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Hofer–Zehnder capacity and Bruhat graph

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We find bounds for the Hofer–Zehnder capacity of spherically monotone coadjoint orbits of compact Lie groups with respect to the Kostant–Kirillov–Souriau symplectic form in terms of the combinatorics of their Bruhat graphs. We show that our bounds are sharp for coadjoint orbits of the unitary group and equal in that case to the diameter of a weighted Cayley graph.

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1 Introduction

The Gromov nonsqueezing theorem in symplectic geometry states that is not possible to embed symplectically a ball into a cylinder of smaller radius, although this can be done with a volume-preserving embedding; see Gromov [10]. Hence, the biggest radius of a ball that can be symplectically embedded into a symplectic manifold can be used as a way to measure the "symplectic size" of the manifold. We call the square of this radius times the number π the *Gromov width* of the symplectic manifold.

The Gromov width as a symplectic invariant is extended through the notion of *symplectic capacity* whose axiomatic formulation is due to Ekeland and Hofer [4; 5]. An important example of capacity is the *Hofer–Zehnder capacity* [14]. The Hofer–Zehnder capacity of a closed symplectic manifold (M,ω) is defined as

$$\mathrm{c}_{\mathrm{HZ}}(M,\omega) := \sup \left\{ \max H - \min H \;\middle|\; \begin{array}{c} H\colon M \to \mathbb{R},\ \text{all periodic trajectories} \\ \text{of the Hamiltonian vector field } X_H \\ \text{of period} < 1 \text{ are constant} \end{array} \right\}.$$

In comparison with the Gromov width, the Hofer–Zehnder capacity measures the size of a symplectic manifold in a Hamiltonian dynamic way.

In this paper, we are interested in computing bounds for the Hofer–Zehnder capacity of coadjoint orbits of compact Lie groups. Recall that given a compact Lie group G, it acts on its dual Lie algebra $\mathfrak{g}^*$ by the coadjoint representation. For $\lambda \in \mathfrak{g}^*$, the coadjoint orbit $\mathcal{O}_\lambda$ passing through λ is endowed with a canonical symplectic form ω_λ, called

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the Kostant–Kirillov–Souriau form, and we bound the Hofer–Zehnder capacity of $\mathcal{O}_\lambda$ with respect to this symplectic form.

We express our bounds for the Hofer–Zehnder capacity in the language of compact simple Lie groups theory. We denote by $T \subset G$ a maximal torus, R a system of roots relative to T, S a choice of simple roots and $W := N_G(T)/T$ the Weyl group. We say that a root $\beta \in R$ is positive if there exist nonnegative integers $n_{\beta\alpha}$ such that

$$\beta = \sum_{\alpha \in S} n_{\beta\alpha}\alpha.$$

We always identify the Lie algebra $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ via an adjoint invariant inner product that is compatible with the naturally defined pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. The main result of this paper is the following theorem:

Main Theorem *Let G be a compact simple Lie group and $(\mathcal{O}_\lambda, \omega_\lambda)$ be a spherically monotone coadjoint orbit passing through $\lambda \in \mathfrak{t}^*$. There exist pairwise orthogonal positive roots $\alpha_1, \dots, \alpha_r \in R$ such that the longest element w_0 of W can be written as*

$$w_0 = s_{\alpha_1} \cdots s_{\alpha_r}.$$

In terms of this decomposition, the Hofer–Zehnder capacity of $(\mathcal{O}_\lambda, \omega_\lambda)$ satisfies the inequalities

$$\max_{\alpha \in S} \left\{ \sum_{k=1}^{r} \frac{n_{\alpha_k\alpha}}{n_{\rho\alpha}} |\langle \lambda, \check{\alpha}_k \rangle| \right\} \le c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \le \sum_{k=1}^{r} |\langle \lambda, \check{\alpha}_k \rangle|,$$

where ρ denotes the highest positive root of R.

In the proof of the nonsqueezing theorem, Gromov noted that the Gromov width of a symplectic manifold is constrained by the existence of pseudoholomorphic curves [10]. The relation between pseudoholomorphic curves and the Hofer–Zehnder capacity was also observed by several authors in the context of the Weinstein conjecture (see eg Floer, Hofer and Viterbo [7], Hofer and Viterbo [13] and G Liu and G Tian [17]). This relation appears more explicit in a theorem of G Lu that bounds the Hofer–Zehnder capacity of a symplectic manifold when it has a nonvanishing Gromov–Witten invariant with two point constraints [19; 20].

Through this paper, we use Lu's theorem to bound from above the Hofer–Zehnder capacity of coadjoint orbits of compact Lie groups. We assume that coadjoint orbits with their Kostant–Kirillov–Souriau form are spherically monotone as we work with the definition of Gromov–Witten invariants defined in this category. It is important to notice that the main theorem can be extended to general coadjoint orbits of compact Lie groups as far as their symplectic Gromov–Witten invariants agree with naive curve-counting. This last statement is believed true by the symplectic community but it has not been fully established in the literature.

We compute Gromov–Witten invariants using a result of Fulton and Woodward that relates the degrees of nonvanishing Gromov–Witten invariants of a coadjoint orbit with paths in its Bruhat graph [9]. The *Bruhat graph* (also known in the literature as *moment graph* or *GKM graph*) of a coadjoint orbit is the graph whose vertices and edges are in one-to-one correspondence with the points and irreducible curves of the coadjoint orbit that are invariant with respect to the action of a maximal torus.

One of the main goals of the present paper is to bound the Hofer–Zehnder capacity of a coadjoint orbit in terms of its *Bruhat graph*. For instance, if we weight the edges of the Bruhat graph of a coadjoint orbit with the symplectic area of the curves that they represent, then the right-hand side of the inequality appearing in the main theorem can be reinterpreted as the inequality

$$\mathrm{c_{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq \text{diameter of the weighted Bruhat graph of } (\mathcal{O}_\lambda, \omega_\lambda).$$

We show that the previous inequality is sharp for coadjoint orbits of the unitary group. The unitary group $U(n)$ acts on its Lie algebra $\mathfrak{u}^*(n)$ by conjugation. According to the spectral theorem, every orbit of this action is parametrized by $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{R}^n$ and it is of the form

$$\mathcal{H}_\lambda := \{A \in \mathfrak{u}(n) : \text{spectrum } A = i\lambda\}.$$

We can identify $\mathcal{H}_\lambda$ with a coadjoint orbit of $U(n)$ via the inner product

$$\mathfrak{u}(n) \times \mathfrak{u}(n) \to \mathbb{R}, \quad (A, B) \mapsto -\mathrm{Trace}(AB),$$

and endow $\mathcal{H}_\lambda$ with a symplectic form ω_λ coming from the Kostant–Kirillov–Souriau form defined on the coadjoint orbit. The main theorem for $(\mathcal{H}_\lambda, \omega_\lambda)$ can be stated as follows:

Theorem *Consider the weighted Cayley graph of the symmetric group S_n where two permutations are joined by an edge of weight $|\lambda_i - \lambda_j|$ if and only if they differ by a transposition (i, j). If $(\mathcal{H}_\lambda, \omega_\lambda)$ is spherically monotone, then*

$$\mathrm{c_{HZ}}(\mathcal{H}_\lambda, \omega_\lambda) = \frac{1}{2}\sum_{k=1}^{n} |\lambda_k - \lambda_{n-k+1}| = \textit{diameter of the weighted Cayley graph of } S_n.$$

The Gromov width of a (not necessarily spherically monotone) coadjoint orbit of the unitary group is equal to the smallest weight of the Cayley graph defined in the previous theorem (see eg Caviedes [3] and Pabiniak [23]). In particular, the Hofer–Zehnder capacity of a coadjoint orbit isomorphic with a projective space coincides with its Gromov width and the Hofer–Zehnder capacity of a coadjoint orbit isomorphic with a Grassmannian manifold is equal to an integer multiple of its Gromov width, and we recover results of Hofer and Viterbo for the projective space [13] and Lu for the Grassmannian manifold [19]. The Gromov width of a coadjoint orbit of the unitary group not isomorphic to a projective space is always strictly smaller than its Hofer–Zehnder capacity.

We suggest that the reader compares our results with the ones of Loi, Mossa and Zuddas [18], where they estimate the Hofer–Zehnder capacity of Hermitian symmetric spaces and with the ones of Hwang and Suh [15], where they compute the Hofer–Zehnder capacity of symplectic manifolds with Hamiltonian semifree circle actions in terms of their moment map.

This paper is organized as follows: In Sections 2 and 3, we recall the definition of symplectic Gromov–Witten invariant that will be used through the text and the definition of Hofer–Zehnder capacity, and state Lu's theorem that bounds the Hofer–Zehnder capacity of a symplectic manifold in terms of its Gromov–Witten invariants. In Section 4, we recall background on the geometry of coadjoint orbits of compact Lie groups. In Section 5, we define the Bruhat graph and indicate its relation with the Hofer–Zehnder capacity of coadjoint orbits. In Section 6, we compute the Hofer–Zehnder capacity of coadjoint orbits of the unitary group. In Section 7, we recall results of Postnikov concerning the minimal degrees of paths in the Bruhat graph, and explain how they can be used to find more optimal upper bounds for the Hofer–Zehnder capacity of regular coadjoint orbits. In Section 8, we explain how to bound from below the Hofer–Zehnder capacity of a coadjoint orbit using the moment map of the Hamiltonian group action of a maximal torus. In Section 9, we write explicitly our bounds for every simple compact Lie group according to its type.

2 J–holomorphic curves

In this section we give a short review of J–holomorphic curves and the definition of Gromov–Witten invariants for spherically monotone symplectic manifolds. Most of the material presented here is adapted from McDuff and Salamon's book [21].

Let (M^{2n}, ω) be a compact symplectic manifold. We denote by $\mathcal{J}(M, \omega)$ the space of ω–compatible almost complex structures.

A symplectic manifold (M^{2n}, ω) is *spherically monotone* if there exists a real constant $\tau > 0$ such that, for all $A \in \pi_2(M)$,

$$c_1(A) = \tau\omega(A).$$

Here c_1 denotes the first Chern class of the bundle (TM, J), where J is any almost complex structure compatible with ω.

Let $(\mathbb{CP}^1, j)$ be the Riemann sphere with its standard complex structure j. Let $J \in \mathcal{J}(M, \omega)$. We call a map $u\colon \mathbb{CP}^1 \to M$ a *J–holomorphic curve of genus zero* or simply a *J–holomorphic sphere* or a *J–holomorphic curve* if

$$J \circ du = du \circ j.$$

A curve $u\colon \mathbb{CP}^1 \to M$ is *multiply covered* if it is the composition of a holomorphic branched covering map $(\mathbb{CP}^1, j) \to (\mathbb{CP}^1, j)$ of degree greater than one with a J–holomorphic map $\mathbb{CP}^1 \to M$. It is simple if it is not multiply covered.

Given $J \in \mathcal{J}(M, \omega)$ and $A \in H_2(M; \mathbb{Z})$, we define the *moduli space of simple J–holomorphic spheres of degree A* as

$$\mathcal{M}^*_A(M, J) := \{u\colon \mathbb{CP}^1 \to M \mid u \text{ is a simple } J\text{–holomorphic sphere and } u_*[\mathbb{CP}^1] = A\}.$$

A *simple J–holomorphic sphere with k–marked points* is a tuple $(u, z_1, \dots, z_k)$ consisting of a simple J–holomorphic sphere $u\colon \mathbb{CP}^1 \to M$ and k pairwise distinct marked points $z_i \in \mathbb{CP}^1$. The group $\mathrm{PSL}(2, \mathbb{C})$ acts on the set of J–holomorphic spheres with k–marked points via

$$\phi \cdot (u, z_1, \dots, z_k) = (u \circ \phi^{-1}, \phi(z_1), \dots, \phi(z_k)).$$

Given $A \in H_2(M; \mathbb{Z})$ and $k \in \mathbb{Z}_{>0}$, we denote by $\mathcal{M}^*_{A,k}(M, J)$ the moduli space of equivalence classes of simple J–holomorphic spheres with k–marked points $(u, z_1, \dots, z_k)$ with $u_*[\mathbb{CP}^1] = A$. The *evaluation map*

$$\mathrm{ev}^k_J = (\mathrm{ev}_1, \dots, \mathrm{ev}_k)\colon \mathcal{M}^*_{A,k}(M, J) \to M^k$$

is defined by

$$\mathrm{ev}^k_J[(u, z_1, \dots, z_k)] = (u(z_1), \dots, u(z_k)).$$

We denote by $\mathcal{J}_{\text{reg}}(M,\omega) \subset \mathcal{J}(M,\omega)$ the set of almost complex structures that are regular in the sense of McDuff and Salamon [21, Definition 3.1.4, Section 6.2, page 150]. If (M^{2n},ω) is spherically monotone, for a regular almost complex structure $J \in \mathcal{J}_{\text{reg}}(M,\omega)$, the evaluation map

$$\text{ev}_J^k\colon \mathcal{M}_{A,k}^*(M,J) \to M^k$$

defines a *pseudocycle* of dimension equal to

$$2n + 2c_1(A) + 2k - 6$$

(for the definition of pseudocycle, we refer the reader to [21, Section 6.5]). The pseudocycle $\text{ev}_J^k\colon \mathcal{M}_{A,k}^*(M,J) \to M^k$, up to cobordism, is independent of the regular almost complex structure $J \in \mathcal{J}_{\text{reg}}(M,\omega)$ (see eg [21, Chapter 6, Theorem 6.6.1]). If $H^*(M)$ denotes the free part of $H^*(M;\mathbb{Z})$ and $\pi_i\colon M^k \to M$ denotes the projection onto the i^{th} factor, the homomorphism

$$\text{GW}_{A,k}^{\text{symp}}\colon H^*(M)^{\otimes k} \to \mathbb{Z}$$

defined by

$$\text{GW}_{A,k}^{\text{symp}}(a_1,\dots,a_k) := (\pi_1^* a_1 \cup \dots \cup \pi_k^* a_k) \cdot \text{ev}_J^k$$

is independent of the almost complex structure $J \in \mathcal{J}_{\text{reg}}(M,\omega)$. The morphism $\text{GW}_{A,k}^{\text{symp}}$ is called the *k–pointed genus zero symplectic Gromov–Witten invariant of (M,ω) in the homology class A* (see eg [21, Theorem 7.1.1]).

3 Hofer–Zehnder capacity and Gromov–Witten invariants

Let (M,ω) be a closed symplectic manifold. A *Hamiltonian* is a smooth function $H\colon M \to \mathbb{R}$. The Hamiltonian vector field of H is the vector field X_H defined by

$$dH = \iota_{X_H}\omega.$$

The *oscillation* of a Hamiltonian $H\colon M \to \mathbb{R}$ is

$$\text{osc}\, H := \max H - \min H.$$

A Hamiltonian function $H\colon M \to \mathbb{R}$ is *slow* if all periodic orbits of the Hamiltonian vector field X_H of period less than one are constant. The *Hofer–Zehnder capacity* of (M,ω) is defined as

$$\text{c}_{\text{HZ}}(M,\omega) := \sup\{\text{osc}\, H \mid H\colon M \to \mathbb{R} \text{ slow}\}.$$

Remark We say that a Hamiltonian function $H\colon M \to \mathbb{R}$ is *admissible* if it is slow and in addition satisfies the following two properties:

- There is a compact set $K \subset M$ (depending on H) such that $H|_K \equiv M(H)$ is constant. If the symplectic manifold M is assumed to be open, we impose that $K \subset M \setminus \partial M$.
- There is an open set $U \subset M$ (depending on H) such that $H|_U \equiv m(H)$ is constant.
- For every $x \in M$,
$$m(H) \le H(x) \le M(H).$$

Usually the Hofer–Zehnder capacity of an arbitrary symplectic manifold (M,ω) is defined as
$$\mathrm{c}_{\mathrm{HZ}}(M,\omega) := \sup\{\operatorname{osc} H \mid H\colon M \to \mathbb{R} \text{ admissible}\}.$$

When the symplectic manifold (M,ω) is closed, both definitions coincide, as the oscillation of a slow Hamiltonian can be approximated with the oscillations of admissible Hamiltonians as follows: Let $H\colon M \to \mathbb{R}$ be a slow Hamiltonian. For small enough $\epsilon > 0$, let $f_\epsilon\colon [\min H, \max H] \to \mathbb{R}$ be a smooth function such that

- $0 \le f'(t) < 1$ for every $t \in [\min H, \max H]$,
- $f(t) = \min H$ for t near $\min H$,
- $f(t) = \max H - \epsilon$ for t near $\max H$.

The composition $H_\epsilon := H \circ f_\epsilon\colon M \to \mathbb{R}$ is an admissible Hamiltonian and
$$\lim_{\epsilon \to 0} \operatorname{osc} H_\epsilon = \operatorname{osc} H.$$

The following theorem, due to Lu, bounds from above the Hofer–Zehnder capacity of a closed symplectic manifold in terms of its Gromov–Witten invariants:

Theorem 3.1 (Lu [19]) *Let (M,ω) be a spherically monotone symplectic manifold. Suppose that there exists a spherical class $A \in H_2(M;\mathbb{Z})$, a nonnegative integer k and cohomology classes $a_3,\dots,a_k \in H^*(M)$ such that*
$$\mathrm{GW}^{\mathrm{symp}}_{A,k}(\mathrm{PD}[\mathrm{pt}], \mathrm{PD}[\mathrm{pt}], a_3,\dots,a_k) \ne 0;$$
then
$$\mathrm{c}_{\mathrm{HZ}}(M,\omega) \le \omega(A).$$

Remark Lu's theorem is stated for general symplectic closed manifolds and its proof relies on the Morse-theoretical definition of Gromov–Witten invariants given by Liu and Tian [17]. A Floer-theoretic proof of Lu's theorem has been obtained by Usher in [27], where he uses his deformed version of Oh–Schwarz spectral invariants and the construction of a virtual fundamental class on the compactification of moduli spaces of J–holomorphic curves endowed with Kuranishi structures.

4 Geometry of coadjoint orbits

In this section we establish the Lie-theoretical convention that is used through the rest of the paper. Most of the material can be found in the classical literature that is concerned about the geometry and topology of coadjoint orbits such as Bernstein, Gelfand and Gelfand [2] and Kirillov [16].

Let G be a compact Lie group, $\mathfrak{g}$ be its Lie algebra and $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. Let $(\cdot,\cdot)$ denote an adjoint invariant inner product defined on $\mathfrak{g}$. We identify the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ via this inner product. Let $\lambda \in \mathfrak{g}^*$ and $\mathcal{O}_\lambda \subset \mathfrak{g}^*$ be the coadjoint orbit passing through λ. Let ω_λ be the *Kostant–Kirillov–Souriau form* defined on $\mathcal{O}_\lambda$ by

$$\omega_\lambda(\hat{X}, \hat{Y}) = \langle \lambda, [X, Y] \rangle, \quad X, Y \in \mathfrak{g},$$

where $\hat{X}$ and $\hat{Y}$ are the vector fields on $\mathfrak{g}^*$ generated by the coadjoint action of G passing through X and Y, respectively. The form ω_λ is closed and nondegenerate, thus defining a symplectic structure on $\mathcal{O}_\lambda$.

We denote by $G_{\mathbb{C}}$ the complexification of the Lie group G. Let $P \subset G_{\mathbb{C}}$ be a parabolic subgroup of $G_{\mathbb{C}}$ such that $\mathcal{O}_\lambda \cong G_{\mathbb{C}}/P$. The quotient of complex Lie groups $G_{\mathbb{C}}/P$ allows us to endow $\mathcal{O}_\lambda$ with a complex structure J_0 compatible with ω_λ, so the triple $(\mathcal{O}_\lambda, \omega_\lambda, J_0)$ is a Kähler manifold. The almost complex structure J_0 is regular (see eg McDuff and Salamon [21, Proposition 7.4.3]).

Let $T \subset G$ be a maximal torus and $\mathfrak{t}$ denote its Lie algebra. Let $R \subset \mathfrak{t}^*$ be the root system of T, so

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g}_{\mathbb{C}} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{t}_{\mathbb{C}}\}$$

is the root space associated with the root $\alpha \in R$. Let $R^+ \subset R$ be a choice of positive roots with simple roots $S \subset R^+$. Let $B \subset G_{\mathbb{C}}$ be the Borel subgroup with Lie algebra

$$\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\alpha}.$$

Each root $\alpha \in R$ has a coroot $\check{\alpha} \in \mathfrak{t}$. The coroot $\check{\alpha}$ is identified with $2\alpha/(\alpha,\alpha) \in \mathfrak{t}$ via the invariant inner product $(\cdot,\cdot)$. The system of coroots is the set $\check{R} = \{\check{\alpha} : \alpha \in R\}$ and the simple coroots is the set $\check{S} = \{\check{\alpha} : \alpha \in S\}$.

Every root $\alpha \in R$ defines a reflection s_α on $\mathfrak{t}^*$ given by

$$s_\alpha\colon \mathfrak{t}^* \to \mathfrak{t}^*, \quad t \mapsto t - \langle t, \check{\alpha}\rangle \alpha,$$

where $\langle \cdot,\cdot\rangle$ denotes the standard pairing $\langle \cdot,\cdot\rangle\colon \mathfrak{t}^* \otimes \mathfrak{t} \to \mathbb{R}$. The group W generated by the set of reflections $\{s_\alpha\}_{\alpha \in R}$ is the *Weyl group* of G relative to T. Recall that the Weyl group W can be canonically identified with $N_G(T)/T$.

We can associate to the parabolic subgroup $P \subset G_{\mathbb{C}}$ a subset of simple roots defined by

$$S_P := \{\alpha \in S : \langle \lambda, \check{\alpha}\rangle \neq 0\}.$$

We denote by W_P the Weyl group of P generated by the set of simple roots S_P. The Weyl group W_P is identified with $N_P(T)/T$. Also, set $R_P := R \cap \mathbb{Z}S_P$ and $R_P^+ := R^+ \cap \mathbb{Z}S_P$, where $\mathbb{Z}S_P = \mathrm{span}_{\mathbb{Z}}(S_P)$.

The *Weyl chamber* relative to the set of simple roots S is the convex polyhedron

$$\mathfrak{t}^*_+ := \{\gamma \in \mathfrak{t}^* : \langle \gamma, \check{\alpha}\rangle \geq 0 \text{ for all } \alpha \in S\}$$

The vector space $\mathfrak{t}^*$ can be decomposed as the union of convex polyhedra

$$\mathfrak{t}^* = \bigcup_{w \in W} w(\mathfrak{t}^*_+),$$

whose interiors are disjoint. For the coadjoint orbit $\mathcal{O}_\lambda$, there exists $\lambda' \in \mathfrak{t}^*_+$ such that $\mathcal{O}_\lambda \cap \mathfrak{t}^* = \{w(\lambda')\}_{w \in W}$. We always assume in what follows that $\lambda \in \mathfrak{t}^*_+$.

For $w \in W$, the *length* $l(w)$ of w is defined as the minimum number of simple reflections $s_\alpha \in W$ with $\alpha \in S$ whose product is w. The Weyl group W has a unique longest element, which we denote by w_0.

Let $B^{\mathrm{op}} := w_0 B w_0 \subset G_{\mathbb{C}}$ be the *Borel subgroup opposite* to B. For $w \in W/W_P$, let $X(w) := \overline{BwP/P} \subset G_{\mathbb{C}}/P$ and $Y(w) := \overline{B^{\mathrm{op}} wP/P} \subset G_{\mathbb{C}}/P$ be the *Schubert* *variety* and the *opposite Schubert variety* associated with w, respectively. We denote by σ_w and $\check{\sigma}_w$ the fundamental classes in the homology group $H_*(G_{\mathbb{C}}/P;\mathbb{Z})$ of $Y(w)$ and $X(w)$, respectively. Note that $\check{\sigma}_w = \sigma_{w_o w} = \sigma_{\check{w}}$, where $\check{w} := w_0 w$. The set of Schubert classes $\{\sigma_w\}_{w\in W/W_P}$ forms a free $\mathbb{Z}$–basis of $H_*(G_{\mathbb{C}}/P;\mathbb{Z})$, and the set of Schubert classes $\{\check{\sigma}_w\}_{w\in W/W_P}$ is the dual basis of $\{\sigma_w\}_{w\in W/W_P}$ with respect to the Poincaré intersection pairing. The *Bruhat order* $\prec$ on W/W_P is defined by saying that $u\prec v$ if $X(u)\subset X(v)$.

5 Bruhat graph

In this section we define the Bruhat graph and indicate its relation with the Hofer–Zehnder capacity of a coadjoint orbit of a compact Lie group.

We keep the convention of the last section. The *Bruhat graph* is the graph on W/W_P where two elements are joined by an edge if they differ by one reflection. More precisely, there is an edge joining u with v if and only if there exists a positive root $\alpha\in R^+ - R_P^+$ such that

$$v = u\cdot s_\alpha \mod W_P.$$

We weight the edges of the Bruhat graph with elements in $\mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P$. If u and v differ by the reflection s_α, then the weight of the corresponding edge is $\check{\alpha}+\mathbb{Z}\check{S}_P$.

For instance, in Figure 1 we show the Bruhat graph of the Weyl group of $U(3)$. The standard set of simple roots of $U(3)$ is the set $\{\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3\}\subset\mathbb{R}^3$, where $\{e_1, e_2, e_3\}$ denotes the standard basis of $\mathbb{R}^3$. The Weyl group of $U(3)$ is the symmetric group S_3 generated by the simple reflections $s_1 := s_{e_1-e_2} = (1\,2)$ and $s_2 := s_{e_2-e_3} = (2\,3)$.

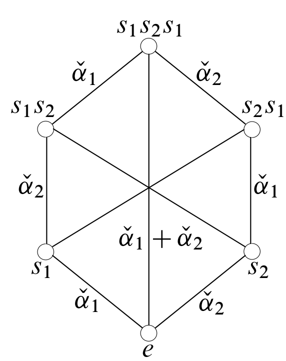

Figure 1: Bruhat graph of S_3.

We define a *chain* from u to v in W/W_P as a sequence $u_0, u_1, \dots, u_r \in W/W_P$ such that u_i and u_{i-1} are adjacent for $1 \le i \le r$, $u \prec u_0$ and $u_r \prec \check{v} = w_0 v$. The *weight* or *degree* of a chain is the sum of its weights. We call a *path* from u to v in W/W_P a sequence $u_0, u_1, \dots, u_r \in W/W_P$ such that u_i and u_{i-1} are adjacent for $1 \le i \le r$, $u = u_0$ and $u_r = v$. A path in the Bruhat graph coincides with the standard notion of path in graph theory. The weight or degree of a path is defined in the same way as the weight of a chain.

We define an ordering on $\mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P$ as follows: for $c, d \in \mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P$, we say that $c \le d$ if there exist $n_\alpha \in \mathbb{Z}_{\ge 0}$ such that

$$d - c = \sum_{\alpha \in S - S_P} n_\alpha \check{\alpha} \mod \mathbb{Z}S_P.$$

This ordering allows us to define chains and paths of minimal weight joining two vertices.

Although the Bruhat graph on W/W_P is defined purely in combinatorial terms, it has a geometric interpretation on $G_{\mathbb{C}}/P$. The vertices and edges of the Bruhat graph are in one-to-one correspondence with the points and irreducible curves that are invariant with respect to the action of the maximal torus. More precisely, the collection of cosets $\{wP\}_{w \in W/W_P}$ is the set of all T–fixed points of $G_{\mathbb{C}}/P$. On the other hand, for each positive root $\alpha \in R^+ - R_P^+$ there is a unique irreducible T–invariant curve C_α that contains $1 \cdot P$ and $s_\alpha \cdot P$. Indeed, $C_\alpha := \mathrm{Sl}(2,\mathbb{C})_\alpha \cdot P/P$, where $\mathrm{Sl}(2,\mathbb{C})_\alpha \subset G_{\mathbb{C}}$ is the subgroup of $G_{\mathbb{C}}$ with Lie algebra

$$\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{t}_{\mathbb{C}}^*.$$

Any other T–invariant curve is of the form $w \cdot C_\alpha$ for some $w \in W$ and $\alpha \in R^+ - R_P^+$. A chain from u to v in the Bruhat graph corresponds to a sequence of T–invariant curves $C_1, C_2, \dots, C_r$ with C_1 meeting $Y(u)$ and C_r meeting $X(\check{v})$. A weight in the Bruhat graph corresponds to a second homology class of $G_{\mathbb{C}}/P$ via the identification

$$\mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P \to H_2(G_{\mathbb{C}}/P;\mathbb{Z}), \quad \check{\alpha} + \mathbb{Z}S_P \mapsto [C_\alpha]$$

(see eg Fulton and Woodward [9] for more details).

Now we go back to our treatment of the Hofer–Zehnder capacity and Gromov–Witten invariants. Let J_0 be the invariant complex structure defined on the quotient of complex Lie groups $G_{\mathbb{C}}/P$. Let $A \in H_2(G_{\mathbb{C}}/P;\mathbb{Z})$ and $k \in \mathbb{Z}_{\ge 1}$. For cohomology classes

$a_1, \dots, a_k \in H^*(G_{\mathbb{C}}/P;\mathbb{Z})$ Poincaré dual to the fundamental classes of Schubert varieties $X_1, \dots, X_k$, the *algebraic Gromov–Witten invariant*

$$\mathrm{GW}^{\mathrm{alg}}_{A,k}(a_1, \dots, a_k)$$

is defined as the number of J_0–holomorphic spheres of degree A passing through $g_1X_1, \dots, g_kX_k$ for generic $g_1, \dots, g_k \in G_{\mathbb{C}}$ (see eg Fulton and Pandharipande [8, Lemma 14]). The group $G_{\mathbb{C}}$ acts transitively on $G_{\mathbb{C}}/P \cong \mathcal{O}_\lambda$ by J_0–holomorphic diffeomorphisms; as a consequence, $J_0 \in \mathcal{J}_{\mathrm{reg}}(\mathcal{O}_\lambda, \omega_\lambda)$ (see eg McDuff and Salamon [21, Proposition 7.4.3]). The evaluation map

$$\mathrm{ev}^k_{J_0}\colon \mathcal{M}^*_{A,k}(\mathcal{O}_\lambda, \omega_\lambda) \to \mathcal{O}^k_\lambda$$

defines a pseudocycle, and, under the assumption that $(\mathcal{O}_\lambda, \omega_\lambda)$ is spherically monotone, the symplectic Gromov–Witten invariant $\mathrm{GW}^{\mathrm{symp}}_{A,3}$ defined by

$$\mathrm{GW}^{\mathrm{symp}}_{A,k}(a_1, \dots, a_k) := \sharp\, \mathrm{ev}^k_{J_0} \pitchfork (X_1 \times \cdots \times X_k)$$

coincides with the algebraic Gromov–Witten invariant $\mathrm{GW}^{\mathrm{alg}}_{A,k}(a_1, \dots, a_k)$; in particular,

$$\mathrm{GW}^{\mathrm{symp}}_{A,k}(a_1, \dots, a_k) \geq 0$$

(see eg McDuff and Salamon [21, Propositions 7.4.3 and 7.4.5] and Fulton and Pandharipande [8, Lemma 14]).

The following result, due to Fulton and Woodward, establishes the relation between chains in the Bruhat graph of W/W_P and the algebraic Gromov–Witten invariants of $G_{\mathbb{C}}/P$. We relate $\mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P$ and $H_2(G_{\mathbb{C}}/P;\mathbb{Z})$ as above.

Theorem 5.1 (Fulton and Woodward [9]) *Let $u, v \in W/W_P$ and $d \in \mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P$. The following are equivalent*:

(1) *There is a chain of weight $c \leq d$ between u and v in the Bruhat graph of W/W_P.*

(2) *There exists a sequence $C_0, C_1, \dots, C_r$ of T–invariant curves with C_0 meeting $Y(u)$ and C_r meeting $X(\check{v})$, with C_{i-1} meeting C_i for $1 \leq i \leq r$, and with $\sum_{i=0}^r [C_i] \leq d$.*

(3) *There exist a degree $c \leq d$ and w in W/W_P such that*

$$\mathrm{GW}^{\mathrm{alg}}_{c,3}(\sigma_u, \sigma_v, \sigma_w) \neq 0.$$

Now we state the relation between chains in the Bruhat graph and the Hofer–Zehnder capacity of coadjoint orbits:

Theorem 5.2 *Assume that $(\mathcal{O}_\lambda, \omega_\lambda)$ is a spherically monotone coadjoint orbit. Then*

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \le \min_d \omega_\lambda(d),$$

where the minimum is taken over all the degrees $d \in H_2(\mathcal{O}_\lambda; \mathbb{Z}) \cong \mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P$ of paths joining the coset $1 \cdot W_P$ with the coset $w_0 \cdot W_P$ in the Bruhat graph of W/W_P.

Proof Let d be minimal among the set of all degrees of paths joining $1 \cdot W_P$ with $w_0 \cdot W_P$ in the Bruhat graph. According to Theorem 5.1, there exists $w \in W/W_P$ such that

$$\mathrm{GW}^{\mathrm{alg}}_{d,3}([\mathrm{pt}], [\mathrm{pt}], \sigma_w) \ne 0.$$

By Theorem 3.1, we conclude that

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \le \omega_\lambda(d),$$

as the algebraic Gromov–Witten invariant $\mathrm{GW}^{\mathrm{alg}}_{d,3}([\mathrm{pt}], [\mathrm{pt}], \sigma_w)$ coincides with the symplectic Gromov–Witten invariant $\mathrm{GW}^{\mathrm{symp}}_{d,3}([\mathrm{pt}], [\mathrm{pt}], \sigma_w)$. □

Remark In the previous theorem, we can drop the spherically monotone assumption if the algebraic Gromov–Witten invariant defined on complex homogeneous projective manifolds coincides with the symplectic Gromov–Witten invariants. Unfortunately, the equivalence of the various definitions of symplectic Gromov–Witten invariants and their algebraic counterparts has not been fully established in the literature for complex projective manifolds, although it is generally believed that they agree. For instance, Siebert showed in [26] that his definition of symplectic Gromov–Witten invariants coincides with the algebraic Gromov–Witten invariants given by Behrend [1].

As this issue is beyond the scope of this paper, we assume through the text that coadjoint orbits are spherically monotone with respect to the Kostant–Kirillov–Souriau form.

Remark A path in the Bruhat graph joining $1 \cdot W_P$ with $w_0 \cdot W_P$ is the same as an ordered sequence of positive roots $\alpha_1, \dots, \alpha_r \in R - R_P$ such that

$$w_0 = s_{\alpha_1} \cdots s_{\alpha_r} \mod W_P.$$

In this case, the path in the Bruhat graph is given by the sequence

$$1 \xrightarrow{\alpha_1} s_{\alpha_1} \xrightarrow{\alpha_2} s_{\alpha_1} s_{\alpha_2} \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{r-1}} s_{\alpha_1} \cdots s_{\alpha_{r-1}} \xrightarrow{\alpha_r} w_0.$$

The degree of the corresponding sequence of T–invariant curves is equal to

$$\sum_{i=1}^{r} \check{\alpha}_i .$$

The symplectic area of the curve C_α with respect to the Kostant–Kirillov–Souriau form is equal to $\langle \lambda, \check{\alpha} \rangle$ (see eg McDuff and Tolman [22, Lemma 3.9]), hence the symplectic area of the above sequence of T–invariant curves is equal to

$$\sum_{i=1}^{r} \langle \lambda, \check{\alpha}_i \rangle .$$

6 Hofer–Zehnder capacity of coadjoint orbits of $U(n)$

In this section we compute the Hofer–Zehnder capacity of coadjoint orbits of the unitary group. We denote by $U(n)$ the set of $n \times n$ unitary matrices and by $\mathfrak{u}(n)$ its Lie algebra. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ and

$$\mathcal{H}_\lambda := \{ A \in \mathfrak{u}(n) : A^* = -A, \text{ spectrum } A = i\lambda \}.$$

The unitary group $U(n)$ acts on $\mathcal{H}_\lambda$ by conjugation. We identify the set of skew-Hermitian matrices $\mathcal{H}_\lambda$ with a regular coadjoint orbit of $U(n)$ via the pairing

$$\mathfrak{u}(n) \times \mathfrak{u}(n) \to \mathbb{R}, \quad (X, Y) \mapsto -\mathrm{Trace}(XY).$$

We denote by ω_λ the symplectic form obtained by identifying $\mathcal{H}_\lambda$ with a coadjoint orbit of $U(n)$.

Let $T = U(1)^n \subset U(n)$ be the maximal torus of diagonal matrices in $U(n)$. We identify the Lie algebra of T with $\mathbb{R}^n$ and we denote by $\{e_1, \dots, e_n\}$ the standard basis of $\mathbb{R}^n$. The system of positive roots associated with the torus T is the set of vectors

$$\{\alpha_{i,j} := e_i - e_j\}_{1 \le i < j \le n} \subset \mathfrak{t} \cong \mathfrak{t}^*.$$

The standard system of simple roots is the set

$$\{\alpha_i := \alpha_{i,i+1}\}_{1 \le i < n}.$$

The corresponding Dynkin diagram is shown in Figure 2.

The Weyl group generated by the reflections

$$s_{\alpha_{ij}} \colon \mathbb{R}^n \to \mathbb{R}^n, \quad (x_1, \dots, x_i, \dots, x_j, \dots, x_n) \mapsto (x_1, \dots, x_j, \dots, x_i, \dots, x_n),$$

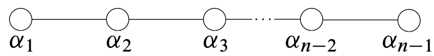

Figure 2: Dynkin diagram of A_{n-1}.

is the symmetric group S_n. The longest element w_0 in S_n relative to the set of simple reflections is the permutation

$$w_0\colon \mathbb{R}^n \to \mathbb{R}^n, \quad (x_1, \dots, x_n) \mapsto (x_n, \dots, x_1).$$

Any T–fixed point of $\mathcal{H}_\lambda$ is a permutation of the diagonal matrix $(i\lambda_1, \dots, i\lambda_n)$ and two T–fixed points of $\mathcal{H}_\lambda$ are joined by one irreducible T–invariant curve if and only if they differ by one transposition.

Theorem 6.1 *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ and assume that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. If $(\mathcal{H}_\lambda, \omega_\lambda)$ is spherically monotone, then*

$$c_{\mathrm{HZ}}(\mathcal{H}_\lambda, \omega_\lambda) = \frac{1}{2} \sum_{k=1}^{n} |\lambda_k - \lambda_{n-k+1}|.$$

Proof According to Theorem 5.1, in order to find an upper bound for the Hofer–Zehnder capacity of $\mathcal{H}_\lambda$, we want to find a path of irreducible T–invariant curves joining the diagonal matrix $i(\lambda_1, \lambda_2, \dots, \lambda_n)$ with the diagonal matrix $i(\lambda_n, \lambda_{n-1}, \dots, \lambda_1)$. Let us consider the path given by the sequence

$$\begin{aligned} i(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n) &\xrightarrow{(1,n)} i(\lambda_n, \lambda_2, \dots, \lambda_{n-1}, \lambda_1) \\ &\xrightarrow{(2,n-1)} \cdots \xrightarrow{([n/2],n-[n/2]+1)} i(\lambda_n, \lambda_{n-1}, \dots, \lambda_2, \lambda_1). \end{aligned}$$

The degree of this path is equal to

$$\sum_{k=1}^{[n/2]} \check{\alpha}_{k,n-k+1}$$

and its symplectic area is equal to

$$\omega_\lambda\biggl(\sum_{k=1}^{[n/2]} \check{\alpha}_{k,n-k+1}\biggr) = \sum_{k=1}^{[n/2]} \langle \lambda, \check{\alpha}_{k,n-k+1} \rangle = \frac{1}{2} \sum_{k=1}^{n} |\lambda_k - \lambda_{n-k+1}|,$$

and thus

$$c_{\mathrm{HZ}}(\mathcal{H}_\lambda, \omega_\lambda) \leq \frac{1}{2} \sum_{k=1}^{n} |\lambda_k - \lambda_{n-k+1}|.$$

Now we show that this inequality is sharp by constructing a slow Hamiltonian function $H\colon \mathcal{H}_\lambda \to \mathbb{R}$ whose oscillation is equal to the right-hand side of the inequality. The conjugation action of the torus T on $\mathcal{H}_\lambda$ is Hamiltonian with moment map given by

$$\mu\colon \mathcal{H}_\lambda \to i\mathbb{R}^n, \quad A \mapsto \text{diagonal}(A).$$

The image of the moment map μ is the convex hull of all possible permutations of the vector $i(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ (see eg Guillemin [11]).

For $t \in U(1)$ and $(m_1, \dots, m_n) \in \mathbb{Z}^n$, we use the convention

$$t^{(m_1,\dots,m_n)} := (t^{m_1}, \dots, t^{m_n}) \in T = U(1)^n \subset U(n).$$

Let

$$\beta = \sum_{k=1}^{[n/2]} (e_k - e_{n-k+1})$$

and $S = \{t^\beta = (t, t, \dots, t^{-1}, t^{-1}) : t \in S^1\} \subset T$. The action of the circle S on $\mathcal{H}_\lambda$ is Hamiltonian with moment map given by

$$\hat{\mu}\colon \mathcal{H}_\lambda \to i\mathbb{R}, \quad A = (a_{ij}) \mapsto \langle \mu(A), \beta\rangle = a_{1,1} - a_{n,n} + a_{2,2} - a_{n-1,n-1} + \cdots .$$

The moment map image of $\hat{\mu}$ is the interval

$$i\left[-\frac{1}{2}\sum_{k=1}^{n} |\lambda_k - \lambda_{n-k+1}|, \frac{1}{2}\sum_{k=1}^{n} |\lambda_k - \lambda_{n-k+1}|\right] \subset i\mathbb{R},$$

and thus the oscillation of $\operatorname{Im}(\hat{\mu})$ is equal to $\sum_{k=1}^n |\lambda_k - \lambda_{n-k+1}|$. Unfortunately, the function $\operatorname{Im}(\hat{\mu})$ is not slow. This is because, under the action of S on $\mathcal{H}_\lambda$, there are elements in $\mathcal{H}_\lambda$ with nontrivial finite stabilizers. All possible stabilizer subgroups of S are either $\{1\}$, $\mathbb{Z}_2$ or S. If the stabilizer subgroup of a skew-Hermitian matrix in $\mathcal{H}_\lambda$ is $\mathbb{Z}_2$, the period of the orbit passing through the skew-Hermitian matrix is one half. Otherwise the skew-Hermitian matrix is either a S–fixed point or the period of the orbit passing through the skew-Hermitian matrix is one.

The Hamiltonian function $H = \frac{1}{2}\operatorname{Im}(\hat{\mu})\colon \mathcal{H}_\lambda \to \mathbb{R}$ fixes this problem. The orbits of H are either constant or their periods are either one or two. Hence the Hamiltonian H is slow, and

$$\operatorname{osc}(H) = \frac{1}{2}\sum_{k=1}^{n} |\lambda_k - \lambda_{n-k+1}| \le c_{\mathrm{HZ}}(\mathcal{H}_\lambda, \omega_\lambda),$$

and we are done. □

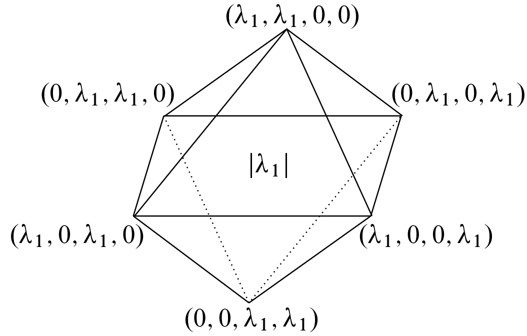

Figure 3: Bruhat graph of $\mathcal{H}_{(\lambda_1,\lambda_1,0,0)}$.

Remark The Hofer–Zehnder capacity of the coadjoint orbit $\mathcal{H}_\lambda$ is the same as the *diameter* of the weighted Cayley graph of S_n where two permutations are joined by an edge of weight $|\lambda_i - \lambda_j|$ if and only if they differ by a transposition (i, j). In this weighted Cayley graph, the distance between the identity permutation 1 and any other permutation σ is given by the expression

$$d(1,\sigma) = \frac{1}{2}\sum_{i=1}^{n} |\lambda_i - \lambda_{\sigma(i)}|,$$

and we have the rearrangement inequality

$$0 \le \frac{1}{2}\sum_i |\lambda_i - \lambda_{\sigma(i)}| \le \frac{1}{2}\sum_i |\lambda_i - \lambda_{n-i+1}|$$

(see eg Farnoud and Milenkovic [6, Theorem 22] for the expression and Rinott [25] and Vince [28, Example 2] for the inequality).

As an illustrative example, Figure 3 shows the Bruhat graph associated with the set of skew-Hermitian matrices $\mathcal{H}_{(\lambda_1,\lambda_1,0,0)}$. We weight the edges of the graph with the symplectic areas of the corresponding T–invariant curves. All the symplectic areas are equal to $|\lambda_1|$. The diameter of this weighted graph, that is, the Hofer–Zehnder capacity of $\mathcal{H}_{(\lambda_1,\lambda_1,0,0)}$, is equal to $2|\lambda_1|$.

7 Upper bounds for the Hofer–Zehnder capacity of regular coadjoint orbits

According to the previous section, we can bound from above the Hofer–Zehnder capacity of a coadjoint orbit of a compact Lie group by considering paths in its Bruhat graph. In order to achieve an optimal upper bound for the Hofer–Zehnder capacity through this method, we want to determine the paths of minimal degree joining the identity element and the longest element of the Weyl group. A theorem due to Postnikov states that the minimal degree of such paths is unique for *regular* coadjoint orbits of compact Lie groups [24]. Recall that a coadjoint orbit of a compact Lie group is regular if the stabilizer subgroup of any element in the coadjoint orbit is a maximal torus.

In this section, we recall Postnikov's theorem and its combinatorial formulation in terms of the *quantum Bruhat graph* as it is done in [24]. We also give a criterion that allow us to identify a path of minimal degree joining the identity element and the longest element in the Bruhat graph.

We follow the same convention as in Section 4. Let G be a compact Lie group. Let $T \subset G$ be a maximal torus and $B \subset G_{\mathbb{C}}$ be a Borel subgroup such that $T \subset B \subset G_{\mathbb{C}}$. Let R and S be the system of roots and simple roots determined by T and B, respectively. Let W be the corresponding Weyl group and w_0 be the longest element in W relative to S. For a positive root α write

$$\check{\alpha} = \sum_{\beta \in S} \check{n}_{\alpha\beta} \check{\beta}$$

for some nonnegative integers $\check{n}_{\alpha\beta}$ and define the height of $\check{\alpha}$ as

$$\mathrm{ht}(\check{\alpha}) := \sum_{\beta \in S} \check{n}_{\alpha\beta}.$$

The *quantum Bruhat graph of W* is a directed graph on the elements of the Weyl group with weighted edges defined as follows: two elements $u, v \in W$ are connected by a directed edge $u \to v$ if and only if $v = us_{\alpha}$ and

$$l(v) = l(u) + 1 \quad \text{or} \quad l(v) = l(u) + 1 - 2\,\mathrm{ht}(\check{\alpha}).$$

If $l(v) = l(u) + 1$ then the degree of the edge equals 0, and if $l(v) = l(u) + 1 - 2\,\mathrm{ht}(\check{\alpha})$ then the degree of the edge equals $\check{\alpha}$. Note that two vertices can be connected with edges going in both directions. The *degree* of a directed path in the quantum Bruhat graph of W is the sum of degrees of its edges. The *length* of a directed path is the number of edges that it uses. A directed path in the quantum Bruhat graph from u to v is *shortest* if it has the minimal possible length among all direct paths from u to v. Figure 4 shows the quantum Bruhat graph of S_3, following the same convention as in Figure 1.

Now we recall the following result about the combinatorics of paths in the quantum Bruhat graph of W:

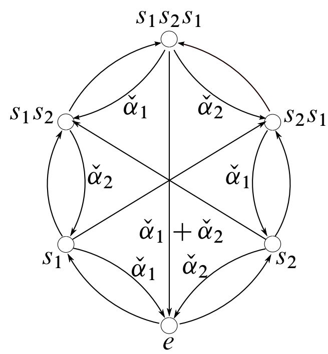

Figure 4: Quantum Bruhat graph of S_3.

Theorem 7.1 (Postnikov [24]) *Let u and v be any two Weyl group elements. There exists a directed path from u to v in the quantum Bruhat graph. All shortest paths from u to v have the same degree $d_{\min}(u,v)$. Moreover, there exists $w \in W$ such that*

$$\mathrm{GW}^{\mathrm{alg}}_{d_{\min}(u,\check v),3}(\sigma_u,\sigma_v,\sigma_w) \neq 0,$$

and $d_{\min}(u,\check v)$ is minimal with respect to this property, ie if there exists a degree d and $w \in W$ such that

$$\mathrm{GW}^{\mathrm{alg}}_{d,3}(\sigma_u,\sigma_v,\sigma_w) \neq 0$$

then

$$d_{\min}(u,\check v) \le d.$$

We consider the following lemma, whose proof we review for completeness:

Lemma 7.2 *For any positive root α, we always have that*

$$l(s_\alpha) \le 2\,\mathrm{ht}(\check\alpha) - 1.$$

Proof First notice that, for every simple root α,

$$\frac{1}{2}\sum_{\gamma\in R^+}\langle\gamma,\check\alpha\rangle = 1.$$

This is because

$$\sum_{\gamma\in R^+-\{\alpha\}}\langle\gamma,\check\alpha\rangle = \sum_{\gamma\in R^+-\{\alpha\}}\frac{\gamma - s_\alpha(\gamma)}{\alpha} = 0$$

and $\langle\alpha,\check\alpha\rangle = 2$. Thus, for any positive root α,

$$\mathrm{ht}(\check\alpha) = \frac{1}{2}\sum_{\gamma\in R^+}\langle\gamma,\check\alpha\rangle.$$

Now, for $w \in W$ let

$$I(w) := \{\beta \in R^+ : w(\beta) \in -R^+\}$$

be the set of *inversions* of w. Since s_α stabilizes the set $R^+ - I(s_\alpha)$,

$$\sum_{\gamma \in R^+ - I(s_\alpha)} \langle \gamma, \check{\alpha} \rangle = \sum_{\gamma \in R^+ - I(s_\alpha)} \frac{\gamma - s_\alpha(\gamma)}{\alpha} = 0.$$

For any root $\gamma \in I(s_\alpha)$ we have $s_\alpha(\gamma) = \gamma - \langle \gamma, \check{\alpha} \rangle \alpha < 0$, hence $\langle \gamma, \check{\alpha} \rangle \geq 1$ and

$$2\,\mathrm{ht}(\check{\alpha}) = \sum_{\gamma \in R^+} \langle \gamma, \check{\alpha} \rangle = \sum_{\gamma \in I(s_\alpha)} \langle \gamma, \check{\alpha} \rangle = 2 + \sum_{\gamma \in I(s_\alpha) - \{\alpha\}} \langle \gamma, \check{\alpha} \rangle \geq l(s_\alpha) + 1. \qquad \square$$

Let $l_T \colon W \to \mathbb{Z}_{\geq 0}$ denote the word length function defined on W with respect to the generating set of reflections $\{s_\alpha\}_{\alpha \in R^+}$, ie for $w \in W$, if we write $w = s_{\alpha_1} \cdots s_{\alpha_r}$ for some positive (not necessarily simple) roots $\alpha_1, \dots, \alpha_r$ and r is minimal with respect to this property, then $r = l_T(w)$. For $w \in W$, we call $l_T(w)$ the *absolute length* of w.

The following statement is a consequence of the previous lemma and Theorem 7.1. It can be used to determine when a decomposition of w_0 into a product of reflections gives rise to a shortest path joining 1 and w_0. We provide in the statement the corresponding upper bound for the Hofer–Zehnder capacity.

Theorem 7.3 *Let w_0 be the longest element in the Weyl group W with respect to the system of simple roots S. If there exist positive roots $\alpha_1, \dots, \alpha_r$ such that $w_0 = s_{\alpha_1} \cdots s_{\alpha_r}$ with $r = l_T(w_0)$ and*

$$\sum_{i=1}^{r} (2\,\mathrm{ht}(\check{\alpha}_i) - 1) = l(w_0) = |R^+|,$$

then

$$d_{\min}(w_0, 1) = \sum_{i=1}^{r} \check{\alpha}_i.$$

In addition, if λ is in the interior of the Weyl chamber relative to the set of simple roots S,

$$\min_d \omega_\lambda(d) = \sum_{i=1}^{r} \langle \lambda, \check{\alpha}_i \rangle,$$

where the minimum is taken over all degrees of **paths** *joining 1 with w_0 in the standard Bruhat graph of W. In particular we obtain the upper bound for the Hofer–Zehnder* *capacity of a spherically monotone regular coadjoint orbit* $(\mathcal{O}_\lambda, \omega_\lambda)$

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq \sum_{i=1}^{r} \langle \lambda, \check{\alpha}_i \rangle.$$

Proof For every $1 \leq i < r$, we have that

$$\begin{aligned} l(w_0) &\leq l(s_{\alpha_1} \cdots s_{\alpha_i}) + l(s_{\alpha_{i+1}} \cdots s_{\alpha_r}) \leq \sum_{j=1}^{i} l(s_{\alpha_j}) + \sum_{j=i+1}^{r} l(s_{\alpha_j}) \\ &\leq \sum_{j=1}^{i} (2\,\mathrm{ht}(\check{\alpha}_j) - 1) + \sum_{j=i+1}^{r} (2\,\mathrm{ht}(\check{\alpha}_j) - 1) = l(w_0). \end{aligned}$$

Thus, for every $1 \leq i < r$,

$$l(s_{\alpha_1} \cdots s_{\alpha_i}) = \sum_{j=1}^{i} (2\,\mathrm{ht}(\check{\alpha}_i) - 1).$$

In particular,

$$l(s_{\alpha_1} \cdots s_{\alpha_i}) = l(s_{\alpha_1} \cdots s_{\alpha_i} \cdot s_{\alpha_{i+1}}) + 1 - 2\,\mathrm{ht}(\check{\alpha}_{i+1}),$$

and

$$w_0 = s_1 \cdot s_2 \cdots s_r \to \cdots \to s_1 \cdot s_2 \to s_1 \to 1$$

represents a directed path in the quantum Bruhat graph from w_0 to 1. This path is a shortest path because of the minimality property of $r = l_T(w_0)$, and hence its degree equals $d_{\min}(w_0, 1)$. The upper bound for the Hofer–Zehnder capacity follows from Theorem 5.2. □

Remark In Section 9, we verify for every type that for the longest element w_0 in the Weyl group W there exist positive roots $\alpha_1, \ldots, \alpha_r$ such that $w_0 = s_{\alpha_1} \cdots s_{\alpha_r}$ with $r = l_T(w_0)$ and

$$\sum_{i=1}^{r} (2\,\mathrm{ht}(\check{\alpha}_i) - 1) = l(w_0).$$

Hence the assumptions made in the last theorem hold for any compact Lie group.

8 Lower bounds for the Hofer–Zehnder capacity and Hamiltonian torus actions

In this section we describe how to compute lower bounds for the Hofer–Zehnder capacity of a symplectic manifold with a Hamiltonian torus action using its moment map. By simplicity, we always assume that the points fixed by the torus action are isolated. We also estimate from below the Hofer–Zehnder capacity of coadjoint orbits of compact Lie groups, as was already done in the proof of Theorem 6.1 for coadjoint orbits of the unitary group.

Let T denote a torus. In what follows we always identify the Lie algebra $\mathfrak{u}(1)$ with $\mathbb{R}$. A *weight* of T is a Lie group morphism $\eta\colon T \to U(1)$. A *coweight* of T is a Lie group morphism $\xi\colon U(1) \to T$. Let $\Lambda \subset \mathfrak{t}$ be the kernel of the exponential map $\exp\colon \mathfrak{t} \to T$ and $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z})$ be its dual. The differential of any weight $\eta\colon T \to U(1)$ is a Lie algebra morphism $\mathfrak{t} \to \mathfrak{u}(1) \cong \mathbb{R}$ that takes Λ into $2\pi\mathbb{Z}$. Conversely, any group morphism $\Lambda \to 2\pi\mathbb{Z}$ arises in this way. Thus, the set of weights $X^*(T) := \mathrm{Hom}(T, U(1))$ can be identified with $2\pi\Lambda^* \subset \mathfrak{t}^*$. Similarly, the set of coweights $X_*(T) := \mathrm{Hom}(U(1), T)$ is identified with $\frac{1}{2\pi}\Lambda \subset \mathfrak{t}$. In this section, we always see weights and coweights as elements of $\mathfrak{t}^*$ and $\mathfrak{t}$, respectively. When we pair a coweight ξ with a weight η, we denote the composition $\eta \circ \xi$ by $\langle \eta, \xi \rangle$. We always identify $\mathrm{Hom}(U(1), U(1))$ with $\mathbb{Z}$.

The Schur lemma implies that for any representation V of T, we can write V as a direct sum

$$V = \bigoplus_{\eta \in X^*(T)} V_\eta,$$

where $V_\eta = \{v \in V \mid t \cdot v = \eta(t)v \text{ for all } t \in T\}$. We call an η such that $V_\eta \neq \{0\}$ a weight of the representation. Any coweight ξ of T defines a representation of $U(1)$ on V by pulling back the action of T on V and the weights of this representation are the set of integers $\{\langle \eta, \xi \rangle : \eta \in X^*(T)\}$.

Let (M^{2n}, ω) be a symplectic manifold with a Hamiltonian action of a torus T generated by a moment map $\phi\colon M \to \mathfrak{t}^*$. The critical points of ϕ are the fixed points of the torus action. In this section, we assume that the number of fixed points is finite.

For every fixed point $p \in M$, the *isotropy weights at* p are the weights $\eta_1, \dots, \eta_n$ such that the tangent space T_pM is linearly symplectomorphic to the action on $(\mathbb{C}^n, \omega_{\mathrm{st}})$ defined by

$$t \cdot (z_1, \dots, z_n) := (\eta_1(t)z_1, \dots, \eta_n(t)z_n)$$

and generated by the moment map

$$\mathbb{C}^n \to \mathfrak{t}^*, \quad (z_1, \dots, z_n) \mapsto \tfrac{1}{2}(|z_1|^2\eta_1 + \dots + |z_n|^2\eta_n).$$

An *isotropy weight of the torus action* is an isotropy weight of the torus action at some fixed point.

For $\xi \in \mathfrak{t}$, define

$$\phi^\xi\colon M \to \mathbb{R}, \quad p \mapsto \langle \phi(p), \xi \rangle.$$

We call $\xi \in \mathfrak{t}$ *generic* if $\langle \eta, \xi \rangle \neq 0$ for every isotropy weight η of the torus action. In this case, the function $\phi^\xi\colon M \to \mathbb{R}$ is Morse and its set of critical points coincides with the set of points fixed by the torus action.

Let ξ be a coweight of T. The coweight ξ defines a Hamiltonian circle action on M with moment map $\phi^\xi\colon M \to \mathbb{R}$. If the isotropy weights of the torus action at a fixed point are $\eta_1, \dots, \eta_n$, the isotropy weights of the circle action defined by the coweight ξ are the integers $\langle \eta_1, \xi \rangle, \dots, \langle \eta_n, \xi \rangle$. The coweight ξ is generic if, for every fixed point, all the integers $\langle \eta_1, \xi \rangle, \dots, \langle \eta_n, \xi \rangle$ are nonzero.

Theorem 8.1 *Let (M^{2n}, ω) be a compact symplectic manifold with a Hamiltonian circle action S^1 generated by a moment map $H\colon M \to \mathbb{R}$. Assume that the number of points in M fixed by the circle action is finite. Let $I \subset \mathbb{Z}$ be the set of all isotropy weights of the circle action and*

$$m^+ = \max_{m \in I} |m|.$$

Then the function

$$H' = \frac{1}{m^+} H\colon M \to \mathbb{R}$$

is slow and, in particular,

$$\operatorname{osc} H' \leq c_{\mathrm{HZ}}(M, \omega).$$

Proof We show that the stabilizer subgroup of S^1 at every nonfixed point of M is a cyclic subgroup of order less than or equal to m^+. This claim implies the theorem.

Let p be a fixed point of S^1 and $m_1, \dots, m_n$ be the isotropy weights at p. The equivariant Darboux theorem asserts that there is a neighborhood U of p in (M, ω) equivariantly symplectomorphic to a neighborhood V of the origin in $(\mathbb{C}^n, \omega_{\mathrm{st}})$ with the circle action defined by

$$t \cdot (z_1, \dots, z_n) = (t^{m_1} z_1, \dots, t^{m_n} z_n).$$

The stabilizer subgroup of $(z_1, \dots, z_n) \in V \setminus \{0\}$ is a cyclic group of order equal to $\gcd\{m_i : z_i \neq 0\}$. Note that $\gcd\{m_i : z_i \neq 0\}$ is less than or equal to m^+, and the claim holds at any point of the Darboux chart.

Finally, the stabilizer group of a point located anywhere in the manifold coincides with the stabilizer group of some point located at some equivariant Darboux chart of some fixed point (see eg Guillemin, Lerman and Sternberg [12, Lemma 3.3.2]). The statement follows from the analysis done in the previous paragraphs. □

Corollary 8.2 *Let (M^{2n},ω) be a compact symplectic manifold with a Hamiltonian torus action T generated by a moment map $H\colon M\to\mathfrak{t}^*$. Let us assume that the set of points in M fixed by the torus action is finite. Let $I\subset X^*(T)$ be the set of isotropy weights of the torus action. For any coweight $\xi\in X_*(T)$, define*

$$m_\xi^+ := \max_{\eta\in I} |\langle\eta,\xi\rangle|.$$

Then

$$\sup\left\{\frac{1}{m_\xi^+}\operatorname{osc}(\phi^\xi)\;\middle|\; \textit{generic } \xi\in X_*(T)\right\}\le c_{\mathrm{HZ}}(M,\omega).$$

We want to apply the previous corollary to bound from below the Hofer–Zehnder capacity of coadjoint orbits of compact Lie groups. We use the same convention as in Section 4. Let G be a compact *simple* Lie group. We identify the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ via an invariant inner product $(\cdot\,,\cdot)$. Let $T\subset G$ be a maximal torus and W be the corresponding Weyl group. Let R be the corresponding system of roots relative to T and S be a choice of simple roots. For any positive root α, we write

$$\alpha=\sum_{\beta\in S} n_{\alpha\beta}\beta$$

for some nonnegative integers $n_{\alpha\beta}$. A positive root β is called the *highest* if $\beta+\alpha$ is not a root for any simple root α. The existence and uniqueness of the highest positive root follows from the fact that G is simple.

In the next statement we give our lower bound for the Hofer–Zehnder capacity of coadjoint orbits (not necessarily spherically monotone). In the proof, we keep the notation of Corollary 8.2.

Theorem 8.3 *Let G be a compact simple Lie group. Let $\lambda\in\mathfrak{t}_+^*$ and $\mathcal{O}_\lambda$ be the coadjoint orbit passing through λ. Assume that the longest element w_0 in W can be decomposed as*

$$w_0=s_{\alpha_1}\cdots s_{\alpha_r},$$

where $\alpha_1, \dots, \alpha_r$ are pairwise orthogonal positive roots. Then

$$\max_{\alpha \in S} \left\{ \sum_{k=1}^{r} \frac{n_{\alpha_k \alpha}}{n_{\rho\alpha}} \langle \lambda, \check{\alpha}_k \rangle \right\} \le c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda),$$

where ρ denotes the highest positive root.

Proof The maximal torus T acts Hamiltonianly on $\mathcal{O}_\lambda$ with moment map

$$\phi\colon \mathcal{O}_\lambda \to \mathfrak{t}^*$$

equal to the composition of the projection map $\mathfrak{g}^* \to \mathfrak{t}^*$ with the inclusion map $\mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$.

The image of ϕ is the convex hull of $\{w(\lambda)\}_{w \in W}$. The set of all isotropy weights of the torus action of T on $\mathcal{O}_\lambda$ is a subset of the set of roots. More precisely, for $w \in W$, the weight decomposition of the tangent space of $\mathcal{O}_\lambda$ at $w(\lambda)$ is

$$T_{w(\lambda)}\mathcal{O}_\lambda = \bigoplus_{\alpha \in R^+ - R^+_P} \mathfrak{g}_{-w(\alpha)}.$$

Thus, the isotropy weights of the circle action defined by a coweight ξ are the set of integers

$$\{-\langle w(\alpha), \xi \rangle : \alpha \in R^+ - R^+_P,\ w \in W\}.$$

We call a coweight ξ *positive* if

$$\langle \alpha, \xi \rangle > 0$$

for every positive root α. We denote the set of positive coweights by $X_*(T)_+$. Every positive coweight ξ is generic by definition and as a consequence the function ϕ^ξ is Morse. We can always assume that a coweight is positive by taking a different system of simple roots if needed.

For a positive coweight ξ, the Morse index of $\phi^\xi\colon \mathcal{O}_\lambda \to \mathbb{R}$ at the critical point $\lambda \in \mathcal{O}_\lambda$ is the maximum possible and λ is a local maximum for ϕ^ξ. Indeed, λ is an absolute maximum for ϕ^ξ. Similarly, the absolute minimum of $\phi^\xi\colon \mathcal{O}_\lambda \to \mathbb{R}$ is achieved at $w_0(\lambda) \in \mathcal{O}_\lambda$. Note that the value

$$m^+_\xi = \max_{\alpha \in R - R_P,\, w \in W} |\langle w(\alpha), \xi \rangle|$$

is achieved when α is the highest positive root ρ.

Our orthogonality assumption implies that

$$w_0(\lambda) = s_{\alpha_1} \cdots s_{\alpha_r}(\lambda) = \lambda - \sum_{k=1}^{r} \langle \lambda, \check{\alpha}_k \rangle \alpha_k .$$

Thus, for a positive coweight ξ,

$$\operatorname{osc}(\phi^{\xi}) = \sum_{k=1}^{r} \langle \lambda, \check{\alpha}_k \rangle \langle \alpha_k, \xi \rangle .$$

Following Corollary 8.2, we want to maximize the expression

$$\frac{1}{\langle \rho, \xi \rangle} \operatorname{osc}(\phi^{\xi}) = \sum_{k=1}^{r} \frac{\langle \alpha_k, \xi \rangle}{\langle \rho, \xi \rangle} \langle \lambda, \check{\alpha}_k \rangle$$

for $\xi \in X_*(T)_+$. The right-hand side of the last equation is scale-invariant and continuous as a function of the variable ξ on the convex cone

$$C = \{\xi \in \mathfrak{t} \setminus \{0\} : \langle \alpha, \xi \rangle \geq 0 \text{ for all } \alpha \in R^+\}.$$

Thus,

$$\sup_{\xi \in X_*(T)_+} \left\{ \sum_{k=1}^{r} \frac{\langle \alpha_k, \xi \rangle}{\langle \rho, \xi \rangle} \langle \lambda, \check{\alpha}_k \rangle \right\} = \sup_{\xi \in C} \left\{ \sum_{k=1}^{r} \frac{\langle \alpha_k, \xi \rangle}{\langle \rho, \xi \rangle} \langle \lambda, \check{\alpha}_k \rangle \right\}.$$

The change of variable

$$y := \frac{\xi}{\langle \rho, \xi \rangle}$$

transforms our problem into the linear optimization problem

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_{k=1}^{r} \langle \alpha_k, y \rangle \langle \lambda, \check{\alpha}_k \rangle \\ \text{subject to} \quad & \langle \rho, y \rangle = 1, \\ & \langle \alpha, y \rangle \geq 0 \text{ for all } \alpha \in S. \end{aligned}$$

The hyperplane in $\mathfrak{t}$ defined by the equation $\langle \rho, y \rangle = 1$ cuts C into the polytope

$$\triangle = \{y \in \mathfrak{t} : \langle \rho, y \rangle = 1,\ \langle \alpha, y \rangle \geq 0 \text{ for all } \alpha \in S\}.$$

The maximum value of the linear expression $\sum_{k=1}^{r} \langle \alpha_k, y \rangle \langle \lambda, \check{\alpha}_k \rangle$ defined on the polytope $\triangle$ is obtained at some of its vertices. Equivalently, the maximum value of the expression

$$\sum_{k=1}^{r} \frac{\langle \alpha_k, \xi \rangle}{\langle \rho, \xi \rangle} \langle \lambda, \check{\alpha}_k \rangle$$

is obtained at some of the one-dimensional faces of C. Each one-dimensional face of C is spanned by some element in the basis dual to the basis of simple roots defined by the relation

$$(\tau_\alpha, \beta) = \delta_{\alpha,\beta} \quad \text{for any } \alpha, \beta \in S.$$

We conclude that

$$\sup_{\xi \in C} \left\{ \sum_{k=1}^{r} \frac{(\alpha_k, \xi)}{(\rho, \xi)} \langle \lambda, \check{\alpha}_k \rangle \right\} = \max_{\alpha \in S} \left\{ \sum_{k=1}^{r} \frac{(\alpha_k, \tau_\alpha)}{(\rho, \tau_\alpha)} \langle \lambda, \check{\alpha}_k \rangle \right\}.$$

By Corollary 8.2, we get that

$$\max_{\alpha \in S} \left\{ \sum_{k=1}^{r} \frac{(\alpha_k, \tau_\alpha)}{(\rho, \tau_\alpha)} \langle \lambda, \check{\alpha}_k \rangle \right\} = \max_{\alpha \in S} \left\{ \sum_{k=1}^{r} \frac{n_{\alpha_k \alpha}}{n_{\rho\alpha}} \langle \lambda, \check{\alpha}_k \rangle \right\} \le \mathrm{c}_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda). \qquad \square$$

Remark The previous statement is compatible with Theorem 5.2, ie with the same notation as in the previous theorem, we have that

$$\max_{\alpha \in S} \left\{ \sum_{k=1}^{r} \frac{n_{\alpha_k \alpha}}{n_{\rho\alpha}} \langle \lambda, \check{\alpha}_k \rangle \right\} \le \sum_{k=1}^{r} \langle \lambda, \check{\alpha}_k \rangle.$$

More generally, let (M, ω) be a symplectic manifold and assume that S^1 acts Hamiltonianly on (M, ω) with a moment map $H\colon M \to \mathbb{R}$. Assume that the fixed points of the circle action are isolated.

Let $p_0, p_1, \dots, p_n$ be a sequence of fixed points such that every consecutive pair p_i and p_{i+1} of critical points is joined by an S^1–invariant sphere S_i and the critical points p_0 and p_n are the minimum and maximum of H, respectively; see Figure 5.

If the isotropy weight of the circle action restricted to the sphere S_i at p_i is the integer k_i, then

$$|H(p_{i+1}) - H(p_i)| = k_i \omega(S_i)$$

(see eg McDuff and Tolman [22, Lemma 3.9]). Thus,

$$\begin{aligned} \operatorname{osc} H = H(p_n) - H(p_0) &\le \sum_i |H(p_{i+1}) - H(p_i)| = \sum_i k_i \omega(S_i) \\ &\le \max_i k_i \sum_i \omega(S_i) \le m^+ \sum_i \omega(S_i), \end{aligned}$$

and

$$\frac{1}{m^+} \operatorname{osc} H \le \sum_i \omega(S_i),$$

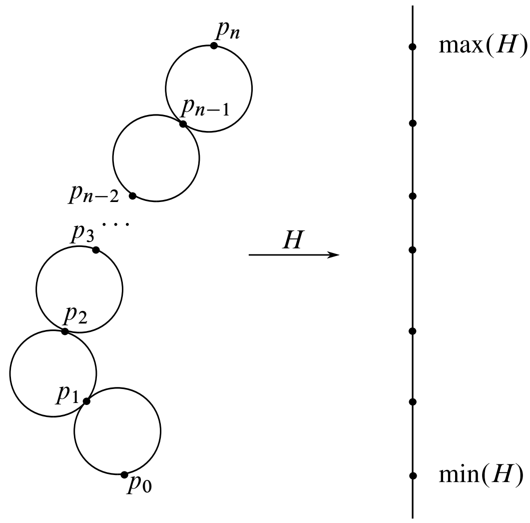

Figure 5

where m^+ is defined as in Theorem 8.1. Roughly speaking, when the symplectic manifold is a coadjoint orbit of a compact Lie group G and the Hamiltonian circle action comes from the restricted action of a circle subgroup $S^1 \hookrightarrow G$, the Hofer–Zehnder capacity of the coadjoint orbit is between the two values of the last inequality (although in principle the sequence of spheres described above should be invariant with respect to the action of a maximal torus containing the circle).

9 Computation of bounds for the Hofer–Zehnder capacity

In this section we show that the assumptions made in Theorems 7.3 and 8.3 hold for any Weyl group and we compute for any compact simple Lie group the corresponding bounds for the Hofer–Zehnder capacity of their coadjoint orbits.

We use the same convention as in Section 4. Let G be a compact *simple* Lie group and $T \subset G$ be a maximal torus. Let R be the set of roots associated with T and S be a choice of simple roots. We denote by ρ the highest positive root. We denote by W the corresponding Weyl group and w_0 the longest element of W relative to S.

Let $\lambda \in \mathfrak{t}^*_+$ and $\mathcal{O}_\lambda$ be the coadjoint orbit passing through λ with its Kostant–Kirillov–Souriau form ω_λ. In the following theorem, we summarize the main results of the paper:

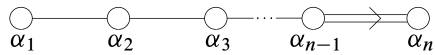

Figure 6: Dynkin diagram of B_n.

Theorem 9.1 *There exist positive roots $\alpha_1, \dots, \alpha_r$ pairwise orthogonal with respect to an invariant inner product such that $l_T(w_0) = r$,*

$$w_0 = s_{\alpha_1} \cdots s_{\alpha_r}$$

and

$$\sum_{i=1}^{r} (2\,\mathrm{ht}(\check{\alpha}_i) - 1) = l(w_0) = |R^+|.$$

In addition, we obtain the following bounds for the Hofer–Zehnder capacity of a spherically monotone coadjoint orbit $(\mathcal{O}_\lambda, \omega_\lambda)$:

$$\max_{\alpha \in S} \left\{ \sum_{k=1}^{r} \frac{n_{\alpha_k \alpha}}{n_{\rho\alpha}} \langle \lambda, \check{\alpha}_k \rangle \right\} \le c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \le \sum_{k=1}^{r} \langle \lambda, \check{\alpha}_k \rangle.$$

We split the proof of the previous statement in several cases according to the type of the Lie group G. We provide the described decomposition of w_0 and the corresponding lower and upper bound for the Hofer–Zehnder capacity of a regular spherically monotone coadjoint orbit $\mathcal{O}_\lambda$. We omit the detailed calculations, although we provide enough information so they can be verified by the reader.

Type *B*

The standard root system for the group $B_n = \mathrm{SO}(2n+1)$ is identified with the set of vectors $R = \{\pm e_i, \pm(e_j \pm e_k) : j \ne k\}_{1 \le i,j \le n} \subset \mathbb{R}^n$ with a choice of simple roots given by $S = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\}$. The Dynkin diagram of B_n is shown in Figure 6

The longest element w_0 of B_n as a map of $\mathbb{R}^n$ is the reflection

$$\mathbb{R}^n \to \mathbb{R}^n, \quad (x_1, \dots, x_n) \mapsto (-x_1, \dots, -x_n).$$

We have that $l_T(w_0) = n$, $l(w_0) = n^2$ and

$$w_0 = \begin{cases} s_{e_1-e_2} s_{e_1+e_2} s_{e_3-e_4} s_{e_3+e_4} \cdots s_{e_{n-1}-e_n} s_{e_{n-1}+e_n} & \text{if } n \text{ is even,} \\ s_{e_1-e_2} s_{e_1+e_2} s_{e_3-e_4} s_{e_3+e_4} \cdots s_{e_{n-2}-e_{n-1}} s_{e_{n-2}+e_{n-1}} s_{e_n} & \text{if } n \text{ is odd.} \end{cases}$$

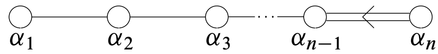

Figure 7: Dynkin diagram of C_n.

Hence,

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \le \begin{cases} 2\lambda_1 + 2\lambda_3 + \cdots + 2\lambda_{n-1} & \text{if } n \text{ is even,} \\ 2\lambda_1 + 2\lambda_3 + \cdots + 2\lambda_{n-2} + 2\lambda_n & \text{if } n \text{ is odd.} \end{cases}$$

The highest root ρ is $e_1 + e_2$ and

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \ge \max\{2\lambda_1, \lambda_1 + \lambda_2 + \cdots + \lambda_n\}.$$

Type C

The standard root system for the group $C_n = \mathrm{Sp}(n)$ is identified with the set of vectors $R = \{\pm 2e_i, \pm(e_j \pm e_k) : j \neq k\}_{1 \le i,j \le n} \subset \mathbb{R}^n$ with a choice of simple roots given by $S = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n\}$. The Dynkin diagram of C_n is shown in Figure 7

Combinatorially speaking, the Weyl group C_n is the same as the Weyl group B_n, however the edges of their Bruhat graphs have different degrees.

As an automorphism of $\mathbb{R}^n$, the longest element w_0 of C_n is the reflection

$$\mathbb{R}^n \to \mathbb{R}^n, \quad (x_1, \dots, x_n) \mapsto (-x_1, \dots, -x_n).$$

We have $l_T(w_0) = n$, $l(w_0) = n^2$ and

$$w_0 = s_{2e_1} s_{2e_2} \cdots s_{2e_n}.$$

Hence,

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \le \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

The longest root is $\rho = 2e_1$ and

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \ge \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

Hence, we get the sharp expression for coadjoint orbits of type C

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

Type D

The standard root system for the group $D_n = \mathrm{SO}(2n)$ is identified with the set of vectors $R = \{\pm(e_j \pm e_k) : j \neq k\}_{1 \le i,j \le n} \subset \mathbb{R}^n$ with a choice of simple roots given

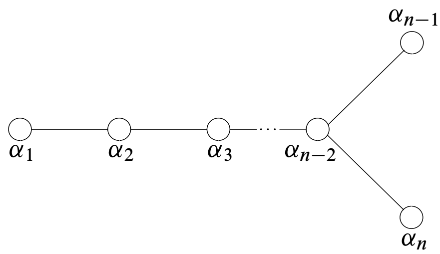

Figure 8: Dynkin diagram of D_n.

by $S = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}$. The Dynkin diagram of D_n is shown in Figure 8.

As a map of $\mathbb{R}^n$, the longest element w_0 is the map

$$\mathbb{R}^n \to \mathbb{R}^n, \qquad (x_1, \dots, x_{n-1}, x_n) \mapsto \begin{cases} (-x_1, \dots, -x_{n-1}, -x_n) & \text{if } n \text{ is even,} \\ (-x_1, \dots, -x_{n-1}, x_n) & \text{if } n \text{ is odd.} \end{cases}$$

We have that

$$l_T(w_0) = \begin{cases} n & \text{if } n \text{ is even,} \\ n-1 & \text{if } n \text{ is odd,} \end{cases}$$

$l(w_0) = n(n-1)$ and

$$w_0 = \begin{cases} s_{e_1-e_2} s_{e_1+e_2} s_{e_3-e_4} s_{e_3+e_4} \cdots s_{e_{n-1}-e_n} s_{e_{n-1}+e_n} & \text{if } n \text{ is even,} \\ s_{e_1-e_2} s_{e_1+e_2} s_{e_3-e_4} s_{e_3+e_4} \cdots s_{e_{n-2}-e_{n-1}} s_{e_{n-2}+e_{n-1}} & \text{if } n \text{ is odd.} \end{cases}$$

Hence,

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \le \begin{cases} 2\lambda_1 + 2\lambda_3 + \cdots + 2\lambda_{n-1} & \text{if } n \text{ is even,} \\ 2\lambda_1 + 2\lambda_3 + \cdots + 2\lambda_{n-2} & \text{if } n \text{ is odd.} \end{cases}$$

On the other hand, $\rho = e_1 + e_2$ and

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \ge \begin{cases} \max\{2\lambda_1, \lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + |\lambda_n|\} & \text{if } n \text{ is even,} \\ \max\{2\lambda_1, \lambda_1 + \lambda_2 + \cdots + \lambda_{n-1}\} & \text{if } n \text{ is odd.} \end{cases}$$

Type *E*

There are three isomorphism classes of compact simple Lie groups of type E: E_6, E_7 and E_8. We start first with E_8. A system of simple roots for E_8 as vectors in $\mathbb{R}^8$ is

$$S = \bigl\{\alpha_1 = \tfrac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8),\ \alpha_2 = e_1 + e_2,\ \alpha_3 = -e_1 + e_2,$$
$$\alpha_4 = -e_2 + e_3,\ \alpha_5 = -e_3 + e_4,\ \alpha_6 = -e_4 + e_5,\ \alpha_7 = -e_5 + e_6,\ \alpha_8 = -e_6 + e_7\bigr\}.$$

The Dynkin diagram of E_8 is shown in Figure 9.

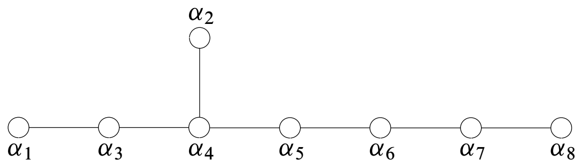

Figure 9: Dynkin diagram of E_8.

As a map of $\mathbb{R}^8$, the longest element of E_8 is the map

$$\mathbb{R}^8 \to \mathbb{R}^8, \quad (x_1, \dots, x_8) \mapsto (-x_1, \dots, -x_8),$$

and its absolute length and length are equal to 8 and 120, respectively. We can write the longest element as the composition of reflections $s_{r_1}, \dots s_{r_7}$ and s_{r_8}, where

$$r_1 = -e_1 + e_2, \quad r_2 = e_1 + e_2, \quad r_3 = -e_3 + e_4, \quad r_4 = e_3 + e_4,$$
$$r_5 = -e_5 + e_6, \quad r_6 = e_5 + e_6, \quad r_7 = -e_7 + e_8, \quad r_8 = e_7 + e_8.$$

The upper bound for the Hofer–Zehnder capacity of a regular coadjoint orbit $(\mathcal{O}_\lambda, \omega_\lambda)$ of E_8 is given by

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq 2\lambda_2 + 2\lambda_4 + 2\lambda_6 + 2\lambda_8.$$

The highest root equals to $e_7 + e_8$ and

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \geq \max\bigl\{2\lambda_8, \tfrac{1}{3}(\lambda_1 + \lambda_2 + \dots + \lambda_7 + 5\lambda_8)\bigr\}.$$

We have finished our analysis for E_8 and now we continue with the one for E_7. We keep the notation used in the previous paragraphs. A system of simple roots for E_7 is the set $\{\alpha_1, \alpha_2, \dots, \alpha_7\}$, so the Dynkin diagram of E_7 is contained in the Dynkin diagram of E_8. The longest element of E_7 is the map

$$\mathbb{R}^8 \to \mathbb{R}^8, \quad (x_1, \dots, x_6, x_7, x_8) \mapsto (-x_1, \dots, -x_6, x_8, x_7),$$

and its absolute length and length are equal to 7 and 63, respectively. We can write the longest element as the composition of the reflections $s_{r_1}, s_{r_2}, \dots, s_{r_7}$. Hence, the upper bound for the Hofer–Zehnder capacity of a regular coadjoint orbit of E_7 is

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq 2\lambda_2 + 2\lambda_4 + 2\lambda_6 + \lambda_8 - \lambda_7 = 2\lambda_2 + 2\lambda_4 + 2\lambda_6 - 2\lambda_7.$$

The highest root is $-e_7 + e_8$ and

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \geq \max\bigl\{2\lambda_6 - 2\lambda_7, \tfrac{1}{2}(\lambda_1 + \dots + \lambda_6 - 4\lambda_7)\bigr\}.$$

Finally, for E_6 a system of simple roots is $\{\alpha_1, \dots, \alpha_6\}$. The longest element of E_6 has absolute length and length equal to 4 and 36, respectively, and it can be written as the composition of the reflections $s_{t_1}, s_{t_2}, s_{t_3}$ and s_{t_4}, where

$$t_1 = -e_2 + e_3, \quad t_3 = \tfrac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8),$$
$$t_2 = -e_1 + e_4, \quad t_4 = \tfrac{1}{2}(-e_1 - e_2 - e_3 - e_4 + e_5 - e_6 - e_7 + e_8).$$

The upper bound for the Hofer–Zehnder capacity is given by

$$\begin{aligned} c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) &\le -\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - \lambda_6 - \lambda_7 + \lambda_8 \\ &= -\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - 3\lambda_6. \end{aligned}$$

The highest root is $\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8)$ and the lower bound for the Hofer–Zehnder capacity is

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \ge \lambda_5 - \lambda_6 - \lambda_7 + \lambda_8 = \lambda_5 - 3\lambda_6.$$

Type F

A system of simple roots for F_4 is

$$S = \left\{\alpha_1 = e_2 - e_3,\ \alpha_2 = e_3 - e_4,\ \alpha_3 = e_4,\ \alpha_4 = \tfrac{1}{2}(e_1 - e_2 - e_3 - e_4)\right\}.$$

The Dynkin diagram of F_4 is shown in Figure 10.

The longest reflection w_0 in F_4 as a map of $\mathbb{R}^4$ is

$$\mathbb{R}^4 \to \mathbb{R}^4, \quad (x_1, x_2, x_3, x_4) \mapsto (-x_1, -x_2, -x_3, -x_4).$$

We have that $l_T(w_0) = 4$, $l(w_0) = 24$ and

$$w_0 = s_{t_1} s_{t_2} s_{t_3} s_{t_4},$$

where

$$t_1 = e_1 + e_2, \quad t_2 = e_1 - e_2, \quad t_3 = e_3 + e_4, \quad t_4 = e_3 - e_4.$$

The upper bound for the Hofer–Zehnder capacity of a regular coadjoint orbit of type F_4 is

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \le 2\lambda_1 + 2\lambda_3.$$

The longest root of F_4 is $\rho = e_1 + e_2$. The lower bound for the Hofer–Zehnder capacity is

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \ge 2\lambda_1.$$

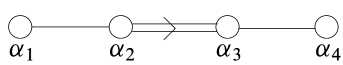

Figure 10: Dynkin diagram of F_4.

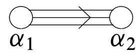

Figure 11: Dynkin diagram of G_2.

Type G

Finally, a system of simple roots for G_2 is

$$S = \{\alpha_1 = e_1 - 2e_2 + e_3,\ \alpha_2 = e_2 - e_3\} \subset \mathbb{R}^3$$

and Dynkin diagram shown in Figure 11

We write

$$w_0 = s_{t_1} s_{t_2},$$

where

$$t_1 = e_2 - e_3, \quad t_2 = 2e_1 - e_2 - e_3.$$

Hence,

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \le \tfrac{2}{3}(\lambda_1 - \lambda_2 - 2\lambda_3) = \tfrac{2}{3}(3\lambda_1 + \lambda_2).$$

The highest root is $\rho = 2e_1 - e_2 - e_3$, and

$$c_{\mathrm{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \ge \tfrac{2}{3}(2\lambda_1 + \lambda_2).$$

All the bounds for the Hofer–Zehnder capacity of spherically monotone coadjoint orbits are summarized in the following table:

G	lower bound	upper bound
$U(n)$	$\frac{1}{2}\sum_{i=1}^n \lvert\lambda_i - \lambda_{n-i+1}\rvert$	$\frac{1}{2}\sum_{i=1}^n \lvert\lambda_i - \lambda_{n-i+1}\rvert$
$\mathrm{Sp}(2n)$	$\lambda_1 + \cdots + \lambda_n$	$\lambda_1 + \cdots + \lambda_n$
$\mathrm{SO}(n)$		
$n = 4m$	$\lambda_1 + \cdots + \lvert\lambda_n\rvert, \quad 2\lambda_1$	$2\lambda_1 + 2\lambda_3 + \cdots + 2\lambda_{n-1}$
$4m+1$	$\lambda_1 + \cdots + \lambda_n, \quad 2\lambda_1$	$2\lambda_1 + 2\lambda_3 + \cdots + 2\lambda_{n-1}$
$4m+2$	$\lambda_1 + \cdots + \lambda_{n-1}, \quad 2\lambda_1$	$2\lambda_1 + 2\lambda_3 + \cdots + 2\lambda_{n-2}$
$4m+3$	$\lambda_1 + \cdots + \lambda_n, \quad 2\lambda_1$	$2\lambda_1 + 2\lambda_3 + \cdots + 2\lambda_n$
E_6	$\lambda_5 - \lambda_6 - \lambda_7 + \lambda_8$	$\lambda_3 + \lambda_4 + \lambda_5 - \lambda_1 - \lambda_2 - 3\lambda_6$
E_7	$\frac{1}{2}(\lambda_1 + \cdots - 4\lambda_7), \quad 2\lambda_6 - 2\lambda_7$	$2\lambda_2 + 2\lambda_4 + 2\lambda_6 - 2\lambda_7$
E_8	$\frac{1}{3}(\lambda_1 + \cdots + 5\lambda_8), \quad 2\lambda_8$	$2\lambda_2 + 2\lambda_4 + 2\lambda_6 + 2\lambda_8$
F_4	$2\lambda_1$	$2\lambda_1 + 2\lambda_3$
G_2	$\frac{2}{3}(2\lambda_1 + \lambda_2)$	$\frac{2}{3}(3\lambda_1 + \lambda_2)$

Remark Regardless of our bounds being sharp or not, we always get the inequality

$$\frac{2}{3}\sum_{k=1}^{r}\langle\lambda,\check{\alpha}_k\rangle \le c_{\mathrm{HZ}}(\mathcal{O}_\lambda,\omega_\lambda) \le \sum_{k=1}^{r}\langle\lambda,\check{\alpha}_k\rangle.$$

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