Splitting formulas for the rational lift of the Kontsevich integral

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Kricker defined an invariant of knots in homology 3–spheres which is a rational lift of the Kontsevich integral and proved with Garoufalidis that this invariant satisfies splitting formulas with respect to a surgery move called null-move. We define a functorial extension of the Kricker invariant and prove splitting formulas for this functorial invariant with respect to null Lagrangian-preserving surgery, a generalization of the null-move. We apply these splitting formulas to the Kricker invariant.

1 Introduction

1.1 Context

This paper presents the construction of a functorial extension of the Kricker rational lift of the Kontsevich integral, which aims at explicating the properties of this invariant as a series of finite-type invariants.

The notion of finite-type invariants first appeared in independent works of Goussarov and Vassiliev involving invariants of knots in the 3–dimensional sphere S^3; in this case, finite-type invariants are also called Vassiliev invariants. Finite-type invariants of

knots in S^3 are defined by their polynomial behaviour with respect to crossing changes. The discovery of the Kontsevich integral, which is a universal invariant among all finite-type invariants of knots in S^3, revealed that this class of invariants is very prolific. It is known, for instance, that it dominates all Witten–Reshetikhin–Turaev quantum invariants. A theory of finite-type invariants can be defined for any kind of topological objects provided that an elementary move on the set of these objects is fixed; the finite-type invariants are defined by their polynomial behaviour with respect to this elementary move. For 3–dimensional manifolds, the notion of finite-type invariants was introduced by Ohtsuki [25], who constructed the first examples for integral homology 3–spheres, and it has been widely developed and generalized since then. In particular, Goussarov and Habiro independently developed a theory which involves any 3–dimensional manifolds — and their knots — and which contains the Ohtsuki theory for $\mathbb{Z}$–spheres; see Garoufalidis, Goussarov and Polyak [6] and Habiro [9].

In [12], Kricker constructed a rational lift of the Kontsevich integral of knots in integral homology 3–spheres ($\mathbb{Z}$–spheres). In [7], he proved with Garoufalidis that his construction provides an invariant of knots in $\mathbb{Z}$–spheres. They also proved that the Kricker invariant satisfies some splitting formulas with respect to the so-called null-move. For knots in $\mathbb{Z}$–spheres with trivial Alexander polynomial, these formulas together with work of Garoufalidis and Rozansky [8] imply that the Kricker invariant is a universal finite-type invariant with respect to the null-move.

Kricker's construction easily generalizes to null-homologous knots in rational homology 3–spheres ($\mathbb{Q}$–spheres); the main goal of this article is to prove splitting formulas for the Kricker invariant of these knots with respect to null Lagrangian-preserving surgery, a move which generalizes the null-move. For null-homologous knots in $\mathbb{Q}$–spheres with trivial Alexander polynomial, these formulas are used in Moussard [24] to prove that this extended Kricker invariant is a universal finite-type invariant with respect to null Lagrangian-preserving surgeries.

Lescop defined in [18] an invariant of null-homologous knots in $\mathbb{Q}$–spheres and proved in [19] splitting formulas for this invariant with respect to null Lagrangian-preserving surgeries, similar to the ones proved in this paper for the Kricker invariant. Lescop conjectured in [19] that her invariant is equivalent to the Kricker invariant. The mentioned results of Garoufalidis, Kricker, Lescop and Rozansky give such an equivalence for knots in $\mathbb{Z}$–spheres with trivial Alexander polynomial and the results of the present paper allow us to generalize this equivalence to null-homologous knots in $\mathbb{Q}$–spheres with trivial Alexander polynomial [24, Theorem 1.1].

A similar situation arises in the study of finite-type invariants of $\mathbb{Q}$–spheres with respect to Lagrangian-preserving surgeries. In this case, the Kontsevich–Kuperberg–Thurston invariant and the Le–Murakami–Ohtsuki invariant are both universal, up to degree 1 invariants deduced from the cardinality of the first homology group, which implies an equivalence result for these two invariants; see Moussard [21]. For the KKT invariant, splitting formulas with respect to Lagrangian-preserving surgeries were proved by Lescop [17]; for the LMO invariant, similar formulas were proved by Massuyeau [20]. Massuyeau's proof of his splitting formulas is based on an extension of the LMO invariant of $\mathbb{Q}$–spheres to a functor defined on a category of Lagrangian cobordisms that he constructed with Cheptea and Habiro [5].

In this paper, we extend the LMO functorial invariant of Cheptea, Habiro and Massuyeau to a category of Lagrangian cobordisms with paths, inserting Kricker's idea in the construction. We obtain a functorial invariant from which the Kricker invariant of null-homologous knots in $\mathbb{Q}$–spheres is recovered. Following Massuyeau, we use the functoriality to obtain splitting formulas for our invariant and, as a consequence, for the Kricker invariant.

Notation and conventions For $\mathbb{K} = \mathbb{Z}, \mathbb{Q}$, a $\mathbb{K}$–*sphere* (resp. a $\mathbb{K}$–*cube*) is a 3–manifold, compact and oriented, which has the same homology with coefficients in $\mathbb{K}$ as the standard 3–sphere (resp. 3–cube). The boundary of an oriented manifold with boundary is oriented with the "outward normal first" convention.

1.2 Statement of the main result

We first give the definitions we need to state our main result.

Null LP–surgeries For $g \in \mathbb{N}$, a *genus g rational homology handlebody* ($\mathbb{Q}$–*handlebody*) is a 3–manifold which is compact, oriented and which has the same homology with rational coefficients as the standard genus g handlebody. Such a $\mathbb{Q}$–handlebody is connected, and its boundary is necessarily homeomorphic to the standard genus g surface.

The *Lagrangian* $\mathcal{L}_C$ of a $\mathbb{Q}$–handlebody C is the kernel of the map $i_*\colon H_1(\partial C; \mathbb{Q}) \to H_1(C; \mathbb{Q})$ induced by the inclusion. The Lagrangian of a $\mathbb{Q}$–handlebody C is indeed a Lagrangian subspace of $H_1(\partial C; \mathbb{Q})$ with respect to the intersection form. A *Lagrangian-preserving pair*, or *LP–pair*, is a pair $\boldsymbol{C} = (C'/C)$ of $\mathbb{Q}$–handlebodies equipped with a homeomorphism $h\colon \partial C \xrightarrow{\cong} \partial C'$ such that $h_*(\mathcal{L}_C) = \mathcal{L}_{C'}$.

Given a 3–manifold M, a *Lagrangian-preserving surgery*, or *LP–surgery*, on M is a family $\boldsymbol{C} = (\boldsymbol{C}_1, \ldots, \boldsymbol{C}_n)$ of LP–pairs such that the C_i are embedded in M and disjoint. The manifold obtained from M by LP–surgery on $\boldsymbol{C}$ is defined as

$$M(\boldsymbol{C}) = \left(M \setminus \left(\bigsqcup_{1 \leq i \leq n} C_i \right) \right) \cup_\partial \left(\bigsqcup_{1 \leq i \leq n} C_i' \right).$$

Let M be a 3–manifold such that $H_1(M; \mathbb{Q}) = 0$ and let K be a disjoint union of knots or paths properly embedded in M. A $\mathbb{Q}$–*handlebody null in* $M \setminus K$ is a $\mathbb{Q}$–handlebody $C \subset M \setminus K$ such that the map $i_*: H_1(C; \mathbb{Q}) \to H_1(M \setminus K; \mathbb{Q})$ induced by the inclusion has a trivial image. A *null LP–surgery* on (M, K) is an LP–surgery $\boldsymbol{C} = (\boldsymbol{C}_1, \ldots, \boldsymbol{C}_n)$ on $M \setminus K$ such that each C_i is null in $M \setminus K$. The pair obtained by surgery is denoted by $(M, K)(\boldsymbol{C})$.

The tensor $\mu(\boldsymbol{C})$ Given an LP–pair $\boldsymbol{C} = (C'/C)$, define the associated *total manifold* $\mathcal{C} = (-C) \cup C'$ and define

$$\mu(\boldsymbol{C}) \in \hom\left({\textstyle\bigwedge^3} H^1(\mathcal{C}; \mathbb{Q}), \mathbb{Q} \right) \cong {\textstyle\bigwedge^3} H_1(\mathcal{C}; \mathbb{Q})$$

by associating with a triple of cohomology classes the evaluation of their triple cup product on the fundamental class of $\mathcal{C}$. For a family $\boldsymbol{C} = (\boldsymbol{C}_1, \ldots, \boldsymbol{C}_n)$ of LP–pairs, let $\mathcal{C} = \mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_n$ and set

$$\mu(\boldsymbol{C}) = \mu(\boldsymbol{C}_1) \otimes \cdots \otimes \mu(\boldsymbol{C}_n) \in \bigotimes_{i=1}^n {\textstyle\bigwedge^3} H_1(\mathcal{C}_i; \mathbb{Q}).$$

The natural identification $H_1(\mathcal{C}; \mathbb{Q}) \cong \bigoplus_{i=1}^n H_1(\mathcal{C}_i; \mathbb{Q})$ shows $\bigotimes_{i=1}^n \bigwedge^3 H_1(\mathcal{C}_i; \mathbb{Q})$ as a subspace of $\left(\bigwedge^3 H_1(\mathcal{C}; \mathbb{Q}) \right)^{\otimes n}$. This subspace injects to $S^n \bigwedge^3 H_1(\mathcal{C}; \mathbb{Q})$ via the canonical surjection $\left(\bigwedge^3 H_1(\mathcal{C}; \mathbb{Q}) \right)^{\otimes n} \twoheadrightarrow S^n \bigwedge^3 H_1(\mathcal{C}; \mathbb{Q})$. Hence we can view $\mu(\boldsymbol{C})$ as an element of $S^n \bigwedge^3 H_1(\mathcal{C}; \mathbb{Q})$.

The bilinear form $\ell_{(S,\kappa)}(\boldsymbol{C})$ Let (S, κ) be a $\mathbb{Q}$SK–*pair*, ie a pair made of a $\mathbb{Q}$–sphere S and a null-homologous knot $\kappa \subset S$. Let $\boldsymbol{C} = (\boldsymbol{C}_1, \ldots, \boldsymbol{C}_n)$ be a null LP–surgery on (S, κ). Let $\mathcal{C} = \mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_n$ be the disjoint union of the associated total manifolds. Fix a lift $\tilde{C}_i$ of each C_i in $\tilde{E}$. We will define a *hermitian* form

$$\ell_{(S,\kappa)}(\boldsymbol{C}): H_1(\mathcal{C}; \mathbb{Q}) \times H_1(\mathcal{C}; \mathbb{Q}) \to \mathbb{Q}(t),$$

ie a $\mathbb{Q}$–bilinear form such that reversing the order of the arguments changes t to t^{-1}. Let $a \in H_1(\mathcal{C}_i; \mathbb{Q})$ and $b \in H_1(\mathcal{C}_j; \mathbb{Q})$ be homology classes that can be represented by simple closed curves $\alpha \subset \partial C_i$ and $\beta \subset \partial C_j$, disjoint if $i = j$. Note that such
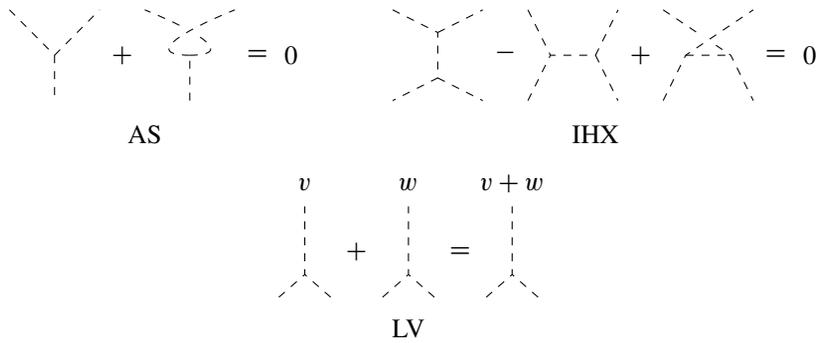
Figure 1: Relations AS, IHX and LV on Jacobi diagrams.

homology classes generate $H_1(\mathcal{C}; \mathbb{Q})$ over $\mathbb{Q}$. Let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be the copies of α and β in $\widetilde{C}_i$ and $\widetilde{C}_j$. Set

$$\ell_{(S,\kappa)}(C)(a, b) = \mathrm{lk}_e(\widetilde{\alpha}, \widetilde{\beta}),$$

where $\mathrm{lk}_e(\cdot, \cdot)$ stands for the equivariant linking number (see for instance Section 2.1 of [24] for a definition). We get a well-defined hermitian form $\ell_{(S,\kappa)}(C)$ associated with a choice of lifts of the C_i. We will keep this choice implicit; the statement of Theorem 1.1 is valid for any such choice.

Diagrammatic representations Let V be a rational vector space. A V–*coloured Jacobi diagram* is a unitrivalent graph whose trivalent vertices are oriented and whose univalent vertices are labelled by V, where an *orientation* of a trivalent vertex is a cyclic order of the three edges that meet at this vertex — fixed as in the pictures. Set

$$\mathcal{A}_{\mathbb{Q}}(V) = \frac{\mathbb{Q}\langle V\text{–coloured Jacobi diagrams}\rangle}{\mathbb{Q}\langle \mathrm{AS},\, \mathrm{IHX},\, \mathrm{LV}\rangle},$$

where the relations are as depicted in Figure 1. A symmetric tensor in $S^n \bigwedge^3 V$ can be represented by a Jacobi diagram via the embedding

$$S^n \bigwedge^3 V \to \mathcal{A}_{\mathbb{Q}}(V), \quad (u_1 \wedge v_1 \wedge w_1) \cdots (u_n \wedge v_n \wedge w_n) \mapsto$$

Now define a $\mathbb{Q}(t)$–*beaded Jacobi diagram* as a trivalent graph whose vertices are oriented and whose edges are oriented and labelled by $\mathbb{Q}(t)$. Set

$$\widetilde{\mathcal{A}}_{\mathbb{Q}(t)}(\varnothing) = \frac{\mathbb{Q}\langle \mathbb{Q}(t)\text{–beaded Jacobi diagrams}\rangle}{\mathbb{Q}\langle \mathrm{AS},\, \mathrm{IHX},\, \mathrm{LE},\, \mathrm{Hol},\, \mathrm{OR}\rangle},$$

$$x \overset{P}{\vert} + y \overset{Q}{\vert} = \overset{xP + yQ}{\vert}$$

LE

$$Q \overset{P}{\underset{R}{\diagup}} = {}^{t}Q \overset{{}^{t}P}{\underset{{}^{t}R}{\diagup}}$$

Hol

$$\overset{P(t)}{\vert} = \overset{P(t^{-1})}{\vert}$$

OR

Figure 2: Relations LE, Hol and OR on Jacobi diagrams.

where the relations are as depicted in Figures 1 and 2, with the IHX relation defined with the central edge labelled by 1. Define the *i–degree*, or *internal degree*, of any Jacobi diagram as its number of trivalent vertices. Given a hermitian form $\ell \colon V \times V \to \mathbb{Q}(t)$, one can *glue with* ℓ some legs of a V–coloured Jacobi diagram as depicted in Figure 3. If n is even, one can pairwise glue all legs of a V–coloured Jacobi diagram of i–degree n in order to get an element of $\widetilde{\mathcal{A}}_{\mathbb{Q}(t)}(\varnothing)$. This latter space is the target space of the Kricker invariant of $\mathbb{Q}$SK–pairs.

We can now state our main result, about the Kricker invariant $\widetilde{Z}$, proved in Section 6. Note that null LP–surgeries define a move on the set of $\mathbb{Q}$SK–pairs.

Theorem 1.1 *Let* (S, κ) *be a* $\mathbb{Q}$SK*–pair. Let* $C = (C_1, \ldots, C_n)$ *be a null LP–surgery on* (S, κ)*. Then*

$$\sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|} \widetilde{Z}((S,\kappa)(C_I)) \equiv_n \left(\begin{array}{c} \text{sum of all ways of gluing all legs} \\ \text{of } \mu(C) \text{ with } \frac{1}{2}\ell_{(S,\kappa)}(C) \end{array} \right),$$

where $\equiv_n$ *means "equal up to terms of i–degree at least* $n + 1$*".*

Example Let $(S^3, \mathcal{O})$ be the $\mathbb{Q}$SK–pair defined by the trivial knot $\mathcal{O}$ in the standard 3–sphere. Let C_1 and C_2 be regular neighbourhoods of the graphs Γ_1 and Γ_2 drawn in Figure 4. One can define an LP–surgery $C = (C_1, C_2)$ by associating with each Γ_i
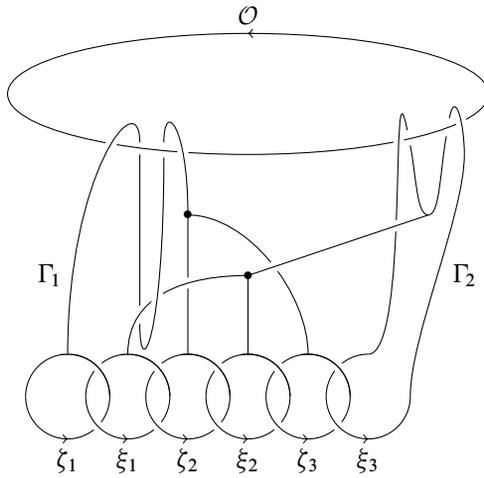

Figure 3: Gluing some legs of a Jacobi diagram with ℓ.

Figure 4: Surgery datum in $(S^3, \mathcal{O})$

a Borromean surgery; see for instance [24, Section 2.2]. The associated tensor is given by $\mu(C_1) = \zeta_1 \wedge \zeta_2 \wedge \zeta_3$ and $\mu(C_2) = \xi_1 \wedge \xi_2 \wedge \xi_3$. There are fifteen ways to glue all legs of

$$\mu(C) = \overset{\zeta_3 \quad \zeta_2 \quad \zeta_1 \quad \xi_3 \quad \xi_2 \quad \xi_1}{\diagdown \ \vdots \ \diagup \quad \diagdown \ \vdots \ \diagup}$$

with $\frac{1}{2}\ell_{(S^3, \mathcal{O})}(C)$; all associated diagrams but one are trivial by the relation LE since they have a trivially labelled edge. Now $\mathrm{lk}_e(\zeta_1, \xi_1) = 1$, $\mathrm{lk}_e(\zeta_2, \xi_2) = 1$ and $\mathrm{lk}_e(\zeta_3, \xi_3) = t^{-1}$, so that we finally get

$$\tilde{Z}((S^3, \mathcal{O})(C)) + \tilde{Z}((S^3, \mathcal{O})) \equiv_2 - \underset{1}{\overset{t}{\bullet \dashrightarrow \bullet}},$$

where the vanishing of two terms in the left-hand side is due to [6, Lemma 2.2].

1.3 Strategy

In this subsection, we give a rough overview of the strategy developed to prove Theorem 1.1.

The main object of this article is the construction of a functorial LMO invariant defined on a category of Lagrangian cobordisms with paths. The morphisms of this category are cobordisms between compact surfaces with one boundary component, satisfying a Lagrangian-preserving condition, with finitely many disjoint paths with
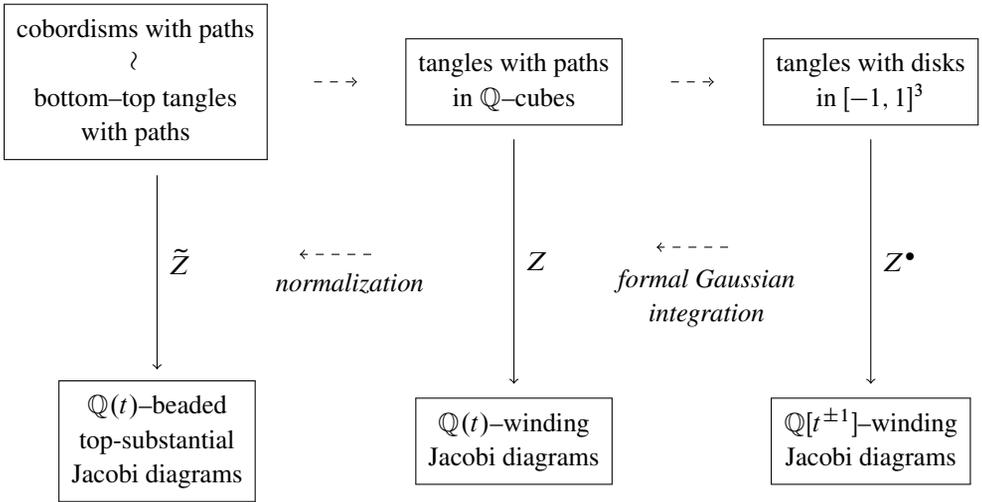
Figure 5: Scheme of construction for the invariant $\widetilde{Z}$.

fixed extremities which we think of as knots with a fixed part on the boundary. This category is equivalent to a category of bottom–top tangles in $\mathbb{Q}$–cubes, whose top part has a trivial linking matrix, with paths with fixed extremities. These bottom–top tangles can be viewed as morphisms in a category of (general) tangles with paths in $\mathbb{Q}$–cubes, with an important difference in the composition law. Now a tangle with paths in a $\mathbb{Q}$–cube can be expressed as the result of a surgery on a link in a tangle with trivial paths — segment lines — in $[-1, 1]^3$. To summarize, with a Lagrangian cobordism with paths we associate a tangle with disks — whose boundaries define the paths — in $[-1, 1]^3$ with a surgery link. This is represented in the first line of the scheme in Figure 5. We initiate the construction of the invariant at the "tangle with disks" level.

On the above-mentioned categories, we define functorial invariants valued in categories of Jacobi diagrams with beads, ie unitrivalent graphs whose univalent vertices are labelled by some finite set or embedded in some 1–manifold — the skeleton — and whose edges are labelled (beaded) by powers of t, polynomials in $\mathbb{Q}[t^{\pm 1}]$ or rational functions in $\mathbb{Q}(t)$. At the first step, we define a functor $Z^\bullet$ on the category of tangles with disks by applying the Kontsevich integral and adding a bead $t^{\pm 1}$ on the skeleton when the corresponding component meets a disk of the tangle. At a second step, we apply the invariant $Z^\bullet$ to surgery presentations of tangles with paths in $\mathbb{Q}$–cubes. We use the formal Gaussian integration methods introduced by Bar-Natan, Garoufalidis, Rozansky and Thurston in [2; 3] and adapted to the beaded setting in [12; 7]. We get a

functor Z on the category of tangles with paths in $\mathbb{Q}$–cubes. At the last step, given a Lagrangian cobordism with paths, we apply Z to the associated bottom–top tangles with paths and normalize it following [5] to obtain a functor $\widetilde{Z}$ on the category of Lagrangian cobordisms with paths. Functoriality allows us to prove splitting formulas for this invariant with respect to null Lagrangian-preserving surgeries.

Given a Lagrangian cobordism with one path between genus 0 surfaces, ie a $\mathbb{Q}$–cube with one path, one can glue a 3–ball to the boundary to get a $\mathbb{Q}$–sphere with a knot. In this way, the functor $\widetilde{Z}$ provides an invariant of $\mathbb{Q}SK$–pairs which coincides with the Kricker invariant for knots in $\mathbb{Z}$–spheres. Splitting formulas for this invariant are deduced from the splitting formulas for the functor $\widetilde{Z}$.

Plan of the paper   We define the domain categories of cobordisms and tangles in Section 2. In Section 3, we define the target categories of Jacobi diagrams and give the tools of formal Gaussian integration. Section 4 is devoted to the introduction of winding matrices, which will play the role of the linking matrices in the Cheptea–Habiro–Massuyeau construction of a functorial LMO invariant. The functors $Z^\bullet$, Z and $\widetilde{Z}$ are constructed in Section 5. At the end of this section, from the functor $\widetilde{Z}$, we deduce our version of the Kricker invariant for $\mathbb{Q}SK$–pairs; the behaviour of this invariant with respect to connected sum is stated. Finally, the splitting formulas are given in Section 6.

2   Domain categories: cobordisms and tangles

2.1   Cobordisms with paths

Given $g \in \mathbb{N}$, we fix a model surface F_g, compact, connected, oriented, of genus g, with one boundary component represented in Figure 6. It is equipped with a fixed basepoint $*$ and a fixed basis $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ of $\pi_1(F_g, *)$. Denote by $C_{g^-}^{g^+}$ the cube $[-1, 1]^3$ with g^+ handles on the top boundary and g^- tunnels in the bottom boundary. We have canonical embeddings $F_{g^+} \hookrightarrow \partial C_{g^-}^{g^+}$ and $F_{g^-} \hookrightarrow \partial C_{g^-}^{g^+}$. For
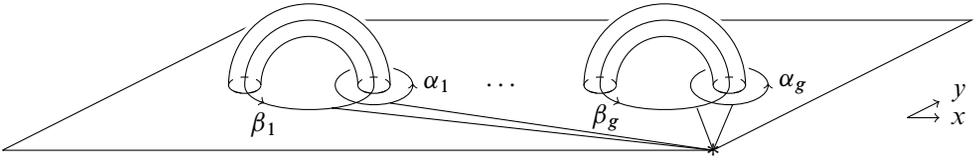
Figure 6: The model surface F_g.

$k \geq 0$ and $1 \leq i \leq k$, set $h_i(k) = \frac{2i}{k+1} - 1$. A *cordism with paths* from F_{g+} to F_{g-} is an equivalence class of triples (M, K, m) where

- M is a compact, connected, oriented 3–manifold,
- $m: \partial C_{g-}^{g+} \overset{\cong}{\Longrightarrow} \partial M$ is an orientation-preserving homeomorphism,
- $K = \bigsqcup_{i=1}^{k} K_i \subset M$ is a union of $k \geq 0$ oriented paths K_i from $m(0, 1, h_i(k))$ to $m(0, -1, h_i(k))$,
- $\hat{K} = \bigsqcup_{i=1}^{k} \hat{K}_i$ is an oriented *boundary link*, ie the $\hat{K}_i$ bound disjoint compact oriented surfaces in M, where $\hat{K}_i$ is the knot defined as the union of K_i with the line segments $[(0, -1, h_i(k)), (1, -1, h_i(k))]$, $[(1, -1, h_i(k)), (1, 1, h_i(k))]$ and $[(1, 1, h_i(k)), (0, 1, h_i(k))]$.

Two such triples are *equivalent* if they are related by an orientation-preserving homeo-morphism which respects the boundary parametrizations and identifies the paths. We get embeddings $m_+: F_{g+} \hookrightarrow \partial M$ and $m_-: F_{g-} \hookrightarrow \partial M$.

Define a category $\widetilde{Cob}$ of cobordisms with paths whose objects are nonnegative integers and whose set of morphisms $\widetilde{Cob}(g^+, g^-)$ is the set of cobordisms with paths from F_{g+} to F_{g-}. The composition of a cobordism (M, K, m) from F_g to F_f with a cobordism (N, J, n) from F_h to F_g is given by gluing N on the top of M, identifying
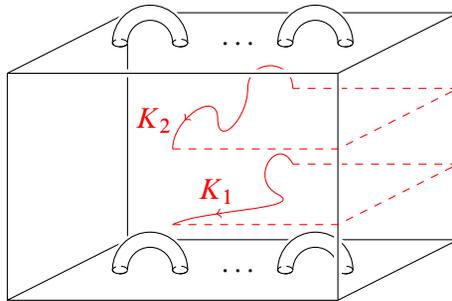

Figure 7: A Lagrangian cobordism with paths.

$m_+(M)$ with $n_-(N)$ and reparametrizing the new manifold. The identity of $g \in \mathbb{N}$ is the cobordism $F_g \times [-1, 1]$ with natural boundary parametrization and no path.

Forgetting the datum of the paths in the cobordisms, one gets the category $\mathcal{C}ob$ described in [5], which we view as the subcategory of $\widetilde{\mathcal{C}ob}$ of cobordisms with no path. For a cobordism (M, m) and a cobordism with paths (N, J, n), define the tensor product $(M, m) \otimes (N, J, n)$ by horizontal juxtaposition in the x direction.

We now define the subcategory of $\widetilde{\mathcal{C}ob}$ of Lagrangian cobordisms with paths. Set $A_g = \ker(\mathrm{incl}_*\colon H_1(F_g; \mathbb{Q}) \to H_1(C_0^g; \mathbb{Q}))$ and $B_g = \ker(\mathrm{incl}_*\colon H_1(F_g; \mathbb{Q}) \to H_1(C_g^0; \mathbb{Q}))$. These are Lagrangian subspaces of $H_1(F_g; \mathbb{Q})$ with respect to the intersection form and A_g (resp. B_g) is generated by the homology classes of the curves α_i (resp. β_i). A cobordism with paths (M, K, m) from F_{g^+} to F_{g^-} is *Lagrangian(-preserving)* if the following conditions are satisfied:

- $H_1(M; \mathbb{Q}) = (m_-)_*(A_{g^-}) \oplus (m_+)_*(B_{g^+})$.

- $(m_+)_*(A_{g^+}) \subset (m_-)_*(A_{g^-})$ as subspaces of $H_1(M; \mathbb{Q})$.

The Lagrangian property is preserved by composition, and we denote by $\widetilde{\mathcal{LCob}}$ the subcategory of $\widetilde{\mathcal{C}ob}$ of Lagrangian cobordisms with paths. The subcategory of Lagrangian cobordisms with no path is the category $\mathcal{LCob}$ — denoted by $\mathbb{Q}\mathcal{LCob}$ in [20].

Define categories $\mathcal{C}ob_q$, $\widetilde{\mathcal{C}ob}_q$, $\mathcal{LCob}_q$ and $\widetilde{\mathcal{LCob}}_q$ of q–cobordisms with objects the noncommutative words in the single letter $\bullet$ and with set of morphisms from a word on g^+ letters to a word on g^- letters the set of morphisms from g^+ to g^- in $\mathcal{C}ob$, $\widetilde{\mathcal{C}ob}$, $\mathcal{LCob}$ and $\widetilde{\mathcal{LCob}}$, respectively.

Let (M, K, m) be a Lagrangian cobordism with paths from g^+ to g^-. Since the space $H_1(M; \mathbb{Q})$ is nontrivial in general, we have to adapt the definition of null LP–surgeries given in the introduction. Let $N(K)$ be a tubular neighbourhood of K in M and set $E = M \setminus \mathrm{Int}(N(K))$. A standard homological computation gives $H_2(M; \mathbb{Q}) = 0$, so that the exact sequence in homology associated with the pair (M, E) provides the short exact sequence

$$0 \to H_2(N(K), N(K) \cap E; \mathbb{Q}) \to H_1(E; \mathbb{Q}) \to H_1(M; \mathbb{Q}) \to 0.$$

The image of $H_2(N(K), N(K) \cap E; \mathbb{Q})$ in $H_1(E; \mathbb{Q})$ is a subspace $H_K \cong \mathbb{Q}^k$ generated by meridians of the components of K, where k is the number of these components. Now, the parametrizations $m_\pm$ of the top and bottom boundaries of M

can be decomposed into injective maps as

$$F_{g\pm} \xrightarrow{m_\pm} M$$
$$e_\pm \searrow \qquad \nearrow$$
$$E$$

Hence we have a canonical decomposition of $H_1(E;\mathbb{Q})$ as

$$H_1(E;\mathbb{Q}) = H_K \oplus (e_-)_*(A_{g-}) \oplus (e_+)_*(B_{g+}),$$

where $(e_-)_*(A_{g-}) \oplus (e_+)_*(B_{g+}) \cong H_1(M;\mathbb{Q})$ via the inclusion map. We say that a $\mathbb{Q}$–handlebody $C \subset M \setminus K$ is *null with respect to* K if the composed map

$$H_1(C;\mathbb{Q}) \xrightarrow{\mathrm{incl}_*} H_1(M \setminus K;\mathbb{Q}) \to H_K$$

has a trivial image, where the second map is the projection on the first factor in the above decomposition of $H_1(M \setminus K;\mathbb{Q}) \cong H_1(E;\mathbb{Q})$. A *null LP–surgery* on (M, K) is an LP–surgery $C = (C_1, \ldots, C_n)$ on $M \setminus K$ such that each C_i is null with respect to K.

2.2 Bottom–top tangles with paths

Let us define the category of bottom–top tangles with paths. For a positive integer $g \geq 0$, let $(p_1, q_1), \ldots, (p_g, q_g)$ be g pairs of points uniformly distributed on $[-1, 1] \times \{0\} \subset [-1, 1]^2 \cong F_0$ as represented in Figure 8. A *bottom–top tangle with paths* of type (g^+, g^-) is an equivalence class of triples (B, K, γ) where

- $(B, K) = (B, K, b)$ is a cobordism with paths from F_0 to F_0,

- $\gamma = (\gamma^+, \gamma^-)$ is a framed oriented tangle in B with g^+ components γ_i^+ from $b(\{p_i\} \times \{1\})$ to $b(\{q_i\} \times \{1\})$ and g^- components γ_i^- from $b(\{q_i\} \times \{-1\})$ to $b(\{p_i\} \times \{-1\})$,

- $\hat{K}$ is a boundary link in $B \setminus \gamma$.

Two such triples (B, K, γ) and (B', K', γ') are *equivalent* if (B, K) and (B', K') are related by an equivalence which identifies γ and γ'.
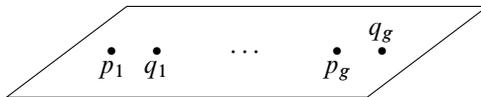

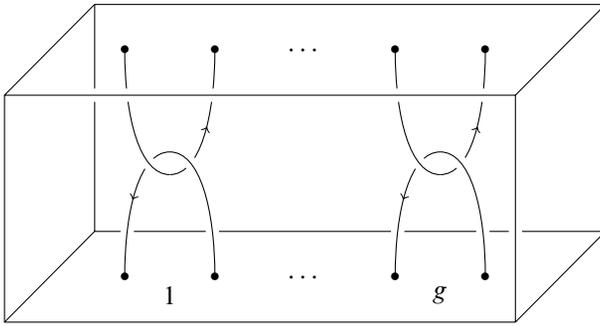Figure 8: The pairs of points (p_i, q_i) on $[-1, 1]^2$.

Figure 9: The bottom–top tangle T_g in $[-1, 1]^3$.

In order to define the composition, we need the bottom–top tangle $([-1, 1]^3, \varnothing, T_g)$ represented in Figure 9. The composition of a bottom–top tangle (B, K, γ) of type (g, f) with a bottom–top tangle (C, J, υ) of type (h, g) is given by first making the composition $(B, K) \circ ([-1, 1]^3, \varnothing) \circ (C, J)$ in the category $\widetilde{\mathcal{Cob}}$ and then performing the surgery on the $2g$ components link $\gamma^+ \cup T_g \cup \upsilon^-$. We get a category ${}^t_b \widetilde{\mathcal{T}}$ whose objects are nonnegative integers and whose set of morphisms ${}^t_b \widetilde{\mathcal{T}}(g^+, g^-)$ is the set of bottom–top tangles with paths of type (g^+, g^-). The identity of $g \in \mathbb{N}$ is the bottom–top tangle in $[-1, 1]^3$ with no path, represented in Figure 10.

Forgetting the datum of the paths, one gets the category ${}^t_b \mathcal{T}$ of bottom–top tangles introduced in [5], which we view as the subcategory of ${}^t_b \widetilde{\mathcal{T}}$ of bottom–top tangles with no path. For a bottom–top tangle (B, γ) and a bottom–top tangle with paths (C, J, υ), define the tensor product $(B, \gamma) \otimes (C, J, \upsilon)$ by horizontal juxtaposition in the x direction. Define categories ${}^t_b \mathcal{T}_q$ and ${}^t_b \widetilde{\mathcal{T}}_q$ of bottom–top q–tangles with objects the noncommutative words in the single letter $\bullet$.
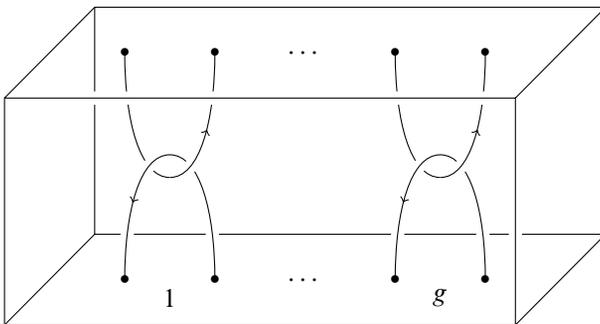

Figure 10: The bottom–top tangle Id_g.

The following result is a direct adaptation of [5, Theorem 2.10], which gives an isomorphism $D\colon {}^t_b\mathcal{T} \to \mathcal{C}ob$. The map D is defined by digging tunnels around the components of the tangle.

Proposition 2.1 *There is an isomorphism $D\colon {}^t_b\widetilde{\mathcal{T}} \to \widetilde{\mathcal{C}ob}$ which identifies ${}^t_b\mathcal{T}$ with $\mathcal{C}ob$ and preserves the tensor product on ${}^t_b\mathcal{T} \otimes {}^t_b\widetilde{\mathcal{T}}$.*

Let (B, K, γ) be a bottom–top tangle with paths in a $\mathbb{Q}$–cube. Let $\bar{\gamma}$ be the link obtained by closing the components of γ with the line segments $[(p_i, \pm 1), (q_i, \pm 1)]$. Define the linking matrix $\mathrm{Lk}(\gamma)$ of γ in B with the linkings of the components of $\bar{\gamma}$. The characterization of the bottom–top tangles sent onto Lagrangian cobordisms by D given in [5, Lemma 2.12] directly generalizes to:

Lemma 2.2 *Given a bottom–top tangle with paths (B, K, γ), the cobordism with paths $D(B, K, \gamma)$ is Lagrangian if and only if B is a $\mathbb{Q}$–cube and $\mathrm{Lk}(\gamma^+)$ is trivial.*

2.3 Tangles with paths and tangles with disks

Given a cobordism (B, b) from F_0 to F_0, a *tangle γ in B* is an isotopy (rel ∂B) class of framed oriented tangles whose boundary points lie on the top and bottom surfaces and are uniformly distributed along the line segments $[-1, 1] \times \{0\} \times \{1\}$ and $[-1, 1] \times \{0\} \times \{-1\}$ in $\partial B = b(\partial[-1, 1]^3)$. Associate with each boundary point of γ the sign $+$ if γ is oriented downwards at that point and the sign $-$ otherwise. This provides two words in the letters $+$ and $-$, one for the top surface and the other for the bottom surface. Lifting these two words into nonassociative words $w_t(\gamma)$ and $w_b(\gamma)$ in the letters $(+, -)$, one gets a *q–tangle*. A *q–tangle γ* in a cobordism with paths (B, K, b) defines a *q–tangle with paths (B, K, γ)* if $\hat{K}$ is a boundary link in $B \setminus \gamma$.

Define two categories $\mathcal{T}_q\mathcal{C}ub$ and $\widetilde{\mathcal{T}}_q\mathcal{C}ub$ with objects the nonassociative words in the letters $(+, -)$ and morphisms the q–tangles in $\mathbb{Q}$–cubes for $\mathcal{T}_q\mathcal{C}ub$ and the q–tangles with paths in $\mathbb{Q}$–cubes for $\widetilde{\mathcal{T}}_q\mathcal{C}ub$, up to orientation-preserving homeomorphism respecting the boundary parametrization. Composition is given by vertical juxtaposition. Given a morphism (C, υ) in $\mathcal{T}_q\mathcal{C}ub$ and a morphism (B, K, γ) in $\widetilde{\mathcal{T}}_q\mathcal{C}ub$, define the tensor product $(C, \upsilon) \otimes (B, K, \gamma)$ by horizontal juxtaposition in the x direction.

Lemma 2.3 *Let (B, K, γ) be a q–tangle with paths in a $\mathbb{Q}$–cube. There exist a q–tangle with paths $([-1, 1]^3, \Xi, \eta)$ and a framed link $L \subset [-1, 1]^3 \setminus (\Xi \cup \eta)$, with Ξ a union of line segments and L null-homotopic in $[-1, 1]^3 \setminus \Xi$, such that (B, K, γ)*
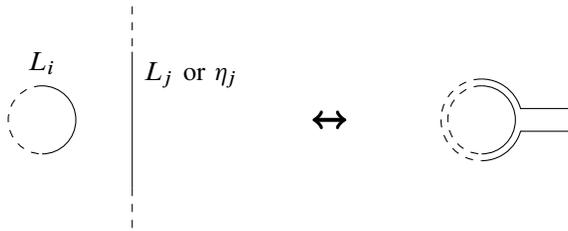
Figure 11: KII move.

is obtained from $([-1,1]^3, \Xi, \eta)$ *by surgery on* L. *Moreover, two such surgery links are related by the following Kirby moves*: *the blow-up/blow-down move KI which adds or removes a split trivial component with framing* ± 1 *unknotted with* η, *and the handleslide move KII which adds a surgery component to another surgery component or to a component of the tangle* η (*see* Figure 11).

Proof Let Σ be a Seifert surface of $\widehat{K}$ which is the disjoint union of Seifert surfaces of the $\widehat{K}_i$. Choose Σ disjoint from γ. Take a link $J \subset B$ such that surgery on J gives $[-1,1]^3$. Performing isotopies on J if necessary, we can assume that J does not meet Σ. The handles of (the image in $[-1,1]^3$ of) Σ can be unlinked by adding surgery components as shown in Figure 12. In this way, $\widehat{K}$ can be turned into a trivial link. This provides a surgery link in $B \setminus (K \cup \gamma)$, disjoint from Σ, such that surgery on this link changes (B, K, γ) to a q–tangle with paths $([-1,1]^3, \Xi, \eta)$, as required. Let L be an inverse surgery link. It is null-homotopic in $[-1,1]^3 \setminus \Xi$ since it is disjoint from Σ.

For the last assertion, apply [10, Theorem 3.1] in $[-1,1]^3 \setminus \Xi$. Note that a split trivial component with framing ± 1 can always be unknotted from η using the KII move. $\square$

A family $(([-1,1]^3, \Xi, \eta), L)$ satisfying the conditions of the lemma with L oriented is a *surgery presentation* of (B, K, γ). When γ (and thus η) is a bottom–top tangle, the components of η can be closed by line segments in the top and bottom surfaces. The obtained curves are null-homotopic in $[-1,1]^3 \setminus \Xi$ since $\widehat{\Xi}$ is a boundary link in $[-1,1]^3 \setminus \eta$.
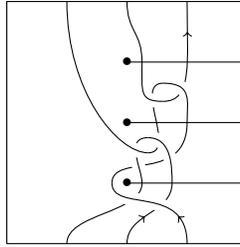

Figure 12: Surgery changing a crossing.

Figure 13: Diagram of a tangle with disks.

The notion of a q–tangle in $[-1,1]^3$ with trivial paths, ie line segments, is equivalent to the following one. A *q–tangle with disks* is an equivalence class of pairs (γ, k) where γ is a q–tangle in $[-1,1]^3$, k is a nonnegative integer understood as the datum of k disks $d_i = [0,1] \times [-1,1] \times \{h_i(k)\}$ and the link $\bigsqcup_{i=1}^k \partial d_i$ associated with the paths $d_i^\partial = \{0\} \times [-1,1] \times \{h_i(k)\}$ is a boundary link in $[-1,1]^3 \setminus \gamma$. An example of such a tangle with disks is drawn in Figure 13, projected in the y direction. Equivalence of such pairs is defined as isotopy relative to $(\partial[-1,1]^3) \cup (\bigcup_{i=1}^k d_i^\partial)$. Define two categories $\mathcal{T}_q$ and $\widetilde{\mathcal{T}}_q$ with objects the nonassociative words in the letters $(+,-)$ and morphisms the q–tangles for $\mathcal{T}_q$ and the q–tangles with disks for $\widetilde{\mathcal{T}}_q$. Composition is given by vertical juxtaposition. Given a q–tangle γ and a q–tangle with disks (υ, k), define the tensor product $\gamma \otimes (\upsilon, k)$ in $\widetilde{\mathcal{T}}_q\big((w_t(\gamma))(w_t(\upsilon)), (w_b(\gamma))(w_b(\upsilon))\big)$ by horizontal juxtaposition in the x direction. Define similarly two categories $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ of tangles and tangles with disks in $[-1,1]^3$ with objects the associative words in the letters $(+,-)$.

3   Target categories: Jacobi diagrams with beads

3.1   Diagram spaces

For a compact oriented 1–manifold X and a finite set C, a *Jacobi diagram on* (X, C) is a unitrivalent graph whose trivalent vertices are oriented and whose univalent vertices are embedded in X or labelled by C, where an orientation of a trivalent vertex is a cyclic order of the three edges that meet at this vertex—fixed as $\vcenter{\hbox{\includegraphics[height=1em]{vertex}}}$ in the pictures. The manifold X is called the *skeleton* of the diagram. Next, let R be the ring $\mathbb{Q}[t^{\pm 1}]$ or $\mathbb{Q}(t)$. An *R–beaded Jacobi diagram on* (X, C) is a Jacobi diagram on (X, C) whose graph edges are oriented and labelled by R. Last, an *R–winding Jacobi diagram on* (X, C) is an R–beaded Jacobi diagram on (X, C) whose skeleton is viewed as a union of edges—defined by the embedded vertices—that are labelled by powers

Figure 14: Relations STU and $\mathrm{Hol_w}$ on Jacobi diagrams.

of t, with the condition that the product of the labels on each component of X is 1. As defined in the introduction, the i–degree of a trivalent diagram is its number of trivalent vertices. Set

$$\mathcal{A}(X, *_C) = \frac{\mathbb{Q}\langle \text{Jacobi diagrams on } (X, C)\rangle}{\mathbb{Q}\langle \text{AS, IHX, STU}\rangle},$$

$$\tilde{\mathcal{A}}_R(X, *_C) = \frac{\mathbb{Q}\langle R\text{–beaded Jacobi diagrams on } (X, C)\rangle}{\mathbb{Q}\langle \text{AS, IHX, STU, LE, OR, Hol}\rangle},$$

$$\tilde{\mathcal{A}}_R^{\mathrm{w}}(X, *_C) = \frac{\mathbb{Q}\langle R\text{–winding Jacobi diagrams on } (X, C)\rangle}{\mathbb{Q}\langle \text{AS, IHX, STU, LE, OR, Hol, Hol}_{\mathrm{w}}\rangle},$$

with the relations in Figures 1, 2 and 14, where the IHX relation for beaded and winding diagrams is defined with the central edge labelled by 1. In the STU relation, the edges corresponding to each other have the same orientation and label. In the pictures, the skeleton is represented with full lines and the graph with dashed lines. We indeed consider the i–degree completion of these vector spaces, keeping the same notation.

Remark For diagrams in $\tilde{\mathcal{A}}_R^{\mathrm{w}}(X, *_C)$, the condition on the labels on the skeleton implies that all labels can be pushed off each component of the skeleton using the $\mathrm{Hol_w}$ relation. When the component is an interval, there is a unique way to do so. Hence, when X contains only intervals, $\tilde{\mathcal{A}}_R^{\mathrm{w}}(X, *_C)$ is isomorphic to $\tilde{\mathcal{A}}_R(X, *_C)$.

For a finite set S, denote by $\uparrow_S$ (resp. $\bigcirc_S$) the manifold made of $|S|$ intervals (resp. circles) indexed by the elements of S. In the following, $\bar{\mathcal{A}}$ stands for $\mathcal{A}$, $\tilde{\mathcal{A}}_R$ or $\tilde{\mathcal{A}}_R^{\mathrm{w}}$. In [1, Theorem 8], Bar-Natan defines a formal PBW isomorphism

$$\chi_S \colon \bar{\mathcal{A}}(X, *_{C \cup S}) \xrightarrow{\cong} \bar{\mathcal{A}}(X \cup \uparrow_S, *_C).$$

For a Jacobi diagram D, the image $\chi_S(D)$ is the average of all possible ways to attach the s–coloured vertices of D on the corresponding s–indexed interval in $\uparrow_S$ for each $s \in S$. The setting of [1] is not exactly the same, but the argument adapts directly. When $|S| = 1$, closing the S–labelled component gives an isomorphism from $\bar{\mathcal{A}}(X \cup \uparrow_S, *_C)$ to $\bar{\mathcal{A}}(X \cup \bigcirc_S, *_C)$ [1, Lemma 3.1]. However, this isomorphism does not hold for $|S| > 1$. To recover an isomorphism onto $\bar{\mathcal{A}}(X \cup \bigcirc_S, *_C)$, some "link relations"
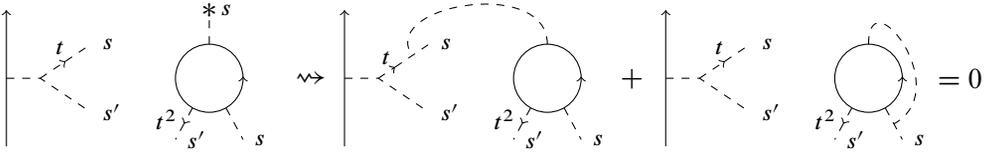
Figure 15: A link relation.

were introduced in [3, Section 5.2]. We recall these relations and introduce additional "winding relations".

Given a (beaded, winding) Jacobi diagram D on $(X, C \cup S)$ and a univalent vertex $*$ of D labelled by $s \in S$, define the associated *link relation* as the vanishing of the sum of all diagrams obtained from D by gluing the vertex $*$ on the edges adjacent to a univalent s–labelled vertex, as follows: $\overset{\bullet\ s}{\underset{{}_{|}}{-\bullet\ *}}$; see Figure 15 (we omit the orientation of the edges when it is not relevant thanks to the OR relation). Given a winding Jacobi diagram D on $(X, C \cup S)$, a label $s \in S$ and an integer k, the associated *winding relation* identifies D with the diagram obtained from D by *pushing t^k at each s–labelled vertex*, ie by multiplying the label of each edge adjacent to a univalent s–labelled vertex by t^k if the orientation of the edge goes backward the vertex and by t^{-k} otherwise; see Figure 16. Denote by $\overline{\mathcal{A}}(X, *_C, \circledast_S)$ (resp. $\overline{\mathcal{A}}(X, *_C, \circledwithdot_S)$) the quotient of $\overline{\mathcal{A}}(X, *_{C \cup S})$ by all link relations (resp. all link and winding relations) on S–labelled vertices. Note that if X contains no closed component, then the spaces $\widetilde{\mathcal{A}}_R^{\mathrm{w}}(X, *_C, \circledwithdot_S)$ and $\widetilde{\mathcal{A}}_R(X, *_C, \circledwithdot_S)$ are isomorphic. When some of the sets X, C and S are empty, we simply drop the corresponding notation, mentioning $\varnothing$ only when they are all empty.

Proposition 3.1 *The isomorphisms* $\chi_S \colon \overline{\mathcal{A}}(X, *_{C \cup S}) \overset{\cong}{\Longrightarrow} \overline{\mathcal{A}}(X \cup \uparrow_S, *_C)$ *descend to isomorphisms*

$$\chi_S \colon \mathcal{A}(X, *_C, \circledast_S) \overset{\cong}{\Longrightarrow} \mathcal{A}(X \cup \circlearrowright_S, *_C),$$
$$\chi_S \colon \widetilde{\mathcal{A}}_R(X, *_C, \circledast_S) \overset{\cong}{\Longrightarrow} \widetilde{\mathcal{A}}_R(X \cup \circlearrowright_S, *_C),$$
$$\chi_S \colon \widetilde{\mathcal{A}}_R^{\mathrm{w}}(X, *_C, \circledwithdot_S) \overset{\cong}{\Longrightarrow} \widetilde{\mathcal{A}}_R^{\mathrm{w}}(X \cup \circlearrowright_S, *_C).$$
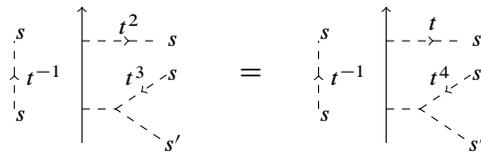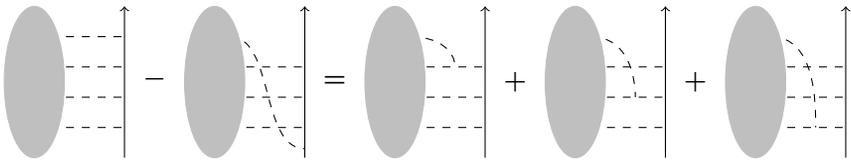

Figure 16: A winding relation.

Figure 17: Recovering the link relation.

Proof In the case $\bar{\mathcal{A}} = \mathcal{A}$ or $\widetilde{\mathcal{A}}_R$, this is [3, Theorem 3]. We recall briefly their argument in order to add the consideration of the winding relations when $\bar{\mathcal{A}} = \widetilde{\mathcal{A}}_R^{\mathrm{w}}$.

The fact that the images by χ_S of the link relations map to 0 in $\bar{\mathcal{A}}(X \cup \bigcirc_S, *_C)$ follows from the STU relation. For the winding relations, it follows from the $\mathrm{Hol_w}$ relation applied at each univalent vertex glued on the s–labelled component, where s is the label involved in the relation. Now, take two diagrams in $\bar{\mathcal{A}}(X \cup \uparrow_S, *_C)$ that are identified when closing an S–labelled component. We have to consider the two situations depicted in Figures 17 and 18, where the gray zone represents a hidden part of the diagram. Equalities are obtained by applying STU relations in the first case and $\mathrm{Hol_w}$ relations in the second case. Application of χ^{-1} to the right members provides linear combinations of the same sum with the skeleton component dropped and possibly trees glued. Using the IHX relation in the first case (resp. the Hol relation in the second case), we obtain link relations (resp. winding relations). $\square$

3.2 Product and coproduct

We first define a coproduct on the diagram spaces of the previous subsection. Given a (beaded, winding) Jacobi diagram D on (X, C), denote by $\ddot{D}$ its graph part, and by $\ddot{D}_i$ for $i \in I$ the connected components of $\ddot{D}$. Set $D_J = D \setminus \left(\bigsqcup_{i \in I \setminus J} \ddot{D}_i \right)$. In the winding case, multiply the labels of the concatenated edges of the skeleton. Define the
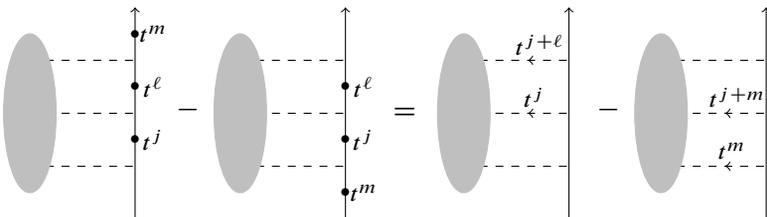

Figure 18: Recovering the winding relation (with $j + \ell + m = 0$).

coproduct of a diagram D by

$$\Delta(D) = \sum_{J \subset I} D_J \otimes D_{I \setminus J}.$$

Note that the different relations on Jacobi diagrams respect the coproduct. This provides a notion of *group-like* elements, ie elements G such that $\Delta(G) = G \otimes G$.

Set $\bar{\mathcal{A}} = \mathcal{A}$ or $\tilde{\mathcal{A}}_R$. We will define a Hopf algebra structure on $\bar{\mathcal{A}}(*_C)$. Define the product of two diagrams as the disjoint union. The unit $\epsilon \colon \mathbb{Q} \to \bar{\mathcal{A}}(*_C)$ is defined by $\epsilon(1) = \varnothing$ and the counit $\varepsilon \colon \bar{\mathcal{A}}(*_C) \to \mathbb{Q}$ is given by $\varepsilon(D) = 0$ if $D \neq \varnothing$ and $\varepsilon(\varnothing) = 1$. The antipode is given by $D \mapsto (-1)^{|I|} D$. We finally have a structure of a graded Hopf algebra on $\bar{\mathcal{A}}(*_C)$, where the grading is given by the i–degree. It is known that an element in a graded Hopf algebra is group-like if and only if it is the exponential of a *primitive* element, ie an element G such that $\Delta(G) = 1 \otimes G + G \otimes 1$. Here, the primitive elements are the series of connected diagrams.

The isomorphisms χ of the previous subsection are not algebra morphisms, but they preserve the coproduct. For the spaces $\bar{\mathcal{A}}(*_C)$ with $\bar{\mathcal{A}} = \mathcal{A}$ or $\tilde{\mathcal{A}}_R$, we have an exponential map associated with the product. For general (beaded, winding) Jacobi diagrams, we will use the notation $\exp_{\sqcup}$, namely exponential with respect to the disjoint union, for linear combination of diagrams with no univalent vertex embedded in the skeleton, where the disjoint union applies only to the graph part.

3.3 Formal Gaussian integration

This part aims at defining a formal Gaussian integration along S on $\tilde{\mathcal{A}}_R^{\mathrm{w}}(X, *_{C \cup S})$.

Definition 3.2 A (beaded, winding) Jacobi diagram on $(X, C \cup S)$ is *substantial* if it has no *strut*, ie no isolated dashed edge. It is *S–substantial* if it has no S–strut, ie no strut with both vertices labelled in S.

Given two (beaded, winding) Jacobi diagrams D and E on $(X, C \cup S)$, one of whose is S–substantial, define $\langle D, E \rangle_S$ as the sum of all diagrams obtained by gluing all s–coloured vertices of D with all s–coloured vertices of E for all $s \in S$ — if the numbers of s–coloured vertices in D and E do not match for some $s \in S$, then $\langle D, E \rangle_S = 0$. In the beaded and winding cases, we must specify the orientation and label of the created edges. Such an edge is the gluing of two or three edges in the initial diagrams. Fix arbitrarily the orientation of the new edge. Let $P(t)$ (resp. $Q(t)$) be the
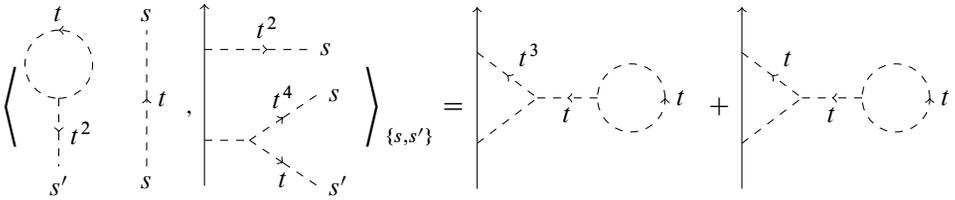
Figure 19: Bracketing diagrams.

product of the labels of the initial edges whose orientation coincides (resp. does not coincide). Define the label of the new edge as $P(t)Q(t^{-1})$; see Figure 19. We have the following immediate lemma:

Lemma 3.3 *If D' and E' are obtained from D and E by applying the same winding relation on s–labelled vertices for some $s \in S$, then $\langle D, E \rangle_S = \langle D', E' \rangle_S$.*

This bracketing defines a $\mathbb{Q}$–bilinear operator on the diagram spaces $\bar{\mathcal{A}}(X, *_{C \cup S})$ for $\bar{\mathcal{A}} = \mathcal{A}$, $\tilde{\mathcal{A}}_R$ or $\tilde{\mathcal{A}}_R^{\mathrm{w}}$.

Theorem 3.4 (Jackson, Moffatt and Morales) *Assume the 1–manifold X is a disjoint union of intervals. If G and H are group-like in $\bar{\mathcal{A}}(X, *_{C \cup S})$, then $\langle G, H \rangle_S$ is also group-like.*

Proof When $X = \varnothing$, it is [11, Theorem 2.4]. The case of a nonempty X follows since the isomorphism $\chi_{\pi_0(X)} \colon \bar{\mathcal{A}}(*_{\pi_0(X) \cup C \cup S}) \overset{\cong}{\longrightarrow} \bar{\mathcal{A}}(X, *_{C \cup S})$ preserves the coproduct and the bracketing $\langle \cdot, \cdot \rangle_S$. $\qquad\square$

Notation If $W = (W_{ij}(t))_{i,j \in S}$ is an (S, S)–matrix with coefficients in $\mathbb{Q}(t)$, we also write $W = \sum_{i,j \in S} {}^j_i W_{ij}(t)$.

Definition 3.5 An element $G \in \tilde{\mathcal{A}}_{\mathbb{Q}[t^{\pm 1}]}^{\mathrm{w}}(X, *_{C \cup S})$ is *Gaussian* if

$$G = \exp_{\sqcup}\left(\tfrac{1}{2} W(t)\right) \sqcup H$$

where $W(t)$ is an (S, S)–matrix with coefficients in $\mathbb{Q}[t^{\pm 1}]$ and H is S–substantial. If $\det(W(t)) \neq 0$, then G is *nondegenerate* and we set

$$\int_S G = \langle \exp_{\sqcup}\left(-\tfrac{1}{2} W^{-1}(t)\right), H \rangle_S \in \tilde{\mathcal{A}}_{\mathbb{Q}(t)}^{\mathrm{w}}(X, *_C).$$

Lemma 3.6 Let $G = \exp_{\sqcup}\left(\frac{1}{2}W(t)\right) \sqcup H$ be a nondegenerate Gaussian in the space $\widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}[t^{\pm 1}]}(X, *_{C \cup S})$.

- If a nondegenerate Gaussian $\exp_{\sqcup}\left(\frac{1}{2}W(t)\right) \sqcup H'$ is equal to G in the space $\widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}[t^{\pm 1}]}(X, *_C, \circledast_S)$, then $\int_S \left(\exp_{\sqcup}\left(\frac{1}{2}W(t)\right) \sqcup H'\right) = \int_S G$.
- If $G' = \exp_{\sqcup}\left(\frac{1}{2}W'(t)\right) \sqcup H'$ is obtained from $G = \exp_{\sqcup}\left(\frac{1}{2}W(t)\right) \sqcup H$ by applying a winding relation, then $\int_S G' = \int_S G$.

Proof The first point is essentially given by the proof of Bar-Natan and Lawrence [4, Proposition 2.2] in the nonbeaded case. Here, multiplication by $\overset{\scriptscriptstyle \shortmid S}{\underset{\scriptscriptstyle \shortmid S}{\lightning}} W_{ss}(t)$ does not preserve the link relations, but the supplementary terms vanish thanks to the AS relation when applying $\left\langle \cdot, \exp_{\sqcup}\left(-\frac{1}{2}\overset{\scriptscriptstyle \shortmid S}{\underset{\scriptscriptstyle \shortmid S}{\lightning}} W_{ss}(t)\right)\right\rangle$.

The second point follows from Lemma 3.3. □

3.4 Categories of diagrams

For $\bar{\mathcal{A}} = \mathcal{A}$, $\widetilde{\mathcal{A}}_R$ or $\widetilde{\mathcal{A}}^{\mathrm{w}}_R$, define a category $\overline{\mathcal{A}}$ whose objects are associative words in the letters $(+, -)$ and whose set of morphisms are $\overline{\mathcal{A}}(v, u) = \bigoplus_X \overline{\mathcal{A}}(X)$, where X runs over all compact oriented 1–manifolds with boundary identified with the set of letters of u and v, with the following sign convention: u has a $+$ when the orientation of X goes towards the boundary point and a $-$ when it goes backward, and the converse for v. Composition is given by vertical juxtaposition, where the label of the created edges in the case of beaded or winding diagrams is defined with the same rule as in the definition of $\langle D, E \rangle$. The tensor product given by disjoint union defines a strict monoidal structure on $\overline{\mathcal{A}}$.

We finally define the target category of our extended Kricker invariant.

Notation Given a positive integer g and a symbol $\natural$, set $\lfloor g \rceil^{\natural} = \{1^{\natural}, \ldots, g^{\natural}\}$. Set $\lfloor 0 \rceil^{\natural} = \varnothing$.

Definition 3.7 Fix nonnegative integers f and g. An R–beaded Jacobi diagram on $(\varnothing, \lfloor g \rceil^+ \cup \lfloor f \rceil^-)$ is *top-substantial* if it is $\lfloor g \rceil^+$–substantial.

Given two such diagrams D and E, define their composition $D \circ E$ as the sum of all ways of gluing all i^+–labelled vertices of D with all i^-–labelled vertices of E, fixing the orientations and labels of the created edges as in the definition of $\langle D, E \rangle_S$. We get

a category ${}^{\mathrm{ts}}\widetilde{\mathcal{A}}$ whose objects are nonnegative integers, with set of morphisms from g to f the subspace of $\widetilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\lfloor g \rceil + \cup \lfloor f \rceil -})$ generated by top-substantial diagrams. The identity of g is $\exp_{\sqcup}(\sum_{i=1}^{g} \vdots \frac{i^{+}}{i^{-}} \vdots)$ — the sum is trivial if $g = 0$. The tensor product defined by disjoint union of diagrams provides ${}^{\mathrm{ts}}\widetilde{\mathcal{A}}$ a strict monoidal structure.

4 Winding matrices

In this section, we define winding matrices associated with tangles with disks and bottom–top tangles with paths and interpret them as equivariant linking matrices in the case of bottom–top tangles. They will be useful in expressions of our invariant and splitting formulas.

4.1 First definition

We first define winding matrices in tangles with disks. Let (γ, k) be a tangle with disks d_ℓ. Write γ as the disjoint union of components γ_i for $i = 1, \ldots, n$. Fix a diagram of (γ, k) and a basepoint $\star_i$ for each closed component γ_i far from the crossings and the disks. Define the associated *winding* $w(\gamma_i, \gamma_j) \in \mathbb{Z}[t^{\pm 1}]$ of γ_i and γ_j in the following way. For a crossing c between γ_i and γ_j, denote by $\varepsilon_{ij}(c)$ the algebraic intersection number of the union of the disks d_ℓ with the path that goes from $\star_i$, or the origin of γ_i, to c along γ_i and then from c to $\star_j$, or the endpoint of γ_j, along γ_j. If $i = j$, change component at the first occurrence of c. Set

$$w(\gamma_i, \gamma_j) = \begin{cases} \frac{1}{2} \sum_c \mathrm{sg}(c) t^{\varepsilon_{ij}(c)} & \text{if } i \neq j, \\ \frac{1}{2} \sum_c \mathrm{sg}(c)(t^{\varepsilon_{ii}(c)} + t^{-\varepsilon_{ii}(c)}) & \text{if } i = j, \end{cases}$$

where the sums are over all crossings between γ_i and γ_j. Note that $w(\gamma_j, \gamma_i)(t) = w(\gamma_i, \gamma_j)(t^{-1})$. Now let I and J be two subsets of $\{1, \ldots, n\}$ and denote by γ_I and γ_J the corresponding subtangles of γ. The *winding matrix* $W_{\gamma_I \gamma_J}$ associated with the fixed diagram and basepoints is the matrix whose coefficients are the windings $w(\gamma_i, \gamma_j)$ for $i \in I$ and $j \in J$ — denote it by W_{γ_I} when $I = J$. In this latter case, note that W_{γ_I} is hermitian.

Lemma 4.1 *The winding matrix is invariant by isotopies which do not allow the basepoints to pass through the disks of the tangle. In particular, when γ contains no closed components, it is an isotopy invariant.*

Figure 20: Basepoint passing through a disk.

Proof First note that the winding matrix is preserved when a crossing passes through a disk d_ℓ. It is also preserved when a basepoint of a closed component passes through a crossing since the algebraic intersection number of this component with the union of the disks d_ℓ is trivial. Hence it only remains to check invariance with respect to framed Reidemeister moves performed far from the basepoints and the disks, which is direct. $\square$

To completely understand the effect of an isotopy on the winding matrix W_{γ_I, γ_J}, we shall describe its modification when a basepoint passes through a disk of the tangle. Fix a closed component γ_i. Fix a diagram of (γ, k) with the basepoint of γ_i located "just before" a disk d_ℓ of the tangle, as shown in the first part of Figure 20. Consider another diagram of (γ, k) which differs from the previous one only by the position of the basepoint $\star_i$, which is as shown on the second part of Figure 20. Let $\varepsilon = \pm 1$ give the sign of the intersection of d_ℓ and γ_i which the basepoint passes through. It is easily seen that the winding matrix of the latter diagram is obtained from the winding matrix of the previous one by multiplication on the left by $T_i(t^{-\varepsilon})$ if $i \in I$ and on the right by $T_i(t^\varepsilon)$ if $i \in J$, where $T_i(t)$ is the diagonal matrix whose diagonal coefficients are all 1 except a t at the ith position.

We now define winding matrices for bottom–top tangles in $\mathbb{Q}$–cubes. Let (B, K, γ) be a bottom–top tangle with paths in a $\mathbb{Q}$–cube. Let $(([-1,1]^3, \Xi, \eta), L)$ be a surgery presentation of (B, K, γ). Denote (η, k) the associated tangle with disks. Fix a diagram of $([-1,1]^3, \Xi, \eta)$ and a basepoint $\star_i$ for each component L_i of L. Define the *winding matrix of* (B, K, γ), with coefficients in $\mathbb{Q}(t)$, as

$$W_\gamma = W_\eta - W_{\eta L} W_L^{-1} W_{L\eta}.$$

Note that $W_L(1)$ is the linking matrix of L, thus $W_L(1) \neq 0$ since B is a homology cube. Hence W_L is invertible over $\mathbb{Q}(t)$.

Lemma 4.2 *The winding matrix W_γ is an isotopy invariant of (B, K, γ).*

Proof First, when the surgery presentation is fixed, the discussion of the previous subsection implies that the winding matrix does not depend on the choice of diagram

and basepoints. Then independence with respect to Kirby moves is easily checked. We detail the less direct, which is the case when a surgery link component is added to a tangle component. Denote by η' the tangle obtained from η by adding the surgery component L_j to η_i. We have

$$W_{\eta'} = W_\eta + {}^t P W_{L\eta} + W_{\eta L} P + {}^t P W_L P \quad \text{and} \quad W_{L\eta'} = W_{L\eta} + W_L P,$$

where P is the $|L| \times |\eta|$ matrix whose only nontrivial term is a 1 at the (j, i) position.

$\square$

4.2 Topological interpretation

Given a bottom–top tangle with paths in a $\mathbb{Q}$–cube, or a tangle with disks defined from a bottom–top tangle with paths in $[-1, 1]^3$ with a surgery link, close the components of the tangle, say γ, by line segments on the top and bottom surfaces to get a link $\overline{\gamma}$. This provides well-defined linking numbers for the tangle components. If there is no path or disk, the winding matrix is the linking matrix. It is clear when working in $[-1, 1]^3$. For bottom–top tangles in $\mathbb{Q}$–cubes, apply the following easy fact, which can be proved by adapting the proof of Proposition 4.5.

Fact 4.3 *Let L be an oriented framed link in $[-1, 1]^3$ whose linking matrix $\mathrm{Lk}(L)$ is nondegenerate. Let ξ and ζ be disjoint oriented knots in $[-1, 1]^3 \setminus L$ and denote by ξ' and ζ' the copies of ξ and ζ in the $\mathbb{Q}$–cube obtained from $[-1, 1]^3$ by surgery on L. Then*

$$\mathrm{lk}(\xi', \zeta') = \mathrm{lk}(\xi, \zeta) - \mathrm{Lk}(\xi, L) \mathrm{Lk}(L)^{-1} \mathrm{Lk}(L, \zeta).$$

More generally, when there are paths or disks, the winding matrix evaluated at $t = 1$ is the linking matrix. We shall give a similar interpretation for the winding matrix at a generic t.

Let (B, K, γ) be a bottom–top tangle in a $\mathbb{Q}$–cube. Let E be the exterior of K in B and let $\widetilde{E}$ be its covering associated with the kernel of the map $\pi_1(E) \to \mathbb{Z} = \langle t \rangle$ which sends the positive meridians of the components of K to t. The automorphism group of the covering is isomorphic to $\mathbb{Z}$; let τ be the generator associated with the action of the positive meridians. Let ζ be a knot in $\widetilde{E}$ such that there are a rational 2–chain Σ in $\widetilde{E}$ and $P \in \mathbb{Q}[t^{\pm 1}]$ which satisfy $\partial \Sigma = P(\tau)(\zeta)$. Let ξ be another knot in $\widetilde{E}$ such that the projections of ζ and ξ in E are disjoint. Define the *equivariant*

linking number of ζ and ξ as

$$\mathrm{lk}_e(\zeta, \xi) = \frac{1}{P(t)} \sum_{j \in \mathbb{Z}} \langle \Sigma, \tau^j(\xi) \rangle t^j \in \mathbb{Q}(t).$$

The equivariant linking number is well-defined since $H_2(\widetilde{E}, \mathbb{Q}) = 0$ (see for instance [23, Lemma 2.1]) and satisfies $\mathrm{lk}_e(\tau(\zeta), \xi) = t\, \mathrm{lk}_e(\zeta, \xi)$.

First consider a tangle with disks (γ, k), with disks d_ℓ associated to the integer k, defined from a surgery presentation of a bottom–top tangle with paths in a $\mathbb{Q}$–cube, so that the closure $\bar{\gamma}_i$ of each component γ_i is well-defined. Fix a diagram of (γ, k) and basepoints $\star_i$ of its components. For an interval component, choose the basepoint to be its origin. Set $d = \bigcup_{\ell=1}^k d_\ell$. Let E be the exterior of ∂d in $[-1, 1]^3$ and let $\widetilde{E}$ be the infinite cyclic covering defined above. Let $\widetilde{E}_0 \subset \widetilde{E}$ be a copy of the exterior of d in $[-1, 1]^3$. Define the lift $\widetilde{\gamma}_i$ of $\bar{\gamma}_i$ in $\widetilde{E}$ by lifting $\star_i$ in $\widetilde{E}_0$. Given two subtangles γ_I and γ_J of γ, define the *equivariant linking matrix* $\mathrm{Lk}_e(\widetilde{\gamma}_I, \widetilde{\gamma}_J)$ of their lifts with the equivariant linking numbers of the $\widetilde{\gamma}_i$.

Lemma 4.4 $W_{\gamma_I \gamma_J} = \mathrm{Lk}_e(\widetilde{\gamma}_I, \widetilde{\gamma}_J).$

Proof   Set $\widetilde{E}_\ell = \tau^\ell(\widetilde{E}_0)$, where τ is the generator of the automorphism group of the covering $\widetilde{E}$ which corresponds to the action of the positive meridians of the ∂d_ℓ. Fix $i \in I$ and $j \in J$. Since $\bar{\gamma}_i$ is null-homotopic in $[-1, 1]^3 \setminus \partial d$, it bounds a disk D immersed in $[-1, 1]^3 \setminus \partial d$. Let $\widetilde{D}$ be the lift of D obtained by lifting $\star_i$ in $\widetilde{E}_0$. Set $\widetilde{D}_\ell = \widetilde{D} \cap \widetilde{E}_\ell$ and let D_ℓ be the image of $\widetilde{D}_\ell$ in E. Set $c_\ell = \partial D_\ell$ and $\widetilde{c}_\ell = \partial \widetilde{D}_\ell$. Similarly, define the c'_ℓ and $\widetilde{c}'_\ell$ from $\bar{\gamma}_j$. Assume the c'_ℓ do not meet the D_ℓ along the disks of the tangle.

Thanks to Lemma 4.1, we have $w(\gamma_i, \gamma_j) = \sum_{\ell,\ell' \in \mathbb{Z}} w(c_\ell, c'_{\ell'}) t^{\ell-\ell'}$ for any choice of basepoints of the c_ℓ and $c'_{\ell'}$. Since these latter curves do not cross the disks of the tangle, we have $w(c_\ell, c'_{\ell'}) = \mathrm{lk}(c_\ell, c'_{\ell'}) = \langle D_\ell, c'_{\ell'} \rangle$, thus $w(\gamma_i, \gamma_j) = \sum_{\ell,\ell' \in \mathbb{Z}} \langle D_\ell, c'_{\ell'} \rangle t^{\ell-\ell'}$. Lifting D_ℓ and $c'_{\ell'}$ in $\widetilde{E}_\ell$ does not change their algebraic intersection number, hence

$$
\begin{aligned}
w(\gamma_i, \gamma_j) &= \sum_{\ell,\ell' \in \mathbb{Z}} \langle \widetilde{D}_\ell, \tau^{\ell-\ell'}(\widetilde{c}'_{\ell'}) \rangle t^{\ell-\ell'} \\
&= \sum_{\ell,\ell' \in \mathbb{Z}} \langle \widetilde{D}_\ell, \tau^{\ell'}(\widetilde{c}'_{\ell-\ell'}) \rangle t^{\ell'} \\
&= \sum_{\ell' \in \mathbb{Z}} \langle \widetilde{D}, \tau^{\ell'}(\widetilde{\gamma}_j) \rangle t^{\ell'} = \mathrm{lk}_e(\widetilde{\gamma}_i, \widetilde{\gamma}_j),
\end{aligned}
$$

where the third equality holds since $\tau^{\ell'}(\widetilde{c}'_{\ell-\ell'}) = \tau^{\ell'}(\widetilde{\gamma}_j) \cap \widetilde{E}_\ell$. $\square$

Now consider a bottom–top tangle with paths in a $\mathbb{Q}$–cube (B, K, γ). Since γ is null-homotopic in $B \setminus K$, we have a well-defined *equivariant linking matrix* $\mathrm{Lk}_e(\widetilde{\gamma})$. Here, all components are intervals, so we have a canonical choice of basepoints.

Proposition 4.5 *Let* (B, K, γ) *be a bottom–top tangle with paths in a* $\mathbb{Q}$*–cube. Then* $W_\gamma = \mathrm{Lk}_e(\widetilde{\gamma})$.

Proof Let $(([-1, 1]^3, \Xi, \eta), L)$ be a surgery presentation of (B, K, γ). Fix a diagram of $([-1, 1]^3, \Xi, \eta \cup L)$ and basepoints for the components of $L = \bigsqcup_{1 \leq i \leq n} L_i$. Let $\widetilde{E}$ be the infinite cyclic covering of the exterior of Ξ in $[-1, 1]^3$. Let d be the disjoint union of disks in $[-1, 1]^3$ bounded by $\widehat{\Xi}$. Let $\widetilde{L} = \bigsqcup_{1 \leq i \leq n} \widetilde{L}_i$, $\widetilde{\gamma}$ and $\widetilde{\eta}$ be the lifts of L, γ and η in $\widetilde{E}$ with all basepoints in the same copy in $\widetilde{E}$ of the exterior of d in $[-1, 1]^3$. We have to prove that

$$\mathrm{Lk}_e(\widetilde{\gamma}) = \mathrm{Lk}_e(\widetilde{\eta}) - \mathrm{Lk}_e(\widetilde{\eta}, \widetilde{L}) \, \mathrm{Lk}_e(\widetilde{L})^{-1} \, \mathrm{Lk}_e(\widetilde{L}, \widetilde{\eta}).$$

The infinite cyclic covering $\widetilde{E}'$ of the exterior of K in B is obtained from $\widetilde{E}$ by surgery on $\bigcup_{\ell \in \mathbb{Z}} \tau^\ell(\widetilde{L})$. Let $\widetilde{\eta}_x$ and $\widetilde{\eta}_y$ be components of $\widetilde{\eta}$, and let $\widetilde{\gamma}_x$ and $\widetilde{\gamma}_y$ be the corresponding components of $\widetilde{\gamma}$. For any knot λ in $\widetilde{E}$ or $\widetilde{E}'$, denote by $m(\lambda)$ a positive meridian. For $1 \leq i \leq n$, let c_i be the parallel of $\widetilde{L}_i$ which bounds a disk after surgery. In the group $H_1\big(\widetilde{E} \setminus \bigcup_{\ell \in \mathbb{Z}}(\tau^\ell(\widetilde{L}) \cup \tau^\ell(\widetilde{\eta}_y)); \mathbb{Z}\big)$, we have

$$\widetilde{\eta}_x = \mathrm{lk}_e(\widetilde{\eta}_x, \widetilde{\eta}_y)m(\widetilde{\eta}_y) + \sum_{i=1}^n \mathrm{lk}_e(\widetilde{\eta}_x, \widetilde{L}_i)m(\widetilde{L}_i)$$

and

$$c_i = \mathrm{lk}_e(\widetilde{L}_i, \widetilde{\eta}_y)m(\widetilde{\eta}_y) + \sum_{j=1}^n \mathrm{lk}_e(\widetilde{L}_i, \widetilde{L}_j)m(\widetilde{L}_j),$$

where multiplication by t is given by the action of τ in homology. In the space $H_1\big(\widetilde{E}' \setminus \bigcup_{\ell \in \mathbb{Z}} \tau^\ell(\widetilde{\gamma}_y); \mathbb{Z}\big)$, this gives

$$\widetilde{\gamma}_x = \mathrm{lk}_e(\widetilde{\eta}_x, \widetilde{\eta}_y)m(\widetilde{\gamma}_y) - \mathrm{Lk}_e(\widetilde{\eta}_x, \widetilde{L}) \, \mathrm{Lk}_e(\widetilde{L})^{-1} \, \mathrm{Lk}_e(\widetilde{L}, \widetilde{\eta}_y)m(\widetilde{\gamma}_y). \qquad \square$$

It is easily checked that the null LP–surgery defined in the introduction provides a move on the set of bottom–top tangles with paths in $\mathbb{Q}$–cubes, which corresponds to the move of null LP–surgery on Lagrangian cobordisms defined at the end of Section 2.1 via the map D of Proposition 2.1. Moreover:

Corollary 4.6 *The winding matrix of a bottom–top tangle with paths in a* $\mathbb{Q}$*–cube is invariant under null LP–surgeries.*

Proof Let (B, K, γ) be a bottom–top tangle with paths in a $\mathbb{Q}$–cube associated with a Lagrangian cobordism with paths (M, K). Let C be a null LP–surgery on (M, K) made of a single $\mathbb{Q}$SK–pair. Let (M', K') be the Lagrangian cobordism with paths obtained by surgery and let (B', K', γ') be the associated bottom–top tangle with paths in a $\mathbb{Q}$–cube. Let $\widetilde{E}$ be the infinite cyclic covering of the exterior of K in B. The nullity condition implies that the preimage of the $\mathbb{Q}$–handlebody C is the disjoint union of $\mathbb{Q}$–handlebodies C_ℓ homeomorphic to C. The infinite cyclic covering $\widetilde{E}'$ of the exterior of K' in B' is obtained from $\widetilde{E}$ by null LP–surgeries on all the C_ℓ. This concludes since LP–surgeries preserve linking numbers (see for instance [22, Lemma 2.1]). $\qquad\square$

5 Construction of an LMO functor on $\widetilde{\mathcal{LCob}}$

5.1 The functor $Z^\bullet \colon \widetilde{\mathcal{T}_q} \to \widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}[t^{\pm 1}]}$

The definition of the functor $Z^\bullet \colon \widetilde{\mathcal{T}_q} \to \widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}[t^{\pm 1}]}$ is based on the functor $Z \colon \mathcal{T}_q \to \mathcal{A}$ of [5], which is a renormalization of the Le–Murakami functor [14; 15]. We recall in Figure 21 the definition of Z on the elementary q–tangles, where $v \in \mathcal{A}(\bigcirc) \cong \mathcal{A}(\uparrow)$ is the value of the Kontsevich integral on the zero-framed unknot, $\Phi \in \mathcal{A}(\downarrow\downarrow\downarrow)$ is a Drinfeld associator with rational coefficients and $\Delta^{+++}_{u_1, u_2, u_3} \colon \mathcal{A}(\downarrow\downarrow\downarrow) \to \mathcal{A}(\downarrow_{u_1 u_2 u_3})$ is obtained by applying $|u_i| - 1$ times Δ on the ith factor.

Let (γ, k) be a q–tangle with disks. Assume γ is transverse to $[-1, 1]^2 \times \{h_i(k)\}$ for all $i \in \{1, \dots, k\}$, and write γ as a composition of q–tangles γ_i by cutting along these

$$Z\left(\begin{smallmatrix}(+ & +)\\ & \\ (+ & +)\end{smallmatrix}\right) = \exp\left(\frac{1}{2}\;\vcenter{\hbox{$\smile$}}\;\right) \in \mathcal{A}\left(\times\right) \qquad Z\left(\begin{smallmatrix}(+ & +)\\ & \\ (+ & +)\end{smallmatrix}\right) = \exp\left(-\frac{1}{2}\;\vcenter{\hbox{$\smile$}}\;\right) \in \mathcal{A}\left(\times\right)$$

$$Z\left(\begin{smallmatrix}\downarrow \frown\\ (+ \quad -)\end{smallmatrix}\right) = \vcenter{\hbox{$\smile^{v}$}} \in \mathcal{A}\left(\frown\right) \qquad Z\left(\begin{smallmatrix}(+ \quad -)\\ \smile\end{smallmatrix}\right) = \cup \in \mathcal{A}\left(\cup\right)$$

$$Z\left(\begin{smallmatrix}(u & & (v\,w))\\ \downarrow & \diagup & \downarrow\\ ((u\,v) & & w)\end{smallmatrix}\right) = \Delta^{+++}_{u,v,w}(\Phi) \in \mathcal{A}(\downarrow_{uvw})$$

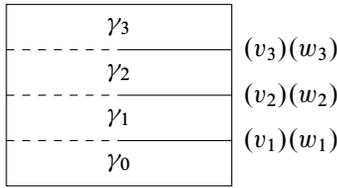Figure 21: The functor $Z \colon \mathcal{T}_q \to \mathcal{A}$.

Figure 22: Cutting a q–tangle with disks $(\gamma, 3)$.

levels; see Figure 22. Write the bottom word of γ_i as $w_b(\gamma_i) = (v_i)(w_i)$, where w_i corresponds to the part of the tangle which meets the disk d_i. Set

$$Z^\bullet(\gamma, k) = Z(\gamma_0) \circ (I_{v_1} \otimes G_{w_1}) \circ Z(\gamma_1) \circ \cdots \circ (I_{v_k} \otimes G_{w_k}) \circ Z(\gamma_k) \in \widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}[t^{\pm 1}]}(\gamma),$$

where I_v is the identity on the word v and G_v is obtained from I_v by adding a label t (resp. t^{-1}) on skeleton components associated with a $-$ sign (resp. a $+$ sign); see Figure 23. At the level of objects, $Z^\bullet$ forgets the parentheses. Invariance with respect to isotopy and to the cutting of γ is due to invariance of the functor Z and the following observation of Kricker [12, Lemma 3.2.4]:

Lemma 5.1 *For a winding Jacobi diagram $D \in \widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}[t^{\pm 1}]}(w, v)$, we have $G_v \circ D = D \circ G_w$.*

Proof Apply the relations Hol and Hol$_\mathrm{w}$ at all vertices of the diagram. $\qquad\square$

Furthermore, $Z^\bullet$ is a clearly a functor and it preserves the tensor product on $\mathcal{T}_q \otimes \widetilde{\mathcal{T}}_q$ since Z is tensor-preserving.

Lemma 5.2 *For any q–tangle with disks (γ, k), $Z^\bullet(\gamma, k)$ is group-like.*

Proof The fact that $Z(\gamma)$ is group-like for a q–tangle γ follows from Theorem 5.1 of [16]. This suffices since the G_v are obviously group-like and the coproduct commutes with the composition. $\qquad\square$

Figure 23: The diagrams I_v and G_v.

5.2 The functor $Z \colon \widetilde{\mathcal{T}_q}\mathcal{C}ub \to \widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}(t)}$

The next step is to evaluate $Z^\bullet$ on the surgery presentation of a q–tangle with paths in a $\mathbb{Q}$–cube. Let $(B, K, \gamma) \in \widetilde{\mathcal{T}_q}\mathcal{C}ub(w, v)$. Let $(([-1, 1]^3, \Xi, \eta), L)$ be a surgery presentation of (B, K, γ). The trivial link $\widehat{\Xi}$ is the union of the boundaries of the disks $d_i = [0, 1] \times [-1, 1] \times \{h_i(k)\}$, where k is the number of components of Ξ. Hence we have a q–tangle with disks $(\eta \cup L, k)$ and $Z^\bullet(\eta \cup L, k) \in \widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}[t^{\pm 1}]}(\eta \cup L)$. Set

$$Zc((\Xi, \eta), L) = \chi^{-1}_{\pi_0(L)}(\nu^{\otimes \pi_0(L)} \sharp_{\pi_0(L)} Z^\bullet(\eta \cup L, k)) \in \widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}[t^{\pm 1}]}(\eta, \circledast_{\pi_0(L)}),$$

where the connected sum means that a copy of ν is summed to each component of L. Note that $Z^\circ((\Xi, \eta), L)$ is group-like since $Z^\bullet(\eta \cup L, k)$ and ν are group-like and $\chi_{\pi_0(L)}$ preserves the coproduct.

We want to apply formal Gaussian integration to $Z^\circ((\Xi, \eta), L)$. We work with a lift $\overline{Z^\circ((\Xi, \eta), L)} \in \widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}[t^{\pm 1}]}(\eta, *_{\pi_0(L)})$. Fix a diagram of the q–tangle with disks $(\eta \cup L, k)$ transverse to the levels $\{h_i(k)\}$, and fix basepoints $\star_i$ on each component L_i of L. Construct $\overline{Z^\circ((\Xi, \eta), L)}$ following the construction from the beginning of Section 5 for this diagram, with the skeleton components corresponding to the components of L defined as intervals by cutting each component L_i at the basepoint $\star_i$.

Lemma 5.3 *The lift $\overline{Z^\circ((\Xi, \eta), L)}$ is group-like, and we have*

$$\overline{Z^\circ((\Xi, \eta), L)} = \exp_{\sqcup}\left(\tfrac{1}{2} W_L\right) \sqcup H,$$

where W_L is the winding matrix associated with our choice of diagram and basepoints and H is $\pi_0(L)$–substantial.

Proof Check as in Lemma 5.2 that $\overline{Z^\circ((\Xi, \eta), L)}$ is group-like. We have to compute the part of $\overline{Z^\circ((\Xi, \eta), L)}$ made of $\pi_0(L)$–struts. We work with

$$\chi^{-1}_{\pi_0(\eta)}(\overline{Z^\circ((\Xi, \eta), L)}) \in \widetilde{\mathcal{A}}_{\mathbb{Q}[t^{\pm 1}]}(*_{\pi_0(\eta) \cup \pi_0(L)}),$$

which is also group-like, in order to have a Hopf algebra structure on our diagram space. In particular, the group-like property implies that $\chi^{-1}_{\pi_0(\eta)}(\overline{Z^\circ((\Xi, \eta), L)})$ is the exponential of a series of connected diagrams. Since ν and the associator Φ have no terms with exactly two vertices, the only contributions to the $\pi_0(L)$–struts part come from the crossings between components of L. For $i \neq j$, the definition of Z and the

Hol_w relation show that the contribution of a crossing c between L_i and L_j is

$$\chi^{-1}_{\pi_0(\eta)}\left(\tfrac{1}{2}\, \mathrm{sg}(c)\ \begin{array}{c} t^{\varepsilon_{ij}(c)} \\ L_i \quad L_j \end{array} \right).$$

Hence the contribution of all crossings between L_i and L_j is

$$\begin{array}{c} L_j \\ (W_L)_{ij} \\ L_i \end{array} \quad = \quad \begin{array}{c} L_i \\ (W_L)_{ji}. \\ L_j \end{array}$$

For $i = j$, the contribution of a self-crossing of L_i is

$$\chi^{-1}_{\pi_0(\eta)}\left(\tfrac{1}{2}\, \mathrm{sg}(c)\ \begin{array}{c} t^{\varepsilon_{ii}(c)} \\ L_i \end{array} \right) = \mathrm{sg}(c)\left(\begin{array}{c} L_i \\ t^{\varepsilon_{ii}(c)} \\ L_i \end{array} + \frac{1}{2}\begin{array}{c} t^{\varepsilon_{ii}(c)} \\ L_i \end{array} \right).$$

Summed over all self-crossings of L_i, we get as strut part:

$$\sum_c \tfrac{1}{2}\, \mathrm{sg}(c)\ \begin{array}{c} L_i \\ t^{\varepsilon_{ii}(c)} \\ L_i \end{array} = \frac{1}{2}\begin{array}{c} L_i \\ (W_L)_{ii}. \\ L_i \end{array}$$

Hence $\chi^{-1}_{\pi_0(\eta)}(\overline{Z^\circ((\Xi,\eta),L)}) = \exp_{\sqcup}\!\left(\tfrac{1}{2}W_L\right) \sqcup H'$, where $H' \in \tilde{\mathcal{A}}_{\mathbb{Q}[t^{\pm1}]}(*_{\pi_0(\eta)\cup\pi_0(L)})$ is $\pi_0(L)$–substantial. Set $H = \chi_{\pi_0(\eta)}(H')$. $\qquad\qquad\square$

The matrix $W_L(1)$ is the linking matrix of the link L, hence it is the presentation matrix of the first homology group of a $\mathbb{Q}$–cube. Thus $\det(W_L(1)) \neq 0$ and $\overline{Z^\circ((\Xi,\eta),L)}$ is a nondegenerate Gaussian. Lemma 3.6 implies:

Lemma 5.4 *The formal Gaussian integral $\int_{\pi_0(L)} \overline{Z^\circ((\Xi,\eta),L)}$ does not depend on the lift* $\overline{Z^\circ((\Xi,\eta),L)} \in \tilde{\mathcal{A}}^w_{\mathbb{Q}[t^{\pm1}]}(\eta, *_{\pi_0(L)})$ *of* $Z^\circ((\Xi,\eta),L) \in \tilde{\mathcal{A}}^w_{\mathbb{Q}[t^{\pm1}]}(\eta, \circledcirc_{\pi_0(L)})$.

This allows us to set

$$\int_{\pi_0(L)} Z^\circ((\Xi,\eta),L) = \int_{\pi_0(L)} \overline{Z^\circ((\Xi,\eta),L)} \in \tilde{\mathcal{A}}^w_{\mathbb{Q}(t)}(\gamma).$$

Proposition 5.5 *Let (B, K, γ) be a q–tangle with paths in a $\mathbb{Q}$–cube. Fix a surgery presentation $(([-1,1]^3, \Xi, \eta), L)$ of (B, K, γ). Then*

$$Z(B, K, \gamma) = U_+^{-\sigma+(L)} \sqcup U_-^{-\sigma-(L)} \sqcup \int_{\pi_0(L)} Z^\circ((\Xi, \eta), L) \in \tilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}(t)}(\gamma),$$

where $U_\pm = Z^\circ\big((\varnothing, \varnothing), \bigcirc\ \pm 1\big)$, defines a functor $Z \colon \tilde{\mathcal{T}}_q \mathcal{C}ub \to \tilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}(t)}$ which preserves the tensor product on $\mathcal{T}_q \mathcal{C}ub \otimes \tilde{\mathcal{T}}_q \mathcal{C}ub$.

Proof We have to check that $Z(B, K, \gamma)$ does not depend on the surgery presentation. Independence with respect to the orientation of the components of L follows from the argument of [3, Proposition 3.1]. The normalization term $U_+^{-\sigma+(L)} \sqcup U_-^{-\sigma-(L)}$ ensures independence with respect to the KI move as usual. Independence with respect to the KII move mainly follows from [7, Section 5.4]. More precisely, the argument of [7, Theorem 4] adapts [13, Proposition 1] to relate the values of $\overline{Z^\circ((\Xi, \eta), L)}$ for surgery links that differ from each other by a KII move. Then [7, Lemma 5.6] shows that this implies the invariance of the formal Gaussian integral. As noted in [3, Section 5.1], the argument remains valid when a surgery component is added to a tangle component since [7, Lemma 5.6] uses integration along the surgery component. $\square$

Restricting the functor $Z \colon \tilde{\mathcal{T}}_q \mathcal{C}ub \to \tilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}(t)}$ to q–tangles in $\mathbb{Q}$–cubes with no path, one recovers the functor $Z \colon \mathcal{T}_q \mathcal{C}ub \to \mathcal{A}$ of [5, Definition 3.16]. When γ is a bottom–top tangle and $K = \varnothing$, $\chi^{-1}_{\pi_0(\gamma)}(Z(B, \varnothing, \gamma))$ is group-like and $\chi^{-1}_{\pi_0(\gamma)}(Z(B, \varnothing, \gamma)) = \exp_{\sqcup}(\mathrm{Lk}(\gamma)) \sqcup H$ for some substantial and group-like H [5, Lemma 3.17]. We generalize this in the next result.

Lemma 5.6 *For any bottom–top q–tangle with paths (B, K, γ) where B is a $\mathbb{Q}$–cube, $\chi^{-1}_{\pi_0(\gamma)}(Z(B, K, \gamma))$ is group-like and*

$$\chi^{-1}_{\pi_0(\gamma)}(Z(B, K, \gamma)) = \exp_{\sqcup}(W_\gamma) \sqcup H \in \tilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\pi_0(\gamma)}),$$

for some substantial and group-like H.

Proof The fact that $Z(B, K, \gamma)$ is group-like follows from the same property for U_+, U_- and $\overline{Z^\circ((\Xi, \eta), L)}$, and Theorem 3.4. It implies that $\chi^{-1}_{\pi_0(\gamma)}(Z(B, K, \gamma))$ is group-like since $\chi_{\pi_0(\gamma)}$ preserves the coproduct. The same computation as in the proof of Lemma 5.3 gives

$$\chi^{-1}_{\pi_0(\eta)}\big(\overline{Z^\circ((\Xi, \eta), L)}\big) = \exp_{\sqcup}\big(\tfrac{1}{2}W_L + \tfrac{1}{2}W_\eta + W_{L\eta}\big) \sqcup H',$$

where H' is substantial — note that $\chi_{\pi_0(\gamma)}$ and $\chi_{\pi_0(\eta)}$ are essentially the same before and after surgery on L. Integrate along $\pi_0(L)$:

$$\chi_{\pi_0(\gamma)}^{-1}\left(\int_{\pi_0(L)} \overline{Z^\circ((\Xi,\eta),L)}\right) = \left\langle \exp_\sqcup\left(-\tfrac{1}{2}W_L^{-1}\right), \exp_\sqcup\left(\tfrac{1}{2}W_\eta + W_{L\eta}\right) \sqcup H'\right\rangle_{\pi_0(L)}$$

$$= \exp_\sqcup\left(\tfrac{1}{2}W_\eta - \tfrac{1}{2}{}^t W_{L\eta}(t^{-1})W_L^{-1}W_{L\eta}\right) \sqcup H$$

$$= \exp_\sqcup\left(\tfrac{1}{2}W_\gamma\right) \sqcup H. \qquad \square$$

5.3 The functor $\tilde{Z} \colon \widetilde{\mathcal{LCob}}_q \to {}^{\mathrm{ts}}\tilde{\mathcal{A}}$

In this section, we define a functor on Lagrangian q–cobordisms with paths by applying the invariant Z on bottom–top q–tangles with paths in $\mathbb{Q}$–cubes. The invariant Z is functorial on q–tangles but not on bottom–top q–tangles, due to the different composition laws. To deal with this, we introduce some specific elements $\mathsf{T}_g \in {}^{\mathrm{ts}}\tilde{\mathcal{A}}(*_{\lfloor g\rfloor^+ \cup \lfloor g\rfloor^-})$ following [5, Section 4]. Set

$$\lambda(x,y;r) = \chi_{\{r\}}^{-1}\left(\left(\sum_{n\geq 0}\frac{1}{n!}\begin{Bmatrix}{}_{\substack{\text{---}\,x \\ \text{---}\,x \\ \vdots \\ \text{---}\,x}}\end{Bmatrix}^n_r\right) \circ \left(\sum_{n\geq 0}\frac{1}{n!}\begin{Bmatrix}{}_{\substack{\text{---}\,y \\ \text{---}\,y \\ \vdots \\ \text{---}\,y}}\end{Bmatrix}^n_r\right)\right) \in \tilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\{x,y,r\}}),$$

$$\mathsf{T}(x^+,x^-) = U_+^{-1} \sqcup U_-^{-1}$$

$$\sqcup \int_{\{r^+,r^-\}} \langle \lambda(1^-,x^-;r^-) \sqcup \lambda(x^+,1^+;r^+), \chi^{-1}(Z(T_1)) \rangle_{\{1^+,1^-\}},$$

$$\mathsf{T}_g = \mathsf{T}(1^+,1^-) \sqcup \cdots \sqcup \mathsf{T}(g^+,g^-) \in \tilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\lfloor g\rfloor^+ \cup \lfloor g\rfloor^-}),$$

where the bottom–top tangle T_1 is drawn in Figure 9. As proven in [5, Lemma 4.9]:

Lemma 5.7 T_g *is a group-like element of* $\tilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\lfloor g\rfloor^+ \cup \lfloor g\rfloor^-})$ *and* $\mathsf{T}_g = \mathrm{Id}_g \sqcup H$ *for some substantial and group-like* H. *In particular,* T_g *is top-substantial and* $\lfloor g\rfloor^-$–*substantial.*

Let (M,K) be a Lagrangian q–cobordism with paths and denote by (B,K,γ) the associated bottom–top q–tangle with paths, of type (g,f). We have $Z(B,K,\gamma) \in \tilde{\mathcal{A}}_{\mathbb{Q}(t)}^{\mathrm{w}}(\gamma) \cong \tilde{\mathcal{A}}_{\mathbb{Q}(t)}(\gamma)$ and we consider $\chi^{-1}(Z(B,K,\gamma)) \in \tilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\lfloor g\rfloor^+ \cup \lfloor f\rfloor^-})$. It may not be top-substantial, but since T_g is $\lfloor g\rfloor^-$–substantial, we can set

$$\tilde{Z}(M,K) = \chi^{-1}(Z(B,K,\gamma)) \circ \mathsf{T}_g.$$

At the level of objects, $\widetilde{Z}$ sends a word on its number of letters. Direct adaptation of the proof of [5, Lemma 4.10] implies that $\widetilde{Z}$ preserves the composition and the next result follows; see [5, Theorem 4.13].

Proposition 5.8 $\widetilde{Z}\colon \widetilde{\mathcal{LCob}}_q \to {}^{\mathrm{ts}}\widetilde{\mathcal{A}}$ *is a functor which preserves the tensor product on* $\mathcal{LCob}_q \otimes \widetilde{\mathcal{LCob}}_q$.

Restricting the functor $\widetilde{Z}\colon \widetilde{\mathcal{LCob}}_q \to {}^{\mathrm{ts}}\widetilde{\mathcal{A}}$ to Lagrangian q–cobordisms with no path, one recovers the functor $\widetilde{Z}$ defined on $\mathcal{LCob}_q$ in [5, Theorem 4.13].

Lemmas 5.6 and 5.7 imply:

Lemma 5.9 *Let* (M, K) *be a Lagrangian* q–*cobordism with paths and let* (B, K, γ) *be the associated bottom–top* q–*tangle with paths. Then* $\widetilde{Z}(M, K)$ *is group-like and* $\widetilde{Z}(M, K) = \exp_{\sqcup}(W_\gamma) \sqcup H$ *for some substantial and group-like* H.

5.4 Application to $\mathbb{Q}\mathrm{SK}$–pairs

Let (S, κ) be a $\mathbb{Q}\mathrm{SK}$–pair. Let M be the $\mathbb{Q}$–cube obtained from S by removing the interior of a ball B^3 disjoint from κ. Isotoping κ in M and fixing a boundary parametrization m of M, one can view κ as the knot $\widehat{K}$ associated with a Lagrangian cobordism with one path (M, K). Since the top and bottom words are empty, we get a Lagrangian q–cobordism with one path.

Proposition 5.10 *Let* (S, κ) *be a* $\mathbb{Q}\mathrm{SK}$–*pair. Define as above an associated Lagrangian* q–*cobordism with one path* (M, K). *Then* $\widetilde{Z}(S, \kappa) = \widetilde{Z}(M, K)$ *defines an invariant of* $\mathbb{Q}\mathrm{SK}$–*pairs, which coincides with the Kricker invariant* Z^{rat} *for knots in* $\mathbb{Z}$–*spheres.*

Proof When associating a cobordism with one path with a $\mathbb{Q}\mathrm{SK}$–pair, we make a choice in the way we isotope the knot to the closure of a path. Once we work with a surgery presentation of our cobordism, this choice corresponds to the sweeping move represented in Figure 24. But the right-hand side diagram of this figure shows this move is trivial — as noted in [7, Lemma 3.26].

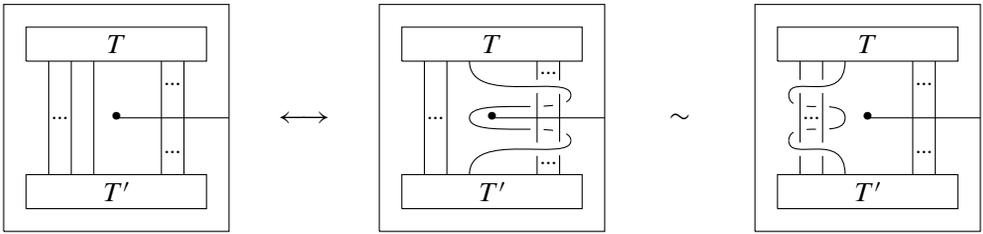Coincidence with Z^{rat} is direct by construction. $\square$

Figure 24: A sweeping move.

Remark The above proof does not work for a cobordism with more than one path, so we do not get an invariant of boundary links in $\mathbb{Q}$–spheres. One may obtain such an invariant by quotienting out the target space by suitable relations; see [7] for a construction of this kind.

Proposition 5.11 *Let (S_1, κ_1) and (S_2, κ_2) be $\mathbb{Q}$SK–pairs. The invariant $\widetilde{Z}$ is given on their connected sum by*

$$\widetilde{Z}((S_1, \kappa_1) \,\natural\, (S_2, \kappa_2)) = \widetilde{Z}(S_1, \kappa_1) \sqcup \widetilde{Z}(S_2, \kappa_2).$$

Proof As previously, associate Lagrangian q–cobordisms with one path (M_1, K_1) and (M_2, K_2) with (S_1, κ_1) and (S_2, κ_2), respectively. Construct a Lagrangian q–cobordism with one path (M, K) associated with $(S, \kappa) = (S_1, \kappa_1) \,\natural\, (S_2, \kappa_2)$ by stacking (M_1, K_1) and (M_2, K_2) together in the y direction. Now (M_1, K_1) and (M_2, K_2) are obtained from the cube $[-1, 1]^3$ with one disk by surgery on links L_1 and L_2, respectively. We obtain a surgery diagram for (M, K) by drawing L_1 "in front" of L_2, or equivalently "around" L_2; see Figure 25. The result follows from this latter diagram since there is no crossing between L_1 and L_2. $\qquad\square$

6 Splitting formulas

We first mention useful lemmas, namely [20, Lemmas 4.3 and 4.4]. Recall the tensor $\mu(C)$ was defined in the introduction.

Lemma 6.1 (Massuyeau) *For a $\mathbb{Q}$–handlebody C of genus g, there exists a boundary parametrization $c \colon \partial C_0^g \to C$ such that $(C, c) \in \mathcal{LCob}(g, 0)$.*
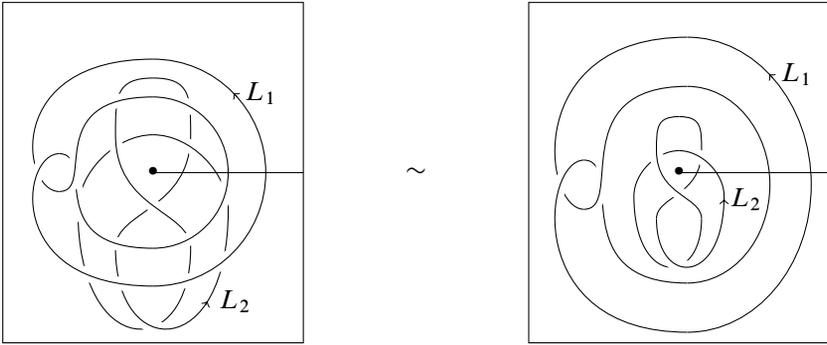
Figure 25: Stacking diagrams.

Lemma 6.2 (Massuyeau) *Let $C = (C'/C)$ be an LP–pair of genus g. Take bound-ary parametrizations $c\colon \partial C_0^g \to C$ and $c'\colon \partial C_0^g \to C'$ compatible with the fixed identification $\partial C \cong \partial C'$ such that $(C, c) \in \mathcal{LCob}(g, 0)$ and $(C', c') \in \mathcal{LCob}(g, 0)$. Then*

$$\mu(C) = \widetilde{Z}_1(C, c) - \widetilde{Z}_1(C', c'),$$

where $\widetilde{Z}_1$ is the i–degree 1 part of $\widetilde{Z}$ and $\mu(C)$ is considered as an element of $\widetilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\lfloor g \rfloor +})$ via the inclusion $\bigwedge^3 H_1(\mathcal{C}; \mathbb{Q}) \hookrightarrow \widetilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\lfloor g \rfloor +})$ defined by

$$[c_+(\beta_i)] \wedge [c_+(\beta_j)] \wedge [c_+(\beta_k)] \mapsto \overset{k^+\,j^+\,i^+}{\underset{\searrow\;\downarrow\;\swarrow}{}} .$$

Let $(M, K) \in \widetilde{\mathcal{LCob}}_q(w, v)$. Let $C = (C_1, \ldots, C_n)$ be a null LP–surgery on (M, K). Let e_i be the genus of C_i. For $1 \le i \le n$, take boundary parametrizations $c_i\colon \partial C_0^{e_i} \to C_i$ and $c_i'\colon \partial C_0^{e_i} \to C_i'$ compatible with the fixed identification $\partial C_i \cong \partial C_i'$ such that $(C_i, c_i) \in \mathcal{LCob}(e_i, 0)$ and $(C_i', c_i') \in \mathcal{LCob}(e_i, 0)$. Set $e = \sum_{i=1}^n e_i$. Take a collar neighbourhood $m_-(F_f) \times [-1, \varepsilon - 1]$ of the bottom surface $m_-(F_f)$. Take pairwise disjoint solid tubes T_i for $i = 1, \ldots, n$ such that T_i connects $(c_i)_-(F_0)$ to a disk in $m_-(F_f) \times \{\varepsilon - 1\}$ in the complement of the C_j, the collar neighbourhood and K. This provides a decomposition of the cobordism (M, K) as

$$(M, K) = ((C_1, \varnothing) \otimes \cdots \otimes (C_n, \varnothing) \otimes \mathrm{Id}_f) \circ (N, J),$$

where f is the number of letters of v (see Figure 26). It is proved in [20, Section 4.4] that N is a Lagrangian cobordism. The nullity condition on the surgery ensures that $\widehat{J}$ is a boundary link. Thus (N, J) is a Lagrangian cobordism with paths.
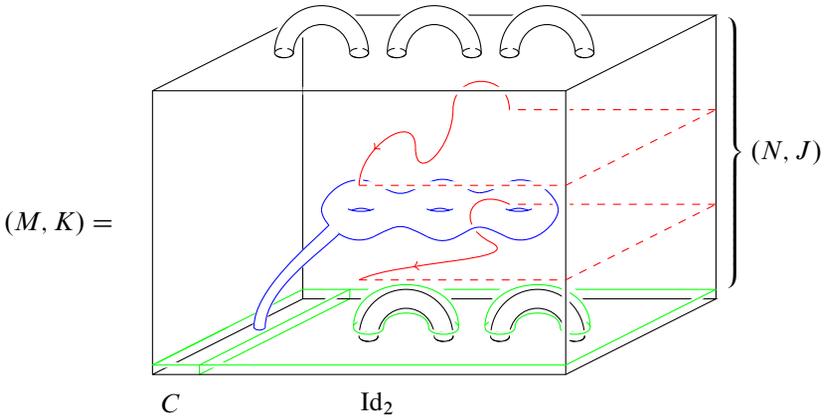
Figure 26: Decomposition of a Lagrangian cobordism with paths

With the surgery C is associated the tensor $\mu(C) \in \mathcal{A}_{\mathbb{Q}}(H_1(\mathcal{C}; \mathbb{Q}))$. Let W be a square matrix of size e with coefficients in $\mathbb{Q}(t)$. Interpret W as a hermitian form on $H_1(\mathcal{C}; \mathbb{Q})$ written in the basis $(([(c_i)_+(\beta_j)])_{1 \le j \le e_i})_{1 \le i \le n}$. Given an $H_1(\mathcal{C}; \mathbb{Q})$–coloured Jacobi diagram, one can glue some legs of the diagram with W; see Figure 3. Changing the labels of the univalent vertices via the bijection

$$[(c_i)_+(\beta_j)] \mapsto \sum_{\ell=1}^{i-1} e_\ell + j$$

onto $\{1, \ldots, e\}$, this provides a diagram in $\tilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\lfloor e \rceil+})$.

The following result is a direct adaptation of [20, Section 4.4], with the winding matrices playing the role of the linking matrices.

Proposition 6.3 *Let (M, K) be a Lagrangian q–cobordism with paths with (B, K, γ) the associated bottom–top q–tangle with paths. Let $C = (C_1, \ldots, C_n)$ be a null LP–surgery on (M, K). Define as above a decomposition of the cobordism (M, K). Choose top and bottom words for (N, J) and the $(C_i, \varnothing)$ in order to get a decomposition of the Lagrangian q–cobordism (M, K) as*

$$(M, K) = ((C_1, \varnothing) \otimes \cdots \otimes (C_n, \varnothing) \otimes \mathrm{Id}_v) \circ (N, J).$$

Let (D, J, ς) be the bottom–top q–tangle with paths associated with (N, J). Let ς^c be the subtangle of ς^- corresponding to the C_i. Let e be the number of components of ς^c. Let $\tilde{\rho}_c \colon \tilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\lfloor e \rceil+}) \to \tilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\lfloor g \rceil+ \cup \lfloor f \rceil-})$ be the linear form which changes

the labels of the univalent vertices as follows:

$$\widetilde{\rho}_{\boldsymbol{c}}(\ell^+) = \sum_{j=1}^{g} W_{\varsigma^c}(\varsigma_j^+, \varsigma_\ell) \cdot j^+ + \sum_{i=1}^{f} W_{\varsigma^c}(\varsigma_{e+i}^-, \varsigma_\ell) \cdot i^-.$$

Then

$$\sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|} \widetilde{Z}((M, K)(\boldsymbol{C}_I))$$
$$\equiv_n \exp_{\sqcup}\left(\tfrac{1}{2} W_\gamma\right) \sqcup \widetilde{\rho}_{\boldsymbol{c}}\left(\begin{array}{c} \text{sum of all ways of gluing} \\ \text{some legs of } \mu(\boldsymbol{C}) \text{ with } \tfrac{1}{2} W_{\varsigma^c} \end{array}\right),$$

where $\boldsymbol{C}_I = ((C_i)_{i \in I})$ and $\equiv_n$ means "equal up to i–degree at least $n+1$ terms".

Note that $W_\gamma = W_{\varsigma \setminus \varsigma^c}$ by [Corollary 4.6](#).

For a cobordism with one path, the next result gives a more intrinsic version of these formulas, which does not refer to a decomposition of the cobordism. A similar result is given by in [20, Lemma 4.1] for a cobordism with no path.

Given a null LP–surgery $\boldsymbol{C} = (C_1, \ldots, C_n)$ on a Lagrangian cobordism with paths (M, K), define a hermitian form $\ell_{(M,K)}(\boldsymbol{C}): H_1(\mathcal{C}; \mathbb{Q}) \times H_1(\mathcal{C}; \mathbb{Q}) \to \mathbb{Q}(t)$ in the same way as $\ell_{(S,\kappa)}(\boldsymbol{C})$ was defined in the introduction. Also define a map

$$\rho_{\boldsymbol{c}}: \mathcal{A}_{\mathbb{Q}}(H_1(\mathcal{C}; \mathbb{Q})) \to \widetilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\lfloor g \rceil^+ \cup \lfloor f \rceil^-})$$

which changes the labels of the univalent vertices by first sending them in $H_1(M; \mathbb{Q})$ via $H_1(\mathcal{C}; \mathbb{Q}) \cong \bigoplus_{i=1}^n H_1(C_i; \mathbb{Q}) \to H_1(M; \mathbb{Q})$, and then writing them in terms of the $[m_+(\beta_i)]$ and $[m_-(\alpha_i)]$.

A direct adaptation of (the end of) [20, Section 4.4] gives:

Proposition 6.4 Let $(M, K) \in \widetilde{\mathcal{LCob}}_q(w, v)$ be a Lagrangian q–cobordism with one path. Let $\boldsymbol{C} = (C_1, \ldots, C_n)$ be a null LP–surgery on (M, K). Let (B, K, γ) be the bottom–top tangle with one path associated with (M, K). Then

$$\sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|} \widetilde{Z}((M, K)(\boldsymbol{C}_I))$$
$$\equiv_n \exp_{\sqcup}\left(\tfrac{1}{2} W_\gamma\right) \sqcup \rho_{\boldsymbol{c}}\left(\begin{array}{c} \text{sum of all ways of gluing} \\ \text{some legs of } \mu(\boldsymbol{C}) \text{ with } \tfrac{1}{2}\ell_{(M,K)}(\boldsymbol{C}) \end{array}\right).$$

Proof of Theorem 1.1 Use Propositions 5.10 and 6.4. The strut part disappears since we deal with a cobordism from F_0 to F_0. The map $\rho_{\boldsymbol{c}}$ kills all terms with at least one univalent vertex since a Lagrangian cobordism from F_0 to F_0 has trivial first homology group over $\mathbb{Q}$. □

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