can write

$$\mathscr{C}=-\delta(\Lambda\check{A})^*(\Lambda\check{A})-B^*B-B_0^*B_0-B_{\mathrm{c}}^*B_{\mathrm{c}}+B_{\mathscr{I}}^*B_{\mathscr{I}}+R,$$

where $B\in\mathcal{A}^{(-\alpha_0,-2\alpha_{\mathscr{I}})}\Psi_{\mathrm{e,b}}^s$ is elliptic at $\mathcal{R}_{\mathrm{c}}$ and such that the sum of the first two terms on the right here is a quantization of the first line in (4-33), and $R\in\mathcal{A}^{(-\alpha_0,-2\alpha_{\mathscr{I}})}\Psi_{\mathrm{e,b}}^{2s-1}$ (with $\mathrm{WF}'_{\mathrm{e,b}}(R)\subset\mathrm{WF}'_{\mathrm{e,b}}(\check{A})$) is the error term not captured by our principal symbol calculations. Plugging this into (4-27) and using (4-29), we obtain

$$\begin{aligned}\delta\|\check{A}u\|^2_{H_{\mathrm{e,b}}^{1/2,(-1,-1)}}+\|Bu\|_{L^2}^2+\|B_0u\|_{L^2}^2+\|B_{\mathrm{c}}u\|_{L^2}^2&\\ \leq C\delta^{-1}\|WPu\|^2_{H_{\mathrm{e,b}}^{s-1,\alpha'}}+\|B_{\mathscr{I}}u\|_{L^2}^2+\delta\|\check{A}u\|^2_{H_{\mathrm{e,b}}^{1/2,(-1,-1)}}+C\|u\|^2_{H_{\mathrm{e,b}}^{-N,\alpha}}+|\langle Ru,u\rangle|.&\end{aligned}$$

Upon canceling the terms involving $\check{A}$, dropping the third and fourth terms on the left, and estimating

$$|\langle Ru,u\rangle|\leq C\|Wu\|^2_{H_{\mathrm{e,b}}^{s-1/2,\alpha}}+C\|u\|^2_{H_{\mathrm{e,b}}^{-N,\alpha}}$$

using elliptic regularity, this gives the estimate

$$\|Bu\|_{L^2}\lesssim\|WPu\|_{H_{\mathrm{e,b}}^{s-1,\alpha'}}+\|B_{\mathscr{I}}u\|_{L^2}+\|Wu\|_{H_{\mathrm{e,b}}^{s-1/2,\alpha}}+\|u\|_{H_{\mathrm{e,b}}^{-N,\alpha}}. \tag{4-35}$$

This gives quantitative $H_{\mathrm{e,b}}^{s,\alpha}$-control of u near $\mathcal{R}_{\mathrm{c}}$ (where B is elliptic), assuming $H_{\mathrm{e,b}}^{s-1/2,\alpha}$-control of u on the elliptic set of W. Note that we can take $\mathrm{WF}'_{\mathrm{e,b}}(W)$ to be contained in any specified neighborhood of $\mathcal{R}_{\mathrm{c}}^+$ by choosing $\epsilon>0$ small enough.

In order to make sense of the integrations by parts in this argument, one needs to use a standard regularization argument [Vasy 2018, §4]; we only give a very brief sketch. One needs to replace $\check{a}$ by $\check{a}_r=(1+r\rho_\infty^{-1})^{-\gamma}\check{a}$, $r\in(0,1]$, where $\gamma>0$. For fixed $r>0$, this effectively replaces s by $s-\gamma$ in the above symbolic calculations, hence the amount γ of regularization one can do is limited in view of the requirement (4-34) for $s_0=s-\gamma$ in place of s, and one needs to assume that $\mathrm{WF}_{\mathrm{e,b}}^{s_0,\alpha}(u)\cap\mathcal{R}_{\mathrm{c}}=\varnothing$ in order to justify the integrations by parts. For such γ then, one obtains an r-dependent version of the estimate (4-35), with an implicit constant that is uniform as $r\searrow 0$; upon letting $r\to 0$, a standard functional analysis argument using the weak compactness of the unit ball in L^2 implies that $Bu\in L^2$ (i.e., one gets to *conclude* regularity of u on $\mathrm{Ell}_{\mathrm{e,b}}(B)$), and the estimate (4-35) holds. This finishes the proof of part (1) of the lemma.

Turning to part (2), the roles of P and P^* are now reversed. We remark that the choice of fiber inner product on E used to define the adjoint P^* in (4-25) is immaterial, as the adjoints with respect to any two choices are related via conjugation by a smooth bundle isomorphism on E; thus, we may as well use the near-optimal inner product already used in the first part of the proof. Thus, in the calculation (4-33) (but now using $\tilde{s}, \tilde{\alpha}_0, \tilde{\alpha}_{\mathscr{I}}$ instead of $s, \alpha_0, \alpha_{\mathscr{I}}$, and writing $(\breve{\alpha}_0, 2\breve{\tilde{a}}_{\mathscr{I}}) = \tilde{\alpha} + (1, 1)$), the terms involving p_1 come with the opposite sign; and propagating backwards, we now want $-\tilde{s} + \frac{1}{2} - \breve{\tilde{\alpha}}_0 + 2\breve{\tilde{\alpha}}_{\mathscr{I}} + \frac{1}{2}(p_1 + p_1^*)$ to be *positive* (thus to have the same sign as the term $b_{\mathscr{I}}^2$ but the opposite sign as $-b_0^2$, $-b_{\mathrm{c}}^2$, which are the places where we need to assume a priori control). This leads to the condition

$$\tilde{s} + \tilde{\alpha}_0 - 2\tilde{\alpha}_{\mathscr{I}} < \tfrac{1}{2} + \underline{p}_1.$$

Note that since this condition remains valid upon decreasing $\tilde{s}$, the allowed amount of regularization is now unlimited; this is the reason for the absence of an a priori regularity assumption on the solution $\tilde{u}$ of $P^*\tilde{u} = \tilde{f}$ at $\mathcal{R}_{\mathrm{c}}$. □

The proofs of Lemmas 4.9–4.11 are similar, hence we shall only explain the symbolic aspects.

Proof of Lemma 4.9. We shall work near $\mathcal{R}^+_{\mathrm{in},-}$ and use the coordinates (4-2), (4-11a) (with $\rho_\infty = \zeta^{-1}$ near $\mathcal{R}^+_{\mathrm{in},-}$, and in addition $\hat{\xi} = 2$ at $\mathcal{R}^+_{\mathrm{in},-}$). We shall define A as in (4-26), where now

$$\breve{a} = \rho_\infty^{-s+1/2}\rho_0^{-\breve{\alpha}_0} x_{\mathscr{I}}^{-2\breve{\alpha}_{\mathscr{I}}}\chi(\rho_0)\chi\left(\frac{x_{\mathscr{I}}}{\delta_{\mathscr{I}}}\right)\chi((2-\hat{\xi})^2)\chi(\rho_\infty^2 G^0_{\mathrm{e,b}}),$$

where χ is a cutoff function (with support near 0) and $0 < \delta_{\mathscr{I}} \ll 1$ as in the proof of Lemma 4.8. Recalling (4-11b), we find

$$\breve{a} H_{G^0_{\mathrm{e,b}}} \breve{a} = (-2\breve{\alpha}_0 + 2\breve{\alpha}_{\mathscr{I}} + \mathcal{O}(\epsilon) + \mathcal{O}(\epsilon^{\ell_0}\epsilon^{2\ell_{\mathscr{I}}}))\rho_\infty^{-1}\breve{a}^2 + \rho_\infty^{-2s}\rho_0^{-2\breve{\alpha}_0} x_{\mathscr{I}}^{-4\breve{\alpha}_{\mathscr{I}}}(-b_0^2 + b_{\mathscr{I}}^2 + b_{\mathrm{in}}^2 + q\rho_\infty G^0_{\mathrm{e,b}}),$$

where b_0, $b_{\mathscr{I}}$, b_{in} arise by differentiation of $\chi(\rho_0)$, $\chi(x_{\mathscr{I}}/\delta_{\mathscr{I}})$, $\chi((2-\hat{\xi})^2)$, respectively; for the latter, note that $2 - \hat{\xi} \geq 0$ is monotonically increasing along $H_{G^0_{\mathrm{e,b}}}$ near $\mathcal{R}^+_{\mathrm{in},-}$. Moreover, by (4-32), we have, over $\mathscr{I}^+$,

$${}^{\mathrm{e,b}}\sigma^1\left(\rho_0^{-2}x_{\mathscr{I}}^{-2}\frac{P-P^*}{2i}\right) = -\tfrac{1}{2}(p_1 + p_1^*)(\xi - 2\zeta) = -\tfrac{1}{2}(p_1 + p_1^*)\rho_\infty^{-1}(\hat{\xi} - 2).$$

At $\mathcal{R}_{\mathrm{in},-}$, this vanishes. Adding these two contributions to ${}^{\mathrm{e,b}}\sigma^{2s,2\alpha}(\mathscr{C})$ in the notation of (4-27) gives, similarly to (4-33),

$$\begin{aligned}{}^{\mathrm{e,b}}\sigma^{2s,2\alpha}(\mathscr{C}) = (-2\breve{\alpha}_0 + 2\breve{\alpha}_{\mathscr{I}} + \mathcal{O}(\epsilon) + \mathcal{O}(\epsilon^{\ell_0}\epsilon^{2\ell_{\mathscr{I}}}))\rho_0^2 x_{\mathscr{I}}^2\rho_\infty^{-1}\breve{a}^2 \\ + \rho_\infty^{-2s}\rho_0^{-2\alpha_0} x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}}(-b_0^2 + b_{\mathscr{I}}^2 + b_{\mathrm{in}}^2 + (q+\tilde{q})\rho_\infty^2 G^0_{\mathrm{e,b}}). \quad (4\text{-}36)\end{aligned}$$

When propagating in the forward direction, we assume a priori control on $\operatorname{supp} b_{\mathscr{I}} \cup \operatorname{supp} b_{\mathrm{in}}$ and obtain control at $\mathcal{R}^+_{\mathrm{in},-}$ (as well as on the elliptic set of b_0), provided that $-\breve{\alpha}_0 + \breve{\alpha}_{\mathscr{I}} < 0$, or equivalently $\alpha_{\mathscr{I}} < \alpha_0 + \frac{1}{2}$. The proof is now finished by quantizing the lower case symbols as before. As for the regularization argument, note that the condition on α_0, $\alpha_{\mathscr{I}}$ does not involve the differential order s, and hence we can do an arbitrary amount of regularization; thus, there is no a priori regularity requirement at $\mathcal{R}_{\mathrm{in},-}$ here. This proves the first part of the lemma.

For the second part concerning propagation in the backward direction, we now need $-\check{\tilde{\alpha}}_0+\check{\tilde{\alpha}}_{\mathscr{I}}>0$ (in the same notation as in the previous proof) to make the main, i.e., first, term of (4-36) (with tildes added) positive, matching the sign of $b_{\mathscr{I}}$, b_{in}, whereas now the term $-b_0^2$ has the opposite sign. Thus, we need microlocal $H_{\mathrm{e,b}}^{s,\alpha}$ regularity of u on $\operatorname{supp} b_0$ in order to conclude regularity at $\mathcal{R}_{\mathrm{in},-}$. □

Proof of Lemma 4.10. We work near $\mathcal{R}_{\mathrm{in},+}^+$ and use the coordinates (4-4), (4-13a) (with $\rho_\infty=\zeta^{-1}$). Using the same notation as in the proof of Lemma 4.8, We use the commutant

$$\check{a}=\rho_\infty^{-s+1/2}x_{\mathscr{I}}^{-2\check{\alpha}_{\mathscr{I}}}\rho_+^{-\check{\alpha}_+}\chi\left(\frac{x_{\mathscr{I}}}{\delta_{\mathscr{I}}}\right)\chi(\rho_+)\chi((2-\hat{\xi})^2)\chi(\rho_\infty^2G_{\mathrm{e,b}}^+).$$

Using (3-18) (and omitting the weight at I^0), we compute the normal operator of the imaginary part of P to be

$$\begin{aligned}{}^{\mathrm{e}}N_{\mathscr{I}^+,y_0}\left(x_{\mathscr{I}}^{-2}\rho_+^{-2}\frac{P-P^*}{2i}\right)&=\tfrac12(p_1+p_1^*)(x_{\mathscr{I}}D_{x_{\mathscr{I}}}-2\rho_+D_{\rho_+})+\widetilde{N},\\ \widetilde{N}&\in(\mathcal{C}^\infty+\mathcal{A}^{\ell_+})(\mathscr{I}^+_{y_0}\setminus I^0;\operatorname{End}(E_{y_0})).\end{aligned}$$

Its principal symbol is $\frac12(p_1+p_1^*)(\xi-2\zeta)=\frac12(p_1+p_1^*)\rho_\infty^{-1}(\hat{\xi}-2)$; this vanishes at $\mathcal{R}_{\mathrm{in},+}^+$. Using (4-13b), we then find (in the notation (4-27))

$$\begin{aligned}{}^{\mathrm{e,b}}\sigma^{2s,2\alpha}(\mathscr{C})=(-2\check{\alpha}_{\mathscr{I}}+2\check{\alpha}_+&+\mathcal{O}(\epsilon)+\mathcal{O}(\epsilon^{2\ell_{\mathscr{I}}}\epsilon^{\ell_+}))x_{\mathscr{I}}^2\rho_+^2\rho_\infty^{-1}\check{a}^2\\&+\rho_\infty^{-2s}x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}}\rho_+^{-2\alpha_+}(-b_{\mathscr{I}}^2+b_+^2-b_{\mathrm{in}}^2+(q+\tilde{q})\rho_\infty^2G_{\mathrm{e,b}}^+).\end{aligned}$$

Here $b_{\mathscr{I}}$, b_+, b_{in} arise from differentiation of $\chi(x_{\mathscr{I}}/\delta_{\mathscr{I}})$, $\chi(\rho_+)$, $\chi((2-\hat{\xi})^2)$, respectively. For forward propagation, and under the threshold condition $-\check{\alpha}_{\mathscr{I}}+\check{\alpha}_+<0$, i.e., $\alpha_+<\alpha_{\mathscr{I}}-\frac12$, we need to assume a priori control on $\operatorname{supp} b_+$. Furthermore, the amount of regularization is again unlimited since the edge-b-regularity s does not enter in the threshold condition. Following the arguments of the proof of Lemma 4.8 yields the desired result. The proof of the backward propagation result is similar. □

Proof of Lemma 4.11. We only give details for the localized forward propagation statement, i.e., part (2). We again work in the coordinates (4-2) on the edge-b-cotangent bundle over $\mathcal{U}_0(T)$, and we use the coordinates (4-11a) on the compactified fibers near $\mathcal{R}_{\mathrm{out}}^+$ (where now $\rho_\infty=-\zeta^{-1}\geq 0$). Fix $\rho_{0,0}\in(\rho_{0,-},\rho_{0,+})$. Recall from Lemma 3.6 that, upon fixing C large and $\delta_0>0$ small, the function

$$\tilde{\rho}_0:=\rho_0-(\rho_{0,0}+Cx_{\mathscr{I}}^{2\ell_{\mathscr{I}}})$$

has past timelike differential near its zero set in $x_{\mathscr{I}}<2\delta_0$. For sufficiently small $\delta\in(0,\delta_0]$ and $c_0>0$, the set $\mathcal{U}_0(T)\cap\{\tilde{\rho}_0\leq\pm c_0, x_{\mathscr{I}}<2\delta\}$ contains $\overline{\mathcal{U}^{\flat}}\cap{}^{\mathrm{e,b}}S^*_{\mathscr{I}^+}M$ (for the "−" sign) and is contained in $\mathcal{U}$ (for the "+" sign).

We again use a cutoff function $\chi\in\mathcal{C}_{\mathrm{c}}^\infty((-\epsilon,\epsilon))$ for small $0<\epsilon<2\delta_0$ which satisfies $\chi\equiv1$ near 0, and which on $[0,\epsilon)$ satisfies $\chi'\leq0$ and $\sqrt{-\chi'\chi}\in\mathcal{C}^\infty$. Let further $\tilde{\chi}\in\mathcal{C}^\infty(\mathbb{R})$ be identically 1 on $(-\infty,-c_0]$, identically 0 on $[c_0,\infty)$, and such that $\tilde{\chi}'\leq0$, $\sqrt{-\tilde{\chi}'\tilde{\chi}}\in\mathcal{C}^\infty$. We then use a commutant $A=\check{A}^*\check{A}$, $\check{A}=\operatorname{Op}_{\mathrm{e,b}}(\check{a})$, where we set

$$\check{a}=\rho_\infty^{-s+1/2}\rho_0^{-\check{\alpha}_0}x_{\mathscr{I}}^{-2\check{\alpha}_{\mathscr{I}}}\tilde{\chi}(\tilde{\rho}_0)\chi\left(\frac{x_{\mathscr{I}}}{\delta_{\mathscr{I}}}\right)\chi(\hat{\xi}^2)\chi(\rho_\infty^2G_{\mathrm{e,b}}^0).$$

In light of (4-32), we have

$$ {}^{\mathrm{e,b}}\sigma^1\Big(\rho_0^{-2}x_{\mathscr{I}}^{-2}\frac{P-P^*}{2i}\Big) = -\tfrac{1}{2}\rho_\infty^{-1}(p_1+p_1^*)(2-\hat{\xi}), $$

which equals $-\rho_\infty^{-1}(p_1+p_1^*)$ at $\mathcal{R}^+_{\mathrm{out}}$. Using (4-11c) and the past timelike nature of $\tilde{\rho}_0$, we can write (in the notation (4-27))

$$ \begin{aligned} {}^{\mathrm{e,b}}\sigma^{2s,2\alpha}(\mathscr{C}) = (2\breve{\alpha}_{\mathscr{I}} - (p_1+p_1)^* + \mathcal{O}(\epsilon) + \mathcal{O}(\epsilon^{\ell_0}\epsilon^{2\ell_{\mathscr{I}}}))\rho_\infty^{-1}\breve{a}^2 \\ + \rho_\infty^{-2s}\rho_0^{-2\alpha_0}x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}}(b_{\mathscr{I}}^2 - \tilde{b}^2 + b_{\mathrm{out}}^2 + q\rho_\infty^2 G^0_{\mathrm{e,b}}), \end{aligned} $$

where $b_{\mathscr{I}}$, $\tilde{b}$, b_{out}, and $q \in \mathcal{A}^{(0,0)}S^0$ (the conormal coefficients necessitated by the merely conormal regularity of $\tilde{\rho}_0$) arise from differentiation of $\chi(x_{\mathscr{I}}/\delta_{\mathscr{I}})$, $\tilde{\chi}(\tilde{\rho}_0)$, $\chi(\hat{\xi}^2)$, and $\chi(\rho_\infty^2 G^0_{\mathrm{e,b}})$, respectively. Since we are considering forward propagation, the terms $b_{\mathscr{I}}^2$ and b_{out}^2 are supported where we make a priori assumptions, and the term $\tilde{b}^2$ has the same sign as the main term (hence in the L^2 estimate arising upon quantization can be dropped) provided $2\breve{\alpha}_{\mathscr{I}} - 2\underline{p}_1 < 0$, i.e., $\alpha_{\mathscr{I}} < -\frac{1}{2} + \underline{p}_1$. Quantization and regularization proves part (2). □

5. Higher b-regularity

As already discussed in Section 1.2, we do not develop any tools in this paper for the microlocal analysis of b-regularity near $\mathscr{I}^+ \subset M$ (using the notation introduced at the beginning of Section 3.2). Recall that, in the coordinates ρ_0, $x_{\mathscr{I}}$, y from (3-5), b-regularity on M is iterated regularity under application of the b-vector fields $\rho_0\partial_{\rho_0}$, $x_{\mathscr{I}}\partial_{x_{\mathscr{I}}}$, and ∂_{y^j}, whereas edge-b-regularity captures regularity under $\rho_0\partial_{\rho_0}$, $x_{\mathscr{I}}\partial_{x_{\mathscr{I}}}$, and $x_{\mathscr{I}}\partial_{y^j}$. However, b-regularity on M (or equivalently on $\widetilde{M}$) is both natural (e.g., on Minkowski space it exactly amounts to regularity under generators of the full Poincaré group, in particular under spatial rotations) and useful in applications; for instance, it is used explicitly as a crucial piece of information in the recovery of sharp asymptotics at $\mathscr{I}^+$ in the stability results [Hintz 2023b, §3.6; Hintz and Vasy 2020, §5.1], and plays a central role (albeit in different terminology) in almost any analysis near null infinity; see [Lindblad 2017] for a sharp example.

In this section, we thus demonstrate how to work microlocally with elements of edge-b-spaces which have additional integer amounts of b-regularity; the main tool will be simple commutator arguments in which we commute appropriate b-vector fields through the edge-b-operators of interest. Since, in applications, higher b-regularity is mainly of interest for the forward solutions of wave equations, we only discuss regularity results for propagation in the forward (i.e., future causal) direction here.

In Sections 5.1–5.2, we work in a general edge-b-setting and discuss the appropriate class of b-vector fields used for commutation, the class of edge-b-Sobolev spaces with extra b-regularity, and basic microlocal estimates (elliptic regularity, real principal-type propagation) on such spaces. In Section 5.3, we return to the analysis near $\mathscr{I}^+$ and prove microlocal propagation results at radial points with extra b-regularity.

5.1. ***Commutator b-vector fields.*** As in Section 2.2, we let M denote a compact manifold with corners and embedded hypersurfaces $H_1, \dots, H_N$, $N \geq 3$, with H_2 being the total space of a fibration $\phi : H_2 \to Y$,

where the base Y is a smooth manifold without boundary and the typical fiber is a manifold Z, possibly with (disconnected) boundary (e.g., a closed interval).

Definition 5.1 (commutator b-vector fields). The space of *commutator b-vector fields* $\mathcal{V}_{[\mathrm{b}]}(M)$ consists of all $V \in \mathcal{V}_{\mathrm{b}}(M)$ such that $[V, W] \in \mathcal{V}_{\mathrm{e,b}}(M)$ for all $W \in \mathcal{V}_{\mathrm{e,b}}(M)$.

Clearly, we have $\mathcal{V}_{\mathrm{e,b}}(M) \subset \mathcal{V}_{[\mathrm{b}]}(M)$. To study this space further, denote by $\mathcal{C}^\infty_\phi(H_2) = \phi^*\mathcal{C}^\infty(Y)$ the space of fiber-constant functions on H_2; thus, $f \in \mathcal{C}^\infty(H_2)$ lies in $\mathcal{C}^\infty_\phi(H_2)$ if and only if $V|_{H_2} f = 0$ for all $V \in \mathcal{V}_{\mathrm{e,b}}(M)$. We also recall that an element $V \in \mathcal{V}_{\mathrm{b}}(M)$ lies in $\mathcal{V}_{\mathrm{e,b}}(M)$ if and only if for all $f \in \mathcal{C}^\infty_\phi(H_2)$ we have $V|_{H_2} f = 0$.

Lemma 5.2 (a simple criterion). *Let $V \in \mathcal{V}_{\mathrm{b}}(M)$. Then $V \in \mathcal{V}_{[\mathrm{b}]}(M)$ if and only if*

$$V|_{H_2} : \mathcal{C}^\infty_\phi(H_2) \to \mathcal{C}^\infty_\phi(H_2).$$

Proof. Let $V \in \mathcal{V}_{\mathrm{b}}(M)$. Given $W \in \mathcal{V}_{\mathrm{e,b}}(M)$, the condition $[V, W] \in \mathcal{V}_{\mathrm{e,b}}(M)$ is equivalent to the vanishing, for all $f \in \mathcal{C}^\infty_\phi(H_2)$, of

$$[V, W]|_{H_2} f = V|_{H_2} W|_{H_2} f - W|_{H_2} V|_{H_2} f = -W|_{H_2}(V|_{H_2} f).$$

The condition that this vanish for all $W \in \mathcal{V}_{\mathrm{e,b}}(M)$ is in turn equivalent to $V|_{H_2} f \in \mathcal{C}^\infty_\phi(H_2)$ for all $f \in \mathcal{C}^\infty_\phi(H_2)$. This proves the lemma. □

If the base Y is a point, then $\mathcal{V}_{\mathrm{e,b}}(M) = \mathcal{V}_{\mathrm{b}}(M) = \mathcal{V}_{[\mathrm{b}]}(M)$; if on the other hand the typical fiber Z is a point, then $\mathcal{C}^\infty_\phi(H_2) = \mathcal{C}^\infty(H_2)$ and thus again $\mathcal{V}_{[\mathrm{b}]}(M) = \mathcal{V}_{\mathrm{b}}(M)$. If both Y and Z have positive dimension, as is the case in our application, one always has $\mathcal{V}_{[\mathrm{b}]}(M) \subsetneq \mathcal{V}_{\mathrm{b}}(M)$. However:

Proposition 5.3 (many commutator b-vector fields). *The space $\mathcal{V}_{[\mathrm{b}]}(M)$ spans $\mathcal{V}_{\mathrm{b}}(M)$ over $\mathcal{C}^\infty(M)$.*

Proof. Define the $\mathcal{C}^\infty_\phi(H_2)$-module

$$\mathcal{V}_{[\mathrm{b}]}(H_2) = \mathcal{V}_{[\mathrm{b}]}(M)|_{H_2} = \{V \in \mathcal{V}_{\mathrm{b}}(H_2) : V(\mathcal{C}^\infty_\phi(H_2)) \subset \mathcal{C}^\infty_\phi(H_2)\}. \tag{5-1}$$

It suffices to show that

$$\mathcal{V}_{[\mathrm{b}]}(H_2) \text{ spans } \mathcal{V}_{\mathrm{b}}(H_2) \text{ over } \mathcal{C}^\infty(H_2). \tag{5-2}$$

Indeed, let $V \in \mathcal{V}_{\mathrm{b}}(M)$; then, assuming (5-2), we can write $V|_{H_2} = \sum f_i V_i$, where $f_i \in \mathcal{C}^\infty(H_2)$ and $V_i \in \mathcal{V}_{[\mathrm{b}]}(H_2)$. Choosing extensions $\tilde{f}_i \in \mathcal{C}^\infty(M)$ of f_i and $\widetilde{V}_i \in \mathcal{V}_{[\mathrm{b}]}(M)$ of V_i, the b-vector field $V - \sum \tilde{f}_i \widetilde{V}_i$ vanishes at H_2 as a b-vector field, and hence a fortiori lies in $\mathcal{V}_{\mathrm{e,b}}(M) \subset \mathcal{V}_{[\mathrm{b}]}(M)$.

Now, the characterization (5-1) for $V \in \mathcal{V}_{\mathrm{b}}(H_2)$ to lie in $\mathcal{V}_{[\mathrm{b}]}(H_2)$ is equivalent to the condition that V pushes forward along ϕ to a well-defined vector field on Y. But since $\phi : H_2 \to Y$ is a fibration, every vector field on Y has a lift to an element $V \in \mathcal{V}_{\mathrm{b}}(H_2)$ (which thus satisfies this condition). It remains to observe that $\mathcal{V}_{\mathrm{b}}(H_2)$ is generated over $\mathcal{C}^\infty(H_2)$ by $\mathcal{V}_{\mathrm{e,b}}(M)|_{H_2} \subset \mathcal{V}_{[\mathrm{b}]}(H_2)$ and the lift of $\mathcal{V}(Y)$. □

In local coordinates as in (2-3a) (or (2-3b)) in the special case that Z is a closed interval, we note that the space $\mathcal{V}_{[\mathrm{b}]}(M)$ is spanned by the vector fields $a(x, y, z)x\partial_x$, $b(x, y, z)\partial_z$ (or $b(x, y, z)z\partial_z$), and $(c(x, y) + xc'(x, y, z))\partial_{y^j}$ for smooth a, b, c, c'. This can be used to give an alternative, local coordinate, proof of Proposition 5.3.

Using induction, one can show that $[\mathcal{V}_{[\mathrm{b}]}(M), \mathrm{Diff}^k_{\mathrm{e,b}}(M)] \subset \mathrm{Diff}^k_{\mathrm{e,b}}(M)$. We generalize this to operators on vector bundles.

Definition 5.4 (commutator b-operators). Let $E \to M$ be a vector bundle. Then the space $\mathrm{Diff}^1_{[\mathrm{b}]}(M; E)$ of *commutator b-operators* denotes the space of all $X \in \mathrm{Diff}^1_{\mathrm{b}}(M; E)$ whose principal symbol is scalar and equal to that of an element $V \in \mathcal{V}_{[\mathrm{b}]}(M)$; that is, for all $f \in \mathcal{C}^\infty(M)$ and $\sigma \in \mathcal{C}^\infty(M; E)$, the Leibniz rule $X(f\sigma) = fX\sigma + (Vf)\sigma$ holds.

Given any commutator b-vector field V, there exists a commutator b-operator X whose principal symbol equals that of V; for example, one can take $X = \nabla_V$, where ∇ is a connection on E.

Lemma 5.5 (commutators: differential operators). *Let $A \in \mathrm{Diff}^k_{\mathrm{e,b}}(M; E)$ (not necessarily with scalar principal symbol) and $X \in \mathrm{Diff}^1_{[\mathrm{b}]}(M; E)$. Then $[X, A] \in \mathrm{Diff}^k_{\mathrm{e,b}}(M; E)$. Similarly, if $\alpha \in \mathbb{R}^N$ is a vector of weights and $A \in \mathcal{A}^{-\alpha}\,\mathrm{Diff}^k_{\mathrm{e,b}}(M; E)$, then $[X, A] \in \mathcal{A}^{-\alpha}\,\mathrm{Diff}^k_{\mathrm{e,b}}(M; E)$.*

Proof. In a local trivialization of E, we have $X = V \otimes 1 + e$, where $V \in \mathcal{V}_{[\mathrm{b}]}(M)$ and $e \in \mathcal{C}^\infty(M; \mathbb{C}^{d\times d})$; here d is the rank of E and 1 is the identity operator on $\mathbb{C}^d$. Moreover, we have $A = (A_{ij})_{1\le i,j\le d}$, where the $A_{ij} \in \mathrm{Diff}^k_{\mathrm{e,b}}(M)$ are scalar operators. Then $[V \otimes 1, A] = ([V, A_{ij}])_{1\le i,j\le d}$ is a matrix of elements of $\mathrm{Diff}^k_{\mathrm{e,b}}(M)$, as is $[e, A]$. (When A is principally scalar, then $[e, A] \in \mathrm{Diff}^{k-1}_{\mathrm{e,b}}(M)$, but this does not hold in general.) This implies the first claim.

For the second claim, it suffices to consider $A = wA_0$, where $w \in \mathcal{A}^{-\alpha}(M)$ and $A_0 \in \mathrm{Diff}^k_{\mathrm{e,b}}(M)$; then $[X, A] = w[X, A_0] + (Vw)A_0$. Since the b-vector field V maps $\mathcal{A}^{-\alpha}(M)$ into itself, the claim follows. □

More generally we have the following.

Lemma 5.6 (commutators: ps.d.o.s). *Let $s \in \mathbb{R}$ and $X \in \mathrm{Diff}^1_{[\mathrm{b}]}(M; E)$. Then, for $A \in \Psi^s_{\mathrm{e,b}}(M; E)$ (resp. $\mathcal{A}^{-\alpha}\Psi^s_{\mathrm{e,b}}(M; E)$), we have $[X, A] \in \Psi^s_{\mathrm{e,b}}(M; E)$ (resp. $\mathcal{A}^{-\alpha}\Psi^s_{\mathrm{e,b}}(M; E)$).*

Proof. Via local trivializations, one can reduce the proof to the case that E is trivial and then to the case that E is the trivial complex rank-1 vector bundle. We work on the level of Schwartz kernels on the edge-b-double space $M^2_{\mathrm{e,b}}$; see (2-19). Denote the lift of X to the left and right factor of $M^2_{\mathrm{e,b}}$ by X_L and X_R, respectively. If K denotes the Schwartz kernel of A, then the Schwartz kernel of $[X, A]$ is given by $D_X K$, where

$$D_X := X_L - X_R^*,$$

with the adjoint acting on edge-b-densities; thus $D_X \in \mathrm{Diff}^1_{\mathrm{b}}(M^2_{\mathrm{e,b}}; \pi^*_{\mathrm{e,b},R}{}^{\mathrm{e,b}}\Omega M)$. Choosing a trivialization of ${}^{\mathrm{e,b}}\Omega M$, the operator D_X is — modulo an element of $\mathcal{C}^\infty(M^2_{\mathrm{e,b}})$ — given by the differentiation along the vector field

$$\widetilde{X} = X_L + X_R.$$

When $X \in \mathcal{V}_{\mathrm{e,b}}(M)$, we have $\widetilde{X} \in \mathcal{V}_{\mathrm{b}}(M^2_{\mathrm{e,b}})$ since $M^2_{\mathrm{e,b}}$ was constructed precisely so that edge-b-vector fields lift to be smooth (and transversal across $\mathrm{diag}_{\mathrm{e,b}}$), and moreover $\widetilde{X}$ is tangent to $\mathrm{diag}_{\mathrm{e,b}}$. We claim that $\widetilde{X} \in \mathcal{V}_{\mathrm{b}}(M^2_{\mathrm{e,b}})$ and $\widetilde{X}$ is tangent to $\mathrm{diag}_{\mathrm{e,b}}$ also for $X \in \mathcal{V}_{[\mathrm{b}]}(M)$; it suffices to check this for a set of generators X of $\mathcal{V}_{[\mathrm{b}]}(M)/\mathcal{V}_{\mathrm{e,b}}(M)$ over $\mathcal{C}^\infty(M)$. We do this in local coordinates on $M^2_{\mathrm{e,b}}$.

We shall give details only in one region, namely near the preimage of the interior of $H_2 \cap H_1$ under the diagonal embedding $\mathrm{diag}_{\mathrm{e,b}} \hookrightarrow M^2_{\mathrm{e,b}}$, where (upon relabeling if necessary) H_1 is a boundary hypersurface with $H_2 \cap H_1 \neq \varnothing$. (The computations in the case that there does not exist such a boundary hypersurface are simpler still than the ones presented here.) Thus, fix local coordinates $x \geq 0,\ y \in \mathbb{R}^{d_Y},\ z_0 \geq 0,\ z_1 \in \mathbb{R}^{d_Z - 1}$ near $(H_2 \cap H_1)^\circ$, where $d_Y = \dim Y$ and $d_Z = \dim Z$, and x (resp. z_0) is a defining function of H_2 (resp. H_1). Denote the corresponding set of product coordinates on M^2 by $(x, y, z_0, z_1, x', y', z_0', z_1')$; then $(H_2)^2_\phi = \{(0, y, z_0, z_1, 0, y, z_0', z_1')\}$ and $H_1^2 = \{(x, y, 0, z_1, x', y', 0, z_1')\}$. It suffices to compute $\widetilde{X}$ in the case $X = \partial_{y^j}$, so $X_L = \partial_{y^j}$ and $X_R = \partial_{(y')^j}$.

To start, upon blowing up $(H_2)^2_\phi$, we replace the coordinates (x, x', y, y') by (x, s, y, Y), where $s = x'/x - 1$ and $Y = (y' - y)/x$, with the lift of the diagonal diag_M given by $s = Y = 0,\ z_0 = z_0'$, $z_1 = z_1'$. Since H_1^2 lifts to $\{(x, s, y, Y, z_0, z_1, z_0', z_1') = (x, s, y, Y, 0, z_1, 0, z_1')\}$, blowing it up amounts, near the lift of diag_M, to replacing the coordinates (z_0, z_0') by (z_0, τ), where $\tau = z_0'/z_0 - 1$, with $\mathrm{diag}_{\mathrm{e,b}}$ now given by $s = Y = \tau = 0,\ z_1 = z_1'$ in the coordinates $(x, s, y, Y, z_0, \tau, z_1, z_1')$. But then the vector field X_L (resp. X_R) reads $\partial_{y^j} - x^{-1}\partial_{Y^j}$ (resp. $x^{-1}\partial_{Y^j}$), and therefore $\widetilde{X} = \partial_{y^j}$ indeed lies in $\mathcal{V}_{\mathrm{b}}(M^2_{\mathrm{e,b}})$ and is tangent to $\mathrm{diag}_{\mathrm{e,b}}$.

Directly from the definition of conormal distributions then, the operator D_X for $X \in \mathcal{V}_{[\mathrm{b}]}(M)$ maps $\Psi^s_{\mathrm{e,b}}(M) \to \Psi^s_{\mathrm{e,b}}(M)$ and $\mathcal{A}^{-\alpha}\Psi^s_{\mathrm{e,b}}(M) \to \mathcal{A}^{-\alpha}\Psi^s_{\mathrm{e,b}}(M)$, finishing the proof of the lemma. □

5.2. ***Edge,b;b-operators and Sobolev spaces.*** In the general setting of Section 5.1, we proceed to describe spaces of operators and distributions with (integer-order) b-regularity on top of (microlocal) edge-b-regularity. This uses a mixed algebra of b-differential edge-b-pseudodifferential operators. Such mixed algebras have been used in a variety of settings, e.g., implicitly in [Melrose 1994] and explicitly in [Vasy 2008; 2010; Melrose et al. 2013]. We shall only consider scalar operators here and leave the (purely notational) addition of vector bundles to the reader.

Definition 5.7 (mixed algebras and Sobolev spaces). Let $s \in \mathbb{R},\ \alpha \in \mathbb{R}^N$, and $k \in \mathbb{N}_0$. Then the space $\mathrm{Diff}^k_{\mathrm{b}}\,\Psi^{s,\alpha}_{\mathrm{e,b}}(M)$ consists of all operators which are finite sums $\sum_i Q_i P_i$, where $Q_i \in \mathrm{Diff}^k_{\mathrm{b}}(M)$ and $P_i \in \Psi^{s,\alpha}_{\mathrm{e,b}}(M)$. The corresponding spaces of *weighted edge,b;b-Sobolev spaces* are defined by

$$H^{(s;k),\alpha}_{\mathrm{e,b;b}}(M) := \{u \in H^{s,\alpha}_{\mathrm{e,b}}(M) : Au \in H^{s,\alpha}_{\mathrm{e,b}}(M)\ \forall\, A \in \mathrm{Diff}^k_{\mathrm{b}}(M)\}.$$

When $\alpha = 0$, we denote these spaces simply by $\mathrm{Diff}^k_{\mathrm{b}}\,\Psi^s_{\mathrm{e,b}}(M)$ and $H^{(s;k)}_{\mathrm{e,b;b}}(M)$, respectively. Operators acting on sections of bundles $E, F \to M$, and Sobolev spaces of E-valued distributions, are defined analogously.

Lemma 5.8 (commuting the differential and pseudodifferential factors). *Let $s \in \mathbb{R},\ \alpha \in \mathbb{R}^N,\ k \in \mathbb{N}_0$, and let $Q \in \mathrm{Diff}^k_{\mathrm{b}}(M)$ and $P \in \Psi^{s,\alpha}_{\mathrm{e,b}}(M)$. Then there exists a finite collection of operators $Q_j^\flat,\ Q_j^\sharp \in \mathrm{Diff}^k_{\mathrm{b}}(M)$ and $P_j^\flat,\ P_j^\sharp \in \Psi^{s,\alpha}_{\mathrm{e,b}}(M)$ such that*

$$QP = \sum_j P_j^\flat Q_j^\flat, \qquad PQ = \sum_j Q_j^\sharp P_j^\sharp. \tag{5-3}$$

This remains true, mutatis mutandis, when P and Q are operators on sections of a vector bundle $E \to M$ and when $P \in \mathcal{A}^{-\alpha}\Psi^s_{\mathrm{e,b}}$.

Proof. We only consider the case $\alpha = 0$. The case $k = 0$ (so $Q \in \mathcal{C}^\infty(M)$) is trivial since $\Psi^s_{\mathrm{e,b}}(M)$ is a right and left module over $\mathcal{C}^\infty(M)$.

Consider next $k = 1$, and let $Q \in \mathcal{V}_{\mathrm{b}}(M)$. If $Q \in \mathcal{V}_{[\mathrm{b}]}(M)$, then $QP = PQ + [P, Q]$, and thus Lemma 5.6 implies (5-3). In general, by Proposition 5.3, the vector field Q is a finite sum of terms fV, where $f \in \mathcal{C}^\infty(M)$ and $V \in \mathcal{V}_{[\mathrm{b}]}(M)$. Suppose thus that $Q = fV$. The first claim in (5-3) then follows from $QP = PQ - f[P, V] - [P, f]V$ since $f[P, V] \in \Psi^s_{\mathrm{e,b}}(M)$ and $[P, f]V \in \Psi^{s-1}_{\mathrm{e,b}} \mathrm{Diff}^1_{\mathrm{b}}$. The second claim follows similarly from

$$PQ = QP + [P, f]V + f[P, V] = QP + V[P, f] + [[P, f], V] + f[P, V],$$

where now $V[P, f] \in \mathrm{Diff}^1_{\mathrm{b}} \Psi^{s-1}_{\mathrm{e,b}}$, $[[P, f], V] \in \Psi^{s-1}_{\mathrm{e,b}}$, and $f[P, V] \in \Psi^s_{\mathrm{e,b}}(M)$.

For $k \geq 2$, we argue inductively. Write Q as a finite sum of terms $Q_1 Q_2$ with $Q_1 \in \mathrm{Diff}^1_{\mathrm{b}}(M)$ and $Q_2 \in \mathrm{Diff}^{k-1}_{\mathrm{b}}(M)$. Then $Q_2 P = \sum_j P^\flat_{2j} Q^\flat_{2j}$ with $Q^\flat_{2j} \in \mathrm{Diff}^{k-1}_{\mathrm{b}}(M)$ and $P^\flat_{2j} \in \Psi^s_{\mathrm{e,b}}(M)$ by the inductive hypothesis, and then by the case $k = 1$, we can further write $Q_1 P^\flat_{2j} = \sum_k P^\flat_{2jk} Q^\flat_{2jk}$ with $Q^\flat_{2jk} \in \mathrm{Diff}^1_{\mathrm{b}}(M)$ and $P^\flat_{2jk} \in \Psi^s_{\mathrm{e,b}}(M)$. This proves the first part of (5-3). The proof of the second part is analogous. □

Lemma 5.9 (further basic properties). *Let $k \in \mathbb{N}_0$ and $s \in \mathbb{R}$. Then* $\mathrm{Diff}^k_{\mathrm{b}} \Psi^s_{\mathrm{e,b}}(M) \subset \mathrm{Diff}^{k+1}_{\mathrm{b}} \Psi^{s-1}_{\mathrm{e,b}}(M)$ *and* $[\mathrm{Diff}^k_{\mathrm{b}}(M), \Psi^s_{\mathrm{e,b}}(M)] \subset \mathrm{Diff}^k_{\mathrm{b}}(M) \Psi^{s-1}_{\mathrm{e,b}}(M)$, *and similarly for spaces of weighted operators.*

Proof. For the first part, we fix a finite collection $V_1, \dots, V_K \in \mathcal{V}_{\mathrm{e,b}}(M)$ which spans $\mathcal{V}_{\mathrm{e,b}}(M)$ over $\mathcal{C}^\infty(M)$. We can then write any $A \in \Psi^s_{\mathrm{e,b}}(M)$ in the form $A = \sum_{j=1}^K V_j A_j + R$, where $A_j, R \in \Psi^{s-1}_{\mathrm{e,b}}(M)$; this follows from an analogous decomposition on the level of principal symbols. Since a fortiori $V_j \in \mathcal{V}_{\mathrm{b}}(M)$, we are done.

In the second part, the case $k = 0$ is trivial. The case $k = 1$ follows by writing an element of $\mathcal{V}_{\mathrm{b}}(M)$ as a finite sum of terms fV, where $f \in \mathcal{C}^\infty(M)$, $V \in \mathcal{V}_{[\mathrm{b}]}(M)$, and using that $[\mathcal{V}_{[\mathrm{b}]}(M), \Psi^s_{\mathrm{e,b}}(M)] \subset \Psi^s_{\mathrm{e,b}}(M) \subset \mathrm{Diff}^1_{\mathrm{b}} \Psi^{s-1}_{\mathrm{e,b}}(M)$ by the first part. The case $k \geq 2$ follows by induction as in the proof of Lemma 5.8. □

Corollary 5.10 (algebra properties of the mixed algebra). *If $A_j \in \mathrm{Diff}^{k_j}_{\mathrm{b}} \Psi^{s_j,\alpha_j}_{\mathrm{e,b}}(M)$ for $j = 1, 2$, then* $A_1 A_2 \in \mathrm{Diff}^{k_1+k_2}_{\mathrm{b}} \Psi^{s_1+s_2,\alpha_1+\alpha_2}_{\mathrm{e,b}}(M)$ *and* $[A_1, A_2] \in \mathrm{Diff}^{k_1+k_2}_{\mathrm{b}} \Psi^{s_1+s_2-1,\alpha_1+\alpha_2}_{\mathrm{e,b}}(M)$.

Proof. The first statement is a consequence of Lemma 5.8, which allows us to commute b-differential through edge-b-pseudodifferential operators at will. The second statement, for $\alpha_1 = \alpha_2 = 0$ for notational simplicity, follows for operators $A_j = Q_j P_j$ with $Q_j \in \mathrm{Diff}^{k_j}_{\mathrm{b}}(M)$ and $P_j \in \Psi^{s_j}_{\mathrm{e,b}}(M)$ from the calculation

$$[A_1, A_2] = Q_1 Q_2 [P_1, P_2] + Q_1 [P_1, Q_2] P_2 + Q_2 [Q_1, P_2] P_1 + [Q_1, Q_2] P_2 P_1.$$

In the first term, we use $[P_1, P_2] \in \Psi^{s_1+s_2-1}_{\mathrm{e,b}}(M)$; for the remaining three terms, the desired conclusion follows from Lemma 5.9. □

Corollary 5.11 (boundedness on Sobolev spaces). *Let $s, m \in \mathbb{R}$, $k, l \in \mathbb{N}_0$, $\alpha, \beta \in \mathbb{R}^N$, with $l \geq k$. Then every $A \in \mathrm{Diff}^k_{\mathrm{b}} \Psi^{m,\alpha}_{\mathrm{e,b}}(M)$ defines a bounded linear operator $A : H^{(s;l),\beta}_{\mathrm{e,b;b}}(M) \to H^{(s-m;l-k),\beta-\alpha}_{\mathrm{e,b;b}}(M)$.*

Proof. This follows from the definitions upon writing $A = \sum_j P^\flat_j Q^\flat_j$ with $P^\flat_j \in \Psi^{s,\alpha}_{\mathrm{e,b}}(M)$ and $Q^\flat_j \in \mathrm{Diff}^k_{\mathrm{b}}(M)$ (cf. Lemma 5.8). □

Next, edge,b;b-regularity can be microlocalized.

Definition 5.12 (edge,b;b-wave front set). Let $s \in \mathbb{R}$, $k \in \mathbb{N}_0$, $\alpha \in \mathbb{R}^N$. Suppose

$$u \in H^{(-\infty;k),\alpha}_{\mathrm{e,b;b}}(M) = \bigcup_{s_0\in\mathbb{R}} H^{(s_0;k),\alpha}_{\mathrm{e,b;b}}(M).$$

Then $\mathrm{WF}^{(s;k),\alpha}_{\mathrm{e,b;b}}(u) \subset {}^{\mathrm{e,b}}S^*M$ is the complement of the set of all $\varpi \in {}^{\mathrm{e,b}}S^*M$ for which there exists $A \in \Psi^0_{\mathrm{e,b}}(M)$, elliptic at ϖ, such that $Au \in H^{(s;k),\alpha}_{\mathrm{e,b;b}}(M)$.

For any other $A' \in \Psi^0_{\mathrm{e,b}}(M)$ with $\mathrm{WF}'_{\mathrm{e,b}}(A') \subset \mathrm{Ell}_{\mathrm{e,b}}(A)$, one then also has $A'u \in H^{(s;k),\alpha}_{\mathrm{e,b;b}}(M)$ by the symbolic parametrix construction in the edge-b-algebra. (The a priori membership of u in a space with b-regularity k and weight α is crucial here.)

Lemma 5.13 (microlocality of edge-b-ps.d.o.s on edge,b;b-Sobolev spaces). *Let $s, m \in \mathbb{R}$, $k \in \mathbb{N}_0$, and $\alpha, \beta \in \mathbb{R}^N$. Let $u \in H^{(-\infty;k),\beta}_{\mathrm{e,b;b}}(M)$ and $A \in \Psi^{m,\alpha}_{\mathrm{e,b}}(M)$. Then*

$$\mathrm{WF}^{(s-m;k),\beta-\alpha}_{\mathrm{e,b;b}}(Au) \subset \mathrm{WF}'_{\mathrm{e,b}}(A) \cap \mathrm{WF}^{(s;k),\beta}_{\mathrm{e,b;b}}(u).$$

Proof. If $\varpi \notin \mathrm{WF}'_{\mathrm{e,b}}(A) \cap \mathrm{WF}^{(s;k),\beta}_{\mathrm{e,b;b}}(u)$, then there exists $B \in \Psi^0_{\mathrm{e,b}}(M)$, elliptic at ϖ, such that either $BA \in \Psi^{-\infty}_{\mathrm{e,b}}(M)$ and hence $BAu \in H^{(\infty;k),\beta}_{\mathrm{e,b;b}}(M)$, or $Bu \in H^{(s;k),\beta}_{\mathrm{e,b;b}}(M)$, in which case we write $I = QB + R$, where Q, $R \in \Psi^0_{\mathrm{e,b}}(M)$ and $\varpi \notin \mathrm{WF}'_{\mathrm{e,b}}(R)$, and then for $A' \in \Psi^0_{\mathrm{e,b}}(M)$ which is elliptic at ϖ but with $\mathrm{WF}'_{\mathrm{e,b}}(A') \cap \mathrm{WF}'_{\mathrm{e,b}}(R) = \varnothing$, we find $A'(Au) = A'AQBu + A'ARu$. The first summand on the right lies in $H^{(s-m;k),\beta-\alpha}_{\mathrm{e,b;b}}(M)$, and the second summand lies in $H^{(\infty;k),\beta-\alpha}_{\mathrm{e,b;b}}(M)$. □

Proposition 5.14 (elliptic regularity). *Let $P \in \Psi^m_{\mathrm{e,b}}(M)$. Then we have*

$$\mathrm{WF}^{(s;k),\beta}_{\mathrm{e,b;b}}(u) \subset \mathrm{WF}^{(s-m;k),\beta}_{\mathrm{e,b;b}}(Pu) \cup \mathrm{Char}_{\mathrm{e,b}}(P).$$

Proof. This follows from the symbolic elliptic parametrix construction in $\Psi_{\mathrm{e,b}}(M)$ together with Corollary 5.11. □

Proposition 5.15 (real principal-type propagation). *Suppose $P \in \Psi^m_{\mathrm{e,b}}(M)$ has a real homogeneous principal symbol. Suppose $B, E, G \in \Psi^0_{\mathrm{e,b}}(M)$ are such that $\mathrm{WF}'_{\mathrm{e,b}}(B) \subset \mathrm{Ell}_{\mathrm{e,b}}(G)$ and such that all backward null-bicharacteristics from $\mathrm{WF}'_{\mathrm{e,b}}(B) \cap \mathrm{Char}_{\mathrm{e,b}}(P)$ reach $\mathrm{Ell}_{\mathrm{e,b}}(E)$ in finite time while remaining in $\mathrm{Ell}_{\mathrm{e,b}}(G)$. Then, for any fixed $N \in \mathbb{R}$, we have*

$$\|Bu\|_{H^{(s;k),\alpha}_{\mathrm{e,b;b}}(M)} \leq C\big(\|GPu\|_{H^{(s-m+1;k),\alpha}_{\mathrm{e,b;b}}(M)} + \|Eu\|_{H^{(s;k),\alpha}_{\mathrm{e,b;b}}(M)} + \|u\|_{H^{(-N;k),\alpha}_{\mathrm{e,b;b}}(M)}\big). \tag{5-4}$$

This holds in the strong sense that if all terms on the right-hand side are finite, then the left-hand side is finite and the estimate holds.

Proof. The case $k = 0$ (which was already tacitly used in Section 4.2) follows from the usual symbolic positive commutator argument: all that is used about edge-b-pseudodifferential operators is their principal symbol calculus, in particular the relationship between commutators and Poisson brackets. See [Hintz 2023c, Theorem 8.7] (and [Baskin and Datchev 2024] for a digest version) for details in the setting of manifolds without boundary and [Vasy 2018, §5.4.1] in yet another (scattering) setting; the original

reference is [Hörmander 1971b]. Arguing by induction, we now apply the estimate (5-4) to Vu in place of u, where $V \in \mathcal{V}_{[\mathrm{b}]}(M)$. We then use elliptic estimates for the commutators with V, such as

$$\|G[P, V]u\|_{H^{(s-m+1;k),\alpha}_{\mathrm{e,b;b}}(M)} \lesssim \|G'u\|_{H^{(s+1;k),\alpha}_{\mathrm{e,b;b}}(M)} + \|u\|_{H^{(-N;k),\alpha}_{\mathrm{e,b;b}}(M)}, \tag{5-5}$$

where $G' \in \Psi^0_{\mathrm{e,b}}(M)$, $\mathrm{WF}'_{\mathrm{e,b}}(G) \subset \mathrm{Ell}_{\mathrm{e,b}}(G')$; this uses the membership $[P, V] \in \Psi^m_{\mathrm{e,b}}(M)$ from Lemma 5.6. The first term on the right can then be estimated, using the inductive hypothesis, by means of (5-4) with B and s replaced by G' and $s+1$ (and correspondingly G replaced by an operator with slightly enlarged elliptic set). The inductive step is then completed by noting that $H^{(s+1;k),\alpha}_{\mathrm{e,b;b}}(M) \subset H^{(s;k+1),\alpha}_{\mathrm{e,b;b}}(M)$ (which follows from Lemma 5.9). □

5.3. ***Propagation estimates near*** $\mathscr{I}^+$**.** The same proof as for Proposition 5.15 gives the higher b-regularity analogues of certain radial point estimates from Section 4.2. We need to use two facts: First, these estimates only lose one edge-b-derivative relative to elliptic estimates, as otherwise the first term on the right in (5-5) would require using a successively larger edge-b-regularity order on the Pu and Eu terms in (5-4) as one increases k. Second, the estimate for $k = 0$ needs to be applicable without an upper bound for the edge-b-regularity s (given fixed weights); note indeed that, in the proof of Proposition 5.15, the proof of the inductive step uses the inductive hypothesis with an increased value for s. (Alternatively, if there is a requirement $s < s_0$ for the case $k = 0$, then the stronger bound $s + k < s_0$ guarantees an estimate on $H^{(s;k),\alpha}_{\mathrm{e,b;b}}$.) Not a single forward propagation estimates imposes an upper bound on s. We thus obtain the following.

Proposition 5.16 (edge,b;b-versions of radial point estimates). *Theorem 4.5 and Lemmas 4.8(1), 4.9, 4.10, and 4.11 remain valid upon replacing* $\mathrm{WF}^{*,*}_{\mathrm{e,b}}$ *by* $\mathrm{WF}^{(*;k),*}_{\mathrm{e,b;b}}$ *throughout. Lemma 4.8(2) remains valid under the same replacement if one requires* $\tilde{s} + k < \frac{1}{2} - \tilde{\alpha}_0 + 2\tilde{\alpha}_{\mathscr{I}} + \underline{p}_1$.

Remark 5.17 (choice of commutator vector fields). In [Hintz and Vasy 2020, Proof of Proposition 4.7], we needed to exercise some care when choosing the b-vector fields V (called G there) for commutations: the commutator $[P, V]$ needed to be an operator whose coefficients decayed in a suitable manner. As far as spherical vector fields are concerned, this required V to be (asymptotic to) a Killing vector field, i.e., a rotation. Here on the other hand, we can use arbitrary elements of $\mathcal{V}_{[\mathrm{b}]}(M)$. The reason is that we already have a sharp edge-b-regularity theory, and the arguments in the present section show how one can inductively upgrade edge-b- to b-regularity. In [Hintz and Vasy 2020] on the other hand, b-regularity was proved directly, without first establishing edge-b-regularity.

6. Energy estimates and solvability near I^0 and $\mathscr{I}^+ \setminus I^+$

We continue to denote by g an $(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)$-admissible metric and P a g-admissible operator on the $(n+1)$-dimensional manifold M, $n \geq 1$; see Section 3.2. We work in a product decomposition $[0, \epsilon)_\varrho \times (-\epsilon, \epsilon)_v \times Y$ near the light cone at infinity Y on the blown-down manifold M_0, in which the underlying metric takes the form (3-10) (or equivalently (3-11)–(3-13)), and write

$$\varrho = \frac{1}{t+r}, \quad v = \frac{t-r}{t+r} \quad \text{(which defines coordinates } t, r \text{ on } M^\circ\text{)}.$$

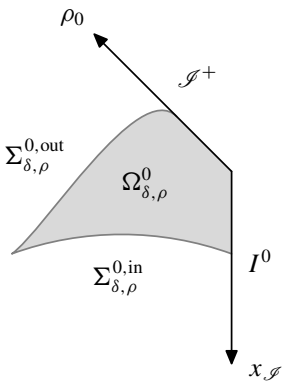

Figure 6.1. The domain (6-1) and its boundary hypersurfaces (6-2) used in the setup of Theorem 6.4.

For every compact subset $K \subset \mathscr{I}^+ \setminus I^+$, there then exists $T \in \mathbb{R}$ such that the domain $\mathcal{U}_0(T)$ from Definition 3.1 contains K. Define the coordinates $\rho_0 = 1/(T - t_*)$, $x_{\mathscr{I}} = \sqrt{(T - t_*)/r}$ on $\mathcal{U}_0(T)$ as in (3-5).

We work away from I^+; thus, we shall only record weights at I^0 and $\mathscr{I}^+$. Lemma 3.6 shows that, given $\underline{\rho}_0 < \bar{\rho}_0 \in (0, 1)$, for large C and small δ_0 the domain

$$\Omega^0_{\delta,\rho} := \{x_{\mathscr{I}} < \delta,\ \rho_0 < \rho + C x_{\mathscr{I}}^{2\ell_{\mathscr{I}}}\} \subset \mathcal{U}_0(T) \subset M \tag{6-1}$$

contains K when $0 < \delta \leq \delta_0$ and $\rho \in [\underline{\rho}_0, \bar{\rho}_0]$, and its boundary hypersurfaces (other than those contained in ∂M)

$$\begin{aligned} \Sigma^{0,\mathrm{in}}_{\delta,\rho} &:= \{x_{\mathscr{I}} = \delta,\ \rho_0 < \rho + C\delta^{2\ell_{\mathscr{I}}}\}, \\ \Sigma^{0,\mathrm{out}}_{\delta,\rho} &:= \{x_{\mathscr{I}} < \delta,\ \rho_0 = \rho + C x_{\mathscr{I}}^{2\ell_{\mathscr{I}}}\} \end{aligned} \tag{6-2}$$

are spacelike by Lemma 3.6. See Figure 6.1.

Since the normal vector field at $\Sigma^{0,\mathrm{in}}_{\delta,\rho}$ (resp. $\Sigma^{0,\mathrm{out}}_{\delta,\rho}$) pointing into (resp. out of) $\Omega^0_{\delta,\rho}$ is future timelike, the Cauchy problem for $Pu = f$ with initial data on $\Sigma^{0,\mathrm{in}}_{\delta,\rho}$ is well-posed in $\Omega^0_{\delta,\rho}$. We shall consider here forward (resp. backward) problems with trivial data at $\Sigma^{0,\mathrm{in}}_{\delta,\rho}$ (resp. $\Sigma^{0,\mathrm{out}}_{\delta,\rho}$). We remark here already that the forward problem is well-posed, whereas the backward problem is not since the backward domain of dependence of $\Sigma^{0,\mathrm{out}}_{\delta,\rho}$ is a proper subset of $\Omega^0_{\delta,\rho}$.

Definition 6.1 (function spaces). Let $\delta < \delta_0$ and $\rho \in (\underline{\rho}_0, \bar{\rho}_0)$, and let $s \in \mathbb{R}$, $\alpha = (\alpha_0, 2\alpha_{\mathscr{I}}) \in \mathbb{R}^2$. Fix on M a volume density of the class (4-16). Then we let

$$\begin{aligned} H^{s,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-} &:= \big\{u|_{\Omega^0_{\delta,\rho}} : u \in \overline{H}^{s,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta_0,\bar{\rho}_0}),\ \operatorname{supp} u \subset \overline{\Omega^0_{\delta,\bar{\rho}_0}}\big\}, \\ H^{s,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{-,\bullet} &:= \big\{u|_{\Omega^0_{\delta,\rho}} : u \in \overline{H}^{s,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta_0,\bar{\rho}_0}),\ \operatorname{supp} u \subset \overline{\Omega^0_{\delta_0,\rho}}\big\}. \end{aligned}$$

Thus, elements in the space $H^{s,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}$ have supported character at $\Sigma^{0,\mathrm{in}}_{\delta,\rho}$ and are extendible at $\Sigma^{0,\mathrm{out}}_{\delta,\rho}$, with the situation reversed for elements of $H^{s,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{-,\bullet}$. Moreover, the L^2-dual of $H^{s,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}$ is $H^{-s,-\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{-,\bullet}$ [Hörmander 2007, Appendix B].

Due to the inflexible nature of energy estimates — they are global in the fibers of ${}^{\mathrm{e,b}}T^*M$ and admit the use of differential operators only — as compared to the microlocal estimates of Section 4.2, we impose here an additional condition on the edge normal operator of P in order to get sharp results later on.

Definition 6.2 (special admissible operators). Let P be an admissible operator, and define

$$p_0 \in \rho_0\rho_+(\mathcal{C}^\infty + \mathcal{A}^{(\ell_0,\ell_+)})(\mathscr{I}^+;\, \beta^*\operatorname{End}(E)),$$
$$p_1 \in (\mathcal{C}^\infty + \mathcal{A}^{(\ell_0,\ell_+)})(\mathscr{I}^+;\, \beta^*\operatorname{End}(E))$$

as in Definition 3.7(3). We then say that P is a *special admissible operator* if there exists a bundle splitting $E|_Y = \bigoplus_{j=1}^J E_j$ such that p_1 is lower triangular, with diagonal entries $p_{1,jj} \in \operatorname{End}(E_j)$ having real spectrum, and such that p_0 is strictly lower triangular.

For scalar operators P, special admissibility is simply the requirement that p_1 be real and $p_0 = 0$.

Remark 6.3 (discussion). While special admissibility is a rather strong requirement on p_0 and p_1, the linearized gauge-fixed Einstein operator considered in [Hintz 2023b, Proposition 3.29 and §3.6] does satisfy it (see Example 3.12 on how to relate the notation in the reference to the one used presently). Similarly, [Hintz 2023b, Appendix A] implies the special admissibility of an appropriate gauge-fixed Maxwell equation on Minkowski space — special admissibility is loosely related to the weak null condition for nonlinear wave equations [Lindblad and Rodnianski 2003] in that in applications both conditions often arise from a hierarchical structure of an asymptotic system of equations (in [Lindblad and Rodnianski 2003; Keir 2018]) or of a normal operator (here and also in [Hintz and Vasy 2020, §1.1.2]). As a simple example (besides the already-discussed Einstein vacuum equations in harmonic gauge or variations thereof), consider a nonlinear wave operator on $\mathbb{R}^{1+3}$ for which the asymptotic system [Lindblad and Rodnianski 2003, (5)] is strictly lower triangular (i.e., the equation for U_i only involves source terms U_j and U_k for $j, k < i$, with i ranging from 1 to the rank of the vector bundle which the sought-after solution of the wave equation is a section of); then the weak null condition of [Lindblad and Rodnianski 2003, Definition 2.1] holds. Linearizing such an operator around a solution with appropriate decay, one obtains a linear wave-type operator for which the edge normal operator (3-18) has the same strictly lower triangular structure, i.e., p_0 and p_1 are strictly lower triangular.

If P is special admissible, then for all $\epsilon > 0$ there exists a positive definite fiber inner product h_E on E so that, in the notation of Definition 4.3 and Lemma 4.4, we have

$$\underline{p}_1 - \epsilon < \underline{p}_1(h_E), \quad -\epsilon < \frac{p_1 - p_1^*}{2i} < \epsilon, \quad \sup_{\mathscr{I}^+} \|p_0\| < \epsilon, \tag{6-3}$$

where the second inequality is an inequality for quadratic forms on $(\beta^*E)|_{\mathscr{I}^+}$ which we require to hold pointwise on $\mathscr{I}^+$, and in the third inequality $\|\cdot\|$ denotes the operator norm on the space of linear maps $(E, h_E) \to (E, h_E)$. Indeed, this holds for h_E equal to the rescaling of any fixed positive definite diagonal inner product on $E = \bigoplus_{j=1}^J E_j$ by $\operatorname{diag}(1, \eta, \dots, \eta^{J-1})$ for sufficiently small $\eta > 0$.[26]

[26] Similar choices of inner products also feature in [Hintz 2017, §3.4] and towards the end of [Hintz and Vasy 2018, §9.1].

Theorem 6.4 (solvability, uniqueness, sharp regularity near $\mathscr{I}^+ \setminus I^+$). *Let P be a special admissible operator, and define $\underline{p}_1$ as in Definition 4.3.*

(1) (forward solution) *Let $s, \alpha_0, \alpha_{\mathscr{I}} \in \mathbb{R}$, and put $\alpha = (\alpha_0, 2\alpha_{\mathscr{I}})$ and $\alpha' = \alpha + (2,2)$. Suppose that*

$$s > \tfrac{1}{2} - \alpha_0 + 2\alpha_{\mathscr{I}} - \underline{p}_1, \quad \alpha_{\mathscr{I}} < \alpha_0 + \tfrac{1}{2}, \quad \alpha_{\mathscr{I}} < -\tfrac{1}{2} + \underline{p}_1. \tag{6-4}$$

Let $f \in H_{\mathrm{e,b}}^{s-1,\alpha'}(\Omega^0_{\delta,\rho}; \beta^*E)^{\bullet,-}$. Then the unique distributional forward solution u of $Pu = f$ in $\Omega^0_{\delta,\rho} \cap M^\circ$ satisfies $u \in H_{\mathrm{e,b}}^{s,\alpha}(\Omega^0_{\delta,\rho}; \beta^*E)^{\bullet,-}$, and

$$\|u\|_{H_{\mathrm{e,b}}^{s,\alpha}(\Omega^0_{\delta,\rho};\beta^*E)^{\bullet,-}} \leq C \|f\|_{H_{\mathrm{e,b}}^{s-1,\alpha'}(\Omega^0_{\delta,\rho};\beta^*E)^{\bullet,-}}, \tag{6-5}$$

where C only depends on P, s, α_0, $\alpha_{\mathscr{I}}$, δ, ρ.

(2) (backward solution) *Let $\tilde{s}, \tilde{\alpha}_0, \tilde{\alpha}_{\mathscr{I}} \in \mathbb{R}$, and put $\tilde{\alpha} = (\tilde{\alpha}_0, 2\tilde{\alpha}_{\mathscr{I}})$ and $\tilde{\alpha}' = \tilde{\alpha} + (2,2)$. Suppose that*

$$\tilde{s} < \tfrac{1}{2} - \tilde{\alpha}_0 + 2\tilde{\alpha}_{\mathscr{I}} + \underline{p}_1, \quad \tilde{\alpha}_0 < \tilde{\alpha}_{\mathscr{I}} - \tfrac{1}{2}, \quad \tilde{\alpha}_{\mathscr{I}} > -\tfrac{1}{2} - \underline{p}_1. \tag{6-6}$$

Let $\tilde{u} \in H_{\mathrm{e,b}}^{\tilde{s},\tilde{\alpha}}(\Omega^0_{\delta,\rho}; \beta^*E)^{-,\bullet}$ be such that $\tilde{f} := P^*\tilde{u} \in H_{\mathrm{e,b}}^{\tilde{s}-1,\tilde{\alpha}'}(\Omega^0_{\delta,\rho}; \beta^*E)^{-,\bullet}$. Then

$$\|\tilde{u}\|_{H_{\mathrm{e,b}}^{\tilde{s},\tilde{\alpha}}(\Omega^0_{\delta,\rho};\beta^*E)^{-,\bullet}} \leq C \|\tilde{f}\|_{H_{\mathrm{e,b}}^{\tilde{s}-1,\tilde{\alpha}'}(\Omega^0_{\delta,\rho};\beta^*E)^{-,\bullet}}, \tag{6-7}$$

where C only depends on P, s, $\tilde{\alpha}_0$, $\tilde{\alpha}_{\mathscr{I}}$, δ, ρ. Conversely, given a forcing term

$$\tilde{f} \in H_{\mathrm{e,b}}^{\tilde{s}-1,\tilde{\alpha}'}(\Omega^0_{\delta,\rho}; \beta^*E)^{-,\bullet},$$

there exists $\tilde{u}$ which satisfies the equation $P^*\tilde{u} = \tilde{f}$ and the estimate (6-7); this $\tilde{u}$ is the unique solution with these weights, in the sense that any other solution $\tilde{u}' \in H_{\mathrm{e,b}}^{-\infty,\tilde{\alpha}}(\Omega^0_{\delta,\rho}; \beta^*E)^{-,\bullet}$ of $P^*\tilde{u}' = \tilde{f}$ is necessarily equal to $\tilde{u}$.

Note that the backward problem is not well-posed without restricting the space in which one seeks the solution, as we are not imposing any (asymptotic) data at $\mathscr{I}^+ \cap \Omega^0_{\delta,\rho}$; in Theorem 6.4(2), the a priori assumption on the decay of $\tilde{u}'$ stands in for the data at null infinity (i.e., in some sense the data are zero).

Remark 6.5 (modifications for nonspecial operators). If one keeps the assumption that P be admissible but drops the special admissibility assumptions on p_0 and p_1 (Definition 6.2), then solvability, uniqueness, and sharp regularity can still be obtained by the arguments used in the proof, except the relationships between α_0 and $\alpha_{\mathscr{I}}$ need to be modified appropriately; roughly speaking, inequalities (6-4) and (6-6) need to hold with enough room to spare so that various error terms arising in the energy (positive commutator) estimate can be absorbed. We shall not make this quantitative here.

Proof of Theorem 6.4. We shall omit the vector bundle β^*E from the notation unless dealing with the case that β^*E is nontrivial requires additional arguments. The proof starts with a basic energy estimate in which we estimate u in a space with one order of edge-b-regularity. The vector field multiplier we will use is

$$V = \rho_0^{-2\breve{\alpha}_0} x_{\mathscr{I}}^{-4\breve{\alpha}_{\mathscr{I}}} W, \quad W := -x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} + (2-c)\rho_0\partial_{\rho_0},$$

where $\breve{\alpha}_0 = \alpha_0 + 1$ and $\breve{\alpha}_{\mathscr{I}} = \alpha_{\mathscr{I}} + \frac{1}{2}$, and $c > 0$ is a small constant. The weights here are forced upon us by our desire to prove an estimate for u in $H_{\mathrm{e,b}}^{s,(\alpha_0,2\alpha_{\mathscr{I}})}$ (the shifts accounting for $\frac{1}{2}$ times the weights of P itself, i.e., 2 at I^0 and 1 at $\mathscr{I}^+$). One way of rewriting the vector field W is

$$W = \Big(1 - \frac{c}{2}\Big)(-x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} + 2\rho_0\partial_{\rho_0}) + \frac{c}{2}\cdot(-x_{\mathscr{I}}\partial_{x_{\mathscr{I}}});$$

this is a convex combination of the vector fields $-x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} + 2\rho_0\partial_{\rho_0}$ and $-x_{\mathscr{I}}\partial_{x_{\mathscr{I}}}$ which are future null at $\mathscr{I}^+$ in view of (3-9), and thus W is future timelike there (and hence nearby). A spacetime energy estimate for the scalar wave equation $\Box_g u = f$ requires the verification that the K-current associated with V (i.e., the contraction of the deformation tensor $\mathcal{L}_V g$ with the stress-energy-momentum tensor of u) is a coercive quadratic form in the first (edge-b-)derivatives of u; we phrase this below as the positive definiteness of $K_V(\mathrm{d}u, \mathrm{d}u)$; cf. (6-11). Since we are working with a wave-type equation which is tensorial and features subprincipal terms at $\mathscr{I}^+$ not present for the scalar wave operator — namely, the terms p_0 and p_1 from Definition 3.7(3) — the K-current needs to be corrected appropriately to $\widetilde{K}$ (see (6-11)). The key computation is then the positivity of $\widetilde{K}$ as a quadratic form on the edge-b-cotangent bundle; this is proved in (6-17) and the subsequent discussion.

Higher regularity follows from the microlocal propagation results proved in Section 4.2. Solvability of the adjoint (backward) problem on negative regularity spaces follows by duality; this can then be upgraded to the maximal amount of regularity allowed in (6-6) via microlocal propagation estimates, which will prove part (2). Another duality argument will then give part (1) in the full range of s.

Part (1): $s = 1$. We shall prove the energy estimate

$$\|u\|_{H_{\mathrm{e,b}}^{1,\alpha}(\Omega^0_{\delta,\rho})^{\bullet,-}} \leq C\|f\|_{H_{\mathrm{e,b}}^{0,\alpha'}(\Omega^0_{\delta,\rho})^{\bullet,-}} \tag{6-8}$$

for the forward solution u of the equation $Pu = f \in H_{\mathrm{e,b}}^{0,\alpha'}(\Omega^0_{\delta,\rho})^{\bullet,-}$ by computing the L^2 pairing (using the volume density $|\mathrm{d}g|$) $2\operatorname{Im}\langle Pu, Zu\rangle$ for $Z = i^{-1}V$, where V is a suitably chosen real vector field on M°.

• Step (1)(i): *Structural considerations for the energy estimate.* For $u \in \mathcal{C}^\infty_{\mathrm{c}}((\Omega^0_{\delta,\rho})^\circ)$, we have

$$2\operatorname{Im}\langle Pu, Zu\rangle = \langle \mathscr{C}u, u\rangle, \quad \mathscr{C} = i(P^*Z - Z^*P) = i[P, Z] - (\operatorname{div}_g iZ)P + 2\frac{P - P^*}{2i}Z, \tag{6-9}$$

where we use that $Z^* = Z + \operatorname{div}_g Z$; here, writing $V = V^j\partial_j$ in local coordinates, we set

$$\operatorname{div}_g(V) = |g|^{-1/2}\partial_j(|g|^{1/2}V^j)$$

(so $V^* = -V - \operatorname{div}_g V$). The principal symbol of $\mathscr{C}$ in M° is

$$\sigma^2(\mathscr{C}) = H_G(\sigma^1(Z)) - (\operatorname{div}_g iZ)G + 2\sigma^1(P_1)\sigma^1(Z), \quad P_1 := \frac{P - P^*}{2i},$$

where G is the dual metric function. A simple calculation gives at a point $\varpi \in T^*_pM^\circ$, $p \in M^\circ$, the expression

$$\sigma^2(\mathscr{C})(\varpi) = K_{iZ}(\varpi, \varpi) + 2\sigma^1(P_1)(\varpi)\cdot\sigma^1(Z)(\varpi) =: \widetilde{K}_{iZ}(\varpi, \varpi), \tag{6-10}$$

where we define for any vector field V the following symmetric 2-cotensors on M°:

$$\begin{aligned} K_V &= {}^V\pi - \tfrac{1}{2} g^{-1} \operatorname{tr}_g({}^V\pi), \quad {}^V\pi := -\mathcal{L}_V(g^{-1}), \\ \widetilde{K}_V &= K_V + 2\sigma^1(P_1) \otimes_s \sigma^1(i^{-1}V), \end{aligned} \tag{6-11}$$

with $\mathcal{L}_V$ denoting the Lie derivative along V.

Concretely now, let $\breve{\alpha} = (\breve{\alpha}_0, 2\breve{\alpha}_{\mathscr{I}}) := (\alpha_0, 2\alpha_{\mathscr{I}}) + (1, 1)$, and define the vector fields

$$W := -x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} + (2-c)\rho_0\partial_{\rho_0}, \quad V := \rho_0^{-2\breve{\alpha}_0} x_{\mathscr{I}}^{-4\breve{\alpha}_{\mathscr{I}}} W, \quad Z = i^{-1}V. \tag{6-12}$$

Note that, for $0 < c < 2$, the vector field W is future timelike at (hence near) $\mathscr{I}^+$ for the rescaled metric $g_{\mathrm{e,b}} := \rho_0^2 x_{\mathscr{I}}^2 g \in (\mathcal{C}^\infty + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}})})(M; S^{2\,\mathrm{e,b}}T^*M)$ by (3-16). Moreover, when the bundle E is trivial, then

$$\begin{aligned} \mathscr{C} &\in \rho_0^{-2\alpha_0} x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}} (\mathcal{C}^\infty + \mathcal{A}^{(\ell_0,(0,0))} + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}})}) \operatorname{Diff}^2_{\mathrm{e,b}}(M; \beta^* E), \\ \widetilde{K}_{iZ} &\in \rho_0^{-2\alpha_0} x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}} (\mathcal{C}^\infty + \mathcal{A}^{(\ell_0,(0,0))} + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}})})(M; S^{2\,\mathrm{e,b}}TM), \end{aligned} \tag{6-13}$$

where the $\mathcal{A}^{(\ell_0,(0,0))}$ terms are contributions from P_1.

When E is not trivial, take $\widetilde{W} \in \operatorname{Diff}^1_{\mathrm{e,b}}(M; \beta^* E)$ to be any operator with scalar principal symbol ${}^{\mathrm{e,b}}\sigma^1(\widetilde{W}) = {}^{\mathrm{e,b}}\sigma^1(W) \otimes 1_E$ and edge normal operator at $\mathscr{I}^+$ equal to that of W. The edge normal operator of $\widetilde{W}$ at a fiber $\mathscr{I}^+_{y_0}$ acts on sections of the trivial bundle E_{y_0}, and differentiation of such sections along W is well-defined; thus, any two choices of $\widetilde{W}$ agree modulo $x_{\mathscr{I}}\mathcal{C}^\infty(M; \operatorname{End}(\beta^* E))$. We now drop the tilde and work with $W \in \operatorname{Diff}^1_{\mathrm{e,b}}(M; \beta^* E)$ simply, with $V, Z \in \operatorname{Diff}^{1,(2\breve{\alpha}_0, 4\breve{\alpha}_{\mathscr{I}})}_{\mathrm{e,b}}(M; \beta^* E)$ given in terms of W as above; we then have (6-13) also when E is not trivial. We shall also fix a connection $\nabla^E \in \operatorname{Diff}^1_{\mathrm{e,b}}(M; \beta^* E, {}^{\mathrm{e,b}}T^*M \otimes \beta^* E)$ on E whose normal operator at $\mathscr{I}^+$ agrees with the canonical flat connection (differentiation of functions valued in a fixed vector space).

On E, we fix a near-optimal positive definite fiber inner product h_E, i.e., the inequalities (6-3) hold for any desired $\epsilon > 0$. Fix moreover a positive definite fiber inner product g_R on ${}^{\mathrm{e,b}}T^*M$. Using g_R, we can identify $\widetilde{K}_{iZ}$ with a section of $\operatorname{End}({}^{\mathrm{e,b}}T^*M)$, and upon tensoring with the identity map on $\beta^* E$ we obtain a section of $\operatorname{End}({}^{\mathrm{e,b}}T^*M \otimes \beta^* E)$ which we shall still denote by $\widetilde{K}_{iZ}$. Define the adjoint $(\nabla^E)^* \in \operatorname{Diff}^1_{\mathrm{e,b}}(M; {}^{\mathrm{e,b}}T^*M \otimes \beta^* E, \beta^* E)$ with respect to the fiber metric $g_R \otimes h_E$. Then $(\nabla^E)^* \widetilde{K}_{iZ} \nabla^E$ and $\mathscr{C}$ have the same principal symbol. Set

$$\widetilde{Q} = \rho_0^2 x_{\mathscr{I}}^2 (x_{\mathscr{I}} D_{x_{\mathscr{I}}} - 2\rho_0 D_{\rho_0}) \equiv \widetilde{Q}^* \mod \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}})}(M),$$

defined as an element of $\rho_0^2 x_{\mathscr{I}}^2 \operatorname{Diff}^1_{\mathrm{e,b}}(M; \beta^* E)$ in the same fashion as W before. We claim that one can then write

$$\begin{aligned} \mathscr{C} &= (\nabla^E)^* \widetilde{K}_{iZ} \nabla^E + \widetilde{\mathscr{C}} + \widetilde{\mathscr{C}}', \\ \widetilde{\mathscr{C}} &\in \rho_0^{-2\alpha_0} x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}} (x_{\mathscr{I}}\mathcal{C}^\infty + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}})}) \operatorname{Diff}^1_{\mathrm{e,b}}(M; \beta^* E), \\ \widetilde{\mathscr{C}}' &= \tfrac{1}{2}(\widetilde{Q}^*(p_1 - p_1^*)Z - Z^*(p_1 - p_1^*)\widetilde{Q}) + \frac{i}{2}\rho_0^2 x_{\mathscr{I}}^2 (p_0^* Z - Z^* p_0) \\ &\in \rho_0^{-2\alpha_0} x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}} (\mathcal{C}^\infty + \mathcal{A}^{(\ell_0,(0,0))} + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}})}) \operatorname{Diff}^1_{\mathrm{e,b}}(M; \beta^* E). \end{aligned} \tag{6-14}$$

Thus, the coefficients of $\widetilde{\mathscr{C}}$ vanish at $\mathscr{I}^+$; similarly, we will regard $\widetilde{\mathscr{C}}'$ as an error term near $\mathscr{I}^+$ since, by assumption, $p_1 - p_1^*$ and $p_0,\ p_0^*$ can be made arbitrarily small for suitable choices of fiber inner products on E.

Only the vanishing factor $x_{\mathscr{I}}$ in the membership statement for $\widetilde{\mathscr{C}}$ requires proof. We have to show the vanishing of the edge normal operator of $\rho_0^{2\alpha_0} x_{\mathscr{I}}^{4\alpha_{\mathscr{I}}} \widetilde{\mathscr{C}}$; for this purpose, we may replace g^{-1} and P in the definition of $\mathscr{C}$ and $\widetilde{K}_{iZ}$ by their normal operators at a fiber $\mathscr{I}^+_{y_0}$:

$$\begin{aligned} g_0^{-1} &:= \rho_0^2 x_{\mathscr{I}}^2 \big(-\tfrac{1}{2} x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} \otimes_s (x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} - 2\rho_0\partial_{\rho_0}) + k^{ij}(y_0)(x_{\mathscr{I}}\partial_{y^i}) \otimes_s (x_{\mathscr{I}}\partial_{y^j})\big), \\ P_0 &:= \tfrac{1}{2}\rho_0^2 x_{\mathscr{I}}^2 \Big(-\Big(x_{\mathscr{I}} D_{x_{\mathscr{I}}} - 2i^{-1}\Big(\frac{n-1}{2} + p_{1,y_0}\Big)\Big)(x_{\mathscr{I}} D_{x_{\mathscr{I}}} - 2\rho_0 D_{\rho_0}) \\ &\qquad + 2k^{ij}(y_0)(x_{\mathscr{I}} D_{y^i})(x_{\mathscr{I}} D_{y^j}) + p^0_{0,y_0}\Big), \end{aligned} \tag{6-15}$$

where we use (3-12), (3-18), and set

$$p^0_{0,y_0} := p_0^0|_{\mathscr{I}^+_{y_0}}, \quad p_{1,y_0} := p_1|_{\mathscr{I}^+_{y_0}} \in (\mathcal{C}^\infty + \mathcal{A}^{\ell_0})([0,1)_{\rho_0}; \operatorname{End}(E_{y_0})).$$

We may moreover work with the metric density

$$|\mathrm{d}g_0| = \rho_0^{-n-1} x_{\mathscr{I}}^{-2n} \left| \frac{\mathrm{d}\rho_0}{\rho_0} \frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}} \mathrm{d}y \right|.$$

In the case that $p^0_{0,y_0} = 0$ and $p_{1,y_0} = 0$, we have $P_0 = P_0^*$. The operators

$$\mathscr{C}_0 := i(P_0^* Z - Z^* P_0)$$

and $(\nabla^E)^* \widetilde{K}_{0,iZ} \nabla^E$ (the subscripts "0" indicating that we are using $g_0,\ P_0,\ P_{0,1} = (P_0 - P_0^*)/2i = 0$ in place of $g,\ P,\ P_1$, respectively, and with ∇^E denoting the trivial connection on $(\beta^* E)|_{\mathscr{I}^+_{y_0}} = \mathscr{I}^+_{y_0} \times E_{y_0}$) have the same principal symbol, real coefficients, and are symmetric. Hence their difference must in fact be an operator of order 0; but since both operators annihilate constant sections of E_{y_0}, this difference is the zero operator. This proves (6-14) in this case, with $\widetilde{\mathscr{C}}' = 0$.

For general p_{1,y_0}, but still assuming $p^0_{0,y_0} = 0$, the operator P_0 gets an extra contribution

$$Q = i^{-1} p_{1,y_0} \widetilde{Q},$$

and $P_{0,1} = (Q - Q^*)/2i$ has principal symbol $-\frac{1}{2}(p_{1,y_0} + p^*_{1,y_0})\sigma^1(\widetilde{Q})$. Hence, compared to the case $p_{1,y_0} = 0$, the difference $\mathscr{C}_0 - (\nabla^E)^* \widetilde{K}_{iZ} \nabla^E$ gets an extra contribution

$$i(Q^* Z - Z^* Q) - (\nabla^E)^* \big(-2\sigma^1\big(\tfrac{1}{2}(p_{1,y_0} + p^*_{1,y_0})\widetilde{Q}\big) \otimes_s \sigma^1(Z)\big)\nabla^E, \tag{6-16}$$

where we regard

$$\sigma^1\big(\tfrac{1}{2}(p_{1,y_0} + p_{1,y_0})^* \widetilde{Q}\big) \in P^{[1]}({}^{\mathrm{e,b}}T^*M \otimes \operatorname{End}(E_{y_0})) \cong {}^{\mathrm{e,b}}TM \otimes \operatorname{End}(E_{y_0})$$

and $\sigma^1(Z) \in P^{[1]}({}^{\mathrm{e,b}}T^*M) \cong {}^{\mathrm{e,b}}TM$ (giving an element of ${}^{\mathrm{e,b}}TM \otimes \operatorname{End}(E_{y_0})$ upon tensoring with the identity on E_{y_0}), and further identify $S^2\,{}^{\mathrm{e,b}}TM \cong {}^{\mathrm{e,b}}T^*M \otimes_s {}^{\mathrm{e,b}}TM$ using the Riemannian edge-b-metric g_R on M. Using that for any vector fields V and W one has

$$(\nabla^E)^*(2\sigma^1(V) \otimes_s \sigma^1(W))\nabla^E \equiv (V + \operatorname{div}_{g_0} V)W + (W + \operatorname{div}_{g_0} W)V,$$

where we write "$\equiv$" for the equality of edge normal operators (this identity depending only on the volume density $|\mathrm{d}g_0|$ and not on the choice of g_R) and noting that $\widetilde{Q} = \rho_0^2 x_{\mathscr{I}}^2(x_{\mathscr{I}} D_{x_{\mathscr{I}}} - 2\rho_0 D_{\rho_0})$ is symmetric with respect to $|\mathrm{d}g_0|$, we find that (6-16) is equal to

$$(-\widetilde{Q} p_{1,y_0}^* Z - Z^* p_{1,y_0} \widetilde{Q}) - \big(-\tfrac{1}{2}\widetilde{Q}(p_{1,y_0} + p_{1,y_0}^*)Z - \tfrac{1}{2}Z^*(p_{1,y_0} + p_{1,y_0}^*)\widetilde{Q}\big) = \widetilde{\mathscr{C}}'.$$

Expanding the larger parentheses and collecting terms gives (6-14) in the case $p_0 = 0$.

Finally, if we allow also p_{0,y_0}^0 to be nonzero, the operator P_0 gets an extra contribution $\frac{1}{2}\rho_0^2 x_{\mathscr{I}}^2 p_{0,y_0}^0$. Relative to the case $p_{0,y_0}^0 = 0$ that we just considered, the map $\widetilde{K}_{iZ}$ is unchanged, whereas $\mathscr{C}_0$ gets an extra contribution $\frac{1}{2} i\rho_0^2 x_{\mathscr{I}}^2((p_{0,y_0}^0)^* Z - Z^* p_{0,y_0}^0)$. Putting this term into $\widetilde{\mathscr{C}}'$ gives (6-14) in general.

• Step (1)(ii): *Energy estimate for compactly supported waves.* We now proceed to compute the leading-order term of $\widetilde{K}_{iZ}$ (see (6-11)) at $\mathscr{I}^+$ for Z defined in (6-12); for this purpose, we may again work with the model metric and edge normal operator (6-15). We compute

$$\begin{aligned}{}^{V}\pi = \rho_0^{-2\alpha_0} x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}} \big(&-4(2-c)\check{\alpha}_{\mathscr{I}}(\rho_0\partial_{\rho_0})^2 + ((c-2)\check{\alpha}_0 + (6-2c)\check{\alpha}_{\mathscr{I}} + c - 1)2\rho_0\partial_{\rho_0} \otimes_s x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} \\ &+ (2\check{\alpha}_0 - 4\check{\alpha}_{\mathscr{I}} + 1 - c)(x_{\mathscr{I}}\partial_{x_{\mathscr{I}}})^2 + 2ck^{ij}(x_{\mathscr{I}}\partial_{y^i}) \otimes_s (x_{\mathscr{I}}\partial_{y^j})\big),\end{aligned}$$

and thus

$$\begin{aligned}\widetilde{K}_{iZ} \equiv \rho_0^{-2\alpha_0} x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}} \Big(&(2-c)(-4\check{\alpha}_{\mathscr{I}} + 2(p_1 + p_1^*))(\rho_0\partial_{\rho_0})^2 \\ &+ \Big(4\check{\alpha}_{\mathscr{I}} - 2(p_1 + p_1^*) + \frac{c}{2}(-n + 1 - 4\check{\alpha}_{\mathscr{I}} + p_1 + p_1^*)\Big)2\rho_0\partial_{\rho_0} \otimes_s x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} \\ &+ \Big(-2\check{\alpha}_{\mathscr{I}} + (p_1 + p_1^*) + \frac{c}{2}(n - 1 + 2\check{\alpha}_0)\Big)(x_{\mathscr{I}}\partial_{x_{\mathscr{I}}})^2 \\ &+ (2 - 4\check{\alpha}_{\mathscr{I}} + 4\check{\alpha}_0 + c(-n + 1 - 2\check{\alpha}_0))k^{ij}(x_{\mathscr{I}}\partial_{y^i}) \otimes_s (x_{\mathscr{I}}\partial_{y^j})\Big) \\ &\mod \rho_0^{-2\alpha_0} x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}} \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}})}(M; S^{2\,\mathrm{e,b}}TM \otimes \mathrm{End}(\beta^* E)). \qquad (6\text{-}17)\end{aligned}$$

Considering the larger parentheses at a point of $\mathscr{I}^+$, we may diagonalize $p_1 + p_1^*$ and thus consider each eigenspace, corresponding to an eigenvalue 2λ, separately; for any desired $\epsilon > 0$, we can ensure that all such eigenvalues satisfy $2\lambda > 2\underline{p}_1 - 2\epsilon$ by choosing a suitable fiber inner product on E. The 2×2 minor of the larger parentheses, with respect to the basis $\mathrm{d}\rho_0/\rho_0$, $\mathrm{d}x_{\mathscr{I}}/x_{\mathscr{I}}$, has trace $-10\check{\alpha}_{\mathscr{I}} + 10\lambda + \mathcal{O}(c)$ as $c \searrow 0$, which is thus positive for small fixed $\epsilon > 0$ and all small $c > 0$ when $\check{\alpha}_{\mathscr{I}} < \underline{p}_1$, i.e., $\alpha_{\mathscr{I}} < -\frac{1}{2} + \underline{p}_1$. The determinant of the 2×2 minor is $8c(\check{\alpha}_{\mathscr{I}} - \lambda)(\check{\alpha}_{\mathscr{I}} - \check{\alpha}_0) + \mathcal{O}(c^2)$, which is positive for small $c > 0$ if, in addition, we also have $\check{\alpha}_{\mathscr{I}} < \check{\alpha}_0$, i.e., $\alpha_{\mathscr{I}} < \frac{1}{2} + \alpha_0$. The $(x_{\mathscr{I}}\partial_y)^2$ term of the larger parentheses finally is $4\big(\check{\alpha}_0 - \check{\alpha}_{\mathscr{I}} + \frac{1}{2}\big) + \mathcal{O}(c)$ times a positive definite symmetric 2-tensor in $x_{\mathscr{I}}\partial_{y^j}$ as $c \searrow 0$, the positivity of which is guaranteed upon reducing $c > 0$ further if necessary.

Given weights α_0, $\alpha_{\mathscr{I}}$ as in the statement of the theorem, we may thus fix $\epsilon > 0$ so that $\alpha_{\mathscr{I}} < -\frac{1}{2} + \underline{p}_1 - \epsilon$, then pick a near-optimal fiber inner product on E to make $p_1 + p_1^* \geq 2\underline{p}_1 - 2\epsilon$, and then choose $c > 0$ sufficiently small in (6-12) so that $\rho_0^{2\alpha_0} x_{\mathscr{I}}^{4\alpha_{\mathscr{I}}} \widetilde{K}_{iZ}$ is positive definite at $\overline{\Omega^0_{\delta,\rho}} \cap \mathscr{I}^+$ (with a lower bound on its eigenvalues that remains positive as $\epsilon \searrow 0$) and hence in $\Omega^0_{\delta,\rho}$ for sufficiently small $\delta > 0$. Thus, if

$\delta > 0$ is sufficiently small, then for $u \in \mathcal{C}^\infty_{\mathrm{c}}((\Omega^0_{\delta,\rho})^\circ)$ we have

$$\|\nabla^E_{\mathrm{e,b}} u\|^2_{H^{0,\alpha}_{\mathrm{e,b}}} \leq C\langle \nabla^E u, \widetilde{K}_{iZ} \nabla^E u\rangle;$$

here, we write $\nabla^E_{\mathrm{e,b}} u = (\nabla^E_{V_j} u)_{j=1,\dots,N}$, where $V_1, \dots, V_N \in \mathcal{V}_{\mathrm{e,b}}(M)$ denotes any fixed collection of vector fields which span $\mathcal{V}_{\mathrm{e,b}}(M)$ over $\mathcal{C}^\infty(M)$. Plugging (6-14) into (6-9), estimating the left-hand side of (6-9) using Cauchy–Schwarz, and likewise for the contributions of $\widetilde{\mathscr{C}}$ and $\widetilde{\mathscr{C}}'$ in (6-14), we thus obtain, for $w = x_{\mathscr{I}} + \rho_0^{\ell_0} x_{\mathscr{I}}^{2\ell_{\mathscr{I}}}$,

$$\|\nabla^E_{\mathrm{e,b}} u\|^2_{H^{0,\alpha}_{\mathrm{e,b}}} \leq C\big(\|Pu\|^2_{H^{0,\alpha'}_{\mathrm{e,b}}} + \|w^{1/2}\nabla^E_{\mathrm{e,b}} u\|_{H^{0,\alpha}_{\mathrm{e,b}}} \|w^{1/2} u\|_{H^{0,\alpha}_{\mathrm{e,b}}} + \epsilon' \|\nabla^E_{\mathrm{e,b}} u\|_{H^{0,\alpha}_{\mathrm{e,b}}} \|u\|_{H^{0,\alpha}_{\mathrm{e,b}}}\big)$$

for all $u \in \mathcal{C}^\infty_{\mathrm{c}}((\Omega^0_{\delta,\rho})^\circ)$ when $\delta > 0$ is sufficiently small; here $\epsilon' = \|p_1 - p_1^*\| + \|p_0\|$ can be made arbitrarily small since P is special admissible, while C is independent of ϵ'. For $\delta, \epsilon' > 0$ sufficiently small, and noting that $w \lesssim x_{\mathscr{I}}^{2\ell_{\mathscr{I}}} \leq \delta^{2\ell_{\mathscr{I}}}$ on $\Omega^0_{\delta,\rho}$ (recalling that $2\ell_{\mathscr{I}} \leq 1$), we may then apply Cauchy–Schwarz to the second and third terms on the right and absorb $\|\nabla^E_{\mathrm{e,b}} u\|^2$ into the left-hand side while retaining a small constant in front of $\|u\|^2_{H^{0,\alpha}_{\mathrm{e,b}}}$. We thus obtain

$$\|\nabla^E_{\mathrm{e,b}} u\|^2_{H^{0,\alpha}_{\mathrm{e,b}}} \leq C(\|Pu\|^2_{H^{0,\alpha'}_{\mathrm{e,b}}} + (\delta^{\ell_{\mathscr{I}}} + \epsilon')\|u\|^2_{H^{0,\alpha}_{\mathrm{e,b}}}), \tag{6-18}$$

with C independent of δ and ϵ'.

Finally then, we claim that one can estimate

$$\|u\|_{H^{0,\alpha}_{\mathrm{e,b}}} \leq C' \|\nabla^E_{\mathrm{e,b}} u\|_{H^{0,\alpha}_{\mathrm{e,b}}} \tag{6-19}$$

with a constant C' independent of δ by integrating the vector field

$$V_0 := \rho_0^{-2\alpha_0} x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}} (-x_{\mathscr{I}} \partial_{x_{\mathscr{I}}} + 2\rho_0 \partial_{\rho_0})$$

starting at $x_{\mathscr{I}} = \delta$ (where u vanishes). Phrased as a commutator estimate, this means computing

$$2\,\mathrm{Re}\langle u, V_0 u\rangle = -\langle (\mathrm{div}\, V_0) u, u\rangle, \tag{6-20}$$

where, with respect to the volume density

$$|\mathrm{d}g_0| = \rho_0^{-n-1} x_{\mathscr{I}}^{-2n} \left| \frac{\mathrm{d}\rho_0}{\rho_0} \frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}}\, \mathrm{d}y \right|$$

of the model metric g_0 in (6-15), we have

$$\begin{aligned} \mathrm{div}_{g_0} V_0 = -(V_0 + V_0^*) &= \rho_0^2 x_{\mathscr{I}}^2 [x_{\mathscr{I}} \partial_{x_{\mathscr{I}}} - 2\rho_0 \partial_{\rho_0}, \rho_0^{-2\breve{\alpha}_0} x_{\mathscr{I}}^{-4\breve{\alpha}_{\mathscr{I}}}] \\ &= (4\breve{\alpha}_0 - 4\breve{\alpha}_{\mathscr{I}}) \rho_0^{-2\alpha_0} x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}}, \end{aligned}$$

which is a positive multiple of $\rho_0^{-2\alpha_0} x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}}$ since $\breve{\alpha}_{\mathscr{I}} < \breve{\alpha}_0$. The Cauchy–Schwarz inequality then implies the estimate (6-19). Plugging this into (6-18) and taking δ and ϵ' sufficiently small, we obtain $\|\nabla^E_{\mathrm{e,b}} u\|_{H^{0,\alpha}_{\mathrm{e,b}}} \leq C\|Pu\|_{H^{0,\alpha'}_{\mathrm{e,b}}}$ provided $\delta > 0$ is sufficiently small, and thus (6-8) by applying (6-19), for $u \in \mathcal{C}^\infty_{\mathrm{c}}((\Omega^0_{\delta,\rho})^\circ)$.

• Step (1)(iii): *Energy estimate for distributions extendible at* $\Sigma^{0,\mathrm{out}}_{\delta,\rho}$. We proceed to remove the assumption that u vanishes near $\Sigma^{0,\mathrm{out}}_{\delta,\rho}$. Thus, in the integration by parts in (6-9), one now has to add a boundary term at $\Sigma^{0,\mathrm{out}}_{\delta,\rho}$. Equivalently, one can multiply the vector field multiplier Z with a sharp cutoff $1_{\Omega^0_{\delta,\rho}}$ (the characteristic function of $\Omega^0_{\delta,\rho}$), in which case (6-9) is valid still, but the operator $\mathscr{C}$ now has distributional coefficients. Using the latter formulation, we note that, for any distribution χ, we have $\mathcal{L}_{\chi V} g^{-1} = \chi(\mathcal{L}_V g^{-1}) - 2V \otimes_s \nabla\chi$ and thus

$$\widetilde{K}_{\chi V} = \chi \widetilde{K}_V + 2T(\nabla\chi, V), \quad T(X,Y) := X \otimes_s Y - \tfrac{1}{2} g(X,Y) g^{-1}. \tag{6-21}$$

Here, the abstract stress-energy-momentum tensor $T(X,Y)$ is positive definite when X and Y are both future timelike. Let now $\chi = 1_{\Omega^0_{\delta,\rho}}$. The extra contribution to $\langle \mathscr{C}u, u\rangle$ in (6-14) arising from differentiation of χ is $2\langle T(\nabla\chi, iZ)\nabla^E u, \nabla^E u\rangle$. This is the sum of two terms: one from integration over $\Sigma^{0,\mathrm{in}}_{\delta,\rho}$ which vanishes since u has supported character (i.e., vanishes) there; and another term, arising from integration over the final Cauchy surface $\Sigma^{0,\mathrm{out}}_{\delta,\rho}$, which is nonnegative due to the future timelike nature of $\nabla\chi$ (using the spacelike nature of $\Sigma^{0,\mathrm{out}}_{\delta,\rho}$) and iZ, and thus of the same sign as the main term (arising from $1_{\Omega^0_{\delta,\rho}} \widetilde{K}_{iZ}$). Hence it can be dropped.

The estimate (6-19) follows similarly for u which are extendible at $\Sigma^{0,\mathrm{out}}_{\delta,\rho}$: in the commutator calculation (6-20), one replaces V_0 by χV_0, $\chi = 1_{\Omega^0_{\delta,\rho}}$, which gives

$$2\,\mathrm{Re}\langle u, \chi V_0 u\rangle = -\langle \chi(\mathrm{div}\, V_0)u, u\rangle - \langle (V_0\chi)u, u\rangle.$$

The second term (including the minus sign) is a sum of two terms; one arising from integration over $\Sigma^{0,\mathrm{in}}_{\delta,\rho}$ which vanishes due to the support assumption on u, and one from integration over $\Sigma^{0,\mathrm{out}}_{\delta,\rho}$ which is nonnegative, hence has the same sign as the main term (from $\mathrm{div}\, V_0$) and hence can be dropped. This establishes (6-8) for smooth elements u of $H^{1,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}$ whose support inside M does not intersect $\mathscr{I}^+$.

• Step (1)(iv): *Unconditional estimate near* $\mathscr{I}^+$. For $u \in H^{2,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}$, the estimate (6-8) now follows by a density argument. Next, given $f \in H^{0,\alpha'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-} \cap H^1_{\mathrm{loc}}(M^\circ)$, the unique forward solution u of $Pu = f$ lies in $H^2_{\mathrm{loc}}((\Omega^0_{\delta,\rho})^\circ)$. In order to show that it satisfies the estimate (6-8), we cannot directly apply the previous estimates, as a priori the growth of u at $\mathscr{I}^+$ is not controlled. Instead, we apply the above energy estimates on $\Omega^0_{\delta,\rho} \setminus \Omega^0_{\delta',\rho}$, i.e., using the sharp cutoff $\chi := 1_{\Omega^0_{\delta,\rho}} - 1_{\Omega^0_{\delta',\rho}}$, for $\delta' \in (0,\delta)$. The boundary hypersurfaces of $\Omega^0_{\delta,\rho} \setminus \Omega^0_{\delta',\rho}$ are $\Sigma^{0,\mathrm{in}}_{\delta,\rho}$ (where u has supported character), $\Sigma^{0,\mathrm{out}}_{\delta,\rho} \setminus \Sigma^{0,\mathrm{out}}_{\delta',\rho}$ (where the boundary term has a good sign, as shown before) and $\Sigma^{0,\mathrm{in}}_{\delta',\rho}$, where the boundary term again has a good sign (since $\nabla\chi$ is future timelike there). Thus, upon dropping these advantageous boundary terms, we have

$$\|u\|_{H^{1,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho}\setminus\Omega^0_{\delta',\rho})^{\bullet,-}} \leq C\|Pu\|_{H^{0,\alpha'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho}\setminus\Omega^0_{\delta',\rho})},$$

where the constant C is independent of δ'; hence, upon taking $\delta' \searrow 0$, we conclude that $u \in H^{1,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}$, and the estimate (6-8) holds.

Finally, for general $f \in H^{0,\alpha'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}$, we pick a sequence $f_j \in H^{1,\alpha'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}$ which converges to f in $H^{0,\alpha'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}$. The forward solution u_j of $Pu_j = f_j$ satisfies $u_j \in H^{1,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}$. Applying (6-8) to

the differences $u_j - u_k$ shows that u_j is a Cauchy sequence, and the limit $u \in H^{1,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}$ satisfies $Pu = f$ together with the estimate (6-8).

• Step (1)(v): *Unconditional estimate in the original domain* $\Omega^0_{\delta,\rho}$. Finally, we need to prove (6-8) for the *original* value of $\delta > 0$, rather than for the (possibly much smaller) value fixed in the course of the above argument, which we denote by $\delta' \in (0, \delta]$ now for clarity. This, however, is straightforward, as one can estimate

$$\|u\|_{H^{1,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho}\setminus\Omega^0_{\delta',\rho})^{\bullet,-}} \leq C\|Pu\|_{H^{0,\alpha'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho}\setminus\Omega^0_{\delta',\rho})^{\bullet,-}}, \tag{6-22}$$

with "•" now indicating the supported character at $\Sigma^{0,\mathrm{in}}_{\delta,\rho}$, and "−" indicating the extendible character at $\Sigma^{0,\mathrm{out}}_{\delta,\rho} \setminus \Sigma^{0,\mathrm{out}}_{\delta',\rho}$ as well as at $\Sigma^{0,\mathrm{in}}_{\delta',\rho}$. Indeed, the domain $\Omega^0_{\delta,\rho} \setminus \Omega^0_{\delta',\rho}$ is disjoint from $\mathscr{I}^+$, and hence the weight at $\mathscr{I}^+$ is irrelevant, and the timelike function $x_{\mathscr{I}}$ is restricted to a compact subset of $(0, \infty)$; thus, (6-22) is a standard finite time energy estimate. (Concretely, one can use the vector field multiplier $\rho_0^{-2\check{\alpha}_0} e^{\mathsf{F}x_{\mathscr{I}}} W$, with $W = -x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} + \rho_0\partial_{\rho_0}$, or any other smooth future timelike edge-b-vector field, and $\mathsf{F} > 1$ sufficiently large — thus, the estimate allows for exponential growth of u as $x_{\mathscr{I}}$ decreases, which is acceptable since we are restricting to a domain where $x_{\mathscr{I}}$ is bounded from below by the fixed positive constant δ'; cf. [Hintz and Vasy 2020, §1.1.1 and §4].)

Part (1): $s > 1$. Fix $\rho' \in (\rho, \bar{\rho}_0)$. Given $f \in H^{s-1,\alpha'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}$, we first extend f to $\tilde{f} \in H^{s-1,\alpha'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho'})^{\bullet,-}$ so that $\|\tilde{f}\|_{H^{s-1,\alpha'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho'})^{\bullet,-}} \leq C\|f\|_{H^{s-1,\alpha'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}}$ (which can be arranged for some constant C depending only on s, α, δ, ρ, ρ' but not on f). The forward solution $\tilde{u}$ of $P\tilde{u} = \tilde{f}$ then satisfies

$$\|\tilde{u}\|_{H^{1,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho'})^{\bullet,-}} \leq C\|\tilde{f}\|_{H^{0,\alpha'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho'})^{\bullet,-}} \tag{6-23}$$

by the first part of the proof. Using the higher regularity of $\tilde{f}$, we can then propagate $H^{s,\alpha}_{\mathrm{e,b}}$ regularity from $x_{\mathscr{I}} > \delta$, where $\tilde{f}$ and $\tilde{u}$ vanish and hence are smooth, using (the quantitative estimate versions of) Lemmas 4.8(1), 4.9(1), and 4.11(2), thus proving local $H^{s,\alpha}_{\mathrm{e,b}}$ regularity in $(\Omega^0_{\delta,\rho'})^\circ$. Note here that, with $s_0 := 1$, the a priori $H^{s_0,\alpha}_{\mathrm{e,b}}$ regularity of u at $\mathcal{R}_{\mathrm{c}}$ is strong enough for an application of Lemma 4.8(1) since, due to the assumptions (6-4), we have

$$\tfrac{1}{2} + (\alpha_{\mathscr{I}} - \alpha_0) + (\alpha_{\mathscr{I}} - \underline{p}_1) < \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{2} = \tfrac{1}{2} < s_0.$$

Restricting to $\Omega^0_{\delta,\rho}$ gives, in view of (6-23), the desired estimate

$$\begin{aligned}\|u\|_{H^{s,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}} = \|\tilde{u}|_{\Omega^0_{\delta,\rho}}\|_{H^{s,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}} &\leq C(\|\tilde{f}\|_{H^{s-1,\alpha'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho'})^{\bullet,-}} + \|\tilde{u}\|_{H^{1,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho'})^{\bullet,-}}) \\ &\leq C'\|f\|_{H^{s-1,\alpha'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}}.\end{aligned}$$

(This extension and restriction procedure is necessitated by the fact that the microlocal estimates are not sharply localized, unlike energy estimates. See also [Hintz and Vasy 2015, §2.1.3].)

Part (2): $\tilde{s} \leq 0$. Let $s = -\tilde{s} + 1$ and $(\alpha_0, 2\alpha_{\mathscr{I}}) = (-\tilde{\alpha}'_0, -2\tilde{\alpha}'_{\mathscr{I}})$. The a priori estimate

$$\|u\|_{H^{s,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}} \leq C\|Pu\|_{H^{s-1,\alpha'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-}} \tag{6-24}$$

(valid for all u for which both norms are finite) implies, by duality, that the equation $P^*\tilde{u} = \tilde{f}$ for $\tilde{f} \in (H^{s,\alpha}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-})^* = H^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{-,\bullet}$ has a solution $\tilde{u} \in (H^{s-1,\alpha'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{\bullet,-})^* = H^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{-,\bullet}$ which moreover satisfies the estimate

$$\|\tilde{u}\|_{H^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{-,\bullet}} \leq C\|\tilde{f}\|_{H^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{-,\bullet}} \tag{6-25}$$

with the same constant C as in (6-24).

As for the uniqueness claim, suppose that $\tilde{u} \in H^{-\infty,\tilde{\alpha}}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{-,\bullet}$ satisfies $P^*\tilde{u} = 0$. Using the microlocal propagation estimates proved in Section 4.2, we find that $\tilde{u} \in H^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b}}(\Omega^0_{\delta',\rho})^{-,\bullet}$ for any $\delta' < \delta$. Given any $f \in (H^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b}}(\Omega^0_{\delta',\rho})^{-,\bullet})^*$, the solvability of $Pu = f$ with $u \in (H^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b}}(\Omega^0_{\delta',\rho})^{-,\bullet})^*$ implies that $\langle \tilde{u}, f\rangle = \langle \tilde{u}, Pu\rangle = \langle P^*\tilde{u}, u\rangle = 0$. Since f is arbitrary, this implies $\tilde{u} = 0$.

In particular, given $\tilde{u} \in H^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{-,\bullet}$ for which $\tilde{f} := P^*\tilde{u}$ satisfies $\tilde{f} \in H^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{-,\bullet}$, the solution $\tilde{u}' \in H^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{-,\bullet}$ of $P^*\tilde{u}' = \tilde{f}$ constructed above by duality must be equal to $\tilde{u}$; but this implies that $\tilde{u} = \tilde{u}'$ obeys the norm bound (6-25). Thus, the estimate (6-25) holds for all $\tilde{u}$ for which both sides are finite.

Part (2): *full range of $\tilde{s}$.* Let now $0 \leq \tilde{s} < \frac{1}{2} - \tilde{\alpha}_0 + 2\tilde{\alpha}_{\mathscr{I}} + \underline{p}_1$. Extending the desired forcing term $\tilde{f} \in H^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{-,\bullet}$ to a slightly larger domain $\Omega^0_{\delta',\rho}$, with $\delta' > \delta$, to a distribution $\tilde{f}'$ with norm bounded by a fixed constant times that of $\tilde{f}$, then finding the solution $P^*\tilde{u}' = \tilde{f}'$ on the larger domain in the space $H^{0,\tilde{\alpha}}_{\mathrm{e,b}}(\Omega^0_{\delta',\rho})^{-,\bullet}$ using the previous step of the proof, subsequently using the propagation results in Section 4.2, and finally restricting back to $\Omega^0_{\delta,\rho}$ produces a solution of $P^*\tilde{u} = \tilde{f}$ with the desired regularity, and gives a quantitative estimate

$$\|\tilde{u}\|_{H^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{-,\bullet}} \leq C\|P^*\tilde{u}\|_{H^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{-,\bullet}}. \tag{6-26}$$

Propagation of regularity and uniqueness of solutions in $H^{-\infty,\tilde{\alpha}}_{\mathrm{e,b}}(\Omega^0_{\delta,\rho})^{-,\bullet}$ of $P^*\tilde{u} = \tilde{f}$ implies, as in the previous step, that (6-26) holds for all $\tilde{u}$ for which both sides are finite.

Part (1): *full range of s.* Letting $\tilde{s} = -s + 1$ and $(\tilde{\alpha}_0, 2\tilde{\alpha}_{\mathscr{I}}) = (-\alpha'_0, -2\alpha'_{\mathscr{I}})$, this now follows by duality from the a priori estimate (6-26). □

To state the forward version with higher b-regularity, we use the Sobolev spaces $H^{(s;k),\alpha}_{\mathrm{e,b;b}}(\Omega^0_{\delta,\rho}; \beta^*E)^{\bullet,-}$ which are defined analogously to Definition 6.1.

Corollary 6.6 (higher b-regularity). *Let P be a special admissible operator, and let $\underline{p}_1$ be as in Definition 4.3. Let $s, \alpha_0, \alpha_{\mathscr{I}} \in \mathbb{R}$ and $k \in \mathbb{N}_0$, put $\alpha = (\alpha_0, 2\alpha_{\mathscr{I}})$ and $\alpha' = \alpha + (2,2)$, and suppose that*

$$s > \tfrac{1}{2} - \alpha_0 + 2\alpha_{\mathscr{I}} - \underline{p}_1, \quad \alpha_{\mathscr{I}} < \alpha_0 + \tfrac{1}{2}, \quad \alpha_{\mathscr{I}} < -\tfrac{1}{2} + \underline{p}_1.$$

Let $f \in H^{(s-1;k),\alpha'}_{\mathrm{e,b;b}}(\Omega^0_{\delta,\rho}; \beta^*E)^{\bullet,-}$. Then the unique distributional forward solution u of $Pu = f$ in $\Omega^0_{\delta,\rho} \cap M^\circ$ satisfies $u \in H^{(s;k),\alpha}_{\mathrm{e,b;b}}(\Omega^0_{\delta,\rho}; \beta^*E)^{\bullet,-}$, and

$$\|u\|_{H^{(s;k),\alpha}_{\mathrm{e,b;b}}(\Omega^0_{\delta,\rho};\beta^*E)^{\bullet,-}} \leq C\|f\|_{H^{(s-1;k),\alpha'}_{\mathrm{e,b;b}}(\Omega^0_{\delta,\rho};\beta^*E)^{\bullet,-}},$$

where C only depends on P, s, k, α_0, $\alpha_{\mathscr{I}}$, δ, ρ.

Proof. For $k = 0$, this is Theorem 6.4(1). Suppose, by induction, that we have proved the corollary for some amount $k \in \mathbb{N}_0$ of b-regularity, and let $f \in H^{(s-1;k+1),\alpha'}_{\mathrm{e,b;b}} \subset H^{((s+1)-1;k),\alpha'}_{\mathrm{e,b;b}}$. Then the inductive hypothesis gives $u \in H^{(s+1;k),\alpha}_{\mathrm{e,b;b}}$. Let $X \in \mathrm{Diff}^1_{[\mathrm{b}]}(M; E)$ be a commutator b-operator (see Definition 5.4). Then

$$P(Xu) = Xf + [P, X]u \in H^{(s-1;k),\alpha'}_{\mathrm{e,b;b}}$$

since $[P, X] \in \mathcal{A}^{(2,2,2)} \mathrm{Diff}^2_{\mathrm{e,b}}(M)$ by Lemma 5.5. Applying the inductive hypothesis again, we find $Xu \in H^{(s;k),\alpha}_{\mathrm{e,b;b}}(M)$. Since X was arbitrary, this implies $u \in H^{(s;k+1),\alpha}_{\mathrm{e,b;b}}(M)$ by Proposition 5.3. □

7. Control of edge-b-decay at $\mathscr{I}^+$

We use the notation M, M_0, Y, the coordinates $\varrho = \frac{1}{t+r}$, $v = \frac{t-r}{t+r}$ as laid out at the beginning of Section 6, and we continue to work with an admissible operator P as in Definition 3.7. In the previous section, we obtained full control of forward solutions of $Pu = f$ near $\mathscr{I}^+ \setminus I^+$ on the scale of weighted edge-b-Sobolev spaces. We shall now work near I^+; we use $t_* = t - r$ and the coordinates $\rho_+ = 1/(t_* - T)$, $x_{\mathscr{I}} = \sqrt{(t_* - T)/r}$ as in (3-6), where $T \in \mathbb{R}$ is arbitrary but fixed. We drop the weight at I^0 from the notation and also the vector bundle $\beta^* E \to M$ unless additional arguments are required in its presence.

As explained in Section 1, the behavior of waves — both regularity and decay — uniformly near $\mathscr{I}^+ \cap I^+$ cannot be analyzed locally, but rather depends on global information which is not captured by Definition 3.7. (See in particular Remark 1.2.) In a situation where regularity is controlled globally near $\mathscr{I}^+$ however (see [Hintz 2023d] for several classes of examples), growth/decay at $\mathscr{I}^+$, as measured in edge-b-Sobolev spaces, is controlled by the edge normal operator of P. In Section 7.1, we prove a priori estimates and solvability results for the localization of the edge normal operator of P near $\mathscr{I}^+ \cap I^+$. These are used in Section 7.2 to prove a priori estimates for P near $\mathscr{I}^+ \cap I^+$ which control a solution u of $Pu = f$ in the sense of decay near $\mathscr{I}^+$ (though with a loss of derivatives); see Theorem 7.3.

7.1. *Analysis of the edge normal operator.*

Fixing $y_0 \in Y$ and writing

$$p^+_{0,y_0} := p^+_0|_{\mathscr{I}^+_{y_0}} \in (\mathcal{C}^\infty + \mathcal{A}^{\ell_+})([0,1)_{\rho_+}; \mathrm{End}(E_{y_0})),$$
$$p_{1,y_0} := p_1|_{\mathscr{I}^+_{y_0}} \in (\mathcal{C}^\infty + \mathcal{A}^{\ell_+})([0,1)_{\rho_+}; \mathrm{End}(E_{y_0})),$$

consider the edge normal operator of P at $\mathscr{I}^+$,[27]

$$P_{y_0} = \tfrac{1}{2} x_{\mathscr{I}}^2 \rho_+^2 \Big(\Big(x_{\mathscr{I}} D_{x_{\mathscr{I}}} - 2i^{-1} \Big(\frac{n-1}{2} + p_{1,y_0} \Big) \Big) (x_{\mathscr{I}} D_{x_{\mathscr{I}}} - 2\rho_+ D_{\rho_+}) + (x_{\mathscr{I}} D_y)^2 + p^+_{0,y_0} \Big),$$

on ${}^+N\mathscr{I}^+_{y_0}$ near $\rho_+ = 0$ (cf. (3-18)), i.e., on the domain[28]

$$\mathcal{N} := [0,1)_{\rho_+} \times [0,\infty)_{x_{\mathscr{I}}} \times \mathbb{R}^{n-1}_y$$

[27] We write $P_{y_0} := {}^{\mathrm{e}}N_{\mathscr{I}^+,y_0}(P)$ for brevity.

[28] We are committing a minor abuse of notation here by writing $x_{\mathscr{I}}$ for a fiber-linear coordinate on ${}^+N\mathscr{I}^+_{y_0}$ whose differential at the zero section we take to agree with the differential of the coordinate function on M denoted by $x_{\mathscr{I}}$ above.

and acting on sections of the trivial bundle E_{y_0}; thus $p_{j,y_0} \in (\mathcal{C}^\infty + \mathcal{A}^{\ell_+})([0,1)_{\rho_+}; \mathrm{End}(E_{y_0}))$ for $j = 0, 1$. Here, we made a linear change of the y-coordinates so that $\partial_{y^1}, \dots, \partial_{y^{n-1}}$ is an orthonormal basis for the metric $\frac{1}{2}k(y_0) \in S^2 T^*_{y_0} Y$. Denote the underlying metric by

$$g_{y_0} := 2x_{\mathscr{I}}^{-2}\rho_+^{-2}\left(-2\frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}} \otimes_s \frac{\mathrm{d}\rho_+}{\rho_+} - \frac{\mathrm{d}\rho_+^2}{\rho_+^2} + \frac{\mathrm{d}y^2}{x_{\mathscr{I}}^2}\right),$$
$$g_{y_0}^{-1} = \tfrac{1}{2}x_{\mathscr{I}}^2\rho_+^2(x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} \otimes_s (x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} - 2\rho_+\partial_{\rho_+}) + (x_{\mathscr{I}}\partial_y)^2).$$

On $\mathcal{N}$, we moreover fix the volume density

$$|\mathrm{d}g_{y_0}| = 2x_{\mathscr{I}}^{-2n}\rho_+^{-n-1}\left|\frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}}\frac{\mathrm{d}\rho_+}{\rho_+}\,\mathrm{d}y\right|;$$

cf. (4-16).

Since $P_{y_0} \in \mathrm{Diff}^2_{\mathrm{e,b},I}(\mathcal{N})$ — i.e., it is invariant under dilations in $(x_{\mathscr{I}}, y)$ and translations in y — its analysis will utilize the invariant edge-b-notions (Sobolev spaces, ps.d.o.s, wave front sets) introduced in Section 2.5. For $\rho \in (0,1)$, let

$$\Omega_\rho := \{\rho_+ < \rho\} \subset \mathcal{N}.$$

Proposition 7.1 (invertibility of the edge normal operator near I^+). *Assume that P is special admissible in the sense of Definition 6.2.*

(1) (forward problem) *Let $s, \alpha_{\mathscr{I}}, \alpha_+ \in \mathbb{R}$, and put $\alpha = (2\alpha_{\mathscr{I}}, \alpha_+)$ and $\alpha' = \alpha + (2,2)$. Suppose that*

$$\alpha_+ + \tfrac{1}{2} < \alpha_{\mathscr{I}} < -\tfrac{1}{2} + \underline{p}_1. \tag{7-1}$$

Let $u \in \dot{H}^{s,\alpha}_{\mathrm{e,b},I}(\Omega_\rho)$ be such that $f := P_{y_0}u \in \dot{H}^{s-1,\alpha'}_{\mathrm{e,b},I}(\Omega_\rho)$. Then

$$\|u\|_{\dot{H}^{s,\alpha}_{\mathrm{e,b},I}(\Omega_\rho)} \leq C\|f\|_{\dot{H}^{s-1,\alpha'}_{\mathrm{e,b},I}(\Omega_\rho)}, \tag{7-2}$$

where C only depends on P, s, $\alpha_{\mathscr{I}}$, α_+, ρ. Moreover, for any $f \in \dot{H}^{s-1,\alpha'}_{\mathrm{e,b},I}(\Omega_\rho)$, there exists u satisfying the equation $P_{y_0}u = f$ and the estimate (7-2); this u is unique in the sense that any other solution $u' \in \dot{H}^{-\infty,\alpha}_{\mathrm{e,b},I}(\Omega_\rho)$ of $P_{y_0}u' = f$ is equal to u.

(2) (backward problem) *Let $\tilde{s}, \tilde{\alpha}_{\mathscr{I}}, \tilde{\alpha}_+ \in \mathbb{R}$, and put $\tilde{\alpha} = (2\tilde{\alpha}_{\mathscr{I}}, \tilde{\alpha}_+)$ and $\tilde{\alpha}' = \tilde{\alpha} + (2,2)$. Suppose that*

$$-\tfrac{1}{2} - \underline{p}_1 < \tilde{\alpha}_{\mathscr{I}} < \tilde{\alpha}_+ + \tfrac{1}{2}. \tag{7-3}$$

Let $\tilde{u} \in \overline{H}^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b},I}(\Omega_\rho)$ be such that $\tilde{f} := P^*_{y_0}\tilde{u} \in \overline{H}^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b},I}(\Omega_\rho)$. Then

$$\|\tilde{u}\|_{\overline{H}^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b},I}(\Omega_\rho)} \leq C\|\tilde{f}\|_{\overline{H}^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b},I}(\Omega_\rho)}, \tag{7-4}$$

where C only depends on P, $\tilde{s}$, $\tilde{\alpha}_{\mathscr{I}}$, $\tilde{\alpha}_+$. Moreover, for any $\tilde{f} \in \overline{H}^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b},I}(\Omega_\rho)$, there exists $\tilde{u}$ which satisfies the equation $P^*_{y_0}\tilde{u} = \tilde{f}$ and the estimate (7-4); this $\tilde{u}$ is unique in the sense that any solution $\tilde{u}' \in \overline{H}^{-\infty,\tilde{\alpha}}_{\mathrm{e,b},I}(\Omega_\rho)$ of $P^*_{y_0}\tilde{u}' = \tilde{f}$ is equal to $\tilde{u}$.

Proof. As in the proof of Theorem 6.4, the main computation concerns the positivity of a modified (by the subprincipal terms of P) K-current associated with a weighted future timelike vector field; we use here a weighted version of the convex combination $\big(1+\frac{1}{2}c\big)(x_{\mathscr{I}}\partial_{x_{\mathscr{I}}}-2\rho_+\partial_{\rho_+})+\frac{1}{2}c\cdot(-x_{\mathscr{I}}\partial_{x_{\mathscr{I}}})$ of the outgoing and incoming future null vector fields over $\mathscr{I}^+$; see (7-5). The positivity of $\widetilde{K}$ is discussed in and after (7-7).

• *Restricted a priori estimate for P_{y_0}, $s=1$; solvability for $P_{y_0}^*$, $\tilde{s}=0$.* Upon imposing slightly stronger conditions on the weights (see (7-8) and (7-9) below), this follows from a variant of the proof of Theorem 6.4. Thus, we first prove part (1) for $s=1$ using an energy estimate, based on

$$2\operatorname{Im}\langle P_{y_0}u, Zu\rangle = \langle \mathscr{C}u, u\rangle, \quad \mathscr{C} := i(P_{y_0}^*Z - Z^*P_{y_0}).$$

Here, we let $\breve{\alpha}=(2\breve{\alpha}_{\mathscr{I}},\breve{\alpha}_+)=(2\alpha_{\mathscr{I}},\alpha_+)+(1,1)$ and define

$$W := x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} - (2+c)\rho_+\partial_{\rho_+}, \quad V := x_{\mathscr{I}}^{-4\breve{\alpha}_{\mathscr{I}}}\rho_+^{-2\breve{\alpha}_+}W, \quad Z := i^{-1}V; \tag{7-5}$$

note that, by (3-16), W is future timelike for $c>0$. Define

$$\begin{aligned}
\widetilde{Q} &:= x_{\mathscr{I}}^2\rho_+^2(x_{\mathscr{I}}D_{x_{\mathscr{I}}} - 2\rho_+D_{\rho_+}) = \widetilde{Q}^*,\\
\widetilde{K}_V &:= K_V + (p_{1,y_0}+p_{1,y_0}^*)\sigma^1(\widetilde{Q})\otimes_s\sigma^1(i^{-1}V),\\
\widetilde{\mathscr{C}}' &:= -\tfrac{1}{2}(\widetilde{Q}(p_{1,y_0}-p_{1,y_0}^*)Z - Z^*(p_{1,y_0}-p_{1,y_0}^*)\widetilde{Q}) + \frac{i}{2}\rho_0^2x_{\mathscr{I}}^2((p_{0,y_0}^0)^*Z - Z^*p_{0,y_0}^0)\\
&\in x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}}\rho_+^{-2\alpha_+}(\operatorname{Diff}^1_{\mathrm{e,b},I}(\mathcal{N};E_{y_0}) + \mathcal{A}^{\ell_+}(Z)\operatorname{Diff}^1_{\mathrm{e,b},I}(\mathcal{N};E_{y_0})).
\end{aligned} \tag{7-6}$$

Here, K_V is given by (6-11) for the metric $g=g_{y_0}$. We then have

$$\mathscr{C} = (\nabla^E)^*\widetilde{K}_{iZ}\nabla^E + \widetilde{\mathscr{C}}'.$$

Indeed, this is proved like (6-14), except in the present invariant setting, there are no terms which are of lower order in the sense of decay at $x_{\mathscr{I}}=0$. (The various sign changes compared to (6-14) and (6-16) are due to the difference of signs in front of the first terms in (3-18).) We compute

$$\begin{aligned}
\widetilde{K}_{iZ} = x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}}\rho_+^{-2\alpha_+}\Big(&\Big(-2\breve{\alpha}_{\mathscr{I}} + (p_{1,y_0}+p_{1,y_0}^*) + \frac{c}{2}(-n+1-2\breve{\alpha}_+)\Big)(x_{\mathscr{I}}\partial_{x_{\mathscr{I}}})^2\\
&+\Big(4\breve{\alpha}_{\mathscr{I}} - 2(p_{1,y_0}+p_{1,y_0}^*) + \frac{c}{2}(n-1+4\breve{\alpha}_{\mathscr{I}} - (p_{1,y_0}+p_{1,y_0}^*))\Big)2x_{\mathscr{I}}\partial_{x_{\mathscr{I}}}\otimes_s\rho_+\partial_{\rho_+}\\
&+(2+c)(-4\breve{\alpha}_{\mathscr{I}} + 2(p_{1,y_0}+p_{1,y_0}^*))(\rho_+\partial_{\rho_+})^2\\
&+(-2+4\breve{\alpha}_{\mathscr{I}} - 4\breve{\alpha}_+ + c(-n+1-2\breve{\alpha}_+))(x_{\mathscr{I}}\partial_y)^2\Big).
\end{aligned} \tag{7-7}$$

In the basis $\mathrm{d}x_{\mathscr{I}}/x_{\mathscr{I}}$, $\mathrm{d}\rho_+/\rho_+$, $\mathrm{d}y/x_{\mathscr{I}}$ and restricting to an eigenspace of $p_{1,y_0}+p_{1,y_0}^*$ with eigenvalue 2λ (satisfying $2\lambda > 2\underline{p}_1 - 2\epsilon$), the trace and determinant of the 2×2 minor of $x_{\mathscr{I}}^{4\breve{\alpha}_{\mathscr{I}}}\rho_+^{2\alpha_+}\widetilde{K}_{iZ}$ are equal to $10(-\breve{\alpha}_{\mathscr{I}}+\lambda)+\mathcal{O}(c)$ and $8(-\breve{\alpha}_{\mathscr{I}}+\lambda)(\breve{\alpha}_{\mathscr{I}}-\breve{\alpha}_+)c+\mathcal{O}(c^2)$, respectively, and therefore positive for $\breve{\alpha}_{\mathscr{I}} < \underline{p}_1$ (provided $\epsilon>0$ is sufficiently small and we choose the fiber inner product on E_{y_0} near-optimally) and $\breve{\alpha}_+ < \breve{\alpha}_{\mathscr{I}}$ (these are precisely the conditions (7-1)), and then $c>0$ sufficiently small. (The term $\widetilde{\mathscr{C}}'$

in (7-6) can at the same time be made arbitrarily small by assumption on p_{1,y_0}.) However, the $(x_\mathscr{I}\partial_y)^2$ term has a positive coefficient (for c near 0) only under the stronger condition $\breve{\alpha}_+ < -\frac{1}{2} + \breve{\alpha}_\mathscr{I}$, i.e.,

$$\alpha_+ + 1 < \alpha_\mathscr{I}. \tag{7-8}$$

Let us write $\nabla^E_{\mathrm{e,b},I} u = (\nabla^E_{V_j} u)_{j=1,\dots,N}$, where $V_1, \dots, V_N$ spans $\mathcal{V}_{\mathrm{e,b},I}(\mathcal{N})$ over $\mathcal{C}^\infty_I(\mathcal{N})$. An application of Cauchy–Schwarz and the invariant Poincaré inequality

$$\|u\|_{\dot{H}^{0,\alpha}_{\mathrm{e,b},I}} \leq C' \|\nabla^E_{\mathrm{e,b},I} u\|_{\dot{H}^{0,\alpha}_{\mathrm{e,b},I}}$$

(proved via (6-20) with $V_0 := x_\mathscr{I}^{-4\alpha_\mathscr{I}} \rho_+^{-2\alpha_+}(x_\mathscr{I}\partial_{x_\mathscr{I}} - 2\rho_+\partial_{\rho_+})$ and using only the assumption that $\breve{\alpha}_+ < \breve{\alpha}_\mathscr{I}$, i.e., $\alpha_+ + \frac{1}{2} < \alpha_\mathscr{I}$), as in the proof of Theorem 6.4, then proves the a priori estimate (7-2) for $s = 1$. By duality, defining $\tilde{\alpha} = -\alpha'$ and $\tilde{s} = -(s-1)$ and under the strengthened assumption

$$\tilde{\alpha}_\mathscr{I} < \tilde{\alpha}_+ \tag{7-9}$$

(which is equivalent to (7-8)), we obtain the solvability of $P^*_{y_0}\tilde{u} = \tilde{f}$ with the estimate (7-4) for $\tilde{s} = 0$.

• *A priori estimate for* $P^*_{y_0}$, $\tilde{s} = 1$; *solvability for* P_{y_0}, $s = 0$. One can also prove an energy estimate for the adjoint problem, with extendible distributions at $\rho_+ = \rho$ (which is a null hypersurface): with W as before, $Z = i^{-1}V$, and $V = x_\mathscr{I}^{-4\breve{\tilde{\alpha}}_\mathscr{I}} \rho_+^{-2\breve{\tilde{\alpha}}_+} W$, where $(2\breve{\tilde{\alpha}}_\mathscr{I}, \breve{\tilde{\alpha}}_+) = (2\tilde{\alpha}_\mathscr{I}, \tilde{\alpha}_+) + (1, 1)$, one now uses $1_{\Omega_\rho} Z$ as a multiplier. The symmetric 2-tensor $\widetilde{K}_{iZ}$ (with p^0_{0,y_0}, p_{1,y_0}, p^*_{1,y_0} replaced by $(p^0_{0,y_0})^*$, $-p^*_{1,y_0}$, $-p_{1,y_0}$, respectively) is then *negative* definite under the assumptions $\tilde{\alpha}_\mathscr{I} > -\underline{p}_1$ and $\breve{\tilde{\alpha}}_+ > \breve{\tilde{\alpha}}_\mathscr{I}$; the condition for the negativity of the $(x_\mathscr{I}\partial_y)^2$ term of $\widetilde{K}_{iZ}$ now reads $\breve{\tilde{\alpha}}_+ > -\frac{1}{2} + \breve{\tilde{\alpha}}_\mathscr{I}$, which now is automatic. The contribution from differentiation of the cutoff 1_{Ω_ρ} at $\rho_+ = \rho$ has the same (negative) sign since $\nabla 1_{\Omega_\rho}$ (resp. W) is past (resp. future) causal, and hence can be discarded. Thus, we obtain the a priori estimate (7-4) under the assumptions (7-3).

By duality, this implies the solvability of $P_{y_0}u = f$, as claimed in part (1), for $s = 0$ and the full range of weights, together with the estimate (7-2).

• *A priori estimate for* $P^*_{y_0}$, $\tilde{s} \geq 1$; *solvability for* P_{y_0}, $s \leq 0$. Let $\tilde{s} \geq 1$. If we are given $\tilde{u} \in \overline{H}^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b},I}(\Omega_\rho)$ with $\tilde{f} = P^*_{y_0}\tilde{u} \in \overline{H}^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b},I}(\Omega_\rho)$, then we have the a priori estimate (7-4) for $\tilde{s} = 1$. We will now let $0 < \rho' < \rho$. We claim that

$$\|\tilde{u}\|_{\overline{H}^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b},I}(\Omega_{\rho'})} \leq C(\|\tilde{f}\|_{\overline{H}^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b},I}(\Omega_\rho)} + \|\tilde{u}\|_{\overline{H}^{1,\tilde{\alpha}}_{\mathrm{e,b},I}(\Omega_\rho)}). \tag{7-10}$$

Indeed, this follows from variants of the microlocal regularity results in Lemmas 4.10(2) and 4.11(4) in which wave front sets and function spaces are replaced by the invariant versions $\mathrm{WF}_{\mathrm{e,b},I}$ and $H_{\mathrm{e,b},I}$, and all localizations to neighborhoods of $x_\mathscr{I} = 0$ are omitted;[29] the proofs of these variants are the same upon removing cutoffs in $x_\mathscr{I}$ and working with symbols and operators which are homogeneous with respect to

[29] For example, the invariant version of Lemma 4.10(2) reads as follows. Suppose that $\tilde{\alpha}_\mathscr{I} < \tilde{\alpha}_+ + \frac{1}{2}$ and $\mathrm{WF}^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b},I}(\tilde{f}) \cap \mathcal{R}_{\mathrm{in},+,I} = \varnothing$, where, in the coordinates on $\mathcal{N}$, we set $\mathcal{R}_{\mathrm{in},+,I} = \{(\rho_+, x_\mathscr{I}, y; \zeta, \xi, \eta) : \rho_+ = 0, \xi = 2\zeta\}$ (similarly to (4-10) but without localization to $x_\mathscr{I} = 0$). If $\mathrm{WF}^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b},I}(\tilde{u}) \cap ({}^{\mathrm{e,b}}S^*_{\{\rho_+=0\}}M \setminus \mathcal{R}_{\mathrm{in},+,I}) = \varnothing$, then $\mathrm{WF}^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b},I}(\tilde{u}) \cap \mathcal{R}_{\mathrm{in},+,I} = \varnothing$.

dilations and invariant under translations on $\mathcal{N}$. Estimating the second term on the right in (7-10) using the already proved a priori estimate for $\tilde{s} = 1$, we obtain

$$\|\tilde{u}\|_{\overline{H}^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b},I}(\Omega_{\rho'})} \leq C\|\tilde{f}\|_{\overline{H}^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b},I}(\Omega_\rho)}. \tag{7-11}$$

It remains to obtain a quantitative estimate of $\tilde{u} \in \overline{H}^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b},I}(\Omega_\rho)$ on the full domain $\Omega_\rho \supsetneq \Omega_{\rho'}$. To this end, fix a cutoff $\phi \in \mathcal{C}^\infty(\mathbb{R})$ so that $\phi \equiv 0$ on $(-\infty, \rho']$ and $\phi \equiv 1$ on $[\rho, \infty)$. Then we have

$$\tilde{f}' := P^*_{y_0}(\phi\tilde{u}) = \phi\tilde{f} + [P^*_{y_0}, \phi]\tilde{u} \in \overline{H}^{\tilde{s}-1,(2\tilde{\alpha}'_{\mathscr{I}},N+2)}_{\mathrm{e,b},I}(\Omega_\rho)$$

for any N since ρ_+ has a positive lower bound on $\operatorname{supp}\tilde{f}'$. We will now extend $\tilde{f}'$ to an element $\tilde{f}'' \in \overline{H}^{\tilde{s}-1,(2\tilde{\alpha}'_{\mathscr{I}},N+2)}_{\mathrm{e,b},I}(\Omega_{\rho''})$, where $\rho'' > \rho$, with norm bounded by a fixed constant times the norm of $\tilde{f}'$. Using the solvability of the adjoint problem stated after (7-9) above, we see that, upon taking $N > \tilde{\alpha}_{\mathscr{I}}$, we can solve $P^*_{y_0}\tilde{u}'' = \tilde{f}''$ with $\tilde{u}'' \in \overline{H}^{0,(2\tilde{\alpha}_{\mathscr{I}},N)}_{\mathrm{e,b},I}(\Omega_{\rho''})$ obeying a norm bound

$$\|\tilde{u}''\|_{\overline{H}^{0,(2\tilde{\alpha}_{\mathscr{I}},N)}_{\mathrm{e,b},I}(\Omega_{\rho''})} \leq C\|\tilde{f}''\|_{\overline{H}^{-1,(2\tilde{\alpha}'_{\mathscr{I}},N+2)}_{\mathrm{e,b},I}(\Omega_{\rho''})} \leq C'\|\tilde{f}'\|_{\overline{H}^{\tilde{s}-1,(2\tilde{\alpha}'_{\mathscr{I}},N+2)}_{\mathrm{e,b},I}(\Omega_\rho)}.$$

But then higher regularity on the smaller domain $\Omega_\rho \subsetneq \Omega_{\rho''}$ follows as above, showing that $\tilde{u}''|_{\Omega_\rho} \in \overline{H}^{\tilde{s},(2\tilde{\alpha}_{\mathscr{I}},N)}_{\mathrm{e,b},I}(\Omega_\rho)$, with an estimate

$$\|\tilde{u}''|_{\Omega_\rho}\|_{\overline{H}^{\tilde{s},(2\tilde{\alpha}_{\mathscr{I}},N)}_{\mathrm{e,b},I}(\Omega_\rho)} \leq C''\|\tilde{f}'\|_{\overline{H}^{\tilde{s}-1,(2\tilde{\alpha}'_{\mathscr{I}},N+2)}_{\mathrm{e,b},I}(\Omega_\rho)} \leq C'''\|\tilde{f}\|_{\overline{H}^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b},I}(\Omega_\rho)}. \tag{7-12}$$

Finally then, we note that $\tilde{u}''|_{\Omega_\rho} = \phi\tilde{u}$, since the difference $w := \tilde{u}''|_{\Omega_\rho} - \phi\tilde{u} \in \overline{H}_{\mathrm{e,b},I}{}^{1,\tilde{\alpha}}(\Omega_\rho)$ satisfies $P^*_{y_0}w = \tilde{f}' - \tilde{f}' = 0$ and thus $w \equiv 0$ in view of the already proven a priori estimate (7-4) in the case $\tilde{s} = 1$. Thus, combining the estimates (7-11) and (7-12) gives the a priori estimate (7-4) for all $\tilde{s} \geq 1$.

By duality, we get the solvability statement in part (1) for all $s \leq 0$.

• *Solvability for P_{y_0}, $s \in \mathbb{R}$; a priori estimate for $P^*_{y_0}$, $\tilde{s} \in \mathbb{R}$.* It remains to prove solvability for $s > 0$. Given $f \in \dot{H}^{s-1,\alpha'}_{\mathrm{e,b},I}(\Omega_\rho)$, we have already proved that there exists a solution $u \in \dot{H}^{0,\alpha}_{\mathrm{e,b},I}(\Omega_\rho)$. But then the invariant versions of the microlocal regularity results, Lemmas 4.10(1) and 4.11(1), imply that in fact $u \in \dot{H}^{s,\alpha}_{\mathrm{e,b},I}(\Omega_\rho)$, together with a quantitative estimate for its norm in terms of the norm of f.

The a priori estimate (7-4) now follows by duality also for $\tilde{s} = -(s-1) < 1$, and hence is now proved for all $\tilde{s}$.

• *A priori estimate for P_{y_0}, $s \in \mathbb{R}$.* Given $u \in \dot{H}^{s,\alpha}_{\mathrm{e,b},I}(\Omega_\rho)$ with $f = P_{y_0}u \in \dot{H}^{s-1,\alpha'}_{\mathrm{e,b},I}(\Omega_\rho)$, we have already shown that the equation $P_{y_0}u' = f$ can be solved with $u' \in \dot{H}^{s,\alpha}_{\mathrm{e,b},I}(\Omega_\rho)$ obeying a bound

$$\|u'\|_{\dot{H}^{s,\alpha}_{\mathrm{e,b},I}(\Omega_\rho)} \leq C\|f\|_{\dot{H}^{s-1,\alpha'}_{\mathrm{e,b},I}(\Omega_\rho)}.$$

But then $P_{y_0}(u - u') = 0$. Propagation of invariant edge-b-regularity shows that $u - u'$ has infinite edge-b-regularity; relaxing the weight at $\rho_+ = 0$, we can thus a fortiori regard $u - u' \in \dot{H}^{1,(2\alpha_{\mathscr{I}},N)}_{\mathrm{e,b},I}(\Omega_\rho)$ for $N < \min(\alpha_+, \alpha_{\mathscr{I}} - 1)$. But then the energy estimate proved in the first step of the proof applies (as $N + 1 < \alpha_{\mathscr{I}}$, cf. (7-8)) and implies that $u - u' \equiv 0$. Therefore, we have $u = u'$, and the proof is complete. □

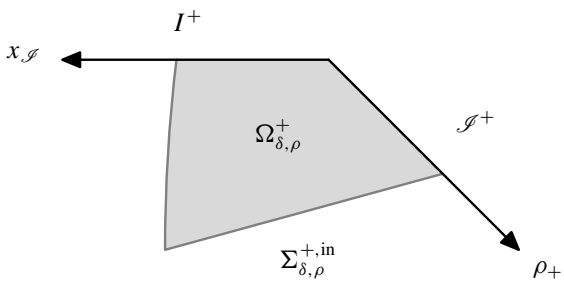

Figure 7.1. The domain $\Omega^+_{\delta,\rho}$ on which Theorem 7.3 takes place and one of its boundary hypersurfaces $\Sigma^{+,\mathrm{in}}_{\delta,\rho}$.

Remark 7.2 (alternative approach). The fact that one can directly prove an energy estimate for $P^*_{y_0}$ in the full range (7-3) of weights allowed in the microlocal propagation estimates is somewhat fortuitous. A conceptually cleaner approach which does not take advantage of this fact starts with energy estimates with sharp weights at $x_{\mathscr{I}} = 0$ but with very weak weights at $\rho_+ = 0$ (i.e., very negative for P_{y_0}, very positive for $P^*_{y_0}$), which can then be improved by exploiting the invertibility properties of the normal operator of P_{y_0} at $\rho_+ = 0$. We leave the implementation of this approach, which can be based on the (semiclassical) 0-analysis in the next section, to the interested reader.

7.2. ***Leading-order control at*** $\mathscr{I}^+$**.** Using Proposition 7.1, we now show how to control solutions of $Pu = f$ or $P^*\tilde{u} = \tilde{f}$ near $\mathscr{I}^+$ to leading order in the sense of *decay*; this complements the regularity results of Section 4. Since we already have full control near $\mathscr{I}^+ \setminus I^+$ by Theorem 6.4, we focus on a neighborhood of $\mathscr{I}^+ \cap I^+$. Fix a compact subset $K \subset \mathscr{I}^+ \setminus I^0$, and choose $T \in \mathbb{R}$ so that $\mathcal{U}_+(T) \supset K$. Using the coordinates ρ_+, $x_{\mathscr{I}}$ on $\mathcal{U}_+(T)$ as in (3-6), we can fix $\delta_0 > 0$ small and $\underline{\rho}_+ < \bar{\rho}_+ \in (0, 1)$ so that

$$\Omega^+_{\delta,\rho} := \{x_{\mathscr{I}} < \delta,\, \rho_+ < \rho\} \subset \mathcal{U}_+(T) \subset M$$

contains K when $0 < \delta \leq \delta_0$ and $\rho \in [\underline{\rho}_+, \bar{\rho}_+]$. We do not arrange for the boundary hypersurfaces of $\Omega^+_{\delta,\rho}$ to have any particular causal structure, as questions of regularity and decay near I^+ have a global character (i.e., answering them requires information about the operator P far from $\mathscr{I}^+$); we can therefore only prove a priori estimates here. As for function spaces, we fix

$$\delta \in (0, \delta_0), \qquad \rho \in (\underline{\rho}_+, \bar{\rho}_+)$$

and weights $\alpha = (2\alpha_{\mathscr{I}}, \alpha_+)$, and work with the space

$$\overline{H}^{s,\alpha}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})$$

of extendible distributions and the space

$$H^{s,\alpha}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})^{\bullet,-} := \{u|_{\Omega^+_{\delta,\rho}} : u \in \overline{H}^{s,\alpha}_{\mathrm{e,b}}(\Omega^+_{\delta_0,\bar{\rho}_+}) : \operatorname{supp} u \subset \overline{\Omega^+_{\delta_0,\rho}}\}$$

of distributions with supported character at $\Sigma^{+,\mathrm{in}}_{\delta,\rho} = \{x_{\mathscr{I}} < \delta,\, \rho_+ = \rho\}$. See Figure 7.1.

Theorem 7.3 (a priori estimate for P near $\mathscr{I}^+$). *Under the same assumptions on P, s, $\alpha = (2\alpha_{\mathscr{I}}, \alpha_+)$, $\tilde{s}$, and $\tilde{\alpha} = (2\tilde{\alpha}_{\mathscr{I}}, \tilde{\alpha}_+)$ as in Proposition 7.1, and defining $\alpha' = \alpha + (2,2)$, $\tilde{\alpha}' = \tilde{\alpha} + (2,2)$, the following estimates hold for some constant C.*

(1) (forward problem) *For all $\epsilon > 0$, there exists C_ϵ such that*

$$\|u\|_{H^{s,\alpha}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})^{\bullet,-}} \leq C\|Pu\|_{H^{s-1,\alpha'}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})^{\bullet,-}} + C_\epsilon \|u\|_{H^{s+1,(\alpha_{\mathscr{I}}-2\ell_{\mathscr{I}},\alpha_+)}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})^{\bullet,-}} + \epsilon\|u\|_{H^{s+1,\alpha}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})^{\bullet,-}} \tag{7-13}$$

for all u for which all norms are finite.

(2) (backward problem) *For all $\epsilon > 0$, there exists C_ϵ such that*

$$\|\tilde{u}\|_{\overline{H}^{\tilde{s},\tilde{\alpha}}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})} \leq C\|P^*\tilde{u}\|_{\overline{H}^{\tilde{s}-1,\tilde{\alpha}'}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})} + C_\epsilon \|\tilde{u}\|_{\overline{H}^{\tilde{s}+1,\tilde{\alpha}-(2\ell_{\mathscr{I}},0)}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})} + \epsilon\|\tilde{u}\|_{\overline{H}^{\tilde{s}+1,\tilde{\alpha}}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})} \tag{7-14}$$

for all $\tilde{u}$ for which all norms are finite.

Thus, if one ignores the edge-b-differential order, the right-hand sides of (7-13) and (7-14) involve norms on u and $\tilde{u}$ which either feature a weaker weight at $\mathscr{I}^+$ or have a small prefactor ϵ. The loss in the edge-b-differential order is acceptable in applications, since regularity is already controlled by the results in Section 4 (if supplemented by propagation estimates that are global near future timelike infinity and thus not covered by our $\mathscr{I}^+$-local theory here). That is, the estimate (7-13) will be applied for a very weak order s, and we will then be able to absorb constant multiples of $\epsilon\|u\|_{H^{s+1,\alpha}_{\mathrm{e,b}}}$; see the discussion following (9-5) for an example. This loss arises from the fact that the edge normal operators of P are nonelliptic and lose one order of regularity upon inversion, whereas the difference of P and any of its normal operators is typically a *second* order operator still.

While the second term on the right in (7-13) has a large constant C_ϵ when $\epsilon > 0$ is small (as required for absorbing the third term), its gain is the improvement (i.e., weakening) of the $\mathscr{I}^+$-weight to $\alpha_{\mathscr{I}} - 2\ell_{\mathscr{I}} < \alpha_{\mathscr{I}}$. This improves the compactness properties of the error term and ultimately can be used to prove the Fredholm property of P on suitable spaces; see (9-2) and the discussion following (9-6).

Remark 7.4 (better error term under stronger assumptions). If $s \geq 1$ and $\alpha_+ + 1 < \alpha_{\mathscr{I}}$, we may replace u and $s+1$ in the final term on the right-hand side of (7-13) by $(1-\chi)u$ and s, respectively, where χ is identically 1 near $\mathscr{I}^+$ and supported in $x_{\mathscr{I}} < \delta'$ for sufficiently small $\delta' \in (0, \delta)$. This follows by writing $u = \chi u + (1-\chi)u$ and using an energy estimate to estimate χu in terms of $P(\chi u) = \chi Pu + [P, \chi]u \in \dot{H}^{s-1,\alpha'}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})$ using the multiplier vector field Z from (7-5); see also (7-8).

The idea behind the proof of Theorem 7.3 is to split u into a sum of pieces localized near individual fibers $\mathscr{I}^+_{y_0}$ of $\mathscr{I}^+$, use the estimates for P_{y_0} provided by Proposition 7.1, and estimate the error terms arising from $P - P_{y_0}$ and commutators of P with the localizers. In order to implement this, we need to relate norms on edge-b-Sobolev spaces to norms of localizations.[30] We first do this in a collar neighborhood

[30] We expect that one can drop the $\mathcal{O}(\epsilon)$ error terms in (7-13) and (7-14) by combining the normal operator inverses controlled by Proposition 7.1 into a single operator on the edge-b-double space. This would however require analyzing the regularity of $P_{y_0}^{-1}$ in the parameter y_0. We do not pursue this further, as the $\mathcal{O}(\epsilon)$ error terms are easily absorbed in applications; see the discussion after (9-5) or [Hintz 2023d, Step (i) in the proof of Proposition 5.19].

$[0,1)_{x_\mathscr{I}} \times \mathscr{I}^+$ of all of $\mathscr{I}^+$; we set

$$\Omega := \big[0, \tfrac{1}{2}\big)_{x_\mathscr{I}} \times \mathscr{I}^+$$

and shall only consider distributions with support in $\overline{\Omega}$. Denote the projection $[0,1)_{x_\mathscr{I}} \times \mathscr{I}^+ \to \mathscr{I}^+$ by π. Cover the (compact) base Y of the fibration $\phi : \mathscr{I}^+ \to Y$ by a finite number of coordinate charts $\psi_1, \dots, \psi_N : B(0,2) \to Y$ (where $B(0,R) \subset \mathbb{R}^{n-1}$ is the open R-ball) so that $Y = \bigcup_{i=1}^N \psi_i(B(0,1))$; let $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(B(0,2))$ be identically 1 on $\overline{B(0,1)}$, and define $\chi_i \in \mathcal{C}^\infty(M)$ to be equal to $\pi^*\phi^*(\chi \circ \psi_i^{-1})$ on $\pi^{-1}\big(\phi^{-1}(\psi_i(B(0,2)))\big)$ and 0 otherwise. Since $\sum_{i=1}^N \chi_i$ has a positive lower bound in the collar neighborhood, we immediately obtain the following.

Lemma 7.5 (localization to coordinate patches on Y). *Let $s \in \mathbb{R}$ and $\alpha \in \mathbb{R}^3$. Then there exists a constant $C > 1$ such that, for all distributions u with* $\operatorname{supp} u \subset \overline{\Omega}$,

$$C^{-1} \sum_{i=1}^N \|\chi_i u\|_{H^{s,\alpha}_{\mathrm{e,b}}(M)} \leq \|u\|_{H^{s,\alpha}_{\mathrm{e,b}}(M)} \leq C \sum_{i=1}^N \|\chi_i u\|_{H^{s,\alpha}_{\mathrm{e,b}}(M)}.$$

Passing to local coordinates $y \in B(0,2) \subset \mathbb{R}^{n-1}$ on $\psi_i(B(0,2)) \subset Y$, we now work in

$$\Omega_i := \big[0, \tfrac{1}{2}\big)_{x_\mathscr{I}} \times B(0,2) \times Z \subset [0,1)_{x_\mathscr{I}} \times \mathbb{R}^{n-1}_y \times Z,$$

where $Z \cong [-1,1]$ is the typical fiber of $\mathscr{I}^+$. Since $\overline{\Omega_i}$ is compact, edge-b-Sobolev spaces of distributions on $[0,1) \times \mathbb{R}^{n-1} \times Z$ with support in $\overline{\Omega_i}$ are well-defined; here, denoting a smooth positive b-density on Z by μ_Z, we fix the volume density

$$\mu := \rho_0^{-n-1} x_\mathscr{I}^{-2n} \rho_+^{-n-1} \left| \frac{\mathrm{d}x_\mathscr{I}}{x_\mathscr{I}} \, \mathrm{d}y \mu_Z \right| \tag{7-15}$$

on Ω_i, mirroring the earlier choice (4-16). For $j \in \mathbb{N}$, put

$$I_j := \{k 2^{-j} : k \in \mathbb{Z},\ -3 \cdot 2^j \leq k < 3 \cdot 2^j\},$$

and denote the $K(j) = (6 \cdot 2^j)^{n-1}$ points in $I_j^{n-1} \subset \mathbb{R}^{n-1}$ by

$$y_{j,1}, \dots, y_{j,K(j)}.$$

Fix cutoff functions

$$\chi_0 \in \mathcal{C}^\infty_{\mathrm{c}}([0,1)), \quad \chi_0 = 1 \text{ on } \big[0, \tfrac{1}{2}\big], \quad \tilde{\chi}_1 \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}), \quad \tilde{\chi}_1 = 1 \text{ on } [-1,1],$$

and define $\chi_1 \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^{n-1})$ by $\chi_1(y) = \tilde{\chi}_1(y_1) \cdots \tilde{\chi}_1(y_{n-1})$. For $j \in \mathbb{N}$ and $1 \leq k \leq K(j)$, we then set

$$\chi_{j,k}(y) := \chi_0(2^j x_\mathscr{I}) \chi_1(2^j(y - y_{j,k})), \quad \chi_j^{\mathrm{tot}}(y) := \sum_{k=1}^{K(j)} \chi_{j,k}(y).$$

Thus, $\chi_{j,k}$ localizes to a size $\sim 2^{-j}$ neighborhood of $\{0\} \times \{y_{j,k}\} \times Z \subset \Omega_i$, and χ_j^{tot} localizes to a size $\sim 2^{-j}$ neighborhood of $\{0\} \times [-3,3]^{n-1} \times Z$.

Lemma 7.6 (properties of $\chi_{j,k}$ and edge-b-Sobolev norms). *Let $s \in \mathbb{R}$. There exists a constant C — only depending on s, the choice of the cutoffs χ_0, $\tilde{\chi}_1$, and the volume density (7-15) — such that the following statements hold for all $j \in \mathbb{N}$:*

(1) *For $u \in \dot{H}^s_{\mathrm{e,b}}(\Omega_i)$ with $2^j x_{\mathscr{I}} < \frac{1}{2}$ on $\operatorname{supp} u$, we have*

$$C^{-1} \sum_{k=1}^{K(j)} \|\chi_{j,k} u\|^2_{H^s_{\mathrm{e,b}}} \le \|u\|^2_{H^s_{\mathrm{e,b}}} \le C \sum_{k=1}^{K(j)} \|\chi_{j,k} u\|^2_{H^s_{\mathrm{e,b}}}. \tag{7-16}$$

(2) *Let $f \in \mathcal{C}^\infty_{\mathrm{c}}([0,\infty) \times \mathbb{R}^{n-1})$. Then*

$$\|f(2^j x_{\mathscr{I}}, 2^j (y - y_{j,k}))u\|_{H^s_{\mathrm{e,b}}} \le C \|u\|_{H^s_{\mathrm{e,b}}}.$$

Analogous estimates hold for weighted edge-b-Sobolev spaces, as well as for spaces consisting of distributions with supported or extendible character on $[0,\delta)_{x_{\mathscr{I}}} \times B(0,2) \times [0,\rho)_{\rho_+} \subset \Omega_i$.

Proof. We note the following three key properties of $\chi_{j,k}$: there exist constants C, L, and C_m, $m \in \mathbb{N}_0$, all independent of j, such that

(1) for all $m \in \mathbb{N}_0$, we have

$$\sup_{l+|\gamma| \le m} |(x_{\mathscr{I}} D_{x_{\mathscr{I}}})^l (x_{\mathscr{I}} D_y)^\gamma \chi_{j,k}| \le C_m; \tag{7-17}$$

(2) whenever $k_1, \dots, k_L \in \{1, \dots, K(j)\}$ are distinct, we have $\bigcap_{l=1}^L \operatorname{supp} \chi_1(2^j(y - y_{j,k_l})) = \varnothing$;

(3) $C^{-1} \le \chi_j^{\mathrm{tot}}(y) \le C$ for $y \in B(0,2) \subset \mathbb{R}^{n-1}$.

This implies uniform bounds

$$\sum_{l+|\gamma| \le m} \left| (x_{\mathscr{I}} D_{x_{\mathscr{I}}})^l (x_{\mathscr{I}} D_y)^\gamma \left(\frac{\chi_0(2^j x_{\mathscr{I}})}{\chi_j^{\mathrm{tot}}(y)} \right) \right| \le C'_m, \quad m \in \mathbb{N}_0.$$

Thus, multiplication by $\chi_0(2^j x_{\mathscr{I}})/\chi_j^{\mathrm{tot}}(y)$ is a uniformly bounded map on the space of $u \in \dot{H}^s_{\mathrm{e,b}}(\Omega_i)$ with $\operatorname{supp} u \subset \{2^j x_{\mathscr{I}} < \frac{1}{2}\}$: for $s \in \mathbb{N}_0$, this follows from the Leibniz rule, and then for general $s \in \mathbb{R}$ by duality and interpolation. The proof of part (2) of the lemma is completely analogous.

Turning to part (1), consider first the case $s = 0$. Then

$$\begin{aligned} \|u\|^2_{L^2} &= \|\chi_0(2^j x_{\mathscr{I}})u\|^2_{L^2} \\ &= \left\| \frac{\chi_0(2^j x_{\mathscr{I}})}{\chi_j^{\mathrm{tot}}(y)} \sum_{k=1}^{K(j)} \chi_{j,k}(y) u(x_{\mathscr{I}}, y) \right\|^2_{L^2} \le C \operatorname{Re} \sum_{k,k'=1}^{K(j)} \langle \chi_{j,k} u, \chi_{j,k'} u \rangle_{L^2}. \end{aligned}$$

But for each k, there are at most $L-1$ values of k' besides $k' = k$ for which the inner product is nonzero; thus, we can apply Cauchy–Schwarz and bound the right-hand side by $C(1 + (L-1)) \sum_{k=1}^{K(j)} \|\chi_{j,k} u\|^2_{L^2}$. More generally, for $s \in \mathbb{N}_0$, similar arguments together with the Leibniz rule imply $\|u\|^2_{H^s_{\mathrm{e,b}}} \le C \sum \|\chi_{j,k} u\|^2_{H^s_{\mathrm{e,b}}}$, with C depending on s but not on j; interpolation then implies this inequality for all real $s \ge 0$.

The converse estimate for $s=0$ follows from the j-independent bound $\sum_{k=1}^{K(j)}\chi_{j,k}^2\leq C'$, which implies

$$\sum_{k=1}^{K(j)}\int\chi_{j,k}^2|u|^2\,\mathrm{d}\mu\leq C'\int|u|^2\,\mathrm{d}\mu=C'\|u\|_{L^2}^2.$$

For $s\in\mathbb{N}$ one uses the Leibniz rule, and for real $s\geq 0$ interpolation. Having proved (7-16) for real $s\geq 0$, the result for $s<0$ follows by duality. □

Proof of Theorem 7.3. We use the edge normal operator in a spirit somewhat similar to how the b-normal operator is used for example in [Hintz and Vasy 2015, §2.1.2], namely by means of estimating the localization of u in terms of the edge normal operators of P, and bounding the error terms arising from localization and passing between P and its normal operators. Rather than assembling the inverses of all edge normal operators[31] into a single object — which in our nonelliptic setting would be technically rather delicate (unlike in the elliptic setting [Mazzeo 1991]) — we only use estimates for the edge normal operators at an ϵ-dense collection of fibers of $\mathscr{I}^+$, with the estimates applied to a localization of u to ϵ-neighborhoods of individual fibers. Thus, the final error term in (7-13) bounds the difference between P and this finite collection of normal operators. An important point is that the commutators of (the edge normal operators of) P with cutoffs to such ϵ-neighborhoods gain a power of x. The relevant local coordinate calculation is $[x\partial_y,\chi(y/\epsilon)]=\epsilon^{-1}x\chi'(y/\epsilon)$; the large constant ϵ^{-1} leads to the second term on the right in (7-13)

We now turn to the details. Assume first that u is supported in some $\overline{\Omega_i}$. For $j\in\mathbb{N}$ to be specified below and putting $\chi_{0,j}(x_{\mathscr{I}}):=\chi_0(2^{j+1}x_{\mathscr{I}})$, we have

$$\begin{aligned}\|u\|^2_{H^{s,\alpha}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})^{\bullet,-}}&\leq 2\|(\chi_0(x_{\mathscr{I}})-\chi_0(2^{j+1}x_{\mathscr{I}}))u\|^2_{H^{s,\alpha}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})^{\bullet,-}}+2\|\chi_0(2^{j+1}x_{\mathscr{I}})u\|^2_{H^{s,\alpha}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})^{\bullet,-}}\\&\leq C_{jN}\|u\|^2_{H^{s,(N,\alpha_+)}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})^{\bullet,-}}+C\sum_{k=1}^{K(j)}\|\chi_{0,j}\chi_{j,k}u\|^2_{H^{s,\alpha}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})^{\bullet,-}}\end{aligned}$$

for any N, with C independent of j. We estimate each term in the sum individually using Proposition 7.1, to wit,

$$\begin{aligned}&\|\chi_{0,j}\chi_{j,k}u\|^2_{H^{s,\alpha}_{\mathrm{e,b}}}\\&\quad\leq C\|P_{y_{j,k}}(\chi_{0,j}\chi_{j,k}u)\|^2_{H^{s-1,\alpha'}_{\mathrm{e,b}}}\\&\quad\leq C\big(\|[P_{y_{j,k}},\chi_{0,j}\chi_{j,k}]u\|^2_{H^{s-1,\alpha'}_{\mathrm{e,b}}}+\|\chi_{0,j}\chi_{j,k}(P-P_{y_{j,k}})u\|^2_{H^{s-1,\alpha'}_{\mathrm{e,b}}}+\|\chi_{0,j}\chi_{j,k}Pu\|^2_{H^{s-1,\alpha'}_{\mathrm{e,b}}}\big).\end{aligned}\tag{7-18}$$

Recall that $\chi_{0,j}(x_{\mathscr{I}})\chi_{j,k}(x_{\mathscr{I}},y)=\chi_{0,j}(x_{\mathscr{I}})\chi_1(2^j(y-y_{j,k}))$. The commutator of $P_{y_{j,k}}$ with $\chi_{0,j}$ is uniformly bounded in j,k as an element of $\mathcal{A}^{(2,2)}\operatorname{Diff}^1_{\mathrm{e,b}}$ (since $\sup|(x_{\mathscr{I}}\partial_{x_{\mathscr{I}}})^l\chi_{0,j}|$ has a j-independent upper bound for all l), with coefficients supported in $x_{\mathscr{I}}\geq 2^{-j-2}>0$. Therefore, for all N, there exists $C_{N,j}$ so that

$$\sum_{k=1}^{K(j)}\|[P_{y_{j,k}},\chi_{0,j}]\chi_1(2^j(y-y_{j,k}))u\|^2_{H^{s-1,\alpha'}_{\mathrm{e,b}}}\leq C_{N,j}\|u\|^2_{H^{s,(2N,\alpha_+)}_{\mathrm{e,b}}}.$$

[31] This is a key difference to the b-setting, where a single normal operator governs the behavior of a b-operator globally near the whole boundary.

On the other hand, if we define $\chi_{j,k}^\sharp = \chi_0(2^{j+1}x_\mathscr{I})\chi_1^\sharp(2^j(y-y_{j,k}))$, where $\chi_1^\sharp \in \mathcal{C}_c^\infty(\mathbb{R}^{n-1})$ is identically 1 on $\operatorname{supp}\chi_1$ (thus Lemma 7.6 applies with $\chi_{j,k}^\sharp$ in place of $\chi_{j,k}$), then

$$\chi_{0,j}[P_{y_{j,k}}, \chi_1(2^j(y-y_{j,k}))] = \chi_{0,j}\chi_{j,k}^\sharp 2^j x_\mathscr{I} Q_{j,k},$$

where the operators $Q_{j,k} \in \mathcal{A}^{(2,2)}\operatorname{Diff}^1_{\mathrm{e,b}}$ are uniformly bounded in j,k (as follows from (7-17)); the factor $2^j x_\mathscr{I}$ here arises via differentiation of $2^j(y-y_{j,k})$ along terms of $P_{y_{j,k}}$ involving $x_\mathscr{I}\partial_y$; thus,

$$\sum_{k=1}^{K(j)} \|\chi_{0,j}[P_{y_{j,k}}, \chi_1(2^j(y-y_{j,k}))]u\|^2_{H^{s-1,\alpha'}_{\mathrm{e,b}}} \le C2^{2j}\|u\|^2_{H^{s,\alpha-(1,0)}_{\mathrm{e,b}}}.$$

We now turn to the second term in (7-18). The definition of $P_{y_{j,k}}$ via freezing the coefficients of $P_0 = P - \widetilde{P}$ (see Definition 3.7(2)) at $(x_\mathscr{I}, y) = (0, y_{j,k})$ as an edge-b-operator implies that we can write

$$\chi_{j,k}(P_0 - P_{y_{j,k}}) = \chi_{j,k}\left(x_\mathscr{I} Q^1_{j,k} + \sum_{l=1}^{n-1}(y-y_{j,k})_l Q^{(l)}_{j,k}\right),$$

where $x_\mathscr{I}^{-2}\rho_+^{-2}Q^1_{j,k}, x_\mathscr{I}^{-2}\rho_+^{-2}Q^{(l)}_{j,k} \in \mathcal{A}^{(0,0)}\operatorname{Diff}^1_{\mathrm{e,b}}$ are bounded independently of j,k upon application of any fixed number of edge-b- (or even b-)derivatives. We can thus estimate

$$\sum_{k=1}^{K(j)} \|\chi_{0,j}\chi_{j,k}x_\mathscr{I} Q^1_{j,k}u\|^2_{H^{s-1,\alpha'}_{\mathrm{e,b}}} \le C\|u\|^2_{H^{s+1,\alpha-(1,0)}_{\mathrm{e,b}}}.$$

Moreover, in view of $\chi_{j,k}\cdot 2^j(y-y_{j,k})_l$ obeying uniform (in j,k) L^∞-bounds (together with any finite number of edge-b-derivatives), we also have

$$\sum_{k=1}^{K(j)} 2^{-2j}\|\chi_{0,j}\chi_{j,k}2^j(y-y_{j,k})_l Q^l_{j,k}u\|^2_{H^{s-1,\alpha'}_{\mathrm{e,b}}} \le 2^{-2j}C\|u\|^2_{H^{s+1,\alpha}_{\mathrm{e,b}}}.$$

The contribution from the lower-order (in the sense of decay) contribution $\widetilde{P}$ to P is estimated simply by

$$\sum_k \|\chi_{0,j}\chi_{j,k}\widetilde{P}u\|^2_{H^{s-1,\alpha'}_{\mathrm{e,b}}} \le C\|u\|^2_{H^{s+1,\alpha-(2\ell_\mathscr{I},\ell_+)}_{\mathrm{e,b}}}.$$

Altogether, we have now proved that

$$\sum_{k=1}^{K(j)} \|\chi_{0,j}\chi_{j,k}u\|^2_{H^{s,\alpha}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})^{\bullet,-}} \le (C_{N,j}+C2^{2j})\|u\|^2_{H^{s+1,\alpha-(2\ell_\mathscr{I},0)}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})^{\bullet,-}} + C2^{-2j}\|u\|^2_{H^{s+1,\alpha}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})^{\bullet,-}} + C\|Pu\|^2_{H^{s-1,\alpha'}_{\mathrm{e,b}}(\Omega^+_{\delta,\rho})^{\bullet,-}},$$

with C independent of j. Upon choosing j large enough so that $2^{-2j}C < \epsilon^2$, this proves the estimate (7-13) for u with $\operatorname{supp} u \subset \overline{\Omega_i}$.

For general u, we apply Lemma 7.5 and note that the arguments given so far apply to $\phi_i u$, $i = 1,\dots,N$; we then only need to observe that $P(\phi_i u) = \phi_i Pu + [P,\phi_i]u$, where the commutator $[P,\phi_i]$ gains a power of $x_\mathscr{I}$ and loses a differential order (relative to P) as an edge-b-differential operator, so

$$\|[P,\phi_i]u\|_{H^{s-1,\alpha'}_{\mathrm{e,b}}} \le C\|u\|_{H^{s,\alpha-(1,0)}_{\mathrm{e,b}}}.$$

The proof of the a priori estimate (7-14) is completely analogous. □

8. Microlocal estimates for the I^+-normal operator

We let P, M, M_0, and Y be as in the beginning of Section 6, and use the coordinates ρ_+, $x_{\mathscr{I}}$ on $\mathcal{U}_+(T)$ ($T \in \mathbb{R}$ arbitrary) as in (3-6); thus, replacing I^+ for simplicity of notation by a collar neighborhood

$$I^+ = [0, \epsilon_{\mathscr{I}})_{x_{\mathscr{I}}} \times Y, \quad \epsilon_{\mathscr{I}} > 0,$$

of $I^+ \cap \mathscr{I}^+ \subset I^+$, we work in a collar neighborhood

$$[0, 1)_{\rho_+} \times I^+$$

of I^+ inside M and hence in a trivialization $[0, \infty)_{\rho_+} \times I^+$ of ${}^+NI^+$ (where we abuse notation by writing the fiber-linear coordinate $\mathrm{d}\rho_+$ as ρ_+ simply). We only record weights at $\mathscr{I}^+$ and I^+, and drop the weight at I^0 from the notation; we shall also drop the bundle $\beta^*E \to M$ from the notation unless it makes a technical difference.

In this section, we demonstrate that the information on P captured by Definition 3.7 is sufficient to obtain some control of the Mellin-transformed normal operator of P at I^+; see Definition 2.5. Concretely, in the decomposition $P = P_0 + \widetilde{P}$ as in (3-17), only the first term of P_0 is of leading order in the sense of decay at I^+. We shall analyze the rescaled normal operator of P_0.

Definition 8.1 (rescaled normal operator at I^+). Denote by $\pi : {}^+NI^+ \to I^+$ the projection to the base. We then set

$$P_+ := N_{I^+}(x_{\mathscr{I}}^{-2}\rho_+^{-2}P) = N_{I^+}(x_{\mathscr{I}}^{-2}\rho_+^{-2}P_0) \in \mathrm{Diff}^2_{\mathrm{e,b},I}({}^+NI^+; \pi^*E) = \mathrm{Diff}^2_{\mathrm{e,b},I}([0, \infty)_{\rho_+} \times I^+; \pi^*E),$$

where the subscript "I" indicates the dilation-invariance of P_+ in the factor $[0, \infty)_{\rho_+}$, and the edge structure is associated with the fibration $[0, \infty)_{\rho_+} \times \partial I^+ \cong [0, \infty)_{\rho_+} \times Y \to Y$. The Mellin-transformed normal operator is denoted by

$$\widehat{P_+}(\lambda) = \rho_+^{-i\lambda} P_+ \rho_+^{i\lambda} \in \mathrm{Diff}^2_0(I^+; \pi^*E).$$

We recall that the principal symbol G of P is the dual metric function $G : \varpi \mapsto g^{-1}(\varpi, \varpi)$. But by Definition 3.2 (see also (3-13)), we have

$$\begin{aligned} g^{-1} &\equiv g_0^{-1} \mod \mathcal{A}^{2+2\ell_{\mathscr{I}}, 2+\ell_+}(M; S^{2\,\mathrm{e,b}}TM), \\ g_0^{-1} &\equiv x_{\mathscr{I}}^2\rho_+^2\big(\tfrac{1}{2}x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} \otimes_s (x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} - 2\rho_+\partial_{\rho_+}) + k^{-1}(y, x_{\mathscr{I}}\partial_y)\big) \mod x_{\mathscr{I}}^3\rho_+^2\mathcal{C}^\infty(M; S^{2\,\mathrm{e,b}}TM); \end{aligned}$$

thus, the principal symbol of P_+ is the quadratic form associated with the dilation-invariant extension of $x_{\mathscr{I}}^{-2}\rho_+^{-2}g_0^{-1}|_{I^+} \in \mathcal{C}^\infty(I^+; S^{2\,\mathrm{e,b}}T_{I^+}M)$. Concretely, this means that

$$ {}^{\mathrm{e,b}}\sigma^2(P_+)(\varpi) =: G_{0,\mathrm{e,b}}(\varpi) := G_+(\varpi) + \widetilde{G}_+(\varpi) \tag{8-1}$$

with $G_+(\varpi) = g_+^{-1}(\varpi, \varpi) \in P^{[2]}({}^{\mathrm{e,b}}T^*_{I^+}M)$ and $\widetilde{G}_+(\varpi) = \tilde{g}_+^{-1}(\varpi, \varpi) \in x_{\mathscr{I}}P^{[2]}({}^{\mathrm{e,b}}T^*_{I^+}M)$, where

$$g_+^{-1} = \tfrac{1}{2}x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} \otimes_s (x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} - 2\rho_+\partial_{\rho_+}) + k^{-1}(y, x_{\mathscr{I}}\partial_y), \tag{8-2}$$

while $\tilde{g}_+$ is a linear combination of symmetric 2-tensors built out of $x_{\mathscr{I}}\partial_{x_{\mathscr{I}}}$, $\rho_+\partial_{\rho_+}$, and $x_{\mathscr{I}}\partial_{y^j}$, with coefficients lying in $x_{\mathscr{I}}\mathcal{C}^\infty([0, \epsilon_{\mathscr{I}})_{x_{\mathscr{I}}} \times Y)$.

We shall work with the b-density

$$\mu_{\mathrm{b}} := \left|\frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}}\,\mathrm{d}k\right| \in \mathcal{C}^\infty(I^+;{}^{\mathrm{b}}\Omega I^+) \tag{8-3}$$

on I^+. On the level of L^2-spaces on $[0,\infty)_{\rho_+}\times I^+$, the relationship of μ_{b} with the density

$$x_{\mathscr{I}}^{-2n}\rho_+^{-n-1}\left|\frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}}\frac{\mathrm{d}\rho_+}{\rho_+}\,\mathrm{d}k\right|$$

(see (4-16)) used in Sections 4–7 is

$$\begin{gathered} x_{\mathscr{I}}^{2\alpha_{\mathscr{I}}}\rho_+^{\alpha_+}L^2\left([0,\infty)\times I^+, x_{\mathscr{I}}^{-2n}\rho_+^{-n-1}\left|\frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}}\frac{\mathrm{d}\rho_+}{\rho_+}\,\mathrm{d}k\right|\right) = x_{\mathscr{I}}^{2\gamma_{\mathscr{I}}}\rho_+^{\gamma_+}L^2\left([0,\infty)\times I^+, \left|\frac{\mathrm{d}\rho_+}{\rho_+}\right|\mu_{\mathrm{b}}\right), \\ \gamma_{\mathscr{I}} = \alpha_{\mathscr{I}} + \frac{n}{2}, \quad \gamma_+ = \alpha_+ + \frac{n+1}{2}. \end{gathered} \tag{8-4}$$

Thus, by Plancherel's theorem, the Mellin transform in ρ_+, defined as in (2-21), gives an isometric isomorphism

$$\mathcal{M}: x_{\mathscr{I}}^{2\alpha_{\mathscr{I}}}\rho_+^{\alpha_+}L^2\left([0,\infty)\times I^+, x_{\mathscr{I}}^{-2n}\rho_+^{-n-1}\left|\frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}}\frac{\mathrm{d}\rho_+}{\rho_+}\,\mathrm{d}y\right|\right) \xrightarrow{\cong} L^2(\{\operatorname{Im}\lambda=-\gamma_+\}; x_{\mathscr{I}}^{2\gamma_{\mathscr{I}}}L^2(I^+,\mu_{\mathrm{b}})). \tag{8-5}$$

8.1. ***Bounded frequency estimates.*** The principal symbol of $\widehat{P_+}(\lambda)$ is an element of $P^{[2]}({}^0T^*I^+)$. Recall that, for $\varpi\in{}^0T^*I^+$, we have ${}^0\sigma_2(\widehat{P_+}(\lambda))(\varpi) = {}^{\mathrm{e,b}}\sigma^2(P_+)(\varpi)$, where we recall the inclusion ${}^0T^*I^+\subset {}^{\mathrm{e,b}}T^*_{I^+}M$. (In local coordinates, every $\varpi\in{}^0T^*I^+$ is of the form $\varpi=\xi(\mathrm{d}x_{\mathscr{I}}/x_{\mathscr{I}})+\eta_j(\mathrm{d}y^j/x_{\mathscr{I}})$.) In view of (8-1)–(8-2), this means

$${}^0\sigma_2(\widehat{P_+}(\lambda))(\varpi) = \tfrac{1}{2}\xi^2 + k^{ij}(y)\eta_i\eta_j + \widetilde{G}_+(\varpi), \quad \varpi = \xi\frac{\mathrm{d}x_{\mathscr{I}}}{x_{\mathscr{I}}} + \eta_j\frac{\mathrm{d}y^j}{x_{\mathscr{I}}},$$

and hence $\widehat{P_+}(\lambda)$ is elliptic near $x_{\mathscr{I}}=0$. Its refined analysis also requires control of its normal operator, which is the conjugation of the edge normal operator of P_+ (i.e., the normal operator at ${}^+N_{\partial I^+}I^+$) by the Mellin transform in ρ_+; but the edge normal operator (at $I^+\cap\mathscr{I}^+_{y_0}$) of the b-normal operator (at I^+) P_+ of P_0 is equal to the b-normal operator (at $\mathscr{I}^+\cap I^+$) of the edge normal operator (at $\mathscr{I}^+_{y_0}$) of P_0, and is thus fixed by Definition 3.7(3). To wit, for $y_0\in Y$,

$$\begin{gathered} {}^0N_{y_0}(\widehat{P_+}(\lambda)) = \tfrac{1}{2}(x_{\mathscr{I}}D_{x_{\mathscr{I}}} - 2i^{-1}q_{1,y_0})(x_{\mathscr{I}}D_{x_{\mathscr{I}}} - 2\lambda) + k^{ij}(y_0)(x_{\mathscr{I}}D_{y^i})(x_{\mathscr{I}}D_{y^j}) + p^+_{0,y_0}, \\ q_{1,y_0} := \tfrac{1}{2}(n-1) + p_1|_{\mathscr{I}^+_{y_0}\cap I^+} \in \operatorname{End}(E_{y_0}), \quad p^+_{0,y_0} := p^+_0|_{\mathscr{I}^+_{y_0}\cap I^+} \in \operatorname{End}(E_{y_0}). \end{gathered}$$

Recalling the definition of $\underline{p}_{1,+}$ from Definition 4.3, we let

$$\underline{q}_{1,+} := \tfrac{1}{2}(n-1) + \underline{p}_{1,+}. \tag{8-6}$$

Theorem 8.2 (estimates for $\widehat{P_+}(\lambda)$ and $\widehat{P_+^*}(\lambda)$ for bounded λ). *If $p_0^+|_{\mathscr{I}^+\cap I^+}\neq 0$, assume that P is special admissible in the sense of Definition 6.2; otherwise assume only that P is admissible. Let χ_1, $\chi_2\in\mathcal{C}^\infty_{\mathrm{c}}(I^+)$, with $\chi_1\equiv 1$ near ∂I^+, $\chi_2\equiv 1$ near $\operatorname{supp}\chi_1$, and such that ${}^0T^*_pI^+$ is spacelike (i.e., $G_{0,\mathrm{e,b}}$ in (8-1) is positive definite on ${}^0T^*_pI^+$) for $p\in\operatorname{supp}\chi_1$. Use the b-density (8-3) on I^+ to define Sobolev spaces.*

(1) (direct problem) *Let $s, \gamma_{\mathscr{I}} \in \mathbb{R}$, and suppose that*

$$-\operatorname{Im}\lambda < \gamma_{\mathscr{I}} < \underline{q}_{1,+}. \tag{8-7a}$$

Then, for all $N \in \mathbb{R}$, there exists a constant C such that

$$\|\chi_1 u\|_{H_0^{s,2\gamma_{\mathscr{I}}}(I^+)} \leq C(\|\chi_2 \widehat{P_+}(\lambda)u\|_{H_0^{s-2,2\gamma_{\mathscr{I}}}(I^+)} + \|\chi_2 u\|_{H_0^{-N,-N}(I^+)}). \tag{8-7b}$$

Moreover, for all u with $\chi_2 u \in H_0^{-\infty,2\gamma_{\mathscr{I}}}(I^+)$ and $\chi_2\widehat{P_+}(\lambda)u = 0$, we have $\chi_1 u \in \mathcal{A}^{2(\underline{q}_{1,+}-\epsilon)}(I^+)$ for all $\epsilon > 0$.

(2) (adjoint problem) *Let $\tilde{s}, \tilde{\gamma}_{\mathscr{I}} \in \mathbb{R}$, and suppose that*

$$-\underline{q}_{1,+} < \tilde{\gamma}_{\mathscr{I}} < -\operatorname{Im}\lambda. \tag{8-8a}$$

Then, for all $N \in \mathbb{R}$, there exists a constant C such that

$$\|\chi_1 \tilde{u}\|_{H_0^{\tilde{s},2\tilde{\gamma}_{\mathscr{I}}}(I^+)} \leq C(\|\chi_2 \widehat{P_+^*}(\lambda)\tilde{u}\|_{H_0^{\tilde{s}-2,2\tilde{\gamma}_{\mathscr{I}}}(I^+)} + \|\chi_2 \tilde{u}\|_{H_0^{-N,-N}(I^+)}). \tag{8-8b}$$

Remark 8.3 (weights). Identifying $-\operatorname{Im}\lambda$ with the weight $\gamma_+ = -\operatorname{Im}\lambda$ (cf. (8-5)) and relating $\gamma_{\mathscr{I}}, \gamma_+$ to $\alpha_{\mathscr{I}}, \alpha_+$ via (8-4), the conditions in (8-7a) are equivalent to $\alpha_+ < -\frac{1}{2} + \alpha_{\mathscr{I}}$ and $\alpha_{\mathscr{I}} < -\frac{1}{2} + \underline{p}_1$, except for the use of $\underline{p}_{1,+}$ (determined from p_1 only at $\mathscr{I}^+ \cap I^+$) instead of $\underline{p}_1$ (determined from p_1 along all of $\mathscr{I}^+$). In this sense, Theorem 8.2(1) applies in the full range of weights allowed in all previous estimates near $\mathscr{I}^+ \cap I^+$ (Lemmas 4.10, 4.11, Proposition 7.1, Theorem 7.3). A similar comment applies to Theorem 8.2(2).

Remark 8.4 (restriction on $\operatorname{Im}\lambda$ and continuation of the resolvent). The condition (8-7a) provides a lower bound $\operatorname{Im}\lambda > -\gamma_{\mathscr{I}}$ of Mellin dual frequencies λ for which one can analyze $\widehat{P_+}(\lambda)$ as a 0-operator in a straightforward manner. Upon inspection of the proof below, one sees that allowing λ to cross the line $\operatorname{Im}\lambda = -\gamma_{\mathscr{I}}$ is closely related to considering the spectral family on an asymptotically hyperbolic space at or across the continuous spectrum, the analysis of which is more delicate; see, e.g., [Mazzeo and Melrose 1987; Guillarmou 2005; Sá Barreto and Wang 2016].

Proof of Theorem 8.2. The proof is based on a standard elliptic parametrix construction in the 0-calculus [Mazzeo and Melrose 1987; Hintz 2023a]; hence we shall be rather brief. By assumption, $\widehat{P_+}(\lambda)$ is elliptic near $\operatorname{supp}\chi_1$.

If q_{1,y_0} is diagonalizable and $p^+_{0,y_0} = 0$, then acting on the individual eigenspaces of q_{1,y_0}, the operator ${}^0N_{y_0}(\widehat{P_+}(\lambda))$ is related to the spectral family of $(n-1)$-dimensional hyperbolic space via conjugation by a suitable power of $x_{\mathscr{I}}$ (which also appears in the proof of the invertibility of the reduced normal operator below, cf. (8-10a)–(8-10b)); its inverse is thus explicit [Mazzeo and Melrose 1987, §6]. This can easily be generalized to the case that q_{1,y_0} has nontrivial Jordan blocks. (In this manner, one can prove (8-7b) for general λ except for an explicitly computable discrete set.)

We analyze ${}^0N_{y_0}(\widehat{P_+}(\lambda))$ here (for $\operatorname{Im}\lambda > -\gamma_{\mathscr{I}}$) instead in a more conceptual manner, using the general machinery of the 0-calculus. Thus, exploiting the translation invariance of ${}^0N_{y_0}(\widehat{P_+}(\lambda))$ by passing to the

Fourier transform in y and then introducing $x := x_{\mathscr{I}}|\eta|_{k^{-1}}$, $\hat{\eta} = \eta/|\eta|_{k^{-1}}$, we obtain the *reduced normal operator* in the terminology of [Lauter 2003], given by

$$\mathcal{P}_{y_0} := \tfrac{1}{2}(xD_x - 2i^{-1}q_{1,y_0})(xD_x - 2\lambda) + x^2 + p^+_{0,y_0}.$$

Considered as an operator on the radial compactification $[0,\infty]_x$ of $[0,\infty)$, it is a b-operator near $x=0$ with normal operator $N_0(\mathcal{P}_{y_0}) = \frac{1}{2}(xD_x - 2i^{-1}q_{1,y_0})(xD_x - 2\lambda) + p^+_{0,y_0}$ and a weighted scattering operator near $R := x^{-1} = 0$ with principal part $R^{-2}\big(\frac{1}{2}(R^2D_R)^2 + 1\big)$, which is elliptic. The Mellin-transformed normal operator of $\mathcal{P}_{y_0}$ at $x=0$ is

$$\widehat{N}_0(\mathcal{P}_{y_0}, \zeta) = \tfrac{1}{2}(\zeta - 2i^{-1}q_{1,y_0})(\zeta - 2\lambda) + p^+_{0,y_0},$$

which is thus invertible for all $\zeta \in \mathbb{C}$ except for those lying in the boundary spectrum $\{2\lambda\} \cup \operatorname{spec}(2i^{-1}q_{1,y_0})$. (For $p^+_{0,y_0} = 0$, this is clear; for $p^+_{0,y_0} \neq 0$ on the other hand, one uses the lower triangular (resp. strictly lower triangular) nature of q_{1,y_0} (resp. p^+_{0,y_0}) coming from the special admissibility assumption on P to obtain the same conclusion.) In particular, the line $\operatorname{Im}\zeta = -2\gamma_{\mathscr{I}}$ does not intersect the boundary spectrum provided $\gamma_{\mathscr{I}} \neq -\operatorname{Im}\lambda$ and $\gamma_{\mathscr{I}} \notin \operatorname{Re}\operatorname{spec} q_{1,y_0}$, which in particular allows for $\gamma_{\mathscr{I}}$ in the range (8-7a).

Define the b-scattering Sobolev spaces

$$H^{s,(2\gamma_{\mathscr{I}},\delta)}_{\mathrm{b,sc}}([0,\infty]_x) := \Big\{u : \chi u \in H^{s,2\gamma_{\mathscr{I}}}_{\mathrm{b}}\Big([0,\infty), \Big|\frac{\mathrm{d}x}{x}\Big|\Big),\ (1-\chi)u \in \langle x\rangle^{-\delta} H^s(\mathbb{R}, |\mathrm{d}x|)\Big\},$$

where $\chi \in \mathcal{C}^\infty_{\mathrm{c}}([0,\infty))$ is identically 1 near 0. Then we claim that, for $\gamma_{\mathscr{I}}$ in the range (8-7a), the operator

$$\mathcal{P}_{y_0} : H^{s,(2\gamma_{\mathscr{I}},\delta)}_{\mathrm{b,sc}}([0,\infty]) \to H^{s-2,(2\gamma_{\mathscr{I}},\delta-2)}_{\mathrm{b,sc}}([0,\infty]) \tag{8-9}$$

is not merely Fredholm (which follows at once from elliptic theory in the b-setting near $s=0$ and in the scattering setting near $s^{-1}=0$) but invertible. For the proof, we first assume that $p^+_{0,y_0} = 0$. Passing to the Jordan block decomposition of q_{1,y_0}, it is sufficient (by exploiting the upper triangular nature of Jordan blocks) to consider the case that $q_{1,y_0} \in \mathbb{C}$ is scalar; and the assumption on $\gamma_{\mathscr{I}}$ reads $-\operatorname{Im}\lambda < \gamma_{\mathscr{I}} < \operatorname{Re} q_{1,y_0}$. If $u \in H^{s,(2\gamma_{\mathscr{I}},\delta)}_{\mathrm{b,sc}}([0,\infty])$ satisfies $\mathcal{P}_{y_0}u = 0$, then $u \in H^{\infty,(2\gamma_{\mathscr{I}},\infty)}_{\mathrm{b,sc}}([0,\infty])$ by elliptic regularity in the scattering calculus, so in particular u is rapidly decaying (together with all x-derivatives) as $x \to \infty$. Moreover,

$$\tilde{u} := x^{-q_{1,y_0}-i\lambda}u \in H^{\infty,(2\tilde{\gamma}_{\mathscr{I}},\infty)}_{\mathrm{b,sc}}([0,\infty]), \quad \tilde{\gamma}_{\mathscr{I}} := \tfrac{1}{2}(2\gamma_{\mathscr{I}} + \operatorname{Im}\lambda - \operatorname{Re} q_{1,y_0}), \tag{8-10a}$$

satisfies the simpler equation

$$\big(\tfrac{1}{2}(xD_x)^2 - \tilde{\lambda}^2 + x^2\big)\tilde{u} = 0, \quad \tilde{\lambda} := \lambda + iq_{1,y_0}. \tag{8-10b}$$

Note that $\operatorname{Im}\tilde{\lambda} = \operatorname{Im}\lambda + \operatorname{Re} q_{1,y_0} > 0$ and $2\tilde{\gamma}_{\mathscr{I}} > -\operatorname{Im}\tilde{\lambda}$; therefore we in fact have $|\tilde{u}|, |xD_x\tilde{u}| \leq Cx^{\operatorname{Im}\tilde{\lambda}}$ for $x \leq 1$. We can therefore multiply (8-10b) by $\bar{\tilde{u}}$ and integrate by parts. If $\operatorname{Re}\tilde{\lambda} \neq 0$, taking imaginary parts then gives $0 = 2(\operatorname{Re}\tilde{\lambda})(\operatorname{Im}\tilde{\lambda})\|\tilde{u}\|^2$, whereas when $\operatorname{Re}\tilde{\lambda} = 0$ one directly obtains

$$0 = \|x\tilde{u}\|^2 + \tfrac{1}{2}\|xD_x\tilde{u}\|^2 + \||\tilde{\lambda}|\tilde{u}\|^2;$$

in both cases, we conclude that $\tilde{u} = 0$ and hence $u = 0$. This proves the injectivity of $\mathcal{P}_{y_0}$. The surjectivity follows from the injectivity of the adjoint $\mathcal{P}^*_{y_0}$, which is proved in the same manner. If $p^+_{0,y_0} \neq 0$, then upon passing to the bundle splitting of E in which q_{1,y_0} and p^+_{0,y_0} are (strictly) lower triangular, these

arguments apply step by step as one considers the projection of $\mathcal{P}_{y_0}$ onto the first, second, etc. summand of the splitting of E.

As shown in [Mazzeo 1991] (see also [Hintz 2023a, §3.2; Albin 2008, §5.5] for detailed expositions, and also [Lauter 2003]), the invertibility of $\mathcal{P}_{y_0}$ implies the existence of a parametrix for $\widehat{P_+}(\lambda)$ acting on $x_{\mathscr{I}}^{2\gamma_{\mathscr{I}}} L^2(I^+)$ in the large 0-calculus with bounds,

$$Q \in \Psi_0^{-2,(\alpha_{\mathrm{lb}},-(n-1),\alpha_{\mathrm{rb}})}(I^+), \quad Q\widehat{P_+}(\lambda) = I + R_L, \quad R_L \in \Psi_0^{-\infty,(\varnothing,\varnothing,\alpha_{\mathrm{rb}})}(I^+).$$

Here, $\Psi_0^{s,(\alpha_{\mathrm{lb}},\alpha_{\mathrm{ff}},\alpha_{\mathrm{rb}})}(I^+)$ is the space of operators whose Schwartz kernels are conormal distributions on the 0-double space $(I^+)_0^2 := [(I^+)^2; \mathrm{diag}_{\partial I^+}]$, where $\mathrm{diag}_{\partial I^+} \subset (\partial I^+)^2$ is the diagonal, valued in (the pullback to $(I^+)_0^2$ of) the right b-density bundle on $(I^+)^2$, which are conormal to the lift of the diagonal $\mathrm{diag}_{I^+} \subset (I^+)^2$, and which are conormal also at the left boundary (the lift of $\partial I^+ \times I^+$) with decay rate α_{lb} (the power to which a boundary-defining function of the left boundary is raised), at the right boundary (the lift of $I^+ \times \partial I^+$) with decay rate α_{rb}, and at the front face with decay rate α_{ff}; the shift by $(n-1)$ of the weight at the front face is due to the fact that we work with a b-density here rather than with a 0-density — which is $x_{\mathscr{I}}^{-(n-1)}$ times a b-density on the n-dimensional manifold I^+. Concretely, the computation of the boundary spectrum of the 0-normal operator above implies that we can take

$$\alpha_{\mathrm{lb}} \geq 2\underline{q}_{1,+} - \epsilon, \quad \alpha_{\mathrm{rb}} \geq 2\,\mathrm{Im}\,\lambda - \epsilon$$

for any $\epsilon > 0$. We conclude that

$$\chi_1 u = \chi_1(\chi_2 u) = \chi_1 Q\widehat{P_+}(\lambda)(\chi_2 u) + \chi_1 R_L(\chi_2 u) = \chi_1 Q(\chi_2 \widehat{P_+}(\lambda)u) + \chi_1 R_L(\chi_2 u) - \chi_1 Q[\widehat{P_+}(\lambda), \chi_2]u.$$

Taking the norm in $H_0^{s,2\gamma_{\mathscr{I}}}(I^+)$ gives the estimate (8-7b) since the operators $\chi_1 R_L \chi_2$ and $\chi_1 Q[\widehat{P_+}(\lambda), \chi_2]$ (noting that $\mathrm{supp}\,\chi_1 \cap \mathrm{supp}\,\mathrm{d}\chi_2 = \varnothing$) are bounded maps $H_0^{-N,-N}(I^+) \to H_0^{s,2\gamma_{\mathscr{I}}}(I^+)$ for all N.

The proof of the adjoint estimate (8-8b) is similar. The adjoint $\widehat{P_+}(\lambda)^* = \widehat{P_+^*}(\bar{\lambda})$ satisfies (8-8b) when $(\tilde{s}-2, 2\tilde{\gamma}_{\mathscr{I}}) = -(s, 2\gamma_{\mathscr{I}})$ (the dual orders of the left-hand side of (8-7b)), which leads to the requirement $-\mathrm{Im}\,\lambda < -\tilde{\gamma}_{\mathscr{I}} < \underline{q}_{1,+}$. Passing from λ to $\bar{\lambda}$ switches the sign of $\mathrm{Im}\,\lambda$ and thus leads to the condition (8-8a). □

8.2. ***High frequency estimates.*** We now prove analogues of the estimates found in Theorem 8.2 when $|\mathrm{Re}\,\lambda| \to \infty$. In the notation of Theorem 8.2, let $\gamma_{\mathscr{I}} < \underline{q}_{1,+}$, and fix $\gamma_+ < \gamma_{\mathscr{I}}$; we shall from now on only consider $\lambda \in \mathbb{C}$ with $\mathrm{Im}\,\lambda = -\gamma_+$. As $|\mathrm{Re}\,\lambda| \to \infty$, the operator $\widehat{P_+}(\lambda)$ can be regarded as a (uniformly degenerate) differential operator with large parameter λ; see [Shubin 1987, §9] for the case of manifolds without boundary. For present purposes, it is sufficient to ignore the symbolic behavior in λ and instead consider the semiclassical rescaling

$$P_{h,\hat{\lambda}} := h^2 \widehat{P_+}(h^{-1}\hat{\lambda}) \in \mathrm{Diff}^2_{0,\hbar}(I^+; \pi^* E), \quad h := \langle\lambda\rangle^{-1}, \quad \hat{\lambda} := \frac{\lambda}{\langle\lambda\rangle}. \tag{8-11}$$

We apply the general considerations of Section 2.6 to the operator $P_+ \in \mathrm{Diff}^2_{\mathrm{e,b}}([0,\infty)_{\rho_+} \times I^+)$ from Definition 8.1. With λ restricted to the line $\mathrm{Im}\,\lambda = -\gamma_+$, we have, as $|\mathrm{Re}\,\lambda| \to \infty$ (and thus $h = |\mathrm{Re}\,\lambda|^{-1} + \mathcal{O}(|\mathrm{Re}\,\lambda|^{-2}) \to 0$),

$$\hat{\lambda} = (\mathrm{sgn}\,\mathrm{Re}\,\lambda) - ih\gamma_+ + \mathcal{O}(h^2). \tag{8-12}$$

Let χ_1, χ_2 be as in the statement of Theorem 8.2, so in particular ${}^0T^*_pI^+$ is spacelike for $p \in \operatorname{supp}\chi_1$. This implies that $P_{h,\hat\lambda}$ is elliptic (as a *semiclassical* 0-operator) outside a compact subset of ${}^0T^*_{\operatorname{supp}\chi_1}I^+$, in particular at fiber infinity; in other words, writing $p_{\hat\lambda} = \sigma^2_{0,\hbar}(P_{h,\hat\lambda})$, the characteristic set

$$ {}^0\Sigma_{\hat\lambda} := p_{\hat\lambda}^{-1}(0) \subset {}^0T^*I^+ $$

has compact intersection with ${}^0T^*_{\operatorname{supp}\chi_1}I^+$. As noted after Lemma 2.10, the Hamiltonian vector field $H_{p_{\pm1}}$ has a critical point at $\varpi \in {}^0\Sigma_{\pm1}$ if and only if $H_{G_{0,\mathrm{e,b}}}$ (see (8-1)) is radial at $\pm\mathrm{d}\rho_+/\rho_+ + \varpi$ (which lies in the characteristic set of P_+). In a sufficiently small neighborhood of $x_{\mathscr{I}} = 0$, this is equivalent to the membership in one of the two sets

$$ \begin{aligned} {}^0\mathcal{R}^{\pm}_{\mathrm{in}} &:= \{(x_{\mathscr{I}}, y; \xi, \eta) : x_{\mathscr{I}} = 0,\ \xi = \pm 2,\ \eta = 0\}, \\ {}^0\mathcal{R}^{\pm}_{\mathrm{out}} &:= \{(x_{\mathscr{I}}, y; \xi, \eta) : x_{\mathscr{I}} = 0,\ \xi = 0,\ \eta = 0\} \end{aligned} \tag{8-13} $$

by Lemma 4.1. This can of course also be checked directly by noting that, at $\zeta = \pm1$ (in the coordinates (4-4) and using (4-5a)), we have

$$ H_{G^+_{\mathrm{e,b}}} = 2(\xi \mp 1)(x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} + \eta\partial_\eta) - 2|\eta|^2\partial_\xi + x_{\mathscr{I}}(4k^{ij}\eta_i\partial_{y^j} - 2(\partial_{y^m}k^{ij})\eta_i\eta_j\partial_{\eta_m}), \tag{8-14} $$

while $H_{\widetilde{G}_+}$ is a lower-order contribution, cf. (4-5b); that is, the expression (8-14) is equal to $H_{p_{\pm1}}$ up to an error term of class $x_{\mathscr{I}}\mathcal{V}_0({}^0T^*I^+)$. Thus, ${}^0\mathcal{R}^{\pm}_{\mathrm{in}}$ is a source, and ${}^0\mathcal{R}^{\pm}_{\mathrm{out}}$ is a sink for the flow of $\pm H_{p_{\pm1}}$ inside ${}^0\Sigma_{\pm1}$.

Theorem 8.5 (semiclassical regularity for $\widehat{P_+}(\lambda)$). *Let $s, \gamma_{\mathscr{I}}, \gamma_+ \in \mathbb{R}$. Consider $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda = -\gamma_+$, and let $h = \langle\lambda\rangle^{-1}$ and $\hat\lambda = \lambda/\langle\lambda\rangle$; thus $\hat\lambda = \pm1 - ih\gamma_+ + \mathcal{O}(h^2)$ when $\pm\operatorname{Re}\lambda > 0$ (see (8-12)). Let*[32] *u, $f \in h^{-N}H^{-N,2\gamma_{\mathscr{I}}}_{0,h,\mathrm{loc}}(I^+)$ and $P_{h,\hat\lambda}u = f$ in the notation of (8-11). Then $\mathrm{WF}^{s,2\gamma_{\mathscr{I}}}_{0,\hbar}(u) \subset \mathrm{WF}^{s-2,2\gamma_{\mathscr{I}}}_{0,\hbar}(f) \cup {}^0\Sigma_{\pm}$.*

Moreover, recalling the quantity $\underline{q}_{1,+} \in \mathbb{R}$ from (8-6), we have the following.

(1) (propagation out of ${}^0\mathcal{R}^{\pm}_{\mathrm{in}}$) *Suppose that $\gamma_+ < \gamma_{\mathscr{I}}$. If $\mathrm{WF}^{s-2,2\gamma_{\mathscr{I}}}_{0,\hbar}(h^{-1}f) \cap {}^0\mathcal{R}^{\pm}_{\mathrm{in}} = \varnothing$, then $\mathrm{WF}^{s,2\gamma_{\mathscr{I}}}_{0,\hbar}(u) \cap {}^0\mathcal{R}^{\pm}_{\mathrm{in}} = \varnothing$.*[33]

(2) (propagation into ${}^0\mathcal{R}^{\pm}_{\mathrm{out}}$) *Suppose $\gamma_{\mathscr{I}} < \underline{q}_{1,+}$. Let $\mathcal{U} \subset {}^0T^*I^+$ denote an open neighborhood of ${}^0\mathcal{R}^{\pm}_{\mathrm{out}}$. If $\mathrm{WF}^{s-2,2\gamma_{\mathscr{I}}}_{0,\hbar}(h^{-1}f) \cap {}^0\mathcal{R}^{\pm}_{\mathrm{out}} = \varnothing$ and $\mathrm{WF}^{s,2\gamma_{\mathscr{I}}}_{0,\hbar}(u) \cap (\mathcal{U} \setminus \mathcal{R}_{\mathrm{out}}) = \varnothing$, then $\mathrm{WF}^{s,2\gamma_{\mathscr{I}}}_{0,\hbar}(u) \cap \mathcal{R}_{\mathrm{out}} = \varnothing$.*

Theorem 8.6 (semiclassical regularity for $\widehat{P^*_+}(\lambda)$). *Let $\tilde s, \tilde\gamma_{\mathscr{I}}, \tilde\gamma_+ \in \mathbb{R}$. Consider $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda = -\tilde\gamma_+$, and let $h = \langle\lambda\rangle^{-1}$, $\hat\lambda = \lambda/\langle\lambda\rangle$. Let $P^\dagger_{h,\hat\lambda} := h^2\widehat{P^*_+}(h^{-1}\hat\lambda)$. Let $\tilde u$, $\tilde f \in h^{-N}H^{-N,2\tilde\gamma_{\mathscr{I}}}_{0,h,\mathrm{loc}}(I^+)$ and $P^\dagger_{h,\hat\lambda}\tilde u = \tilde f$. Then*

$$ \mathrm{WF}^{\tilde s,2\tilde\gamma_{\mathscr{I}}}_{0,\hbar}(\tilde u) \subset \mathrm{WF}^{\tilde s-2,2\tilde\gamma_{\mathscr{I}}}_{0,\hbar}(\tilde f) \cup {}^0\Sigma_{\pm}. $$

[32] The subscript "loc" means membership in the space $h^{-N}H^{-N,2\gamma_{\mathscr{I}}}_{0,h}$ upon multiplication by any function in $\mathcal{C}^\infty_{\mathrm{c}}(I^+)$. This is merely a technical detail; we are only interested in local (but uniform down to ∂I^+) semiclassical regularity here, for which the noncompact nature of I^+ is irrelevant.

[33] Recall that, since ${}^0\mathcal{R}^{\pm}_{\mathrm{in}} \subset {}^0T^*I^+$ is disjoint from fiber infinity, the wave front set conditions here are in fact independent of the differential orders.

Moreover, we have the following:

(1) (propagation into ${}^0\mathcal{R}^{\pm}_{\mathrm{in}}$) *Suppose that* $\tilde\gamma_+ > \tilde\gamma_{\mathscr{I}}$. *Let* $\mathcal{U} \subset {}^0T^*I^+$ *denote an open neighborhood of* ${}^0\mathcal{R}^{\pm}_{\mathrm{in}}$. *If* $\mathrm{WF}^{\tilde s-2,2\tilde\gamma_{\mathscr{I}}}_{0,\hbar}(h^{-1}\tilde f) = \varnothing$ *and* $\mathrm{WF}^{\tilde s,2\tilde\gamma_{\mathscr{I}}}_{0,\hbar}(\tilde u) \cap (\mathcal{U} \setminus {}^0\mathcal{R}^{\pm}_{\mathrm{in}}) = \varnothing$, *then* $\mathrm{WF}^{\tilde s,2\tilde\gamma_{\mathscr{I}}}_{0,\hbar}(\tilde u) \cap {}^0\mathcal{R}^{\pm}_{\mathrm{in}} = \varnothing$.

(2) (propagation out of ${}^0\mathcal{R}^{\pm}_{\mathrm{out}}$) *Suppose that* $\tilde\gamma_{\mathscr{I}} > -\underline{q}_{1,+}$. *If* $\mathrm{WF}^{\tilde s-2,2\tilde\gamma_{\mathscr{I}}}_{0,\hbar}(h^{-1}\tilde f) \cap {}^0\mathcal{R}^{\pm}_{\mathrm{out}} = \varnothing$, *then* $\mathrm{WF}^{\tilde s,2\tilde\gamma_{\mathscr{I}}}_{0,\hbar}(\tilde u) \cap {}^0\mathcal{R}^{\pm}_{\mathrm{out}} = \varnothing$.

Mirroring Remark 8.3, we recall the relationships

$$\alpha_{\mathscr{I}} = \gamma_{\mathscr{I}} - \tfrac{1}{2}n, \quad \alpha_+ = \gamma_+ - \tfrac{1}{2}(n+1), \quad \underline{p}_{1,+} = \underline{q}_{1,+} - \tfrac{1}{2}(n-1)$$

from (8-4) and (8-6); then the conditions on $\gamma_+, \gamma_{\mathscr{I}}, \underline{q}_{1,+}$ in Theorem 8.5(1) (resp. Theorem 8.5(2)) are precisely those on $\alpha_+, \alpha_{\mathscr{I}}, \underline{p}_1$ in Lemmas 4.10(1) (resp. Lemma 4.11(1)) upon replacing $\underline{p}_1$ by $\underline{p}_{1,+}$. A similar comment applies to Theorem 8.6 and Lemmas 4.10(2) and 4.11(3).

Proof of Theorems 8.5 and 8.6. These results follow again from positive commutator arguments. One can in fact simply copy the arguments from the proofs of Lemmas 4.8–4.11 but factor out the overall weights of the differential operators under study (cf. Definition 8.1) and drop localizations in ρ_+ and its dual momentum ζ in the commutator constructions and calculations. We omit the details here but point out that the subprincipal terms which enter in threshold conditions for radial point estimates for $P_{h,\hat\lambda}$ can be computed from those of P via the formula (2-24), where $\operatorname{Im}\hat\lambda/h = -\gamma_+ + \mathcal{O}(h)$. The upshot is that the term

$$\frac{\operatorname{Im}\hat\lambda}{h}\rho_+^{-1}H_p\rho_+ \equiv -\gamma_+\rho_+^{-1}H_p\rho_+ + \mathcal{O}(h)$$

has precisely the same effect as the weight $\rho_+^{-2\gamma_+}$ of $\check a^2$ in the positive commutator proof of, for example, Lemma 4.10 (where we are using $\check\alpha_+$ in place of γ_+). □

As in Section 4.2, the positive commutator proofs give quantitative estimates. We state this for Theorem 8.5(2): suppose $B, W \in \Psi^0_{0,\hbar}(I^+)$ have Schwartz kernels supported inside $(\operatorname{supp}\chi_1)^2$, where $\chi_1 \in \mathcal{C}^\infty_{\mathrm{c}}(I^+)$ is identically 1 near ∂I^+, and their operator wave front sets are compact subsets of ${}^0T^*I^+$ (thus, differential orders are irrelevant below). Furthermore, suppose that

- all backward (for the "+" sign) and forward (for the "−" sign) null-bicharacteristics starting at $\mathrm{WF}'_{0,\hbar}(B) \cap \Sigma_{\pm 1}$ tend to ${}^0\mathcal{R}^{\pm}_{\mathrm{out}}$ while remaining inside $\mathrm{Ell}^0_{0,\hbar}(W)$;
- W is elliptic on $\mathrm{WF}'_{0,\hbar}(B)$ and at ${}^0\mathcal{R}^{\pm}_{\mathrm{out}}$.

Then, for all $N \in \mathbb{R}$ and under the assumptions of Theorem 8.5(2), there exists a constant C so that

$$\|Bu\|_{H^{s,2\gamma_{\mathscr{I}}}_{0,h}(I^+)} \leq C(h^{-1}\|WP_{h,\hat\lambda}u\|_{H^{s-2,2\gamma_{\mathscr{I}}}_{0,h}(I^+)} + h^N\|\chi u\|_{H^{-N,2\gamma_{\mathscr{I}}}_{0,h}(I^+)}).$$

We leave the statements of estimates corresponding to the wave front set statements of the remaining parts of the above two theorems to the reader.

9. Application to linear waves on asymptotically flat spacetimes

We now sketch how to use the black box results proved in Sections 4–8 to obtain a Fredholm and solvability theory for linear wave operators on a simple class of (future geodesically complete) asymptotically Minkowskian spacetimes. The setting we consider here is closely related to that presented by Baskin–Vasy–Wunsch [Baskin et al. 2015; 2018] insofar as the metrics and wave operators are asymptotically homogeneous of degree ± 2 with respect to dilations in the forward timelike cone. However, compared to the reference, we allow here for significantly more general behavior near null infinity, in that the light cone at infinity (in the terminology of [Baskin et al. 2015]) is blown up here to resolve the metric and operator coefficients. On the flipside, we shall only record basic solvability and decay results here and do not extract sharp asymptotics of solutions of wave equations.

We work on $\mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x$. We denote by $Y = \left\{\frac{r}{t} = 1, r^{-1} = 0\right\} \subset \partial\overline{\mathbb{R}^{n+1}}$ the light cone at infinity and let M denote the square root blow-up of $\widetilde{M} = [\overline{\mathbb{R}^{n+1}}; Y]$ at the front face $\widetilde{\mathscr{I}}^+$ (as in Section 3.1.1; see also Figure 3.1). Denote by $\rho_0, x_{\mathscr{I}}, \rho_+ \in \mathcal{C}^\infty(M)$ defining functions of the boundary hypersurfaces I^0 (the closure of $\frac{t}{r} < 1,\ r^{-1} = 0$), $\mathscr{I}^+$ (the front face of the square root blow-up, i.e., the closure of $|t-r| < \infty$, $r^{-1} = 0$), $I^+ \subset M$ (the closure of $\frac{r}{t} < 1,\ t^{-1} = 0$), respectively.[34] We shall consider forcing problems in the domain

$$\Omega := \overline{\{t \geq 0\}} \subset M.$$

Let then g be a smooth Lorentzian metric on $\mathbb{R}^{n+1}$ (with signature $(-,+,\dots,+)$) with the following properties:

(1) The conformal rescaling $g_{\mathrm{e,b}} := \rho_0^2 x_{\mathscr{I}}^2 \rho_+^2 g$ is a nondegenerate edge-b-metric on M,

$$g_{\mathrm{e,b}} \in (\mathcal{C}^\infty + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)})(M; S^{2\,\mathrm{e,b}}T^*M),$$
$$g_{\mathrm{e,b}}^{-1} \in (\mathcal{C}^\infty + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)})(M; S^{2\,\mathrm{e,b}}TM),$$

where $\ell_0 \in (0,1]$ and $\ell_{\mathscr{I}} \in \left(0, \frac{1}{2}\right]$, and moreover g is an $(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)$-admissible metric (Definition 3.2).

(2) The level sets of t are timelike, with $\mathrm{d}t$ past timelike.

(3) There exists a smooth function $\tau \in \mathcal{C}^\infty(M \setminus (\mathscr{I}^+ \cup I^+))$ which is equal to t/r near I^0 and such that $\tau \geq 0$ has past timelike differential in Ω°.

(4) The b-normal (or *scaling*) vector field[35] $\rho_+\partial_{\rho_+}$ is past timelike for $g_{\mathrm{e,b}}$ at $(I^+)^\circ = I^+ \setminus \mathscr{I}^+$, and there exists a particular choice of ρ_+ such that $\nabla\rho_+/\rho_+$ is future timelike in a neighborhood of $(I^+)^\circ$.

(5) g is nontrapping in the following sense: every past null-bicharacteristic (i.e., lift of a past causal null-geodesic) of $g_{\mathrm{e,b}}$ starting over a point in $\Omega \cap (I^0)^\circ$ (where $(I^0)^\circ = I^0 \setminus \mathscr{I}^+$) reaches the closure of $t^{-1}(0)$; and any past null-bicharacteristic of $g_{\mathrm{e,b}}$ starting over a point in $\Omega \cap (I^+)^\circ$ tends to the radially compactified edge-b-cotangent bundle over $\mathscr{I}^+$.

[34] Thus, one can, for example, take $\rho_+ = t^{-1}$ in $t > 1,\ r/t < c_0 < 1$, further $\rho_0 = r^{-1}$ in $r > 1,\ 0 \leq t/r < c_0 < 1$; and possible choices near $\mathscr{I}^+$ are given in (3-5) and (3-6).

[35] This is defined in any collar neighborhood of $I^+ \subset M$, and its restriction to I^+ as a b-vector field is independent of choices.

Moreover, we consider a wave-type operator $P \in \mathrm{Diff}^2(\mathbb{R}^{n+1})$ with the following properties:

(6) P is a weighted edge-b-operator, $P \in \rho_0^2 x_{\mathscr{I}}^2 \rho_+^2 (\mathcal{C}^\infty + \mathcal{A}^{(\ell_0, 2\ell_{\mathscr{I}}, \ell_+)})\,\mathrm{Diff}^2_{\mathrm{e,b}}(M)$, whose principal symbol is equal to the dual metric function $\zeta \mapsto g^{-1}(\zeta,\zeta)$ of g.

(7) P is g-admissible (Definition 3.7). Define $\underline{p}_1$ and $\underline{p}_{1,+}$ in terms of p_1 from Definition 3.7(3) as in Definition 4.3.

(8) The Mellin-transformed normal operator family $\widehat{P_+}(\lambda)$ of the b-normal operator $N_{I^+}(x_{\mathscr{I}}^{-2}\rho_+^{-2}P) \in \mathrm{Diff}^2_{\mathrm{e,b},I}({}^+NI^+)$ (see Definition 8.1) has trivial kernel on $\mathcal{A}^{2\gamma_{\mathscr{I}}}(I^+)$ for all $\lambda \in \mathbb{C}$, $\gamma_{\mathscr{I}} \in \mathbb{R}$ with $-\operatorname{Im}\lambda < \gamma_{\mathscr{I}} < \bar{q}$ for some fixed $\bar{q} \leq \underline{p}_{1,+} + \frac{1}{2}(n-1)$.

One can also consider special admissible operators P (see Definition 6.2) acting on sections of a vector bundle over M which is the pullback of a bundle $E \to \overline{\mathbb{R}^{n+1}}$; we leave the required notational modifications to the reader.

Example 9.1 (the model example). The Minkowski metric (3-1) satisfies assumptions (1)–(5). Indeed, it served as the motivating example for admissible metrics near $\mathscr{I}^+$ in Section 3.1. Also, the nontrapping condition is satisfied, since the backward null-bicharacteristics of $g_{\mathrm{e,b}}$ are limits of appropriate reparametrizations of backwards null-geodesics (lifted to $T^*\mathbb{R}^{n+1}$) at $\partial\overline{\mathbb{R}^{n+1}}$, which indeed have the required property; note here that differently (near $\mathscr{I}^+$) rescaled maximally extended backwards null-geodesics on $\overline{\mathbb{R}^{n+1}}$ start at the light cone at future infinity and tend to the light cone at past infinity, and on their journey cross the light cone at future infinity, or the closure of $t^{-1}(0)$, or both. Furthermore, the scalar wave operator on Minkowski space satisfies assumptions (6)–(7) with $\underline{p}_1 = \underline{p}_{1,+} = 0$ as shown in Example 3.10. Furthermore, the operator family $\widehat{P_+}(\lambda)$ is a conjugation of the spectral family on hyperbolic n-space; see [Baskin et al. 2015, §§7 and 10.1]. Using this, one can then show that (8) is satisfied for $\bar{q} = \frac{1}{2}(n-1)$. See also [Vasy 2014; Baskin and Marzuola 2019].

Remark 9.2 (nonexample: asymptotically Schwarzschild spacetimes and related settings). For $n = 3$ and in the region $r \geq \max(3\mathfrak{m}, c|t|)$ for any $c > 0$, the Schwarzschild metric $g_{\mathfrak{m}}$ (see Example 3.5) satisfies conditions (1) $\big($with $\ell_0 = 1$, $\ell_{\mathscr{I}} = \frac{1}{2}\big)$ and (2)–(5). The scalar wave operator $\Box_{g_{\mathfrak{m}}}$ satisfies conditions (6)–(7) in the same region. In order to control solutions of the wave equation globally in $t \geq 0$, one needs to capture the full structure of $g_{\mathfrak{m}}$ and $\Box_{g_{\mathfrak{m}}}$ also in $r \leq c|t|$. Not only are there trapped null-geodesics in this region, but even more severely the metric does not have smooth coefficients on $\overline{\mathbb{R}^{1+3}_{t,x}}$ (in the region where $r > 2\mathfrak{m}$): smoothness on this space amounts, near the "north pole" $N = \big\{\frac{x}{t} = 0, t^{-1} = 0\big\}$, to smoothness in x/t and $1/t$, whereas the metric, and the coefficients of the wave operator, are smooth only as functions of $x = \frac{x}{t}/\frac{1}{t}$, i.e., on the blow-up of $\overline{\mathbb{R}^{1+3}}$ at N.

Another setting which is a nonexample for the same lack-of-smoothness reason is the wave equation $\Box_{g_0} + V$ on Minkowski space coupled to a stationary potential $0 \neq V = V(x) \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3_x)$. Such operators, and indeed time-dependent and geometric generalizations thereof, are analyzed in [Hintz 2023d] (with the main results being regularity and decay estimates). Hintz [2023d] uses the present paper as an input for the microlocal description of a neighborhood of null infinity and combines it with microlocal estimates and spectral information far from null infinity in order to obtain global Fredholm and invertibility properties.

It is exactly for problems like those of [Hintz 2023d] that the microlocal description is key. See also Remark 1.2(3).

Theorem 9.3 (solving wave-type equations). *Let $\Omega \subset M$, P, and g be as above. Define edge-b-Sobolev spaces on M with respect to the volume density $|\mathrm{d}g|$. Let $s, \alpha_0, \alpha_{\mathscr{I}}, \alpha_+ \in \mathbb{R}$ and $k \in \mathbb{N}_0$, and suppose that*

$$\alpha_+ + \tfrac{1}{2} < \alpha_{\mathscr{I}} < \min\big(\alpha_0 + \tfrac{1}{2}, -\tfrac{1}{2} + \underline{p}_1\big), \quad s > -(\alpha_0 - \alpha_{\mathscr{I}}) - \big(-\tfrac{1}{2} + \underline{p}_1 - \alpha_{\mathscr{I}}\big).$$

Then the operator

$$P : \{u \in \dot{H}_{\mathrm{e,b;b}}^{(s;k),(\alpha_0,2\alpha_{\mathscr{I}},\alpha_+)}(\Omega) : Pu \in \dot{H}_{\mathrm{e,b;b}}^{(s-1;k),(\alpha_0+2,2\alpha_{\mathscr{I}}+2,\alpha_++2)}(\Omega)\} \to \dot{H}_{\mathrm{e,b;b}}^{(s-1;k),(\alpha_0+2,2\alpha_{\mathscr{I}}+2,\alpha_++2)}(\Omega)$$

is invertible, where $\dot{H}_{\mathrm{e,b;b}}^{(s;k),(\alpha_0,2\alpha_{\mathscr{I}},\alpha_+)}(\Omega)$ is the subspace of $H_{\mathrm{e,b;b}}^{(s;k),(\alpha_0,2\alpha_{\mathscr{I}},\alpha_+)}(M)$ consisting of those elements with support contained in Ω. That is,[36] for each forcing term $f \in \dot{H}_{\mathrm{e,b;b}}^{(s-1;k),(\alpha_0+2,2\alpha_{\mathscr{I}}+2,\alpha_++2)}(\Omega)$, the unique forward solution u of $Pu = f$ satisfies $u \in \dot{H}_{\mathrm{e,b;b}}^{(s;k),(\alpha_0,2\alpha_{\mathscr{I}},\alpha_+)}(\Omega)$.

Proof. We first consider the case $k = 0$.

• Step (1): *Solution up to a cross section of $\mathscr{I}^+$.* Let $f \in \dot{H}_{\mathrm{e,b}}^{s-1,(\alpha_0+2,2\alpha_{\mathscr{I}}+2,\alpha_++2)}(\Omega)$. Using the timelike nature of t and τ, one can solve $Pu_0 = f$ in the region $t \geq 0$, $\tau \leq \tau_0$ for any fixed $\tau_0 < 1$, with the solution u_0 lying in $H_{\mathrm{b}}^{s,\alpha_0}$ near $I^0 \cap \{t \geq 0, \tau \leq \tau_0\}$. Once this is shown for $s = 1$, one can infer this for $s \geq 1$ using real principal-type propagation of regularity in the b-setting; to get the full range of $s \in \mathbb{R}$, one uses duality arguments completely analogous to those in the proof of Theorem 6.4. The case $s = 1$ follows from an energy estimate with multiplier $-w\nabla\tau$, $w := e^{-\mathsf{F}\tau}\rho_0^{-2(\alpha_0+1)}$ for large $\mathsf{F} > 1$; see the proof of [Hintz and Vasy 2020, Proposition 4.3] for a very similar argument. In brief, note that $-w\nabla\tau$ is future timelike; the key computation then concerns the positive definiteness of $\rho_0^{2\alpha_0}\widetilde{K}_{-w\nabla\tau}$ (see (6-11)) as a quadratic form on ${}^{\mathrm{e,b}}T^*M$ in $\Omega^{\tau_0} := \{t \geq 0, \tau \leq \tau_0\}$. By (6-21), we have

$$\begin{aligned}\rho_0^{2\alpha_0}\widetilde{K}_{-w\nabla\tau} &= \rho_0^{2\alpha_0}(w\widetilde{K}_{-\nabla\tau} + 2T(\nabla w, -\nabla\tau)) \\ &= e^{-\mathsf{F}\tau}\left[\left(\rho_0^{-2}\widetilde{K}_{-\nabla\tau} - 4(\alpha_0+1)\rho_0^{-2}T\left(\frac{\nabla\rho_0}{\rho_0}, -\nabla\tau\right)\right) + 2\mathsf{F}\rho_0^{-2}T(-\nabla\tau, -\nabla\tau)\right].\end{aligned}$$

Both

$$\rho_0^{-2}\widetilde{K}_{-\nabla\tau} \quad \text{and} \quad \rho_0^{-2}T\left(\frac{\nabla\rho_0}{\rho_0}, -\nabla\tau\right)$$

are of class $(\mathcal{C}^\infty + \mathcal{A}^{\ell_0})(M; S^2\,{}^{\mathrm{e,b}}TM)$ in Ω^{τ_0}; and also $\rho_0^{-2}T(-\nabla\tau, -\nabla\tau)$ is of this class *and positive definite*. For sufficiently large F, $\rho_0^{2\alpha_0}\widetilde{K}_{-w\nabla\tau}$ is thus positive definite. Proceeding as in Section 6 then gives the energy estimate

$$\|u\|_{\rho_0^{\alpha_0}\dot{H}_{\mathrm{b}}^1(\Omega^{\tau_0})} \leq C\|f\|_{\rho_0^{\alpha_0+2}L^2(\Omega^{\tau_0})}.$$

Choosing τ_0 close enough to 1, Theorem 6.4 can be used to extend u_0 as a solution of $Pu_0 = f$ to a neighborhood of $I^0 \cap \mathscr{I}^+$, i.e., to a domain $\Omega^0_{\delta,\rho}$ (see (6-1)) for suitable δ and ρ. Indeed, for a cutoff χ which is 0 for $\tau \leq \tau_0$ and 1 for $\tau \geq \frac{1}{2}(1+\tau_0)$, Theorem 6.4 produces a forward solution u_1 of

[36] The equivalence of this statement with the invertibility of P follows from the uniqueness of (distributional) forward solutions of $Pu = f$.

$Pu_1 = f_1 := P(\chi u_0) = \chi f + [P, \chi]u_0$, which thus has supported character (i.e., vanishes) at $\Sigma^{0,\mathrm{in}}_{\delta,\rho}$; and necessarily $u_0 = (1-\chi)u_0 + u_1$, with the right-hand side defining the desired extension of u_0 of class $\dot{H}^{s,(\alpha_0,2\alpha_{\mathscr{I}})}_{\mathrm{e,b}}$ (omitting the weight at I^+ for now).

Fix now a smooth function $\mathfrak{t}$ on M° whose level sets $\mathfrak{t}^{-1}(c)$ for $c \in [-1,1]$ are spacelike and intersect $\mathscr{I}^+$ in its interior; we further arrange that $\mathrm{d}\mathfrak{t}$ is past timelike, with $\mathfrak{t}$ an increasing function of t except in a neighborhood of $\mathscr{I}^+$ where we demand $\mathfrak{t} = \rho_0 - (\rho + Cx_{\mathscr{I}}^{2\ell_{\mathscr{I}}})$ near $\mathscr{I}^+$ for some $\rho \in (0,1)$ and large C (cf. Lemma 3.6). Thus, the level sets of $\mathfrak{t}$ interpolate between the level sets of t and the surfaces $\Sigma^{0,\mathrm{out}}_{\delta,\rho}$; see (6-2) and Figure 6.1. Let now $\chi \in \mathcal{C}^\infty(M)$ denote a cutoff which is 0 in the causal past of $\mathfrak{t} = 0$ and 1 in the closure of the causal future $\Omega^+ \subset M$ of $\mathfrak{t} \geq \frac{1}{2}$. Then the global forward solution u of $Pu = f$ can be written as $u = (1-\chi)u_0 + u'$, where u' is the forward solution of

$$Pu' = f' := \chi f + [P,\chi]u_0 \in \dot{H}^{s-1,(\alpha_0+2,2\alpha_{\mathscr{I}}+2,\alpha_++2)}_{\mathrm{e,b}}(\Omega). \tag{9-1}$$

Since $(1-\chi)u_0 \in \dot{H}^{s,(\alpha_0,2\alpha_{\mathscr{I}},\alpha_+)}_{\mathrm{e,b}}(\Omega)$ (with the weight at I^+ in fact arbitrary), we need to show that the forward solution u' lies in this space as well (now with the weight at I^0 arbitrary).

• Step (2): *Solution near I^+.* Dropping the weight at I^0 from the notation, we shall show that

$$P : \{u \in \dot{H}^{s,(2\alpha_{\mathscr{I}},\alpha_+)}_{\mathrm{e,b}}(\Omega^+) : Pu \in \dot{H}^{s-1,(2\alpha_{\mathscr{I}}+2,\alpha_++2)}_{\mathrm{e,b}}(\Omega^+)\} \to \dot{H}^{s-1,(2\alpha_{\mathscr{I}}+2,\alpha_++2)}_{\mathrm{e,b}}(\Omega^+) \tag{9-2}$$

is invertible, i.e., that (9-1) has a solution in $\dot{H}^{s,(\alpha_0+2,2\alpha_{\mathscr{I}}+2,\alpha_++2)}_{\mathrm{e,b}}(\Omega)$. We shall in fact show the stronger statement that (9-2) is invertible for all $s \in \mathbb{R}$. (This is possible since the only microlocal propagation result placing a restriction on s is the radial point estimate at $\mathcal{R}_{\mathrm{c}}$ — Lemma 4.8 — which lies outside of Ω^+.) Our argument proceeds as follows.

- Step (2)(i). In the spirit of [Hintz and Vasy 2020, Proposition 4.11], we first prove the surjectivity of (9-2) for $s = 1$ when the weight α_+ at I^+ is replaced by a very negative weight α'_+ (thus allowing for fast polynomial growth as $t - r \to \infty$) using an energy estimate which is global near I^+.
- Step (2)(i) still. When $s \geq 1$, we can upgrade the regularity from $H^1_{\mathrm{e,b}}$ to $H^s_{\mathrm{e,b}}$ using microlocal propagation estimates: this uses the radial point estimates from Lemmas 4.10 and 4.11 and real principal-type propagation estimates everywhere else; the nontrapping assumption (5) on the metric g implies that a concatenation of these estimates gives microlocal control everywhere in Ω^+.
- Step (2)(ii). To improve the I^+-weight to α_+, we use a Mellin transform and contour shifting argument which exploits the invertibility properties of the I^+-normal operator (assumption (8)). This is akin to the arguments at the end of [Hintz and Vasy 2020, §5.2].
- Step (2)(iii). The full range of s is then obtained using an approximation argument. See Remark 9.4 for an alternative argument which avoids the use of a global energy estimate.

• Step (2)(i): *Solution for $s \geq 1$ and very negative I^+-weight.* When $s = 1$ and if one replaces α_+ by $\alpha'_+ \leq \alpha_+$ with α'_+ sufficiently negative, a solution can be shown to exist via an energy estimate on Ω^+. A key point is that $-\rho_+\partial_{\rho_+}$ is uniformly future timelike near compact subsets of $I^+ \setminus \mathscr{I}^+$; this implies that weights $\rho_+^{-(2\alpha'_+ + 1)}$ in vector field multipliers can be used to give positive bulk terms in energy estimates.

Concretely, if $\chi \in \mathcal{C}^\infty(M)$ equals 1 in a small neighborhood of $\mathscr{I}^+$ and 0 outside a slightly larger neighborhood, one can work with the (future timelike) vector field multiplier

$$\rho_+^{-2(\alpha'_+-\alpha_+)}[\chi(\rho_+^{-2(\alpha_++1)}x_{\mathscr{I}}^{-4\alpha_{\mathscr{I}}-2}(x_{\mathscr{I}}\partial_{x_{\mathscr{I}}}-(2+c)\rho_+\partial_{\rho_+}))+(1-\chi)\cdot(-\rho_+^{-2(\alpha_++1)}\rho_+\partial_{\rho_+})],$$

cf. (7-5). Near $\mathscr{I}^+$, the positivity of the associated (modified) K-current follows (for arbitrary $\alpha'_+ \geq \alpha_+$) from (7-7) and the subsequent discussion. Choosing α'_+ so that $\alpha'_+ - \alpha_+$ is large and exploiting the timelike nature of $-\nabla\rho_+/\rho_+$ away from $\mathscr{I}^+$ implies this positivity globally near I^+. See [Hintz and Vasy 2020, Proposition 4.11] for details of this argument.

Higher regularity $s \geq 1$ then follows by propagating edge-b-regularity through $\mathscr{I}^+$ using Theorem 4.5(1) (or indeed just Lemma 4.10), then along the characteristic set of P in the b-cotangent bundle over $(I^+)^\circ$, and then into $\mathscr{I}^+$ using Theorem 4.5(2) (or indeed just Lemma 4.11(2)); the control of u in a punctured neighborhood of the outgoing radial set $\mathcal{R}_{\mathrm{out}}$ required for this final step uses the nontrapping assumption. (If one constructs the solution u via a duality argument, one can cover all $s \geq 0$ in this fashion.)

• Step (2)(ii): *Solution for $s \geq 1$ and the I^+-weight α_+.* Thus far, we have obtained a solution $u \in \dot{H}_{\mathrm{e,b}}^{s,(2\alpha_{\mathscr{I}},\alpha'_+)}(\Omega^+)$. We shall now show that it has better decay by utilizing a standard normal operator, Mellin transform, and contour shifting argument near I^+. The key ingredient is the fact that

$$\widehat{P_+}(\lambda)^{-1} : H_0^{s-2,2\gamma_{\mathscr{I}}}(I^+) \to H_0^{s,2\gamma_{\mathscr{I}}}(I^+) \tag{9-3}$$

(where $\gamma_{\mathscr{I}} = \alpha_{\mathscr{I}} + \frac{1}{2}n$ as in (8-4), and the Sobolev spaces are defined with respect to a positive b-density on I^+) is an analytic family of bounded operators for $-\operatorname{Im}\lambda < \gamma_{\mathscr{I}} < \bar{q}$, which moreover satisfies high energy estimates

$$\|v\|_{H_{0,h}^{s,2\gamma_{\mathscr{I}}}(I^+)} \leq Ch^{-1}\|\widehat{P_+}(\lambda)v\|_{H_{0,h}^{s-2,2\gamma_{\mathscr{I}}}(I^+)}, \quad h = \langle\lambda\rangle^{-1}, \tag{9-4}$$

when $-\operatorname{Im}\lambda = \gamma_+$, with $\gamma_+ \in (-\infty, \gamma_{\mathscr{I}})$ contained in a compact subinterval. To begin the proof of (9-3), one combines Theorem 8.2(1) and elliptic estimates for $\widehat{P_+}(\lambda)$ in $(I^+)^\circ$; the ellipticity of $\widehat{P_+}(\lambda)$ (as a differential operator on $(I^+)^\circ$) follows from assumption (4) above. This gives the estimate

$$\|v\|_{H_0^{s,2\gamma_{\mathscr{I}}}(I^+)} \leq C(\|\widehat{P_+}(\lambda)v\|_{H_0^{s-2,2\gamma_{\mathscr{I}}}(I^+)} + \|v\|_{H_0^{-N,-N}(I^+)}).$$

Moreover, an application of the parametrix used in the proof of Theorem 8.2 shows that every $v \in H_0^{s,2\gamma_{\mathscr{I}}}(I^+)$ with $\widehat{P_+}(\lambda)v = 0$ automatically satisfies $v \in \mathcal{A}^{2\gamma_{\mathscr{I}}}(I^+)$; since by assumption (8) this implies $v = 0$, we can drop the compact error term in this estimate upon increasing C. (One can choose C uniformly when $\lambda \in \mathbb{C}$ is restricted to a compact set.) In order to get (9-3), one also needs the surjectivity of $\widehat{P_+}(\lambda)$; this is a consequence of the invertibility of $\widehat{P_+}(\lambda)$ for large $|\operatorname{Re}\lambda|$ (when $\operatorname{Im}\lambda$ is contained in a compact subinterval of $(-\infty, \gamma_{\mathscr{I}})$), to which we turn now.

To wit, the estimate (9-4) follows by applying Theorem 8.5(1) to control semiclassical 0-regularity at ${}^0\mathcal{R}_{\mathrm{in}}^{\pm}$ (the sign corresponding to the sign of $\operatorname{Re}\lambda$), which due to the nontrapping assumption on P can be propagated into a full punctured neighborhood of ${}^0\mathcal{R}_{\mathrm{out}}^{\pm}$. There, Theorem 8.5(2) applies. Thus, one gets (9-4) with an error term $Ch^N\|v\|_{H_{0,h}^{-N,2\gamma_{\mathscr{I}}}(I^+)}$ for any fixed N, which for sufficiently small h can be absorbed into the left-hand side.

By analogous means, but propagating in the reverse direction using Theorem 8.6, one can prove the adjoint estimate

$$\|v\|_{H_{0,h}^{-s+2,-2\gamma_{\mathscr{I}}}(I^+)} \leq Ch^{-1}\|\widehat{P_+^*}(\lambda)v\|_{H_{0,h}^{-s,-2\gamma_{\mathscr{I}}}(I^+)}, \quad h = \langle\lambda\rangle^{-1},$$

which in particular implies the triviality of the cokernel of $\widehat{P_+}(\lambda)$ for sufficiently large $|\operatorname{Re}\lambda|$. Since the Fredholm index of

$$\widehat{P_+}(\lambda) : H_0^{s,2\gamma_{\mathscr{I}}}(I^+) \to H_0^{s-2,2\gamma_{\mathscr{I}}}(I^+)$$

is constant in λ, this completes the proof of (9-3).

Returning to the task of improving the decay of $u \in \dot{H}_{\mathrm{e,b}}^{s,(2\alpha_{\mathscr{I}},\alpha'_+)}(\Omega^+)$, one writes

$$N_{I^+}(\rho_+^{-2}x_{\mathscr{I}}^{-2}P)u = \rho_+^{-2}x_{\mathscr{I}}^{-2}f - (P - N_{I^+}(\rho_+^{-2}x_{\mathscr{I}}^{-2}P))u,$$

with the second term on the right lying in $\dot{H}_{\mathrm{e,b}}^{s-2,(2\alpha_{\mathscr{I}},\alpha'_+ + \ell_+)}(\Omega^+)$. Inverting $N_{I^+}(\rho_+^{-2}x_{\mathscr{I}}^{-2}P)$ using the Mellin transform (see Lemma 2.9) and its inverse gives

$$u \in \dot{H}_{\mathrm{e,b}}^{s-1,(2\alpha_{\mathscr{I}},\min(\alpha_+,\alpha'_+ + \ell_+))}(\Omega^+),$$

which improves on the weight of u at I^+, at the expense of one edge-b-derivative. The edge-b-regularity can then be improved to s again using the microlocal propagation results as before (note that the propagation results near $\mathscr{I}^+ \cap I^+$ do not require any conditions on s). This improves the decay of u at I^+ by ℓ_+ until the decay rate α_+ is obtained after a finite number of steps. This proves the theorem for $s \geq 1$ and $k = 0$.

• Step (2)(iii): *Full range of s.* Consider now the remaining case that $s < 1$. The microlocal propagation estimates imply an a priori estimate for P,

$$\|u\|_{\dot{H}_{\mathrm{e,b}}^{s,(2\alpha_{\mathscr{I}},\alpha_+)}(\Omega^+)} \leq C(\|Pu\|_{\dot{H}_{\mathrm{e,b}}^{s-1,(2\alpha_{\mathscr{I}}+2,\alpha_++2)}(\Omega^+)} + \|u\|_{\dot{H}_{\mathrm{e,b}}^{-N,(2\alpha_{\mathscr{I}},\alpha_+)}(\Omega^+)}), \tag{9-5}$$

where we fix N so that $-N < s-2$. We then apply Theorem 7.3(1) to the second term on the right, with ϵ in (7-14) chosen so small that $C\epsilon < \frac{1}{2}$, and therefore the error term $\epsilon\|u\|_{\dot{H}_{\mathrm{e,b}}^{-N+1,(2\alpha_{\mathscr{I}},\alpha_+)}(\Omega^+)}$ from (7-13) can be absorbed into the left-hand side of (9-5). This implies that (9-5) holds (for a different constant) with the error term $\|u\|_{\dot{H}_{\mathrm{e,b}}^{-N+1,(2\alpha_{\mathscr{I}}-2\ell_{\mathscr{I}},\alpha_+)}(\Omega^+)}$ which is improved (i.e., weaker) at $\mathscr{I}^+$. One next estimates u, localized to a collar neighborhood of I^+, in terms of the I^+-normal operator of $\rho_+^{-2}x_{\mathscr{I}}^{-2}P$ applied to u using the (inverse) Mellin transform. Since $\rho_+^{-2}x_{\mathscr{I}}^{-2}P$ differs from its I^+-normal operator by an element of $\mathcal{A}^{(0,\ell_+)}\operatorname{Diff}_{\mathrm{e,b}}^2$, this improves the error term further to $\|u\|_{\dot{H}_{\mathrm{e,b}}^{-N+2,(2\alpha_{\mathscr{I}}-2\ell_{\mathscr{I}},\alpha_+-\ell_+)}(\Omega^+)}$. Since the embedding

$$\dot{H}_{\mathrm{e,b}}^{s,(2\alpha_{\mathscr{I}},\alpha_+)}(\Omega^+) \hookrightarrow \dot{H}_{\mathrm{e,b}}^{-N+2,(2\alpha_{\mathscr{I}}-2\ell_{\mathscr{I}},\alpha_+-\ell_+)}(\Omega^+) \tag{9-6}$$

is compact, this now implies that P, as a map (9-2), has closed range. But we have already shown that the range includes $\dot{H}_{\mathrm{e,b}}^{0,(2\alpha_{\mathscr{I}},\alpha_+)}(\Omega^+)$, which is a dense subspace. Therefore, P in (9-2) is surjective. The proof is complete in the case $k = 0$.

• Step (3): *Higher b-regularity.* The proof of the theorem for $k \in \mathbb{N}$ follows from the same inductive argument as in the proof of Corollary 6.6. □

Remark 9.4 (alternative proof of solvability). An alternative proof of the invertibility of (9-2) proceeds as follows. First, one shows that P in (9-2) is Fredholm (which again uses the I^+-normal operator and the Mellin transform as in Step (2)(iii) of the above proof). The I^+-normal operator is surjective on the spaces (9-2) since solutions can be written down using the (inverse) Mellin transform (using the Paley–Wiener theorem for the support condition). But the localization of the difference of P and its I^+-normal operator to a neighborhood $\{\rho_+ < \epsilon\}$ of I^+ tends to 0 in $x_{\mathscr{I}}^2\rho_+^2\mathcal{A}^{(0,\delta)}\operatorname{Diff}^2_{\mathrm{c,b}}$ as $\epsilon \searrow 0$ for any fixed $\delta \in (0, \ell_+)$, and one can then upgrade the Fredholm property of P to invertibility on $\{\rho_+ < \epsilon\}$ for small enough ϵ. The *finite time* solvability of P on the remaining region $\Omega^+ \setminus \{\rho_+ < \epsilon\}$ is clear, and thus one obtains the invertibility of (9-2). For a detailed implementation of this approach, see [Hintz 2023d, Proof of Theorem 5.23, and Appendix A].

Remark 9.5 (conormal regularity and pointwise decay). If we work with $k > \frac{1}{2}(n+1)$ degrees of b-regularity, the pointwise bound on u provided by Theorem 9.3 is $\mathcal{O}(\rho_{\mathscr{I}}^{\alpha_{\mathscr{I}}+n/2}\rho_+^{\alpha_+ +(n+1)/2})$ (cf. (8-4)) by Sobolev embedding (for b-Sobolev spaces). In Example 9.1, the strongest possible bound arises by taking $\alpha_{\mathscr{I}} = -\frac{1}{2} - \epsilon$ and $\alpha_+ = -1 - 2\epsilon$, and hence one gets almost (namely, up to an r^ϵ loss for any $\epsilon > 0$) the sharp $\mathcal{O}(r^{-(n-1)/2})$ decay towards $\mathscr{I}^+$, and almost $\mathcal{O}((t-r)^{-(n-1)/2})$ decay towards future timelike infinity (which is far from the sharp $(t-r)^{-(n-1)}$ bound on Minkowski spacetimes with odd spacetime dimension, but matches what simple vector field methods give; see, e.g., [Klainerman 1985]). Improved decay at I^+ requires estimates on the meromorphic continuation of $\widehat{P_+}(\lambda)^{-1}$ across the line $-\operatorname{Im}\lambda = \underline{p}_{1,+} + \frac{1}{2}(n-1)$, which is a delicate problem, cf. Remark 8.4.

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COMPLEX HESSIAN MEASURES WITH RESPECT TO A BACKGROUND HERMITIAN FORM

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To the memory of Jean-Pierre Demailly

We develop potential theory for m-subharmonic functions with respect to a Hermitian metric on a Hermitian manifold. First, we show that the complex Hessian operator is well defined for bounded functions in this class. This allows us to define the m-capacity and then show the quasicontinuity of m-subharmonic functions. Thanks to this we derive other results parallel to those in pluripotential theory such as the equivalence between polar sets and negligible sets. The theory is then used to study the complex Hessian equation on compact Hermitian manifold with boundary, with the right-hand side of the equation admitting a bounded subsolution. This is an extension of a recent result of Collins and Picard dealing with classical solutions.

1. Introduction

The m-Hessian operator is defined in terms of elementary symmetric polynomials of degree m of eigenvalues of the Hessian matrix of the given function. If the degree is equal to the dimension of the space then one deals with the most important case of the Monge–Ampère operator. One can also consider more general symmetric functions of eigenvalues. The nonlinear equations involving such operators will be called in this article *Hessian-type* equations. They do appear in geometry in problems involving curvatures, like the prescribed Gauss curvature equation or the Lagrangian mean curvature equation. The m-Hessian equations in $\mathbb{R}^n$ were first solved by Caffarelli, Nirenberg and Spruck [Caffarelli et al. 1985] for smooth, nondegenerate data. The study of weak solutions for measures on the right-hand side was initiated by Trudinger and Wang [1997; 1999; 2002] (see also [Wang 2009]).

Here we are interested in the complex setting and weak solutions. For smooth data the first solutions in complex variables were obtained by Vinacua [1988] and S.-Y. Li [2004] who followed the method of [Caffarelli et al. 1985]. Błocki [2005] adopted the methods of pluripotential theory (initiated by Bedford and Taylor [1976; 1982] in relation to the complex Monge–Ampère equation) to define the action of the m-Hessian operator on nonsmooth functions and study weak solutions of the associated equation.

Let $\Omega \subset \mathbb{C}^n$ be an open set and ω be a positive Hermitian $(1,1)$-form on Ω. Let $1 \le m \le n$ be an integer and consider a function $u \in C^2(\Omega, \mathbb{R})$. The complex Hessian operator with respect to ω acts on u by

$$H_m(u) = (dd^c u)^m \wedge \omega^{n-m}.$$

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The operator is elliptic if we restrict ourselves to functions u whose eigenvectors $\lambda = (\lambda_1, \dots, \lambda_n)$ of the complex Hessian matrix $[u_{i\bar{j}}]_{1\le i,j\le n}$, with respect to ω, belong to the Gårding cone

$$\Gamma_m = \{\lambda \in \mathbb{R}^n : S_1(\lambda) > 0, \dots, S_m(\lambda) > 0\},$$

where $S_k(\lambda)$ is the k-th elementary symmetric polynomial on λ. Such a function is called m-ω-subharmonic (or m-ω-sh for short).

For $\omega = dd^c|z|^2$ the standard Kähler form on $\mathbb{C}^n$ Błocki defined nonsmooth m-subharmonic functions. He showed that the Hessian operator acting on a bounded m-subharmonic function is a well-defined positive Radon measure, that the operator is stable under decreasing sequences, and that the homogeneous Dirichlet problem is solvable. The nonhomogeneous one with the right-hand side in L^p, $p > n/m$, was solved by Dinew and Kołodziej in [2014].

On a compact Hermitian manifold (X, ω) the right m-Hessian operator to consider is

$$H_m(u) = (dd^c u + \omega)^m \wedge \omega^{n-m},$$

or more generally

$$H_{m,\alpha}(u) = (dd^c u + \alpha)^m \wedge \omega^{n-m},$$

where α is another $(1,1)$-form.

For ω Kähler the counterpart of the Calabi–Yau theorem was shown by Dinew and Kołodziej [2017], with a use of earlier C^2 estimates of Hou, Ma and Wu [Hou et al. 2010]. Having this result an analogue of pluripotential theory yields weak solutions (see [Dinew and Kołodziej 2014]). We refer the readers to [Dinew and Lu 2015; Lu 2013a; 2013b; 2015; Lu and Nguyen 2015] for results in potential theory for m-ω-sh functions on a compact Kähler manifold.

Our first goal here is to develop potential theory for m-subharmonic functions (with respect to a Hermitian metric) on a Hermitian manifold. The results often parallel those of pluripotential theory.

Now we assume that ω is a general Hermitian metric. The complex m-Hessian equation on compact manifolds was solved independently by Székelyhidi [2018] and Zhang [2017]. The current authors obtained weak continuous solutions for the right-hand side in L^p, $p > n/m$ [Kołodziej and Nguyen 2016]. This partially motivates the development of potential theory for m-ω-sh functions on Hermitian manifolds. Unlike in the Kähler case, we have to deal with the nonzero torsion terms $dd^c\omega$ and $d\omega \wedge d^c\omega$. A direct computation shows that for a smooth m-ω-sh function u, $0 \le p \le n-m-1$ and $k \ge 1$, the form $(dd^c u)^k \wedge \omega^p$ may not be positive. Those terms appear when we perform integration by parts. This makes the proofs of basic potential estimates in the Hermitian setting substantially more difficult. For example, we need to fully exploit the properties of the positive cone Γ_m, and show new inequalities on elementary symmetric polynomials to prove the Chern–Levine–Nirenberg (CLN) inequality [Kołodziej and Nguyen 2016] and a variant of the Cauchy–Schwarz inequality in this paper. This coupled with the uniform convergence allows us to define the complex Hessian measure of a continuous m-ω-sh function u as the weak limit of

$$H_m(u) := \lim_{\delta\to 0} H_m(u^\delta) = \lim_{\delta\to 0} (dd^c u^\delta)^m \wedge \omega^{n-m}, \tag{1-1}$$

where $\{u^\delta\}$ is a sequence of smooth m-ω-sh functions converging uniformly to u.

However, if u is a bounded m-ω-sh function, up until now we have not been able to define its complex Hessian measure. Thanks to a basic observation (Lemma 3.1) that $dd^c u$ is a current of order zero we found a very natural way to extend the classical approach of Bedford–Taylor in this case. The first main result of the paper (Theorem 3.3) shows that the measure on the left-hand side of (1-1) can be defined by taking the weak limit inductively. Thus, for bounded m-ω-subharmonic functions u and $u^\delta \downarrow u$ pointwise as $\delta \to 0$,

$$H_m(u) := \lim_{\delta_m \to 0} \cdots \lim_{\delta_1 \to 0} dd^c u^{\delta_m} \wedge \cdots \wedge dd^c u^{\delta_1} \wedge \omega^{n-m}$$

exists in the sense of currents of order zero. The proof is based on the CLN inequality in [Kołodziej and Nguyen 2016]. This is the starting point for proving analogues of Bedford–Taylor results presented in Chapter 1 of [Kołodziej 2005]. The main difference is that we no longer have a nice integration by parts formula for closed positive currents. Instead we need to work with the (nonclosed) currents of order zero in general. This makes our proof of weak convergence of currents under decreasing limits of m-ω-sh bounded functions in Proposition 3.20 more complicated than the one in [Bedford and Taylor 1982]. Next, we obtain (Theorem 4.9) the quasicontinuity of m-ω-sh functions with respect to a suitable m-capacity: for a Borel set $E \subset \Omega$,

$$\operatorname{cap}_m(E) = \sup\Big\{ \int_E H_m(u) : u \text{ is } m\text{-}\omega\text{-sh in } \Omega,\ -1 \le u \le 0 \Big\}. \tag{1-2}$$

To define this capacity we needed Theorem 3.3. Once quasicontinuity is proven, one obtains weak convergence of mixed wedge products of the forms $dd^c u_j$ for m-ω-sh functions u_j under monotone convergence (Lemmas 5.1 and 5.4) following the classical arguments in [Bedford and Taylor 1982; 1987]. Next we study the polar sets and negligible sets of m-ω-sh functions. In this setting it seems impossible to obtain nice formulae for the capacity of compact or open sets in terms of Hessian measures of extremal functions as it is the case for Monge–Ampère measures. Exploiting further the properties of Γ_m in Section 2.4, especially Proposition 2.15 we are able to compare the outer capacity and the Hessian measures of relative extremal functions in Lemma 7.5. This suffices to give a characterization of a polar set by $\operatorname{cap}_m^*(E) = 0$ in Proposition 7.7. Consequently, we conclude the equivalence of polar sets and negligible sets (Theorem 7.8).

In the last sections we apply the above results to the complex m-Hessian equation. Recently, there is a lot of active research on fully nonlinear elliptic equations on compact Hermitian manifolds with or without boundary (see [Chu and McCleerey 2021; Collins and Picard 2022; Dong 2021; Dong and Li 2021; Guan 1998; Guan and Li 2010; Guedj and Lu 2025; Guo and Phong 2024; Li and Shen 2020; Phong and Tô 2021; Székelyhidi 2018; Székelyhidi et al. 2017; Tosatti and Weinkove 2010; Yuan 2021]) in various geometric contexts.

In particular Collins and Picard [2022] solved the Dirichlet problem for the m-Hessian equation in an open subset of a Hermitian manifold under the hypothesis of existence of a subsolution and smooth data. We extend it in Sections 8 and 9 showing that the existence of a bounded subsolution implies the existence of a bounded solution for both bounded domains (Theorem 8.7) and compact complex manifolds with boundary (Theorem 9.1). In the proof the above equivalence of polar and negligible sets will play a crucial role (see Lemma 8.3). The homogeneous m-Hessian equation on a (Kähler) manifold with boundary was recently solved in a particular case in [Wu 2023] in relation to the Wess–Zumino–Witten-type equation proposed by Donaldson [1999].

2. Generalized m-subharmonic functions

2.1. ***Elementary symmetric positive cones.*** In this section we prove important pointwise estimates for elementary symmetric polynomials. Let $1 \le m \le n$ be two integers. The positive cone Γ_m is given by

$$\Gamma_m = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : S_1(\lambda) > 0, \dots, S_m(\lambda) > 0\}, \tag{2-1}$$

where $S_k(\lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}$; and conventionally, $S_0(\lambda) = 1$ and $S_k(\lambda) = 0$ for $k < 0$ or $k > n$. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma_m$ be arranged in the decreasing manner, i.e.,

$$\lambda_1 \ge \cdots \ge \lambda_m \ge \cdots \ge \lambda_n.$$

Then, we know from [Ivochkina 1983, Lemma 8] that $\lambda_m > 0$ which is a consequence of a characterization of the cone Γ_m. Namely, for $\{i_1, \dots, i_t\} \subset \{1, \dots, n\}$ such that $k + t \le m$, one has

$$S_{k;i_1 \cdots i_t}(\lambda) > 0, \tag{2-2}$$

where $S_{k;i_1 \cdots i_t}(\lambda) = S_k|_{\lambda_{i_1} = \cdots = \lambda_{i_t} = 0}$. This implies also that for $1 \le k \le m$,

$$S_{k-1}(\lambda) \ge \lambda_1 \cdots \lambda_{k-1}. \tag{2-3}$$

Lemma 2.1. *There exists $\theta = \theta(n, m) > 0$ such that the following statements hold.*

(a) *For $1 \le j \le m$, $\lambda_j S_{m-1;j} \ge \theta S_m$.*

(b) *For $1 \le i \le m-1$, $\lambda_i S_{m-2;im} \ge \theta S_{m-1;m}$.*

Proof. The item (a) follows from [Kołodziej and Nguyen 2016, equation (2.7)], while (b) follows from (a) if we replace n, m and S_m by $n-1$, $m-1$ and $S_{m-1;m}$, respectively. □

Lemma 2.2. *There exists a uniform constant C, depending on n, m, such that the following inequalities are satisfied.*

(a) *For $1 \le i \le m-1$ and $\lambda \in \Gamma_m$,*

$$\frac{\lambda_1 \cdots \lambda_m}{\lambda_i} \le C(S_{m-1;i} S_{m-1})^{\frac{1}{2}}.$$

(b) *Generally, for $1 \le \ell \le n$ and increasing multi-indices $(i_1, \dots, i_{m-1})$,*

$$\prod_{\substack{i_s \ne \ell \\ s=1}}^{m-1} |\lambda_{i_s}| \le C(S_{m-1;\ell} S_{m-1})^{\frac{1}{2}}.$$

Proof. (a) The inequality is equivalent to saying that there exist uniform constants $c_1, c_2 > 0$ such that for every positive number a,

$$a \frac{\lambda_1 \cdots \lambda_m}{\lambda_i} \le c_1 a^2 S_{m-1;i} + c_2 S_{m-1}, \tag{2-4}$$

where $1 \le i \le m-1$. In fact as we will see in the proof c_1, c_2 are explicitly given constants. We observe that if $a \le 1$, then we can easily get the claim as

$$\frac{\lambda_1 \cdots \lambda_m}{\lambda_i} \le \lambda_1 \cdots \lambda_{m-1} \le S_{m-1}.$$

Now we consider $a > 1$. We prove (a) for the case $i = 1$, the other cases $1 \le i \le m-1$ follow in the same way. The basic identities/inequalities are

$$\begin{aligned} S_{m;1m} + \lambda_1 S_{m-1;1m} &= S_{m;m} = S_m - \lambda_m S_{m-1;m} \\ &\ge -\lambda_m S_{m-1;m}, \end{aligned} \tag{2-5}$$

and

$$S_{m-1;1m} + \lambda_1 S_{m-2;1m} = S_{m-1;m}. \tag{2-6}$$

Multiplying (2-5) by $S_{m-2;1m}$ and (2-6) by $S_{m-1;1m}$ to eliminate λ_1 we get that

$$\begin{aligned} S_{m-1;1m}^2 - S_{m;1m} S_{m-2;1m} &\le S_{m-1;m}(S_{m-1;1m} + \lambda_m S_{m-2;1m}) \\ &= S_{m-1;m} S_{m-1;1}. \end{aligned} \tag{2-7}$$

The Newton inequality holds for every $\lambda \in \mathbb{R}^n$ and tells us

$$S_{m;1m} S_{m-2;1m} \le \frac{(m-1)(n-m+1)}{m(n-m+2)} [S_{m-1;1m}]^2 =: c_m [S_{m-1;1m}]^2.$$

Notice that $0 < c_m < 1$. Hence, we derive from the above and (2-7) that

$$S_{m-1;1m}^2 - c_m [S_{m-1;1m}]^2 \le S_{m-1;m} S_{m-1;1}.$$

Therefore,

$$S_{m-1;1m}^2 \le \frac{1}{1-c_m} S_{m-1;m} S_{m-1;1}. \tag{2-8}$$

Using the Cauchy–Schwarz inequality, (2-8) and

$$S_{m-1;1m} = S_{m-1;1} - \lambda_m S_{m-2;1m},$$

we get

$$\begin{aligned} a^2 S_{m-1;1} + \frac{1}{4(1-c_m)} S_{m-1;m} &\ge a \left[\frac{S_{m-1;1} S_{m-1;m}}{1-c_m} \right]^{\frac{1}{2}} \\ &\ge a |S_{m-1;1m}| \\ &\ge a(-S_{m-1;1} + \lambda_m S_{m-2;1m}). \end{aligned}$$

This implies

$$(a^2 + a) S_{m-1;1} + \frac{1}{4(1-c_m)} S_{m-1;m} \ge a \lambda_m S_{m-2;1m}. \tag{2-9}$$

Since $a \geq 1$, it follows that $2a^2 S_{m-1;1} + C S_{m-1} \geq a\lambda_m S_{m-2;1m}$. So, using Lemma 2.1 and (2-3) we have

$$\begin{aligned}(2a^2 S_{m-1;1} + C S_{m-1})\lambda_1 &\geq a\lambda_m \lambda_1 S_{m-2;1m} \\ &\geq a\theta^2 \lambda_m S_{m-1} \\ &\geq a\theta^2 \lambda_1 \cdots \lambda_m. \end{aligned} \tag{2-10}$$

The proof of the lemma is completed with $c_1 = 2$, $c_2 = 1/(4(1-c_m))$ and $C = \sqrt{2/(1-c_m)}$.

(b) The characterization (2-2) implies that a sum of any $n-m+1$ entries of λ is positive. Hence, we have for $\lambda_{i_s} \leq 0$,

$$|\lambda_{i_s}| \leq (n-m)\lambda_m.$$

So, for $1 \leq \ell \leq m-1$,

$$\prod_{\substack{i_s \neq \ell \\ s=1}}^{m-1} |\lambda_{i_s}| \leq (n-m)^{m-1} \frac{\lambda_1 \cdots \lambda_m}{\lambda_\ell} \leq C[S_{m-1;\ell} S_{m-1}]^{\frac{1}{2}}, \tag{2-11}$$

where we used (a) for the second inequality.

Now we treat the remaining range $m \leq \ell \leq n$. By a result of Lin and Trudinger [1994, Theorem 1.1], we know that $S_{m-1;\ell} \geq \theta S_{m-1}$ for a constant $\theta = \theta(n, m)$ depending only on n, m. This implies

$$S_{m-1} \leq \frac{(S_{m-1;\ell} S_{m-1})^{\frac{1}{2}}}{\sqrt{\theta}}.$$

Thus, the desired inequality easily follows from this and the bound

$$\prod_{\substack{i_s \neq \ell \\ s=1}}^{m-1} |\lambda_{i_s}| \leq (n-m)^{m-1} \lambda_1 \cdots \lambda_{m-1} \tag{2-12}$$

for $m \leq \ell \leq n$. □

2.2. ***The Cauchy–Schwarz inequality.*** Let ω be a Hermitian metric on $\mathbb{C}^n$ and let Ω be a bounded open set in $\mathbb{C}^n$. The positive cone $\Gamma_m(\omega)$, associated to ω, of real $(1,1)$-forms is defined as follows. A real form γ is said to belong to $\Gamma_m(\omega)$ if at any point $z \in \Omega$,

$$\gamma^k \wedge \omega^{n-k}(z) > 0 \quad \text{for } k = 1, \ldots, m.$$

Equivalently, in the normal coordinate system with respect to ω at z, diagonalizing $\gamma = \sqrt{-1} \sum_i \lambda_i dz_i \wedge d\bar{z}_i$, we have $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_m$. Now we will translate the estimates in Section 2.1 into the integral forms.

We can state the following versions of the Cauchy–Schwarz inequality in this setting. Let h be a smooth real-valued function and ϕ, ψ be Borel functions. Let T be a positive current of bidegree $(n-2, n-2)$.

Lemma 2.3. *There exists a uniform constant C depending on ω such that*

$$\left| \int \phi\psi \, dh \wedge d^c\omega \wedge T \right|^2 \leq C \int |\phi|^2 \, dh \wedge d^c h \wedge \omega \wedge T \int |\psi|^2 \, \omega^2 \wedge T.$$

Proof. The proof of [Nguyen 2016, Proposition 1.4] can be easily adapted. □

The above lemma can be applied for the case $T = \gamma^s \wedge \omega^{n-m+\ell}$, where $\gamma \in \Gamma_m(\omega)$ and $0 \le s, \ell \le m-1$ and $s + \ell = m - 1$. Next, we also need to deal with possible nonpositive forms $T' = \gamma^{m-1} \wedge \omega^{n-m-1}$ where the classical Cauchy–Schwarz is not immediately applicable. However, we still have:

Lemma 2.4. *There exists a uniform constant C depending on ω, n, m such for every $\gamma \in \Gamma_m(\omega)$,*

$$\left| \int \phi\psi \, dh \wedge d^c\omega \wedge \gamma^{m-1} \wedge \omega^{n-m-1} \right|^2 \le C \int |\phi|^2 \, dh \wedge d^c h \wedge \gamma^{m-1} \wedge \omega^{n-m} \times \int |\psi|^2 \, \gamma^{m-1} \wedge \omega^{n-m+1}.$$

Proof. We express the integrands of both sides as follows.

$$\begin{aligned} dh \wedge d^c\omega \wedge \gamma^{m-1} \wedge \omega^{n-m-1} &= f_1(z)\omega^n, \\ dh \wedge d^c h \wedge \gamma^{m-1} \wedge \omega^{n-m} &= [f_2(z)]^2\omega^n, \\ \gamma^{m-1} \wedge \omega^{n-m+1} &= [f_3(z)]^2\omega^n. \end{aligned}$$

Thus, the inequality will follow from the classical Cauchy–Schwarz inequality if we have pointwise $|f_1(z)| \le C f_2(z) f_3(z)$ for every $z \in \Omega$. This is proved by using the normal coordinate system at a given point z with respect to ω which diagonalizes also γ, i.e.,

$$\omega = \sqrt{-1} \sum_{i=1}^n dz_i \wedge d\bar{z}_i, \quad \gamma = \sqrt{-1} \sum_i \lambda_i \, dz_i \wedge d\bar{z}_i,$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_m$. Define $h_i = \partial h / \partial z_i$. Then, at the point z,

$$\binom{n}{m-1}(f_2)^2 = \sum_{i=1}^n |h_i|^2 S_{m-1;i}, \quad \binom{n}{m-1}(f_3)^2 = S_{m-1}. \tag{2-13}$$

Now, observe that $\gamma^{m-1} \wedge \omega^{n-m-1}$ is an $(n-2, n-2)$-form, so after taking the wedge product with $dh \wedge d^c\omega$ the nonzero contribution give only $\sqrt{-1}\partial h \wedge \bar{\partial}\omega$ and $\sqrt{-1}\,\bar{\partial} h \wedge \partial\omega$. As h is a real-valued function, these two forms are mutually conjugate. Let us write

$$\bar{\partial}\omega = \sum \omega_{i\bar{j}\bar{k}} \, d\bar{z}_k \wedge dz_i \wedge d\bar{z}_j.$$

Define $dV = \left(\sqrt{-1}\right)^{n^2} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{\wedge} z_1 \wedge \cdots \wedge d\bar{z}_n$. Let $J = (j_1, \ldots, j_{n-m-1})$ be an increasing multi-index. Then,

$$\frac{1}{(m-1)!} \partial h \wedge \bar{\partial}\omega \wedge \gamma^{m-1} \wedge dz_J \wedge d\bar{z}_J / dV = \sum_{\substack{j, \ell \notin J \\ j \ne \ell}} c_{j\bar{j}\bar{\ell}} \, h_\ell \prod_{i_s \notin J \cup \{j,\ell\}} \lambda_{i_s},$$

where $c_{i\bar{i}\bar{\ell}}$ is $\omega_{i\bar{i}\bar{\ell}}$ or $\omega_{i\bar{\ell}\bar{i}}$, and $i_s \in I = (i_1, \ldots, i_{m-1})$ which is an increasing multi-index satisfying $I \cap J = \varnothing$. Then, at the point z,

$$|f_1(z)| \le c_0 \sum_{|I|=m-1} \sum_{\ell=1}^n |h_\ell| \prod_{\substack{i_s \ne \ell \\ s=1}}^{m-1} |\lambda_{i_s}|, \tag{2-14}$$

where c_0 is a uniform constant depending only on ω.

By (2-13) and (2-14) we have reduced $|f_1| \le C f_2 f_3$ to the one for symmetric polynomials. To show the latter, by Lemma 2.2(b) for $1 \le \ell \le n$, we have

$$|h_\ell| \prod_{\substack{i_s \ne \ell \\ s=1}}^{m-1} |\lambda_{i_s}| \le C[|h_\ell|^2 S_{m-1;\ell} S_{m-1}]^{\frac{1}{2}} \le C\bigg(\sum_{i=1}^{n} |h_i|^2 S_{m-1;i}\bigg)^{\frac{1}{2}} [S_{m-1}]^{\frac{1}{2}}.$$

Taking the sum of (finitely many) terms on the left-hand side of (2-14) the proof of the theorem follows. □

We also need this inequality for wedge products of two forms and more. This is done by solving a linear system of inequalities as in [Kołodziej and Nguyen 2016, page 2226].

Corollary 2.5. *There exists a uniform constant C, depending on ω, n, m, such that the following inequalities hold.*

(a) *For $\eta, \gamma \in \Gamma_m(\omega)$,*

$$\bigg|\int \phi\psi \, dh \wedge d^c\omega \wedge \eta^k \wedge \gamma^{m-k-1} \wedge \omega^{n-m-1}\bigg|^2 \le C \int |\phi|^2 \, dh \wedge d^c h \wedge (\eta+\gamma)^{m-1} \wedge \omega^{n-m} \int |\psi|^2 \, (\eta+\gamma)^{m-1} \wedge \omega^{n-m+1}.$$

(b) *Generally, for $\gamma_1, \dots, \gamma_{m-1} \in \Gamma_m(\omega)$,*

$$\bigg|\int \phi\psi \, dh \wedge d^c\omega \wedge \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \omega^{n-m-1}\bigg|^2 \le C \int |\phi|^2 \, dh \wedge d^c h \wedge \bigg(\sum_{i=1}^{m-1} \gamma_i\bigg)^{m-1} \wedge \omega^{n-m} \times \int |\psi|^2 \bigg(\sum_{i=1}^{m-1} \gamma_i\bigg)^{m-1} \wedge \omega^{n-m+1}.$$

2.3. ***m-subharmonic functions on Hermitian manifolds.*** Let us recall the definition of generalized m-subharmonic function in the Hermitian setting (see [Błocki 2005; Kołodziej and Nguyen 2016; Gu and Nguyen 2018]). Let Ω be a bounded open set in $\mathbb{C}^n$ and let ω be a Hermitian metric on Ω.

Definition 2.6. An upper semicontinuous function $u : \Omega \to [-\infty, +\infty)$ is called m-ω-subharmonic if $u \in L^1_{\mathrm{loc}}(\Omega)$ and for any collection $\gamma_1, \dots, \gamma_{m-1} \in \Gamma_m(\omega)$,

$$dd^c u \wedge \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \omega^{n-m} \ge 0$$

in the sense of currents.

Remark 2.7. By [Gårding 1959] a function $u \in C^2(\Omega)$ is m-ω-sh if and only if $dd^c u \in \overline{\Gamma_m(\omega)}$, that is $dd^c u \wedge \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \ge 0$ pointwise in Ω. Thus, the estimates for forms in $\Gamma_m(\omega)$ are applicable to $dd^c u$ if u is a smooth m-ω-sh function.

It follows from [Michelsohn 1982] that for $\gamma_1, \dots, \gamma_{m-1} \in \Gamma_m(\omega)$ there is a unique $(1,1)$ positive form α such that

$$\alpha^{n-1} = \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \omega^{n-m}.$$

The above definition of m-ω-sh function can be expressed more familiarly, in terms of potential theory, using the notion of α-subharmonicity (see, e.g., [Gu and Nguyen 2018, Definition 2.1, Lemma 9.10]). Thanks to this, many potential-theoretic properties of m-ω-sh functions can be derived from those of α-sh functions. For example, if two m-ω-sh functions are equal almost everywhere (with respect to the Lebesgue measure), then they are equal everywhere [Gu and Nguyen 2018, Corollary 9.7]. One also has the "gluing property" that allows one to modify the function outside a compact subset. This is an immediate consequence of [Gu and Nguyen 2018, Lemma 9.5].

Lemma 2.8. *Let $U \subset \Omega$ be two open sets in $\mathbb{C}^n$. Let u be m-ω-sh in U and v be m-ω-sh in Ω. Assume that $\limsup_{z\to x} u(z) \le v(x)$ for every $x \in \partial U \cap \Omega$. Then,*

$$\widetilde{u} = \begin{cases} \max\{u, v\} & \text{on } U, \\ v & \text{on } \Omega \setminus U, \end{cases}$$

is m-ω-sh in Ω.

Because of this we have the following way of reducing a proof to a simpler case (see [Kołodziej 2005, page 7] for the proof).

Localization principle. *If we are to prove the weak convergence or local estimate for a family of locally uniformly bounded m-ω-sh functions, it is no loss of generality to assume that the functions are defined in a ball and are all equal on some neighborhood of the boundary.*

For a bounded psh function its convolution with a radial smoothing kernel provides locally a smooth, decreasing approximation of this function. It is no longer the case for generalized m-ω-sh. However, in a strictly m-pseudoconvex domain Ω, that is for $\Omega = \{\rho < 0\}$, where $\rho \in C^2(\overline{\Omega})$ is strictly m-ω-sh and $d\rho \neq 0$ on $\partial\Omega$, we still have (nonexplicit) way of approximation by smooth m-ω-sh functions.

Proposition 2.9. *Let $\Omega \Subset \mathbb{C}^n$ be strictly m-pseudoconvex domain. Let u be a bounded m-ω-sh function in a neighborhood of $\overline{\Omega}$. Then, there exists a sequence of smooth m-ω-sh functions $u_j \in C^\infty(\overline{\Omega})$ that decreases to u pointwise in $\overline{\Omega}$.*

Proof. The proof is the same as the one of [Gu and Nguyen 2018, Theorem 3.18] if we replace the ball by a strictly m-pseudoconvex domain and invoke [Collins and Picard 2022, Theorem 1.1] for the smooth solution of the Dirichlet problem on a strictly m-pseudoconvex domain. □

2.4. ***Integral estimates for smooth functions.*** Let Ω be a bounded open set in $\mathbb{C}^n$. Let $-1 \le v \le w \le 0$ be smooth m-ω-sh functions in Ω such that $v = w$ in a neighborhood of $\partial\Omega$. Let ρ be a smooth m-ω-sh function such that $-1 \le \rho \le 0$. Using the notation $\gamma := dd^c\rho$, $h = w - v$ we consider

$$e_{(q,k,s)} = \int h^{q+1}\gamma^k \wedge (dd^c v)^s \wedge \omega^{n-k-s}, \tag{2-15}$$

where $q \ge 0$, the integers $0 \le k \le m$ and $0 \le s \le m-k$. We wish to bound

$$e_{(q,m,0)} = \int h^{q+1}\gamma^m \wedge \omega^{n-m},$$

by the integrals

$$e_{(r,0,i)} = \int h^{r+1}(dd^c v)^i \wedge \omega^{n-i},$$

where $i = 0, \dots, m$ and $1 \le r < q$. This is done via repeated use of the integration by parts to replace γ by $dd^c v$. However, there are three different cases depending on the total degree $k+s$ of the form $\gamma^k \wedge (dd^c v)^s$ that we need to deal with separately.

- Case 1: $k+s = m$.
- Case 2: $k+s = m-1$.
- Case 3: $k+s \le m-2$.

Let us start with an auxiliary inequality that we use frequently below.

Lemma 2.10. *Let $p \ge 1$ and $0 \le k \le m-1$. There exists a constant C depending on ω, n, m such that*:

(a) *For $0 \le s \le m-1-k$,*

$$\int h^{p-1}\, dh \wedge d^c h \wedge \gamma^k \wedge (dd^c v)^s \wedge \omega^{n-k-s-1} \le \int h^p \gamma^k \wedge (dd^c v)^{s+1} \wedge \omega^{n-k-s-1} + C \int h^{p+1}(\gamma + dd^c v)^{k+s} \wedge \omega^{n-k-s}.$$

(b) *For $0 \le s \le m-3-k$,*

$$\int h^{p-1}\, dh \wedge d^c h \wedge \gamma^k \wedge (dd^c v)^s \wedge \omega^{n-k-s-1} \le \int h^p \gamma^k \wedge (dd^c v)^{s+1} \wedge \omega^{n-k-s-1} + C \int h^{p+1}\gamma^k \wedge (dd^c v)^s \wedge \omega^{n-k-s}.$$

Proof. (a) Note first that $0 \le h \le 1$ and also $T := \gamma^k \wedge (dd^c v)^s \wedge \omega^{n-k-s-1}$ and $dd^c w \wedge T$ are positive forms for $n-s-k-1 \ge n-m$. Therefore,

$$\begin{aligned} p(p+1)h^{p-1} dh \wedge d^c h \wedge T &= [dd^c h^{p+1} - (p+1)h^p dd^c h] \wedge T \\ &\le [dd^c h^{p+1} + (p+1)h^p dd^c v] \wedge T. \end{aligned}$$

Hence,

$$\int h^{p-1}\, dh \wedge d^c h \wedge \gamma^k \wedge (dd^c v)^s \wedge \omega^{n-s-k-1} \le \int (dd^c h^{p+1} + h^p dd^c v) \wedge \gamma^k \wedge (dd^c v)^s \wedge \omega^{n-s-k-1}. \quad (2\text{-}16)$$

It remains to estimate the product involving the first term in the bracket. By integration by parts and [Kołodziej and Nguyen 2016, Lemma 2.3],

$$\begin{aligned} \int dd^c h^{p+1} \wedge \omega^{n-s-k-1} \wedge \gamma^k \wedge (dd^c v)^s &= \int h^{p+1} dd^c(\omega^{n-s-k-1}) \wedge \gamma^k \wedge (dd^c v)^s \\ &\le C \int h^{p+1}(\gamma + dd^c v)^{k+s} \wedge \omega^{n-m+1}. \end{aligned} \quad (2\text{-}17)$$

Combining the last two inequalities the proof of the lemma follows.

(b) The proof is very similar, we first have (2-16). Then, in the middle integral of (2-17) one can express $dd^c(\omega^{n-s-k-1}) = \eta \wedge \omega^{n-m}$ for a smooth $(m-s-k, m-s-k)$-form η. Then, since $\gamma, dd^c v \in \Gamma_m(\omega)$, in this case the inequality has a better form

$$\left| \int h^{p+1} \eta \wedge \gamma^k \wedge (dd^c v)^s \wedge \omega^{n-m} \right| \leq C \int h^{p+1} \gamma^k \wedge (dd^c v)^s \wedge \omega^{n-k-s}. \qquad \square$$

Let us start with the simplest situation in Case 1. We are going to show that

$$e_{(q,m,0)} \leq C(e_{(q-1,m-1,1)} + e_{(q-1,m-1,0)}), \tag{2-18}$$

where C is a uniform constant depending on ω, m, n. Equivalently:

Lemma 2.11. *Let $q \geq 1$. Then,*

$$\int_\Omega (w-v)^{q+1} \gamma^m \wedge \omega^{n-m} \leq C \int_\Omega (w-v)^q \gamma^{m-1} \wedge dd^c v \wedge \omega^{n-m} + C \int_\Omega (w-v)^q \gamma^{m-1} \wedge \omega^{n-m+1}.$$

Proof. Recall that $h := w - v \geq 0$ and $h = 0$ near $\partial\Omega$. A direct computation gives

$$\begin{aligned} dd^c(h^{q+1}\omega^{n-m}) &= q(q+1)h^{q-1} dh \wedge d^c h \wedge \omega^{n-m} \\ &\quad + (q+1)h^q dd^c h \wedge \omega^{n-m} \\ &\quad + (q+1)(n-m)h^q d\omega \wedge d^c h \wedge \omega^{n-m-1} \\ &\quad + (q+1)(n-m)h^q dh \wedge d^c \omega \wedge \omega^{n-m-1} \\ &\quad + h^{q+1} dd^c \omega^{n-m} \\ &=: T_0 + T_1 + T_2 + T_3 + T_4. \end{aligned} \tag{2-19}$$

By integration by parts,

$$\begin{aligned} \int h^{q+1} dd^c \rho \wedge \gamma^{m-1} \wedge \omega^{n-m} &= \int \rho dd^c(h^{q+1}\omega^{n-m}) \wedge \gamma^{m-1} \\ &= \int \rho (T_0 + T_1 + T_2 + T_3 + T_4) \wedge \gamma^{m-1}. \end{aligned} \tag{2-20}$$

Case 1a: *estimate of* T_0. Since $-1 \leq \rho \leq 0$ and T_0 is a positive current,

$$\rho T_0 \wedge \gamma^{m-1} \leq 0. \tag{2-21}$$

Case 1b: *estimate of* T_1. Similarly, because v is a m-ω-sh function,

$$\begin{aligned} \rho T_1 \wedge \gamma^{m-1} &= (q+1)\rho h^q (dd^c w - dd^c v) \wedge \omega^{n-m} \wedge \gamma^{m-1} \\ &\leq (q+1) h^q dd^c v \wedge \gamma^{m-1} \wedge \omega^{n-m}. \end{aligned} \tag{2-22}$$

Case 1c: *estimate of* T_4. Using the inequality [Kołodziej and Nguyen 2016, Lemma 2.3]

$$\gamma^{m-1} \wedge dd^c \omega^{n-m} \leq C_{m,n} \gamma^{m-1} \wedge \omega^{n-m+1}, \tag{2-23}$$

we can estimate the last term T_4,

$$\left|\int \rho h^{q+1}\gamma^{m-1}\wedge dd^c\omega^{n-m}\right| \le C\int |\rho| h^{q+1}\gamma^{m-1}\wedge\omega^{n-m+1}$$
$$\le Ce_{(q,m-1,0)}, \tag{2-24}$$

where we used again the fact that $|\rho|\le 1$.

Case 1d: *estimate of* T_2 *and* T_3. We use the Cauchy–Schwarz inequality in Lemma 2.4, where an extra uniform constant appears. Let us estimate T_2 (for T_3 the estimate is completely the same). By the Cauchy–Schwarz inequality,

$$\left|\int \rho h^q\, d\omega\wedge d^c h\wedge\omega^{n-m-1}\wedge\gamma^{m-1}\right|^2$$
$$\le C\int |\rho| h^{q-1}\, dh\wedge d^c h\wedge\omega^{n-m}\wedge\gamma^{m-1}\int |\rho| h^{q+1}\omega^{n-m+1}\wedge\gamma^{m-1}$$
$$\le C\left(\int h^{q-1}\, dh\wedge d^c h\wedge\gamma^{m-1}\wedge\omega^{n-m}+\int h^{q+1}\gamma^{m-1}\wedge\omega^{n-m+1}\right)^2$$
$$\le C(e_{(q-1,m-1,1)}+e_{(q,m-1,0)})^2, \tag{2-25}$$

where we used Lemma 2.10 with $s=0$, $k=m-1$ and $p=q$ for the first integral in the last inequality, namely,

$$\int h^{q-1}\, dh\wedge d^c h\wedge\omega^{n-m}\wedge\gamma^{m-1}\le C(e_{(q-1,m-1,1)}+e_{(q,m-1,0)}).$$

Combining the estimates (2-21), (2-22), (2-24) and (2-25) and the fact that $e_{(q,\bullet,\bullet)}\le e_{(q-1,\bullet,\bullet)}\le e_{(q-2,\bullet,\bullet)}$ we conclude the proof of the lemma. □

We can now state the general inequality in Case 1.

Corollary 2.12. *For* $1\le k\le m$ *and* $s=m-k\ge 0$ *and* $q\ge 2$,

$$e_{(q,k,s)}\le c_k\sum_{i=0}^{k-1} e_{(q-1,i,m-i)}+C\sum_{i=0}^{m-1} e_{(q-2,i,m-1-i)}.$$

Proof. The proof is by induction in k but "downward". For $k=m$ it is Lemma 2.11. Assume that it is true for every $k+1\le\ell\le m$, i.e., we have

$$e_{(q,\ell,m-\ell)}\le c_\ell\sum_{i=0}^{k} e_{(q-1,i,m-i)}+c_\ell\sum_{i=0}^{m-1} e_{(q-2,i,m-1-i)}. \tag{2-26}$$

We proceed to prove the conclusion holds for k. The strategy is the same as in the proof of the last lemma, however we need to estimate T_2 and T_3 more carefully. We repeat the steps of the proof of Lemma 2.11 replacing $dd^c\rho\wedge\Gamma$ by $dd^c\rho\wedge\Gamma^{(s)}$, where

$$\Gamma=\gamma^{m-1}\wedge\omega^{n-m}\quad\text{and}\quad\Gamma^{(s)}=\gamma^{m-1-s}\wedge(dd^c v)^s\wedge\omega^{n-m},$$

in the integrand on the left-hand side. The corresponding estimates for T_0, T_1 are similar. Namely,

$$\rho T_0 \wedge \gamma^{m-1-s} \wedge (dd^c v)^s \le 0, \tag{2-21′}$$

and

$$\rho T_1 \wedge \gamma^{m-s-1} \le (q+1)h^q (dd^c v)^{s+1} \wedge \gamma^{m-1-s} \wedge \omega^{n-m}. \tag{2-22′}$$

The one for T_4 is

$$\begin{aligned}\gamma^{m-1-s} \wedge (dd^c v)^s \wedge dd^c \omega^{n-m} &\le C_{m,n}(\gamma + dd^c v)^{m-1} \wedge \omega^{n-m+1} \\ &\le C_{m,n}[\gamma^{m-1} + \gamma^{m-2} \wedge dd^c v + \cdots + (dd^c v)^{m-1}] \wedge \omega^{n-m+1}. \end{aligned} \tag{2-23′}$$

Hence, integrating both sides and using $0 \le h \le 1$, leads to

$$\left| \int h^{q+1} \gamma^{m-1-s} \wedge (dd^c v)^s \wedge dd^c \omega^{n-m} \right| \le C \sum_{i=0}^{m-1} e_{(q,i,m-1-i)}. \tag{2-24′}$$

Lastly for T_2 and T_3, we need to use Corollary 2.5,

$$\begin{aligned} I^2 &:= \left| \int \rho h^q d\omega \wedge d^c h \wedge \omega^{n-m-1} \wedge \gamma^{m-1-s} \wedge (dd^c v)^s \right|^2 \\ &\le C \int h^{q-1} (\gamma + dd^c v)^{m-1} \wedge \omega^{n-m+1} \times \int h^{q+1} dh \wedge d^c h \wedge (\gamma + dd^c v)^{m-1} \wedge \omega^{n-m}. \end{aligned}$$

The standard Cauchy–Schwarz inequality (Lemma 2.3) gives, for $\varepsilon > 0$ to be determined later,

$$I \le \frac{C}{\varepsilon} \int h^{q-1} (\gamma + dd^c v)^{m-1} \wedge \omega^{n-m+1} + \varepsilon \int h^{q+1}\, dh \wedge d^c h \wedge (\gamma + dd^c v)^{m-1} \wedge \omega^{n-m}.$$

By the last inequality in (2-23′) the first integral is bounded by

$$\frac{C}{\varepsilon} \sum_{i=0}^{m-1} e_{(q-2,i,m-1-i)}.$$

To bound the second integral we use

$$(\gamma + dd^c v)^{m-1} \wedge \omega^{n-m} \le C_{m,n} \sum_{i=0}^{m-1} \gamma^i \wedge (dd^c v)^{m-1-i} \wedge \omega^{n-m},$$

and then Lemma 2.10(a) for $k = i$, $s = m-1-i$ and $p = q+2$. This gives a bound for the second integral by

$$C\varepsilon \sum_{i=0}^{m-1} e_{(q+1,i,m-i)} + C\varepsilon \sum_{i=0}^{m-1} e_{(q+2,i,m-1-i)}.$$

Let us consider the first sum above:

$$\varepsilon \sum_{i=0}^{m-1} e_{(q+1,i,m-i)} = \varepsilon \sum_{i \ge k+1} e_{(q+1,i,m-i)} + \varepsilon \sum_{i=0}^{k-1} e_{(q+1,i,m-i)} + \varepsilon e_{(q+1,k,m-k)}.$$

Applying the induction hypothesis (2-26) to the first term on the right, we derive

$$\varepsilon\sum_{i=0}^{m-1} e_{(q+1,i,m-i)} \le \varepsilon b_k\left(e_{(q,k,m-k)} + \sum_{i=0}^{k-1} e_{(q,i,m-i)}\right) + \varepsilon b_k \sum_{i=0}^{m-1} e_{(q-1,i,m-1-i)} + \varepsilon\sum_{i=0}^{k-1} e_{(q+1,i,m-i)} + \varepsilon e_{(q+1,k,m-k)},$$

where $b_k = c_m + \cdots + c_{k+1}$.

Since $e_{(q+2,\bullet,\bullet)} \le e_{(q+1,\bullet,\bullet)} \le e_{(q,\bullet,\bullet)} \le e_{(q-1,\bullet,\bullet)}$, it follows from the above estimates that

$$I \le \varepsilon(1+b_k)e_{(q,k,m-k)} + \varepsilon\sum_{i=0}^{k-1} e_{(q-1,i,m-i)} + \left[\varepsilon(b_k+1) + \frac{C}{\varepsilon} + C\varepsilon\right]\sum_{i=0}^{m-1} e_{(q-2,i,m-1-i)}. \tag{2-25$'$}$$

Thus, combining (2-21′), (2-22′), (2-24′) and (2-25′) we have

$$e_{(q,k,s)} \le Ce_{(q-1,k-1,s+1)} + C\sum_{i=0}^{m-1} e_{(q,i,m-1-i)} + \varepsilon(1+b_k)e_{(q,k,s)} + \varepsilon\sum_{i=0}^{k-1} e_{(q-1,i,m-i)} + \left[\varepsilon(b_k+1) + \frac{C}{\varepsilon} + C\varepsilon\right]\sum_{i=0}^{m-1} e_{(q-2,i,m-1-i)}.$$

Now we can choose ε so that $\varepsilon(1+b_k) = \frac{1}{2}$. Since $s = m-k$, regrouping terms on the right-hand side (decreasing the first parameter in $e_{(q,i,m-1-i)}$ if necessary) we get for possibly larger $C > 0$ that

$$\tfrac{1}{2}e_{(q,k,m-k)} \le (C+\varepsilon)\sum_{i=0}^{k-1} e_{(q-1,i,m-i)} + \left(C + \frac{C}{\varepsilon}\right)\sum_{i=0}^{m-1} e_{(q-2,i,m-1-i)}. \qquad \square$$

Next, we consider Case 2.

Lemma 2.13. *For $1 \le k \le m-1$ and $s = m-1-k \ge 0$ and $q \ge 1$, we have*

$$e_{(q,k,s)} \le C\left(e_{(q-1,k-1,s+1)} + \sum_{i=0}^{m-2} e_{(q-1,i,m-2-i)}\right).$$

Proof. The basic computation using integration by parts that corresponds to (2-19) starts with

$$dd^c(h^{q+1}\omega^{n-m+1}) := T_0 + T_1 + T_2 + T_3 + T_4, \tag{2-19$''$}$$

where each term has higher exponent of ω. The estimates for T_0, T_1 are the same as the ones in (2-21′) and (2-22′). Precisely,

$$\rho T_0 \wedge \gamma^{m-2-s} \wedge (dd^c v)^s \le 0, \tag{2-21$''$}$$

and

$$\rho T_1 \wedge \gamma^{m-2-s} \le (q+1)h^q\gamma^{m-2-s} \wedge (dd^c v)^{s+1} \wedge \omega^{n-m+1}. \tag{2-22$''$}$$

The one for T_4 is

$$\gamma^{m-2-s}\wedge(dd^c v)^s\wedge dd^c(\omega^{n-m+1}) \le C_{m,n}(\gamma+dd^c v)^{m-2}\wedge\omega^{n-m+2}$$
$$\le C_{m,n}[\gamma^{m-2}+\gamma^{m-3}\wedge dd^c v+\cdots+(dd^c v)^{m-2}]\wedge\omega^{n-m+2}. \quad (2\text{-}23'')$$

Integrating both sides and using the fact that $0\le h\le 1$ yield

$$\left|\int \rho h^{q+1}\gamma^{m-2-s}\wedge(dd^c v)^s\wedge dd^c(\omega^{n-m+1})\right| \le C\sum_{i=0}^{m-2} e_{(q,i,m-2-i)}. \quad (2\text{-}24'')$$

Next, the corresponding inequalities for T_2 and T_3 are easier. This is due to the fact that

$$T_2\wedge\gamma^{m-2-s}\wedge(dd^c v)^s = C_0 h^q d\omega\wedge d^c h\wedge\omega^{n-m}\wedge\gamma^{m-2-s}\wedge(dd^c v)^s.$$

Therefore, the classical Cauchy–Schwarz inequality (Lemma 2.3) is sufficient. Namely,

$$I^2 := \left|\int \rho h^q d\omega\wedge d^c h\wedge\omega^{n-m}\wedge\gamma^{m-2-s}\wedge(dd^c v)^s\right|^2$$
$$\le C\int h^q\gamma^{m-2-s}\wedge(dd^c v)^s\wedge\omega^{n-m+2}\times\int h^q\, dh\wedge d^c h\wedge\gamma^{m-2-s}\wedge(dd^c v)^s\wedge\omega^{n-m+1}.$$

The Cauchy–Schwarz inequality gives

$$I\le C\int h^q\gamma^{m-2-s}\wedge(dd^c v)^s\wedge\omega^{n-m+2}+C\int h^q\, dh\wedge d^c h\wedge\gamma^{m-2-s}\wedge(dd^c v)^s\wedge\omega^{n-m+1}.$$

Here, the first integral in the sum is $e_{(q-1,k-1,s)}$. By applying Lemma 2.10(a) for $k-1$, s and $p=q+1$ one gets a bound for the second integral by

$$\int h^{q+1}\gamma^{m-2-s}\wedge(dd^c v)^{s+1}\wedge\omega^{n-m+1}+C\int h^{q+2}(\gamma+dd^c v)^{m-2}\wedge\omega^{n-m+2}$$
$$\le e_{(q,k-1,s+1)}+C\sum_{i=0}^{m-2} e_{(q+1,i,m-2-i)}.$$

Combining this with the decreasing property in the first parameter of $e_{(q,\bullet,\bullet)}$ we get

$$I\le C\left[e_{(q-1,k-1,s)}+e_{(q,k-1,s+1)}+\sum_{i=0}^{m-2} e_{(q+1,i,m-2-i)}\right]$$
$$\le C\left[e_{(q-1,k-1,s+1)}+\sum_{i=0}^{m-2} e_{(q-1,i,m-2-i)}\right]. \quad (2\text{-}25'')$$

Finally, combining (2-21″), (2-22″), (2-24″) and (2-25″) one completes the proof of the lemma. □

Lastly, we consider Case 3.

Lemma 2.14. *For $1\le k\le m-2$ and $0\le s\le m-2-k$ and $q\ge 1$ we have*

$$e_{(q,k,s)}\le C[e_{(q-1,k-1,s+1)}+e_{(q,k-1,s)}].$$

Proof. We need to estimate $dd^c\rho\wedge\gamma^{k-1}\wedge(dd^cv)^s\wedge\omega^{n-k-s}$, where $n-k-s\geq n-m+2$. Then, there is a significant change in basic computation of

$$dd^c(h^{q+1}\omega^{n-k-s})=T_0+T_1+T_2+T_3+T_4, \tag{2-19‴}$$

where all forms T_i contain powers of ω with the exponent at least $n-m$. The estimates for T_0, T_1 are the same as in Corollary 2.12 and improved estimates for T_2, T_3 are as in Lemma 2.13. Moreover the bound for T_4 is easier. Namely, since $\gamma^{k-1}\wedge(dd^cv)^s\wedge\omega^{n-m}$ is a positive form, one obtains

$$\gamma^{k-1}\wedge(dd^cv)^s\wedge dd^c(\omega^{n-k-s})\leq C\gamma^{k-1}\wedge(dd^cv)^s\wedge\omega^{n-k-s+1}. \tag{2-23‴}$$

Hence, multiplying both sides by ρh^{q+1} and then integrating we get

$$\left|\int\rho h^{q+1}\gamma^{k-1}\wedge(dd^cv)^s\wedge dd^c(\omega^{n-k-s})\right|\leq Ce_{(q,k-1,s)}, \tag{2-24‴}$$

where we used the fact that $-1\leq\rho\leq 0$.

Next, the estimates for T_2 and T_3 are

$$\begin{aligned}I&:=\left|\int\rho h^q\,d\omega\wedge d^ch\wedge\omega^{n-k-s-1}\wedge\gamma^{k-1}\wedge(dd^cv)^s\right|\\&\leq C\int h^{q+1}\gamma^{k-1}\wedge(dd^cv)^s\wedge\omega^{n-k-s}+C\int h^{q-1}\,dh\wedge d^ch\wedge\gamma^{k-1}\wedge(dd^cv)^s\wedge\omega^{n-k-s}.\end{aligned}$$

Using Lemma 2.10(b) for $k-1$, s and $p=q-1$ in the last integral above yields

$$\begin{aligned}I&\leq Ce_{(q,k-1,s)}+C[e_{(q-1,k-1,s+1)}+e_{(q,k-1,s)}]\\&\leq C[e_{(q-1,k-1,s+1)}+e_{(q,k-1,s)}].\end{aligned} \tag{2-25‴}$$

Combining the estimates for T_0, T_1 in (2-19‴), (2-24‴) and (2-25‴) we complete the proof of lemma. □

We are ready to state the main inequality.

Proposition 2.15. *Let $e_{(q,k,s)}$ be the numbers defined by (2-15). Then, for $q=3m$,*

$$e_{(q,m,0)}\leq C\sum_{s=0}^{m}e_{(0,0,s)},$$

where $C=C(\omega,n,m)$ is a uniform constant.

Proof. We start with Lemma 2.11 which gives

$$e_{(q,m,0)}\leq C[e_{(q-1,m-1,1)}+e_{(q-1,m-1,0)}]. \tag{2-27}$$

Then, the first term in the bracket is estimated via Corollary 2.12. Applying this corollary $m-1$ times and using decreasing property of $e_{(p,k,s)}$ in the first parameter, we get

$$e_{(q-1,m-1,1)}\leq Ce_{(q-m,0,m)}+C\sum_{i=0}^{m-1}e_{(q-m-2,i,m-1-i)}. \tag{2-28}$$

The second term in the bracket in (2-27) satisfies $e_{(q-1,m-1,0)} \le e_{(q-m-2,m-1,0)}$. Next, we use Lemma 2.13 for each term $e_{(q',\ell,m-1-\ell)}$ with $q' = q-m-2$ in the sum above. Namely, applying the lemma ℓ times and using the decreasing property again, we get

$$e_{(q',\ell,m-1-\ell)} \le C e_{(q'-\ell,0,m-1)} + C \sum_{i=0}^{m-2} e_{(q'-\ell-1,i,m-2-i)}.$$

The smallest value of the first parameter in the last sum is $q'-m$ for $\ell = m-1$. Hence,

$$\sum_{i=0}^{m-1} e_{(q',i,m-1-i)} \le C e_{(q'-\ell,0,m-1)} + C \sum_{i=0}^{m-2} e_{(q'-m,i,m-2-i)}. \tag{2-29}$$

It remains to apply Lemma 2.14 for each term $e_{(q'',\ell,m-2-\ell)}$ in the sum on the right-hand side, where $q'' = q'-m = q-2m-2$. Again, we have

$$\begin{aligned} e_{(q'',\ell,m-2-\ell)} &\le C e_{(q''-1,\ell-1,m-\ell-1)} + C e_{(q'',\ell-1,m-\ell-2)} \\ &\le \cdots \\ &\le C e_{(q''-\ell,0,m-2)} + \sum_{i=0}^{\ell-1} e_{(q''-i,\ell-1-i,m-\ell-2+i)}. \end{aligned}$$

Therefore, an easy induction argument gives us

$$e_{(q'',\ell,m-2-\ell)} \le C \sum_{s=0}^{m-2} e_{(q''-\ell,0,s)}. \tag{2-30}$$

Combining (2-27), (2-28), (2-29) and (2-30) we arrive at

$$e_{(q,m,0)} \le C \sum_{s=0}^{m} e_{(q''-m+2,0,s)} = C \sum_{s=0}^{m} e_{(0,0,s)}$$

as $q''-m+2 = q-3m = 0$. □

So far all considered functions were smooth, however, by [Kołodziej and Nguyen 2016, Proposition 2.11] we know that the integrands on both sides of the above statements (Lemmas 2.10–2.14, Corollary 2.12 and Proposition 2.15) are well defined for continuous m-ω-sh functions. Let us record the following observation.

Remark 2.16. Let Ω be a strictly m-pseudoconvex domain. The statements above are still valid for continuous m-ω-sh functions v, w and $-1 \le \rho \le 0$ satisfying $-1 \le v \le w \le 0$ and $v = w$ in a neighborhood of $\partial\Omega$.

In fact, there exist decreasing sequences of m-ω-sh functions v_j, w_j and ρ_j belonging to $C^\infty(\overline{\Omega})$ such that $v_j \downarrow v$, $w_j \downarrow w$ and $\rho_j \downarrow \rho$ (uniformly) in $\overline{\Omega}$, and moreover,

$$-1 \le v_j \le w_j \le 0, \quad -1 \le \rho_j \le 0.$$

For plurisubharmonic functions the usual convolution with standard kernels produces the approximating sequence and hence, the property $v_j = w_j$ near the boundary is preserved. Then, the integration by parts

is not affected and passing to the limit as $j \to +\infty$ gives the desired inequalities. However, in this new setting we used a different way to obtain the approximating sequence. The property that $v_j = w_j$ near the boundary of Ω needs to be verified, which is possible via the stability estimates for complex Hessian equations. However, we can get around this by showing the uniform convergence to zero of the sequence $h_j = w_j - v_j$ near the boundary.

Let $\Omega' \Subset \Omega$ be a smooth domain such that $v = w$ outside Ω'. Let $T_j = (dd^c\rho_j)^k \wedge (dd^c v_j)^s$, where $k + s \leq m$. Then, it follows from the weak convergence [Kołodziej and Nguyen 2016, Proposition 2.11] and the CLN inequality [Kołodziej and Nguyen 2016, Proposition 2.9] that for $p \geq 1$,

$$\lim_{j\to\infty} \int_{\Omega'} h_j^p \, T_j \wedge \omega^{n-k-s} = \int_{\Omega} h^p \, T \wedge \omega^{k-s}.$$

Therefore, we reduce the required inequality to the case of smooth functions v_j, w_j and ρ_j. However the integration by parts in (2-20) will contain the extra boundary terms:

$$\int_{\Omega'} h_j^p \, dd^c\rho_j \wedge T_j \wedge \omega^{n-k-s} = \int_{\Omega'} \rho_j \, dd^c(h_j^p \omega^{n-k-s}) \wedge T_j + E_1 + E_2,$$

where

$$\begin{aligned} E_1 &= \int_{\partial\Omega'} h_j^p \, d^c\rho_j \wedge \omega^{n-k-s} \wedge T_j; \\ E_2 &= -\int_{\partial\Omega'} \rho_j \, d^c(h_j^p \omega^{n-k-s}) \wedge T_j \\ &= -\int_{\partial\Omega'} \rho_j h_j^{p-1} \, (p d^c h_j \wedge \omega^{n-k-s} + h_j d^c\omega^{n-k-s}) \wedge T_j. \end{aligned}$$

By the CLN inequality and $h_j \to 0$ uniformly on a neighborhood of $\partial\Omega'$ as $j \to \infty$, the two boundary terms go to zero when we pass to the limit.

Remark 2.17. We will see later that the above statements also hold for bounded m-ω-sh functions once we define the wedge product for currents related to such functions and prove the weak convergence under decreasing sequences.

3. Wedge product for bounded functions

In this section we prove the existence of the wedge product of currents where the dd^c operator is applied to bounded m-ω-sh functions. Then, we introduce a weighted Sobolev space associated to these resulting positive currents. This is a crucial technical tool that allows us to establish the integration by parts formulae for (nonclosed) currents of order zero.

3.1. ***The wedge product for bounded m-ω-sh functions.*** Let $\Omega \subset \mathbb{C}^n$ be a bounded open set. We are going to show that the wedge product of bounded m-ω-sh functions can be defined inductively by

$$dd^c v \wedge dd^c u = dd^c(v dd^c u)$$

and similarly for more terms. Let us start with the following observation.

Lemma 3.1. *Let u be a bounded m-ω-sh function. Then, $dd^c u$ is a real $(1,1)$-current of order zero whose coefficients are signed Radon measures.*

Proof. Without loss of generality we may assume $-1 \le u \le 0$. We will show that there is a signed measure μ such that

$$dd^c u(\phi) = \int \phi \, d\mu$$

for every (smooth) test $(n-1, n-1)$-form. By a theorem in distribution theory in Federer's book [1969, Section 4.15] it is enough to show that for all $K \subset \Omega$ compact there exists $C = C(K, \Omega)$ such that

$$|dd^c u(\phi)| \le C \|\phi\|_K$$

for any smooth test form ϕ with $\operatorname{supp} \phi \subset K$. Here we define

$$\|\phi\|_K = \sum_{i,j} \sup_K |\phi_{i\bar{j}}|,$$

where the $\phi_{i\bar{j}}$ are coefficients of the form ϕ. By the definition of dd^c-operator for currents

$$dd^c u(\phi) = u(dd^c \phi) = \int u \wedge dd^c \phi.$$

Thus, for a smooth sequence of m-ω-sh functions $\{u_s\}$ that decrease to u we have

$$dd^c u(\phi) = \lim_{s\to\infty} (dd^c u_s)(\phi) = \lim_{s\to\infty} \int dd^c u_s \wedge \phi.$$

Since $dd^c u_s \in \Gamma_m(\omega)$, it follows from [Kołodziej and Nguyen 2016, Corollary 2.4] that

$$|dd^c u_s \wedge \phi| \le C \|\phi\|_K dd^c u_s \wedge \omega^{n-1} \tag{3-1}$$

for some universal constant depending on n, m. It follows that

$$\left| \int dd^c u_s \wedge \phi \right| \le C \|\phi\|_K \int_K dd^c u_s \wedge \omega^{n-1}.$$

Thanks to the CLN inequality [Kołodziej and Nguyen 2016, Proposition 2.9] we know that the last integral on the right-hand side is bounded by a constant $C(K, \Omega)\|u_s\|_{L^\infty}$. This finishes the proof.

Furthermore, for any test function $\chi \ge 0$,

$$0 \le \lim_{s\to\infty} \int \chi \, \omega^{n-1} \wedge dd^c u_s = dd^c u(\chi \omega^{n-1}).$$

In other words, $dd^c u \wedge \omega^{n-1}$ is a (positive) Radon measure. Moreover, it follows from (3-1) that the total variation of coefficients of $dd^c u$ is dominated by $dd^c u \wedge \omega^{n-1}$. Therefore, the coefficients of $T = dd^c u$ are signed Radon measures. □

It follows from Lemma 3.1 that we can multiply $dd^c u$ by a bounded function v. In fact, we can continue to define inductively the wedge product in this way and obtain a real current of order zero for up to $m-1$ bounded functions.

Lemma 3.2. *Let $u_1, \dots, u_p$ with $1 \le p \le m-1$ be bounded m-ω-sh functions. Assume $T_0 = 1$ and $T_{p-1} = dd^c u_{p-1} \wedge \cdots \wedge dd^c u_1$. Then, the current*

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 := dd^c[u_p T_{p-1}]$$

is a well-defined real (p, p)-current of order zero whose coefficients are signed Radon measures. Moreover,

$$\mathscr{L}_p(u_p, \dots, u_1) := dd^c u_p \wedge \cdots \wedge dd^c u_1 \wedge \omega^{n-m}$$

is a positive $(n-m+p, n-m+p)$-current.

Proof. We argue by induction in p. If $p=1$, then this is Lemma 3.1. Assume that it holds for $1 \le p \le m-2$, we need to show that the lemma holds for $p+1$. For the simplicity of notation we present the argument for $p+1=2$ and note that the proof in the general case is completely the same. Let us write $u_1 = u$ and $u_2 = v$ and assume that $-1 \le u, v \le 0$. Since $dd^c u$ has signed measure coefficients, the currents

$$S = v dd^c u \quad \text{and} \quad dd^c v \wedge dd^c u := dd^c S = dd^c(v dd^c u)$$

are well defined. Our next goal is to show that the real (closed) current $dd^c S$ of bidegree $(2, 2)$ has order zero, thus its coefficients are signed Radon measures.

As above we will show that for each compact set $K \subset \Omega$, there is a uniform constant C such that

$$|dd^c S(\phi)| \le C\|\phi\|_K \tag{3-2}$$

for every smooth test form with $\operatorname{supp}\phi \subset K$, where $\|\phi\|_K$ denotes the C^0-norm of the form ϕ. To this end, let $\{v_\ell\}_{\ell \ge 1}$ and $\{u_s\}_{s \ge 1}$ be sequences of smooth m-ω-sh functions such that

$$v_\ell \downarrow v \quad \text{and} \quad u_s \downarrow u.$$

We may also assume that $-1 \le u_s, v_\ell \le 0$. Fix ϕ a test form. By definition we have

$$dd^c S(\phi) = S(dd^c \phi) = \int v\, dd^c u \wedge dd^c \phi. \tag{3-3}$$

Since $dd^c u \wedge dd^c \phi$ has signed Radon measure coefficients, it follows from the dominated convergence theorem that

$$\int v\, dd^c u \wedge dd^c \phi = \lim_{\ell \to \infty} \int v_\ell\, dd^c u \wedge dd^c \phi = \lim_{\ell \to \infty} \lim_{s \to \infty} \int v_\ell\, dd^c u_s \wedge dd^c \phi.$$

By the integration by parts formula,

$$\int v_\ell\, dd^c u_s \wedge dd^c \phi = \int dd^c v_\ell \wedge dd^c u_s \wedge \phi.$$

Notice that $m-1 \ge 2$. Applying [Kołodziej and Nguyen 2016, Corollary 2.4] for $\gamma_1 = dd^c u_s$, $\gamma_2 = dd^c v_\ell$ belonging to $\Gamma_m(\omega)$, we get

$$|dd^c v_\ell \wedge dd^c u_s \wedge \phi| \le C\|\phi\|_K [dd^c(u_s + v_\ell)]^2 \wedge \omega^{n-2}. \tag{3-4}$$

Integrating both sides and then applying the CLN inequality for the right-hand side we get

$$\left|\int dd^c v_\ell \wedge dd^c u_s \wedge \phi\right| \leq C(K,\Omega)\|\phi\|_K.$$

Letting $s \to \infty$ and then $\ell \to \infty$ in this order, we conclude from (3-2) the desired estimate (3-3). Hence, $dd^c S$ is a $(2,2)$ current of order zero.

Furthermore, for a test function $\chi \geq 0$,

$$dd^c S \wedge \omega^{n-2}(\chi) = \lim_{\ell\to\infty}\lim_{s\to\infty}\int \chi\, dd^c v_\ell \wedge dd^c u_s \wedge \omega^{n-2} \geq 0.$$

Thus, $dd^c S \wedge \omega^{n-2}$ is a positive Radon measure. It follows from (3-4) (by passing to the limit) that the coefficients of $dd^c S$ are signed Radon measures.

Finally, for smooth m-ω-sh functions $u_1^{j_1}, \ldots, u_p^{j_p}$, the form

$$\mathscr{L}_p(u_p^{j_p}, \ldots, u_1^{j_1}) = dd^c u_p^{j_p} \wedge \cdots \wedge dd^c u_1^{j_1} \wedge \omega^{n-m}$$

is positive by Gårding's inequality [1959]. Letting $j_1 \to \infty, \ldots, j_p \to \infty$ in this order, for decreasing sequences approximating the functions $u_p, \ldots, u_1$ we obtain that $\mathscr{L}_p(u_p, \ldots, u_1)$ is a positive current of bidegree $(n-m+p, n-m+p)$. □

Thus, we defined inductively for bounded m-ω-sh functions, $u_1, \ldots, u_{m-1}$, a real closed current

$$S := dd^c u_{m-1} \wedge \cdots \wedge dd^c u_1 \tag{3-5}$$

of order zero whose coefficients are signed Radon measures. Furthermore,

$$S \wedge \omega^{n-m+1} \tag{3-6}$$

is a positive Radon measure. In the last step we multiply S by a bounded function $v = u_m$.

Theorem 3.3. *Let S be as in (3-5) and let v be a bounded m-ω-sh function. Then,*

$$T := dd^c(vS)$$

is a real closed (m,m)-current and $T \wedge \omega^{n-m}$ is a positive Radon measure.

Proof. Let $\chi \geq 0$ be a test function. By definition

$$T \wedge \omega^{n-m}(\chi) = T(\chi\omega^{n-m}) = \int vS \wedge dd^c(\chi\omega^{n-m}).$$

Since S is of order zero, it follows from the dominated convergence theorem that for a sequence of smooth m-ω-sh functions $v_\ell \downarrow v$,

$$\int vS \wedge dd^c(\chi\omega^{n-m}) = \lim_{\ell\to\infty}\int v_\ell S \wedge dd^c(\chi\omega^{n-m}).$$

Using the inductive definition of S and the integration by parts, we get

$$\begin{aligned}\int v_\ell S \wedge dd^c(\chi\omega^{n-m}) &= \lim_{j_{m-1}\to\infty} \cdots \lim_{j_1\to\infty} \int dd^c v_\ell \wedge dd^c u^{j_{m-1}} \wedge \cdots \wedge u^{j_1} \wedge dd^c(\chi\omega^{n-m}) \\ &= \lim_{j_{m-1}\to\infty} \cdots \lim_{j_1\to\infty} \int dd^c v_\ell \wedge dd^c u^{j_{m-1}} \wedge \cdots \wedge dd^c u^{j_1} \wedge (\chi\omega^{n-m}).\end{aligned}$$

Since $v_\ell, u^{j_1}, \ldots, u^{j_{m-1}}$ are smooth m-ω-sh functions, the last integrand is positive and the proof follows. □

Consequently the following definition is justified.

Definition 3.4. Let $u_1, \ldots, u_p$ with $1 \le p \le m$ be bounded m-ω-sh functions in Ω. The wedge product is given inductively by

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 := dd^c[u_p T_{p-1}],$$

where $T_0 = 1$ and $T_{p-1} = dd^c u_{p-1} \wedge \cdots \wedge dd^c u_1$.

We are most interested in the following positive Radon measures.

Definition 3.5. Let u be a bounded m-ω-sh function. Then, the Hessian operator $H_m(u)$ is defined by

$$H_m(u) := (dd^c u)^m \wedge \omega^{n-m} = \mathscr{L}_m(u, \ldots, u).$$

Moreover, for $1 \le s \le m$, we also write

$$H_s(u) = (dd^c u)^s \wedge \omega^{n-s}.$$

For convenience, we summarize the results in Lemma 3.2 and Theorem 3.3 as follows. Let $\{u_s^j\}_{j\ge1}$ be decreasing sequences such that $u_s^j \downarrow u_s$ for each $1 \le s \le m$. Define

$$T^{j_p \cdots j_1} = dd^c u_p^{j_p} \wedge \cdots \wedge dd^c u_1^{j_1}. \tag{3-7}$$

Then, we have $T^{j_p \cdots j_2} \wedge dd^c u_1 = \lim_{j_1\to\infty} T^{j_p \cdots j_1}$, and consequently

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 = \lim_{j_p\to\infty} \cdots \lim_{j_1\to\infty} T^{j_p \cdots j_1}. \tag{3-8}$$

For $1 \le p \le m-1$, each limit in (3-8) is in the sense of currents of order zero. For $p = m$, we have

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 \wedge \omega^{n-m} = \lim_{j_p\to\infty} \cdots \lim_{j_1\to\infty} T^{j_p \cdots j_1} \wedge \omega^{n-m}, \tag{3-9}$$

where each limit is also in the sense of currents of order zero.

Remark 3.6. At this point the definition of the wedge product in Definition 3.4 is not symmetric with respect to $u_1, \ldots, u_p$. Furthermore, if the metric ω is Kähler or just the standard Kähler form $\beta = dd^c|z|^2$ in $\mathbb{C}^n$, then by (3-9) it follows that Definition 3.4 coincides with the ones given before in [Błocki 2005; Dinew and Kołodziej 2014].

By taking the limit inductively as in (3-8) and (3-9) we can state now the CLN inequality for bounded functions.

Proposition 3.7 (CLN inequality). *Let $K \Subset U \Subset \Omega$, where K is compact and U is open. Let $u, u_1, \dots, u_p$, be bounded m-ω-sh functions in Ω, where $1 \le p \le m$. Then, there exists a constant C depending on K, U, ω such that*

(a) $\int_K (dd^c u)^p \wedge \omega^{n-p} \le C(1 + \|u\|_{L^\infty(U)})^p$;

(b) $\int_K dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge \omega^{n-p} \le C\big(1 + \sum_{s=1}^p \|u_s\|_{L^\infty(U)}\big)^p$.

Proof. (a) Without loss of generality we may assume that $\|u\|_{L^\infty(\Omega)} \le 1$. We can cover K by finitely many small balls, hence by the localization principle we can assume that K and U are concentric balls. Let $\{u^j\}_{j\ge1}$ be sequences of smooth m-ω-sh functions such that $u^j \downarrow u$ as $j \to \infty$. Let $0 \le \chi \le 1$ be a cut-off function such that $\chi \equiv 1$ on K and $\operatorname{supp}\chi \subset U$. By the CLN inequality for smooth functions [Kołodziej and Nguyen 2016, Proposition 2.9]

$$\int \chi \mathscr{L}_p(u^{j_p}, \dots, u^{j_1}) \wedge \omega^{m-p} \le C\bigg(1 + \|\sum_{s=1}^p u^{j_s}\|^p_{L^\infty(U)}\bigg).$$

By the Hartogs lemma for ω-sh functions [Gu and Nguyen 2018, Lemma 9.14] it follows that

$$\lim_{j\to\infty} \|1 + u^j\|_{L^\infty(U)} = \lim_{j\to\infty} \sup_{\overline{U}}(1 + u^j) = 1 + \sup_{\overline{U}} u \le 1 + \|u\|_{L^\infty(U)}.$$

Letting $j_1 \to \infty, \dots, j_p \to \infty$ in this order, we get

$$\begin{aligned}\int_K \mathscr{L}_p(u) \wedge \omega^{m-p} &\le \lim_{j_p\to\infty} \cdots \lim_{j_1\to\infty} \int \chi \mathscr{L}_p(u^{j_p}, \dots, u^{j_1}) \wedge \omega^{m-p} \\ &\le C \lim_{j_p\to\infty} \cdots \lim_{j_1\to\infty} \bigg(1 + \bigg\|\sum_{s=1}^p u^{j_s}\bigg\|_{L^\infty}\bigg)^p \\ &= C(1 + \|u\|_{L^\infty(U)})^p.\end{aligned}$$

(b) Observe that for $v := u_1 + \cdots + u_p$ we have $\mathscr{L}_p(v) \ge \mathscr{L}(u_1, \dots, u_p)$ as positive currents. So, (b) is an immediate consequence of (a). □

We also need a simplified version of Lemma 2.3 for particular positive currents. Let $-1 \le u_1, \dots, u_{m-2} \le 0$ be bounded m-ω-sh functions in Ω. Denote by T the current

$$T = dd^c u_1 \wedge \cdots \wedge dd^c u_{m-2} \wedge \omega^{n-m}. \tag{3-10}$$

Corollary 3.8. *Let ϕ be a Borel function such that $\operatorname{supp}\phi = K \Subset \Omega$. Assume also $|\phi| \le 1$. Let $0 \le h \le 1$ be a smooth m-ω-sh function in Ω. There exists a constant $C = C(\omega, K)$ such that*

$$\bigg|\int \phi\, dh \wedge d^c\omega \wedge T\bigg|^2 \le C \int |\phi|\, T \wedge \omega^2.$$

Proof. By Lemma 2.3, it is enough to show that $\int |\phi|\, dh \wedge d^c h \wedge T \wedge \omega \le C$. In fact, since h^2 and h are m-ω-sh, we have

$$2dh \wedge d^c h \wedge T \wedge \omega = [dd^c h^2 - 2h dd^c h] \wedge T \wedge \omega \le dd^c h^2 \wedge T \wedge \omega.$$

Multiplying both sides by $|\phi|$ and integrating them, we get

$$\int |\phi|\, dh \wedge d^c h \wedge T \wedge \omega \leq \int |\phi|\, dd^c h^2 \wedge T \wedge \omega.$$

Using the assumption on ϕ the right-hand side is bound by

$$\int_K dd^c h^2 \wedge T \wedge \omega.$$

The desired estimate follows from the CLN inequality for bounded functions (Proposition 3.7). □

3.2. ***Weighted Sobolev space.*** Let $K \subset \Omega$ be a compact subset and let ψ be a strictly plurisubharmonic defining function for Ω. Let us denote by $P^* = P^*(\Omega, K, \psi)$ the set of bounded m-ω-sh functions that

$$u(z) = \psi, \quad z \in \Omega \setminus K.$$

Let $u_1, \ldots, u_p \in P^*$ with $1 \leq p \leq m-1$ and define

$$\tau = dd^c u_1 \wedge \cdots \wedge dd^c u_p, \tag{3-11}$$

and conventionally $\tau = 1$ for $p = 0$. By definition this is a closed (p, p)-current of order zero. We first make an additional assumption on the current τ. By considering $u_i = u_i + \delta|z|^2$ for some $\delta > 0$ small, we may assume that there is $c_0 > 0$ such that

$$\tau \wedge \omega^{n-p-1} \geq c_0 \omega^{n-1}. \tag{3-12}$$

Hence, the trace measure (with respect to ω) of the positive current $\tau \wedge \omega^{n-p}$ satisfies

$$\mu_p := \tau \wedge \omega^{n-p} \geq c_0 \omega^n. \tag{3-13}$$

Let us introduce a norm associated to τ on the space smooth functions $C^\infty(\Omega, \mathbb{R})$. If v, w are smooth functions in Ω, then we can define

$$\langle v, w\rangle_\tau = \int dv \wedge d^c w \wedge \tau \wedge \omega^{n-p-1} \quad \text{and} \quad \|v\|_\tau^2 = \langle v, v\rangle_\tau = \int dv \wedge d^c v \wedge \tau \wedge \omega^{n-p}.$$

Let us consider the inner product

$$\langle v, w\rangle = \int v w \beta^n + \langle v, w\rangle_\tau,$$

where $\beta = dd^c|z|^2$, and the associated norm

$$\|v\|^2 = \langle v, v\rangle = \|v\|_{L^2}^2 + \|v\|_\tau^2.$$

The assumption makes this inner product positive definite. In fact, it is clear that $\langle \bullet, \bullet \rangle$ satisfies the properties of an inner product, namely

(i) $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$;

(ii) $\langle a v_1 + b v_2, w\rangle = a\langle v_1, w\rangle + b\langle v_2, w\rangle$;

(iii) $\langle v, w\rangle = \langle w, v\rangle$.

This implies also that it satisfies the triangle inequality

$$|\langle v, w\rangle| \leq \sqrt{\langle v, v\rangle\langle w, w\rangle}.$$

Let us denote by $W^{1,2}(\Omega, \mu_p)$ the real Hilbert space completion of $C^\infty(\Omega, \mathbb{R})$ with respect to this norm. Let us state a property of this space immediately following from the definition.

Lemma 3.9. *$W^{1,2}(\Omega, \mu_p)$ is uniformly convex. Consequently, it is reflexive.*

Our goal is to show that for each $0 \leq p \leq m-1$,

$$P^* \subset W^{1,2}(\Omega, \mu_p).$$

If we take $u_p = \psi$ the strictly psh defining function for Ω, then clearly

$$W^{1,2}(\Omega, \mu_p) \subset W^{1,2}(\Omega, \mu_{p-1}).$$

Therefore, we only need to prove two special cases for $p = 0$ and $p = m-1$. Let us start with this simpler case $p = 0$, i.e., $\tau = 1$, $\tau \wedge \omega^{n-1} = \omega^{n-1}$ and $\mu_0 = \omega^n$ which is equivalent to the Lebesgue measure $dV = \beta^n$. This is a classical result if $\omega = \beta$ is the standard Kähler metric (see, e.g., [Hörmander 1994, Proposition 3.4.19]).

Lemma 3.10. *All functions in P^* belong to the Sobolev space $W^{1,2}(\Omega)$.*

Proof. Let $u \in P^*$. By the approximation theorem there is $\{u_j\}_{j\geq 1} \subset P^*$ a sequence of smooth functions such that $u_j \downarrow u$ pointwise. We may assume that $0 \leq u, u_j \leq 1$. Observe that the sequence $\{u_j\}_{j\geq 1}$ is uniformly bounded in $W^{1,2}(\Omega)$. Indeed, by the ω-subharmonicity

$$\begin{aligned}\int du_j \wedge d^c u_j \wedge \omega^{n-1} &= \int \left(\tfrac{1}{2} dd^c u_j^2 - 2u_j dd^c u_j\right) \wedge \omega^{n-1} \\ &\leq \frac{1}{2}\int dd^c u_j^2 \wedge \omega^{n-1}.\end{aligned}$$

The last integral is bounded by a constant $C = C(K, \psi, \Omega)$ by an application of the CLN inequality. Therefore, the sequence of (vector-valued) functions ∇u_j is uniformly bounded in $L^2(\Omega)$. Thus, by the weak compactness theorem, ∇u_j converges weakly to a vector-valued function $\boldsymbol{v} \in L^2(\Omega)$.

To complete the proof we are going to show that $\boldsymbol{v} = \nabla u$. First, note that $u_j \to u$ in $L^2(\Omega)$ by the dominated convergence theorem. So, $(u_j, \nabla u_j) \to (u, \boldsymbol{v})$ weakly in the Hilbert space $L^2(\Omega) \times L^2(\Omega, \mathbb{R}^n)$. The Mazur lemma [Yosida 1995, Chapter V, Theorem 2] implies that there is a sequence of convex combinations

$$\widetilde{u}_\ell = \sum_{j=1}^{\ell} \lambda_{\ell,j}(u_j, \nabla u_j), \quad \lambda_{\ell,j} \geq 0 \text{ and } \sum_{j=1}^{\ell} \lambda_{\ell,j} = 1,$$

which converges in norm to $(u, \boldsymbol{v})$ in $L^2(\Omega) \times L^2(\Omega, \mathbb{R}^n)$. It follows that

$$w_\ell = \sum_{j=1}^{\ell} \lambda_{\ell,j} u_j$$

is a Cauchy sequence in $W^{1,2}(\Omega)$. Hence, there exist a limit function

$$\widetilde{u} = \lim_{\ell\to\infty} w_\ell \in W^{1,2}(\Omega).$$

It follows that $u = \widetilde{u}$ and $\boldsymbol{v} = \nabla\widetilde{u} = \nabla u$. □

We proceed to prove the most general case $1 \le p = m-1$ when

$$\mu = \mu_p := \tau \wedge \omega^{n-m+1}$$

is a positive Radon measure. Equivalently, we wish to show that if $v \in P^*$, then the form dv has coefficients in $L^2(\Omega, d\mu)$. For a smooth function v we have

$$\int |\nabla v|^2 \, dV \le \frac{1}{c_0} \int dv \wedge d^c v \wedge \tau \wedge \omega^{n-m}.$$

This implies that $W^{1,2}(\Omega, \mu) \subset W^{1,2}(\Omega)$. In particular, the gradient ∇u is well defined in $W^{1,2}(\Omega, \mu)$.

Proposition 3.11. *Let $v \in P^*$ and $\{v_\ell\}_{\ell\ge 0} \subset P^*$ be a sequence of smooth m-ω-sh functions such that $v_\ell \downarrow v$ pointwise in Ω. Then v_ℓ converges weakly to v in $W^{1,2}(\Omega, \mu)$, i.e., for every $w \in W^{1,2}(\Omega, \mu)$,*

$$\lim_{\ell\to\infty} \langle v_\ell, w\rangle = \langle v, w\rangle.$$

In particular, $P^* \subset W^{1,2}(\Omega, \mu)$.

Proof. We may assume that $0 \le v \le v_\ell \le 1$. Then, there exists a uniform constant $C = C(\omega, n, m, \psi, \Omega)$ such that

$$\|\nabla v_\ell\|_\tau^2 = \int dv_\ell \wedge d^c v_\ell \wedge \tau \wedge \omega^{n-m} \le C\|v_\ell\|_{L^\infty}^2.$$

In fact, since $dv_\ell \wedge d^c v_\ell = \frac{1}{2} dd^c v_\ell^2 - v_\ell dd^c v_\ell$, and $dd^c v_\ell \wedge \tau \wedge \omega^{n-m}$ is a positive measure and $v_\ell \ge 0$, it follows that

$$\int dv_\ell \wedge d^c v_\ell \wedge \tau \wedge \omega^{n-m} \le \int dd^c v_\ell^2 \wedge \tau \wedge \omega^{n-m}.$$

Since v_ℓ^2 is also m-ω-sh, the desired inequality follows from the CLN inequality.

In the same way as in the previous proof, via Mazur's lemma, one proves that $v \in W^{1,2}(\Omega, \mu)$. □

Remark 3.12. The proofs given in Lemma 3.10 and Proposition 3.11 are inspired by the one in the book by Heinonen, Kilpeläinen and Martio [Heinonen et al. 2006, Theorem 1.30].

3.3. ***Integration by parts for bounded functions.*** In this section we show "integration by parts inequalities" for bounded m-ω-sh functions which are smooth near the boundary. In particular, they hold for functions in the class P^*. First we deal with the case of the wedge product of $m-2$ bounded functions and consider

$$\eta = dd^c u_1 \wedge \cdots \wedge dd^c u_{m-2}, \tag{3-14}$$

where $u_1, \ldots, u_{m-2} \in P^*$ as in Section 3.2. Recall that we also impose the assumption (3-12) for η, namely

$$\eta \wedge \omega^{n-m+1} \ge c_0 \omega^{n-1}, \tag{3-15}$$

and denote the corresponding trace measure by μ:

$$\mu = \eta \wedge \omega^{n-m+2}. \tag{3-16}$$

The following integration by parts formula for currents of order zero is used frequently below. Let ϕ and ρ be smooth functions such that $\phi = 0$ near $\partial\Omega$. Then,

$$\int \phi\, dd^c\rho \wedge \eta \wedge \omega^{n-m+1} = -\int d\phi \wedge d^c\rho \wedge \eta \wedge \omega^{n-m+1} + \int \phi\, d^c\rho \wedge d\omega \wedge \eta \wedge \omega^{n-m}. \tag{3-17}$$

If $v, w \in P^* \subset W^{1,2}(\Omega, \mu)$ are smooth, then

$$\langle v, w\rangle_\eta = \int dv \wedge d^c w \wedge \eta \wedge \omega^{n-m+1}.$$

By abusing the notation for general $v, w \in P^*$ we will write the integral on the right understanding that it is the inner product $\langle v, w\rangle_\eta$. The first version of the "integration by parts inequality" in the class P^* reads as follows.

Lemma 3.13. *Let $\{v_\ell\}_{\ell\geq 1}$ be a sequence of smooth functions from P^* and let $w \in P^*$ be smooth. Assume $v_\ell \leq w$ and $v_\ell \downarrow v \in P^*$. Then,*

$$-\int d(w-v) \wedge d^c v \wedge \eta \wedge \omega^{n-m+1} \leq \int (w-v) dd^c v \wedge \eta \wedge \omega^{n-m+1} + C\bigg(\int (w-v)\eta \wedge \omega^{n-m+2}\bigg)^{\frac{1}{2}}.$$

The constant C depends on ω, K, ψ and the uniform norm of functions, but it is independent of δ in (3-12).

Proof. Without loss of generality we may assume that $0 \leq v, v_\ell, w \leq 1$ and $0 \leq u_i \leq 1$. By the weak convergence in the space $W^{1,2}(\Omega, \mu)$ (Proposition 3.11) and the dominated convergence theorem it is enough to prove that

$$-\int d(w-v_\ell) \wedge d^c v \wedge \eta \wedge \omega^{n-m+1} \leq \int (w-v_\ell) dd^c v \wedge \eta \wedge \omega^{n-m+1} + C\bigg(\int (w-v_\ell)\eta \wedge \omega^{n-m+2}\bigg)^{\frac{1}{2}}.$$

To this end we use once more the weak convergence property when ℓ is fixed. Note that $w - v_\ell$ is a continuous function whose support is compact in Ω. It follows from the weak convergence of currents (of order zero) that

$$\int (w-v_\ell) dd^c v \wedge \eta \wedge \omega^{n-m+1} = \lim_{j\to\infty} \int (w-v_\ell) dd^c v_j \wedge \eta \wedge \omega^{n-m+1}.$$

Now for each ℓ and j, the integration by parts formula (3-17) gives

$$-\int d(w-v_\ell) \wedge d^c v_j \wedge \eta \wedge \omega^{n-m+1} = \int (w-v_\ell) dd^c v_j \wedge \eta \wedge \omega^{n-m+1} + \int (w-v_\ell) d^c v_j \wedge d\omega \wedge \eta \wedge \omega^{n-m}.$$

By Corollary 3.8 applied for $\phi = w - v_\ell$ and CLN inequality (Proposition 3.7) for bounded functions, the second integral on the right-hand side is bounded by

$$C\bigg(\int (w-v_\ell)\eta \wedge \omega^{n-m+2}\bigg)^{\frac{1}{2}}.$$

The desired inequality follows by letting $j \to \infty$. □

Lemma 3.14. *Let v, w and $\{v_\ell\}$ be as in Lemma 3.13. Let ρ be a bounded m-ω-sh in Ω such that $-1 \le \rho \le 0$. Then,*

$$\int_\Omega (w-v)dd^c\rho\wedge\eta\wedge\omega^{n-m+1} \le -\int_\Omega d(w-v)\wedge d^c\rho\wedge\eta\wedge\omega^{n-m+1} + C\bigg(\int_\Omega (w-v)\wedge\eta\wedge\omega^{n-m+2}\bigg)^{\frac{1}{2}}.$$

The dependence of C is the same as in the previous lemma.

Proof. Since the supports of the integrands on both sides are contained in K, we replace ρ by $\max\{\rho, A\psi\}$ for some $A > 0$ large such that $A\psi \le -1$ on a neighborhood of K. Thus, we may assume that $\rho \in P^*(\Omega, K', A\psi)$ for a compact subset $K \subset K' \Subset \Omega$. By the dominated convergence theorem for Radon measures and the weak convergence in Proposition 3.11 it is enough to prove the inequality for v_ℓ in the place of v, that is

$$\int_\Omega (w-v_\ell)dd^c\rho\wedge\eta\wedge\omega^{n-m+1} \le -\int_\Omega d(w-v_\ell)\wedge d^c\rho\wedge\eta\wedge\omega^{n-m+1} + C\bigg(\int_\Omega (w-v_\ell)\eta\wedge\omega^{n-m+2}\bigg)^{\frac{1}{2}}. \quad (3\text{-}18)$$

For a fixed ℓ the function $w - v_\ell$ is continuous, with its support in Ω. We can apply the limit argument once more to assume that ρ also is smooth as follows. By the approximation theorem for m-ω-sh functions we can find a decreasing sequence $\{\rho_j\}_{j\ge1} \subset P^*$ of smooth m-ω-sh functions such that $\rho_j \downarrow \rho$. We may assume that $-A \le \rho_j \le 0$. Then,

$$dd^c\rho_j\wedge\eta\wedge\omega^{n-m+1} \to dd^c\rho\wedge\eta\wedge\omega^{n-m+1}$$

weakly as measures. So, if the inequality holds for ρ_j, then so it does for ρ.

Using again the integration by parts formula (3-17) we have

$$\int_\Omega (w-v_\ell)dd^c\rho_j\wedge\eta\wedge\omega^{n-m+1} = -\int_\Omega d(w-v_\ell)\wedge d^c\rho_j\wedge\eta\wedge\omega^{n-m+1} + \int_\Omega (w-v_\ell)d^c\rho_j\wedge d\omega\wedge\eta\wedge\omega^{n-m}.$$

As in the proof of Lemma 3.13, by an application of Cauchy–Schwarz inequality (Corollary 3.8) and then the CLN inequality for the second integral on the right-hand side, we conclude that the inequality holds for ρ_j and so does (3-18). □

Lastly we can state the important estimate for functions in P^* and for $\eta = dd^cu_1\wedge\cdots\wedge dd^cu_{m-2}$, where $u_1, \dots, u_{m-2} \in P^*$ without strict positive assumption on η.

Lemma 3.15. *Let v, w and $\{v_\ell\}$ be as in Lemma 3.13. Let ρ be a bounded m-ω-sh in Ω such that $-1 \le \rho \le 0$. Then,*

$$\begin{aligned}&\int_\Omega (w-v)dd^c\rho\wedge\eta\wedge\omega^{n-m+1}\\ &\le C\bigg(\int_\Omega (w-v)dd^cv\wedge\eta\wedge\omega^{n-m+1}\bigg)^{\frac{1}{2}} + C\bigg(\int_\Omega (w-v)\eta\wedge\omega^{n-m+2}\bigg)^{\frac{1}{2}} + C\bigg(\int_\Omega (w-v)\eta\wedge\omega^{n-m+2}\bigg)^{\frac{1}{4}},\end{aligned}$$

with the same dependence of C as in previous lemmas.

Proof. Observe that it is enough to prove the inequality for $u_i := u_i + \delta|z|^2$ and then let $\delta \to 0^+$. Therefore without loss of generality we assume that η satisfies conditions (3-15) and (3-16). As in the proof of Lemma 3.14 we may assume that $\rho \in P^*$ and it is smooth. It follows from (3-18) that

$$\int_\Omega (w - v_\ell) dd^c\rho \wedge \eta \wedge \omega^{n-m+1} \leq -\int_\Omega d(w - v_\ell) \wedge d^c\rho \wedge \eta \wedge \omega^{n-m+1} + C\left(\int_\Omega (w - v_\ell)\eta \wedge \omega^{n-m+2}\right)^{\frac{1}{2}}. \quad (3\text{-}19)$$

It remains to bound the first integral on the right-hand side. Using the Cauchy–Schwarz inequality (Lemma 2.3) for $T = \eta \wedge \omega^{n-m+1}$ we get

$$\left|\int_\Omega d(w - v_\ell) \wedge d^c\rho \wedge T\right|^2 \leq C_1 \int_\Omega d(w - v_\ell) \wedge d^c(w - v_\ell) \wedge T, \quad (3\text{-}20)$$

where C_1 is the bound for

$$C\int_K d\rho \wedge d^c\rho \wedge T$$

and so the dependence of C_1 is as in the statement which is a simple consequence of the CLN inequality.

Next, we have

$$\int_\Omega d(w - v_\ell) \wedge d^c(w - v_\ell) \wedge T = -\int_\Omega (w - v_\ell) dd^c(w - v_\ell) \wedge T + \int_\Omega (w - v_\ell) d^c(w - v_\ell) \wedge d\omega \wedge \eta \wedge \omega^{n-m}. \quad (3\text{-}21)$$

Let us estimate the last integral. The Cauchy–Schwarz inequality and the CLN inequality imply that

$$\left|\int_\Omega (w - v_\ell) d^c(w - v_\ell) \wedge d\omega \wedge \eta \wedge \omega^{n-m}\right|^2 \leq C_2 \int_\Omega (w - v_\ell)\eta \wedge \omega^{n-m+2}, \quad (3\text{-}22)$$

where C_2 is the bound for

$$\int_\Omega (w - v_\ell) d(w - v_\ell) \wedge d^c(w - v_\ell) \wedge \eta \wedge \omega^{n-m+1}.$$

Notice also that as $w - v_\ell \geq 0$,

$$-(w - v_\ell) dd^c(w - v_\ell) \wedge T \leq (w - v_\ell) \wedge dd^c v_\ell \wedge T. \quad (3\text{-}23)$$

Combining (3-19), (3-20), (3-21), (3-22) and (3-23) we get

$$\int_\Omega (w - v_\ell) dd^c\rho \wedge \eta \wedge \omega^{n-m+1} \leq C\left(\int_\Omega (w - v_\ell) dd^c v_\ell \wedge T\right)^{\frac{1}{2}} + C\left(\int_\Omega (w - v_\ell)\eta \wedge \omega^{n-m+2}\right)^{\frac{1}{2}} + C\left(\int_\Omega (w - v_\ell)\eta \wedge \omega^{n-m+2}\right)^{\frac{1}{4}}. \quad (3\text{-}24)$$

Notice that by smooth sequence $\rho_j \downarrow \rho$ we obtain this inequality for $\rho \in P^*$. To finish the proof we need to know that passing to the limit as $\ell \to \infty$ we get the inequality in the statement. This is known for the

limits of all integrals above except for the first integral on the right-hand side. To handle this, we use the fact that $v_\ell \geq v$. Then

$$\int_\Omega (w - v_\ell) dd^c v_\ell \wedge \eta \wedge \omega^{n-m+1} \leq \int_\Omega (w - v) dd^c v_\ell \wedge \eta \wedge \omega^{n-m+1}. \tag{3-25}$$

Applying Lemma 3.14 for the right-hand side, we have

$$\int_\Omega (w - v) dd^c v_\ell \wedge \eta \wedge \omega^{n-m+1} \leq -\int_\Omega d(w - v) \wedge d^c v_\ell \wedge \eta \wedge \omega^{n-m+1} + C\left(\int_\Omega (w - v_\ell)\tau \wedge \omega^{n-m+2} \right)^{\frac{1}{2}}, \tag{3-26}$$

Combining three inequalities (3-24), (3-25) and (3-26), we obtain

$$\int_\Omega (w - v_\ell) dd^c \rho \wedge \eta \wedge \omega^{n-m+1} \leq C\left(-\int_\Omega d(w-v) \wedge d^c v_\ell \wedge \eta \wedge \omega^{n-m+1} \right)^{\frac{1}{2}} + C\left(\int_\Omega (w-v_\ell)\eta \wedge \omega^{n-m+2} \right)^{\frac{1}{2}} + C\left(\int_\Omega (w-v_\ell)\eta \wedge \omega^{n-m+2} \right)^{\frac{1}{4}}.$$

Letting $\ell \to \infty$ and invoking Proposition 3.11 gives

$$\int_\Omega (w - v) dd^c \rho \wedge \eta \wedge \omega^{n-m+1} \leq C\left(-\int_\Omega d(w-v) \wedge d^c v \wedge \eta \wedge \omega^{n-m+1} \right)^{\frac{1}{2}} + C\left(\int_\Omega (w-v)\eta \wedge \omega^{n-m+2} \right)^{\frac{1}{2}} + C\left(\int_\Omega (w-v)\eta \wedge \omega^{n-m+2} \right)^{\frac{1}{4}}.$$

Now the desired estimate follows from Lemma 3.13. □

Remark 3.16. In Lemmas 3.13, 3.14, 3.15 we only need to assume that w is continuous after using one more approximation argument. In fact, it is enough to assume $w \in P^*$.

For further applications we need the following lemma.

Lemma 3.17. *Let u be a bounded m-ω-sh. Suppose $\{u_j\}_{j\geq 1}$ is a decreasing sequence of smooth m-ω-sh functions with $\|u_j\|_{L^\infty} \leq 1$ such that $u_j \downarrow u$ pointwise. Assume also that all $u_j = u$ on a neighborhood of $\partial\Omega$. Let $-1 \leq \rho \leq 0$ be m-ω-sh functions. Then, for every $0 \leq s \leq m-1$,*

$$\lim_{j\to+\infty} \sup_\rho \left\{ \int_\Omega (u_j - u)(dd^c\rho)^s \wedge \omega^{n-s} \right\} = 0.$$

Proof. Let $K \Subset \Omega$ be the compact set such that $u_j = u$ on $\Omega \setminus K$ for all j. First we can modify ρ outside K by $A\psi$ without changing the integral value. Hence, we may assume that $\rho \in P^*$. Then, applying repeatedly Lemma 3.15 to replace $dd^c\rho$ by $dd^c u$ we get

$$\int_\Omega (u_j - u)(dd^c\rho)^{m-1} \wedge \omega^{n-m+1} \leq C \sum_{s=1}^{m-1} [e_{(0,0,s)}]^{1/2^s},$$

where

$$e_{(0,0,s)} = \int_\Omega (u_j - u)(dd^c u)^s \wedge \omega^{n-s}.$$

Thus, the proof of the lemma follows by the dominated convergence theorem applied to each term on the right-hand side. □

We get immediate consequences by the classical arguments. Let us define the $(m-1)$-capacity in the class of m-ω-sh functions. For a Borel set $E \subset \Omega$,

$$\mathrm{cap}_{m-1}(E) = \sup\left\{\int_E H_{m-1}(\rho) : -1 \le \rho \le 0,\ \rho \text{ is } m-\omega\text{-sh in } \Omega\right\}.$$

Corollary 3.18. *If $\{u_j\}_{j\ge1}$ is a uniformly bounded decreasing sequence of smooth m-ω-sh functions such that $u_j \downarrow u$ (m-ω-sh function), then $u_j \to u$ in $\mathrm{cap}_{m-1}(\bullet)$.*

Corollary 3.19. *Every bounded m-ω-sh function is quasicontinuous with respect to $\mathrm{cap}_{m-1}(\bullet)$.*

The quasicontinuity in $(m-1)$-capacity allows us to obtain the weak convergence for the wedge product of m-ω-sh bounded functions (compare [Bedford and Taylor 1987, Proposition 3.2]).

Proposition 3.20. *Let $0 \le p \le m-1$. Let $v, u_1, \dots, u_p$ be uniformly bounded m-ω-sh functions in Ω. Let $\{v_\ell\}_{\ell\ge1}$, $\{u_s^j\}_{j\ge1}$ be decreasing sequences of smooth m-ω-sh functions such that $v_\ell \downarrow v$ and $u_s^j \downarrow u_s$ for each $1 \le s \le p$. Then*

$$v_j dd^c u_1^j \wedge \cdots \wedge dd^c u_p^j \to v dd^c u_1 \wedge \cdots \wedge dd^c u_p$$

as $j \to \infty$ in the sense of currents of order zero. Consequently,

$$dd^c v_j \wedge dd^c u_1^j \wedge \cdots \wedge dd^c u_p^j \wedge \omega^{n-p} \to dd^c v \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge \omega^{n-p}$$

in the sense of currents.

Proof. By the localization principle we may assume Ω is a ball and all functions belong to P^*. Without loss of generality we may assume that $-1 \le v_j, u_s^j, v, u \le 0$. Define $S^j = dd^c u_1^j \wedge \cdots \wedge dd^c u_p^j$ and $S = dd^c u_1 \wedge \cdots \wedge dd^c u_p$. Both of them are currents of order zero. Let χ be a test form with $\mathrm{supp}\,\chi = K$. We need to show that

$$\lim_{j\to\infty} \left| \int (v_j S^j - vS) \wedge \chi \right| = 0.$$

We argue by induction over p. It is obvious for $p = 0$. Assume that it is true for $p-1 \le m-2$. This means that

$$u_1^j dd^c u_2^j \wedge \cdots \wedge dd^c u_p^j \to u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_p$$

in the sense of currents of order zero. Hence,

$$S^j = dd^c[u_1^j dd^c u_2^j \wedge \cdots \wedge dd^c u_p^j] \to dd^c[u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_p] = S.$$

Furthermore, by [Kołodziej and Nguyen 2016, Corollary 2.4], for a test form ϕ, we have

$$|S^j \wedge \phi| \le C\|\phi\|_K \widetilde{S}^j \wedge \omega^{n-p}, \tag{3-27}$$

where $\widetilde{S}^j = \big(dd^c \sum_{s=1}^p u_s^j\big)^p$ and $\|\phi\|_K$ is the C^0-norm of ϕ on K. Note that $\widetilde{S}^j \to \widetilde{S} = \big(dd^c \sum_{s=1}^p u_s\big)^p$ as the weak convergence holds for the wedge product of currents associated to p bounded functions. Letting $j \to \infty$ one obtains

$$|S \wedge \phi| \le C\|\phi\|_K \widetilde{S} \wedge \omega^{n-p}. \tag{3-28}$$

We are ready to show that the conclusion holds for $p \le m-1$. The uniform constants C below depend additionally on the uniform norm of coefficients of χ.

Let $\varepsilon > 0$. By the quasicontinuity with respect to $(m-1)$-capacity in Corollary 3.19, we can find an open set G such that $\mathrm{cap}_{m-1}(G) < \varepsilon$ and u_s^j, u_s, v are continuous on $\Omega \setminus G$. Note that $\Omega \setminus G$ is compact in Ω. It follows from Dini's theorem that $u_s^j \to u_s$ uniformly as $j \to \infty$ on that set. We first write

$$\int (v_j S^j - vS) \wedge \chi = \int (v_j - v) S^j \wedge \chi + \int v(S^j - S) \wedge \chi. \tag{3-29}$$

For the first term on the right, it follows from (3-27) that

$$\left| \int (v_j - v) S^j \wedge \chi \right| \le C \bigg(\int_{\Omega \setminus G} (v_j - v) \widetilde{S}^j \wedge \omega^{n-p} + n^p \,\mathrm{cap}_{m-1}(G) \bigg), \tag{3-30}$$

where we used the fact that

$$\int_G (v_j - v) \widetilde{S}^j \wedge \omega^{n-p} \le n^p \,\mathrm{cap}_{m-1}(G).$$

Next, we estimate the second integral in (3-29). Let $-1 \le \widetilde{v} \le 0$ be a continuous extension of v from $\Omega \setminus G$ to Ω. By induction hypothesis we know that $S^j \to S$ weakly, as currents. So,

$$\lim_{j \to \infty} \int \widetilde{v}(S^j - S) \wedge \chi = 0. \tag{3-31}$$

Moreover, similarly as above

$$\left| \int_G \widetilde{v} S^j \wedge \chi \right| \le C n^p \,\mathrm{cap}_{m-1}(G),$$

and by (3-28)

$$\left| \int_G \widetilde{v} S \wedge \chi \right| \le C \left| \int_G \widetilde{S} \wedge \omega^{n-p} \right| \le C n^p \,\mathrm{cap}_{m-1}(G).$$

Then,

$$\left| \int v(S^j - S) \wedge \chi \right| \le \left| \int \widetilde{v}(S^j - S) \wedge \chi \right| + 4C n^p \,\mathrm{cap}_{m-1}(G). \tag{3-32}$$

Combining (3-29), (3-30), (3-31) and (3-32) we conclude that

$$\lim_{j \to \infty} \left| \int (v_j S^j - vS) \wedge \chi \right| \le C\varepsilon.$$

This holds for arbitrary ε, so the proof of the proposition follows. □

Remark 3.21. The proposition showed that the wedge product of currents associated to continuous m-ω-sh functions given in [Kołodziej and Nguyen 2016, Proposition 2.11] coincides with the one in Definition 3.4.

Corollary 3.22. *The wedge products*

$$dd^c u_1 \wedge \cdots \wedge dd^c u_{m-1}, \quad dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge \omega^{n-m}$$

are symmetric with respect to bounded m-ω-sh functions $u_1, \ldots, u_{m-1}, u_m$.

We can now extend the generalized Cauchy–Schwarz inequality in Lemma 2.4 and Corollary 2.5 to the case of bounded functions. We state here a simple version which is essential to prove the quasicontinuity later on. This is a stronger version of Corollary 3.8. Let $-1 \le u_1, \ldots, u_{m-1} \le 0$ be bounded m-ω-sh functions in Ω. Define

$$\tau = dd^c u_1 \wedge \cdots \wedge dd^c u_{m-1}. \tag{3-33}$$

Corollary 3.23. *Let $0 \le \phi \le 1$ be a continuous function such that $\operatorname{supp}\phi = K \Subset \Omega$. Let $0 \le h \le 1$ be a smooth m-ω-sh function. There exists a constant $C = C(\omega, K)$ such that*

$$\left| \int \phi \, dh \wedge d^c\omega \wedge \tau \wedge \omega^{n-m-1} \right|^2 \le C \int \phi \, \widetilde{\tau} \wedge \omega^{n-m+1},$$

where $\widetilde{\tau} = \big(dd^c \sum_{s=1}^{m-1} u_s\big)^{m-1}$.

Proof. The left-hand side is well defined because the coefficients of S are signed Radon measures. Let $\{u_s^j\}_{j\ge1}$ be a sequence of smooth m-ω-sh functions such that $u_s^j \downarrow u_s$ for each $1 \le s \le m-1$. We may assume $-1 \le u_s^j \le 0$. Define

$$S^j = dd^c u_1^j \wedge \cdots \wedge dd^c u_{m-1}^j, \quad \widetilde{S}^j = (dd^c u_1^j + \cdots + dd^c u_{m-1}^k)^{m-1}.$$

Applying Corollary 2.5 for $\gamma_s = dd^c u_s^j$, $s = 1, \ldots, p$, we get

$$\left| \int \phi \, dh \wedge d^c\omega \wedge S^j \wedge \omega^{n-m-1} \right|^2 \le C \int \phi \, dh \wedge d^c h \wedge \left(\sum_{s=1}^{m-1} \gamma_s \right)^{m-1} \wedge \omega^{n-m} \times \int \phi \, \widetilde{S}^j \wedge \omega^{n-m+1}.$$

As in the proof of Corollary 3.8, the CLN inequality implies that the first integral inequality on the right-hand side is bounded by a constant $C = C(\omega, K)$. By Proposition 3.20 the last integral converges to $\int \phi \, \widetilde{\tau} \wedge \omega^{n-m+1}$. □

4. Quasicontinuity

Having the Hessian measure defined for bounded m-ω-sh functions, we can introduce the m-capacity (see [Bedford and Taylor 1982]): for a Borel subset $E \subset \Omega$,

$$\operatorname{cap}_m(E) = \sup\left\{ \int_E (dd^c \rho)^m \wedge \omega^{n-m} : \rho \text{ is } m-\omega\text{-sh in } \Omega, -1 \le \rho \le 0 \right\}. \tag{4-1}$$

Here in fact $\mathrm{cap}_m(E) = \mathrm{cap}_m(E, \Omega)$ but we shall often suppress Ω in the notation if the domain is fixed. Then, this is an inner capacity, namely,

$$\mathrm{cap}_m(E) = \sup\{\mathrm{cap}_m(K) : K \text{ is compact subset of } E\}.$$

Proposition 4.1. *Let Ω be a open set in $\mathbb{C}^n$ and $\mathrm{cap}_m(E) = \mathrm{cap}_m(E, \Omega)$.*

(a) *If $E_1 \subset E_2$, then $\mathrm{cap}_m(E_1) \le \mathrm{cap}_m(E_2)$.*

(b) *If $E \subset \Omega_1 \subset \Omega_2$, then $\mathrm{cap}_m(E, \Omega_2) \le \mathrm{cap}_m(E, \Omega_1)$.*

(c) $\mathrm{cap}_m\big(\bigcup_j E_j\big) \le \sum_j \mathrm{cap}_m(E_j)$.

(d) *If $E_1 \subset E_2 \subset \cdots$ are Borel sets in Ω and $E := \bigcup_j E_j$, then $\mathrm{cap}_m(E) = \lim_j \mathrm{cap}_m(E_j)$.*

Definition 4.2 (convergence in capacity). A sequence of Borel functions u_j in Ω is said to converge in capacity (or in $\mathrm{cap}_m(\bullet)$) to u if for any $\delta > 0$ and $K \Subset \Omega$,

$$\lim_{j\to\infty} \mathrm{cap}_m(K \cap |u_j - u| \ge \delta) = 0.$$

Proposition 4.3. *If $\{u_j\}_{j\ge1}$ is a uniformly bounded sequence of continuous m-ω-sh functions that decreases to a bounded m-ω-sh function u in Ω, then u_j converges to u in $\mathrm{cap}_m(\bullet)$.*

The proof of this proposition will need the improved versions of Lemma 3.13, 3.14, 3.15 in which $\eta \wedge \omega$ is replaced by

$$\tau = dd^c u_1 \wedge \cdots \wedge dd^c u_{m-1},$$

where $u_1, \ldots, u_{m-1} \in P^*$ in (3-11) and the positivity assumption (3-12) is satisfied. This is done thanks to Corollary 3.23. Since the proofs follow line by line those from the previous section after replacing $\eta \wedge \omega$ by τ and applying Corollary 3.23 instead of Corollary 3.8, we only state the lemmas.

Lemma 4.4. *Let $\{v_\ell\}_{\ell\ge1}$ be a sequence of smooth functions from P^* and let $w \in P^*$ be smooth. Assume $v_\ell \le w$ and $v_\ell \downarrow v \in P^*$. Then,*

$$-\int d(w-v) \wedge d^c v \wedge \tau \wedge \omega^{n-m} \le \int (w-v) dd^c v \wedge \tau \wedge \omega^{n-m} + C\bigg(\int (w-v)\tau \wedge \omega^{n-m+1}\bigg)^{\frac{1}{2}},$$

where the constant C depends on ω, K, ψ and the uniform norm of functions, but it is independent of δ in (3-12).

Lemma 4.5. *Let v, w and $\{v_\ell\}$ be as in Lemma 4.4. Let ρ be a bounded m-ω-sh in Ω such that $-1 \le \rho \le 0$. Then,*

$$\int_\Omega (w-v) dd^c \rho \wedge \tau \wedge \omega^{n-m} \le -\int_\Omega d(w-v) \wedge d^c \rho \wedge \tau \wedge \omega^{n-m} + C\bigg(\int_\Omega (w-v) \wedge \tau \wedge \omega^{n-m+1}\bigg)^{\frac{1}{2}}.$$

The dependence of C is the same as in the previous lemma.

As in Lemma 3.15 the strict positivity assumption (3-12) is not needed in the following statement.

Lemma 4.6. *Let v, w and $\{v_\ell\}$ be as in Lemma 4.4. Let ρ be a bounded m-ω-sh in Ω such that $-1 \le \rho \le 0$. Then,*

$$\int_\Omega (w-v)dd^c\rho\wedge\tau\wedge\omega^{n-m} \le C\left(\int_\Omega (w-v)dd^c v\wedge\tau\wedge\omega^{n-m}\right)^{\frac{1}{2}} + C\left(\int_\Omega (w-v)\tau\wedge\omega^{n-m+1}\right)^{\frac{1}{2}} + C\left(\int_\Omega (w-v)\tau\wedge\omega^{n-m+1}\right)^{\frac{1}{4}}$$

with the same dependence of C as in previous lemmas.

Remark 4.7. In Lemmas 4.4, 4.5, 4.6 it is enough to assume that $w \in P^*$ after using one more approximation argument.

Thanks to Lemma 4.6, we can easily prove the last case $s = m$ of Lemma 3.17.

Lemma 4.8. *Let u be a bounded m-ω-sh. Suppose $\{u_j\}_{j\ge1}$ is a decreasing sequence of smooth m-ω-sh functions with $\|u_j\|_{L^\infty} \le 1$ such that $u_j \downarrow u$ pointwise. Assume also that all $u_j = u$ on a neighborhood of $\partial\Omega$. Let $-1 \le \rho \le 0$ be m-ω-sh functions. Then,*

$$\lim_{j\to+\infty} \sup_\rho \left\{\int_\Omega (u_j-u)(dd^c\rho)^m\wedge\omega^{n-m}\right\} = 0. \tag{4-2}$$

End of proof of Proposition 4.3. Because of (b) and (c) in Proposition 4.1 we can assume that Ω is a ball and all functions are equal near the boundary. Let $\delta > 0$. We wish to show that

$$\lim_{j\to\infty} \mathrm{cap}_m(\{u_j - u > \delta\}) = 0.$$

Argument by contradiction. Suppose that the statement were not true. Then, there would exist $\varepsilon > 0$, a subsequence $\{u_{j_s}\} \subset \{u_j\}$ and a sequence of m-ω-sh functions ρ_{j_s} with $-1 \le \rho_{j_s} \le 0$ such that

$$\limsup_{j_s\to+\infty} \int_{\{u_{j_s}-u>\delta\}} H_m(\rho_{j_s}) \ge \varepsilon.$$

On the other hand, by Markov's inequality,

$$\int_{\{u_{j_s}-u>\delta\}} H_m(\rho_{j_s}) \le \frac{1}{\delta}\int_\Omega (u_{j_s}-u)H_m(\rho_{j_s}).$$

Lemma 4.8 shows that the right-hand side converges to zero. This leads to a contradiction. □

Theorem 4.9. *Let Ω be a bounded open set in $\mathbb{C}^n$. Let u be a m-ω-sh function in Ω. Then, for every $\varepsilon > 0$, there exists an open subset $U \subset \Omega$ with $\mathrm{cap}_m(U,\Omega) < \varepsilon$ such that u restricted to $\Omega\setminus U$ is continuous.*

Proof. The result is local and, moreover, it can be reduced to the case of bounded functions by the property (7-5) whose proof will use only bounded functions. We can use the classical argument in [Bedford and Taylor 1982, Theorem 3.5] since there exists a sequence of smooth m-ω-sh functions that decrease to u pointwise (Proposition 2.9) and our capacity is subadditive by Proposition 4.1. □

We obtain the convergence in capacity for monotone sequences of uniformly bounded functions. In particular, the smoothness assumption in Proposition 4.3 can be relaxed by yet another approximation. We recall first a classical result in measure theory, giving a short proof for the reader's convenience.

Lemma 4.10. *Let μ_j be a sequence of positive Radon measures with compact support in Ω. Assume that μ_j converges weakly to μ. Let $\Omega \Supset F_1 \supset F_2 \supset \cdots$ be a sequence of decreasing closed subsets in Ω satisfying*

$$\lim_{j\to\infty} \mu(F_j) = 0.$$

Then, $\lim_{j\to+\infty} \mu_j(F_j) = 0$.

Proof. Fix $\varepsilon > 0$. By the assumption there exists $j_0 > 0$ such that $\mu(F_{j_0}) < \varepsilon$. Using the inclusions we have $\mu_j(F_j) \le \mu_j(F_{j_0})$, for $j > j_0$. The weak convergence implies

$$\limsup_{j\to\infty} \mu_j(F_j) \le \limsup_{j\to\infty} \mu_j(F_{j_0}) \le \mu(F_{j_0}) < \varepsilon.$$

It follows that

$$0 \le \liminf_{j\to\infty} \mu_j(F_j) \le \limsup_{j\to\infty} \mu_j(F_j) \le \varepsilon.$$

This holds for every $\varepsilon > 0$. □

Corollary 4.11. *Let Ω be a bounded open set in $\mathbb{C}^n$. Let $\{u_j\}_{j\ge1}$ be a uniformly bounded and monotone sequence of m-ω-sh functions such that either $u_j \downarrow u$ pointwise or $u_j \uparrow u$ almost everywhere for a bounded m-ω-sh function u in Ω. Then, u_j converges to u in m-capacity.*

Proof. The localization principle applies in both cases. We can assume that Ω is a ball and the u_j are equal to a fixed smooth psh function in a neighborhood of the boundary. Hence, $u - u_j = 0$ on $\Omega \setminus K$ for a fixed compact subset K. Let $\delta > 0$. We wish to show that

$$\lim_{j\to\infty} \operatorname{cap}_m(|u - u_j| \ge \delta) = 0.$$

Arguing by contradiction, suppose that this were not true. Then, there would exist $\varepsilon > 0$ and a sequence of m-ω-sh functions ρ_j with $-1 \le \rho_j \le 0$ such that

$$\limsup_{j\to\infty} \int_{\{|u-u_j|\ge\delta\}} H_m(\rho_j) \ge \varepsilon.$$

Fix a cut-off function χ with compact support in Ω and equal 1 on K. By the CLN inequality for bounded functions (Proposition 3.7) we may assume that $\mu_j := \chi H_m(\rho_j)$ converges weakly to a positive Radon measure μ.

We use the quasicontinuity. Find an open set $G \subset \Omega$ such that $\operatorname{cap}_m(G) \le \varepsilon/2$ and the restrictions of u_j, u to $\Omega \setminus G$ are continuous functions. Since u_j is a monotone sequence, it follows from Dini's lemma that it converges uniformly on $\Omega \setminus G$. In particular, the sets $F_j := \{|u - u_j| \ge \delta\} \setminus G$ are closed in Ω and satisfy $\lim_{j\to\infty} \mu(F_j) = 0$. Hence,

$$\int_{\{|u-u_j|\ge\delta\}} H_m(\rho_j) \le \int_{F_j} \chi H_m(\rho_j) + \operatorname{cap}_m(G) \le \mu_j(F_j) + \tfrac{1}{2}\varepsilon.$$

Letting $j \to \infty$ and using Lemma 4.10 this leads to a contradiction. □

5. Weak convergence

5.1. ***Convergence theorems for decreasing sequences.*** Let Ω be a bounded open set in $\mathbb{C}^n$. We have continuity of wedge products of m-ω-sh functions under decreasing sequences of bounded functions.

Lemma 5.1. *Let $v, u_1, \dots, u_p$, $1 \le p \le m$, be a bounded m-ω-sh functions in Ω. Let $\{v^j\}_{j\ge1}$ and $\{u_s^j\}_{j\ge1}$ be uniformly bounded sequences of m-ω-sh such that $v^j \downarrow v$ and $u_s^j \downarrow u_s$ as $j \to +\infty$, for each $s = 1, \dots, p$. Then,*

(a) $\lim_{j\to\infty} \mathscr{L}(u_1^j, \dots, u_p^j) = \mathscr{L}(u_1, \dots, u_p)$;

(b) $\lim_{j\to+\infty} v^j \mathscr{L}(u_1^j, \dots, u_p^j) = v\mathscr{L}(u_1, \dots, u_p)$;

where the convergence is understood in the sense of currents of order zero.

Proof. (a) By the localization principle we may assume that all functions are defined in a ball Ω and they are equal to a fixed smooth psh function ψ outside $\Omega' \Subset \Omega$. Let $\{u_s^{j,\delta}\}_{\delta>0}$ be decreasing sequences of smooth m-ω-sh functions such that $u_s^{j,\delta} \downarrow u_s^j$ as $\delta \to 0$. Similarly, let $\{u_s^\delta\}_{\delta>0}$ be approximating sequences for u_s. We may assume that all involved functions are negative and of uniform norm less than one.

Let χ be a test form such that $\operatorname{supp}\chi = K \Subset \Omega$. We consider the difference

$$M_{(j,\delta)} = \int \chi[\mathscr{L}(u_1^{j,\delta}, \dots, u_p^{j,\delta}) - \mathscr{L}(u_1^\delta, \dots, u_p^\delta)].$$

By Proposition 3.20 $\lim_{\delta\to0} \mathscr{L}(u_1^\delta, \dots, u_p^\delta) = \mathscr{L}(u_1, \dots, u_p)$. So, the proof is completed as soon as we show that

$$\lim_{j\to\infty} \lim_{\delta\to0} |M_{(j,\delta)}| = 0.$$

We claim that

$$|M_{(j,\delta)}| \le C \sum_{s=1}^{p} \int_K |u_s^{j,\delta} - u_s^\delta| (dd^c\rho^{j,\delta})^{p-1} \wedge \omega^{n-p+1}, \tag{5-1}$$

where $\rho^{j,\delta} = \frac{1}{2p}\sum_{s=1}^p (u_s^{j,\delta} + u_s^\delta)$. In fact,

$$[\mathscr{L}(u_1^{j,\delta}, \dots, u_p^{j,\delta}) - \mathscr{L}(u_1^\delta, \dots, u_p^\delta)] = \sum_{s=1}^{p} dd^c(u_s^{j,\delta} - u_s^\delta) \wedge T_s \wedge \omega^{n-m},$$

where

$$T_s(j,\delta) = dd^c u_1^{j,\delta} \wedge \cdots \wedge dd^c u_{s-1}^{j,\delta} \wedge dd^c u_{s+1}^\delta \wedge \cdots \wedge dd^c u_p^\delta.$$

(Here one should use the obvious modifications for $s = 1$ and $s = p$.) Notice that T_s is a smooth closed $(p-1, p-1)$-form. By integration by parts,

$$\int \chi\omega^{n-m} \wedge dd^c(u_s^{j,\delta} - u_s^\delta) \wedge T_p = \int (u_s^{j,\delta} - u_s^\delta) dd^c(\chi\omega^{n-m}) \wedge T_s.$$

Since $0 \le p-1 \le m-1$, it follows from [Kołodziej and Nguyen 2016, Corollary 2.4] that

$$|dd^c(\chi\omega^{n-m}) \wedge T_s| \le 2^m C[dd^c(u_1^{j,\delta} + \cdots + u_p^\delta)]^{p-1} \wedge \omega^{n-p+1}$$

for a uniform constant C depending only on ω and χ. This proves (5-1).

At this point we no longer have the continuity of u_s^j and u_s, however, we can make use of the quasicontinuity. Let $\varepsilon > 0$. Find an open set G such that all $u_s^{j,\delta}$, u_s^j and also u_s^j, u_s are continuous on $\Omega \setminus G$ and $\mathrm{cap}_m(G) < \varepsilon$. We know that $\Omega \setminus G$ is compact in Ω and by Dini's theorem for $s = 1, \dots, p$ we have $u_s^{j,\delta} \to u_s^j$ and $u_s^\delta \to u_s$ uniformly as $\delta \to 0$ on that set. Therefore,

$$\int_K |u_s^{j,\delta} - u_s^\delta| (dd^c \rho^{j,\delta})^{p-1} \wedge \omega^{n-p+1} \le \int_{\Omega\setminus G} |u_s^{j,\delta} - u_s^\delta| (dd^c \rho^{j,\delta})^{p-1} \wedge \omega^{n-p+1} + \mathrm{cap}_m(G).$$

To estimate the integral on the right-hand side we use

$$\begin{aligned}\int_{\Omega\setminus G} |u_s^{j,\delta} - u_s^\delta| (dd^c \rho^{j,\delta})^{p-1} \wedge \omega^{n-p+1} \le{}& \int_{\Omega\setminus G} (u_s^{j,\delta} - u_s^j)(dd^c \rho^{j,\delta})^{p-1} \wedge \omega^{n-p+1} \\ &+ \int_{\Omega\setminus G} (u_s^\delta - u_s)(dd^c \rho^{j,\delta})^{p-1} \wedge \omega^{n-p+1} \\ &+ \int_{\Omega\setminus G} (u_s^j - u_s)(dd^c \rho^{j,\delta})^{p-1} \wedge \omega^{n-p+1}.\end{aligned}$$

By the uniform convergence and then the CLN inequality (Proposition 3.7) the first two terms go to zero as $\delta \to 0$. Moreover, $\rho^{j,\delta}$ is a sequence of smooth m-ω-sh functions decreasing to $\rho^j = \frac{1}{2p} \sum_{s=1}^p (u_s^j + u_s)$ as $\delta \to 0$. This implies by Proposition 3.20 that $\mathscr{L}_{p-1}(\rho^{j,\delta})$ converges weakly to $\mathscr{L}_{p-1}(\rho^j)$ as $\delta \to 0$. Combining this with the continuity on $\Omega \setminus G$ we get

$$\lim_{\delta\to 0} \int_{\Omega\setminus G} (u_s^j - u_s)\mathscr{L}_{p-1}(\rho^{j,\delta}) = \int_{\Omega\setminus G} (u_s^j - u_s)\mathscr{L}_{p-1}(\rho^j).$$

It follows that

$$\lim_{\delta\to 0} |M_{(j,\delta)}| \le \int_{\Omega\setminus G} (u_s^j - u_s)\mathscr{L}_{p-1}(\rho^j) + \varepsilon.$$

Letting $j \to +\infty$ we have by the uniform convergence,

$$\lim_{j\to\infty} \lim_{\delta\to 0} |M_{(j,\delta)}| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof is completed.

(b) For simplicity we assume that $u_1 = \cdots = u_p = u$ and also $\{u_s^j\} = \{u^j\}$. The general case follows in the same way. Then we write

$$\mathscr{L}_p(u) = \mathscr{L}_p(u, \dots, u).$$

Since v^j decreases to v and $\mathscr{L}_p(u^j)$ converges weakly to $\mathscr{L}_p(u)$ thanks to (a), any weak limit Θ of the sequence $v^j \mathscr{L}_p(u^j)$ satisfies

$$\Theta \le v\mathscr{L}_p(u).$$

Hence, $v\mathscr{L}_p(u) - \Theta$ is a positive current. In particular,

$$v\mathscr{L}_p(u) \wedge \omega^{m-p} - \Theta \wedge \omega^{m-p} = vH_p(u) - \Theta \wedge \omega^{m-p}$$

is a positive Radon measure. To show the converse let $\chi \geq 0$ be a test function in Ω. We will show that

$$\int \chi \Theta \wedge \omega^{m-p} \geq \int \chi v H_p(u).$$

In fact, since $v^j H_p(u^j)$ converges weakly to $\Theta \wedge \omega^{m-p}$, it is enough to show

$$\lim_{j\to+\infty} \int \chi v^j H_p(u^j) \geq \int \chi v H_p(u). \tag{5-2}$$

Let $\varepsilon > 0$ and choose an open set $G \subset \Omega$ such that $\mathrm{cap}_m(G, \Omega) \leq \varepsilon$ and v, v_s are all continuous on $F = \Omega \setminus G$. Since v is continuous on F, there is a continuous extension g to Ω such that $v = g$ on F with the same uniform norm. Without loss of generality we also assume that $0 \leq v, v_s, g \leq 1$. Hence,

$$\begin{aligned}
\int \chi v H_p(u) &\leq \int_F \chi v H_p(u) + \mathrm{cap}_m(G) \\
&= \int_F \chi g H_p(u) + \mathrm{cap}_m(G) \\
&\leq \int \chi g H_p(u) + \mathrm{cap}_m(G) \\
&= \lim_{j\to\infty} \int \chi g H_p(u^j) + \mathrm{cap}_m(G) \\
&\leq \lim_{j\to\infty} \int_F \chi g H_p(u^j) + 2\,\mathrm{cap}_m(G).
\end{aligned}$$

By Dini's theorem v^j converges to $v = g$ uniformly on F, so the last integral does not exceed

$$\lim_{j\to\infty} \int_F \chi v^j H_p(u^j) + \varepsilon \leq \lim_{j\to\infty} \int \chi v^j H_p(u^j) + \varepsilon.$$

Therefore, we have proved that

$$\int \chi v H_p(u) \leq \lim_{j\to\infty} \int \chi v^j H_p(u^j) + 2\,\mathrm{cap}_m(G) + \varepsilon.$$

Since $\mathrm{cap}_m(G) \leq \varepsilon$ and $\varepsilon > 0$ is arbitrary, the inequality (5-2) follows. □

Corollary 5.2. *Let u, v be bounded m-ω-sh functions and $T = dd^c v_1 \wedge \cdots \wedge dd^c v_{m-p} \wedge \omega^{n-m}$ for bounded m-ω-sh functions $v_1, \ldots, v_{m-p}$, where $1 \leq p \leq m$. Then,*

$$\mathbb{1}_{\{u<v\}} (dd^c \max\{u, v\})^p \wedge T = \mathbb{1}_{\{u<v\}} (dd^c v)^p \wedge T.$$

Consequently,

$$(dd^c \max\{u, v\})^p \wedge T \geq \mathbb{1}_{\{u\geq v\}} (dd^c u)^p \wedge T + \mathbb{1}_{\{u<v\}} (dd^c v)^p \wedge T.$$

Proof. Given the weak convergence results under decreasing sequences in the above lemma, the proof of [Guedj and Zeriahi 2017, Theorem 3.27] can be easily adapted to the current case. □

We conclude this section by going back to the extensions of results in Section 2.4, noted in Remark 2.17. We give here only the most important statement that will be used later.

Corollary 5.3. *Let $\Omega \Subset \mathbb{C}^n$ be strictly m-pseudoconvex. Let $-1 \le v \le w \le 0$ be bounded m-ω-sh functions such that $\lim_{z\to\partial\Omega}(w-v)=0$. Let ρ be a bounded m-ω-sh function such that $-1 \le \rho \le 0$. There is a constant $C = C(\omega, n, m)$ such that*

$$\int_\Omega (w-v)^{3m}(dd^c\rho)^m \wedge \omega^{n-m} \le C\sum_{s=0}^{m}\int_\Omega (w-v)(dd^c v)^s \wedge \omega^{n-s}.$$

Proof. Let us replace w by $w_\varepsilon = \max\{w-\varepsilon, v\}$ for $\varepsilon > 0$, so that $w_\varepsilon = v$ in a neighborhood of $\partial\Omega$. If we could prove the inequality for w_ε and v, then by letting $\varepsilon \to 0$, the domination convergence theorem would imply the required inequality. Let $\Omega' \Subset \Omega$ be a smooth subdomain such that $w = v$ on $\Omega \setminus \Omega'$. Then the integrals on both sides will not change if we modify v, w outside Ω'. Hence, we may further assume that $w = v = \psi$ on $\Omega \setminus \Omega'$ with ψ a smooth m-ω-sh defining function for Ω.

Using the quasicontinuity it is easy to see from Lemma 5.1(b) that for smooth decreasing sequences $w_j \downarrow w$, $v_j \downarrow v$ and $\rho_j \downarrow \rho$ we have

$$\lim_{j\to\infty}\int_\Omega (w_j - v_j)^{3m} H_m(\rho_j) = \int_\Omega (w-v)^{3m} H_m(\rho),$$

and for $0 \le s \le m$,

$$\lim_{j\to\infty}\int_\Omega (w_j - v_j) H_s(v_j) = \int_\Omega (w-v) H_s(v).$$

Therefore, it is enough to prove the inequality for smooth functions $v_j \le w_j \le 0$ and $-1 \le \rho \le 0$. Notice that $w_j \to w$ and $v_j \to v$ uniformly on $\Omega \setminus \Omega'$ (this is the reason why we modify w, v near the boundary). Thus, we can follow the argument in Remark 2.16 and conclude that the extra terms will vanish after passing to the limit as $j \to +\infty$. Hence, the proof for the bounded functions case follows. □

5.2. ***Convergence theorems for increasing sequences.*** With a similar proof as that of Lemma 5.1 we get:

Lemma 5.4. *Let $1 \le p \le m$ and $v, u_1, \dots, u_p$ be bounded m-ω-sh functions. Suppose that $\{v^j\}_{j\ge1}$, $\{u_s^j\}_{j\ge1}$ are uniformly bounded increasing sequences of m-ω-sh functions such that $v^j \uparrow v$ and $u_s^j \uparrow u_s$ (almost everywhere) as $j \to \infty$ for $s = 1, \dots, p$. Then,*

$$\lim_{j\to+\infty} v^j \mathscr{L}(u_1^j, \dots, u_p^j) = v\mathscr{L}(u_1, \dots, u_p) \tag{5-3}$$

in the sense of currents of order zero.

Corollary 5.5. *Let Ω be a bounded open set. Let $\mathcal{U}_m$ be a uniformly bounded family of m-ω-sh functions in Ω. Define $v(x) = \sup\{v_\alpha(x) : v_\alpha \in \mathcal{U}_m\}$. Then, the set*

$$N := \{v < v^*\}$$

has zero measure with respect to any measure $\mathscr{L}(u_1, \dots, u_m) = dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge \omega^{n-m}$, where the u_i are bounded m-ω-sh functions. In particular, $\mathrm{cap}_m(N, \Omega) = 0$.

Proof. By Choquet's lemma we can reduce the argument to the case when $\mathcal{U}_m$ is an increasing sequence $\{v_j\}_{j\ge1}$ with $w = \sup_j v_j$ and $N = \{w < v^*\}$. It follows from the proof of [Gu and Nguyen 2018,

Corollary 9.9] that $w = v^*$ almost everywhere. Therefore, Lemma 5.4 implies $v_j \mathscr{L}(u_1, \dots, u_m)$ converges weakly to $v^* \mathscr{L}(u_1, \dots, u_m)$. Then, the positive currents $(v^* - v_j)\mathscr{L}(u_1, \dots, u_m)$ converge weakly to zero and hence, for any compact set $K \subset \Omega$,

$$\lim_{j\to\infty} \int_K (v^* - v_j)\mathscr{L}(u_1, \dots, u_m) = 0.$$

By the monotone convergence theorem, $\lim_{j\to\infty} \int_K (w - v_j)\mathscr{L}(u_1, \dots, u_m) = 0$. Therefore,

$$\int_K (v^* - w)\mathscr{L}(u_1, \dots, u_m) = 0.$$

In other words, $v^* = w$ a.e. on K with respect to $\mathscr{L}(u_1, \dots, u_m)$. The last conclusion follows from the inner regularity of capacity. □

6. Comparison principle

Let Ω be a bounded open set which is relatively compact in a strictly m-pseudoconvex bounded domain D in $\mathbb{C}^n$. Fix a constant $\boldsymbol{B}$ such that on $\overline{\Omega}$,

$$-\boldsymbol{B}\omega^2 \le dd^c\omega \le \boldsymbol{B}\omega^2, \quad -\boldsymbol{B}\omega^3 \le d\omega \wedge d^c\omega \le \boldsymbol{B}\omega^3.$$

Let ρ be a strictly psh function satisfying $\rho \le 0$ and $dd^c\rho \ge \omega$ in D. In this section we assume all functions are defined in D which means that they can be approximated by a decreasing sequence of smooth m-ω-sh functions in a neighborhood of $\overline{\Omega}$.

Theorem 6.1. *Let u, v be bounded m-ω-sh functions in Ω such that $d = \sup_\Omega(v - u) > 0$, and $\liminf_{z\to\partial\Omega}(u - v)(z) \ge 0$. Fix $0 < \varepsilon < \min\{\frac{1}{2}, d/(2\|\rho\|_\infty)\}$. Let us denote for $0 < s < \varepsilon_0 := \varepsilon^3/16\boldsymbol{B}$,*

$$U(\varepsilon, s) := \{u < (v + \varepsilon\rho) + S(\varepsilon) + s\}, \quad \textit{where } S(\varepsilon) = \inf_\Omega[u - (v + \varepsilon\rho)].$$

Then,

$$\int_{U(\varepsilon,s)} H_m(v + \varepsilon\rho) \le \left(1 + \frac{Cs}{\varepsilon^m}\right) \int_{U(\varepsilon,s)} H_m(u),$$

where C is a uniform constant depending on m, n, ω.

Proof. If u, v are smooth, then the proof follows from [Gu and Nguyen 2018, Lemmas 3.8, 3.9 and 3.10]. To pass from the smooth case to the bounded case we use the quasicontinuity of m-ω-sh functions and the argument as the one in [Bedford and Taylor 1982, Theorem 4.1] (see also [Kołodziej 2005, Theorem 1.16]). The proof is readily adaptable with obvious changes of notation. Here we only indicate the points of difference that we need to take care of. Firstly, replacing u by $u + \delta$ with $\delta > 0$ and then letting $\delta \downarrow 0$ we may assume that $\{u < v\} \Subset \Omega' \Subset \Omega$ and $u \ge v + \delta$ on $\Omega \setminus \Omega'$. By restricting u, v to a smaller domain we may assume that u, v are defined in a neighborhood of $\overline{\Omega}$.

Let $\{u_k\}_{k\ge1}$, $\{v_j\}_{j\ge1}$ be sequences of smooth m-ω-sh functions in a neighborhood of $\overline{\Omega}$ (Proposition 2.9) such that $u_k \downarrow u$ and $v_j \downarrow v$ pointwise in $\overline{\Omega}$. Define $d_{jk} = \sup_{\overline{\Omega}}(v_j - u_k)$. Then, for $j \ge k > 0$ large we have

$$d_{jk} \ge \tfrac{1}{2}d > 0.$$

In fact, for small $\epsilon > 0$ there exists $x \in \Omega$ such that $d - \epsilon \le v(x) - u(x)$. So, for $k > k_0$ large enough,

$$d - 2\epsilon \le v(x) - u_k(x) \le v_j - u_k \le d_{jk}.$$

We get the desired inequality by letting $j \to \infty$ and then $\epsilon \to 0$. Next, since $u \ge v + \delta$ on a compact set $K = \overline{\Omega} \setminus \Omega'$, we have $u_k \ge v + \delta$ for every $k \ge 1$. Since u_k is continuous, by Hartogs' lemma for ω-sh functions [Gu and Nguyen 2018, Lemma 9.14], there is $j_k \ge k > 0$ large enough such that for $j \ge j_k$,

$$v_j + \delta \le u_k \quad \text{on } K.$$

Thus, there exist subsequences of $\{u_k\}$ and $\{v_j\}$, which can be used in the argument from [Bedford and Taylor 1982, Theorem 4.1]. □

Corollary 6.2. *Let u, v be bounded m-ω-sh functions in a neighborhood of $\overline{\Omega}$ such that*

$$\liminf_{z \to \partial\Omega} (u - v)(z) \ge 0.$$

Assume that $H_m(v) \ge H_m(u)$ in Ω. Then, $u \ge v$ on Ω.

Proof. Arguing by contradiction, suppose that $\sup_\Omega (v - u) = d > 0$. Hence, there exist $\delta, a > 0$ so small that $\sup_\Omega [(1+a)v - (u+\delta)] > d/2$ and $\liminf_{z \to \partial\Omega} [(u+\delta) - (1+a)v](z) \ge 0$. Applying Theorem 6.1 for $\widetilde{u} = u + \delta$ and $\widetilde{v} = (1+a)v$, we have for $0 < s < \varepsilon_0$,

$$\int_{U(\varepsilon,s)} H_m(\widetilde{v} + \varepsilon\rho) \le \left(1 + \frac{Cs}{\varepsilon^m}\right) \int_{U(\varepsilon,s)} H_m(u).$$

Observe that

$$H_m(\widetilde{v} + \varepsilon\rho) \ge (1+a)^m H_m(v) + \varepsilon^m H_m(\rho) \ge (1+a)^m H_m(u) + \varepsilon^m H_m(\rho).$$

Hence, we derive from the above inequality that

$$\varepsilon^m \int_{U(\varepsilon,s)} H_m(\rho) \le 0$$

for $s > 0$ so small that $(1+a)^m \ge 1 + Cs/\varepsilon^m$. Therefore, the Lebesgue measure of $U(\varepsilon, s)$ is zero. This is impossible as it is a nonempty quasiopen set for $0 < s < \varepsilon_0$. □

The above argument also gives:

Corollary 6.3 (domination principle). *Let u, v be bounded m-ω-sh such that* $\limsup_{z \to \partial\Omega} |u(z) - v(z)| = 0$ *and $\int_{\{u<v\}} H_m(u) = 0$. Then, $u \ge v$ in Ω.*

7. Polar sets and negligible sets

In this section we study the polar sets and negligible sets of m-ω-sh functions. We obtain here results analogous to those in pluripotential theory from [Bedford and Taylor 1982]. Let us first give the definitions.

Definition 7.1 (m-polar sets). A set E in $\mathbb{C}^n$ is m-polar if for each $z \in E$ there is an open set $z \in U$ and a m-ω-sh function u in U such that $E \cap U \subset \{u = -\infty\}$.

Let $\{u_\alpha\}$ be a family of m-ω-sh functions in Ω which is locally bounded from above. Then, the function

$$u(z) = \sup_\alpha u_\alpha(z)$$

need not be m-ω-sh, but its upper semicontinuous regularization

$$u^*(z) = \limsup_{x\to z} u(x) \geq u(z)$$

is m-ω-sh (see [Gu and Nguyen 2018, Proposition 2.6-(c)]). A set of the form

$$N = \{z \in \Omega : u(z) < u^*(z)\} \tag{7-1}$$

is called *m-negligible*.

An n-polar/n-negligible set is pluripolar/negligible. Clearly a pluripolar set is a m-polar (or m-negligible sets) for every $1 \leq m \leq n$. More generally, a m-polar (resp. m-negligible) is a $(m-1)$-polar (resp. $(m-1)$-negligible) set. An effective way to study these sets is by extremal functions.

Definition 7.2. Let E be a subset of a bounded open set $\Omega \subset \mathbb{C}^n$. We define

$$u_E = u_{E,\Omega} = \sup\{v(x) : v \text{ is } m\text{-}\omega\text{-sh in } \Omega,\ u \leq 0,\ u \leq -1 \text{ on } E\}$$

By Choquet's lemma u_E is the limit of an increasing sequence of m-ω-sh functions. It follows from [Gu and Nguyen 2018, Corollary 9.9] that u_E^* is m-ω-sh and $u_E = u_E^*$ almost everywhere. Moreover, $u_E^* \equiv 0$ if and only if there exists an increasing sequence of m-ω-sh functions $\{v_j\}_{j\geq 1}$ satisfying

$$v_j \leq 0, \quad v_j \leq -1 \text{ on } E, \quad \int_\Omega |v_j|\, dV_{2n} \leq 2^{-j}. \tag{7-2}$$

Lemma 7.3. *Let Ω be a bounded open set in $\mathbb{C}^n$.*

(i) *If $E_1 \subset E_2$, then $u_{E_2} \leq u_{E_2}$.*

(ii) *If $E \subset \Omega_1 \subset \Omega_2$, then $u_{E,\Omega_2} \leq u_{E,\Omega_1}$.*

(iii) *Let K_j be nonincreasing sequence of compact subset in Ω and $K = \bigcap_j K_j$. Then, $u_{K_j}^*$ increases almost everywhere to u_K^*.*

(iv) *If $u_{E_j}^* \equiv 0$ and $E = \bigcup_{j=1}^\infty E_j$, then $u_E^* \equiv 0$.*

Suppose moreover that Ω is strictly m-pseudoconvex.

(v) *If $E \Subset \Omega$, then $\lim_{z\to\partial\Omega} u_E^* = 0$.*

(vi) *For every set $E \subset \Omega$, $H_m(u_E^*) \equiv 0$ on $\Omega \setminus \overline{E}$.*

Proof. The properties (i) and (ii) are obvious from the definition, and also $\lim_j u_{K_j} \leq u_K^*$ in (iii). To prove the reverse inequality let v be a m-ω-sh with $v \leq 0$ and $u \leq -1$ on K. For $\varepsilon > 0$, the open set $U_\varepsilon = \{u < -1+\varepsilon\}$ contains K. Hence, $K_j \subset U_\varepsilon$ for j large enough. So, $v - \varepsilon \leq u_{K_j}^*$. Taking the supremum over all such functions v we get $u_K - \varepsilon \leq u := \lim_j u_{K_j}$. Letting $\varepsilon \to 0$ we obtain the conclusion. The statement that $u = u^*$ almost everywhere follows from [Gu and Nguyen 2018, Corollary 9.9].

(iv) Let $\varepsilon > 0$. By (7-2) we can choose a sequence $v_j \le 0$, $v_j \le -1$ on E_j and $\int_\Omega |v_j|\, dV_{2n} \le \varepsilon 2^{-j}$. Then, $v = \sum_j v_j$ is a m-ω-sh function satisfying $v \le 0$, $v \le -1$ on E and $\int_\Omega |v|\, dV_{2n} \le \varepsilon$. Hence, $u_E^* \equiv 0$.

(v) Let ψ be a strictly m-ω-sh defining function of Ω. Then, for $A > 1$ large enough, $A\psi \le u_E^*$. This finishes the proof.

(vi) Given the unique continuous solution of the Dirichlet problem for the homogeneous Hessian equation [Gu and Nguyen 2018, Theorem 3.15] in small balls, the result follows from a classical balayage argument. □

The outer capacity $\mathrm{cap}_m^*(\bullet)$ is defined as

$$\mathrm{cap}_m^*(E) = \inf\{\mathrm{cap}_m(U) : E \subset U,\ U \subset \Omega \text{ is open}\}. \tag{7-3}$$

Then, we have basic properties which follow easily from the corresponding ones of the capacity cap_m.

Proposition 7.4. *Let Ω be a bounded open set in $\mathbb{C}^n$. Then,*

(i) $\mathrm{cap}_m^*(E_1) \le \mathrm{cap}_m^*(E_2)$ *if* $E_1 \subset E_2 \subset \Omega$;

(ii) $\mathrm{cap}_m^*(E_1, \Omega_1) \ge \mathrm{cap}_m^*(E, \Omega_2)$ *if* $E \subset \Omega_1 \subset \Omega_2$;

(iii) $\mathrm{cap}_m^*\big(\bigcup_j E_j\big) \le \sum_j \mathrm{cap}_m^*(E_j)$.

Lemma 7.5. *Let $\Omega \Subset \mathbb{C}^n$ be a strictly m-pseudoconvex domain. Let $E \Subset \Omega$ a Borel subset. Then*

$$\int_\Omega H_m(u_E^*) \le \mathrm{cap}_m^*(E) \le C \sum_{s=0}^{m} \int_\Omega (-u_E^*) H_s(u_E^*).$$

Proof. We prove first the left-hand side inequality. Assume that $E = \overline{E}$ is compact. Lemma 7.3(vi) implies

$$\int_\Omega H_m(u_K^*) = \int_K H_m(u_K^*) \le \mathrm{cap}_m(K) \le \mathrm{cap}_m^*(K).$$

Assume $E = G$ is an open subset. We can find an increasing sequence of compact sets K_j such that $\bigcup_j K_j = G$. It is easy to see that $u_{K_j}^*$ decreases to $u_G = u_G^*$ on Ω. Hence, by the weak convergence theorem for decreasing sequences, $H_m(u_{K_j}^*) \to H_m(u_G)$ weakly. This implies

$$\int_\Omega H_m(u_G) = \lim_{j\to\infty} \int_\Omega H_m(u_{K_j}^*) \le \lim_{j\to\infty} \mathrm{cap}_m(K_j) \le \mathrm{cap}_m(G).$$

Since $\mathrm{cap}_m(G) = \mathrm{cap}_m^*(G)$, the conclusion follows.

Now let E be a Borel subset. By definition there exists a sequence of open sets $\{O_j\}$ in Ω containing E such that $\mathrm{cap}_m^*(E) = \lim_j \mathrm{cap}_m(O_j)$. Replacing O_j by $\bigcap_{1\le s\le j} O_j$ we may assume that $\{O_j\}_{j\ge 1}$ is decreasing. Moreover, by Choquet's lemma there exists an increasing sequence $\{v_j\}$ of negative m-ω-sh functions in Ω such that $v_j = -1$ on E and $\lim_j v_j = u_E$ almost everywhere on Ω. Set $G_j = O_j \cap \{v_j < -1 + 1/j\}$. Then, $E \subset G_j \subset O_j$ and

$$v_j - \frac{1}{j} \le u_{G_j} \le u_E.$$

So, $\lim_{j\to\infty} \mathrm{cap}_m(G_j) = \mathrm{cap}_m^*(E)$ and u_{G_j} increases to u_E almost everywhere on Ω. Therefore, by the weak convergence for increasing sequences (Lemma 5.4),

$$\int_\Omega H_m(u_E^*) = \lim_{j\to\infty} \int_\Omega H_m(u_{G_j}) \le \lim_{j\to\infty} \mathrm{cap}_m(G_j) = \mathrm{cap}_m^*(E).$$

Thus, the proof of left-hand side inequality is completed.

Next we prove the other one. Let $E \Subset \Omega$ be a Borel subset and consider the sets G_j defined as above. Then, $\lim_j \mathrm{cap}_m(G_j) = \mathrm{cap}_m^*(E)$. We also have for $0 \le s \le m$,

$$\lim_{j\to\infty} \int_\Omega (-u_{G_j}) H_s(u_{G_j}) = \int_\Omega (-u_E^*) H_s(u_E^*)$$

by the weak convergence for increasing sequence again. Thus, it is enough to prove the inequality for $E = G \Subset \Omega$ an open subset.

To this end let $-1 \le \rho \le 0$ be a m-ω-sh function in Ω. Since $G \subset \{u_G = -1\}$ and $u_G = u_G^*$ it follows that for $q \ge 1$,

$$\int_G H_m(\rho) \le \int (-u_G)^q H_m(\rho).$$

Applying Corollary 5.3 for $v = 0$ and $u = u_G$ we get

$$\int_\Omega (-u_G)^{3m} H_m(\rho) \le C \sum_{s=0}^{m} \int_\Omega (-u_G) H_s(u_G).$$

Taking the supremum over all such functions ρ, we get the desired inequality. □

Remark 7.6. For a compact set K in a strictly m-pseudoconvex domain Ω, $\mathrm{cap}(K, \Omega) = 0$ if and only if $\mathrm{cap}_m^*(K, \Omega) = 0$.

Proposition 7.7. *In a strictly m-pseudoconvex domain Ω the following are equivalent*:

(a) $u_{E,\Omega}^* = 0$.

(b) $E \subset \{u = -\infty\}$ *for an m-ω-sh function $u < 0$ in Ω.*

(c) $\mathrm{cap}_m^*(E, \Omega) = 0$.

Proof. The implication (a) ⇒ (b) follows from (7-2) by setting $u = \sum_{j\ge1} v_j$. Conversely, $E \subset \{v = -\infty\}$, where $v < 0$ and m-ω-sh, implies $u_E \ge v/j$ for $j = 1, 2 \ldots$. So $u_E = 0$ outside $\{v = -\infty\}$ whose Lebesgue measure is zero. Hence, $u_E^* = 0$ by [Gu and Nguyen 2018, Corollary 9.7]. The implication (c) ⇒ (a) follows from the fact that $H_m(u_E^*) \equiv 0$ and the domination principle (Corollary 6.2) if E is relatively compact in Ω. The general case follows from the countable subadditivity of cap_m^* and the corresponding property of u_E^* above.

To prove (b) ⇒ (c) let us fix an open subset $V \Subset \Omega$ and define $\mathcal{O}_j = \{u < -j\} \cap V$. Let $\varepsilon > 0$. We wish to find an open subset $E \subset G \subset \Omega$ with $\mathrm{cap}_m(G) < \varepsilon$. Indeed, we have $0 \ge u_{\mathcal{O}_j} \ge \max\{u/j, -1\}$, where $u_{\mathcal{O}_j}$ is the relative extremal function. Then, $u_{\mathcal{O}_j} \uparrow 0$ a.e. on Ω by using ω-subharmonicity. Now the

right-hand side inequality in Lemma 7.5 gives

$$\operatorname{cap}_m(\mathcal{O}_j) \le C \sum_{s=0}^{m} e_{(0,0,s)},$$

where $e_{(0,0,s)} = \int_\Omega (-u_{\mathcal{O}_j}) H_s(u_{\mathcal{O}_j})$. Applying the weak convergence theorem for increasing convergence sequences we get that $H_s(u_{\mathcal{O}_j}) \to 0$ weakly in Ω, $1 \le s \le m$. Furthermore, for $s = m$ and $s = 0$,

$$\lim_{j\to\infty} \int_\Omega (-u_{\mathcal{O}_j}) H_m(u_{\mathcal{O}_j}) = 0 = \lim_{j\to\infty} \int_\Omega (-u_{\mathcal{O}_j}) \omega^n.$$

Now we claim that for $1 \le s \le m-1$,

$$\lim_{j\to\infty} \int_\Omega (-u_{\mathcal{O}_j}) H_s(u_{\mathcal{O}_j}) = 0. \tag{7-4}$$

Assume this is true for a moment and let us finish the proof. The above facts imply that

$$\lim_{j\to\infty} \operatorname{cap}_m(\mathcal{O}_j) = 0. \tag{7-5}$$

Take a sequence of open sets V_s exhausting Ω. Choose $\mathcal{O}_{j_s} = \{u < -j_s\} \cap V_s$ such that $\operatorname{cap}_m(\mathcal{O}_{j_s}) < \varepsilon/2^s$. Define $G = \bigcup_{s\ge 1} \mathcal{O}_{j_s}$ which is an open set containing E and which has capacity less than ε.

Finally, let us verify (7-4). Let ρ be strictly m-ω-sh defining function for Ω. Since $\mathcal{O}_j \subset V \Subset \Omega$, we have $u_{\mathcal{O}_j} \ge u_V \ge A\rho$ for a constant $A > 0$ depending only on V, Ω by the proof of Lemma 7.3(v). Hence, the $u_{\mathcal{O}_j}$ can be extended to a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$ by $A\rho$ (see, e.g., (8-9)). By the CLN inequality there is a uniform constant $C = C(\Omega, \widetilde{\Omega})$ such that for every $j \ge 1$,

$$\int_\Omega H_s(u_{\mathcal{O}_j}) \le C.$$

Then, for a fixed $\varepsilon > 0$,

$$\begin{aligned}
\int_\Omega (-u_{\mathcal{O}_j}) H_s(u_{\mathcal{O}_j}) &\le A \int_\Omega |\rho| H_s(u_{\mathcal{O}_j}) \\
&\le A\varepsilon \int_{\{|\rho|<\varepsilon\}} H_s(u_{\mathcal{O}_j}) + A \int_{\{|\rho|\ge\varepsilon\}} H_s(u_{\mathcal{O}_j}) \\
&\le AC\varepsilon + A \int_{\{|\rho|\ge\varepsilon\}} H_s(u_{\mathcal{O}_j}).
\end{aligned}$$

Since $H_s(u_{\mathcal{O}_j}) \to 0$ weakly as $j \to \infty$ and $\{|\rho| \ge \varepsilon\} \subset \Omega$ is compact, letting $j \to \infty$ we get

$$\lim_{j\to\infty} \int_\Omega (-u_{\mathcal{O}_j}) H_s(u_{\mathcal{O}_j}) \le AC\varepsilon.$$

This holds for arbitrary $\varepsilon > 0$, where A, C are uniform constants independent of ε. □

Theorem 7.8. *m-negligible sets are m-polar.*

Proof. The result is local, so we may assume that all functions are defined on a bounded strictly m-pseudoconvex domain Ω. Thanks to the characterization in Proposition 7.7 it is enough to show that a

negligible set E has outer capacity zero. Let $\{u_j\}$ be the sequence in the definition of the negligible set and put $u = \sup_j u_j$. By Choquet's lemma we may assume this is an increasing sequence. Let $\varepsilon > 0$. By quasi-continuity we can find an open set $G \subset \Omega$ such that $\mathrm{cap}_m(G) < \varepsilon$ and u, the u_j are continuous on $F := \Omega \setminus G$.

Since $\mathrm{cap}_m^*(\bullet)$ is countably subadditive, it is enough to show that

$$\mathrm{cap}_m^*(E \cap K) = 0$$

for a fixed compact subset $K \subset \Omega$. Observe that for all rational numbers $r < t$, the sets

$$K_{rt} = K \cap F \cap \{u \le r < t \le u^*\}$$

are compact, because u is lower semicontinuous in $K \cap F$ and u^* is upper-semicontinuous. Also, $(K \cap E) \setminus G$ is contained in the countable union of such compact sets. Thus, by countable subadditivity, it remains to verify $\mathrm{cap}_m^*(K_{rt}) = 0$.

Since u is lower semicontinuous on K, there exists a constant c such that $u \ge c$ on K. Define $u_c = \max\{u, c\}$ and notice that $K_{rt} \subset K'_{rt}$, where

$$K'_{rt} = K \cap F \cap \{u_c \le r < t \le u_c^*\}$$

are compact sets. Since $\{u_c < u_c^*\}$ has inner capacity cap_m zero, we have $\mathrm{cap}_m(K'_{rt}) = 0$. This implies that $\mathrm{cap}_m^*(K'_{rt}) = 0$ by Remark 7.6. □

We have the following analogue of Josefson's theorem whose proof is the same as the one of [Kołodziej 2005, Theorem 1.23].

Theorem 7.9. *For any m-polar subset E of $\mathbb{C}^n$, there exists a m-ω-sh function h on $\mathbb{C}^n$ such that $E \subset \{h = -\infty\}$.*

8. Dirichlet problem in domains in $\mathbb{C}^n$

Let Ω be a bounded strictly m-pseudoconvex domain in $\mathbb{C}^n$. The comparison principle in Corollary 6.2 coupled with the proof of [Gu and Nguyen 2018, Lemma 3.13] gives the following stability estimate for the complex Hessian equation:

Proposition 8.1. *Let $u, v \in C^0(\overline{\Omega})$ be m-ω-sh in Ω and satisfy*

$$H_m(u) = f dV_{2n}, \quad H_m(v) = g dV_{2n}$$

with $0 \le f, g \in L^p(\Omega)$ and $p > n/m$. Then

$$\|u - v\|_{L^\infty} \le \sup_{\partial\Omega} |u - v| + C \|f - g\|_{L^p(\Omega)}^{\frac{1}{m}},$$

where $C = C(m, n, p, \Omega)$.

Let $\psi \in C^\infty(\partial\Omega)$. Given a smooth positive function $f \in C^\infty(\overline{\Omega}, \mathbb{R})$, there is always a smooth m-ω-sh subsolution $\underline{u} \in C^\infty(\overline{\Omega})$, that is

$$H_m(\underline{u}) \ge f(z), \qquad \underline{u} = \psi \quad \text{on } \partial\Omega. \tag{8-1}$$

The stability estimate and an easy approximation argument imply that we can solve the Dirichlet problem when the right-hand side is in L^p, $p > n/m$, after invoking the solution for the smooth data due to Collins and Picard [2022].

Theorem 8.2. *Let $0 \le f \in L^p(\Omega)$ for some $p > n/m$. Suppose $\varphi \in C^0(\partial\Omega)$. Then there exists a unique continuous m-ω-sh function $u \in C^0(\overline{\Omega})$ solving the Dirichlet problem*

$$H_m(u) = f\omega^n, \qquad u = \varphi \quad \text{on } \partial\Omega.$$

Now we wish to solve the equation with the right-hand side just being a positive Radon measure assuming the existence of a subsolution. We use ideas from [Kołodziej and Nguyen 2023a], however several steps require very different proofs.

Assume $\widetilde{\Omega}$ is a neighborhood of $\overline{\Omega}$. Let us define a slightly modified Cegrell class

$$\widetilde{\mathcal{E}}_0(\Omega) = \big\{u \text{ is bounded and } m\text{-}\omega\text{-sh in } \widetilde{\Omega} : \lim_{z\to\partial\Omega} u(z) = 0\big\}. \tag{8-2}$$

The set $\widetilde{\Omega}$ is suppressed in this notation. By the CLN inequality for $u \in \widetilde{\mathcal{E}}_0(\Omega)$ we have $\int_\Omega H_m(u) < +\infty$. We introduce this modified class to control the integrals of the wedge products of currents associated to bounded m-ω-sh functions.

Now we follow the steps in [Kołodziej and Nguyen 2023a]. The first one corresponds to [Kołodziej and Nguyen 2023a, Lemma 2.1] which in turn was inspired by [Cegrell 1998, Lemma 5.2].

Lemma 8.3. *Let λ be a finite positive Radon measure on Ω which vanishes on m-polar sets. Let $\{u_j\}_{j\ge1} \subset \widetilde{\mathcal{E}}_0(\Omega)$ be a uniformly bounded in $\widetilde{\Omega}$ sequence that converges dV-a.e. to $u \in \widetilde{\mathcal{E}}_0(\Omega)$. Then, there exists a subsequence u_{j_s} such that*

$$\lim_{j_s\to\infty} \int_\Omega u_{j_s}\, d\lambda = \int_\Omega u\, d\lambda.$$

Proof. By the comparison principle, all functions are negative on Ω. The proof of [Kołodziej and Nguyen 2023a, Lemma 2.1] is applicable provided that the m-negligible sets are m-polar and this is the content of Theorem 7.8. □

Applying the lemma twice for the sequences $\{u_j\}$, $\max\{u_j, u\}$ and combining with the identity $2\max\{u_j, u\} = u_j + u + |u_j - u|$ we easily get a corollary.

Corollary 8.4. *Let λ and $\{u_j\}_{j\ge1}$ be as in Lemma 8.3. Then, there is a subsequence, still denoted by $\{u_j\}$, such that $\lim_{j\to\infty} \int_\Omega |u_j - u|\, d\lambda = 0$.*

The following result is crucial for proving the weak convergence later.

Lemma 8.5. *Let $d\lambda$ and $\{u_j\}_{j\ge1}$ be as in Lemma 8.3. Let $\{w_j\}_{j\ge1} \subset \widetilde{\mathcal{E}}_0(\Omega)$ be uniformly bounded in $\widetilde{\Omega}$. Suppose that w_j converges in capacity — $\mathrm{cap}_m(\bullet, \Omega)$ — to $w \in \widetilde{\mathcal{E}}_0(\Omega)$. Then,*

$$\lim_{j\to\infty} \int_\Omega |u - u_j| H_m(w_j) = 0.$$

Remark 8.6. Since all functions are uniformly bounded on the fixed neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$, with $\|u_j\|_{L^\infty}, \|w_j\|_{L^\infty} \leq A$, by Proposition 3.7 there exist two positive constants C_1, C_2 depending only on sup-norm of the u_j and the w_j (and the domains $\Omega, \widetilde{\Omega}$), such that

$$\sup_j \int_\Omega H_m(u_j) \leq C_1, \quad \sup_j \int_\Omega H_m(w_j) \leq C_2.$$

Proof. Note that $|u-u_j| = (\max\{u,u_j\}-u_j)+(\max\{u,u_j\}-u)$. Observe first that by the Hartogs lemma and quasicontinuity of u ([Gu and Nguyen 2018, Lemma 9.14] and Theorem 4.9) $\phi_j := \max\{u,u_j\} \to u$ in capacity. Fix $\varepsilon > 0$. We have for j large,

$$\begin{aligned}\int_\Omega (\max\{u,u_j\}-u)H_m(w_j) &\leq \int_{\{|\phi_j-u|>\varepsilon\}} H_m(w_j) + \varepsilon \int_\Omega H_m(w_j) \\ &\leq A^m \operatorname{cap}_m(|\phi_j-u|>\varepsilon) + C_2\varepsilon.\end{aligned}$$

Therefore, $\lim_{j\to\infty} \int (\phi_j - u)H_m(w_j) = 0$. Here and in what follows we drop the domain Ω in the integrals if no confusion arises.

Next, we consider for $j > k$,

$$\int (\phi_j - u_j)H_m(w_j) - \int (\phi_j - u_j)H_m(w_k) = \int (\phi_j - u_j) dd^c(w_j - w_k) \wedge T \wedge \omega^{n-m},$$

where $T = T(j,k) = \sum_{s=1}^{n-1} (dd^c w_j)^s \wedge (dd^c w_k)^{m-1-s}$. Let us write $h_j = \phi_j - u_j$. Now the proof gets more complicated than the one in [Kołodziej and Nguyen 2023a, Lemma 2.3] as the integration by parts produces more terms involving the torsion of ω.

By the integration by parts

$$\int h_j dd^c(w_j - w_k) \wedge T \wedge \omega^{n-m} = \int (w_j - w_k) dd^c(h_j \omega^{n-m}) \wedge T. \tag{8-3}$$

This integration by parts formula is justified by an approximation argument as follows. All functions are continuous on the boundary $\partial\Omega$ with zero value there, so the approximating sequences of smooth functions converge to zero uniformly on $\partial\Omega$. Moreover, the functions are defined and uniformly bounded on a neighborhood $\widetilde{\Omega}$ so the total masses of wedge products are uniformly bounded by the CLN inequality. Hence, the boundary terms vanish after passing to the limit (see Remark 2.16 and also [Guedj and Zeriahi 2017, Proposition 3.7]).

By a direct calculation,

$$dd^c(h_j\omega^{n-m}) = dd^c h_j \wedge \omega^{n-m} + h_j dd^c \omega^{n-m} + (n-m)[dh_j \wedge d^c\omega + d\omega \wedge d^c h_j] \wedge \omega^{n-m-1}. \tag{8-4}$$

For the first term we obtain a bound

$$\int (w_j - w_k) dd^c h_j \wedge \omega^{n-m} \wedge T \leq \int |w_j - w_k| dd^c(\phi_j + u_j) \wedge T \wedge \omega^{n-m}, \tag{8-5}$$

and using inequality [Kołodziej and Nguyen 2016, Lemma 2.3] for the second term

$$\int (w_j - w_k) h_j dd^c \omega^{n-m} \wedge T \leq C \int |w_j - w_k| h_j [dd^c(w_j + w_k)]^{m-1} \wedge \omega^{n-m+1}. \tag{8-6}$$

Next, since two terms in the bracket of (8-4) are mutually conjugate, we only estimate the first one. To this end we will use the Cauchy–Schwarz inequality (Corollary 2.5):

$$\left|\int (w_j - w_k) dh_j \wedge d^c\omega \wedge \omega^{n-m-1} \wedge T\right|^2$$
$$\leq C \int |w_j - w_k| dh_j \wedge d^c h_j \wedge [dd^c(w_j + w_k)]^{m-1} \wedge \omega^{n-m} \times \int |w_j - w_k| [dd^c(w_j + w_k)]^{m-1} \wedge \omega^{n-m+1}.$$

Let us consider the second factor in the product. Since $\|w_j\|_{L^\infty}, \|u_j\|_{L^\infty} \leq A$ in Ω, it follows that

$$\begin{aligned}\int_\Omega |w_j - w_k| [dd^c(\phi_j + u_j)]^{m-1} \wedge \omega^{n-m+1}& \\ \leq A \int_{\{|w_j - w_k| > \varepsilon\}} [dd^c(\phi_j + u_j)]^{m-1} \wedge \omega^{n-m+1} + \varepsilon \int_{\{|w_j - w_k| \leq \varepsilon\}} [dd^c(\phi_j + u_j)]^{m-1} \wedge \omega^{n-m+1}& \\ \leq (2A)^m \operatorname{cap}_m(|w_j - w_k| > \varepsilon) + C\varepsilon,& \qquad (8\text{-}7)\end{aligned}$$

where the uniform bound for the integral in the second line follows from the CLN inequality as all functions are uniformly bounded in $\widetilde{\Omega}$. It means that the left-hand side of (8-7) is less than $2C\varepsilon$ for some k_0 and every $j > k \geq k_0$.

As for the first factor in the product we observe that

$$dh_j \wedge d^c h_j \leq 2du_j \wedge d^c u_j + 2d\phi_j \wedge d^c \phi_j,$$

and $2d\phi_j \wedge d^c\phi_j = dd^c\phi_j^2 - 2\phi_j dd^c\phi_j$ and similarly for u_j. Therefore, we can apply the estimate as in (8-7) for this integral. The same is true for the integrals on the right-hand sides of (8-5) and (8-6).

Thus,

$$\begin{aligned}\int (\phi_j - u_j) H_m(w_j) &\leq \int (\phi_j - u_j) H_m(w_k) + \left|\int (\phi_j - u_j) H_m(w_j) - \int (\phi_j - u_j) H_m(w_k)\right| \\ &\leq \int (\phi_j - u_j) H_m(w_k) + 8C\varepsilon \\ &\leq \int |u - u_j| H_m(w_k) + 8C\varepsilon.\end{aligned}$$

Fix $k = k_0$ and apply Corollary 8.4 for $d\lambda = H_m(w_{k_0})$ to get that, for $j \geq k_1 \geq k_0$,

$$\int (\phi_j - u_j) H_m(w_j) \leq (8C + 1)\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the proof of the lemma is completed. □

Let μ be a positive Radon measure on a bounded strictly m-pseudoconvex domain Ω. Assume that there exists a bounded m-ω-sh function $\underline{u}$ in Ω such that

$$H_m(\underline{u}) \geq \mu, \qquad \lim_{x \to z \in \partial\Omega} \underline{u}(x) = 0. \qquad (8\text{-}8)$$

This function $\underline{u}$ is called a subsolution for $d\mu$. Our goal is to prove the following.

Theorem 8.7. *Let $\varphi \in C^0(\partial\Omega)$ and let μ be a positive Radon measure in Ω. Assume that μ admits a bounded subsolution $\underline{u}$ as in (8-8). Then, there exists a unique bounded m-ω-sh function u solving $\lim_{z\to x} u(z) = \varphi(x)$ for $x \in \partial\Omega$,*

$$H_m(u) = \mu \quad \text{in } \Omega.$$

We first make some reduction steps. In fact it is enough to prove the statement under the following additional assumptions on the measure $d\mu$, the boundary data φ and the subsolution $\underline{u}$.

Lemma 8.8. *We may assume additionally that*

(a) *φ is smooth on $\partial\Omega$;*

(b) *μ has compact support in Ω;*

(c) *the support of $\nu = (dd^c\underline{u})^m \wedge \omega^{n-m}$ is compact in Ω;*

(d) *$\underline{u}$ can be extended as a m-ω-sh function to a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$.*

Proof. Step 1: (c) $\Rightarrow$ (d). Define $K := \operatorname{supp}\nu$. Let ρ be a strictly m-ω-sh defining function for Ω. In particular, ρ is defined in a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$. For $A > 0$ (to be chosen), we define

$$v = \begin{cases} \max\{\underline{u}, A\rho\} & \text{in } \Omega, \\ A\rho & \text{on } \widetilde{\Omega} \setminus \Omega. \end{cases} \tag{8-9}$$

Notice that $\lim_{x\to\partial\Omega}(\underline{u} - A\rho) = 0$, so the function v is well defined. We claim that for a sufficiently large A we have $\underline{u} \geq A\rho$ on Ω, and then v is a required extension. Indeed, let $U \Subset \Omega$ be a neighborhood of K. Since $\sup_{\overline{U}} \rho \leq -\delta$ for some $\delta > 0$ and $\underline{u}$ is bounded, we can choose $A > 0$ large enough so that $v \geq A\rho$ on $\overline{U}$. Furthermore, $(dd^c v)^m \wedge \omega^{n-m} \equiv 0$ on $\Omega \setminus \overline{U}$ and $v \geq A\rho$ on the boundary $\partial(\Omega \setminus \overline{U})$. The domination principle implies that $v \geq A\rho$ on $\Omega \setminus \overline{U}$.

Remark 8.9. The constant $A > 0$ chosen in this argument depends only on the defining function ρ, the support of ν and the sup-norm of $\underline{u}$.

Step 2: (b) $\Rightarrow$ (c). This follows from the classical balayage argument. However, several ingredients are only available recently. We give a detailed argument. Assume that $\operatorname{supp}\mu \subset U$ which is an open subset relatively compact in Ω. We define an envelope

$$v = \sup\{w : w \text{ is } m\text{-}\omega\text{-sh in } \Omega,\ w \leq \underline{u} \text{ on } U,\ w \leq 0\}.$$

It follows from [Gu and Nguyen 2018, Proposition 2.6] that the upper semicontinuous regularization $v^* \geq v$ is also m-ω-sh function in Ω. Hence, $v^* = v$ belongs to the family in the definition of the envelope. So $\underline{u} \leq v$. Thus, $\underline{u} = v$ on U containing $\operatorname{supp}\mu$. Therefore,

$$H_m(v) \geq \mu \quad \text{in } \Omega.$$

Now we verify that

$$H_m(v) \equiv 0 \quad \text{on } \Omega \setminus \overline{U}.$$

Let $B(a,r) \Subset \Omega \setminus \overline{U}$ be a small ball. By using the solution to the homogeneous Hessian equation (Theorem 8.2) one can find a continuous m-ω-sh function $h \geq v$ in Ω which is maximal in $B(a,r)$ in the sense that $H_m(h) \equiv 0$ in $B(a,r)$ and $h = v$ on $\Omega \setminus B(a,r)$.

Observe also $U \subset \Omega \setminus B(a,r)$ and the function h is a candidate in the envelope. So, $h \leq v$ in Ω. Hence, $h = v$ everywhere. We have $H_m(v) \equiv 0$ on $B(a,r)$. The ball is arbitrary so we get the desired property for v.

Step 3: (b). If the problem is solvable for measures with compact support, then it is solvable for a general measure. In fact, let $\eta_j \uparrow 1$ be a sequence of cut-off functions. Then, $\eta_j \mu$ admits $\underline{u}$ as a bounded subsolution. Solve $H_m(w) = \eta_j \mu$ to obtain a bounded m-ω-sh function u_j with $u_j = \varphi$ on $\partial\Omega$. Denote by $h \in C^0(\overline{\Omega})$ a unique m-ω-sh solution to $H_m(h) \equiv 0$ in Ω with $h = \varphi$ on $\partial\Omega$. Then, $H_m(\underline{u} + h) \geq H_m(u_j)$. The domination principle gives

$$\underline{u} + h \leq u_j \leq h,$$

and the sequence u_j is decreasing. Define $u = \lim_j u_j$. Then, it is a solution to $H_m(u) = \mu$ with the boundary data φ.

Step 4: (a). If we can solve the Dirichlet problem for smooth boundary data, then it is solvable for the continuous one. Indeed, let φ_j be a sequence of smooth functions decreasing to φ. Find a bounded m-ω-sh function u_j in Ω such that

$$H_m(u_j) = \mu, \qquad \lim_{z \to x} u_j(z) = \varphi_j(x) \quad \text{for every } x \in \partial\Omega.$$

Let h_j be the solution the homogeneous equation $H_m(h_j) \equiv 0$ with the boundary data φ_j. By the domination principle, $u_j \geq h_j + \underline{u}$ and u_j is a decreasing sequence. Since φ_j is uniformly bounded, so is h_j. Therefore, the function $u = \lim_j u_j$ is a required solution to the equation. □

We proceed to prove the theorem under the assumptions (a)–(d).

Proof. Recall that we assumed that subsolution $\underline{u}$ is defined in a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$. Hence, $\underline{u} \in \widetilde{\mathcal{E}}_0(\Omega)$ in the sense of (8-2). By Proposition 2.9 we can find a decreasing sequence of smooth m-ω-sh functions v_j defined in $\widetilde{\Omega}$ and such that $v_j \downarrow \underline{u}$ pointwise. Since $\underline{u}$ is bounded, $\{v_j\}$ is uniformly bounded. Next, let us write

$$H_m(v_j) = f_j dV \quad \text{in } \widetilde{\Omega},$$

where dV is the Euclidean volume form on $\mathbb{C}^n$. Observe that v_j is no longer zero on the boundary of Ω, however we can modify it by solving the Dirichlet problem to find $\bar{v}_j \in C^0(\overline{\Omega})$, m-ω-sh satisfying

$$H_m(\bar{v}_j) = f_j dV \quad \text{in } \Omega, \qquad \bar{v}_j = 0 \quad \text{on } \partial\Omega.$$

Since $\underline{u}$ continuous on $\partial\Omega$, by Dini's theorem, v_j converges uniformly to $\underline{u}$ on $\partial\Omega$. As a consequence of the stability estimate for the right-hand side in L^p, we have

$$\|\bar{v}_j - v_j\|_{L^\infty(\Omega)} \leq \sup_{\partial\Omega} |\bar{v}_j - v_j| = \sup_{\partial\Omega} |v_j|.$$

Hence, $\{\bar{v}_j\}$ is also uniformly bounded on $\overline{\Omega}$. Also by Dini's theorem $v_j \to \underline{u}$ uniformly on compact sets where $\underline{u}$ is continuous. By the stability estimate above $\bar{v}_j \to \underline{u}$ uniformly on such compact sets. Combining this with the quasicontinuity of $\underline{u}$, we get also that

$$\bar{v}_j \to \underline{u} \quad \text{in capacity as } \ j \to \infty.$$

Thus the $\bar{v}_j$ have zero boundary values, but in general those are only continuous functions. Also $H_m(\bar{v}_j)$ converges weakly to ν whose support is compact in Ω. Thus,

$$\sup_j \int_\Omega H_m(\bar{v}_j) \le C.$$

Moreover all $\bar{v}_j$ can be extended so that they form a uniformly bounded sequence in $\widetilde{\mathcal{E}}_0(\Omega)$ as follows. Set $\check{v}_j := \max\{\bar{v}_j, A\rho\}$ where A is the same as in (8-2). We verify that

$$\check{v}_j \to \underline{u} \quad \text{in capacity as } \ j \to \infty. \tag{8-10}$$

In fact, we fix an $\varepsilon > 0$. Then,

$$\{z \in \Omega : |\check{v}_j - \underline{u}| > \varepsilon\} = \{z \in \Omega : \check{v}_j - \underline{u} > \varepsilon\} \cup \{z \in \Omega : \check{v}_j - \underline{u} < -\varepsilon\}.$$

Note that $A\rho \le \underline{u}$ in Ω, which implies $\{\check{v}_j - \underline{u} > \varepsilon\} = \{\bar{v}_j - \underline{u} > \varepsilon\}$. So,

$$\mathrm{cap}_m(\check{v}_j - \underline{u} > \varepsilon) \to 0 \quad \text{as } \ j \to \infty.$$

On the other, since $\bar{v}_j \le \check{v}_j$ we have

$$\{\check{v}_j - \underline{u} < -\varepsilon\} \subset \{\bar{v}_j - \underline{u} < -\varepsilon\}.$$

The capacity of the latter set tends to zero, so the same is true for the former. This finishes the proof of the claim $\check{v}_j \to \underline{u}$ in the capacity cap_m.

We are now ready to produce a sequence of functions whose limit point will be the desired solution. By the Radon–Nikodym theorem we can write $\mu = g\nu$ for a Borel measurable function $0 \le g \le 1$. Assume first that g is continuous (we will relax the assumption on g at the end of the proof) and solve for $u_j \in \mathrm{SH}_m(\omega) \cap C^0(\overline{\Omega})$, the equation

$$H_m(u_j) = g\chi f_j dV, \qquad u_j = \varphi \quad \text{on } \partial\Omega,$$

where $0 \le \chi \le 1$ is the cut-off function in Ω such that $\chi \equiv 1$ on a neighborhood of $\mathrm{supp}\,\nu$ and $\mathrm{supp}\,\chi \Subset \Omega$. Set

$$u = \big(\limsup_{j\to\infty} u_j\big)^*. \tag{8-11}$$

Let us show first that $\{u_j\}$ is uniformly bounded. Indeed, let ψ be a smooth m-ω-sh solution to $(dd^c\psi)^m \wedge \omega^{n-m} = 1$ in $\overline{\Omega}$ with $\psi = \varphi$ on $\partial\Omega$. Thus, we may assume that ψ is m-ω-sh on $\widetilde{\Omega}$. At this point we used the smoothness of φ. Furthermore, let $h \in C^0(\overline{\Omega})$ be a m-ω-sh solution to

$$H_m(h) = 0 \quad \text{in } \Omega, \qquad h = \varphi \quad \text{on } \partial\Omega.$$

It follows from the domination principle that

$$\psi + \check{v}_j \le u_j \le h.$$

(Here we may increase A so that $\check{v}_j = \max\{\bar{v}_j, A\rho\} = \bar{v}_j$ in a neighborhood of $\operatorname{supp} \nu$ and $\operatorname{supp} \chi$.) The above lower and upper bounds of u_j are continuous on the boundary $\partial\Omega$ and equal to φ there. So u_j and u have the same property. Thus passing to a subsequence we may assume that

$$u_j \to u \quad \text{in } L^1(\Omega), \qquad u_j \to u \quad \text{a.e. in } dV. \tag{8-12}$$

The next step is to prove that $H_m(u) = \mu$. Observe that $\psi + \check{v}_j$ is defined in the neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$. This combined with $\psi + \check{v}_j \le u_j$ allows us to extend u_j to $\widetilde{\Omega}$ by setting

$$\widetilde{u}_j = \begin{cases} \max\{u_j, \psi + \check{v}_j\} & \text{on } \Omega, \\ \psi + \check{v}_j & \text{on } \widetilde{\Omega} \setminus \Omega. \end{cases}$$

Using again the smoothness of φ, we can find ψ' a strictly m-sh function on a (possibly smaller) neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$ satisfying

$$\psi' = -\varphi \quad \text{on } \partial\Omega.$$

Then, we have clearly

$$\widehat{u}_j := u_j + \psi' \in \widetilde{\mathcal{E}}_0(\Omega) \tag{8-13}$$

for all $j \ge 1$. Consequently, we get the important uniform bound for the total mass of mixed Hessian operators.

Lemma 8.10. *Let $T_{j,k} = (dd^c \widehat{u}_j)^s \wedge (dd^c \check{v}_k)^\ell \wedge \omega^{n-s-\ell}$, where $0 \le s + \ell \le m$. There exists a uniform constant C independent of j, k such that*

$$\int_\Omega T_{j,k} \le C.$$

Proof. All functions belong to $\widetilde{\mathcal{E}}_0(\Omega)$ defined on the fixed neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$. Thanks to Remark 8.9 we know that the extensions of $\check{v}_i$ are uniformly bounded on $\widetilde{\Omega}$. Hence, the extensions of u_j above are uniformly bounded as well. The proof follows from the CLN inequality. □

The next result corresponds to [Kołodziej and Nguyen 2023a, Lemma 3.5].

Lemma 8.11. *There exists a subsequence $\{u_{j_s}\}$ such that for*

$$w_s := \max\left\{u_{j_s}, u - \frac{1}{s}\right\}$$

the following claims hold:

(a) $\lim_{s\to\infty} \int_\Omega |u_{j_s} - u| H_m(u) = 0.$

(b) $\lim_{s\to\infty} \int_\Omega |u_{j_s} - u| H_m(w_s) = 0.$

(c) $\lim_{s\to\infty} \int_\Omega |u_{j_s} - u| H_m(u_{j_s}) = 0.$

Proof. (a) Since u is bounded in Ω by Proposition 7.7 the measure $H_m(u)$ vanishes on m-polar sets. Define $\widehat{u} = u + \psi'$. Then $\widehat{u}$ is the limit (a.e.-dV) of the sequence $\{\widehat{u}_j\}_{j\geq 1} \subset \widetilde{\mathcal{E}}_0(\Omega)$ from (8-13). Thus, $\widehat{u} \in \widetilde{\mathcal{E}}_0(\Omega)$. Notice that $\widehat{u}_j - u = u_j - u$, so the assumptions of Corollary 8.4 are satisfied.

(b) Clearly, $\widehat{w}_s = w_s + \psi' \in \widetilde{\mathcal{E}}_0(\Omega)$. Since $w_s \to u$ in capacity as $s \to \infty$, the same convergence holds for $\widehat{w}_s \to \widehat{u}$. It follows from Lemma 8.5 that

$$0 = \lim_{s\to\infty} \int_\Omega |\widehat{w}_s - \widehat{u}| H_m(\widehat{w}_s) \geq \lim_{s\to\infty} \int_\Omega |u_{j_s} - u| H_m(w_s).$$

(c) We use

$$H_m(u_{j_s}) = g\chi f_{j_s} dV \leq \chi H_m(\bar{v}_{j_s}) \leq H_m(\check{v}_{j_s}),$$

where the first inequality follows from $0 \leq g \leq 1$ and the last one is by $\bar{v}_j = \check{v}_j$ in a neighborhood of $\operatorname{supp} \nu$ which can be taken to be the neighborhood of $\operatorname{supp} \chi$. Taking into account the convergence in capacity of $\check{v}_{j_s}$ to $\underline{u}$, as $j_s \to \infty$, the proof of (c) follows again from Lemma 8.5. □

We are in the position to conclude that u from (8-11) is indeed the solution.

The argument of [Kołodziej and Nguyen 2023a, Lemma 3.6] is readily applicable to conclude that there exists a subsequence $\{u_{j_s}\}_{s\geq 1}$ of $\{u_j\}_{j\geq 1}$ such that

$$H_m(u_{j_s}) \to H_m(u) \quad \text{weakly}.$$

Hence, if $0 \leq g \leq 1$ is a continuous function whose support is compact in Ω, then there exists a unique bounded m-ω-sh function with $u = \varphi$ on $\partial\Omega$ and $H_m(u) = g H_m(\underline{u})$. The general case of a Borel function $0 \leq g \leq 1$ follows from the argument in [Kołodziej and Nguyen 2023a, page 11] at the end of the proof of Theorem 3.1. □

9. Hessian equations on Hermitian manifolds with boundary

Let $(\overline{M}, \omega)$ be a smooth compact Hermitian manifold of dimension n with nonempty boundary ∂M. Then, $\overline{M} = M \cup \partial M$, where M is a complex manifold of dimension n. Let $1 \leq m \leq n$ be an integer and $\alpha \in \Gamma_m(\omega)$ be a real $(1,1)$-form.

Recently Collins and Picard [2022] solved the Dirichlet problem in M for the Hessian equation $(\alpha + dd^c u)^m \wedge \omega^{n-m} = f\omega^n$, for smooth data, assuming the existence of a subsolution. The goal of this section is to extend this result to the case of bounded functions. The special case of the Monge–Ampère equation was treated in [Kołodziej and Nguyen 2023a, Theorem 1.2]. The theorem below is also a significant improvement of [Gu and Nguyen 2018, Theorem 1.3].

Recall from [Gu and Nguyen 2018, Definition 2.4, Lemma 9.10] that a function $u : M \to [-\infty, +\infty)$ is called (α, m)-ω-subharmonic if it can be written locally as a sum of a smooth function and a ω-sh function, and globally for any collection $\gamma_1, \ldots, \gamma_{m-1} \in \Gamma_m(M, \omega)$,

$$(\alpha + dd^c u) \wedge \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \omega^{n-m} \geq 0 \quad \text{on } M \tag{9-1}$$

in the weak sense of currents. Denote by $\mathrm{SH}_{\alpha,m}(M, \omega)$ or $\mathrm{SH}_{\alpha,m}(\omega)$ the set of all (α, m)-ω-sh functions on M.

If Ω is a local coordinate chart on M and ρ is a strictly psh function on Ω such that

$$dd^c\rho \geq \alpha \quad \text{on } \Omega,$$

then $u+\rho$ is a m-ω-sh function on Ω. Using this fact, we can easily extend the definition of the wedge product for currents associated to bounded (α, m)-ω-sh functions by using partition of unity and the local one (Definition 3.4). Namely, write $\tau = \alpha - dd^c\rho$ which is a smooth $(1,1)$-form. Then, $\alpha + dd^c u = dd^c(u+\rho) + \tau$. We define

$$(\alpha + dd^c u)^m \wedge \omega^{n-m} := \sum_{k=0}^{m} \binom{m}{k} [dd^c(u+\rho)]^k \wedge \tau^{m-k} \wedge \omega^{n-m} = \sum_{k=0}^{m} \binom{m}{k} \mathcal{L}_k(u+\rho) \wedge \tau^{m-k}.$$

This gives a positive Radon measure on M by the weak convergence theorem. Similarly, the wedge product for bounded (α, m)-ω-sh functions $u_1, \ldots, u_m$,

$$(\alpha + dd^c u_1) \wedge \cdots \wedge (\alpha + dd^c u_m) \wedge \omega^{n-m},$$

is a well-defined positive Radon measure. Since the definition is local, all local results for m-ω-sh functions in a local coordinate chart transfer to (α, m)-ω-sh functions on the manifold M. For simplicity we define $\alpha_u := \alpha + dd^c u$ and

$$H_{m,\alpha}(u) = (\alpha + dd^c u)^m \wedge \omega^{n-m}.$$

Now, given a positive Radon measure μ on M and a continuous boundary data $\varphi \in C^0(\partial M, \mathbb{R})$ we wish to solve the Dirichlet problem

$$\begin{cases} u \in \mathrm{SH}_{\alpha,m}(\omega) \cap L^\infty(\overline{M}), \\ H_{m,\alpha}(u) = \mu, \\ \lim_{z\to x} u(x) = \varphi(x) \qquad \text{for } x \in \partial M. \end{cases} \tag{9-2}$$

Let us state a general existence result.

Theorem 9.1. *Assume there exists a bounded (α, m)-ω-sh function $\underline{u}$ on M such that $\lim_{z\to x} \underline{u}(z) = \varphi(x)$ for $x \in \partial M$ and $H_{m,\alpha}(\underline{u}) \geq \mu$ on M. Then, there is a solution to the Dirichlet problem (9-2).*

Remark 9.2. In the general setting the uniqueness is not known, unlike in a bounded strictly m-pseudoconvex domain (Theorem 8.7). On the other hand, if we assume further that either the manifold M is Stein, or both ω and α are closed forms, then the solution will be unique.

As we use the Perron envelope method to show the theorem the most important ingredient is the proof of the special case $M \equiv \Omega$ a ball in $\mathbb{C}^n$.

Lemma 9.3. *Let $\varphi \in C^0(\partial\Omega, \mathbb{R})$. Suppose $\mu \leq H_m(v)$ for some bounded m-ω-sh function v in Ω with $\lim_{z\to x} v(z) = 0$ for $x \in \partial\Omega$. Then, there exists a unique (α, m)-ω-sh function u in Ω solving*

$$H_{m,\alpha}(u) = \mu \quad \text{in } \Omega, \qquad \lim_{z\to x} u(z) = \varphi(x) \quad \text{for } x \in \partial\Omega. \tag{9-3}$$

The proof of this lemma is a straightforward extension of Theorem 8.7 so we omit the proof.

Proof of Theorem 9.1. Let us proceed with the proof of the bounded subsolution theorem on $\overline{M}$. Consider the set of functions

$$\mathcal{B}(\varphi, \mu) := \{w \in \mathrm{SH}_{\alpha,m}(M, \omega) \cap L^\infty(\overline{M}) : H_{m,\alpha}(w) \geq \mu,\ w^*_{|\partial M} \leq \varphi\}, \tag{9-4}$$

where $w^*(x) = \limsup_{M \ni z \to x} w(z)$ for every $x \in \partial M$. Clearly, $\underline{u} \in \mathcal{B}(\varphi, \mu)$. Let us solve the linear PDE finding $h_1 \in C^0(\overline{M}, \mathbb{R})$ such that

$$(\alpha + dd^c h_1) \wedge \omega^{n-1} = 0, \qquad h_1 = \varphi \quad \text{on } \partial M. \tag{9-5}$$

Since $(\alpha + dd^c w) \wedge \omega^{n-1} \geq 0$ for $w \in \mathrm{SH}_{\alpha,m}(M, \omega)$, the maximum principle for the Laplace operator with respect to ω gives

$$w \leq h_1 \quad \text{for all } w \in \mathcal{B}(\varphi, \mu).$$

Set

$$u(z) = \sup_{w \in \mathcal{B}(\varphi,\mu)} w(z) \quad \text{for every } z \in M. \tag{9-6}$$

Then, by Choquet's lemma and the fact that $\mathcal{B}(\varphi, \mu)$ satisfies the lattice property, $u = u^* \in \mathcal{B}(\varphi, \mu)$. Again by the definition of u, we have $\underline{u} \leq u \leq h_1$. It follows that

$$\lim_{z \to x} u(z) = \varphi(x) \quad \text{for every } x \in \partial M. \tag{9-7}$$

Lemma 9.4 (lift). *Let $v \in \mathcal{B}(\varphi, \mu)$. Let $B \Subset M$ be a small coordinate ball (a chart biholomorphic to a ball in $\mathbb{C}^n$). Then, there exists $\widetilde{v} \in \mathcal{B}(\varphi, \mu)$ such that $v \leq \widetilde{v}$ and $H_{m,\alpha}(\widetilde{v}) = \mu$ on B.*

Proof. Given the solution in a small coordinate ball in Lemma 9.3, the proof from [Kołodziej and Nguyen 2023a, Lemma 3.7] is readily adaptable here. □

By (9-7) it remains to show that the function u above satisfies $H_{m,\alpha}(u) = \mu$. Let $B \Subset M$ be a small coordinate ball. It is enough to check $H_{m,\alpha}(u) = \mu$ on B. Let $\widetilde{u}$ be the lift of u as in Lemma 9.4. It follows that $\widetilde{u} \geq u$ and $H_{m,\alpha}(\widetilde{u}) = \mu$ on B. However, by the definition $\widetilde{u} \leq u$ on M. Thus, $\widetilde{u} = u$ on B, in particular on B we have $H_{m,\alpha}(\widetilde{u}) = H_{m,\alpha}(u) = \mu$. □

Remark 9.5. We can also study the continuity of the solution to the Dirichlet problem (9-2) for a measure that is well dominated by capacity as in [Kołodziej and Nguyen 2023a, Section 4] and the weak solution to the complex Hessian-type equations such as a generalization of the Monge–Ampère equation in [Kołodziej and Nguyen 2023b]. We leave these to future projects.

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ℓ^2 DECOUPLING THEOREM FOR SURFACES IN $\mathbb{R}^3$

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We identify a new way to divide the δ-neighborhood of surfaces $\mathcal{M} \subset \mathbb{R}^3$ into a finitely overlapping collection of rectangular boxes S. We obtain a sharp (ℓ^2, L^p) decoupling estimate using this decomposition for the sharp range of exponents $2 \le p \le 4$. Our decoupling inequality leads to new exponential sum estimates where the frequencies lie on surfaces which do not contain a line.

1. Introduction

Consider the manifold

$$\mathcal{M}_\phi := \{(\xi_1, \xi_2, \phi(\xi_1, \xi_2)) : \xi_1, \xi_2 \in [0, 1]\}, \tag{1-1}$$

where $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Denote by $N_\delta(\mathcal{M}_\phi) \subset \mathbb{R}^3$ the δ-neighborhood of the manifold $\mathcal{M}_\phi$. Our main theorem is an ℓ^2 decoupling theorem for $\mathcal{M}_\phi$ into (ϕ, δ)-flat sets. This type of theorem was introduced by [Bourgain et al. 2020].

Definition 1.1. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be smooth. We say that $S \subset [0, 1]^2$ is (ϕ, δ)-flat if

$$\sup_{u,v \in S} |\phi(u) - \phi(v) - \nabla\phi(u) \cdot (u - v)| \le \delta. \tag{1-2}$$

For a measurable function $f : \mathbb{R}^3 \to \mathbb{C}$ and a measurable set $S \subset \mathbb{R}^2$, denote by f_S a smooth Fourier restriction of f on the set $S \times \mathbb{R}$. We refer to Definition 2.1 for the rigorous definition.

Theorem 1.2. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Fix $\epsilon > 0$. Then there exists a sufficiently large number A depending on ϵ and ϕ satisfying the following.*

For any $\delta > 0$, there exists a collection $\mathcal{S}_\delta$ of finitely overlapping sets S such that:

(1) *The overlapping number is $O(\delta^{-\epsilon})$ in the sense that*

$$\sum_{S \in \mathcal{S}_\delta} \chi_S \le C_{\epsilon,\phi} \delta^{-\epsilon}. \tag{1-3}$$

(2) *S is $(\phi, A\delta)$-flat and of the form of a parallelogram.*

(3) *For $2 \le p \le 4$, we have*

$$\|f\|_{L^p} \le C_{\epsilon,\phi} \delta^{-\epsilon} \left(\sum_{S \in \mathcal{S}_\delta} \|f_S\|_{L^p}^2 \right)^{1/2} \tag{1-4}$$

for all functions f whose Fourier supports are in $N_\delta(\mathcal{M}_\phi)$.

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One important aspect of our theorem is that the collection $\mathcal{S}_\delta$ is not a strict partition of $[0,1]^2$ since the sets $S \in \mathcal{S}_\delta$ are $O(\delta^{-\epsilon})$-overlapping. The decoupling inequality (1-4) for the hyperbolic paraboloid is not true if a family $\mathcal{S}_\delta$ is a partition of $[0,1]^2$. In other words, the $O(\delta^{-\epsilon})$-overlapping property is necessary for the theorem to hold; see Appendix A. The situation is different for curves $(\xi, \phi(\xi))$ in $\mathbb{R}^2$, where Yang [2021] proved a version of Theorem 1.2 using a strict partition. We realized that to prove an ℓ^2 decoupling inequality of the form (1-4), it is crucial to understand the number of possible choices of (ϕ, δ)-flat sets. For a curve $(\xi, \phi(\xi)) \in \mathbb{R}^2$ and a given point $p \in [0,1]$, there is essentially one choice of a (ϕ, δ)-flat set containing the point p with the maximal size. On the other hand, for the hyperbolic paraboloid in $\mathbb{R}^3$ and a given point $p \in [0,1]^2$, there are essentially $O(\log \delta^{-1})$ many choices of (ϕ, δ)-flat sets, which is the form of a rectangle with dyadic sidelengths, containing the point p and with maximal size. This turns out to be the reason why a partition is not enough and we need (1-3). For the hyperbolic paraboloid in $\mathbb{R}^n$ with $n \geq 4$, the possible choices of (ϕ, δ)-flat sets containing a point is $\sim \delta^{-A_n}$ for some $A_n > 0$, which is why we do not have an analogous theorem without any δ^{-1}-power loss; see Appendix B.

Next we describe our main application of Theorem 1.2, which is to prove discrete restriction estimates for manifolds $\mathcal{M}_\phi$ not containing a line segment. For a function F and a set A, define

$$\|F\|^p_{L^p_\#(A)} := \frac{1}{|A|}\|F\|^p_{L^p(A)}. \tag{1-5}$$

Denote by Λ_δ a collection of δ-separated points in $[0,1]^2$. Define $e(t) := e^{2\pi i t}$.

Corollary 1.3. *Let ϕ be a polynomial of degree d with coefficients bounded by 1. Suppose that the manifold $\mathcal{M}_\phi$ does not contain a line segment. Then, for $2 \leq p \leq 4$, $\epsilon > 0$, and any sequence $\{a_\xi\}_{\xi \in \Lambda_\delta}$, we have*

$$\left\| \sum_{\xi \in \Lambda_\delta} a_\xi e(x \cdot (\xi, \phi(\xi))) \right\|_{L^p_\#(B_{\delta^{-d}})} \leq C_{\phi,\epsilon} \delta^{-\epsilon} \left(\sum_{\xi \in \Lambda_\delta} |a_\xi|^2 \right)^{1/2}. \tag{1-6}$$

An (ℓ^2, L^p) discrete restriction estimate is an inequality of the form (1-6). Corollary 1.3 is sharp in the following senses: First, the range of p of (1-6) is optimal for the paraboloid, though we expect that a wider range of p is possible for generic polynomials ϕ. Second, the inequality (1-6) is false for any $p > 2$ for some collection Λ_δ if the manifold $\mathcal{M}_\phi$ contains a line segment.

The (ℓ^2, L^p) discrete restriction estimate (1-6) implies the sharp (ℓ^p, L^p) discrete restriction estimate. Li and Yang [2021] obtained a sharp ℓ^p decoupling inequality for smooth manifolds, but their decoupling does not seem to imply (ℓ^p, L^p) discrete restriction estimates.

Our second application is about the Strichartz estimate for the nonelliptic Schrödinger equation on irrational tori. Let us introduce the conjecture. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\alpha < 0$, define $\tilde{\Delta} := \frac{1}{2\pi}(\partial_{11} + \alpha \partial_{22})$. Let v be a solution of the partial differential equation

$$i\partial_t v - \tilde{\Delta} v = 0, \quad v(0,x) = f(x), \tag{1-7}$$

where the initial data f is defined on the torus $\mathbb{T}^2$. We let $e^{it\tilde{\Delta}} f(x) := v(x,t)$. The Strichartz estimate for the elliptic Schrödinger equation on irrational tori, which corresponds to the case $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\alpha > 0$,

was studied in [Deng et al. 2017; 2022]. As in the case of the elliptic Schrödinger equation on irrational tori in [Deng et al. 2017, Conjecture 1.2], we make the following conjecture.

Conjecture 1.4. *Let α be an irrational number. Then*

$$\|e^{it\tilde{\Delta}} f\|_{L^p([0,T]\times\mathbb{T}^2)} \le C_{\epsilon,\alpha} N^{\epsilon} \|f\|_{L^2(\mathbb{T}^2)} \begin{cases} T^{1/p} & \text{for } 2 \le p \le 4, \\ T^{1/p} + N^{1-4/p} & \text{for } 4 \le p \le 6, \\ T^{1/p} N^{1-6/p} + N^{1-4/p} & \text{for } 6 \le p \end{cases} \tag{1-8}$$

for functions f whose Fourier support is in $[-N, N]^2$.

Corollary 1.5 gives partial progress on the conjecture for the range $2 \le p \le 4$.

Corollary 1.5. *Let α be an irrational number. Consider $2 \le p \le 4$. For $T \ge N$,*

$$\|e^{it\tilde{\Delta}} f\|_{L^p([0,T]\times\mathbb{T}^2)} \le C_{\epsilon,\alpha} N^{\epsilon} T^{1/p} \|f\|_{L^2(\mathbb{T}^2)} \tag{1-9}$$

for functions f whose Fourier support is in $[-N, N]^2$.

A theorem analogous to Corollary 1.5 for the elliptic Schrödinger equation is obtained as a corollary of an ℓ^2 decoupling theorem [Bourgain and Demeter 2015] for the paraboloid, which is observed by [Deng et al. 2017]. Corollary 1.5 is not always true if α is not irrational. We refer to [Sun and Tzvetkov 2020; Mizutani and Tzvetkov 2015] for related topics.

We prove the corollaries in Section 4.

1.1. ***Comparison to decoupling inequalities in the literature.*** Let $\mathcal{M} \subset \mathbb{R}^3$ be a manifold in $\mathbb{R}^3$ with nonvanishing Gaussian curvature, and let $\mathcal{P}_\delta(\mathcal{M})$ be a partition of $N_\delta(\mathcal{M})$ into approximate $\delta^{1/2} \times \delta^{1/2} \times \delta$ caps θ. In their foundational decoupling paper, Bourgain and Demeter [2015] proved that if $\mathcal{M}$ has everywhere positive Gaussian curvature and $2 \le p \le 4$, then

$$\|f\|_{L^p(\mathbb{R}^3)} \lesssim_\epsilon \delta^{-\epsilon} \Bigg(\sum_{\theta \in \mathcal{P}_\delta(\mathcal{M})} \|f_\theta\|^2_{L^4(\mathbb{R}^3)} \Bigg)^{1/2} \tag{1-10}$$

for all f with $\operatorname{supp} \hat{f} \subset N_\delta(\mathcal{M})$. It is observed in [Bourgain and Demeter 2017a] that (1-10) is false for the hyperbolic paraboloid. The counterexample comes from the geometric fact that the hyperbolic paraboloid contains a line. Based on this observation, one might expect (1-10) to hold for manifolds avoiding a line segment.

Conjecture 1.6. *Let ϕ be a polynomial of degree d. Assume that $\mathcal{M}_\phi$ does not contain a line segment and has everywhere negative Gaussian curvature. Define*

$$\mathcal{P}_\delta := \{[a, a+\delta^{1/d}] \times [b, b+\delta^{1/d}] : a, b \in \delta^{1/d}\mathbb{Z} \cap [0, 1-\delta^{1/d}]\}. \tag{1-11}$$

Then, for $2 \le p \le 4$,

$$\|f\|_{L^p(\mathbb{R}^3)} \lesssim_\epsilon \delta^{-\epsilon} \Bigg(\sum_{\theta \in \mathcal{P}_\delta} \|f_\theta\|^2_{L^p(\mathbb{R}^3)} \Bigg)^{1/2} \tag{1-12}$$

for functions f whose Fourier supports are in $N_\delta(\mathcal{M})$.

Conjecture 1.6 implies Corollary 1.3 in the case that the determinant of the Hessian matrix of ϕ is negative. Unfortunately, it is not clear how to prove Conjecture 1.6 using the current decoupling techniques. Instead of proving Conjecture 1.6, we introduce a modified version of the decoupling inequality, which is inspired by (ϕ, δ)-flat sets, and prove that this decoupling still implies Corollary 1.3. Both Conjecture 1.6 and Theorem 1.2 imply Corollary 1.3, but neither of them seems to imply the other.

Let us review decoupling for smooth manifolds to motivate the use of (ϕ, δ)-sets. Bourgain and Demeter [2015; 2017a] proved that if $\mathcal{M}$ has everywhere nonzero Gaussian curvature and $2 \le p \le 4$, then

$$\|f\|_{L^p(\mathbb{R}^3)} \lesssim_\epsilon \delta^{-\epsilon}(\delta^{-1})^{1/2-1/p}\Big(\sum_{\theta\in\mathcal{P}_\delta(\mathcal{M})} \|f_\theta\|^p_{L^p(\mathbb{R}^3)}\Big)^{1/p} \tag{1-13}$$

for all f with $\operatorname{supp} \hat{f} \subset N_\delta(\mathcal{M})$. As mentioned before, if the Gaussian curvature is positive, then the above decoupling inequality may be refined to an (ℓ^2, L^p) decoupling (see [Bourgain and Demeter 2015]), which has the form

$$\|f\|_{L^p(\mathbb{R}^3)} \lesssim_\epsilon \delta^{-\epsilon}\Big(\sum_{\theta\in\mathcal{P}_\delta(\mathcal{M})} \|f_\theta\|^2_{L^p(\mathbb{R}^3)}\Big)^{1/2}. \tag{1-14}$$

When $\mathcal{M}$ is the truncated paraboloid $\mathbb{P}^2 = \{(\xi, |\xi|^2) \in \mathbb{R}^3 : |\xi| \le 1\}$, we have that $\mathcal{P}_\delta(\mathbb{P}^2)$ is the coarsest partition of $N_\delta(\mathbb{P}^2)$ such that each piece θ is essentially flat. Bourgain and Demeter [2015] also proved sharp decoupling estimates for the cylinder $\mathrm{Cyl} = \{(\xi_1, \xi_2, \xi_3) : \xi_1^2 + \xi_2^2 = 1, |\xi_3| \lesssim 1\}$ and the cone $\mathcal{C} = \big\{(\xi_1, \xi_2, \sqrt{\xi_1^2 + \xi_2^2}) : \frac{1}{4} \le \xi_1^2 + \xi_2^2 \le 4\big\}$, where their δ-neighborhoods are partitioned into planks θ of dimensions about $1 \times \delta^{1/2} \times \delta$. They observed that these are the coarsest (ϕ, δ)-flat sets. Based on this, Bourgain, Demeter, and Kemp [Bourgain et al. 2020] introduced the notion of (ϕ, δ)-flat sets (Definition 1.1) for a general manifold $\mathcal{M}_\phi$ and asked if there is a partition of $N_\delta(\mathcal{M}_\phi)$, denoted by $\mathcal{P}_\delta(\mathcal{M}_\phi)$, such that each element is (ϕ, δ)-flat and for $2 \le p \le 4$ the following ℓ^p decoupling is true:

$$\|f\|_{L^p(\mathbb{R}^3)} \lesssim_\epsilon \delta^{-\epsilon}|\mathcal{P}_\delta(\mathcal{M}_\phi)|^{1/2-1/p}\Big(\sum_{\theta\in\mathcal{P}_\delta(\mathcal{M}_\phi)} \|f_\theta\|^p_{L^p(\mathbb{R}^3)}\Big)^{1/p} \tag{1-15}$$

for all functions f whose Fourier supports are in $N_\delta(\mathcal{M}_\phi)$. This question is answered affirmatively in [Li and Yang 2021].

One may ask if there is a partition of $N_\delta(\mathcal{M}_\phi)$ such that each element is (ϕ, δ)-flat and for $2 \le p \le 4$ the following ℓ^2 decoupling is true:

$$\|f\|_{L^p(\mathbb{R}^3)} \lesssim_\epsilon \delta^{-\epsilon}\Big(\sum_{\theta\in\mathcal{P}_\delta(\mathcal{M}_\phi)} \|f_\theta\|^2_{L^p(\mathbb{R}^3)}\Big)^{1/2} \tag{1-16}$$

for all functions f whose Fourier supports are in $N_\delta(\mathcal{M}_\phi)$. Note that (1-16) implies (1-15) by Hölder's inequality. When S is the truncated hyperbolic paraboloid, Bourgain and Demeter proved that

$$\|f\|_{L^p(\mathbb{R}^3)} \lesssim_\epsilon \delta^{-\epsilon}\delta^{-1/2(1/2-1/p)}\Big(\sum_{\theta\in\mathcal{P}_\delta(\mathcal{M}_\phi)} \|f_\theta\|^2_{L^p(\mathbb{R}^3)}\Big)^{1/2}. \tag{1-17}$$

They also proved that the loss of δ on the right-hand side is necessary up to the loss of $\delta^{-\epsilon}$, and the sharp example comes from the geometric fact that the hyperbolic paraboloid contains a line. At first, this might look like a counterexample to the question raised in (1-16). However, since there are multiple choices of (ϕ, δ)-flat sets, it does not give a counterexample. Also, given that the ℓ^2 decoupling theorem for a curve in $\mathbb{R}^2$ is proved by [Yang 2021] (which is the two-dimensional version of (1-16)), one might still expect that (1-16) is true for some collection of (ϕ, δ)-flat sets. In Appendix A, we prove Theorem A.1, which says that, for the hyperbolic paraboloid, there is no partition, whose elements are (ϕ, δ)-flat, such that (1-16) holds. The proof of Theorem A.1 crucially uses the assumption that a collection of (ϕ, δ)-flat sets is a partition. However, in all of the known applications of decouplings we are aware of, it does not matter if a collection of (ϕ, δ)-flat sets is a partition or $O(\delta^{-\epsilon})$-finitely overlapping. Theorem 1.2 says the ℓ^2 decoupling theorem is true after allowing the collection $\mathcal{S}_\delta$ to be $O(\log \delta^{-1})$-finitely overlapping; see (1-3). Our decoupling theorem is still useful in the sense that it implies Corollaries 1.3 and 1.5 and Proposition 1.10. Also, by Hölder's inequality, our decoupling "essentially" implies the ℓ^p decoupling theorem (1-15) of Yang and Li.

Let us finish this subsection by mentioning higher-dimensional manifolds. It is conceivable that the ℓ^p decoupling conjecture is still true in high dimensions. Let us introduce some notation to state the conjecture. For a function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$, we say $S \subset [0,1]^{n-1}$ is a (ϕ, δ)-set if (1-2) is true. Consider the manifold

$$\mathcal{M}_\phi := \{(\xi, \phi(\xi)) : \xi \in [0,1]^{n-1}\}. \tag{1-18}$$

For a measurable function $f : \mathbb{R}^n \to \mathbb{C}$ and a measurable set $S \subset \mathbb{R}^{n-1}$, denote by f_S the Fourier restriction of f on the set $S \times \mathbb{R}$.

Conjecture 1.7. *Let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a smooth function. Fix $\epsilon > 0$. Then there exists a sufficiently large number A depending on ϵ satisfying the following.*

For any $\delta > 0$, there exists a collection $\mathcal{S}_\delta$ of finitely overlapping sets $S \subset [0,1]^{n-1}$ such that:

(1) *The overlapping number is $O(\log \delta^{-1})$ in the sense that*

$$\sum_{S \in \mathcal{S}_\delta} \chi_S \le C_\epsilon \log(\delta^{-1}). \tag{1-19}$$

(2) *S is $(\phi, A\delta)$-flat.*

(3) *For $2 \le p \le \frac{2(n+1)}{n-1}$, we have*

$$\|f\|_{L^p} \le C_\epsilon \delta^{-\epsilon} (\#\mathcal{S}_\delta)^{1/2-1/p} \Big(\sum_{S \in \mathcal{S}_\delta} \|f_S\|_{L^p}^p \Big)^{1/p} \tag{1-20}$$

for all functions f whose Fourier supports are in $N_\delta(\mathcal{M}_\phi)$.

Conjecture 1.7 is known for the paraboloid and hyperbolic paraboloid but would be new for general manifolds. The ℓ^p decoupling inequality for manifolds in $\mathbb{R}^2$ and $\mathbb{R}^3$ are proved in [Yang 2021] and [Li and Yang 2021]. It remains open for higher dimensions. One main ingredient of ℓ^p decoupling for manifolds in $\mathbb{R}^3$ is ℓ^2 decoupling for a curve in $\mathbb{R}^2$. Since we now have ℓ^2 decoupling for manifolds in $\mathbb{R}^3$, it might be possible to prove ℓ^p decoupling for manifolds in $\mathbb{R}^4$.

1.2. ***Example*: *hyperbolic paraboloid.*** Our theorem is even new for the hyperbolic paraboloid. Let us state this special case.

Theorem 1.8. *Consider* $\phi(\xi_1, \xi_2) = \xi_1\xi_2$. *For* $2 \le p \le 4$,

$$\|f\|_{L^4} \le C_\epsilon \delta^{-\epsilon} \Bigg(\sum_{m=-2^{-1}\log\delta^{-1}}^{2^{-1}\log\delta^{-1}} \sum_{S \in \mathcal{R}_{2^m\delta^{1/2},2^{-m}\delta^{1/2},0}} \|f_S\|_{L^4}^2 \Bigg)^{1/2} \tag{1-21}$$

for all functions f *whose Fourier support is in* $N_\delta(\mathcal{M}_\phi)$.

The set $\mathcal{R}_{2^m\delta^{1/2},2^{-m}\delta^{1/2},0}$ is defined in Section 2.1.

While the ℓ^p decoupling for the hyperbolic paraboloid does not imply the Tomas–Stein theorem for the manifold, our theorem implies the Tomas–Stein theorem for the manifold. As an application of Theorem 1.8 to exponential sum estimates, we will prove Corollary 1.5 in Section 4.

Theorem 1.8 itself does not imply ℓ^p decoupling by [Bourgain and Demeter 2017a]. However, in the proof of the theorem, we proved a slightly stronger inequality, which is a mixture of ℓ^2 and ℓ^p norms for all the intermediate scales. We do not state this as a theorem as it is very involved. But this inequality likely implies the ℓ^p decoupling by [Bourgain and Demeter 2017a].

We state the following version of the Stein–Tomas restriction theorem, which is recorded in [Demeter 2020, Theorem 1.16 and Proposition 1.27]. Let $\mathbb{H}$ be the truncated hyperbolic paraboloid.

Theorem 1.9 [Tomas 1975; Stein 1986]. *For* $4 \le p \le \infty$, *we have*

$$\|\widehat{F}\|_{L^p(B_{\delta^{-1}})} \lesssim \delta^{1/2} \|F\|_{L^2(N_\delta(\mathbb{H}))}$$

for each $0 < \delta < 1$, *each ball* $B_{\delta^{-1}} \subset \mathbb{R}^3$, *and each* $F \in L^2(N_\delta(\mathbb{H}))$ *supported in* $N_\delta(\mathbb{H})$.

Proposition 1.10. *Theorem 1.8 implies Theorem 1.9 up to a* $\delta^{-\epsilon}$ *loss.*

Proof of Proposition 1.10. Theorem 1.9 with $p = \infty$ is trivial, so it suffices to prove the $p = 4$ case and invoke interpolation for the remaining exponents $p \in (4, \infty)$. Apply Theorem 1.8 with $p = 4$ to obtain the estimate

$$\|\widehat{F}\|_{L^4(B_{\delta^{-1}})} \lesssim_\epsilon \delta^{-\epsilon} \Bigg(\sum_{m=-2^{-1}\log\delta^{-1}}^{2^{-1}\log\delta^{-1}} \sum_{S \in \mathcal{R}_{2^m\delta^{1/2},2^{-m}\delta^{1/2},0}} \|(\widehat{F})_S\|_{L^4(\mathbb{R}^3)}^2 \Bigg)^{1/2}. \tag{1-22}$$

Note that $(\widehat{F})_S = \widehat{F\chi_S}$. Using Hölder's inequality, we have, for each $S \in \mathcal{R}_{2^m\delta^{1/2},2^{-m}\delta^{1/2},0}$, that

$$\|\widehat{F\chi_S}\|_{L^4(\mathbb{R}^3)} \le \|\widehat{F\chi_S}\|_{L^\infty(\mathbb{R}^3)}^{1/2} \|\widehat{F\chi_S}\|_{L^2(\mathbb{R}^3)}^{1/2} \le \|F\|_{L^1(N_\delta(S))}^{1/2} \|F\|_{L^2(N_\delta(S))}^{1/2} \le |N_\delta(S)|^{1/4} \|F\|_{L^2(N_\delta(S))}.$$

Using the above inequality to bound the right-hand side of (1-22), we have

$$\begin{aligned} \|\widehat{F}\|_{L^4(B_{\delta^{-1}})} &\lesssim_\epsilon \delta^{-\epsilon} \Bigg(\sum_{m=-2^{-1}\log\delta^{-1}}^{2^{-1}\log\delta^{-1}} \sum_{S \in \mathcal{R}_{2^m\delta^{1/2},2^{-m}\delta^{1/2},0}} |N_\delta(S)|^{1/2} \|F\|_{L^2(N_\delta(S))}^2 \Bigg)^{1/2} \\ &\lesssim_\epsilon \delta^{-\epsilon} \max_{\substack{|m| \lesssim \log\delta^{-1} \\ S \in \mathcal{R}_{2^m\delta^{1/2},2^{-m}\delta^{1/2},0}}} |N_\delta(S)|^{1/4} \|F\|_{L^2(N_\delta(\mathbb{H}))}. \end{aligned}$$

It remains to note that $|N_\delta(S)| \le \delta^2$ for each $S \in \mathcal{R}_{2^m\delta^{1/2},2^{-m}\delta^{1/2},0}$ and each $|m| \lesssim \log\delta^{-1}$. □

1.3. ***Proof strategy and structure of the paper.*** In Section 2, we prove ℓ^2 decoupling inequalities for perturbed hyperbolic paraboloids using a collection of $O(\log \delta^{-1})$-overlapping (ϕ, δ)-flat sets. Our result is new even for the hyperbolic paraboloid. The main new idea for the hyperbolic paraboloid is an iterative way of using broad-narrow analysis, which is combined with the induction-on-scale argument by [Bourgain and Demeter 2015; Bourgain et al. 2016]. To prove the theorem for perturbed hyperbolic paraboloids (Theorem 2.2), we combine the aforementioned argument with a restriction theory for perturbed hyperbolic paraboloids developed by [Buschenhenke et al. 2020; 2022; 2023; Guo and Oh 2024]. In Section 3, we use the ℓ^2 decoupling for the perturbed hyperbolic paraboloids as a black box and prove the ℓ^2 decoupling for general polynomials. This argument is essentially the same as that of [Li and Yang 2021]. In Section 4, we prove Corollaries 1.3 and 1.5 from the decoupling theorem. In Appendix A, we prove that there is no ℓ^2 decoupling for the hyperbolic paraboloid in $\mathbb{R}^3$ using a partition. This explains why it is necessary to introduce $\log(\delta^{-1})$ many partitions in Theorem 1.2. In Appendix B, we prove that there does not exist a collection of (ϕ, δ)-flat rectangles such that ℓ^2 decoupling for the collection is true in high dimensions without any additional power. In Appendix C, we prove the sharpness of Theorem 1.2.

1.4. ***Notation.*** For a function F and a set A, define

$$\|F\|^p_{L^p_\#(A)} := \frac{1}{|A|}\|F\|^p_{L^p(A)}. \tag{1-23}$$

For real numbers a_i, define

$$\left|\prod_{i=1}^{3} a_i\right| := \prod_{i=1}^{3} |a_i|^{1/3}. \tag{1-24}$$

For a ball B_K of radius K centered at $c(B_K)$, define

$$w_{B_K}(x) := \left(1 + \left|\frac{x - c(B_K)}{K}\right|^2\right)^{-100}. \tag{1-25}$$

For a measurable set A and a function f, define

$$\fint_A |f(x)|\,dx := \frac{1}{|A|}\int_A |f(x)|\,dx. \tag{1-26}$$

For a ball B, we define

$$\|f\|^p_{L^p_\#(w_B)} := |B|^{-1}\|f\|^p_{L^p(w_B)}. \tag{1-27}$$

For a rectangular box T, we denote by CT the dilation of T by a factor of C with respect to its centroid.

For two nonnegative numbers A_1 and A_2, we write $A_1 \lesssim A_2$ to mean that there exists a constant C such that $A_1 \le CA_2$. We write $A_1 \sim A_2$ if $A_1 \lesssim A_2$ and $A_2 \lesssim A_1$. We also write $A_1 \lesssim_\epsilon A_2$ if there exists C_ϵ depending on a parameter ϵ such that $A_1 \le C_\epsilon A_2$.

2. Perturbed hyperbolic paraboloids

Consider the manifold $\mathcal{M}_\phi$ associated with

$$\phi(\xi_1, \xi_2) := \xi_1\xi_2 + a_{2,0}\xi_1^2 + a_{0,2}\xi_2^2 + \sum_{3\le j+k\le d} a_{j,k}\xi_1^j\xi_2^k, \tag{2-1}$$

where the coefficients $a_{j,k}$ satisfy

$$|a_{j,k}| \le 10^{-10d} \tag{2-2}$$

for $d \ge 3$.

When $d = 2$, the manifold $\mathcal{M}_\phi$ is associated with

$$\phi(\xi_1, \xi_2) := \xi_1 \xi_2.$$

Note that this is a special of case of (2-1) as we allow $a_{j,k}$ to be zero.

Definition 2.1. Let S be a rectangle in $\mathbb{R}^2$. Take a smooth function $\Xi_S : \mathbb{R}^2 \to \mathbb{R}$ such that:

(1) The support of $\widehat{\Xi_S}$ is contained in $2S$.

(2) $0 \le \widehat{\Xi_S}(\xi_1, \xi_2) \le 1$ for all $(\xi_1, \xi_2) \in \mathbb{R}^2$.

(3) $\widehat{\Xi_S}$ is greater than $\frac{1}{10}$ on the set S.

(4) $\|\Xi_S\|_{L^1} \le 1000$.

Given a function $f : \mathbb{R}^3 \to \mathbb{C}$, define $f_S : \mathbb{R}^3 \to \mathbb{C}$ by

$$f_S(x_1, x_2, x_3) := \int_{\mathbb{R}^2} f(x_1 - y_1, x_2 - y_2, x_3)\, \Xi_S(y_1, y_2)\, dy_1\, dy_2. \tag{2-3}$$

Note that this is a convolution of f and Ξ_S for the first two variables.

Theorem 2.2 (uniform ℓ^2 decoupling for perturbed hyperbolic paraboloids). *Fix $d \ge 2$ and $\epsilon > 0$. Then there exists a sufficiently large number A depending on d and ϵ satisfying the following.*

Let $\mathcal{M}_\phi$ be a manifold of the form (2-1) satisfying (2-2). Then, for any $\delta > 0$, there exists a family $\mathcal{S}_\delta$ of rectangles $S \subset \mathbb{R}^2$ such that:

(1) *The overlapping number is $O(\log \delta^{-1})$ in the sense that*

$$\sum_{S \in \mathcal{S}_\delta} \chi_S \le C_{d,\epsilon} \log(\delta^{-1}). \tag{2-4}$$

(2) *Every S is $(\phi, A\delta)$-flat and of the form of a parallelogram.*

(3) *For all f whose Fourier support is in $N_\delta(\mathcal{M}_\phi)$,*

$$\|f\|_{L^4} \le C_{d,\epsilon} \delta^{-\epsilon} \left(\sum_{S \in \mathcal{S}_\delta} \|f_S\|_{L^4}^2 \right)^{1/2}. \tag{2-5}$$

The constant $C_{d,\epsilon}$ is independent of the choice of ϕ.

The first step of the proof is to construct the family $\mathcal{S}_\delta$. A restriction theory for perturbed hyperbolic paraboloids is developed by [Buschenhenke et al. 2020; 2022; 2023; Guo and Oh 2024]. We use this theory, in particular, the language in [Buschenhenke et al. 2023] to construct the family.

2.1. ***Construction of the family $\mathcal{S}_\delta$.*** To construct a family $\mathcal{S}_\delta$, we will first define null vectors of the Hessian matrix of ϕ. Let us recall the notions in Section 3.1.1 of [Buschenhenke et al. 2023]. Consider two vectors at $\xi \in [0,1]^2$ defined by

$$w_\xi := (-A(\xi), 1) \quad \text{and} \quad v_\xi := (1, -B(\xi)). \tag{2-6}$$

Here A and B are given by

$$A(\xi) = \frac{\phi_{22}(\xi)}{\phi_{12}(\xi) + \sqrt{|H_\phi(\xi)|}} \quad \text{and} \quad B(\xi) = \frac{\phi_{11}(\xi)}{\phi_{12}(\xi) + \sqrt{|H_\phi(\xi)|}}, \tag{2-7}$$

and $H_\phi(\xi)$ is the determinant of the Hessian matrix of ϕ. The functions $\phi_{ij}(\xi)$ are defined by $\partial_i \partial_j \phi(\xi)$. Note that by (2-2) we have

$$|A(\xi)| \le 10^{-5d} \quad \text{and} \quad |B(\xi)| \le 10^{-5d}. \tag{2-8}$$

By (3.7) of [Buschenhenke et al. 2023], the vectors w_ξ and v_ξ have the following properties:

$$(w_\xi) H_\phi(\xi) (w_\xi)^T = 0 \quad \text{and} \quad (v_\xi) H_\phi(\xi) (v_\xi)^T = 0. \tag{2-9}$$

Moreover, by (2-2), w_ξ and v_ξ are linearly independent.

We are now ready to define a family $\mathcal{S}_\delta$. Assume that δ^{-1} is a dyadic number. Let us first define $\mathcal{R}_\delta$; the family $\mathcal{S}_\delta$ will be a subset of it:

$$\mathcal{R}_\delta := \bigcup_{\{1 \le \alpha \le \delta^{-1/2} : \alpha \in 2^{\mathbb{Z}}\}} \mathcal{R}_{\delta\alpha,\alpha^{-1}}. \tag{2-10}$$

We need to define $\mathcal{R}_{\delta\alpha,\alpha^{-1}}$. Every element in the set will be a rectangle with dimension $\delta\alpha \times \alpha^{-1}$. Introduce a parameter β which will be related to the angle of the long direction of the rectangle and the ξ_1-axis. Define

$$\mathcal{R}_{\delta\alpha,\alpha^{-1}} := \bigcup_{\{\beta \in \mathbb{Z} : 0 \le \beta \le \delta^{-1}\alpha^{-2}\pi\}} \mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}. \tag{2-11}$$

Here $\mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$ is a collection of the translated copies of the rectangle of dimension $\delta\alpha \times \alpha^{-1}$ with the angle $\delta\alpha^2\beta$ with respect to the ξ_1-axis, so that the elements of $\mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$ are disjoint and their union contains $[0,1]^2$. So we have defined

$$\mathcal{R}_\delta = \bigcup_{\{1 \le \alpha \le \delta^{-1/2} : \alpha \in 2^{\mathbb{Z}}\}} \bigcup_{\{\beta \in \mathbb{Z} : 0 \le \beta \le \delta^{-1}\alpha^{-2}\pi\}} \mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}. \tag{2-12}$$

The family $\mathcal{S}_\delta$ in Theorem 2.2 will be a subcollection of $\mathcal{R}_\delta$.

Definition 2.3. Let A be a sufficiently large constant, which will be the constant A in Theorem 2.2. We say two rectangles R_1 and R_2 are comparable if

$$R_1 \subset 2AR_2 \quad \text{and} \quad R_2 \subset 2AR_1. \tag{2-13}$$

Remark 2.4. Suppose that R_1 and R_2 are comparable. Let L be an affine transformation such that $L(R_1)$ and $L(R_2)$ are rectangles. Then $L(R_1)$ and $L(R_2)$ are also comparable.

Let us now explain how to choose rectangles. Let A be a large number. Let $R \in \mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$. If the two following conditions are satisfied, we add R to $\mathcal{S}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$:

(1) R is $(\phi, A\delta)$-flat.

(2) For any point $z \in R$, we consider a rectangle centered at z, of dimension $\delta\alpha \times \alpha^{-1}$, with a long direction parallel to the vector w_z. This rectangle is comparable to R.

If the two following conditions are satisfied, we also add R to $\mathcal{S}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$:

(1) R is $(\phi, A\delta)$-flat.

(2) For any point $z \in R$, we consider a rectangle centered at z, of dimension $\delta\alpha \times \alpha^{-1}$, with a long direction parallel to the vector v_z. This rectangle is comparable to R.

Recall that v_z and w_z are defined in (2-6). After this process, we finally have

$$\mathcal{S}_\delta := \bigcup_\alpha \bigcup_\beta \mathcal{S}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}. \tag{2-14}$$

We define a partition of unity associated with $\mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$. Namely, we take smooth functions $\{\Xi_R\}_{R\in\mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}}$ such that

(1) the support of $\widehat{\Xi_R}$ is contained in $2R$,

(2) $0 \le \widehat{\Xi_R}(\xi_1, \xi_2) \le 1$ for all $(\xi_1, \xi_2) \in \mathbb{R}^2$,

(3) $\widehat{\Xi_R}$ is greater than $\frac{1}{10}$ on the set R,

(4) for any $\xi \in [0, 1]^2$, we have

$$\sum_{R\in\mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}} \widehat{\Xi_R}(\xi) = 1, \tag{2-15}$$

(5) $\|\Xi_R\|_{L^1} \le 1000$.

For a function f, define f_R by

$$f_R(x_1, x_2, x_3) := \int_{\mathbb{R}^2} f(x_1 - y_1, x_2 - y_2, x_3)\,\Xi_R(y_1, y_2)\,dy_1\,dy_2. \tag{2-16}$$

2.2. ***Properties of partitions.*** We have defined partitions $\mathcal{S}_\delta$; see (2-14). Now let us study some properties of these partitions.

Proposition 2.5. *Let A be a sufficiently large constant. Then*

$$\mathcal{R}_{\delta^{1/2},\delta^{1/2},0} \subset \mathcal{S}_\delta. \tag{2-17}$$

This says that our partitions contain the squares with the canonical scale. As a remark, for the hyperbolic paraboloid, note that the decoupling

$$\|f\|_{L^4} \le C_\epsilon \delta^{-\epsilon} \left(\sum_{S\in\mathcal{R}_{\delta^{1/2},\delta^{1/2},0}} \|f_S\|_{L^4}^2 \right)^{1/2} \tag{2-18}$$

is false for some functions f whose Fourier transforms are supported on the δ-neighborhood of the hyperbolic paraboloid.

Proof. Take $R \in \mathcal{R}_{\delta^{1/2},\delta^{1/2},0}$. Since R is a square, by the definition of $\mathcal{S}_\delta$, it suffices to prove that R is $(\phi, A\delta)$-flat. By definition, we need to prove that

$$\sup_{u,v\in R} |\phi(u) - \phi(v) - \nabla\phi(u)\cdot(u-v)| \le A\delta. \tag{2-19}$$

Since R is a square of side length $\delta^{1/2}$ and ϕ has the Hessian matrix whose determinant is bounded by 2 (see (2-1) and (2-2)), (2-19) follows from an application of Taylor's theorem. □

Proposition 2.6 (finitely overlapping property). *For any* $\xi \in [0,1]^2$,

$$\#\{S \in \mathcal{S}_\delta : \xi \in S\} \lesssim A\log(\delta^{-1}). \tag{2-20}$$

This proposition gives a proof of the first property of the family $\mathcal{S}_\delta$ in Theorem 2.2 (recall that A is a constant depending only on d and ϵ). Let us give a proof.

Proof. Fix $\xi \in [0,1]^2$. Since the number of dyadic numbers α with $1 \le \alpha \le \delta^{-1/2}$ is $O(\log(\delta^{-1}))$, it suffices to show that, given α,

$$\#\{\beta \in \mathbb{Z} : \text{there exist } S \in \mathcal{S}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta} \text{ s.t. } \xi \in S\} \lesssim A. \tag{2-21}$$

Denote by $S_{1,\xi}$ (resp. $S_{2,\xi}$) the rectangle centered at z, of dimension $\delta\alpha \times \alpha^{-1}$, with a long direction parallel to the vector w_ξ (resp. v_ξ).

Suppose that $\xi \in S$ for some $S \in \mathcal{S}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$. Then S is comparable to either $S_{1,\xi}$ or $S_{2,\xi}$. Without loss of generality, we may assume that S is comparable to $S_{1,\xi}$. Then the angle between the long direction of S and w_ξ is $\lesssim A\delta\alpha^2$. Since $\{\delta\alpha^2\beta\}_{\beta\in\mathbb{Z}}$ is $\delta\alpha^2$-separated, there are only $\sim A$ many β satisfying the property. This gives the proof. □

What follows is a technical lemma, which says that the angle between the longest direction of a rectangle S and the ξ_1-axis (or ξ_2-axis) is small. If S is a square, it is not clear how to define "the longest direction of S". So we prove such a statement under the condition (2-22) to guarantee that S looks like a rectangle quantitatively.

Lemma 2.7. *Let A be a sufficiently large number. Let $S \in \mathcal{S}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$. Suppose that*

$$\frac{\textit{length of the long direction of } S}{\textit{length of the short direction of } S} \ge A^2. \tag{2-22}$$

Then we have

$$\delta\alpha^2\beta \le \frac{1}{10^{5d}}. \tag{2-23}$$

Proof. Suppose that $S \in \mathcal{S}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$. Let us follow the proof of Proposition 2.6. Fix $\xi \in S$. Without loss of generality, we may assume that S is comparable to $S_{1,\xi}$. Then, by Euclidean geometry, the angle between the long direction of S and w_ξ is $\lesssim A\delta\alpha^2$. The condition (2-22) says that $\delta\alpha^2 \le A^{-2}$, so the angle is bounded by $\lesssim A^{-1}$. On the other hand, by (2-6) and the normalization condition (2-2), the angle between w_ξ and the ξ_1-axis is bounded by 10^{-9d}. Since A is sufficiently large, the angle between the long direction of S and the ξ_1-axis is bounded by 10^{-5d}. Therefore, $\delta\alpha^2\beta$, which indicates the angle, is bounded by 10^{-5d}. □

Definition 2.8. Define $D(\delta)$ to be the smallest constant such that

$$\|f\|_{L^4} \le D(\delta)\Big(\sum_{R\in\mathcal{S}_\delta} \|f_R\|_{L^4}^2\Big)^{1/2} \tag{2-24}$$

for all manifolds $\mathcal{M}_\phi$ of the form (2-1) satisfying (2-2) and functions f whose Fourier supports are in $N_\delta(\mathcal{M}_\phi)$.

One of the key properties of the partitions is the parabolic rescaling lemma. This type of rescaling is very crucial in [Bourgain and Demeter 2015; Li and Yang 2021]. We observe that the same rescaling lemma still holds for our collections. The proof uses a basic property of perturbed hyperbolic paraboloids; for example, see Lemma 3.2 of [Guo and Oh 2024].

Lemma 2.9 (parabolic rescaling). *Let $\delta \le \sigma \le 1$. Let $R' \in \mathcal{S}_\sigma$. Then we have*

$$\|f_{R'}\|_{L^4} \le C\,D(\sigma^{-1}\delta)\Big(\sum_{R\in\mathcal{S}_\delta} \|f_R\|_{L^4}^2\Big)^{1/2} \tag{2-25}$$

for all manifolds $\mathcal{M}_\phi$ of the form (2-1) satisfying (2-2) and functions f whose Fourier supports are in $N_\delta(\mathcal{M}_\phi)$.

Proof. Suppose that R' has dimension $\sigma\alpha \times \alpha^{-1}$ for some $1 \le \alpha \le \sigma^{-1}$. Then the Fourier support of $f_{R'}$ is contained in $2R' \times \mathbb{R}$. By using the triangle inequality, we may assume that the Fourier support of $f_{R'}$ is contained in $B \times \mathbb{R}$, where B is a box with dimension $\sigma\alpha \times \alpha^{-1}$. By translation and rotation, we may assume that the Fourier support of $f_{R'}$ is $[0, \alpha^{-1}] \times [0, \sigma\alpha]$. By abusing notation, let us still call this box R'. After the change of variables, our phase function changes. Let us denote the new phase function by

$$\phi_0(\xi_1, \xi_2) := \xi_1\xi_2 + b_{2,0}\xi_1^2 + b_{0,2}\xi_2^2 + \sum_{3\le j+k\le d} b_{j,k}\xi_1^j\xi_2^k. \tag{2-26}$$

For convenience, we introduce $b_{1,1} = 0$. Note that

$$|b_{j,k}| \lesssim 1, \quad 2 \le j+k \le d. \tag{2-27}$$

Since R' is $(\phi_0, A\sigma)$-flat, by definition,

$$\sup_{u,v\in R'} |\phi_0(u) - \phi_0(v) - \nabla\phi_0(u)\cdot(u-v)| \le A\sigma. \tag{2-28}$$

Take $u = (0,0)$ and $v = (t,0)$. Since $\nabla\phi_0(u) = 0$, this inequality becomes

$$\sup_{0\le t\le\alpha^{-1}} \Big|\sum_{j=2}^{d} b_{j,0}t^j\Big| \le A\sigma. \tag{2-29}$$

By [Li and Yang 2021, Proposition 7.1], this inequality gives

$$|b_{j,0}| \lesssim \sigma\alpha^j, \quad 2 \le j \le d. \tag{2-30}$$

Similarly, we take $u = (0,0)$ and $v = (0,t)$, where $0 \le t \le \sigma\alpha$. Then, by the same reasoning, we obtain

$$|b_{0,j}| \lesssim \sigma(\sigma\alpha)^{-j}, \quad 2 \le j \le d. \tag{2-31}$$

We next do rescaling. Define $L(\xi,\eta) := (\alpha^{-1}\xi, \sigma\alpha\eta)$ and

$$\tilde{\phi}(\xi,\eta) := \sigma^{-1}\phi_0(L(\xi,\eta)). \tag{2-32}$$

The function $\tilde{\phi}(\xi,\eta)$ can be rewritten as

$$\xi_1\xi_2 + (\sigma^{-1}\alpha^{-2}b_{2,0})\xi_1^2 + (\sigma\alpha^2 b_{0,2})\xi_2^2 + \sum_{3\le j+k\le d} (\sigma^{k-1}\alpha^{k-j}b_{j,k})\xi_1^j\xi_2^k. \tag{2-33}$$

To apply the induction hypothesis, we will prove

$$|b_{j,k}| \lesssim \sigma^{-k+1}\alpha^{-k+j}, \quad 2\le j+k\le d. \tag{2-34}$$

The cases for $j=0$ or $k=0$ follow from (2-30) and (2-31). Hence we may assume that $j\ge 1$ and $k\ge 1$.

Let us consider the subcase $k\le j$. By the inequality $1\le\alpha\le\sigma^{-1}$, we have $1\le\alpha^{-k+j}$. So what we need to prove follows from

$$|b_{j,k}| \lesssim \sigma^{-k+1}. \tag{2-35}$$

Since $k\ge 1$ and $\sigma\le 1$, this follows from (2-27). Let us next consider the subcase $k\ge j$. By the inequality $1\le\alpha\le\sigma^{-1}$, we have $\sigma^{k-j}\le\alpha^{-k+j}$. So what we need to prove follows from

$$|b_{j,k}| \lesssim \sigma^{-j+1}. \tag{2-36}$$

Since $j\ge 1$ and $\sigma\le 1$, this follows from (2-27).

We have now proved (2-34). So the function (2-33) can be rewritten as

$$\xi_1\xi_2 + c_{2,0}\xi_1^2 + c_{0,2}\xi_2^2 + \sum_{3\le j+k\le d} c_{j,k}\xi_1^j\xi_2^k, \tag{2-37}$$

where

$$|c_{j,k}| \lesssim 1, \quad 2\le j+k\le d. \tag{2-38}$$

By applying a linear transformation, we may assume that $c_{2,0}=c_{0,2}=0$. Moreover, by using the triangle inequality, we may assume that $|c_{j,k}|\le 10^{-10d}$. By abusing the notation, let us still denote by $\tilde{\phi}$ the new phase function. By (2-32) and a change of variables on physical variables, the δ-neighborhood of $\mathcal{M}_\phi$ becomes the $\sigma^{-1}\delta$-neighborhood of $\mathcal{M}_{\tilde{\phi}}$. For functions g whose Fourier supports are in $N_{\sigma^{-1}\delta}(\mathcal{M}_{\tilde{\phi}})$, we have

$$\|g\|_{L^4} \le D(\sigma^{-1}\delta)\Big(\sum_{R''\in\mathcal{S}_{\sigma^{-1}\delta}} \|g_{R''}\|_{L^4}^2\Big)^{1/2}. \tag{2-39}$$

We next rescale back to the original variables. Given $R''\in\mathcal{S}_{\sigma^{-1}\delta}$, by definition, we have

$$\sup_{u,v\in R''} |\tilde{\phi}(u) - \tilde{\phi}(v) - \nabla\tilde{\phi}(u)\cdot(u-v)| \le A\sigma^{-1}\delta. \tag{2-40}$$

By (2-32), we can see that $L^{-1}(R'')$ is $(\phi, A\delta)$-flat. To conclude that $L^{-1}(R'')\in\mathcal{S}_\delta$, it remains to show the following property: For any point $z\in L^{-1}(R'')$, we consider a rectangle centered at z, of dimension $\sigma\alpha\times\alpha^{-1}$, with a long direction parallel to the vector v_z; let us denote this by R_z. This rectangle is comparable to $L^{-1}(R'')$.

Let us prove the property. Fix $z \in L^{-1}(R'')$, and let v_z be a vector associated with the phase ϕ. By Remark 2.4, it suffices to show that $L(R_z)$ is comparable to R''. After calculating the Hessian matrix, we see that $L(v_z)$ is comparable to the vector $v_{L(z)}$ associated with the phase function $\tilde{\phi}$. Since R'' belongs to $\mathcal{S}_{\sigma^{-1}\delta}$, this gives the desired property. □

2.3. ***The broad-narrow analysis.*** In this subsection, we introduce a multilinear decoupling and show that a linear decoupling constant is comparable to a multilinear decoupling constant up to epsilon loss (Theorem 2.12). This framework is introduced by [Bourgain and Demeter 2015].

Definition 2.10. Let $n(\xi)$ be the unit normal vector of $\mathcal{M}_\phi$ at the point $(\xi, \phi(\xi))$. We say three points $\xi_{(1)}, \xi_{(2)}, \xi_{(3)} \in [0,1]^2$ are N^{-1}-transverse if

$$|n(\xi_{(1)}) \wedge n(\xi_{(2)}) \wedge n(\xi_{(3)})| \geq N^{-1}. \tag{2-41}$$

Three squares $\tau_1, \tau_2, \tau_3 \subset [0,1]^2$ are called N^{-1}-transverse if every triple $\xi_{(i)} \in \tau_i$ is N^{-1}-transverse.

Definition 2.11. Define $D_{\text{mul}}(\delta, N^{-1})$ to be the smallest constant satisfying the following: for any N^{-1}-transverse squares $\tau_1, \tau_2, \tau_3 \in \mathcal{R}_{a,a,0}$ for $\delta^{1/2} \leq a \leq 1$,

$$\left\| \prod_{i=1}^{3} f_{\tau_i} \right\|_{L^4} \leq D_{\text{mul}}(\delta, N^{-1}) \Big(\sum_{R \in \mathcal{S}_\delta} \|f_R\|_{L^4}^2 \Big)^{1/2} \tag{2-42}$$

for all manifolds $\mathcal{M}_\phi$ of the form (2-1) satisfying (2-2) and functions f whose Fourier supports are in $N_\delta(\mathcal{M}_\phi)$.

Recall that the definition of f_{τ_i} is given in the paragraph below (2-14). The main theorem of this subsection follows.

Theorem 2.12. *Given $\epsilon > 0$, there exists a sufficiently large number N depending on ϵ such that, for all $\delta > 0$, we have*

$$D(\delta) \leq C_\epsilon \delta^{-\epsilon} \big(\sup_{\delta \leq \delta' \leq 1} D_{\text{mul}}(\delta', N^{-1}) + 1 \big). \tag{2-43}$$

Proof. We do a broad-narrow analysis by [Bourgain and Guth 2011]. Let K be a sufficiently large number, which will be determined later. Take a sufficiently large constant K_1 such that K is sufficiently large compared to K_1. Fix a ball B_K. We write

$$f = \sum_{\tau \in \mathcal{R}_{K^{-1/2}, K^{-1/2}, 0}} f_\tau. \tag{2-44}$$

Recall the definition in (2-11). Each τ is a square of side length $K^{-1/2}$. Consider a collection of squares

$$\mathcal{C} := \{\tau \in \mathcal{R}_{K^{-1/2}, K^{-1/2}, 0} : \|f_\tau\|_{L^4(B_K)} \geq K^{-100} \|f\|_{L^4(B_K)}\}. \tag{2-45}$$

We will consider several cases. By a pigeonholing argument, we can see that $\mathcal{C}$ is nonempty. Take $\tau_1 \in \mathcal{C}$. Consider the first case that all the squares $\tau \in \mathcal{C}$ are in the $(K_1)^{-1}$-neighborhood of τ_1. Denote

by τ_1' the neighborhood. Then, by the triangle inequality, we have

$$\|f\|_{L^4(B_K)} \le \Big\| \sum_{\{\tau\in\mathcal{R}_{K^{-1/2},K^{-1/2},0}:\tau\cap\tau_1'\neq\varnothing\}} f_\tau \Big\|_{L^4(B_K)} + \sum_{\tau\notin\mathcal{C}} \|f_\tau\|_{L^4(B_K)}. \tag{2-46}$$

By the definition of $\mathcal{C}$, this gives

$$\|f\|_{L^4(B_K)} \le 2\Big\| \sum_{\{\tau\in\mathcal{R}_{K^{-1/2},K^{-1/2},0}:\tau\cap\tau_1'\neq\varnothing\}} f_\tau \Big\|_{L^4(B_K)}. \tag{2-47}$$

The set τ_1' depends on a choice of B_K. Hence, by summing over all possible squares, we have

$$\|f\|_{L^4(B_K)} \lesssim \Big(\sum_{\tau'\in\mathcal{R}_{K_1^{-1},K_1^{-1},0}} \|f_{\tau'}\|_{L^4(B_K)}^4 \Big)^{1/4}. \tag{2-48}$$

We next consider the case that there exists $\tau_2\in\mathcal{C}$ outside of the $(K_1)^{-1}$-neighborhood of τ_1. There are two subcases. Suppose that there is $\tau_3\in\mathcal{C}$ such that τ_1, τ_2, τ_3 are $10K^{-1}$-transverse. Then we have

$$\|f\|_{L^4(B_K)} \lesssim K^{100} \prod \|f_{\tau_i}\|_{L^4(B_K)}. \tag{2-49}$$

By an application of a randomization argument and the uncertainty principle, we have

$$\|f\|_{L^4(B_K)} \lesssim K^{200} \Big\| \prod \tilde{f}_{\tau_i} \Big\|_{L^4(w_{B_K})}, \tag{2-50}$$

where $\tilde{f}_{\tau_i}$ is a modulation of f_{τ_i}. The definition of w_{B_K} is given in (1-25). We refer to (2.5)–(2.8) of [Guth et al. 2023] for the details of the argument.

Lastly, suppose that there does not exist such a τ_3. Denote by $\xi_{(i)}$ the center of τ_i for $i=1,2$. By Definition 2.10, all the elements $\tau\in\mathcal{C}$ are contained in the set

$$Z_0 := \{\xi\in[0,1]^2 : |n(\xi_{(1)})\wedge n(\xi_{(2)})\wedge n(\xi)| < 100K^{-1}\}. \tag{2-51}$$

Here $n(\xi)$ is the unit normal vector of the manifold (2-1) at the point $(\xi,\phi(\xi))$. By the definition of $\mathcal{C}$, we can obtain

$$\|f\|_{L^4(B_K)} \lesssim \Big\| \sum_{\tau\subset Z_0} f_\tau \Big\|_{L^4(B_K)}. \tag{2-52}$$

We apply a uniform ℓ^2 decoupling theorem for the set Z_0 of [Li and Yang 2021, Theorem 3.1] and obtain

$$\|f\|_{L^4(B_K)} \lesssim_\epsilon K^\epsilon \Big(\sum_S \|f_S\|_{L^4(w_{B_K})}^2 \Big)^{1/2} \tag{2-53}$$

for any $\epsilon>0$. Here $S\subset[0,1]^2$ is a rectangular box, and the collection $\{S\}_S$ has bounded overlap. Moreover,

$$\bigcup_S S \subset \{\xi\in[0,1]^2 : |n(\xi_{(1)})\wedge n(\xi_{(2)})\wedge n(\xi)| \lesssim K^{-1}\}. \tag{2-54}$$

The function f_S is the Fourier restriction of f to the rectangle $S \times \mathbb{R}$.[1] Define

$$Z := \{\xi \in [0,1]^2 : |n(\xi_{(1)}) \wedge n(\xi_{(2)}) \wedge n(\xi)| = 0\}. \tag{2-55}$$

Then, by the separation between τ_1 and τ_2 and by the normalization conditions (2-1) and (2-2), one can see that

$$(2\text{-}54) \subset N_{CK_1/K}(Z). \tag{2-56}$$

Let us state this as a lemma.

Lemma 2.13. $\{\xi \in [0,1]^2 : |n(\xi_{(1)}) \wedge n(\xi_{(2)}) \wedge n(\xi)| \lesssim K^{-1}\} \subset N_{CK_1/K}(Z)$.

We postpone the proof of this lemma and continue the proof of Theorem 2.12. By the lemma, each S in (2-54) is contained in a rectangle with dimension $1 \times CK_1K^{-1}$. Let us fix S. Suppose that

$$\frac{\text{length of the long direction of S}}{\text{length of the short direction of S}} \le K_1. \tag{2-57}$$

Then we decompose S into squares of side length equal to the length of the short direction of S. Then the number of squares is bounded by $O(K_1)$. By Proposition 2.5, we have

$$\|f_S\|_{L^4} \lesssim (K_1)^{1/4} \left(\sum_{\{S' \in \mathcal{S}_X : S' \cap 2S \neq \varnothing\}} \|f_{S'}\|_{L^4}^2 \right)^{1/2}, \tag{2-58}$$

where $K^{-2} \lesssim X \lesssim (K_1K^{-1})^2$. Since K is sufficiently large compared to K_1, the term $(K_1)^{1/4}$ on the right-hand side will not make any trouble.

Let us consider the case that

$$\frac{\text{length of the long direction of S}}{\text{length of the short direction of S}} \ge K_1. \tag{2-59}$$

Assume that K_1 is large enough that $K_1 > A^2$. By translation and rotation, we may assume that S is contained in $S' := [0,1] \times [0, CK_1K^{-1}]$. By Lemma 2.7, after some change of variables, we may write the new phase function as

$$\xi_1\xi_2 + a\xi_1^2 + b\xi_2^2 + \sum_{3 \le j+k \le d} b_{j,k}\xi_1^j\xi_2^k, \tag{2-60}$$

where

$$|a| + |b| + \sum_{3 \le j+k \le d} |b_{j,k}| \le 10^{-d}. \tag{2-61}$$

We will apply the following lemma.

Lemma 2.14. *Let K_2 be a number such that $1 \le K_2 \le \delta^{-1/2}$. Let $S' = [0,1] \times [0, K_2^{-1}]$. Suppose that the phase function is*

$$\Phi(\xi_1, \xi_2) := \xi_1\xi_2 + a\xi_1^2 + b\xi_2^2 + \sum_{3 \le j+k \le d} b_{j,k}\xi_1^j\xi_2^k, \tag{2-62}$$

[1] There is a minor technical issue. In the work of [Li and Yang 2021], the authors used a characteristic function for the cutoff function. The proof can be modified so that the cutoff is smooth. In our application, we use a smooth cutoff function.

where

$$|a| + |b| + \sum_{3 \le j+k \le d} |b_{j,k}| \le 10^{-d}. \tag{2-63}$$

Then, for any $\epsilon > 0$, *we have*

$$\|f_{S'}\|_{L^4(B_{K_2})} \le C_{\epsilon,d,A} K_2^{\epsilon} \sup_{K_2 \le L \le (K_2)^2} \left(\sum_{\{T \in \mathcal{S}_{L^{-1}} : T \subset 2S'\}} \|f_T\|^2_{L^4(w_{B_{K_2}})} \right)^{1/2} \tag{2-64}$$

for all functions f *whose Fourier support is in* $N_\delta(\mathcal{M}_\Phi)$. *Here* $f_{S'}$ *is the Fourier restriction of* f *to the rectangle* $S' \times \mathbb{R}$.

Let us assume this lemma and finish the proof. By (2-53) and Lemma 2.14, we have

$$\|f\|_{L^4(B_K)} \le C_\epsilon (K_1^{-1}K)^{\epsilon} K^{\epsilon} \sup_{(K_1^{-1}K) \le L \le (K)^2} \left(\sum_{T \in \mathcal{S}_{L^{-1}}} \|f_T\|^2_{L^4(w_{B_K})} \right)^{1/2}. \tag{2-65}$$

To summarize, by considering all possible cases, we have

$$\|f\|_{L^4} \lesssim_\epsilon \left(\sum_{\tau' \in \mathcal{R}_{K_1^{-1},K_1^{-1},0}} \|f_{\tau'}\|^4_{L^4} \right)^{1/4} + K^{200} \left\| \prod_{i=1}^{3} \tilde{f}_{\tau_i} \right\|_{L^4} + K^{\epsilon} \sup_{K_1^{-1}K \le L \le (K)^2} \left(\sum_{T \in \mathcal{S}_L} \|f_T\|^2_{L^4} \right)^{1/2} \tag{2-66}$$

for some transverse τ_i. By Definition 2.11 and Lemma 2.9, we have

$$D(\delta) \lesssim_\epsilon D(K_1^2\delta) + K^{200} D_{\mathrm{mul}}(\delta, 10K^{-1}) + K^{\epsilon} \sup_{(K_1^{-1}K) \le L \le (K)^2} D(L\delta). \tag{2-67}$$

We apply this inequality to the first and third terms on the right-hand side of (2-67) repeatedly and obtain the desired result. □

Proof of Lemma 2.13. For simplicity, we introduce $F(\xi) := n(\xi_{(1)}) \wedge n(\xi_{(2)}) \wedge n(\xi)$. By the definition of the normal vector $n(\xi)$, this can be rewritten as

$$F(\xi) = c \begin{vmatrix} \partial_1\phi(\xi_{(1)}) & \partial_2\phi(\xi_{(1)}) & -1 \\ \partial_1\phi(\xi_{(2)}) & \partial_2\phi(\xi_{(2)}) & -1 \\ \partial_1\phi(\xi) & \partial_2\phi(\xi) & -1 \end{vmatrix} \tag{2-68}$$

for some $|c| \sim 1$. After routine computations, this is equal to

$$-\partial_1\phi(\xi)(\partial_2\phi(\xi_{(1)}) - \partial_2\phi(\xi_{(2)})) + \partial_2\phi(\xi)(\partial_1\phi(\xi_{(1)}) - \partial_1\phi(\xi_{(2)})) \\ - \partial_1\phi(\xi_{(1)})\partial_2\phi(\xi_{(2)}) + \partial_1\phi(\xi_{(2)})\partial_2\phi(\xi_{(1)}). \tag{2-69}$$

Recall that $\xi_{(1)}$ and $\xi_{(2)}$ are fixed. For simplicity, we write the function as

$$F(\xi) = A\partial_1\phi(\xi) + B\partial_2\phi(\xi) + C. \tag{2-70}$$

We first claim that

$$\max(|\partial_2\phi(\xi_{(1)}) - \partial_2\phi(\xi_{(2)})|, |\partial_1\phi(\xi_{(1)}) - \partial_1\phi(\xi_{(2)})|) \gtrsim \frac{1}{K_1}. \tag{2-71}$$

The claim is equivalent to

$$|\nabla\phi(\xi_{(1)}) - \nabla\phi(\xi_{(2)})| \gtrsim \frac{1}{K_1}. \tag{2-72}$$

By Taylor's theorem, note that

$$|\nabla^2\phi(\xi_{(1)}) \cdot (\xi_{(1)} - \xi_{(2)})| \sim |\nabla\phi(\xi_{(1)}) - \nabla\phi(\xi_{(2)})|. \tag{2-73}$$

Since the absolute value of eigenvalues of the Hessian matrix of ϕ is comparable to 1 and by the separation $|\xi_{(1)} - \xi_{(2)}| \gtrsim 1/K_1$, we have

$$|\nabla^2\phi(\xi_{(1)}) \cdot (\xi_{(1)} - \xi_{(2)})| \gtrsim \frac{1}{K_1}. \tag{2-74}$$

This completes the proof of the claim.

Let us continue the proof of Lemma 2.13. Recall (2-70). Let us consider the case $|A| \le |B|$. The case $|A| \ge |B|$ can be dealt with similarly. To prove Lemma 2.13, let us fix $\xi \in [0,1]^2$ and suppose that

$$F(\xi) = X, \quad |X| \lesssim K^{-1}. \tag{2-75}$$

It suffices to show that there exists $\alpha \in \mathbb{R}$ such that

- $|\alpha| \lesssim K_1 K^{-1}$,
- $F(\xi + (\alpha, 0)) = 0$.

By Taylor's theorem, since F is a polynomial of degree $d-1$,

$$F(\xi + (\alpha, 0)) = F(\xi) + \partial_1 F(\xi)\alpha + \sum_{2 \le j \le d-1} \frac{\partial_1^j F(\xi)}{j!}\alpha^j. \tag{2-76}$$

By the expression (2-70),

$$\partial_1^j F(\xi) = A\partial_1^{j+1}\phi(\xi) + B\partial_1^j\partial_2\phi(\xi). \tag{2-77}$$

By the normalization (2-1) and (2-2), we have

$$\begin{aligned} F(\xi + (\alpha, 0)) &= F(\xi) + (\partial_1 F(\xi))\alpha + O(B\alpha^2) \\ &= X + (\partial_1 F(\xi))\alpha + O(B\alpha^2). \end{aligned} \tag{2-78}$$

Note that $|\partial_1 F(\xi)| \sim |B|$. By (2-71), we have $|B| \gtrsim K_1^{-1}$, and we can find $|\alpha| \lesssim K_1 K^{-1}$ such that

$$X + (\partial_1 F(\xi))\alpha + O(B\alpha^2) = 0. \qquad \square$$

Proof of Lemma 2.14. We use an induction on K_2. By taking $C_{\epsilon,d,A}$ sufficiently large, we may assume that Lemma 2.14 is true for sufficiently large K_2 (compared to A).

We write our phase function (2-62) as

$$A(\xi_1) + B(\xi_1, \xi_2) := \left(a\xi_1^2 + \sum_{3 \le j \le d} b_{j,0}\xi_1^j\right) + (\xi_1\xi_2 + b\xi_2^2 + \cdots). \tag{2-79}$$

On the set $(\xi_1, \xi_2) \in S'$,

$$A(\xi_1) + B(\xi_1, \xi_2) = A(\xi_1) + O(K_2^{-1}). \tag{2-80}$$

By [Li and Yang 2021, Proposition 7.1],

$$\sup_{0\le\xi_1\le 1} |A(\xi_1)| \sim_d \max\Big(|a|, \max_j |b_{j,0}|\Big). \tag{2-81}$$

Let us consider the case that the left-hand side of (2-81) is smaller than AK_2^{-1} for some fixed large number A. We would like to show that $S' \in \mathcal{S}_{(K_2)^{-1}}$. First, note that S' is (ϕ, K_2^{-1})-flat. Also, for any $z \in S'$, the angle between the ξ_1-axis and v_z is smaller than $O(AK_2^{-1})$. Hence the rectangle, centered at z of dimension $1 \times K_2^{-1}$ with a long direction parallel to the vector v_z, is comparable to S'. So we see that $S' \in \mathcal{S}_{(K_2)^{-1}}$. This already gives the desired result (2-64).

We next consider the case that the left-hand side of (2-81) is larger than $\frac{1}{2}AK_2^{-1}$. We write our function as

$$f(x) = \int_{\mathbb{R}^3} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi. \tag{2-82}$$

We do a change of variables

$$x_3 \mapsto M^{-1}x_3 \quad \text{and} \quad \xi_3 \mapsto M\xi_3,$$

so that the Fourier support of f is contained in the δM-neighborhood of

$$\{M(A(\xi_1) + B(\xi_1, \xi_2)) : \xi_1 \in [0, 1],\ \xi_2 \in [0, K_2^{-1}]\} \tag{2-83}$$

for some $M^{-1} \gtrsim AK_2^{-1}$ such that

$$M \sup_{0\le\xi_1\le 1} |A(\xi_1)| \sim 1.$$

Note that

$$M(A(\xi_1) + B(\xi_1, \xi_2)) = MA(\xi_1) + O(A^{-1}). \tag{2-84}$$

After the change of variables, the ball B_{K_2} on the physical ball becomes a ball of radius K_2/M, which is still larger than A. Then we use a uniform ℓ^2 decoupling for a curve of [Yang 2021, Theorem 1.4] by ignoring the ξ_2-variable and obtain

$$\|f_{S'}\|_{L^4} \le C^{\text{curve}}_{\epsilon/2}(A')^{\epsilon/2}\Big(\sum_{T'} \|f_{T'}\|^2_{L^4}\Big)^{1/2}. \tag{2-85}$$

Here the sidelength of the longest direction of T' is $(A')^{-1}$, which is smaller than or equal to $A^{-1/d}$. The set T' has dimension $(A')^{-1} \times (K_2)^{-1}$. We next do a isotropic rescaling (with translation), and T' becomes a rectangle with dimension $1 \times A'/K_2$. Since $A'/K_2 > 1/K_2$, by applying the induction hypothesis on K_2 and rescaling back, the left-hand side of (2-85) is bounded by

$$C^{\text{curve}}_{\epsilon/2} C_\epsilon (A')^{\epsilon/2}\Big(\frac{K_2}{A'}\Big)^{\epsilon}\Big(\sum_{T} \|f_T\|^2_{L^4}\Big)^{1/2}. \tag{2-86}$$

Recall that $A' \ge A^{1/d}$. Since A is sufficiently large, we have

$$(A')^{-\epsilon/2} \le (C^{\text{curve}}_{\epsilon/2})^{-1},$$

and we can close the induction. □

2.4. ***Bourgain–Demeter-type iteration.*** We have shown that the linear decoupling constant is bounded by the multilinear decoupling constant (Theorem 2.12). The next step is to bound the multilinear decoupling constant. Here is an ingredient.

Lemma 2.15 (ball inflation lemma). *Let ϕ be a function (2-1) satisfying (2-2). Let $p = 4$, $q = \frac{8}{3}$, and $N > 0$. Let τ_1, τ_2, τ_3 be N^{-1}-transverse squares. For any $\rho > 0$, $\epsilon > 0$, and $x_0 \in \mathbb{R}^3$,*

$$\left(\fint_{B(x_0,\rho^{-2})}\left(\prod_{i=1}^{3}\left(\sum_{\{J\in\mathcal{R}_{\rho,\rho,0}:J\subset\tau_i\}}\|f_J\|^2_{L^q_\#(w_{B_{\rho^{-1}}(x)})}\right)^{1/2}\right)^p dx\right)^{1/p}$$
$$\le C_{\epsilon,N}\rho^{-\epsilon}\prod_{i=1}^{3}\left(\sum_{\{J\in\mathcal{R}_{\rho,\rho,0}:J\subset\tau_i\}}\|f_J\|^2_{L^q_\#(w_{B_{\rho^{-2}}(x_0)})}\right)^{1/2} \tag{2-87}$$

for all functions f whose Fourier supports are in $N_{\rho^2}(\mathcal{M}_\phi)$.

The proof of the ball inflation lemma is standard. We refer to [Bourgain et al. 2016, Theorem 6.6] for the details; see also [Bourgain and Demeter 2017b, Theorem 9.2].

We are ready to give a proof of Theorem 2.2. Let Γ be the smallest constant such that, for every $\epsilon > 0$, we have

$$D(\delta) \le C_\epsilon \delta^{-\Gamma-\epsilon} \quad \text{for every dyadic } \delta < 1, \tag{2-88}$$

where C_ϵ is a constant depending on ϵ. Our goal is to prove $\Gamma = 0$.

We introduce some notation. Take $q := \frac{8}{3}$, and note that

$$\frac{1}{q} = \frac{1/2}{2} + \frac{1/2}{4}. \tag{2-89}$$

Define

$$\begin{aligned}
\tilde{A}_2(b) &:= \left\|\prod_{i=1}^{3}\left(\sum_{\{J\in\mathcal{R}_{\delta^{b/2},\delta^{b/2},0}:J\subset\tau_i\}}\|f_J\|^2_{L^2_\#(w_{B(x,\delta^{-b})})}\right)^{1/2}\right\|_{L^4_{x\in\mathbb{R}^3}},\\
\tilde{A}_q(b) &:= \left\|\prod_{i=1}^{3}\left(\sum_{\{J\in\mathcal{R}_{\delta^{b/2},\delta^{b/2},0}:J\subset\tau_i\}}\|f_J\|^2_{L^q_\#(w_{B(x,\delta^{-b/2})})}\right)^{1/2}\right\|_{L^4_{x\in\mathbb{R}^3}},\\
\tilde{A}_4(b) &:= \left\|\prod_{i=1}^{3}\left(\sum_{\{J\in\mathcal{R}_{\delta^{b/2},\delta^{b/2},0}:J\subset\tau_i\}}\|f_J\|^2_{L^4_\#(w_{B(x,\delta^{-b})})}\right)^{1/2}\right\|_{L^4_{x\in\mathbb{R}^3}},
\end{aligned} \tag{2-90}$$

where $0 < b < 1$. For $0 < b < 1$ and $* = 2, q, 4$, we let $a_*(b)$ be the infimum over all exponents a satisfying

$$\tilde{A}_*(b) \lesssim_{a,N} \delta^{-a}\left(\sum_{R\in\mathcal{S}_\delta}\|f_R\|^2_{L^4(\mathbb{R}^3)}\right)^{1/2} \tag{2-91}$$

for every $\delta > 0$, every N^{-1}-transverse squares τ_i, and every choice of a function f. Lastly, define

$$a_* := \liminf_{b\to 0}\frac{\Gamma - a_*(b)}{b} \tag{2-92}$$

for $* = 2, q, 4$.

Lemma 2.16. *We have the following inequalities*:

(1) (*setup*) $a_* < \infty$ *for* $* = 2, q, 4$,

(2) (*rescaling*) $a_4 \geq \Gamma$,

(3) (L^2-*orthogonality*) $a_2 \geq 2a_q$,

(4) (*ball inflation*) $a_q \geq \frac{1}{2}a_4 + \frac{1}{2}a_2$.

Notice that by combining all the inequalities in the lemma, we obtain

$$\Gamma \leq a_4 \leq 2a_q - a_2 \leq 0. \tag{2-93}$$

Hence, in order to prove Theorem 2.2, it suffices to prove Lemma 2.16.

Proof of Lemma 2.16. Let us show inequality (1). It suffices to show

$$\Gamma \leq Cb + a_*(b) \tag{2-94}$$

for some constant C. By the essentially constant property and Bernstein's inequality, we have

$$\left\| \prod_{i=1}^{3} f_{\tau_i} \right\|_{L^4} \lesssim \delta^{-Cb} \tilde{A}_*(b) \tag{2-95}$$

for any $* = 2, q, 4$ and $0 < b < 1$. By the definition of $a_*(b)$ (see (2-91)), we have

$$\left\| \prod_{i=1}^{3} f_{\tau_i} \right\|_{L^4} \lesssim \delta^{-Cb-a_*(b)} \left(\sum_{R \in \mathcal{S}_\delta} \|f_R\|^2_{L^4(\mathbb{R}^3)} \right)^{1/2}. \tag{2-96}$$

Hence we have

$$D_{\mathrm{mul}}(\delta, N^{-1}) \lesssim \delta^{-Cb-a_*(b)}. \tag{2-97}$$

Combining this with Theorem 2.12 gives

$$D(\delta) \lesssim_\epsilon \delta^{-\epsilon} \delta^{-Cb-a_*(b)}. \tag{2-98}$$

Since $\epsilon > 0$ is arbitrary, by the definition of Γ, we obtain (2-94).

Let us next show inequality (2). By the definition of a_4, it follows from

$$a_4(b) \leq \Gamma \cdot (1 - b). \tag{2-99}$$

By Hölder's inequality and Minkowski's inequality, we have

$$\tilde{A}_4(b) \lesssim \prod_{i=1}^{3} \left(\sum_{\{J \in \mathcal{R}_{\delta^{b/2}, \delta^{b/2}, 0} : J \subset \tau_i\}} \|f_J\|^2_{L^4(\mathbb{R}^3)} \right)^{1/2}. \tag{2-100}$$

By the parabolic rescaling lemma (Lemma 2.9), this is further bounded above by

$$D(\delta^{1-b}) \prod_{i=1}^{3} \left(\sum_{\{J \in \mathcal{S}_\delta : J \subset \tau_i\}} \|f_J\|^2_{L^4(\mathbb{R}^3)} \right)^{1/2} \lesssim D(\delta^{1-b}) \left(\sum_{J \in \mathcal{S}_\delta} \|f_J\|^2_{L^4(\mathbb{R}^3)} \right)^{1/2}. \tag{2-101}$$

By (2-88) and (2-91), we have

$$\delta^{-a_4(b)} \lesssim \delta^{-\Gamma\cdot(1-b)}. \tag{2-102}$$

This gives (2-99).

Let us move onto inequality (3). By the L^2-orthogonality and Hölder's inequality, we obtain

$$\begin{aligned}\tilde{A}_2(b) &\lesssim \left\| \prod_{i=1}^{3} \Bigg(\sum_{\{J\in\mathcal{R}_{\delta^b,\delta^b,0}: J\subset\tau_i\}} \|f_J\|^2_{L^2_\#(w_{B(x,\delta^{-b})})} \Bigg)^{1/2} \right\|_{L^4_{x\in\mathbb{R}^3}} \\ &\lesssim \left\| \prod_{i=1}^{3} \Bigg(\sum_{\{J\in\mathcal{R}_{\delta^b,\delta^b,0}: J\subset\tau_i\}} \|f_J\|^2_{L^q_\#(w_{B(x,\delta^{-b})})} \Bigg)^{1/2} \right\|_{L^4_{x\in\mathbb{R}^3}}. \end{aligned} \tag{2-103}$$

The last expression is $\tilde{A}_q(2b)$, so we have

$$\delta^{-a_2(b)} \lesssim \delta^{-a_q(2b)}, \tag{2-104}$$

or equivalently,

$$a_2(b) \le a_q(2b). \tag{2-105}$$

After some computations, this gives

$$a_2 \ge 2a_q. \tag{2-106}$$

This completes the proof of inequality (3).

Lastly, let us show inequality (4). By the ball inflation lemma (Lemma 2.15),

$$\tilde{A}_q(b) \lesssim_\epsilon \delta^{-\epsilon} \left\| \prod_{i=1}^{3} \Bigg(\sum_{\{J\in\mathcal{R}_{\delta^{b/2},\delta^{b/2},0}: J\subset\tau_i\}} \|f_J\|^2_{L^q_\#(w_{B(x,\delta^{-b})})} \Bigg)^{1/2} \right\|_{L^4_{x\in\mathbb{R}^3}}. \tag{2-107}$$

By Hölder's inequality and (2-89),

$$\tilde{A}_q(b) \lesssim_\epsilon \delta^{-\epsilon} \tilde{A}_2(b)^{1/2} \tilde{A}_4(b)^{1/2}. \tag{2-108}$$

By the definition of $a_*(b)$, this gives

$$\delta^{-a_q(b)} \lesssim \delta^{-\epsilon} \delta^{-(a_2(b)+a_4(b))/2}. \tag{2-109}$$

After some computations, this gives

$$a_q \ge \tfrac{1}{2}a_4 + \tfrac{1}{2}a_2. \tag{2-110}$$

This completes the proof of inequality (4). □

3. General manifolds

We have proved the ℓ^2 decoupling for perturbed hyperbolic paraboloids. We will use this decoupling as a black box and prove Theorem 3.1. This section does not contain any novelty, and we simply follow the argument of [Li and Yang 2021] in Appendix A.

The proof of Theorem 1.2 can be reduced to that of ℓ^2 decoupling for polynomials (Theorem 3.1). We refer to Section 2.3 of [Li and Yang 2021] for the details.

Theorem 3.1. *Fix $d \geq 2$ and $\epsilon > 0$. Then there exists a sufficiently large number A depending on d and ϵ satisfying the following.*

Let ϕ be a polynomial of two variables of degree d with coefficients bounded by 1. For any $\delta > 0$, there exists a collection $\mathcal{S}_\delta$ of finitely overlapping sets S such that:

(1) *The overlapping number is $O(\delta^{-\epsilon})$ in the sense that*

$$\sum_{S \in \mathcal{S}_\delta} \chi_S \leq C_{d,\epsilon} \delta^{-\epsilon}. \tag{3-1}$$

(2) *S is $(\phi, A\delta)$-flat and of the form of a parallelogram.*

(3) *We have*

$$\| f \|_{L^4} \leq C_{d,\epsilon} \delta^{-\epsilon} \Big(\sum_{S \in \mathcal{S}_\delta} \| f_S \|_{L^4}^2 \Big)^{1/2} \tag{3-2}$$

for all f whose Fourier support is in $N_\delta(\mathcal{M}_\phi)$.

The constant $C_{d,\epsilon}$ is independent of the choice of ϕ.

To prove Theorem 3.1, we use Theorem 2.2 as a black box. Theorem 2.2 is a decoupling theorem for a phase function (2-1) satisfying (2-2), but by a simple change of variables, this theorem can be generalized to that for a polynomial function ϕ with bounded coefficients such that

$$|H_\phi(\xi)| \gtrsim 1 \tag{3-3}$$

for all $\xi \in [0,1]^2$. Here H_ϕ is the determinant of the Hessian matrix of ϕ.

3.1. ***Sketch of the proof of Theorem 3.1.*** Let M be a sufficiently large number, which will be determined later; see the end of Section 3. The proof is composed of three steps.

Step 1: *Dichotomy*: *a curved part and a flat part.* Decompose $[0,1]^2$ according to the size of $|H_\phi(\xi)|$:

$$\begin{aligned} S_{\text{curved}} &:= \{(\xi_1, \xi_2) \in [0,1]^2 : |H_\phi(\xi_1, \xi_2)| > M^{-1}\}, \\ S_{\text{flat}} &:= \{(\xi_1, \xi_2) \in [0,1]^2 : |H_\phi(\xi_1, \xi_2)| < M^{-1}\}. \end{aligned} \tag{3-4}$$

Introduce a parameter $M \ll M_1$. Decompose $[0,1]^2$ into squares of side length M_1. Then on each square, one can see that $|H_\phi| > \frac{1}{2} M^{-1}$ because $M_1^{-1} \ll M^{-1}$ and ∇H_ϕ is bounded. By the triangle inequality,

$$\| f \|_{L^4} \leq \Big\| \sum_{\{|\tau| = M_1^{-1} : \tau \cap S_{\text{curved}} \neq \varnothing\}} f_\tau \Big\|_{L^4} + \Big\| \sum_{\{|\tau| = M_1^{-1} : \tau \subset S_{\text{flat}}\}} f_\tau \Big\|_{L^4}. \tag{3-5}$$

After applying the triangle inequality, we apply Theorem 2.2 (see also the discussion below Theorem 3.1) to the first term on the right-hand side, and this gives

$$\Big\| \sum_{\{|\tau| = M_1^{-1} : \tau \cap S_{\text{curved}} \neq \varnothing\}} f_\tau \Big\|_{L^4} \lesssim M_1 \Big(\sum_{\{|\tau| = M_1^{-1} : \tau \cap S_{\text{curved}} \neq \varnothing\}} \| f_\tau \|_{L^4}^2 \Big)^{1/2} \lesssim_\epsilon M_1 \delta^{-\epsilon} \Big(\sum_{\{S \in \mathcal{S}_\delta\}} \| f_S \|_{L^4}^2 \Big)^{1/2}. \tag{3-6}$$

Since M_1 is a fixed number independent of δ, this already gives the desired bound. It remains to bound the second term on the right-hand side of (3-5).

Step 2: *Analysis of the flat part.* We next bound the term

$$\left\| \sum_{\{|\tau|=M_1^{-1}:\tau\subset S_{\text{flat}}\}} f_\tau \right\|_{L^4}. \tag{3-7}$$

By the generalized two-dimensional uniform ℓ^2 decoupling for polynomials, see Theorem 3.1 in [Li and Yang 2021], we can cover S_{flat} by rectangles T satisfying

$$T \subset \{(\xi_1,\xi_2)\in[0,1]^2 : |H_\phi(\xi_1,\xi_2)| \lesssim M^{-1}\} \tag{3-8}$$

and

$$\left\| \sum_{\{|\tau|=M_1^{-1}:\tau\subset S_{\text{flat}}\}} f_\tau \right\|_{L^4} \lesssim_\epsilon M^\epsilon \left(\sum_T \|f_T\|_{L^4}^2\right)^{1/2}. \tag{3-9}$$

Let us fix T. Take an affine transformation L such that it maps $[0,1]^2$ to T. Take $\phi_1 := \phi\circ L$. Then one can see that

$$|H_{\phi_1}(\xi)| \lesssim M^{-1} \tag{3-10}$$

for all $\xi\in[0,1]^2$. By [loc. cit., Proposition 7.1], all the coefficients of $H_{\phi_1}(\xi)$ are bounded above by $\sim M^{-1}$. Hence we can apply [loc. cit., Theorem 3.2] and obtain a rotation ρ such that

$$\phi_2(\xi_1,\xi_2) := \phi_1\circ\rho(\xi_1,\xi_2) = a(\xi_1) + M^{-\alpha}b(\xi_1,\xi_2). \tag{3-11}$$

Here a and b have coefficients bounded above by ~ 1, and α is a positive number. By the uncertainty principle, the term $M^{-\alpha}B(\xi_1,\xi_2)$ is negligible on the ball of radius M^α on the physical side. Hence we apply the two-dimensional uniform ℓ^2 decoupling of [loc. cit., Theorem 4.4] to the one variable polynomial $a(\xi_1)$ and partition $[0,1]$ into intervals I such that each rectangle is $(\phi_2, AM^{-\alpha})$-flat. Combining all the inequalities we have obtained so far, we have

$$\left\| \sum_{\{|\tau|=M_1^{-1}:\tau\subset S_{\text{flat}}\}} f_\tau \right\|_{L^4} \lesssim_\epsilon M^\epsilon \left(\sum_T \|f_T\|_{L^4}^2\right)^{1/2} \lesssim_\epsilon M^\epsilon \left(\sum_{S'\in\mathcal{S}_{M^{-\alpha}}} \|f_{S'}\|_{L^4}^2\right)^{1/2}. \tag{3-12}$$

Step 3: *Iteration.* By (3-5), (3-6), and (3-12), we obtain

$$\|f\|_{L^4} \le C_\epsilon \delta^{-\epsilon}\left(\sum_{S\in\mathcal{S}_\delta}\|f_S\|_{L^4}^2\right)^{1/2} + C_\epsilon (M^\alpha)^\epsilon \left(\sum_{S'\in\mathcal{S}_{M^{-\alpha}}}\|f_{S'}\|_{L^4}^2\right)^{1/2}. \tag{3-13}$$

The first term is already of the desired form. To bound the second term, we fix S' and do rescaling. Let L be an affine transformation mapping $[0,1]^2$ to S'. Since S' is $(\phi, AM^{-\alpha})$-flat, we have

$$\sup_{u,v\in S'} |\phi(u)-\phi(v)-\nabla\phi(u)\cdot(u-v)| \le AM^{-\alpha}. \tag{3-14}$$

Fix any point $u_0\in S'$, and define

$$\tilde\phi(\xi) := A^{-1}M^\alpha(\phi(Lu_0)-\phi(L\xi)-\nabla\phi(Lu_0)\cdot(Lu_0-L\xi)). \tag{3-15}$$

Then by (3-14), we have $\sup_{\xi\in[0,1]^2}|\tilde{\phi}(\xi)|\le 1$. By [Li and Yang 2021, Proposition 7.1], all the coefficients of $\tilde{\phi}$ are bounded by 1. So this phase function satisfies the hypothesis of Theorem 3.1. We apply (3-13) to the function and obtain

$$\|f_{S'}\|_{L^4}\le C_\epsilon\delta^{-\epsilon}\bigg(\sum_{\{S\in\mathcal{S}_\delta:S\cap S'\neq\varnothing\}}\|f_S\|_{L^4}^2\bigg)^{1/2}+C_\epsilon(M^\alpha)^\epsilon\bigg(\sum_{\{S'\in\mathcal{S}_{M^{-2\alpha}}:S\cap S'\neq\varnothing\}}\|f_{S'}\|_{L^4}^2\bigg)^{1/2}. \tag{3-16}$$

By (3-13) and (3-16), we have

$$\|f\|_{L^4}\le(C_\epsilon+C_\epsilon^2(M^\alpha)^\epsilon)\delta^{-\epsilon}\bigg(\sum_{S\in\mathcal{S}_\delta}\|f_S\|_{L^4}^2\bigg)^{1/2}+C_\epsilon^2(M^{2\alpha})^\epsilon\bigg(\sum_{S'\in\mathcal{S}_{M^{-2\alpha}}}\|f_{S'}\|_{L^4}^2\bigg)^{1/2}. \tag{3-17}$$

We repeat this process $O(\log_M\delta^{-1})$ times, and we obtain

$$\|f\|_{L^4}\lesssim_\epsilon(\delta^{-2\epsilon}+\delta^{-\log_M C_\epsilon})\delta^{-\epsilon}\bigg(\sum_{S\in\mathcal{S}_\delta}\|f_S\|_{L^4}^2\bigg)^{1/2}. \tag{3-18}$$

We take M large enough that $\log_M C_\epsilon\le\epsilon$. This completes the proof.

4. Proof of Corollaries 1.3 and 1.5

We prove Corollaries 1.3 and 1.5. We first remark that Corollary 1.3 holds under a general condition.

Remark 4.1. Fix $d\ge 2$. Let $\phi:\mathbb{R}^2\to\mathbb{R}$ be a smooth function. Given a straight line l intersecting $[0,1]^2$, we parametrize the line by $\gamma(t)$ with the unit speed. Assume that

$$\phi(\gamma(t))=a_0+a_1t+a_2t^2+\cdots+a_dt^d+E(t), \tag{4-1}$$

where

$$|E(t)|\le c_\phi t^{d+1} \tag{4-2}$$

and

$$|a_2|+\cdots+|a_d|\ge C_\phi>0. \tag{4-3}$$

Here, c_ϕ and C_ϕ are independent of the choice of the line l. Then (1-6) is true.

Corollary 1.5 is stated using the language of a partial differential equation. By Fourier series, we can write the operator $e^{it\tilde{\Delta}}f$ as

$$e^{it\tilde{\Delta}}f(x)=\sum_{\xi\in[-N,N]^2\cap\mathbb{Z}^2}\hat{f}(\xi)e((x_1,x_2,x_3)\cdot(\xi_1,\xi_2,\xi_1^2-\alpha\xi_2^2)). \tag{4-4}$$

Also by Parseval's identity, we have

$$\|f\|_{L^2(\mathbb{T}^2)}^2\sim\sum_{\xi\in[-N,N]^2\cap\mathbb{Z}^2}|\hat{f}(\xi)|^2. \tag{4-5}$$

Hence Corollary 1.5 can be rephrased as follows.

Corollary 4.2. *Let α be irrational. Suppose that*

$$\phi(\xi_1, \xi_2) = \xi_1^2 - \xi_2^2 \quad \text{and} \quad \Lambda_\delta = (\delta\mathbb{Z} \times \alpha\delta\mathbb{Z}) \cap [0,1]^2. \tag{4-6}$$

Then, for $2 \le p \le 4$ and $\epsilon > 0$, we have

$$\left\| \sum_{\xi \in \Lambda_\delta} a_\xi e(x \cdot (\xi, \phi(\xi))) \right\|_{L^p_\#(B_{\delta^{-3}})} \le C_\epsilon \delta^{-\epsilon} \left(\sum_{\xi \in \Lambda_\delta} |a_\xi|^2 \right)^{1/2}. \tag{4-7}$$

4.1. ***Proof of Corollary 1.3.*** Suppose that the manifold $\mathcal{M}_\phi$ does not contain a line. Fix $\epsilon > 0$. By Theorem 3.1 with δ replaced by δ^d, we have

$$\|f\|_{L^4} \le C_{d,\epsilon} \delta^{-\epsilon} \left(\sum_{S \in \mathcal{S}_{\delta^d}} \|f_S\|_{L^4}^2 \right)^{1/2} \tag{4-8}$$

for all f whose Fourier support is in $N_{\delta^d}(\mathcal{M}_\phi)$. Since ϕ does not contain a line, by a compactness argument, for any l, we can parametrize it as a function of t and write

$$\phi(t, at + b) = a_0 + a_1 t + a_2 t^2 + \cdots + a_d t^d, \tag{4-9}$$

where

$$\max(|a_2|, |a_3|, \ldots, |a_d|) \sim 1. \tag{4-10}$$

Suppose that S is a rectangle and is $(\phi, A\delta^d)$-flat. We claim that the length of a long direction of S is smaller than or equal to $C_\phi \delta$ for some constant C_ϕ depending on the choice of ϕ. We may assume that the angle between the long direction of S and the ξ_1-axis is smaller than or equal to $\frac{\pi}{4}$. Let l be a line segment passing through the center of S and parallel to the long direction of S but contained in S. For convenience, we introduce a parametrization of the line l: $\gamma(t) = (t, at + b)$, where $t \in [b_0, b_1]$. To prove the claim, it suffices to show that

$$|b_1 - b_0| \le C_{\phi, A} \delta. \tag{4-11}$$

By the definition of $(\phi, A\delta^d)$-flat, we have

$$\sup_{t \in [b_0, b_1]} |\phi(\gamma(b_0)) - \phi(\gamma(t)) - \nabla\phi(\gamma(b_0)) \cdot (\gamma(b_0) - \gamma(t))| \le A\delta^d. \tag{4-12}$$

By (4-9) and (4-10), after some computations, this can be rewritten as

$$\sup_{t \in [b_0, b_1]} |\tilde{a}_1(t - b_0) + \tilde{a}_2(t - b_0)^2 + \cdots + \tilde{a}_d(t - b_0)^d| \le A\delta^d, \tag{4-13}$$

where

$$\max(|\tilde{a}_2|, |\tilde{a}_3|, \ldots, |\tilde{a}_d|) \sim 1. \tag{4-14}$$

By [Li and Yang 2021, Proposition 7.1], (4-13) gives

$$\max(|\tilde{a}_2(b_0 - b_1)^2|, |\tilde{a}_3(b_0 - b_1)^3|, \ldots, |\tilde{a}_d(b_0 - b_1)^d|) \lesssim A\delta^d. \tag{4-15}$$

The condition (4-14) gives $|b_1 - b_0| \lesssim \delta$, and this finishes the proof of the claim.

We have proved the claim. Corollary 1.3 will follow by counting the number of frequencies in Λ_δ contained in a δ^d-neighborhood of each $S \in \mathcal{S}_\delta$. The claim says that each S contains at most $\lesssim 1$ many frequencies in Λ_δ, and this gives the desired result. Let us give more details. We take f to be a Fourier transform of

$$\sum_{\xi \in \Lambda_\delta} \psi_\xi, \tag{4-16}$$

where ψ_ξ is a smooth bump function supported on the ball of radius δ^d centered at the point $(\xi, \phi(\xi))$. By the claim, we have

$$\Big(\sum_{S \in \mathcal{S}_{\delta^d}} \|f_S\|_{L^4}^2 \Big)^{1/2} \lesssim \Big(\sum_{\xi \in \Lambda_\delta} \|\widehat{\psi_\xi}\|_{L^4}^2 \Big)^{1/2}. \tag{4-17}$$

On the other hand, by the definition of f, we have

$$\|f\|_{L^4} \sim \Big\| \sum_{\xi} \widehat{\psi_\xi} \Big\|_{L^4}. \tag{4-18}$$

Note that

$$\widehat{\psi_\xi}(x) = \delta^{3d} e(\xi, \phi(\xi)) \psi_{B(0,\delta^{-d})}, \tag{4-19}$$

where ψ_B is a smooth function essentially supported on the ball B. Corollary 1.3 follows from this, (4-8), (4-9), and (4-10). This completes the proof.

4.2. ***Proof of Corollary 1.5.*** As we discussed, it suffices to prove Corollary 4.2. For simplicity, we only consider $\alpha = \sqrt{2}$. The general case can be proved identically. For $\phi(\xi_1, \xi_2) = \xi_1^2 - \xi_2^2$ and $\Lambda_\delta = \delta\mathbb{Z} \times \sqrt{2}\delta\mathbb{Z}$, our goal is to show that

$$\Big\| \sum_{\xi \in \Lambda_\delta} a_\xi e(x \cdot (\xi, \phi(\xi))) \Big\|_{L^p_\#(B_{\delta^{-3}})} \le C_\epsilon \delta^{-\epsilon} \Big(\sum_{\xi \in \Lambda_\delta} |a_\xi|^2 \Big)^{1/2}. \tag{4-20}$$

This will follow easily from Theorem 1.8 by counting the number of frequencies in Λ_δ contained in a δ^3-neighborhood of each $S \in \mathcal{S}_\delta$. Suppose that $(m_i\delta, n_i\sqrt{2}\delta) \in \Lambda_\delta \cap N_{\delta^3}(S)$ for $i = 1, 2$. Then using the definition of (ϕ, δ^3)-flat, we have

$$|m_1^2\delta^2 - 2n_1^2\delta^2 - (m_2^2\delta^2 - 2n_2^2\delta^2) - 2(m_1\delta, -\sqrt{2}n_1\delta) \cdot ((m_1 - m_2)\delta, \sqrt{2}(n_1 - n_2)\delta)| \lesssim \delta^3.$$

This simplifies to

$$|(m_1 - m_2)^2 - 2(n_1 - n_2)^2| \lesssim \delta,$$

so

$$|[m_1 - m_2 - \sqrt{2}(n_1 - n_2)][m_1 - m_2 + \sqrt{2}(n_1 - n_2)]| \lesssim \delta.$$

By Lemma 4 of Section 2, Chapter 2 of [Cassels 1972], we have $1/b^{1+\epsilon'} \lesssim_{\epsilon'} |a + \sqrt{2}b|$ for any $a, b \in \mathbb{Z}$. Therefore, if $(m_1, n_1) \neq (m_2, n_2)$, the above displayed inequality implies that $|n_1 - n_2| \gtrsim_{\epsilon'} \delta^{-1+\epsilon'}$. Since the elements of $\Lambda_\delta \cap N_{\delta^3}(S)$ are $\gtrsim \delta$-separated, there are fewer than $\lesssim_{\epsilon'} \delta^{-\epsilon'}$ frequencies in $\Lambda_\delta \cap N_{\delta^3}(S)$, as desired.

Appendix A: Partition is not enough

In this appendix, we prove that there is no ℓ^2 decoupling for the hyperbolic paraboloid using a partition. Note that in Theorem 1.8 we introduce $O(\log \delta^{-1})$ many partitions to obtain ℓ^2 decoupling for the hyperbolic paraboloid. The following theorem shows that it is necessary to introduce many partitions.

Theorem A.1. *Let $\mathcal{M}_\phi$ be the hyperbolic paraboloid given by $\phi(\xi_1, \xi_2) = \xi_1\xi_2$. Fix $0 < \epsilon < \frac{1}{100}$ and $A \geq 1$. Then the following is false: for any $\delta > 0$, there exists a family $\mathcal{S}_\delta$ of rectangles $S \subset \mathbb{R}^2$ such that*

(1) *the interiors of rectangles are disjoint*:

$$\operatorname{int}(S_1) \cap \operatorname{int}(S_2) = \varnothing, \quad S_1, S_2 \in \mathcal{S}_\delta, \tag{A-1}$$

(2) *every S is $(\phi, A\delta)$-flat*,

(3) *for all f whose Fourier support is in $N_\delta(\mathcal{M}_\phi)$*,

$$\|f\|_{L^4} \leq C_{\epsilon,A}\delta^{-\epsilon}\left(\sum_{S\in\mathcal{S}_\delta} \|f_S\|_{L^4}^2\right)^{1/2}. \tag{A-2}$$

Proof. Fix $\delta > 0$. For simplicity, assume that δ^{-1} is a dyadic number. Suppose that such a family $\mathcal{S}_\delta$ exists for a contradiction. Let us first consider the partition

$$\begin{aligned} [0,1]^2 &= \bigcup_{a\in\mathbb{Z}\cap[0,\delta^{-1}-1]} ([0,1]\times[a\delta, a\delta+\delta]) =: \bigcup_a C_a, \\ [0,1]^2 &= \bigcup_{b\in\mathbb{Z}\cap[0,\delta^{-1}-1]} ([b\delta, b\delta+\delta]\times[0,1]) =: \bigcup_b D_b. \end{aligned} \tag{A-3}$$

Let us fix C_a. Consider a collection of elements of $\mathcal{S}_\delta$ which have a large intersection with C_a:

$$\mathcal{S}_{\delta,C_a} := \{S \in \mathcal{S}_\delta : |S \cap C_a| \geq \delta^1\delta^{1/2-\epsilon}\}. \tag{A-4}$$

Define $\mathcal{S}_{\delta,C_b}$ similarly. By abusing notation, let us denote by $\mathcal{S}_{\delta,a}$ and $\mathcal{S}_{\delta,b}$ the sets $\mathcal{S}_{\delta,C_a}$ and $\mathcal{S}_{\delta,D_b}$. Recall that elements of the set $\mathcal{S}_{\delta,a}$ are disjoint by the first condition (A-1).

We claim that

$$\left|\bigcup_{S\in\mathcal{S}_{\delta,a}} S\cap C_a\right| \geq \tfrac{99}{100}\delta \quad \text{and} \quad \left|\bigcup_{S\in\mathcal{S}_{\delta,b}} S\cap D_b\right| \geq \tfrac{99}{100}\delta. \tag{A-5}$$

Let us assume this claim for a moment and finish the proof of the theorem. By taking the union over a in (A-5), we have

$$\left|\bigcup_{a\in\mathbb{Z}\cap[0,\delta^{-1}-1]}\ \bigcup_{S\in\mathcal{S}_{\delta,a}} S\cap C_a\right| \geq \tfrac{99}{100}. \tag{A-6}$$

By pigeonholing, there exists $D_b \in \mathcal{S}_{\delta,b}$ such that

$$\left|D_b \cap \left(\bigcup_{a\in\mathbb{Z}\cap[0,\delta^{-1}-1]}\ \bigcup_{S\in\mathcal{S}_{\delta,a}} S\cap C_a\right)\right| \geq \tfrac{99}{100}\delta. \tag{A-7}$$

This implies

$$\left| D_b \cap \left(\bigcup_{a \in \mathbb{Z} \cap [0, \delta^{-1}-1]} \bigcup_{S \in \mathcal{S}_{\delta,a}} S \right) \right| \geq \tfrac{99}{100}\delta. \tag{A-8}$$

We claim that if $S \in \mathcal{S}_{\delta,a}$ for some a, then

$$|D_b \cap S| \lesssim \delta^1 \delta^{1/2+\epsilon}. \tag{A-9}$$

Note that this means that S does not belong to $\mathcal{S}_{\delta,D_b}$. Since the area of D_b is δ, the first inequality of (A-5) gives

$$\left| \bigcup_{S \in \mathcal{S}_{\delta,D_b}} S \cap D_b \right| \leq \tfrac{1}{100}\delta. \tag{A-10}$$

This contradicts the second inequality of (A-5). Let us give a proof of (A-9). Suppose that $S \in \mathcal{S}_{\delta,a}$. It suffices to prove that

$$S \subset (\alpha, \beta) + ([0,1] \times [0, C\delta^{1/2+\epsilon}]) \tag{A-11}$$

for some $\alpha, \beta \in [0,1]$. To prove this, we use the assumption that S is $(\phi, A\delta)$-flat. By the definition of $\mathcal{S}_{\delta,a}$, the rectangle S contains a line segment parallel to the ξ_1-axis with length $\delta^{1/2-\epsilon}$. Since our manifold is translation invariant, for simplicity, assume that the line is $[0, \delta^{1/2-\epsilon}] \times \{0\}$. We will show that if $p = (p_1, p_2)$ is an element of S, then $|p_2| \lesssim \delta^{1/2+\epsilon}$. This gives the proof of (A-9). To show the bound of p_2, we set $u = (0,0)$. Then by the definition of $(\phi, A\delta)$-flat (Definition 1.1),

$$|p_1 p_2| \leq A\delta. \tag{A-12}$$

Similarly, we use $u = (\delta^{1/2-\epsilon}, 0)$, and this gives

$$|p_1 p_2 - \delta^{1/2-\epsilon} p_2| \leq A\delta. \tag{A-13}$$

Combining these two gives $|p_2| \lesssim \delta^{1/2+\epsilon}$.

It remains to prove (A-5). By symmetry, let's show only the first inequality. We will use assumption (A-2). Fix C_a. Let us first show that there exists $S \in \mathcal{S}_\delta$ such that

$$|S \cap C_a| \geq (\log \delta^{-1})^{-10} \delta^{1+4\epsilon}. \tag{A-14}$$

Suppose that such an S does not exist for a contradiction. We take f such that

(1) $\hat{f}$ is supported on the δ-neighborhood of the set $\mathcal{M}_\phi \cap (C_a \times \mathbb{R})$,

(2) $\hat{f}$ is equal to 1 on the $\frac{1}{2}\delta$-neighborhood of the set $\mathcal{M}_\phi \cap (C_a \times \mathbb{R})$.

Here $\mathcal{M}_\phi$ is the hyperbolic paraboloid. Then f is essentially supported on a box with dimension $\delta^{-1} \times 1 \times \delta^{-1}$ and has amplitude $\sim \delta^2$. So we have

$$\|f\|_{L^4} \sim \delta^2 \delta^{-2/4} \sim \delta^{3/2}. \tag{A-15}$$

On the other hand, by pigeonholing, there exists j such that

$$\Big(\sum_{S\in\mathcal{S}_\delta}\|f_S\|_{L^4}^2\Big)^{1/2}\lesssim(\log\delta^{-1})^{1/2}\Big(\sum_{S\in\mathcal{S}_\delta^j}\|f_S\|_{L^4}^2\Big)^{1/2},\tag{A-16}$$

where the elements S of $\mathcal{S}_\delta^j$ satisfy

$$|S\cap C_a|\sim\delta 2^{-j}.\tag{A-17}$$

Here, we used the fact that, for large j, the right-hand side of (A-16) is negligible (which is proved in (A-18)). Since we are assuming that there does not exist S satisfying (A-14), (A-16) is true for $j>4\epsilon\log_2\delta^{-1}+10\log_2\log\delta^{-1}$. Note that f_S is essentially constant on a box with volume $(\delta\delta 2^{-j})^{-1}$ and has amplitude $\delta\delta 2^{-j}$. Since $\mathcal{S}_\delta$ is a partition (see (A-1)), the cardinality of $\mathcal{S}_\delta^j$ is bounded by 2^j. So we have

$$\Big(\sum_{S\in\mathcal{S}_\delta^j}\|f_S\|_{L^4}^2\Big)^{1/2}\lesssim(\#\mathcal{S}_\delta^j)^{1/2}\max_S\|f_S\|_{L^4}\lesssim 2^{j/2}(\delta^2 2^{-j})(\delta^2 2^{-j})^{-1/4}\sim\delta^{3/2}2^{-j/4}.\tag{A-18}$$

Inequality (A-2), (A-15), and (A-18) give

$$\delta^{3/2}\lesssim(\log\delta^{-1})^{1/2}\delta^{\epsilon}\delta^{3/2}2^{-j/4}.\tag{A-19}$$

We get a contradiction from the assumption that $j>4\epsilon\log_2\delta^{-1}+10\log_2\log\delta^{-1}$. The same argument works for a general situation. Let $C\subset C_a\subset[0,1]^2$ be a convex set (polygon). Then we can show that there exists $S\in\mathcal{S}_\delta$ such that

$$|S\cap C|\gtrsim(\log\delta^{-1})^{-10}\delta^{4\epsilon}|C|.\tag{A-20}$$

Let us explain the proof. First, since C is a convex set, we can find a rectangle $\widetilde{C}\subset C$ such that $|\widetilde{C}|\gtrsim|C|$; for example, by the Kovner–Besicovitch theorem. Since $\widetilde{C}$ is a rectangle in C_a, we can repeat the proof of (A-14) and find $S\in\mathcal{S}_\delta$ such that

$$|S\cap C|\geq|S\cap\widetilde{C}|\geq(\log\delta^{-1})^{-10}\delta^{4\epsilon}|\widetilde{C}|\gtrsim(\log\delta^{-1})^{-10}\delta^{4\epsilon}|C|.\tag{A-21}$$

This gives (A-20).

We have shown that there exists $S_1\in\mathcal{S}_\delta$ such that (A-14) holds. In particular, $S_1\in\mathcal{S}_{\delta,a}$; see (A-4) for the definition of $\mathcal{S}_{\delta,a}$ and note that we used the assumption that ϵ is small. If we have $|S_1\cap C_a|\geq\frac{99}{100}\delta$, then this gives (A-5). So suppose that

$$\delta^{1+4\epsilon}\leq|S_1\cap C_a|\leq\tfrac{99}{100}\delta.\tag{A-22}$$

Since S_1 is a rectangle, the set $C_a\setminus(S_1)^c$ is either a convex set or a union of two convex sets. Let us write it as

$$C_a\setminus(S_1)^c=C_{11}\cup C_{12}.\tag{A-23}$$

Here C_{11} and C_{12} are disjoint convex sets and

$$|C_{11}|+|C_{12}|\leq|C_a|(1-\delta^{4\epsilon}).\tag{A-24}$$

This finishes the first round of the iteration. Let us explain how to proceed. If C_{11} satisfies $|C_{11}| \le \delta\delta^{1/4}$, then we leave this set. If $|C_{11}| > \delta\delta^{1/4}$, then we apply (A-20) to the convex set C_{11} and obtain the set $S_2 \in \mathcal{S}_\delta$ such that

$$|S_2 \cap C_{11}| \gtrsim (\log \delta^{-1})^{-10}\delta^{4\epsilon}|C_{11}|. \tag{A-25}$$

By the stopping time condition $|C_{11}| > \delta\delta^{1/4}$ and the assumption that ϵ is small, we have $S_2 \in \mathcal{S}_{\delta,C_a}$; see (A-4) for the definition. We write $C_{11} \setminus (S_2)^c = C_{111} \cup C_{112}$. Then we have

$$|C_{111}| + |C_{112}| \le |C_{11}|(1-\delta^{4\epsilon}), \tag{A-26}$$

where the C_{11j} are convex sets. Repeat this process for C_{12}. If $|C_{12}| < \delta\delta^{1/4}$, then we have

$$|C_{111}| + |C_{112}| + |C_{12}| \le |C_a|(1-\delta^{4\epsilon})^2 + |C_{12}|\delta^{4\epsilon}. \tag{A-27}$$

If $|C_{12}| > \delta\delta^{1/4}$, then we have

$$|C_{111}| + |C_{112}| + |C_{113}| + |C_{114}| \le |C_a|(1-\delta^{4\epsilon})^2. \tag{A-28}$$

This finishes the second round of the iteration. We repeat this process M times with $M \gtrsim \delta^{-6\epsilon}$. Denote by $\{C_X\}_X$ and $\{S_Y\}_Y$ a collection of convex sets and a collection of rectangles $S \in \mathcal{S}_\delta$, respectively, that we obtained via this process. Note that

$$\sum_X |C_X| + \sum_Y |S_Y \cap C_a| = |C_a| = \delta. \tag{A-29}$$

We have

$$\begin{aligned}\sum_X |C_X| &= \sum_{\{X:|C_X|\ge\delta\delta^{1/4}\}} |C_X| + \sum_{\{X:|C_X|<\delta\delta^{1/4}\}} |C_X| \\ &\le |C_a|(1-\delta^{4\epsilon})^M + \Big|\bigcup_{\{X:|C_X|<\delta\delta^{1/4}\}} C_X\Big|\delta^{4\epsilon} \le \tfrac{1}{100}|C_a|.\end{aligned} \tag{A-30}$$

The last two inequalities follow from the fact that all C_X are disjoint and $C_X \subset C_a$. This completes the proof of the claim (A-5). □

Appendix B: Higher dimensions

In this appendix, we prove that an analogous result to Theorem 1.2 is false in higher dimensions. For convenience, let us consider only a manifold in $\mathbb{R}^4$. Higher-dimensional manifolds can be handled in a similar way. Definitions 1.1 and 2.1 can be naturally generalized to higher dimensions. We will not state them. Recall that the critical exponent p of decoupling for the hyperbolic paraboloid in $\mathbb{R}^4$ is $\frac{10}{3}$.

Theorem B.1. *Consider $\phi(\xi_1,\xi_2,\xi_3) = \xi_1^2+\xi_2^2-\xi_3^2$. Fix $0<\epsilon<\frac{1}{1000}$. Let A be a constant. Then the following statement is false.*

For any $\delta > 0$, there exists a collection $\mathcal{S}_\delta$ of rectangular boxes $S \subset [0,1]^3$ such that:

(1) *The overlapping number is $O(\log \delta^{-1})$ in the sense that*

$$\sum_{S\in\mathcal{S}_\delta} \chi_S \le C_\epsilon \log(\delta^{-1}). \tag{B-1}$$

(2) *S is $(\phi, A\delta)$-flat and of the form of a parallelogram.*

(3) *We have*

$$\|f\|_{L^{10/3}} \le C_\epsilon \delta^{-\epsilon} \left(\sum_{S \in \mathcal{S}_\delta} \|f_S\|^2_{L^{10/3}} \right)^{1/2} \tag{B-2}$$

for all f whose Fourier support is in $N_\delta(\mathcal{M}_\phi)$.

The same conclusion of Theorem B.1 occurs if (B-2) is replaced with

$$\|f\|_{L^p} \le C_\epsilon \delta^{-\epsilon} \left(\sum_{S \in \mathcal{S}_\delta} \|f_S\|^2_{L^p} \right)^{1/2} \tag{B-3}$$

for $2 < p \le \frac{10}{3}$. Inequality (B-6) is the only place where (B-2) is used. One may check that (B-3) still implies the desired result.

Proof. Let us prove the result by contradiction. Fix $0 < \epsilon < 10^{-10}$. Define

$$\mathcal{C} := \{(\xi_1, \xi_2, \xi_3) \in [0,1]^3 : \xi_1^2 + \xi_2^2 - \xi_3^2 = 0\}. \tag{B-4}$$

We cover $\mathcal{C}$ by $\sim \delta^{-1/2}$ many canonical blocks P with dimension $1 \times \delta^{1/2} \times \delta$. Let us denote by $\mathcal{P}$ the collection of blocks. Note that each block is $(\phi, A\delta)$-flat for some large constant A. For given $P \in \mathcal{P}$, we take translated copies and cover $[0,1]^3$ so that they are disjoint. Denote by $\mathcal{P}_P$ the collection of translated copies of P. Note that the cardinality of $\mathcal{P}_P$ is comparable to $\delta^{-3/2}$.

Fix $P' \in \mathcal{P}_P$ for some $P \in \mathcal{P}$. We next decompose P' into parallel tubes with radius δ^X, where X is sufficiently large number. Let us denote by $\mathcal{P}_{P,P'}$ the collections of tubes P''. We claim that, for given $P'' \in \mathcal{P}_{P,P'}$, there is an element S of $\mathcal{S}_\delta$ such that the diameter of the set $S \cap P''$ is larger than or equal to $(\log \delta^{-1})\delta^{5\epsilon}$. To prove the claim, let us use the hypothesis (B-2). Take a function f such that

$$\hat{f}(\xi) := \sum_{j=1}^{(\log \delta^{-1})\delta^{-5\epsilon}} f_j(\xi) := \sum_{j=1}^{(\log \delta^{-1})\delta^{-5\epsilon}} 1_{N_{\delta^X}(B_j)}(\xi), \tag{B-5}$$

where $\{B_j\}_{j=1}^{(\log \delta^{-1})\delta^{-5\epsilon}}$ is an arithmetic progression with difference $(\log \delta^{-1})\delta^{5\epsilon}$ and $\bigcup_j B_j \subset P''$. By direct computations, one can see that

$$\|f\|_{L^{10/3}} \gtrsim (\log \delta^{-1})^{1/5} \delta^{-5\epsilon(1/2 - 1/(10/3))} \left(\sum_j \|f_j\|^2_{L^{10/3}} \right)^{1/2}. \tag{B-6}$$

Hence inequality (B-2) says that there must be an element S of $\mathcal{S}_\delta$ containing at least two B_j. This means that the diameter of $S \cap P''$ is larger than or equal to $(\log \delta^{-1})\delta^{5\epsilon}$. This completes the proof of the claim.

For given $P' \in \mathcal{P}_P$, denote by $\mathcal{S}_{P'}$ the elements of $S \in \mathcal{S}_\delta$ satisfying the following property: for some $P'' \in \mathcal{P}_{P,P'}$ the diameter of $S \cap P''$ is larger than or equal to $(\log \delta^{-1})\delta^{5\epsilon}$. The claim says that $\mathcal{S}_{P'}$ is nonempty.

We next claim that if S is $(\phi, A\delta)$-flat and $S \in \mathcal{S}_\delta$, then S is contained in a box with dimension $\delta^{1/2-5\epsilon/2} \times \delta^{1-5\epsilon} \times \delta^{5\epsilon}$. Let us give a proof. First of all, by an affine transformation, we may assume that S

contains a line $\{0\}\times\{0\}\times[0,\delta^{5\epsilon}]$, and our new phase function is $\tilde{\phi}(\xi)=\xi_1^2+\xi_2\xi_3$. Let $v=(v_1,v_2,v_3)\in S$. We will show that

$$|v_1|\lesssim\delta^{1/2-5\epsilon/2}\quad\text{and}\quad|v_2|\lesssim\delta^{1-5\epsilon}. \tag{B-7}$$

By the definition of a $(\phi, A\delta)$-flat set with $u=(0,0,0)$, we obtain

$$|v_1^2+v_2v_3|\le A\delta. \tag{B-8}$$

We next use $u=(0,0,\delta^{5\epsilon})$ and obtain

$$|v_1^2+v_2v_3-\delta^{5\epsilon}v_2|\le A\delta. \tag{B-9}$$

These two inequalities give $|v_2|\lesssim\delta^{1-5\epsilon}$. This bound and (B-8) give $|v_1|\lesssim\delta^{1/2-5\epsilon/2}$. This proves the claim.

By the claim, if $P_1'\in\mathcal{P}_{P_1}$ and $P_2'\in\mathcal{P}_{P_2}$ and the angle of the longest directions of P_1 and P_2 is greater than $\delta^{1/2-100\epsilon}$, then any two sets $S_1\in\mathcal{S}_{P_1'}$ and $S_2\in\mathcal{S}_{P_2'}$ are distinct. Note also that

$$\Big|\bigcup_{S'\in\mathcal{S}_{P'}}S'\cap P'\Big|\gtrsim\delta^{5\epsilon}|P'|. \tag{B-10}$$

Let us finish the proof of the theorem. To get a contradiction, we need to prove that there exists $\xi\in[0,1]^3$ such that the number of $S\in\mathcal{S}_\delta$ containing ξ is greater than the right-hand side of (B-1). To prove this, it suffices to show that there exists $\xi\in[0,1]^3$ satisfying

$$\sum_{P\in\mathcal{P}}\sum_{P'\in\mathcal{P}_P}\sum_{S'\in\mathcal{S}_{P'}}1_{S'\cap P'}(\xi)\gtrsim\delta^{-\alpha-100\epsilon} \tag{B-11}$$

for some number $\alpha>0$. This follows from a pigeonholing argument with

$$\begin{aligned}\int_{[0,1]^3}\sum_{P\in\mathcal{P}}\sum_{P'\in\mathcal{P}_P}\sum_{S'\in\mathcal{S}_{P'}}1_{S'\cap P'}(\xi)\,d\xi&=\sum_{P\in\mathcal{P}}\sum_{P'\in\mathcal{P}_P}\int_{[0,1]^3}\sum_{S'\in\mathcal{S}_{P'}}1_{S'\cap P'}(\xi)\,d\xi\\&\gtrsim\delta^{-3/2}\delta^{-1/2}\delta^{5\epsilon}\delta^{3/2}\sim\delta^{-1/2+5\epsilon}.\end{aligned}$$

□

Appendix C: Sharpness of Theorem 1.2

Let us discuss the sharpness of Theorem 1.2. More precisely, we would like to discuss if the range of p for which (1-4) holds is sharp. An example of a manifold where the aforementioned range of p is not sharp is given by

$$\phi(\xi_1,\xi_2)=\xi_1^2. \tag{C-1}$$

The ℓ^2 decoupling for (C-1) is proved for $2\le p\le 6$ by [Bourgain and Demeter 2015], and this range of p is sharp. Motivated by this example, let us introduce the following definition.

Definition C.1. We say $\phi(\xi_1,\xi_2)$ depends only on a variable if there exists an affine transformation L such that

$$\phi(L(\xi_1,\xi_2))=\psi(\xi_1)+a\xi_2 \tag{C-2}$$

for some function ψ and some $a \in \mathbb{R}$. Second, we say $\phi(\xi_1, \xi_2)$ does not depend on any variable if

$$\phi(\xi_1, \xi_2) = a + b\xi_1 + c\xi_2 \tag{C-3}$$

for some $a, b, c \in \mathbb{R}$. Lastly, we say $\phi(\xi_1, \xi_2)$ depends on two variables if it is not of the form of (C-3) and there does not exist L such that (C-2) holds.

Here is a complete characterization of the ℓ^2 decoupling theorem.

Proposition C.2. *Let $p \geq 2$. Consider a polynomial $\phi := \phi(\xi_1, \xi_2)$. Suppose that $\mathcal{S}_\delta$ is a family constructed in Section 2.1 and Section 3.*

(1) *Let ϕ depend on two variables. Then (1-4) is true for $2 \leq p \leq 4$.*

(2) *Let ϕ depend only on one variable. Then (1-4) is true for $2 \leq p \leq 6$.*

(3) *Let ϕ not depend on any variable. Then (1-4) is true for $2 \leq p < \infty$.*

The ranges of p stated in items (1), (2), *and* (3) *are sharp.*

Proof. Let us first prove item (1). Theorem 1.2 gives (1-4) for $2 \leq p \leq 4$, so it suffices to prove that the range is sharp. Let $H_\phi(\xi)$ be the Hessian matrix of ϕ. By [de Bondt and van den Essen 2004, Theorem 3.1], $H_\phi(\xi)$ is not identically zero. Consider

$$Z_\phi := \{\xi \in \mathbb{R}^2 : \det(H_\phi(\xi)) = 0\}. \tag{C-4}$$

Since $\det H_\phi(\xi)$ is not identically zero, it is a zero set of a polynomial. So there exists a square τ not intersecting Z_ϕ. Since $H_\phi(\xi)$ is independent of the parameter δ, we may assume that the size of τ is independent of δ. Let us fix such a τ. Assume that δ is smaller than the sidelength of τ. Let $\hat{f}$ be a smooth bump function of the δ-neighborhood of

$$\{(\xi_1, \xi_2, \phi(\xi_1, \xi_2)) : \xi \in \tau\}. \tag{C-5}$$

One can see that if $\xi \in \tau$, then

$$|\det H_\phi(\xi)| \gtrsim 1. \tag{C-6}$$

So the multiplication of eigenvalues of $H_\phi(\xi)$ does not change the sign over $\xi \in \tau$.

Suppose that the eigenvalues of $H_\phi(\xi)$ have the same sign for all $\xi \in \tau$. Then each $S \in \mathcal{S}_\delta$ intersecting τ is a square with length $\delta^{1/2} \times \delta^{1/2}$. So we have

$$f = \sum_S f_S, \quad |f_S(x)| = \delta^2 1_{T_S}, \tag{C-7}$$

where T_S is a tube with dimension $\delta^{-1/2} \times \delta^{-1/2} \times \delta^{-1}$. Then

$$\delta^2 \delta^{-1} \lesssim \|f\|_{L^p(B_1)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}. \tag{C-8}$$

On the other hand,

$$\left(\sum_S \|f_S\|_{L^p}^2\right)^{1/2} \lesssim \delta^{-1/2}\delta^2 |T_S|^{1/p} \sim \delta^{-1/2}\delta^2\delta^{-2/p}. \tag{C-9}$$

This shows that the decoupling inequality is false for $p > 4$.

Suppose that the eigenvalues have different signs. Then, as in the previous case, we have

$$\delta \lesssim \|f\|_{L^p(\mathbb{R}^3)}. \tag{C-10}$$

We need to calculate

$$\left(\sum_{S\in\mathcal{S}_\delta} \|f_S\|_{L^p}^2\right)^{1/2}. \tag{C-11}$$

By the construction of $\mathcal{S}_\delta$, we have $|f_S| = \delta^2 1_{T_S}$, where T_S has dimension $\delta^A \times \delta^{1-A} \times \delta$ for some $0 \le A \le 1$. Here the number A depends on the choice of S. Also, the cardinality of $\mathcal{S}_\delta$ is comparable to δ^{-1} up to $\delta^{-\epsilon}$ losses. So we have

$$\left(\sum_{S\in\mathcal{S}_\delta} \|f_S\|_{L^p}^2\right)^{1/2} \lesssim_\epsilon \delta^{-\epsilon}\delta^{-1/2}\delta^2|T_S|^{1/p} \sim \delta^{-\epsilon}\delta^{-1/2}\delta^2\delta^{-2/p}. \tag{C-12}$$

This shows that the decoupling inequality is false for $p > 4$, and completes the proof of item (1).

Let us move on to item (2). Since ϕ depends only on one variable, after some change of variables and abusing notation, we may assume that $\phi(\xi_1, \xi_2) = \psi(\xi_1)$. Fix x_2, and define

$$g(x_1, x_3) := f(x_1, x_2, x_3). \tag{C-13}$$

Then (1-4) simply follows from [Yang 2021, Theorem 1.4]. Let us show the sharpness. Since ψ is a one-variable polynomial, the zeros of ψ are finite. So we can find an open interval $I \subset [0, 1]$ such that, for every $\xi_1 \in I$, we have $|\psi''(\xi_1)| \gtrsim 1$. We write I as a union of intervals J with length $\delta^{1/2}$. Take $\hat{f}$ to be a smooth bump function of the δ-neighborhood of

$$\{(\xi_1, \xi_2, \phi(\xi_1, \xi_2)) : \xi_1 \in I, \xi_2 \in [0, 1]\}. \tag{C-14}$$

Then similar calculations in the proof of item (1) give the sharpness for p. We leave out the details.

Lastly, let us prove item (3). If ϕ does not depend on any variable, then, according to our construction, $f_S = f$. So (1-4) is true for $2 \le p < \infty$. This completes the proof of Proposition C.2. □

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Analysis & PDE

Volume 19 No. 1 2026

Microlocal analysis near null infinity in asymptotically flat spacetimes 1
Peter Hintz and András Vasy

Complex Hessian measures with respect to a background Hermitian form 107
Sławomir Kołodziej and Ngoc Cuong Nguyen

ℓ^2 decoupling theorem for surfaces in $\mathbb{R}^3$ 167
Larry Guth, Dominique Maldague and Changkeun Oh